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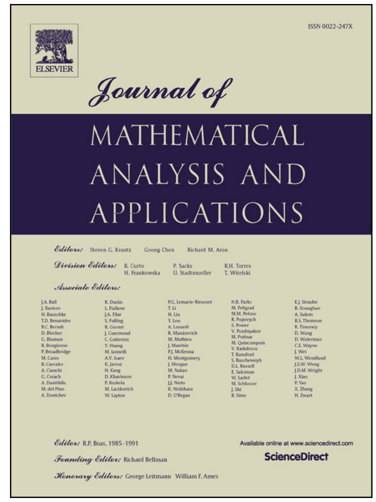

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UNIQUENESS OF POSITIVE SOLUTIONS TO SOME NONLINEAR NEUMANN PROBLEMS

YOUYAN WAN AND CHANG-LIN XIANG

ABSTRACT. Using the moving plane method, we obtain a Liouville type theorem for nonnegative solutions of the Neumann problem

$$\begin{cases} \operatorname{div}(y^a \nabla u(x,y)) = 0, & x \in \mathbb{R}^n, y > 0, \\ \lim\limits_{y \to 0+} y^a u_y(x,y) = -f(u(x,0)), & x \in \mathbb{R}^n, \end{cases}$$

under general nonlinearity assumptions on the function $f : \mathbb{R} \to \mathbb{R}$ for any constant $a \in (-1,1)$.

Keywords: Neumann problem; Liouville type theorem; Moving plane method

2010 Mathematics Subject Classification: 35J20 · 35J25 · 35J65

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Let $a \in (-1,1)$, $n \geq 1$ and $\mathbb{H}$ denote the upper half space

$$\mathbb{H} = \{(x,y) : x \in \mathbb{R}^n, y > 0\}.$$

In this paper, we consider the Neumann problem

$$\begin{cases} \operatorname{div}(y^a \nabla u(x,y)) = 0 & \text{in } \mathbb{H}, \\ \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial\mathbb{H}, \end{cases} \tag{1.1}$$

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where$f:\mathbb{R}\to[0,\infty)$ is a nonnegative function, $\nabla=(\partial_{x_1},\cdots,\partial_{x_n},\partial_y)$ is the full gradient operator in $\mathbb{H}$, and

$$\frac{\partial u}{\partial \nu^a}=-\lim_{y\to 0+}y^a\partial_y u(x,y).$$

Equation (1.1) has been studied extensively in the literature. Indeed, equation (1.1) is closely related to the fractional Laplacian equation

$$(-\Delta)^s u=f(u)\qquad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $(-\Delta)^s$ is the usual fractional Laplacian operator defined via its multiplier $|\xi|^{2s}$ in Fourier space, where we denote

$$s=(1-a)/2$$

throughout the paper. This connection has been highlighted by Caffarelli and Silvestre [4] and by related applications such as Cabré and Sire [2, 3], Frank and Lenzmann [15] and Frank et al. [16]. More precisely, let $y>0$ and let $P_y^a:\mathbb{R}^n\to\mathbb{R}$ be the kernel given by

$$P_y^a(x)=k_a y^{-n}\left(1+(|x|/y)^2\right)^{-(n+1-a)/2},\qquad x\in\mathbb{R}^n,$$

where the positive constant k_a is chosen such that $\int_{\mathbb{R}^n}P_y^a(x)\mathrm{d}x=1$. It was proven in Caffarelli and Silvestre [4] that for sufficiently regular function ϕ in $\mathbb{R}^n$ (e.g., ϕ belongs to the fractional Sobolev space $\dot{H}^s(\mathbb{R}^n)$), the function $\Phi:\mathbb{H}\to\mathbb{R}$ defined as

$$\Phi(x,y)=P_y^a*\phi(x)=\int_{\mathbb{R}^n}P_y^a(x-z)\phi(z)\mathrm{d}z,\qquad (x,y)\in\mathbb{H},$$

is an extension of ϕ to the upper half plane, such that $\lim_{y\to 0+}\Phi(x,y)=\phi(x)$ holds on $\partial\mathbb{H}$ in some sense. Moreover, Φ solves the boundary problem

$$\begin{cases}\operatorname{div}\left(y^a\nabla\Phi(x,y)\right)=0, & (x,y)\in\mathbb{H},\\ \partial\Phi/\partial\nu^a=d_s(-\Delta)^s\phi, & \text{on }\partial\mathbb{H},\end{cases}$$

with $d_s=2^{1-2s}\Gamma\left(1-s\right)/\Gamma\left(s\right)$. In particular, the following identity holds for all functions $\phi\in C_0^\infty(\overline{\mathbb{H}})$, the space of smooth functions on $\bar{\mathbb{H}}$ with compact support,

$$d_s\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\phi|^2\mathrm{d}x=\int_{\mathbb{H}}y^{1-2s}|\nabla\Phi(x,y)|^2\mathrm{d}x\mathrm{d}y.$$

Another equivalence of fractional Laplacian operators given in the form of difference quotients, such as

$$(-\Delta)^s u(x)=C_{n,s}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y, \tag{1.3}$$

are also used quite often in the literature, see e.g. Cabré and Sire [2, 3] and Chen et al. [9, 11].

On the other hand, nonlinear Neumann boundary value problems of type (1.1) have their own independent interests. In the case $a=0$, equation (1.1) is reduced to

$$\begin{cases}\Delta u(x,y)=0, & (x,y)\in\mathbb{H},\\ \frac{\partial u}{\partial\nu}=f(u) & \text{on }\partial\mathbb{H},\end{cases} \tag{1.4}$$

where Δ is the usual Laplacian operator in $\mathbb{R}^{n+1}$ and ν is the unit outward normal on $\partial\mathbb{H}$. Equation (1.4) has been studied considerably, see e.g. [13, 19, 24, 25, 26] and the references therein. In particular, Hu [19] established nonexistence results for positive solutions to equation (1.4) with $f(u) = u^p$ for $1 \le p < n/(n-1)$, which have found applications in the study of heat equations with nonlinear boundary condition in Hu and Yin [20]. Ou [26] exteneded the result of Hu [19] to the range $-\infty < p \le n/(n-1)$ by the moving plane method. Quite recently, Jin, Li and Xiong [21] studied equation (1.1) for all $a \in (-1,1)$ with $f(u) = u^{(n+2s)/(n-2s)}$ in the weak sense and classified all the positive solutions to equation (1.1). They also pointed out the nonexistence of positive solutions to equation (1.1) with $f(u) = u^p$ for $p < (n+2s)/(n-2s)$. The same results to [21] was also obtained by de Pablo and Sánchez [12] in the case $-1 < a < 0$ and $f(u) = u^p$ with $1 < p < (n+2s)/(n-2s)$.

1.2. **Main result.** In this paper, our aim is to extend above results in a more general setting. This is motivated naturally by the effort of gaining a better understanding on the role played by the nonlinear term $f(u)$ in problems of type (1.1). To state our main results, assume throughout the paper that $f : [0,\infty) \to [0,\infty)$ is a continuous function satisfying

(H1) $f(t)$ is nondecreasing for $t \ge 0$ and $f(0) = 0$; and

(H2) $g(t) \equiv f(t)/t^{2_s^*-1}$ is a nonincreasing function in $t > 0$, where

$$2_s^* = 2n/(n-2s)$$

is the so called fractional critical Sobolev exponent.

As examples, it is straightforward to verify that both functions $f(t) = t^p$ for $0 < p < 2_s^* - 1$, and $f(t) = \sum_{k=1}^m c_k t^{p_k}$ for $0 < p_k < 2_s^* - 1$, $c_k > 0$, satisfy (H1) and (H2).

Due to the regularity theory developed in Cabré and Sire [2], we can prove that weak solutions of problem (1.1) are also classical solutions. Thus, we will restrict ourselves to classical solutions of problem (1.1). Our result reads as follows.

Theorem 1.1. *Suppose that f satisfies (H1) and (H2), and that $u \ge 0$ is a classical solution to the Neumann problem (1.1). Then $u \not\equiv 0$ holds on $\mathbb{H}$ if and only if $g(t) \equiv f(t)/t^{2_s^*-1}$ is a constant function for $t > 0$, in which case the following hold:*

(1) If $g \equiv 0$, that is, $f(t) \equiv 0$ for all $t > 0$, then

$$u(x,y) = C_1 y^{1-a} + C_2, \qquad (x,y) \in \overline{\mathbb{H}},$$

for some nonnegative constants C_1, C_2 with $C_1^2 + C_2^2 > 0$.

(2) If $g \equiv g_0 > 0$ is a constant function, that is, $f(t) = g_0 t^{2_s^*-1}$ for all $t > 0$, then

$$u(x,y) = \left(P_y^a * u_0\right)(x), \qquad \forall (x,y) \in \overline{\mathbb{H}}, \tag{1.5}$$

where

$$u_0(x) = \left(\frac{cd}{d^2 + |x-x_0|^2}\right)^{\frac{n-2s}{2}}$$

for some $x_0 \in \mathbb{R}^n$, $d > 0$, and $c > 0$ is a constant depending only on n and g_0.

In view of the extension principle of Caffarelli and Silvestre [4] aforementioned, the following theorem is a direct consequence of of Theorem 1.1. Recall that $s = (1-a)/2$.

Theorem 1.2. *Assume that f satisfies the conditions (H1) and (H2). Then there exists a nontrivial nonnegative solution to the fractional Laplacian equation (1.2) in $\dot{H}^s(\mathbb{R}^n)$ if and only if $f(t) = Ct^{2_s^*-1}$ for $t > 0$ with some constant $C > 0$, in which case, solutions of equation (1.2) are of the form*

$$Q(x) = \left(\frac{b}{|x-x_0|^2+c}\right)^{(n-2s)/2}, \qquad x \in \mathbb{R}^n, \tag{1.6}$$

for some constants $b, c > 0$ and a point $x_0 \in \mathbb{R}^n$.

We remark that in Theorem 1.2, we restricted the solutions to the fractional Sobolev space $\dot{H}^s(\mathbb{R}^n)$, the completion of $C_0^\infty(\mathbb{R}^n)$ under the quadratic form

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 \mathrm{d}x = \int_{\mathbb{R}^n} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi.$$

This is due to the fact that solutions of problem (1.2) in $\dot{H}^s(\mathbb{R}^n)$ can be considered in the setting of problem (1.1), in view of the extension principle of Caffarelli and Silvestre [4]. As a matter of fact, Theorem 1.2 holds for nonnegative functions under far more general conditions, see e.g. [10, 11, 23]. In particular, in the most recent paper Chen, Li and Zhang [11], by introducing a direct method of moving spheres for fractional Laplacian operators given by the difference form (1.3), the authors proved that if $u \in L_s(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a nonnegative solution to equation (1.2), where $L_s(\mathbb{R}^n)$ is given by

$$L_s(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : (1+|x|)^{-n-2s} f \in L^1(\mathbb{R}^n)\},$$

and $f : (0, \infty) \to [0, \infty)$ is a locally bounded function satisfying (H2), then the results of Theorem 1.2 holds, see Theorem 2 of [11] for more details. However, since we do not need so general results in the present paper, we refer the interested readers to [10, 11, 23] for more details.

As another consequence, consider the variational problem

$$S = \inf\left\{\int_{\mathbb{H}} y^a |\nabla\phi(x,y)|^2 \mathrm{d}x\mathrm{d}y : \phi \in C_0^\infty(\bar{\mathbb{H}}), \int_{\partial\mathbb{H}} |\phi(x,0)|^{\frac{2n}{n-1+a}} \mathrm{d}x = 1\right\}. \tag{1.7}$$

The constant S in problem (1.7) is well defined, due to the trace inequality

$$\int_{\partial\mathbb{H}} |\phi(x,0)|^{\frac{2n}{n-1+a}} \mathrm{d}x \le C_{n,a} \int_{\mathbb{H}} y^a |\nabla\phi(x,y)|^2 \mathrm{d}x\mathrm{d}y, \qquad \forall\, \phi \in C_0^\infty(\bar{\mathbb{H}}),$$

where $C_{n,a} > 0$ is a constant depending only on n and a, see e.g. Frank et al. [15, 16]. In the case $a = 0$, minimizers of problem (1.7) was classified by Escobar [14], in which the author showed that minimizers are of the form (1.5) with $s = 1/2$, for some $x_0 \in \partial\mathbb{H}$, $y_0 < 0$. By Theorem 1.2 and the extension principle of Caffarelli and Silvestre [4], we have the following.

Corollary 1.3. *Minimizers of problem (1.7) are of the form (1.5) for all $-1 < a < 1$.*

To prove Theorem 1.1, we apply the famous moving plane method which was invented by the Soviet mathematician Alexanderoff in the early 1950s, and later developed by Serrin [27], Gidas, Ni and Nirenberg [17], Caffarelli, Gidas and Spruck [4], Li [22], Chen and Li [6, 7], Chang and Yang [5], Chen, Li and Ou [10], Li [23], Chen et al. [8, 9, 11] and many others. Now this method has been developed to study more classes of problems, such as integral systems, subelliptic equations on Heisenberg groups, see e.g. [1, 28], and even on fully nonlinear nonlocal problems (see e.g. [8]). In this paper, we will mainly use the moving plane method in integral form developed in [10]. We also combine some useful result in Li [23] so as to simplify the arguments.

2. Classifications of positive solutions

2.1. **Some basic facts and notations.** We collect some useful properties of equation (1.1) in this subsection. First we have the following comparison principle.

Lemma 2.1. *(Comparison principle) Let $\Omega \subset \mathbb{H}$ be an open set with a part of flat boundary $\Gamma \subset \partial\mathbb{H}$. Let $u \geq 0$, $u \not\equiv 0$, be a classical solution to equation*

$$\begin{cases} \operatorname{div}\left(y^a \nabla u(x,y)\right) = 0 & \text{in } \Omega, \\ \lim\limits_{y\to 0+} y^a u_y(x,y) \leq 0, & x \in \Gamma. \end{cases}$$

Then

$$u > 0 \qquad \text{on } \Omega \cup \Gamma.$$

Proof. The result holds in Ω by the maximum principle for uniform elliptic equations, see e.g. Gilbert and Trudinger [18]. We need to show that $u > 0$ holds on Γ. Fix an arbitrary point $X_0 = (x_0, 0) \in \Gamma$. Suppose that $\lim_{y\to 0+} y^a u_y(x_0, y) < 0$. By continuity, we have $u_y(x_0, y) < 0$ for (x_0, y) close to X_0 enough, which implies that $u(x_0, 0) \geq u(x_0, y) > 0$ if we choose y small enough. In the general case, choose $\phi(y) = y^{1-a}$. The function $u - \epsilon\phi$ is also a solution of the same equation in Ω but with boundary condition

$$y^a \left(u - \epsilon\phi\right)_y \leq -(1-a)\epsilon < 0 \quad \text{on } \Gamma.$$

Hence we deduce that $u(x,0) = (u - \epsilon\phi)(x,0) > u(x,y) - \epsilon y^{1-a}$ for sufficiently small y. Letting $\epsilon \to 0$ we obtain $u(X) \geq u(x,y) > 0$. The proof of Lemma 2.1 is complete. □

As an application of Lemma 2.1, we have the following corollary.

Corollary 2.2. *If $u \geq 0$ is a nontrivial classical solution to equation (1.1), then $u > 0$ on $\bar{\mathbb{H}}$.*

Next we introduce some notations that will be used in the proof of Theorem 1.1. Let $\lambda \in \mathbb{R}$ and $X = (x_1, x_2, \cdots, x_n, y) \in \mathbb{H}$. We write

$$\begin{aligned} T_\lambda &= \left\{X \in \mathbb{H} : x_1 = \lambda\right\}, \\ \Sigma_\lambda &= \left\{X \in \mathbb{H} : x_1 > \lambda\right\}, \\ p_\lambda &= (2\lambda, 0, \cdots, 0, 0) \in \partial\mathbb{H}, \\ X_\lambda &= \left(2\lambda - x_1, x_2, \cdots, x_n, y\right). \end{aligned}$$

We note that if u is a nonnegative solution to problem (1.1), then the function v defined by

$$v(X) = \frac{1}{|X|^{n-2s}} u\left(\frac{X}{|X|^2}\right), \qquad X = (x,y) \in \overline{\mathbb{H}} \backslash \{0\}, \tag{2.1}$$

is also nonnegative in $\mathbb{H}$ and satisfies

$$\begin{cases} \operatorname{div}\left(y^{1-2s}\nabla v(x,y)\right)=0 & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0+} y^{1-2s}v_y(x,y)=-g(|x|^{n-2s}v(x))v^{2_s^*-1}(x) & \text{on } \partial\mathbb{H}\backslash\{0\}. \end{cases} \tag{2.2}$$

Moreover v satisfies

$$\lim_{|X|\to\infty}|X|^{n-2s}v(X)=u(0). \tag{2.3}$$

Thus $v\in L^{\frac{q}{n-2s}}(\Sigma_\lambda)$ for any $n+1<q\le\infty$ and any $\lambda>0$. Since v seems to have better properties than that of u in the neighborhood of infinity, we turn to study the function v instead of u in the following. Remark that it is possible that v has singularity at $X=0$.

Furthermore, write $v_\lambda(X)=v(X_\lambda)$. It is straightforward to verify that $v_\lambda\ge 0$ solves the equation

$$\begin{cases} \operatorname{div}\left(y^{1-2s}\nabla v_\lambda(x,y)\right)=0 & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0+} y^{1-2s}\partial_y v_\lambda(x,y)=-g(|x_\lambda|^{n-2s}v_\lambda(x))v_\lambda^{2_s^*-1}(x) & \text{on } \partial\mathbb{H}\backslash\{p_\lambda\}. \end{cases} \tag{2.4}$$

Now let us start the proof of Theorem 1.1 with the following homogeneous case, which is also the simplest case that we can expect.

2.2. **Homogeneous case.** In this subsection we consider the case $f\equiv 0$. In this case the result of Theorem 1.1 can be viewed as an analogue of the classical Liouville theorem for nonnegative harmonic functions in Euclidean spaces. There are many different ways to study this homogeneous case, such as by Harnack type inequality. But here we prefer to use the moving plane method, since the essential point of the moving plane method is already contained in this case.

Fix $\lambda>0$ and let $0<2\epsilon<\lambda$. Choose a cut-off function $\eta_\epsilon\in C_0^\infty(\mathbb{R}^{n+1})$ such that $0\le\eta_\epsilon\le 1$ in $\mathbb{R}^{n+1}$, $\eta_\epsilon\equiv 1$ for $2\epsilon\le|X-p_\lambda|\le\epsilon^{-1}$ and $\eta_\epsilon=0$ for $|X-p_\lambda|\le\epsilon$ or $|X-p_\lambda|\ge 2\epsilon^{-1}$, $|\nabla\eta_\epsilon(X)|\le C\epsilon^{-1}$ for $\{\epsilon\le|X-p_\lambda|\le 2\epsilon\}$ and $|\nabla\eta_\epsilon(X)|\le C\epsilon$ for $\epsilon^{-1}\le|X-p_\lambda|\le 2\epsilon^{-1}$. Here $C>0$ is independent of ϵ. Multiply both sides of equations (2.2) and (2.4) by $\phi_\epsilon=(v-v_\lambda)_+\eta_\epsilon^2$. Here $c_+=\max\{c,0\}$. We deduce that

$$\begin{aligned} &\int_{\Sigma_\lambda\cap\{2\epsilon\le|X-p_\lambda|\le 1/\epsilon\}} y^a|\nabla(v-v_\lambda)_+|^2\mathrm{d}X \\ &\le\int_{\Sigma_\lambda} y^a|\nabla\left(\eta_\epsilon(v-v_\lambda)_+\right)|^2\mathrm{d}X \\ &=\int_{\Sigma_\lambda} y^a\nabla(v-v_\lambda)_+\cdot\nabla\phi_\epsilon\mathrm{d}X+\int_{\Sigma_\lambda} y^a(v-v_\lambda)_+^2|\nabla\eta_\epsilon|^2\mathrm{d}X \\ &=\int_{\Sigma_\lambda} y^a(v-v_\lambda)_+^2|\nabla\eta_\epsilon|^2\mathrm{d}X=:J. \end{aligned} \tag{2.5}$$

The last equality holds since we are considering the case $f \equiv 0$. Estimate J as below. Write $R_r = \{X \in \mathbb{H} : r \le |X - p_\lambda| \le 2r\}$ for $r > 0$. Then

$$\begin{aligned} J &\le C\epsilon^{-2} \int_{\Sigma_\lambda \cap R_\epsilon} y^a (v - v_\lambda)_+^2 \mathrm{d}X + C\epsilon^2 \int_{\Sigma_\lambda \cap R_{1/\epsilon}} y^a (v - v_\lambda)_+^2 \mathrm{d}X \\ &\le C\epsilon^{-2} \int_{R_\epsilon} y^a |v|^2 \mathrm{d}X + C\epsilon^2 \int_{R_{1/\epsilon}} y^a |v|^2 \mathrm{d}X, \end{aligned}$$

where $C > 0$ is independent of ϵ. For $\epsilon > 0$ sufficiently small, we derive from (2.3) that

$$\epsilon^{-2} \int_{R_\epsilon} y^a v^2 \mathrm{d}X \le C_\lambda \epsilon^{-2} \int_{\{X \in \mathbb{H} : |X - p_\lambda| \le 2\epsilon\}} y^a \mathrm{d}X = O(\epsilon^{n-2s}),$$

and that

$$\begin{aligned} \epsilon^2 \int_{\Sigma_\lambda \cap R_{1/\epsilon}} y^a v^2 \mathrm{d}X &\le C_\lambda \epsilon^2 \int_{\{X \in \mathbb{H} : 1/\epsilon \le |X - p_\lambda| \le 2/\epsilon\}} y^a |X|^{2(2s-n)} \mathrm{d}X \\ &\le C_\lambda \epsilon^{2+2(n-2s)} \int_{\{X \in \mathbb{H} : |X - p_\lambda| \le 2/\epsilon\}} y^a \mathrm{d}X \\ &= O(\epsilon^{n-2s}) \end{aligned}$$

for some constants $C_\lambda > 0$. Hence

$$J = O(\epsilon^{n-2s}) \to 0 \quad \text{as } \epsilon \to 0.$$

Therefore, combining above estimate for J and (2.5) yields that

$$v \le v_\lambda \qquad \text{in } \Sigma_\lambda.$$

Since $\lambda > 0$ is an arbitrary constant, we derive by sending $\lambda \to 0$ that

$$v(x_1, x_2, \cdots, x_n, y) \le v(-x_1, x_2, \cdots, x_n, y) \qquad \forall (x, y) \in \mathbb{H}.$$

Note that in the above arguments x_1 could denote any direction on $\partial\mathbb{H}$. We conclude that v is radially symmetric with respect to the variable $x \in \partial\mathbb{H}$. That is, $u = u(|x|, y)$. Moreover, since we can apply the Kelvin transform centered at any point of $\partial\mathbb{H}$, we infer from the same procedure that u is symmetric with respect to any point on $\partial\mathbb{H}$, which implies $u(x, y) = u(y)$ for all $(x, y) \in \mathbb{H}$. By substituting $u = u(y)$ into the equation, we obtain

$$u(x, y) = C_1 y^{1-a} + C_2, \qquad (x, y) \in \overline{\mathbb{H}}$$

for some constants $C_1, C_2 \ge 0$. This finishes the proof of Theorem 1.1 in the case $f \equiv 0$.

2.3. **Nonhomogeneous case.** Now we consider the case $f \not\equiv 0$. We divide the proof of Theorem 1.1 into several steps.

Step 1. We show that the procedure of moving plane can be started for sufficiently large λ. The essential idea is already contained in the proof for homogeneous case. We start with the following integral inequality.

Lemma 2.3. *For any fixed $\lambda > 0$, there holds*

$$\int_{\Sigma_\lambda} y^a |\nabla(v - v_\lambda)_+|^2 \mathrm{d}X \le C_\lambda \left(\int_{\partial\mathbb{H} \cap \partial A_\lambda} v^{2_s^*} \mathrm{d}x \right)^{\frac{2_s^*-2}{2_s^*}} \int_{\Sigma_\lambda} y^a |\nabla(v - v_\lambda)_+|^2 \mathrm{d}X, \tag{2.6}$$

where $A_\lambda = \{X \in \Sigma_\lambda : v(X) > v_\lambda(X)\}$, C_λ *is a constant which is bounded when* λ *is away from zero.*

Proof. Formula (2.6) is a consequence of Caccioppoli type inequality. Indeed, let $0 < \epsilon < \lambda$ and $\eta_\epsilon \in C_0^\infty(\mathbb{R}^{n+1})$ be given as in the homogeneous case. Multiply both sides of equations (2.2) and (2.4) by $\phi_\epsilon = (v - v_\lambda)_+ \eta_\epsilon^2$. We obtain that

$$\begin{aligned} &\int_{\Sigma_\lambda \cap \{2\epsilon \le |X - p_\lambda| \le \frac{1}{\epsilon}\}} y^a |\nabla (v - v_\lambda)_+|^2 \mathrm{d}X \\ \le\ & \int_{\Sigma_\lambda} y^a |\nabla \left(\eta_\epsilon (v - v_\lambda)_+\right)|^2 \mathrm{d}X \\ =\ & \int_{\Sigma_\lambda} y^a \nabla (v - v_\lambda)_+ \cdot \nabla \phi_\epsilon \mathrm{d}X + \int_{\Sigma_\lambda} y^a (v - v_\lambda)_+^2 |\nabla \eta_\epsilon|^2 \mathrm{d}X \\ =:\ & I + J. \end{aligned}$$

As in the previous subsection, we have

$$J = O(\epsilon^{n-2s}) \to 0 \qquad \text{as } \epsilon \to 0.$$

Thus, we only need to estimate I.

By equations (2.2) and (2.4), we have

$$\begin{aligned} I &= \int_{\Sigma_\lambda} y^a \nabla (v - v_\lambda)_+ \cdot \nabla \phi_\epsilon \\ &= \int_{\partial(\Sigma_\lambda \cap \mathrm{supp}\eta_\epsilon)} \phi_\epsilon y^a \nabla (v - v_\lambda)_+ \cdot \nu \\ &= \int_{\{x \in \mathbb{R}^n : x_1 > \lambda, \epsilon \le |x - p_\lambda| \le 2/\epsilon\}} \left(g(|x|^{n-2s} v(x)) v^{2_s^* - 1}(x) - g(|x_\lambda|^{n-2s} v_\lambda(x)) v_\lambda^{2_s^* - 1}(x) \right) \phi_\epsilon. \end{aligned}$$

Since $|x| \ge |x_\lambda|$ if $x_1 > \lambda$ and $v > v_\lambda$ on A_λ, and since g is nonincreasing by assumption, it follows that

$$\begin{aligned} I &\le \int_{\{x \in \mathbb{R}^n : x_1 > \lambda, \epsilon \le |x - p_\lambda| \le 2/\epsilon\}} g(|x|^{n-2s} v(x)) (v^{2_s^* - 1} - v_\lambda^{2_s^* - 1}) \phi_\epsilon \mathrm{d}x \\ &\le C'_\lambda \int_{\partial\mathbb{H} \cap \partial A_\lambda} v^{2_s^* - 2}(x) (v - v_\lambda)_+^2 \mathrm{d}x \\ &\le C'_\lambda \left(\int_{\partial\mathbb{H} \cap \partial A_\lambda} v^{2_s^*}(x) \mathrm{d}x \right)^{\frac{2_s^* - 2}{2_s^*}} \left(\int_{\partial\mathbb{H} \cap \partial\Sigma_\lambda} (v - v_\lambda)_+^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}}, \end{aligned}$$

where $C'_\lambda := (2_s^* - 1) \sup_{x_1 > \lambda} g(|x|^{n-2s} v(x))$. Recall that $|x|^{n-2s} v(x) \to u(0)$ as $|x| \to \infty$. So $C'_\lambda \to (2_s^* - 1) g(u(0)) > 0$ as $\lambda \to \infty$, which implies that C'_λ is bounded for λ being away from zero. By virtue of the trace inequality (1.7), we deduce

$$S \left(\int_{\partial\mathbb{H} \cap \partial\Sigma_\lambda} (v - v_\lambda)_+^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}} \le \int_{\Sigma_\lambda} y^a |\nabla (v - v_\lambda)_+|^2 \mathrm{d}X,$$

where S is the constant defined as in (1.7). Therefore,

$$I \le C_\lambda \left(\int_{\partial\mathbb{H} \cap \partial A_\lambda} v^{2_s^*}(x) \mathrm{d}x \right)^{\frac{2_s^* - 2}{2_s^*}} \int_{\Sigma_\lambda} y^a |\nabla (v - v_\lambda)_+|^2 \mathrm{d}X,$$

where C_λ is a positive constant which is bounded when λ is away from zero.

Finally, combining the estimate of I and J together and letting $\epsilon \to 0$, we obtain (2.6). The proof is complete. □

As a consequence of (2.6), we infer immediately that for $\lambda > 0$ large enough, there holds

$$C_\lambda \left(\int_{\partial H \cap \partial A_\lambda} v^{2_s^*} \right)^{2_s^*-2} \leq \frac{1}{2},$$

since $v(x,0) \in L^{2_s^*}(\partial \mathbb{H} \cap \partial \Sigma_\lambda)$. Hence, for λ large enough we deduce that

$$\int_{\Sigma_\lambda} y^a |\nabla (v - v_\lambda)_+|^2 = 0.$$

Thus for λ large enough we obtain

$$v \leq v_\lambda \qquad \text{in } \Sigma_\lambda.$$

Step 2. Now we can move the plane. Define

$$\mu = \inf \left\{ \lambda > 0 : v \leq v_\lambda \text{ in } \Sigma_\lambda \right\}.$$

Lemma 2.4. *If $\mu > 0$, then $v \equiv v_\mu$ in Σ_μ.*

Proof. By continuity, we have $v \leq v_\mu$ in Σ_μ. Suppose on the contrary that $v \not\equiv v_\mu$ in Σ_μ. Then for $(x,0) \in \partial \mathbb{H} \cap \partial \Sigma_\mu$, we have that

$$\begin{aligned} g(|x|^{n-2s} v(x)) v^\tau(x) &= \frac{f(|x|^{n-2s} v(x))}{|x|^{n+2s}} \\ &\leq \frac{f(|x|^{n-2s} v_\mu(x))}{|x|^{n+2s}} \\ &= g\left(|x|^{n-2s} v_\mu(x)\right) v_\mu^{2_s^*-1}(x) \\ &\leq g\left(|x_\mu|^{n-2s} v_\mu(x)\right) v_\mu^{2_s^*-1}(x). \end{aligned}$$

Applying the comparison principle of lemma 2.1, we deduce that

$$v(x,y) < v_\mu(x,y) \qquad \text{for } (x,y) \in \Sigma_\mu \cup \{X \in \partial \mathbb{H}, x_1 > \mu\}.$$

By virtue of the strict inequality, we find that the characteristic function $\chi_{\partial A_\lambda} \to 0$ a.e. in $\mathbb{R}^n$ as $\lambda \to \mu$. Thus the Dominated convergence theorem implies

$$\lim_{\lambda \to \mu} C_\lambda \left(\int_{\partial \mathbb{H} \cap \partial A_\lambda} v^{2_s^*} \right)^{2_s^*-2} = 0.$$

Combining above limit together with the inequality (2.6), we conclude that there exists a sufficiently small positive constant $\delta > 0$ such that for all $\lambda \in [\mu - \delta, \mu]$

$$v \leq v_\lambda \qquad \text{in } \Sigma_\lambda.$$

However, this is against the choice of μ. The proof of Lemma 2.4 is complete. □

Now we can prove Theorem 1.1 in the nonhomogeneous case.

Case 1: *g is not a constant function.*

We show that u vanishes everywhere. To this end, first we claim that $\mu = 0$. For otherwise, $\mu > 0$ implies that $v \equiv v_\mu$ in Σ_μ. But then substituting the equality into equations (2.2) and (2.4) leads to the identity

$$g(|x|^{n-2s}v(x)) \equiv g(|x_\mu|^{n-2s}v(x)) \text{ for all } x \in \partial\Sigma_\mu,$$

which is impossible since g is monotone and nonconstant. Hence $\mu = 0$. Then we deduce that

$$v(x_1, x_{2,} \cdots, x_n, y) \leq v(-x_1, x_{2,} \cdots, x_n, y) \quad \text{for all } (x_1, x_{2,} \cdots, x_n, y) \in \Sigma_0.$$

By the same argument as that of the homogeneous case, we conclude u depends only on the variable y. But then the equation can be explicitly solved by

$$u(x, y) = -\frac{f(m)}{1-a}y^{1-a} + m$$

for some constant $m > 0$. Thus u cannot be nonnegative for y large enough if $f(m) > 0$. However, this happens, for otherwise if $f(t_0) = 0$ for some $t_0 > 0$, the monotonicity assumptions (H1) and (H2) implies that $g \equiv 0$, which is against our assumption. Therefore, there is no nontrivial nonnegative solution to equation (1.1).

Case 2: *$g \equiv$ constant > 0.*

In this case, $f(u) = f_0 u^{2_s^*-1}$ for some $f_0 > 0$. With no loss of generality, we assume that $f_0 = 1$ so that $g \equiv 1$. The proof in this case is essentially the same as that of [10, 26] but with some simplifications. We give a sketch of proof below. First we prove

Lemma 2.5. *There exists a constant $u_\infty > 0$ such that*

$$\lim_{|X|\to\infty} |X|^{n-2s}u(X) = u_\infty. \tag{2.7}$$

Proof. In fact, if $\mu > 0$, then Lemma 2.4 shows that u has no singularity at infinity, and so the result holds. Indeed, suppose that (2.7) does not hold. Then for any two different point $a, b \in \partial\mathbb{H}$, let $c = (a + b)/2$ and consider the Kelvin transform centered at c:

$$v(X) = \frac{1}{|X-c|^{n-2s}}u\left(\frac{X-c}{|X-c|^2} + c\right).$$

Then v has singularity at $X = c$. Repeat the same argument in the above. We conclude that $\mu = 0$. Thus, v is radially symmetric about the axis that passes $X = c$ and parallels to $y-$axis. In particular, we have $u(a) = u(b)$. Since a, b are two arbitrary points on ∂H, u must depend only on y. We obtain a contradiction as in Case 1. The proof of Lemma 2.5 is complete. □

The following lemma is very useful to derive the formula (1.5).

Lemma 2.6. *Let u be a solution to equation (1.1) and $a \in \mathbb{R}^n$, $\lambda^{n-2s} = u_\infty/u(a, 0)$. Then we have*

$$u(x, 0) = \left(\frac{\lambda}{|x-a|}\right)^{n-2s} u\left(a + \frac{\lambda^2(x-a)}{|x-a|^2}, 0\right), \qquad x \in \mathbb{R}^n.$$

Proof. First notice that in our case for any $(a, 0) \in \partial H$ and $\delta > 0$, the translation $u(\cdot - (a, 0))$ and the scaling $u_\delta(X) = \delta^{\frac{n-2s}{2}} u(\delta X)$ are solutions to equation (1.1) as well. Consider $a = 0$, $\lambda^{n-2s} = u_\infty / u(0)$. Let e be any unit vector on $\partial\mathbb{H}$, and set

$$v(X) = \frac{1}{|X - e|^{n-2s}} u_\lambda \left(\frac{X - e}{|X - e|^2} + e \right).$$

Then v is a solution to (1.1) with $v(0) = \lambda^{\frac{n-2s}{2}} u(0)$, $v(e) = \lambda^{-\frac{n-2s}{2}} u_\infty$ by (2.7). By the choice of λ, we also have $v(0) = v(e)$. Hence v is radially symmetric with respect to $x = e/2$. In particular, we have for any $h \in \mathbb{R}$ that

$$\left(\frac{\sqrt{\lambda}}{|h|} \right)^{n-2s} u \left(\lambda \frac{h-1}{h} e \right) = \left(\frac{\sqrt{\lambda}}{|h-1|} \right)^{n-2s} u \left(\lambda \frac{h}{h-1} e \right).$$

Letting $t = \frac{h}{h-1}$, we arrive at

$$u(\lambda t e) = \frac{1}{|t|^{n-2s}} u \left(\lambda \frac{e}{t} \right).$$

Thus for any $(x, 0) \in \partial\mathbb{H}$, we achieve

$$u(x, 0) = \left(\frac{\lambda}{|x|} \right)^{n-2s} u \left(\frac{\lambda^2 x}{|x|^2}, 0 \right).$$

Now the lemma follows from a translation. □

We are quite close to our result now. Combining Lemma 5.8 [23] and Lemma 2.6 yields

$$u(x, 0) = \left(\frac{cd}{d^2 + |x - x_0|^2} \right)^{\frac{n-2s}{2}}, \qquad x \in \mathbb{R}^n$$

for some $c, d > 0$. Then, using a standard Caccioppoli type inequality, combining Lemma 2.5 and the above explicit formula of $u(\cdot, 0)$, we deduce that u has finite energy in the upper plane in the sense that

$$\int_{\mathbb{H}} y^a |\nabla u|^2 \mathrm{d}X < \infty.$$

Next, let

$$\Gamma(x, y) = \left(P_y^a * u(\cdot, 0) \right)(x).$$

Then Γ is a solution to equation (1.1) with $\Gamma = u$ on $\partial\mathbb{H}$ as aforementioned in the introduction. Moreover, notice that $u(\cdot, 0) \in \dot{H}^s(\mathbb{R}^n)$ holds since it is the function that achieves the best constant for the fractional Sobolev inequality. Hence it follows from Proposition 3.5 of Frank and Lenzmann [15] (see also [16] for a higher dimensional analog) that Γ satisfies

$$\int_{\mathbb{H}} y^a |\nabla \Gamma|^2 \mathrm{d}X < \infty.$$

Hence both u and Γ are finite energy solutions to equation (1.1) with the same boundary value. This fact implies that $u \equiv \Gamma$ in $\mathbb{H}$. This finishes the proof for case 2. The proof of Theorem 1.1 is complete.

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