Author(s): Haslinger, Jaroslav; Mäkinen, Raino; Stebel, Jan

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SHAPE OPTIMIZATION FOR STOKES PROBLEM WITH THRESHOLD SLIP BOUNDARY CONDITIONS

Jaroslav Haslinger

Department of Numerical Mathematics, Faculty of Mathematics and Physics
Charles University Prague, Sokolovská 83, 186 75 Prague 8, Czech Republic

Raino A. E. Mäkinen*

Department of Mathematical Information Technology
University of Jyvaskyla, P.O. Box 35 (Agora)
FIN-40014 Jyvaskyla, Finland

Jan Stebel

Department of Nanotechnology and Informatics
Institute for Nanomaterials, Advanced Technology and Innovation
Technical University of Liberec
Studentská 1402/2, 461 17 Liberec 1, Czech Republic

This paper is dedicated to Prof. Tomáš Roubíček in the occasion of his 60th birthday.

Abstract. This paper deals with shape optimization of systems governed by the Stokes flow with threshold slip boundary conditions. The stability of solutions to the state problem with respect to a class of domains is studied. For computational purposes the slip term and impermeability condition are handled by a regularization. To get a finite dimensional optimization problem, the optimized part of the boundary is described by Bézier polynomials. Numerical examples illustrate the computational efficiency.

1. **Introduction.** The standard kinematic boundary condition in mathematical models of fluid mechanics is represented by the no-slip condition, namely the fluid has the zero velocity $\boldsymbol{u}$ on the boundary of a solid impermeable wall. This condition however does not always hold. In many real problems a fluid slip along the boundary has been observed. In particular, this effect occurs on hydrophobic surfaces, i.e. surfaces coated by a thick film of a non-wettable material from which the fluid (water) is repelled [24]. The Navier boundary condition is the classical one which takes into account the fluid slip [20]. It says that the shear stress σ_t is proportional to the tangential velocity u_t: $\sigma_t = -ku_t$, where k is the adhesive coefficient. Consequently, a slip occurs whenever σ_t is non-vanishing. This model is not able to describe frequent situations when the slip has a threshold character, i.e. it may come only if the

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* Corresponding author.

shear stress attains certain value which is either given a-priori or depends in some way on the solution itself. For the physical justification of different slip laws we refer to [15, 23]. Due to their non-smooth character, resulting mathematical models lead to an inequality type problem. For the steady Stokes flow with slip conditions and a given slip bound we refer to [6], [25], [17], and to [18] for the steady Navier-Stokes flow. The Stokes problem with a solution dependent slip bound has been studied in [13]. Recently, the stick-slip condition has been considered in [3] as an implicit constitutive equation on the boundary having a monotone 2-graph property. The existence of weak solutions to Bingham and Navier–Stokes fluids is proven there.

The present paper deals with optimal shape design problems governed by the Stokes system subject to the threshold boundary conditions. Such problems are of a great practical importance. Indeed, using appropriate shapes of hydrophobic surfaces, one can control (among others) the velocity profile to reduce the energy losses. The stability of solutions to the state problem with respect to an appropriate class of domains is the key property used in the existence analysis. This subject has been studied in [26] for the Navier boundary condition, in [14] for the slip bound given a priori and in [13] for the solution dependent slip bound. Due to the threshold character of the slip boundary conditions, the respective control-to-state mapping which with any admissible domain associates the solution to the state problem $(\mathcal{M})$ is non-differentiable in the classical sense. Therefore the resulting optimization problem $(\mathbb{P})$ formulated and analyzed in [13] and [14] is generally non-smooth, as well. It can be solved numerically by non-smooth optimization methods. The main drawback (to some extent) of this approach is the fact that it requires knowledge of the non-smooth differential calculus to perform sensitivity analysis ([21]) needed in computations. A possible way how to overcome this difficulty is to approximate the nonsmooth slip term j in $(\mathcal{M})$ by an appropriate sequence of smooth functionals j_ε, $\varepsilon \to 0+$ to get a sequence of smooth nonlinear equations $(\mathcal{M}_\varepsilon)$. Denoting by $(\mathbb{P}_\varepsilon)$ the shape optimization problem with $(\mathcal{M}_\varepsilon)$ as the state problem a natural question arises, namely if exists a relation between solutions to $(\mathbb{P}_\varepsilon)$ and $(\mathbb{P})$ for $\varepsilon \to 0+$? This is one of subjects analyzed here.

The paper is organized as follows: in Section 2 we present the velocity-pressure formulation $(\mathcal{M})$ of the Stokes system with a class of slip boundary conditions. Besides the regularization of the slip term we use for computational purposes also a penalization of the impermeability condition to define the regularized-penalized problems $(\mathcal{M}_\varepsilon)$. Section 3 is devoted to the stability analysis of solutions to $(\mathcal{M}_\varepsilon)$ with respect to domains Ω and the parameter $\varepsilon \to 0+$. The assumptions are formulated in an abstract way enabling us to use them in other problems, too. On the basis of these results one can easily prove the existence of solutions to $(\mathbb{P}_\varepsilon)$ and $(\mathbb{P})$ and to establish the mutual relation between their solutions if $\varepsilon \to 0+$. This is done in Section 4. Section 5 introduces an appropriate family of admissible domains, and penalty/regularization functionals satisfying all the assumptions formulated in the previous sections. Section 6 deals with numerical aspects. Optimized part of the boundary is described by Bézier polynomials, while the regularized-penalized problem $(\mathcal{M}_\varepsilon)$ is discretized by P1-bubble/P1 elements. Sensitivity analysis uses the standard adjoint state approach. Finally, Section 7 presents numerical results of three model examples.

2. **State problem.**

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega = \bar{\Gamma} \cup \bar{S}$, where Γ, S are disjoint, non-empty and open in $\partial\Omega$. In Ω we consider the Stokes system with a slip-type boundary condition prescribed on S:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \\ u_\nu = 0 & \text{on } S \\ \sigma_t \in \partial j(-u_t) & \text{on } S, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure and $\boldsymbol{f}$ is an external force. Further, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\boldsymbol{t} = (-\nu_2, \nu_1)$ stand for the unit outward normal, and tangential vector to S, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $u_t = \boldsymbol{u} \cdot \boldsymbol{t}$ denote the normal, and tangential component of $\boldsymbol{u}$, $\sigma_t = 2\mathbb{D}(\boldsymbol{u})\boldsymbol{\nu} \cdot \boldsymbol{t}$ is the shear stress on S and $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{u}$. Finally, $\partial j(\bullet)$ stands for the subgradient of a convex functional j at a point $\bullet$.

To give the weak *velocity* and *velocity-pressure* formulation we first introduce several function spaces:

$$\begin{aligned} \mathbb{V}(\Omega) &= \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\ v_\nu = 0 \text{ on } S\} \\ \mathbb{V}_{\text{div}}(\Omega) &= \{\boldsymbol{v} \in \mathbb{V}(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\} \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) \mid \textstyle\int_\Omega q\, dx = 0\}. \end{aligned}$$

The weak velocity formulation of (1) is defined by the following minimization problem:

$$\text{Find } \boldsymbol{u} = \operatorname*{argmin}_{\boldsymbol{v} \in \mathbb{V}_{\text{div}}(\Omega)} \left\{ J(\boldsymbol{v}) := \tfrac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + j(v_t) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \right\}, \tag{$\mathcal{P}(\Omega)$}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2 \int_\Omega \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v})\, dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in (H^1(\Omega))^2.$$

Further $j : L^2(S) \to \mathbb{R}_+$ is a *non-negative, convex, lower semicontinuous* functional, and $(\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{f}, \boldsymbol{v} \in (L^2(\Omega))^2$.

It is well-known that ($\mathcal{P}(\Omega)$) has a unique solution $\boldsymbol{u}$ and, in addition, ($\mathcal{P}(\Omega)$) is equivalent to the following variational inequality of the second kind:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V}_{\text{div}}(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(v_t) - j(u_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}_{\text{div}}(\Omega). \end{cases} \tag{$\mathcal{P}'(\Omega)$}$$

The velocity-pressure variational formulation of (1) reads as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V}(\Omega) \times L_0^2(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j(v_t) - j(u_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega}\ \forall \boldsymbol{v} \in \mathbb{V}(\Omega) \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{$\mathcal{M}(\Omega)$}$$

where $b : (H^1(\Omega))^2 \times L_0^2(\Omega) \to \mathbb{R}$ is defined by $b(\boldsymbol{v}, q) = \int_\Omega \operatorname{div} \boldsymbol{v}\, q\, dx$. Also ($\mathcal{M}(\Omega)$) has a unique solution $(\boldsymbol{u}, p)$ as a consequence of the inf-sup condition satisfied by b (see [8, Th. 3.7]):

$$\exists \beta = \text{const.} > 0 : \quad \sup_{\boldsymbol{v} \in (H_0^1(\Omega))^2} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega} \quad \forall q \in L_0^2(\Omega). \tag{2}$$

In addition, the first component $\boldsymbol{u}$ solves ($\mathcal{P}(\Omega)$).

Since the functional j is generally non-differentiable, we use a regularization approach together with a penalization of the impermeability condition $v_\nu = 0$ on S. To this end we introduce the spaces

$$\mathbb{W}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}$$
$$\mathbb{W}_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in \mathbb{W}(\Omega) \mid b(\boldsymbol{v}, q) = 0 \quad \forall q \in L^2_0(\Omega)\}$$

and a system of functionals $\{j_\varepsilon\}, \varepsilon \to 0+$ with the following properties:

– $j_\varepsilon : L^2(S) \to \mathbb{R}_+$ is non-negative, convex, and differentiable $\forall \varepsilon > 0$; (3)

– $\lim\limits_{\varepsilon \to 0+} j_\varepsilon(q) = j(q) \quad \forall q \in L^2(S)$; (4)

– $\liminf\limits_{\varepsilon \to 0+} j_\varepsilon(q_\varepsilon) \geq j(q)$ holds for any $\{q_\varepsilon\}$, $q_\varepsilon \in L^2(S)$ s.t. $q_\varepsilon \to q$ in $L^2(S)$. (5)

The condition $v_\nu = 0$ on S will be penalized by the functional

$$g(v_\nu) = \tfrac{1}{2} \int_S (v_\nu)^2 \, ds, \quad \boldsymbol{v} \in \mathbb{W}(\Omega). \tag{6}$$

The *penalized-regularized* formulation of $(\mathcal{P}(\Omega))$, $(\mathcal{M}(\Omega))$ reads as follows:

$$\text{Find } \boldsymbol{u}_\varepsilon = \operatorname*{argmin}_{\boldsymbol{v} \in \mathbb{W}_{\mathrm{div}}(\Omega)} \left\{ J_\varepsilon(\boldsymbol{v}) := \tfrac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + j_\varepsilon(v_t) + \frac{1}{\varepsilon} g(v_\nu) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \right\} \tag{$\mathcal{P}_\varepsilon(\Omega)$}$$

and

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbb{W}(\Omega) \times L^2_0(\Omega) \text{ such that} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - b(\boldsymbol{v}, p_\varepsilon) + \langle \nabla j_\varepsilon(u_{\varepsilon t}), v_t \rangle \\ \qquad\qquad + \dfrac{1}{\varepsilon} \langle Dg(u_{\varepsilon\nu}), v_\nu \rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega) \\ b(\boldsymbol{u}_\varepsilon, q) = 0 \quad \forall q \in L^2_0(\Omega), \end{cases} \tag{$\mathcal{M}_\varepsilon(\Omega)$}$$

respectively. From (6) we see that

$$\langle Dg(u_\nu), v_\nu \rangle = \int_S u_\nu v_\nu \, ds. \tag{7}$$

Problems $(\mathcal{P}_\varepsilon(\Omega))$, $(\mathcal{M}_\varepsilon(\Omega))$ have unique solutions $\boldsymbol{u}_\varepsilon$, and $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$, respectively for every $\varepsilon > 0$. In addition, the first component $\boldsymbol{u}_\varepsilon$ of the solution to $(\mathcal{M}_\varepsilon(\Omega))$ solves $(\mathcal{P}_\varepsilon(\Omega))$. Using techniques from [9, Chpt. I, Th. 7.1 and Chpt. II, Th. 6.3] one can show that

$$(\boldsymbol{u}_\varepsilon, p_\varepsilon) \rightharpoonup (\boldsymbol{u}, p) \quad \text{in } (H^1(\Omega))^2 \times L^2(\Omega), \tag{8}$$

as $\varepsilon \to 0+$ where $(\boldsymbol{u}, p)$ is the solution of $(\mathcal{M}(\Omega))$. In the next section we shall study the stability of solutions to $(\mathcal{M}_\varepsilon(\Omega))$ with respect to Ω and $\varepsilon \to 0+$. Convergence (8) will be a special case of this result.

3. **Stability of $(\mathcal{M}_\varepsilon(\Omega))$ with respect to $\varepsilon > 0$ and Ω.** Now we shall consider problems $(\mathcal{P}_\varepsilon(\Omega))$ and $(\mathcal{M}_\varepsilon(\Omega))$ parametrized simultaneously by $\varepsilon > 0$ and Ω. To this end we introduce a system $\mathcal{O}$ of bounded domains Ω with the Lipschitz boundaries $\partial\Omega = \overline{\Gamma^\Omega} \cup \overline{S^\Omega}$, where Γ^Ω, S^Ω are parts of $\partial\Omega$ where the no-slip, and slip boundary conditions, respectively are prescribed. We shall suppose that $|\Gamma^\Omega| \geq \delta$, $|S^\Omega| \geq \delta$, where $\delta > 0$ does not depend on $\Omega \in \mathcal{O}$ and $|S^\Omega|, |\Gamma^\Omega|$ stand for the length of S^Ω, and Γ^Ω, respectively. Furthermore let there exist two bounded domains C, $\hat{\Omega}$ such that $C \subset \overline{\Omega} \subset \hat{\Omega}$ and $\mathrm{dist}(\partial\Omega, \partial\hat{\Omega}) \geq \delta_0$ for all $\Omega \in \mathcal{O}$, where $\delta_0 > 0$ is independent of $\Omega \in \mathcal{O}$.

To analyze the stability with respect to $\Omega \in \mathcal{O}$ one has to define convergence $\stackrel{\mathcal{O}}{\to}$ in $\mathcal{O}$. The system $\mathcal{O}$ with a concrete choice of $\stackrel{\mathcal{O}}{\to}$ will be denoted by $\{\mathcal{O}, \stackrel{\mathcal{O}}{\to}\}$.

In this abstract setting we do not specify explicitly the choice of $\{\mathcal{O}, \stackrel{\mathcal{O}}{\to}\}$. This will be done implicitly, namely we shall consider such $\{\mathcal{O}, \stackrel{\mathcal{O}}{\to}\}$ for which the assumptions formulated below will be satisfied. Only what we require a-priori is that any subsequence of a convergent sequence of domains from $\{\mathcal{O}, \stackrel{\mathcal{O}}{\to}\}$ converges to the same element.

First we suppose that $\mathcal{O}$ possesses a *uniform extension property*: for any $\Omega \in \mathcal{O}$ there exists an extension mapping $E_\Omega \in \mathcal{L}((H^1(\Omega))^2, (H^1(\hat{\Omega}))^2)$ and a constant $c > 0$ such that

$$\|E_\Omega \boldsymbol{v}\|_{1,\hat{\Omega}} \leq c\|\boldsymbol{v}\|_{1,\Omega} \tag{9}$$

holds for every $\boldsymbol{v} \in (H^1(\Omega))^2$ and any $\Omega \in \mathcal{O}$.

To emphasize the fact that Ω is one of the parameters of the problem, it will be appended to all data as a superscript. Thus we shall write $a^\Omega, b^\Omega, \boldsymbol{u}^\Omega, \boldsymbol{u}_\varepsilon^\Omega$,... instead of $a, b, \boldsymbol{u}, \boldsymbol{u}_\varepsilon$,... which are defined in the same way as in Section 2. In particular the functionals, $j^\Omega, j_\varepsilon^\Omega, g^\Omega : L^2(S^\Omega) \to \mathbb{R}_+$. To simplify notation we shall also write $\hat{\boldsymbol{v}}^\Omega := E_\Omega \boldsymbol{v}$, $\boldsymbol{v} \in (H^1(\Omega))^2$, in what follows while $\mathring{q}$ stands for the extension of a function $q \in L^2(\Omega)$ by zero outside of Ω.

To guarantee uniform boundedness of solutions to $(\mathcal{M}_\varepsilon(\Omega))$ with respect to $\Omega \in \mathcal{O}$ and $\varepsilon > 0$, the system $\mathcal{O}$ will be chosen in such a way that the following assumptions are satisfied:

$$\exists\, \alpha = \text{const.} > 0 : \quad a^\Omega(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|_{1,\Omega}^2 \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega)\ \forall \Omega \in \mathcal{O}; \tag{10}$$

$$\exists\, \beta = \text{const.} > 0 : \quad \sup_{\boldsymbol{v} \in (H_0^1(\Omega))^2} \frac{b^\Omega(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta\|q\|_{0,\Omega} \quad \forall q \in L_0^2(\Omega)\ \forall \Omega \in \mathcal{O}, \tag{11}$$

i.e. a^Ω is $\mathbb{W}(\Omega)$-elliptic and b^Ω satisfies the inf-sup condition, both uniformly with respect to $\Omega \in \mathcal{O}$.

Further we shall suppose that

$$\exists\, c = \text{const.} > 0\ \exists\, \varepsilon_0 > 0 : \quad j_\varepsilon^\Omega(0) \leq c \quad \forall \varepsilon \in]0, \varepsilon_0],\ \forall \Omega \in \mathcal{O}, \tag{12}$$

and the right hand side of the Stokes system in Ω is the restriction of a function $\boldsymbol{f} \in (L^2(\hat{\Omega}))^2$.

Lemma 3.1. *Let (10)–(12) be satisfied. Then there exists a constant $c > 0$ such that*

$$\|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega} + \|p_\varepsilon^\Omega\|_{0,\Omega} \leq c \tag{13}$$

and

$$0 \leq g^\Omega(u_{\varepsilon\nu}^\Omega) \leq c\,\varepsilon \tag{14}$$

hold for any $\varepsilon \in]0, \varepsilon_0]$ and $\Omega \in \mathcal{O}$.

Proof. From the definition of $(\mathcal{P}_\varepsilon(\Omega))$ we have:

$$\begin{aligned} \frac{1}{2}a^\Omega(\boldsymbol{u}_\varepsilon^\Omega, \boldsymbol{u}_\varepsilon^\Omega) + \frac{1}{\varepsilon}g^\Omega(u_{\varepsilon\nu}^\Omega) &\leq J_\varepsilon^\Omega(\boldsymbol{u}_\varepsilon^\Omega) + (\boldsymbol{f}, \boldsymbol{u}_\varepsilon^\Omega)_{0,\Omega} \\ &\leq J_\varepsilon^\Omega(\boldsymbol{0}) + (\boldsymbol{f}, \boldsymbol{u}_\varepsilon^\Omega)_{0,\Omega} \leq j_\varepsilon^\Omega(0) + \|\boldsymbol{f}\|_{0,\hat{\Omega}}\|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega} \end{aligned}$$

From this, (10) and (12) the boundedness of $\{\boldsymbol{u}_\varepsilon^\Omega\}$ and (14) follow. Using test functions $\boldsymbol{v} \in (H_0^1(\Omega))^2$ in $(\mathcal{M}_\varepsilon(\Omega))$ together with (11) we obtain:

$$\beta \|p_\varepsilon^\Omega\|_{0,\Omega} \le \sup_{\boldsymbol{v}\in(H_0^1(\Omega))^2} \frac{b^\Omega(\boldsymbol{v}, p_\varepsilon^\Omega)}{\|\boldsymbol{v}\|_{1,\Omega}} \le \|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega} + \|\boldsymbol{f}\|_{0,\hat{\Omega}}$$

and hence (13) holds true. $\square$

Owing to (9) and (13), the extensions of $(\boldsymbol{u}_\varepsilon^\Omega, p_\varepsilon^\Omega)$ from Ω to $\hat{\Omega}$ are bounded, as well:

$$\exists\, c = \text{const.} > 0: \quad \|\hat{\boldsymbol{u}}_\varepsilon^\Omega\|_{1,\hat{\Omega}} + \|\mathring{p}_\varepsilon^\Omega\|_{0,\hat{\Omega}} \le c \quad \forall \varepsilon \in]0, \varepsilon_0]\ \forall \Omega \in \mathcal{O}. \tag{15}$$

Let $\{\Omega_k\}, \Omega_k \in \mathcal{O}$ be such that $\Omega_k \xrightarrow{\mathcal{O}} \Omega \in \mathcal{O}$ and consider problems $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$, where $\varepsilon_k \to 0+$ as $k \to \infty$. Next we will study the relation between solutions of $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$ and $(\mathcal{M}(\Omega))$ when $k \to \infty$. To this end we shall suppose that $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$ is chosen in such a way that the following assumptions are satisfied:

– for any $\{\boldsymbol{v}^k\}$ such that $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, $\boldsymbol{v}^k|_{\Omega_k} \in \mathbb{W}(\Omega_k)$ it follows that

$$\boldsymbol{v}|_\Omega \in \mathbb{W}(\Omega); \tag{16}$$

– $\forall \boldsymbol{v} \in \mathbb{V}(\Omega)$ there exists a sequence $\{\boldsymbol{v}^k\}$, $\boldsymbol{v}^k \in (H^1(\hat{\Omega}))^2$ and a function $\bar{\boldsymbol{v}} \in (H^1(\hat{\Omega}))^2$, $\bar{\boldsymbol{v}}|_\Omega = \boldsymbol{v}$ such that

$$\boldsymbol{v}^k \to \bar{\boldsymbol{v}} \quad \text{in } (H^1(\hat{\Omega}))^2 \tag{17}$$

and for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ for which

$$\boldsymbol{v}^k|_{\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k}); \tag{18}$$

– if $\{\boldsymbol{v}^k\}$, $\{\boldsymbol{w}^k\}$, and $\{q_k\}$, $\{z_k\}$ are such that $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$, $\boldsymbol{w}^k \to \boldsymbol{w}$ in $(H^1(\hat{\Omega}))^2$, and $q_k \rightharpoonup q$, $z_k \to z$ in $L^2(\hat{\Omega})$ then

$$\limsup_{k\to\infty} a^{\Omega_k}(\boldsymbol{v}^k|_{\Omega_k}, \boldsymbol{w}^k|_{\Omega_k} - \boldsymbol{v}^k|_{\Omega_k}) \le a^\Omega(\boldsymbol{v}|_\Omega, \boldsymbol{w}|_\Omega - \boldsymbol{v}|_\Omega) \tag{19}$$

$$\lim_{k\to\infty} b^{\Omega_k}(\boldsymbol{w}^k|_{\Omega_k}, q_k|_{\Omega_k}) = b^\Omega(\boldsymbol{w}|_\Omega, q|_\Omega) \tag{20}$$

$$\lim_{k\to\infty} b^{\Omega_k}(\boldsymbol{v}^k|_{\Omega_k}, z_k|_{\Omega_k}) = b^\Omega(\boldsymbol{v}|_\Omega, z|_\Omega) \tag{21}$$

$$\lim_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v}|_{\Omega_k})_{0,\Omega_k} = (\boldsymbol{f}, \boldsymbol{v}|_\Omega)_{0,\Omega}. \tag{22}$$

– if $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ then[1]

$$g^{\Omega_k}(v_\nu^k) \to g^\Omega(v_\nu) \tag{23}$$

and

$$j_{\varepsilon_k}^{\Omega_k}(v_t^k) \to j^\Omega(v_t), \quad k \to \infty. \tag{24}$$

Theorem 3.2. *Let $\varepsilon_k \to 0+$, $\Omega_k \xrightarrow{\mathcal{O}} \Omega$ as $k \to \infty$, $\Omega_k, \Omega \in \mathcal{O}$ and (16)–(24) be satisfied. Let the sequence of solutions $\{(\boldsymbol{u}_{\varepsilon_k}^{\Omega_k}, p_{\varepsilon_k}^{\Omega_k})\}$ to $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$ be such that*

$$\hat{\boldsymbol{u}}_{\varepsilon_k}^{\Omega_k} \rightharpoonup \bar{\boldsymbol{u}} \quad \textit{in } (H^1(\hat{\Omega}))^2, \tag{25}$$

$$\mathring{p}_{\varepsilon_k}^{\Omega_k} \rightharpoonup \bar{p} \quad \textit{in } L^2(\hat{\Omega}), \quad k \to \infty \tag{26}$$

for some $(\bar{\boldsymbol{u}}, \bar{p}) \in (H^1(\hat{\Omega}))^2 \times L_0^2(\hat{\Omega})$. Then $(\bar{\boldsymbol{u}}|_\Omega, \bar{p}|_\Omega)$ solves $(\mathcal{M}(\Omega))$.

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[1] Recall that $v_\nu^k = \boldsymbol{v}^k|_{S^{\Omega_k}} \cdot \boldsymbol{\nu}^k$, $v_\nu = \boldsymbol{v}|_{S^\Omega} \cdot \boldsymbol{\nu}$, and $\boldsymbol{\nu}^k, \boldsymbol{\nu}$ is the outward unit normal vector to S^{Ω_k}, and S^Ω, respectively (similarly v_t^k and v_t).

Proof. To simplify notation, instead of the superscript Ω_k we shall write simply k in what follows. Thus $a^k := a^{\Omega_k}$, $j^k_{\varepsilon_k} := j^{\Omega_k}_{\varepsilon_k}$, etc. Using this convention, $(\boldsymbol{u}^k_{\varepsilon_k}, p^k_{\varepsilon_k})$ satisfies:

$$\begin{cases} a^k(\boldsymbol{u}^k_{\varepsilon_k}, \boldsymbol{v}) - b^k(\boldsymbol{v}, p^k_{\varepsilon_k}) + \langle \nabla j^k_{\varepsilon_k}(u^k_{\varepsilon_k t}), v_t\rangle \\ \qquad\qquad + \dfrac{1}{\varepsilon_k}\langle \nabla g^k(u^k_{\varepsilon_k \nu}), v_\nu\rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega_k} \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega_k) \\ b^k(\boldsymbol{u}^k_{\varepsilon_k}, q) = 0 \quad \forall q \in L^2_0(\Omega_k). \end{cases} \tag{$\mathcal{M}_{\varepsilon_k}(\Omega_k)$}$$

First we show that $\bar{\boldsymbol{u}}_{|\Omega} \in \mathbb{V}_{\rm div}(\Omega)$. The fact that $\bar{\boldsymbol{u}} = \boldsymbol{0}$ on Γ^Ω and $\bar{\boldsymbol{u}}_{|\Omega}\cdot\boldsymbol{\nu} = 0$ on S^Ω follows from (16), and (14)+(23). Let $z \in L^2(\hat{\Omega})$ be arbitrary, denote $z_k := z_{|\Omega_k}$ and decompose z_k: $z_k = \bar{z}_k + c_k$, where $c_k = \left(\int_{\Omega_k} z_k\, dx\right)/\text{meas}\,\Omega_k$. Since $\bar{z}_k \in L^2_0(\Omega_k)$ we have:

$$\int_{\Omega_k} \operatorname{div} \boldsymbol{u}^k_{\varepsilon_k} z_k\, dx = \int_{\Omega_k} \operatorname{div} \boldsymbol{u}^k_{\varepsilon_k} \bar{z}_k\, dx + c_k \int_{\Omega_k} \operatorname{div}\boldsymbol{u}^k_{\varepsilon_k}\, dx = c_k \int_{S^k} u^k_{\varepsilon_k \nu}\, ds.$$

Passing to the limit with $k \to \infty$, using (14), (21), (25) and the definition of z_k we see that

$$\int_\Omega \operatorname{div} \bar{\boldsymbol{u}}\, z\, dx = 0 \quad \forall z \in L^2(\hat{\Omega}), \tag{27}$$

i.e. $\operatorname{div} \bar{\boldsymbol{u}}_{|\Omega} = 0$.

The fact that $\bar{p}_{|\Omega} \in L^2_0(\Omega)$ follows from (20), hence $(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) \in \mathbb{V}_{\rm div}(\Omega) \times L^2_0(\Omega)$.

Let $\boldsymbol{v} \in \mathbb{V}(\Omega)$ be arbitrary and $\{\boldsymbol{v}^k\}$ be a sequence satisfying (17) and (18). Using $\boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_k}$ as a test function in $(\mathcal{M}_{\varepsilon_{n_k}}(\Omega_{n_k}))$[2], where n_k is the filter of indices for which (18) holds, we obtain:

$$\begin{aligned} a^{n_k}(\boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}) - b^{n_k}(\boldsymbol{v}^k, p^{n_k}_{\varepsilon_{n_k}}) + \\ \langle \nabla j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k} t}), v^k_t - u^{n_k}_{\varepsilon_{n_k} t}\rangle \geq (\boldsymbol{f}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}})_{0,\Omega_{n_k}} \end{aligned} \tag{28}$$

taking into account that $\boldsymbol{v}^k{}_{|\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k})$ and the second equation in $(\mathcal{M}_{\varepsilon_{n_k}}(\Omega_{n_k}))$. Adding the term $j^{n_k}_{\varepsilon_{n_k}}(v^k_t) - j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k} t})$ to both sides of (28) and using convexity of $j^{n_k}_{\varepsilon_{n_k}}$ we arrive at

$$\begin{aligned} a^{n_k}(\boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}) - b^{n_k}(\boldsymbol{v}^k, p^{n_k}_{\varepsilon_{n_k}}) \\ + j^{n_k}_{\varepsilon_{n_k}}(v^k_t) - j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k} t}) \geq (\boldsymbol{f}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}})_{0,\Omega_{n_k}}. \end{aligned} \tag{29}$$

If $k \to \infty$ in (29) then

$$a^\Omega(\bar{\boldsymbol{u}}_{|\Omega}, \boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega}) - b^\Omega(\boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) + j^\Omega(v_t) - j^\Omega(\bar{u}_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega).$$

making use of (19)–(22), (24). From this and the inf-sup condition it follows that $\bar{p}_{|\Omega} = p^\Omega$. Consequently, $(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) := (\boldsymbol{u}^\Omega, p^\Omega)$ is the solution of $(\mathcal{M}(\Omega))$. □

Let us comment on the assumptions formulated above. It is known that if the system $\mathcal{O}$ consists of domains with the *uniform cone property* (it will be denoted

[2]for simplicity of notation we write $\boldsymbol{v}^k$ instead of $\boldsymbol{v}^k{}_{|\Omega_{n_k}}$

by $\mathcal{O}_{\mathrm{cone}}$) then (9) is satisfied ([4]). $\mathcal{O}_{\mathrm{cone}}$ has yet other properties which guarantee that some of the previous assumptions are automatically satisfied. It holds:

– $\mathcal{O}_{\mathrm{cone}}$ is compact with respect to the Hausdorff metric; (30)

– if $\Omega_k \overset{\mathcal{H}}{\to} \Omega,\ \Omega_k \in \mathcal{O}_{\mathrm{cone}}$ then

$$\partial\Omega_k \overset{\mathcal{H}}{\to} \partial\Omega \tag{31}$$

and

$$\chi_k \to \chi \quad \text{in } L^2(\hat{\Omega}), \tag{32}$$

where $\overset{\mathcal{H}}{\to}$ stands for convergence in the Hausdorff metric, χ_k, χ are the characteristic functions of Ω_k and Ω, respectively (see [22], [16]). From (32) we easily obtain (19)–(22). Also the uniform ellipticity of a^{Ω} with respect to $\Omega \in \mathcal{O}_{\mathrm{cone}}$, i.e. (10), is satisfied (see [11]). The next property is an easy consequence of (32), too:

– if $\Omega_k \overset{\mathcal{H}}{\to} \Omega$, $\Omega_k \in \mathcal{O}_{\mathrm{cone}}$ and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ then

$$\liminf_{k\to\infty} |\mathbb{D}\boldsymbol{v}^k|_{0,\Omega_k} \geq |\mathbb{D}\boldsymbol{v}|_{0,\Omega}. \tag{33}$$

On the other hand, in order to satisfy (16), (17), (18), (23), and (24) we usually need appropriate subsets of $\mathcal{O}_{\mathrm{cone}}$ which consist of domains with more regular boundaries. One example of such a system will be presented in Section 5.

4. **Optimal shape design problems.**

First we present a class of optimal shape design problems we want to solve with the velocity-pressure formulation $(\mathcal{M}(\Omega))$ of (1) as the state relation.

Let $\{\mathcal{O}, \overset{\mathcal{O}}{\to}\}$ be a system of admissible domains introduced in Section 3. Further we choose an objective functional I which depends on $(\Omega, \boldsymbol{u}^{\Omega}, p^{\Omega})$, $\Omega \in \mathcal{O}$, with $(\boldsymbol{u}^{\Omega}, p^{\Omega})$ being the solution to $(\mathcal{M}(\Omega))$ and denote $\mathcal{J}(\Omega) := I(\Omega, \boldsymbol{u}^{\Omega}, p^{\Omega})$.

Optimal shape design problems we shall deal with read as follows:

$$\text{Find } \Omega^* \in \arg\min \{\mathcal{J}(\Omega) \mid \Omega \in \mathcal{O}\}. \tag{$\mathbb{P}$}$$

To prove the existence of solutions to $(\mathbb{P})$ we shall need the following assumptions:

– (*sequential compactness of* $\mathcal{O}$)
in any sequence $\{\Omega_k\}$, $\Omega_k \in \mathcal{O}$ there exist: a subsequence $\{\Omega_{k_j}\}$ and $\Omega \in \mathcal{O}$ such that

$$\Omega_{k_j} \overset{\mathcal{O}}{\to} \Omega \in \mathcal{O}, \quad j \to \infty; \tag{34}$$

– (*uniform boundedness of* $\{(\boldsymbol{u}^{\Omega}, p^{\Omega})\}$)

$$\exists\, c = \text{const.} > 0 : \quad \|\hat{\boldsymbol{u}}^{\Omega}\|_{1,\hat{\Omega}} + \|\mathring{p}^{\Omega}\|_{0,\hat{\Omega}} \leq c \quad \forall \Omega \in \mathcal{O}; \tag{35}$$

– (*lower semicontinuity of* I)
if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$, $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, and $q_k \rightharpoonup q$ in $L^2(\hat{\Omega})$ then

$$\liminf_{k\to\infty} I(\Omega_k, \boldsymbol{v}^k|_{\Omega_k}, q_k|_{\Omega_k}) \geq I(\Omega, \boldsymbol{v}|_{\Omega}, q|_{\Omega}). \tag{36}$$

Finally we suppose that the following assumption is satisfied:

– if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$ and $\{(\boldsymbol{u}^{\Omega_k}, p^{\Omega_k})\}$ is the sequence of solutions to $(\mathcal{M}(\Omega_k))$ such that $(\hat{\boldsymbol{u}}^{\Omega_k}, \mathring{p}^{\Omega_k}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ then

$$(\bar{\boldsymbol{u}}|_{\Omega}, \bar{p}|_{\Omega}) \text{ solves } (\mathcal{M}(\Omega)). \tag{37}$$

Theorem 4.1. *Let (34)–(37) be satisfied. Then* $(\mathbb{P})$ *has a solution.*

Proof. Let $\{\Omega_k\}$, $\Omega_k \in \mathcal{O}$ be a minimizing sequence in $(\mathbb{P})$:

$$\lim_{k\to\infty} \mathcal{J}(\Omega_k) = \inf_{\mathcal{O}} \mathcal{J}(\Omega).$$

Owing to (34), (35), and (37) we may pass to a subsequence (denoted by the same symbol) such that $\Omega_k \overset{\mathcal{O}}{\to} \Omega^* \in \mathcal{O}$, $(\hat{\boldsymbol{u}}^{\Omega_k}, \mathring{p}^{\Omega_k}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ and $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ solves $(\mathcal{M}(\Omega^*))$. The rest of the proof follows from (36). $\square$

As we have mentioned in Introduction, problem $(\mathbb{P})$ is generally non-smooth due to a possible non-differentiability of the control-to-state mapping $\phi : \Omega \mapsto (\boldsymbol{u}^\Omega, p^\Omega)$. For this reason we shall approximate problem $(\mathbb{P})$ by a sequence of problems $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$ which utilize $(\mathcal{M}_\varepsilon(\Omega))$ as the state problem.

Problem $(\mathbb{P}_\varepsilon), \varepsilon > 0$, reads as follows:

$$\text{Find } \Omega^*_\varepsilon \in \arg\min \{\mathcal{J}_\varepsilon(\Omega) \mid \Omega \in \mathcal{O}\}, \tag{$\mathbb{P}_\varepsilon$}$$

where $\mathcal{J}_\varepsilon(\Omega) := I(\Omega, \boldsymbol{u}^\Omega_\varepsilon, p^\Omega_\varepsilon)$ and $(\boldsymbol{u}^\Omega_\varepsilon, p^\Omega_\varepsilon)$ solves $(\mathcal{M}_\varepsilon(\Omega))$.

Next we shall analyze if and under which conditions there exists a relation between solutions to $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ when $\varepsilon \to 0+$.

First of all we have to guarantee that $(\mathbb{P}_\varepsilon)$ has a solution for any $\varepsilon \in]0, \varepsilon_0]$, $\varepsilon_0 > 0$ sufficiently small. To this end we need the following minor modification of (37):

– if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$ and $\{(\boldsymbol{u}^{\Omega_k}_{\overline{\varepsilon}}, p^{\Omega_k}_{\overline{\varepsilon}})\}$ is the sequence of solutions to $(\mathcal{M}_{\overline{\varepsilon}}(\Omega_k))$ such that $(\hat{\boldsymbol{u}}^{\Omega_k}_{\overline{\varepsilon}}, \mathring{p}^{\Omega_k}_{\overline{\varepsilon}}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$, $k \to \infty$, then

$$(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) \text{ solves } (\mathcal{M}_{\overline{\varepsilon}}(\Omega)) \tag{38}$$

and this holds for any $\overline{\varepsilon} \in]0, \varepsilon_0]$.

Theorem 4.2. *Let (15), (34), (36), and (38) be satisfied. Then* $(\mathbb{P}_\varepsilon)$ *has a solution for any* $\varepsilon > 0$.

Proof. It can be omitted. $\square$

To prove the next theorem we have to replace (36) by the following stronger *continuity assumption*:

– if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$, and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, $q_k \rightharpoonup q$ in $L^2(\hat{\Omega})$, $k \to \infty$, then

$$\lim_{k\to\infty} I(\Omega_k, \boldsymbol{v}^k{}_{|\Omega_k}, q_k{}_{|\Omega_k}) = I(\Omega, \boldsymbol{v}_{|\Omega}, q_{|\Omega}). \tag{39}$$

Theorem 4.3. *Let (15), (34), (39) and all the assumptions of Theorem 3.2 be satisfied. Then for any sequence of solutions* $\{\Omega^*_{\varepsilon_k}\}$ *to* $(\mathbb{P}_{\varepsilon_k})$, $\varepsilon_k \to 0+$ *as* $k \to \infty$, *there exist: its subsequence (denoted by the same symbol) and* $\Omega^* \in \mathcal{O}$ *such that*

$$\begin{cases} \Omega^*_{\varepsilon_k} \overset{\mathcal{O}}{\to} \Omega^*, \\ (\hat{\boldsymbol{u}}^*_k, \mathring{p}^*_k) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p}) \quad \textit{in } (H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega}), \quad k \to \infty, \end{cases} \tag{40}$$

where $(\boldsymbol{u}^*_k, p^*_k)$ *is the solution of* $(\mathcal{M}_{\varepsilon_k}(\Omega^*_{\varepsilon_k}))$. *In addition,* Ω^* *is a solution to* $(\mathbb{P})$ *and* $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ *solves* $(\mathcal{M}(\Omega^*))$. *Any accumulation point of* $\{(\boldsymbol{u}^*_k, p^*_k)\}$ *in the sense of (40) is a solution to* $(\mathbb{P})$.

Proof. From (15), (34) and Theorem 3.2 it follows that there exists $\Omega^* \in \mathcal{O}$ and a subsequence of $\{(\Omega^*_{\varepsilon_k}, \hat{\boldsymbol{u}}^*_k, \mathring{p}^*_k)\}$ (denoted by the same symbol) satisfying (40) and such that $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ solves $(\mathcal{M}(\Omega^*))$. It remains to show that Ω^* solves $(\mathbb{P})$. Indeed, let $\widetilde{\Omega} \in \mathcal{O}$ be arbitrary but fixed and $\{(\widetilde{\boldsymbol{u}}_k, \widetilde{p}_k)\}$ be the sequence of solutions to $(\mathcal{M}_{\varepsilon_k}(\widetilde{\Omega}))$, $k \to \infty$. Since

$$(\widetilde{\boldsymbol{u}}_k, \widetilde{p}_k) \rightharpoonup (\widetilde{\boldsymbol{u}}, \widetilde{p}) \quad \text{in } (H^1(\widetilde{\Omega}))^2 \times L^2(\widetilde{\Omega}), \quad k \to \infty, \tag{41}$$

where $(\widetilde{\boldsymbol{u}}, \widetilde{p})$ is a solution of $(\mathcal{M}(\widetilde{\Omega}))$, the definition of $(\mathbb{P}_{\varepsilon_k})$ yields:

$$I(\Omega^*_{\varepsilon_k}, \boldsymbol{u}^*_k, p^*_k) \leq I(\widetilde{\Omega}, \widetilde{\boldsymbol{u}}_k, \widetilde{p}_k).$$

Letting $k \to \infty$ and using (39), (40), and (41) we obtain that $\mathcal{J}(\Omega^*) \leq \mathcal{J}(\widetilde{\Omega}) \quad \forall \widetilde{\Omega} \in \mathcal{O}$. $\square$

To conclude the theoretical part we formulate assumptions under which (37) and (38) are satisfied. Since the proof is only a minor modification of the one of Theorem 3.2 it will be omitted.

Theorem 4.4. *Let* (16) *and* (19)–(22) *be satisfied. If, in addition*

a) (17), (18) *are satisfied and from* $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ *in* $(H^1(\hat{\Omega}))^2$, $\Omega_k \xrightarrow{\mathcal{O}} \Omega$ *it follows that* $j^{\Omega_k}(v^k_t) \to j^{\Omega}(v_t)$, *then* (37) *holds;*

b) (23) *and* (24) *with* $\varepsilon_k = \overline{\varepsilon} \in]0, \varepsilon_0]$ $\forall k \in \mathbb{N}$ *are satisfied, then* (38) *holds.*

5. **Model shape optimization problems.** The aim of this section is to apply the previous theoretical results to a class of optimization problems that will be used in numerical experiments. The system $\mathcal{O}$ consists of domains with a simple shape, namely a part of the boundary to be optimized with the prescribed stick-slip condition is represented by the graph of a function.

The system $\mathcal{O}$ is defined as follows:

$$\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in \mathcal{U}_{ad}\},$$

where

$$\Omega(\alpha) = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in]0,1[,\ x_2 \in]\alpha(x_1), 1[\big\}$$

and

$$\mathcal{U}_{ad} = \left\{\alpha \in C^{1,1}([0,1]) \mid \alpha_{\min} \leq \alpha \leq \alpha_{\max} < 1,\ |\alpha^{(j)}| \leq C_j, j{=}1,2 \text{ a.e. in }]0,1[\right\}, \tag{42}$$

i.e. $\mathcal{U}_{ad}$ is the set of functions which are together with their first derivatives equi-bounded and equi-Lipschitz continuous in $[0,1]$. The constants $\alpha_{\min}$, $\alpha_{\max}$, C_1, and C_2 are chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. The boundary $\partial\Omega(\alpha) = \overline{\Gamma(\alpha)} \cup \overline{S(\alpha)}$, and $S(\alpha)$ is the graph of $\alpha \in \mathcal{U}_{ad}$ (see Figure 1).

In $\mathcal{O}$ we introduce convergence as follows:

$$\Omega_k := \Omega(\alpha_k) \xrightarrow{\mathcal{O}} \Omega(\alpha),\ \alpha_k \in \mathcal{U}_{ad} \iff \alpha_k \to \alpha \text{ in } C^1([0,1]).$$

On any $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ we shall consider the Stokes system with the no-slip, stick-slip boundary condition prescribed on $\Gamma(\alpha)$, and $S(\alpha)$, respectively. The fact that the shape of $\Omega(\alpha)$ is fully determined by the function $\alpha \in \mathcal{U}_{ad}$ enables us to simplify notation. Instead of $\mathbb{V}(\Omega(\alpha))$, $\mathbb{V}_{div}(\Omega(\alpha))$, $L^2_0(\Omega(\alpha))$,... we shall write $\mathbb{V}(\alpha)$, $\mathbb{V}_{div}(\alpha)$, $L^2_0(\alpha)$,... Similarly, $a^\alpha, b^\alpha, j^\alpha$,... is used in place of $a^{\Omega(\alpha)}$, $b^{\Omega(\alpha)}$, $j^{\Omega(\alpha)}$,...

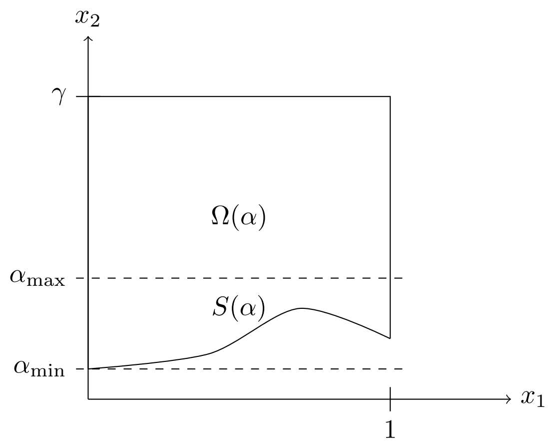

FIGURE 1. Shape of admissible domains.

Using this convention of notation, the velocity-pressure formulation of (1) reads as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}^\alpha, p^\alpha) \in \mathbb{V}(\alpha) \times L^2_0(\alpha) \text{ such that} \\ a^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v} - \boldsymbol{u}^\alpha) - b^\alpha(\boldsymbol{v} - \boldsymbol{u}^\alpha, p^\alpha) \\ \qquad + j^\alpha(v_t) - j^\alpha(u_t^\alpha) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^\alpha)_{0,\Omega(\alpha)} \quad \forall \boldsymbol{v} \in \mathbb{V}(\alpha) \\ b^\alpha(\boldsymbol{u}^\alpha, q) = 0 \quad \forall q \in L^2_0(\alpha). \end{cases} \tag{$\mathcal{M}(\alpha)$}$$

Now we shall verify all the assumptions of Section 3 and 4. Owing to the definition of $\mathcal{U}_{ad}$, all domains from $\mathcal{O}$ enjoy the uniform cone property and consequently (9), (10), (19)–(22) are satisfied. The constant β in the inf-sup condition (11) depends only on $\|\alpha\|_{W^{1,\infty}(0,1)}$ and it can be chosen independently of $\alpha \in \mathcal{U}_{ad}$ (see [2], [7]). Clearly, (16) is satisfied as well due to the special shape of $\Omega(\alpha), \alpha \in \mathcal{U}_{ad}$. Let us notice that all these assumptions hold true for any α belonging to an appropriate subset of $C^{0,1}([0,1])$. The reason why we ask $\alpha \in C^{1,1}([0,1])$ is to satisfy the remaining conditions of Section 3 and 4. Some of them have been already proven in [14], namely:

– (17) and (18) (Lemma 3 in [14]) ;
– (23) for the penalty functional

$$g^\alpha(v_\nu) = \int_0^1 (v_\nu \circ \alpha)^2 \, dx_1 = \int_0^1 (\boldsymbol{v}(x_1, \alpha(x_1)) \cdot \boldsymbol{\nu}^\alpha)^2 \, dx_1,$$

where $\boldsymbol{\nu}^\alpha$ stands for the unit outward normal vector to $S(\alpha)$.

In the next section we use the slip functional j^α, $\alpha \in \mathcal{U}_{ad}$, of the following form:

$$j^\alpha(q) = \int_{S(\alpha)} \varphi(q) \, ds, \quad \varphi(q) = \sigma_0 |q| + \frac{\sigma_1}{2} |q|^2, \; q \in L^2(S(\alpha)), \tag{43}$$

where σ_0 and σ_1 are given non-negative constants such that $\sigma_0 + \sigma_1 > 0$. With this choice of φ, the boundary condition $(1)_5$ can be rewritten as follows:

$$\left. \begin{aligned} &|\sigma_t| < \sigma_0 \Rightarrow u_t = 0, \\ &|\sigma_t| \geq \sigma_0 \Rightarrow -\sigma_t = \sigma_0 \frac{u_t}{|u_t|} + \sigma_1 u_t \end{aligned} \right\} \text{ on } S(\alpha). \tag{44}$$

If $\sigma_0 = 0$ then (44) is usually referred to as the Navier boundary condition, while for $\sigma_1 = 0$ it reminds the Tresca friction law known from solid mechanics.

Lemma 5.1. *The functional j^α defined by (43) is non-negative, convex, continuous and weakly lower semicontinuous in $L^2(S(\alpha))$ $\forall \alpha \in \mathcal{U}_{ad}$.*

Proof. Clearly, φ is non-negative and convex, so is j^α. Since

$$j^\alpha(q) = \sigma_0 \|q\|_{L^1(S(\alpha))} + \frac{\sigma_1}{2}\|q\|^2_{0,S(\alpha)},$$

its continuity and the weak lower semicontinuity, respectively, follows from the corresponding properties of the norms $\|\cdot\|_{L^1(S(\alpha))}$ and $\|\cdot\|_{0,S(\alpha)}$. $\square$

For any $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon > 0$ we define the regularization functional

$$j^\alpha_\varepsilon(q) = \int_{S(\alpha)} \varphi_\varepsilon(q)\, ds,$$

where

$$\varphi_\varepsilon(q) = \begin{cases} \varphi(q) & \text{if } |q| \geq \varepsilon, \\ \sigma_0 \frac{|q|^2+\varepsilon^2}{2\varepsilon} + \frac{\sigma_1}{2}|q|^2 & \text{if } |q| < \varepsilon, \end{cases} \quad q \in L^2(S(\alpha)). \tag{45}$$

The behavior of j^α_ε and j^α with respect to $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon \to 0+$ is summarized in the next lemma.

Lemma 5.2. *The functionals $j^\alpha, j^\alpha_\varepsilon$ defined by (43), and (45), respectively, have the following properties:*

(i) for every $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon > 0$, j^α_ε is non-negative, convex and continuously differentiable in $L^2(S(\alpha))$;

(ii) condition (12) is satisfied;

(iii) for every $\alpha \in \mathcal{U}_{ad}$ it holds:

$$q_\varepsilon \to q \text{ in } L^2(S(\alpha)) \Rightarrow j^\alpha_\varepsilon(q_\varepsilon) \to j^\alpha(q),\ \varepsilon \to 0+; \tag{46}$$

(iv) for every $\alpha \in \mathcal{U}_{ad}$ it holds:

$$q_\varepsilon \rightharpoonup q \text{ weakly in } L^2(S(\alpha)) \Rightarrow \liminf_{\varepsilon \to 0+} j^\alpha_\varepsilon(q_\varepsilon) \geq j^\alpha(q); \tag{47}$$

(v) if $\alpha_k \to \alpha$ in $C^1([0,1])$, $\alpha_k, \alpha \in \mathcal{U}_{ad}$, $\varepsilon_k \to 0+$, and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, then

$$j^{\alpha_k}_{\varepsilon_k}(v^k_t) \to j^\alpha(v_t), \quad k \to \infty$$

and

$$j^{\alpha_k}(v^k_t) \to j^\alpha(v_t), \quad k \to \infty.$$

Proof. It is readily seen that *(i)* holds. From the definition of φ_ε and φ we have:

$$\|\varphi_\varepsilon - \varphi\|_{\infty,\mathbb{R}} \leq \varphi_\varepsilon(0) = \frac{\sigma_0 \varepsilon}{2}, \tag{48}$$

and

$$\varphi_\varepsilon(x) \geq \varphi(x)\ \forall x \in \mathbb{R}; \tag{49}$$

ad (ii) From (48) it follows that

$$\forall \alpha \in \mathcal{U}_{ad}\ j^\alpha_\varepsilon(0) \leq \sqrt{1 + C_1^2}\frac{\sigma_0}{2}\varepsilon, \tag{50}$$

making use of the definition of $\mathcal{U}_{ad}$, i.e. (12) holds with $c := \sqrt{1+C_1^2}\frac{\sigma_0}{2}\varepsilon_0$;

ad (iii) If $q_\varepsilon \to q$ in $L^2(S(\alpha))$, $\varepsilon \to 0+$, $\alpha \in \mathcal{U}_{ad}$, then

$$|j^\alpha_\varepsilon(q_\varepsilon) - j^\alpha(q)| \leq |j^\alpha_\varepsilon(q_\varepsilon) - j^\alpha(q_\varepsilon)| + |j^\alpha(q_\varepsilon) - j^\alpha(q)| =: J_1 + J_2. \tag{51}$$

But

$$J_1 \leq \int_{S(\alpha)} |\varphi_\varepsilon(q_\varepsilon) - \varphi(q_\varepsilon)| \, ds \leq |S(\alpha)| \|\varphi_\varepsilon - \varphi\|_{\infty,\mathbb{R}} \overset{(48)}{\leq} \frac{|S(\alpha)|\sigma_0\varepsilon}{2}$$

and

$$J_2 \to 0 \text{ for } \varepsilon \to 0+,$$

as follows from continuity of j^α, i.e. (46) holds true;

ad (iv) Let $q_\varepsilon \rightharpoonup q$ weakly in $L^2(S(\alpha))$, $\varepsilon \to 0+$, $\alpha \in \mathcal{U}_{ad}$. Then

$$\liminf_{\varepsilon \to 0+} j^\alpha_\varepsilon(q_\varepsilon) \overset{(49)}{\geq} \liminf_{\varepsilon \to 0+} j^\alpha(q_\varepsilon) \geq j^\alpha(q),$$

making use of the weak lower semicontinuity of j^α;

ad (v) It holds:

$$\begin{aligned}
|j^{\alpha_k}_{\varepsilon_k}(v^k_t) - j^\alpha(v_t)| &= \left| \int_0^1 \varphi_{\varepsilon_k}(v^k_t \circ \alpha_k)\sqrt{1+\alpha_k'^2} - \varphi(v_t \circ \alpha)\sqrt{1+\alpha'^2} \right| dx_1 \\
&\leq \int_0^1 \left|\varphi_{\varepsilon_k}(v^k_t \circ \alpha_k) - \varphi(v^k_t \circ \alpha_k)\right| \sqrt{1+\alpha_k'^2}\, dx_1 \\
&+ \int_0^1 \left| \varphi(v^k_t \circ \alpha_k)\sqrt{1+\alpha_k'^2} - \varphi(v_t \circ \alpha)\sqrt{1+\alpha'^2} \right| dx_1 =: J_3 + J_4.
\end{aligned}$$

Clearly

$$J_3 \leq \|\sqrt{1+\alpha_k'^2}\|_{\infty,(0,1)} \|\varphi_{\varepsilon_k} - \varphi\|_{\infty,\mathbb{R}} \overset{(48)}{\leq} \sqrt{1+C_1^2}\frac{\sigma_0}{2}\varepsilon_k. \tag{52}$$

From $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ and $\alpha_k \to \alpha$ in $C^1([0,1])$ it follows that (see Theorem 3 in [14])

$$v^k_t \circ \alpha_k \to v_t \circ \alpha \quad \text{in } L^2(0,1), \quad k \to \infty$$

and also

$$\varphi(v^k_t \circ \alpha_k) \to \varphi(v_t \circ \alpha) \quad \text{in } L^1(0,1).$$

Hence

$$J_4 \to 0 \text{ as } k \to \infty.$$

From this and (52), the first limit in *(v)* follows. The second limit can be proven analogously. □

Owing to the definition of $\mathcal{U}_{ad}$, the system $\mathcal{O}$ is compact with respect to the C^1 norm, i.e. (34) holds. If the cost functional I is lower semicontinous as in (36) then all the assumptions of Theorem 4.1 and 4.2 are satisfied. Consequently, problems $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ have a solution. If, in addition, I satisfies (39), then Theorem 4.3 can be applied. It says that solutions to $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$ are close to the ones of $(\mathbb{P})$ in the sense of (40).

6. **Approximation and numerical realization of** $(\mathbb{P}_\varepsilon)$**.** In this section we describe how to discretize and realize shape optimization problems governed by the Stokes system with the regularized, penalized threshold, and impermeablity condition, respectively. The system of admissible domains Ω is as in Section 5, i.e. the shapes of Ω are uniquely determined by functions $\alpha \in \mathcal{U}_{ad}$ defined by (42). The control variables $\alpha \in \mathcal{U}_{ad}$ will be discretized by Bézier functions, while a finite element method will be used to discretize the state equation $(\mathcal{M}_\varepsilon(\alpha))$.

6.1. **Discrete design parametrization and a finite element approximation of the state problem.** We define the following finite dimensional parametrization of the slip boundary $S(\alpha) = \{(x_1, x_2) \mid x_1 \in [0,1],\ x_2{=}\alpha(x_1)\}$, $\alpha \in \mathcal{U}_{ad}$ using a Bézier polynomial of degree m:

$$\alpha_m(x_1) = \sum_{i=0}^{m} a_i B_i^{(m)}(x_1), \quad x_1 \in [0,1], \tag{53}$$

where $B_i^{(m)}(t) = \binom{m}{i} t^i (1-t)^{(m-i)}$, $i{=}0,...,m$ are the Bernstein polynomials on $[0,1]$. Thus, the discrete design variable is the vector of the x_2-coordinates $\mathbf{a} = (a_0, a_1, ..., a_m)$ of the Bézier control points $(\frac{i}{m}, a_i)$, $i{=}0,...,m$.

Next we discretize the state problem $(\mathcal{M}_\varepsilon(\alpha_m))$ by the P1-bubble/P1 elements satisfying the Ladyzhenskaya-Babuška-Brezzi condition [1]. Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}_h(\alpha_m)$ (a polygonal approximation of $\Omega(\alpha_m)$) and

$$\mathcal{V}_h(\alpha_m) = \{v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_h\}$$

be the space of piecewise linear finite elements of Lagrange type. The space of bubble functions is defined by

$$B_h(\alpha_m) = \left\{ v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in \operatorname{span}(b_T)\ \forall T \in \mathcal{T}_h \right\},$$

where $b_T = \lambda_{1,T}\lambda_{2,T}\lambda_{3,T} \in P_3(T)$ is the "bubble" function and $\lambda_{1,T}$, $\lambda_{2,T}$, and $\lambda_{3,T}$ are the barycentric coordinates of points with respect to the vertices of T.

Then we introduce the following finite element spaces:

$$\begin{aligned}
&\mathbb{W}_h(\alpha_m) = [\mathcal{V}_h(\alpha_m) + B_h(\alpha_m)]^2 \\
&Q_h(\alpha_m) = \left\{ q_h \in C(\overline{\Omega}_h(\alpha_m)) \mid q_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h,\ \textstyle\int_{\Omega_h(\alpha_m)} q_h\, dx = 0 \right\},
\end{aligned}$$

which are the discretizations of the spaces $\mathbb{W}(\Omega(\alpha_m))$ and $L_0^2(\Omega(\alpha_m))$, respectively.

The finite element approximation of the state problem in the parametrized domain $\Omega(\alpha_m)$ then reads (for simplicity of notation, the superscript α_m is omitted):

$$\begin{cases}
\text{Find } (\boldsymbol{u}_\varepsilon^h, p_\varepsilon^h) \in \mathbb{W}_h(\alpha_m) \times Q_h(\alpha_m) \text{ such that} \\
a(\boldsymbol{u}_\varepsilon^h, \boldsymbol{v}) - b(\boldsymbol{v}, p_\varepsilon^h) + \langle \nabla j_\varepsilon(u_{\varepsilon t}^h), v_t \rangle \\
\qquad\qquad + \dfrac{1}{\varepsilon} \langle \nabla g(u_{\varepsilon\nu}^h), v_\nu \rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha_m)} \quad \forall \boldsymbol{v} \in \mathbb{W}_h(\alpha_m) \\
b(\boldsymbol{u}_\varepsilon^h, q) = 0 \quad \forall q \in Q_h(\alpha_m).
\end{cases} \tag{$\mathcal{M}_\varepsilon^h(\alpha_m)$}$$

Finally we present a way how to construct a finite element mesh $\mathcal{T}_h$ in $\overline{\Omega}_h(\alpha_m)$ in such a way that the coordinates of its nodes $\{N^{(i)}\}_{i=1}^{n_p}$ depend smoothly on the design parameter vector $\mathbf{a}$. Let $\widehat{\mathcal{T}_h}$ be a (not necessarily structured) reference triangulation of the square $[0,1] \times [0,1]$ with the nodes $\{\hat{N}^{(i)}\}_{i=1}^{n_p}$. Then we set

$$N_1^{(i)} = \hat{N}_1^{(i)}, \quad N_2^{(i)} = \hat{N}_2^{(i)} + \alpha_m(\hat{N}_1^{(i)})(1 - \hat{N}_2^{(i)}),\ i = 1, ..., n_p,$$

where $N^{(i)} = (N_1^{(i)}, N_2^{(i)})$ and similarly for $\hat{N}^{(i)}$. This simple transformation is efficient and works well in case of moderate mesh deformations as shown in Figure 2.

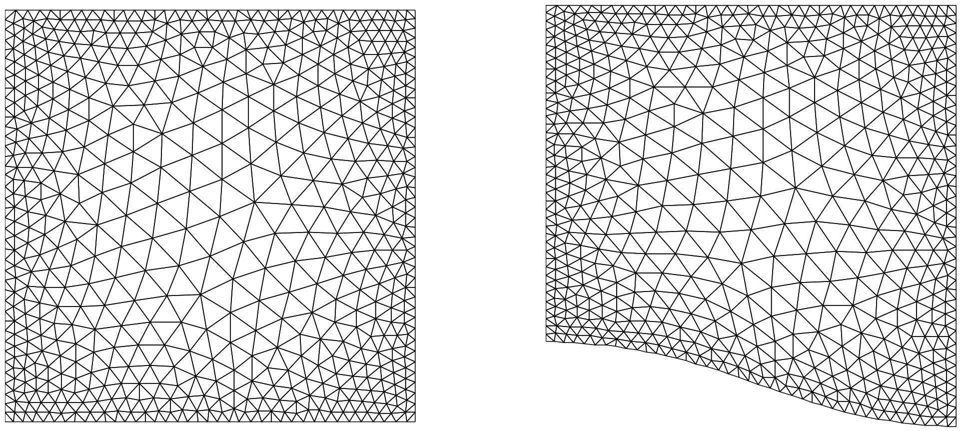

FIGURE 2. Left: reference triangulation $\widehat{\mathcal{T}_h}$. Right: Mapped triangulation $\mathcal{T}_h$.

6.2. **Nonlinear programming problem and sensitivity analysis.** After performing the finite element discretization of $(\mathcal{M}_\varepsilon^h(\alpha_m))$, the algebraic form of the state problem is given by the following system of nonlinear algebraic equations:

$$\mathbf{r}([\mathbf{u},\mathbf{p}]^{\mathrm{T}}) := \begin{bmatrix} \mathbf{A}+\mathbf{C}_\varepsilon(\mathbf{u})+\frac{1}{\varepsilon}\mathbf{G} & -\mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \tag{54}$$

where $\mathbf{u} \in \mathbb{R}^{n_u}, \mathbf{p} \in \mathbb{R}^{n_p}$ is the vector of the nodal values of the velocity $\boldsymbol{u}$ and the pressure p, respectively, $\mathbf{A} \in \mathbb{R}^{n_u \times n_u}$ is a symmetric and positive definite matrix, $\mathbf{B} \in \mathbb{R}^{n_p \times n_u}$ is the velocity-pressure coupling matrix, $\frac{1}{\varepsilon}\mathbf{G} \in \mathbb{R}^{n_c \times n_u}$ is a matrix representation of the penalized impermeability condition, and $\mathbf{C}(\mathbf{u}) \in \mathbb{R}^{n_c \times n_u}$ is a matrix function representation of the smoothed slip term. Further n_p is the total number of the nodes in $\mathcal{T}_h$, n_c is the number of the nodes lying on the slip boundary $\overline{S}(\alpha_m)$, and n_u is the dimension of the solution component representing the velocity. The system (54) can be solved iteratively by a standard way by using e.g. Newton's method.

Let

$$\boldsymbol{\mathcal{U}} = \big\{\mathbf{a}\in\mathbb{R}^{m+1} \mid \alpha_{\min}\le a_i\le\alpha_{\max},\ i=0,...,m;\ |a_{i+1}-a_i| \le \tfrac{C_1}{m},\ i=0,...,m-1,$$
$$|a_{i+2}-2a_{i+1}+a_i| \le \tfrac{C_2}{m^2},\ i=0,...,m-2\big\},$$

where C_1, C_2 are the same as in (42), be the set of admissible discrete design variables. From the properties of the Bernstein polynomials ([5]) it easily follows that if $\mathbf{a} \in \boldsymbol{\mathcal{U}}$ then $\alpha_m \in \mathcal{U}_{ad}$, where α_m is defined by (53).

As the residual vector $\mathbf{r}$ in (54) depends also on the design variable $\mathbf{a}$, we write the algebraic state problem (54) in the form

$$\mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a})) = \mathbf{0}, \quad \mathbf{q}(\mathbf{a}) = [\mathbf{u}(\mathbf{a}),\mathbf{p}(\mathbf{a})]^{\mathrm{T}}.$$

Denote $\mathfrak{J} : \boldsymbol{\mathcal{U}} \to \mathbb{R}$, $\mathfrak{J}(\mathbf{a}) := \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a}))$, where $\mathcal{I}$ is a discretization of the cost functional I. Then the discrete optimization problem to be realized reads as follows:

$$\mathbf{a}^* \in \operatorname*{argmin}_{\mathbf{a}\in\boldsymbol{\mathcal{U}}} \{\mathfrak{J}(\mathbf{a}) \mid \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a})) = \mathbf{0}\}. \tag{55}$$

In order to be able to use gradient-based nonlinear programming algorithms for solving (55) we need to evaluate the gradient of $\mathfrak{J}$ with respect to the design variable vector $\mathbf{a}$. The cost function $\mathfrak{J}$ is continuously differentiable provided that $\mathcal{I}$ is so owing to the fact that $\mathcal{T}_h$ is a smooth topologically equivalent deformation of $\widehat{\mathcal{T}}_h$ (see [12]). Then, it is well-known that the partial derivatives of $\mathfrak{J}$ with respect to the design variables are given by

$$\frac{\mathrm{d}\mathfrak{J}(\mathbf{a})}{\mathrm{d}a_i} = \frac{\partial \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_i} + \boldsymbol{\eta}^{\mathrm{T}} \left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_i}\right], \quad i = 0, ..., m, \tag{56}$$

where $\boldsymbol{\eta}$ is the solution to the adjoint equation

$$\left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial \mathbf{q}}\right]^{\mathrm{T}} \boldsymbol{\eta} = \nabla_{\mathbf{q}} \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a})). \tag{57}$$

The partial derivatives in (56), (57) can be computed by hand or using automatic differentiation of computer programs. For details we refer to [10] and [12].

Remark 1. The evaluation of the Jacobian matrix on the left hand side of (57) requires in fact C^2 continuity of φ_ε and this is not the case when φ_ε is defined by (45). To get such smoothness, the piecewise quadratic approximation of the absolute value function in $(45)_2$ has to be replaced by a piecewise quartic approximation resulting in

$$\varphi_\varepsilon(q) = \begin{cases} \varphi(q) & \text{if } |q| \geq \varepsilon. \\ \sigma_0 \left[-\frac{1}{8\varepsilon^3}|q|^4 + \frac{3}{4\varepsilon}|q|^2 + \frac{3}{8}\varepsilon\right] + \frac{1}{2}\sigma_1 |q|^2 & \text{if } |q| < \varepsilon. \end{cases}$$

The functional j_ε defined using this approximation clearly satisfies all the assumptions of Lemma 5.2.

7. **Numerical examples.** In this section we present numerical results of three model examples in which for the sake of simplicity of computations we use the bilinear form a^Ω defined by the full velocity gradients, i.e. $a^\Omega(\boldsymbol{u},\boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{0,\Omega}$. Let us mention that for this definition of a^Ω, the assumptions (10) and (19) remain valid. We consider the following cost functionals of the least squares type:

$$I_1(\Omega(\alpha), \boldsymbol{u}^\alpha, p^\alpha) = \tfrac{1}{2} \int_0^1 (u_t^\alpha \circ \alpha - u_t^0)^2 \, dx_1$$

and

$$I_2(\Omega(\alpha), \boldsymbol{u}^\alpha, p^\alpha) = \tfrac{1}{2} \int_{\Omega(\alpha)} (p^\alpha - p_0)^2 \, d\boldsymbol{x},$$

where $u_t^0 \in C([0,1]),\ p_0 \in L^2(\hat{\Omega})$ are given.

The state solver as well as the cost function evaluation were implemented using MATLAB [19]. The partial derivatives in (56), (57) of the MATLAB code were easy enough to be computed by hand. Minimization was carried out by `fmincon` with `'interior-point'` option from the MATLAB Optimization Toolbox. The parameters defining the stopping criterion were chosen as `TolX`$=10^{-4}$, `TolFun`$=10^{-4}$, `TolCon`$=10^{-5}$.

Example 1. (Tresca) Let σ_0=1, σ_1=0 in (45) and

$$\boldsymbol{f}(\boldsymbol{x}) = (10\sin(2\pi(\tfrac{1}{2} - x_2)),\, 0).$$

Our aim is to minimize the objective functional I_1 with

$$u_t^0(x_1) = 0.036 \cdot \left[\max\{\sin(2\pi x_1 - \tfrac{\pi}{5}), 0\}\right]^2.$$

The parameters defining the set $\boldsymbol{\mathcal{U}}$ are m=10, $\alpha_{\min}$=-0.05, $\alpha_{\max}$=0.25, C_1=1, and C_2=10.

We solved the shape optimization problem using a reference mesh $\widehat{\mathcal{T}}_h$ consisting of 4759 elements for three different penalty and smoothing parameters ε=10^{-3}, 10^{-4}, 10^{-5} to implement the non-penetration and slip terms. In all cases $\mathbf{a}^0$=$\mathbf{0} \in \mathbb{R}^{m+1}$ was used as the initial guess.

The zoomed optimized shapes of the slip boundaries and convergence histories of the objective function values are shown in Figure 3. Gradient based (descent) optimization methods are guaranteed to find only local minima. In this case the found local minima are close to the global ones, too. Moreover, the behaviour is stable with respect to ε. There is almost no difference between the optimized shapes for ε=10^{-4} and ε=10^{-5}.

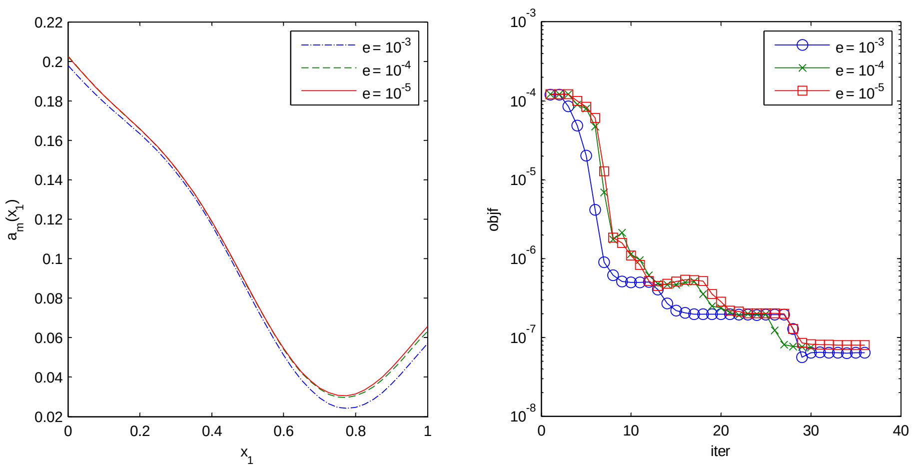

FIGURE 3. Optimized shapes (left) and convergence histories (right) for different values of the penalty/smoothing parameter ε.

The streamlines and pressure contours as well as the distributions of the tangential velocity u_t and the shear stress σ_t on $S(\alpha_{opt})$ corresponding to the state solution in the optimized domain for $\varepsilon = 10^{-5}$ are shown in Figures 4 and 5.

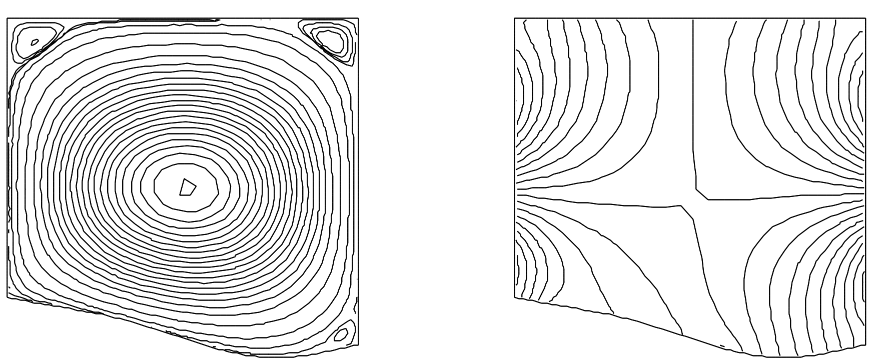

FIGURE 4. Streamlines (left) and pressure contours (right) for $\varepsilon = 10^{-5}$.

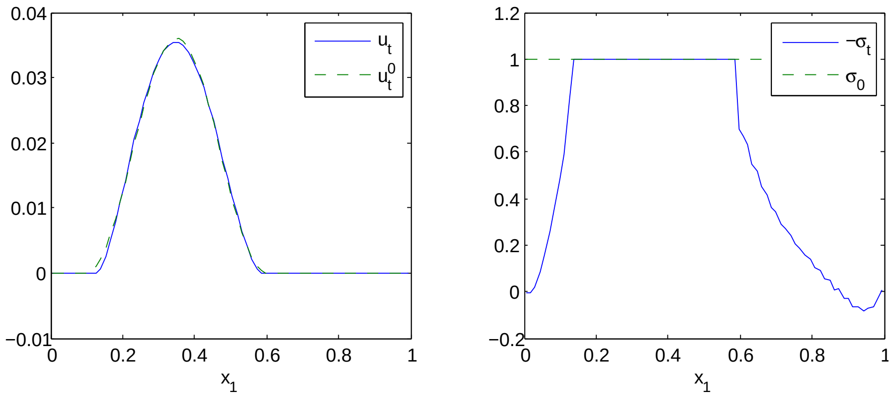

FIGURE 5. Tangential velocity and shear stress for $\varepsilon = 10^{-5}$

Example 2. (Threshold Navier) In this example we assume the threshold Navier boundary condition (44) with $\sigma_0 = 1, \sigma_1 = 10$, and $\varepsilon = 10^{-5}$. The cost functional I_1, the external force $\boldsymbol{f}$, and the reference mesh are the same as in Example 1.

The parameters defining the set $\boldsymbol{\mathcal{U}}$ are $m = 10, \alpha_{\min}{=}{-0.05}, \alpha_{\max} = 0.25, C_1 = 1$, and $C_2 = 10$. The value of the cost functional corresponding the initial guess $\mathbf{a}^0 = \mathbf{0}$ was 1.21×10^{-4}. After 28 optimization iterations (29 function evaluations) it was reduced to 1.30×10^{-7}. The streamlines and pressure contours as well as the tangential velocity and shear stress distributions on $S(\alpha_{opt})$ in the optimized domain are shown in Figures 6 and 7.

The computed optimal shapes corresponding to this example and the previous one (for $\varepsilon = 10^{-5}$) are compared in Figure 8.

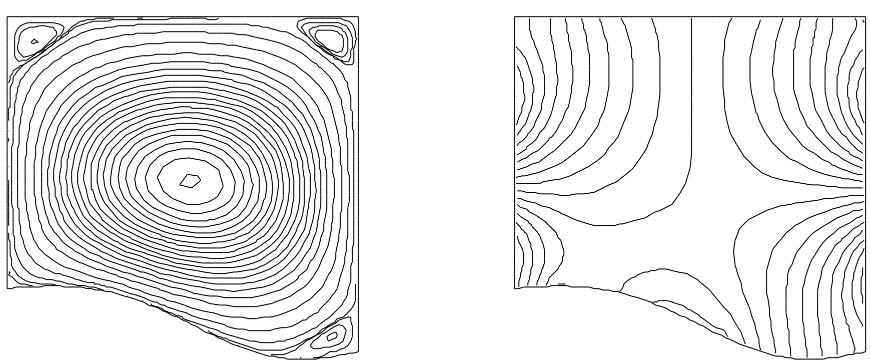

FIGURE 6. Streamlines (left) and pressure contours (right).

Example 3. In this example we wish to solve a pressure reconstruction problem, by minimizing the cost functional I_2. As the pressure is uniquely determined up to a constant, we set $p(1, 1) = 0$.

We consider the Tresca-type model with $\sigma_0{=}10,\ \sigma_1{=}0$, and $\varepsilon{=}10^{-5}$. The parameters defining $\boldsymbol{\mathcal{U}}$ are $m{=}10,\ \alpha_{\min}{=}{-0.05},\ \alpha_{\max}{=}0.25,\ C_1{=}2$, and $C_2{=}10$. Further,

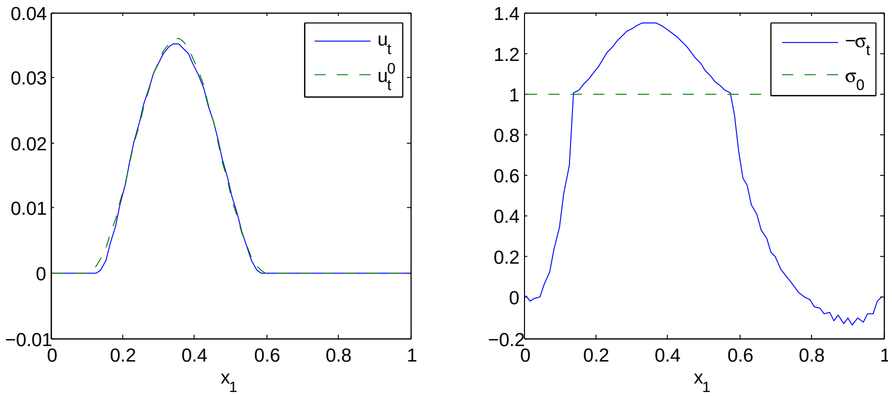

FIGURE 7. Tangential velocity and shear stress.

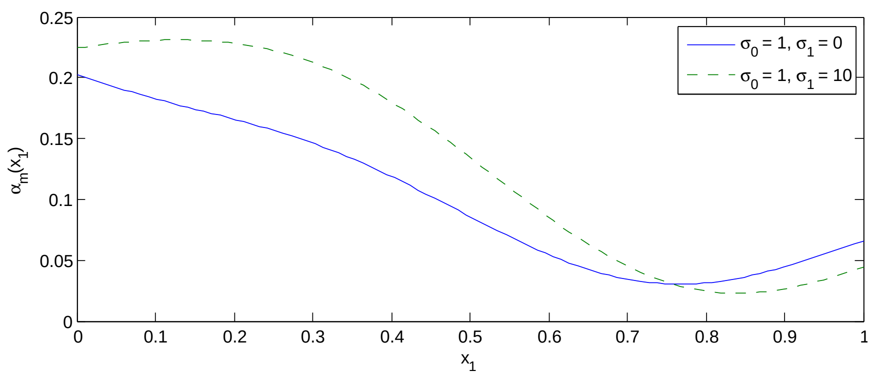

FIGURE 8. Optimized Bézier functions α_m for two different values of σ_1.

let $\boldsymbol{f} = (f_1, f_2)$, where

$$
\begin{aligned}
f_1(x_1, x_2) &= 4\pi^2(\sin(2\pi x_1) + \sin(2\pi x_2) - 2\cos(2\pi x_1)\sin(2\pi x_2)) \\
f_2(x_1, x_2) &= -4\pi^2(\sin(2\pi x_1) + \sin(2\pi x_2) - 2\cos(2\pi x_2)\sin(2\pi x_1)) \\
p_0 &= 2\pi(\cos(2\pi x_2) - \cos(2\pi x_1))
\end{aligned}
$$

be the external force and the target pressure.

The objective function value corresponding to the initial guess $\alpha_i = 0.1,\ i = 0, ..., 10$ was 1.26×10^0. After 31 optimization iterations (48 function evaluations) it was reduced to 1.06×10^{-2}.

The contours of the target pressure p_0 and the computed pressure in the optimized geometry are shown in Figure 9. The shear stress and tangential velocity distributions are shown in Figure 10.

8. **Conclusions.** In this paper we have considered shape optimization with the state constraint given by the Stokes system with the threshold slip boundary conditions on a part of the computational domain. In numerical realization, the part of boundary to be optimized is parametrized using a Bézier function. The problem is discretized using stable P1-bubble/P1 elements. The slip boundary condition

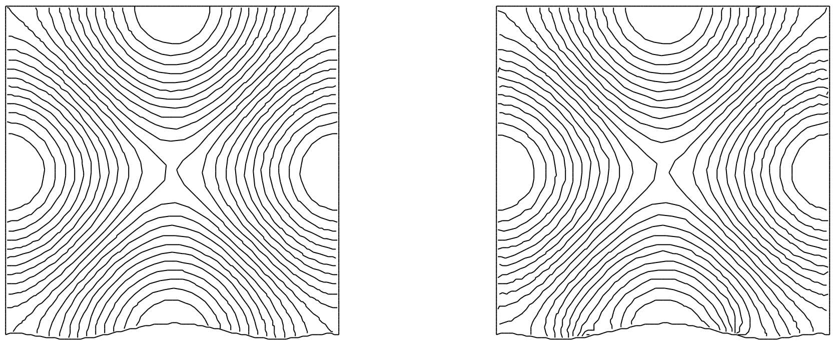

FIGURE 9. Contours of the target pressure p_0 (left) and computed pressure (right).

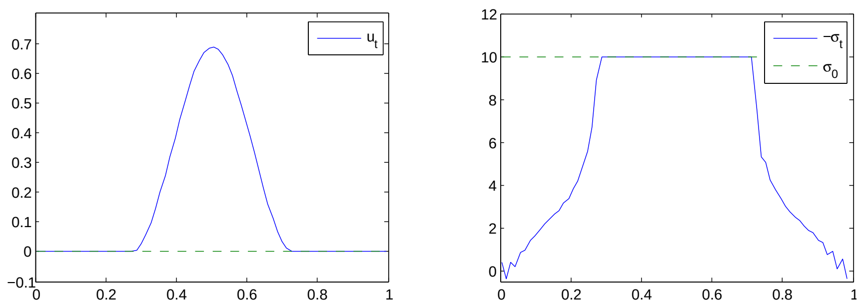

FIGURE 10. Tangential velocity and shear stress on $S(\alpha_{opt})$.

is realized approximately using a combination of the penalty method and smoothing of the nondifferentiable slip term. The numerical examples demonstrate the effectiveness of our approach.

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E-mail address: hasling@karlin.mff.cuni.cz
E-mail address: raino.a.e.makinen@jyu.fi
E-mail address: jan.stebel@tul.cz