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REGULARITY PROPERTIES OF TUG-OF-WAR GAMES AND NORMALIZED EQUATIONS

EERO RUOSTEENOJA

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Jyväskylä, March 2017

Eero Ruosteenoja

List of included articles

This dissertation consists of an introductory part and the following articles:

[A] E. Ruosteenoja. Local regularity results for value functions of tug-of-war with noise and running payoff. *Adv. Calc. Var.*, 9(1):1–17, 2016.

[B] M. Parviainen and E. Ruosteenoja. Local regularity for time-dependent tug-of-war games with varying probabilities. *J. Differential Equations*, 261(2):1357–1398, 2016.

[C] A. Attouchi, M. Parviainen, and E. Ruosteenoja. $C^{1,\alpha}$ regularity for the normalized p-Poisson problem. To appear in *J. Math. Pures Appl.*

In the introduction these articles will be referred to as [A], [B], and [C], whereas other references will be referred as [APSS12], [AS12],...

The author of this dissertation has actively taken part in the research of the joint articles [B, C].

Introduction

This dissertation addresses stochastic game theory and partial differential equations (PDEs for short). A basic observation in the theories of linear PDEs and probability is that harmonic functions and martingales share similar mean value properties. To illustrate this connection, we fix $\varepsilon > 0$ and denote by $\Omega \subset \mathbb{R}^n$ a bounded domain with the smooth boundary $\partial\Omega$. We put a token at $x_0 \in \Omega$ with $B_\varepsilon(x_0) \subset \Omega$, where $B_\varepsilon(x_0)$ is an open ball centered at x_0 with the radius ε. In general, when the token is at $x_k \in \Omega$, it is moved to $x_{k+1} \in B_\varepsilon(x_k)$ according to the uniform probability density. The process stops when the token hits the complement of Ω, and the final payoff is given by a continuous function $F : \Omega^c \to [-M, M]$. The expected payoff at x_0 is denoted by $u_\varepsilon(x_0)$, and $u_\varepsilon : \Omega \to [-M, M]$ is called a *value function.* The value function satisfies the mean value property over the ball $B_\varepsilon(x_0)$,

$$u_\varepsilon(x_0) = \fint_{B_\varepsilon(x_0)} u_\varepsilon(y)\,\mathrm{d}y := \frac{1}{|B_\varepsilon(x_0)|}\int_{B_\varepsilon(x_0)} u_\varepsilon(y)\,\mathrm{d}y. \tag{1}$$

When $\varepsilon \to 0$, value functions converge uniformly towards the unique solution of the Dirichlet problem of the Laplace equation $-\Delta u := -\mathrm{div}\,(Du) = 0$ with boundary data F. We recall that the solution of this linear PDE is called *harmonic* and satisfies a similar mean value property.

Probabilistic interpretations for nonlinear PDEs arose in 1950s and 1960s from optimal control problems and differential games. Bellman [Bel57] showed that a continuously differentiable value function of a certain optimal control problem is a solution to the first order nonlinear PDE that is now called the Hamilton-Jacobi-Bellman equation,

$$u_t(x,t) + \inf_\alpha(f(x,t,\alpha)\cdot Du(x,t) + r(x,t,\alpha)) = 0,$$

where $Du(x,t)$ is the gradient of u in space, α is the control, r is the running cost, and f gives the state dynamics. Isaacs [Isa65] studied connections between two-player zero-sum differential games and the corresponding nonlinear second order PDE that is now called Isaac's equation,

$$\sup_\alpha \inf_\beta (L_{\alpha\beta}u - f_{\alpha\beta}) = 0,$$

where $L_{\alpha\beta}$ is a family of elliptic operators with bounded measurable coefficients, and $f_{\alpha\beta}$ are real valued functions. These connections were based on the concept of *dynamic programming principle* (DPP for short), which describes Bellman's Principle of optimality ([Bel57, III.3]): Whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Equation (1) is an example of DPP.

The Hamilton-Jacobi-Bellman equation does not necessarily have continuously differentiable solutions. Since this equation is in non-divergence form, distributional weak theory based on integration by parts is not applicable. It was not clear how to relax the concept of solution. In the early 1980s Evans, Crandall and Lions [Eva80, CL84] introduced the concept of *viscosity solution* as an appropriately defined weak solution to PDEs in non-divergence form. The name viscosity solution refers to the method of vanishing viscosity to prove existence results. The method of vanishing viscosity is no longer central, but it lead to the modern definition.

1. Tug-of-war games

1.1. **Background.** In 2006 Peres, Schramm, Sheffield and Wilson introduced a two-player zero-sum stochastic game which they called *tug-of-war*, and showed that for a fixed $\varepsilon > 0$ and a bounded final payoff F, the value function u_ε of this game satisfies the DPP

$$u_\varepsilon(x) = \frac{1}{2}\Big(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon\Big).$$

In the seminal article [PSSW09], they showed that when $\varepsilon \to 0$, value functions converge uniformly towards the unique viscosity solution u of the homogeneous infinity Laplace equation

$$-\Delta_\infty u := -\langle D^2 u\, Du, Du\rangle = 0$$

with boundary data F. The infinity Laplace equation is a nonlinear PDE in non-divergence form, and the solution u is called *infinity harmonic.* This equation was introduced by Aronsson [Aro67] in the context of absolutely minimizing Lipschitz extensions. Existence, uniqueness and regularity properties of this equation gained increasing interest in 1980s and 1990s when the viscosity theory developed. After [PSSW09], game-theoretic interpretations for different variants of the infinity Laplace type equations were studied for example by Atar and Budhiraja [AB10], Antunović, Peres, Sheffield and Somersille [APSS12], Armstrong and Smart [AS12], as well as Bjorland, Caffarelli and Figalli [BCF12].

Peres and Sheffield [PS08] developed a new variant *tug-of-war with noise* and showed that it has close connections to the nonlinear PDEs that are sometimes called normalized or game-theoretic p-Laplace equations, $1 < p < \infty$. The case $p = 1$ is related to the mean curvature flow [KS06]. The version of tug-of-war with noise we present in Section 1.2 below was formulated by Manfredi, Parviainen and Rossi in [MPR12]. Ferrari, Liu and Manfredi [FLM14] extended game-theoretic methods to Heisenberg groups, whereas the game related to the obstacle problem of the p-Laplacian was studied by Lewicka and Manfredi [LM]. In [MPR10a], the authors introduced time-dependent tug-of-war games related to certain parabolic equations.

The dynamic programming principle is often the link between differential games and PDEs. In the spirit of tug-of-war games, Liu and Schikorra [LS15] obtained solutions to discrete DPPs satisfying certain general conditions.

1.2. **Tug-of-war with noise and the normalized p-Laplacian.** A central variant of tug-of-war games for this thesis is the tug-of-war with noise, played between Max and Minnie in $\Omega \subset \mathbb{R}^n$ as follows. First we fix $p > 2$ and $\varepsilon > 0$. The game token is placed at $x_0 \in \Omega$. With the probability $\alpha = (p-2)/(n+p) \in (0,1)$, the players flip a fair coin and the winner can move the token anywhere in $B_\varepsilon(x_0)$. With the probability $\beta = 1 - \alpha$, the token is moved randomly to a point in $B_\varepsilon(x_0)$ according to the uniform probability density. The game continues until the token hits the ε-boundary strip

$$\Gamma_\varepsilon := \{x \in \mathbb{R}^n \setminus \Omega \ : \ \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\}$$

for the first time at, say x_τ. Then Minnie pays Max the amount $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a bounded, Borel-measurable payoff-function. Max tries to maximize the payoff and Minnie tries to minimize it. We define the value of the game for Max in $x_0 \in \Omega$ as

$$u_{\mathrm{I}}^\varepsilon(x_0) := \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{x_0} \left[F(x_\tau)\right],$$

while the value of the game for Minnie is

$$u_{\mathrm{II}}^\varepsilon(x_0) := \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{x_0} \left[F(x_\tau)\right].$$

Here S_{I} and S_{II} are the strategies of the players.

It turns out that there is a unique value function $u_\varepsilon := u_{\mathrm{I}}^\varepsilon = u_{\mathrm{II}}^\varepsilon$ in $\Omega_\varepsilon := \Omega \cup \Gamma_\varepsilon$, $u_\varepsilon = F$ on Γ_ε, satisfying the DPP

$$u_\varepsilon(x) = \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon\right) + \beta \fint_{B_\varepsilon(x)} u_\varepsilon(y)\,\mathrm{d}y. \tag{2}$$

Heuristically, the rules of the game can be read from this equation. Moreover, when $\varepsilon \to 0$, value functions converge uniformly towards the unique viscosity solution of the normalized p-Laplace equation

$$-\Delta_p^N u = 0 \tag{3}$$

with boundary data F. Here

$$\Delta_p^N u := \frac{1}{p}|Du|^{2-p} \operatorname{div}(|Du|^{p-2}Du) = \frac{1}{p}\Delta u + \frac{p-2}{p}\Delta_\infty^N u,$$

where $\Delta_\infty^N u := \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$ is the normalized infinity Laplacian.

The definition of viscosity solution for equation (3) is based on the following observation: If u is a smooth solution of (3), $Du(x_0) \neq 0$ for $x_0 \in \Omega$, and $u - \phi$ attains a local maximum at x_0 for some function $\phi \in C^2(\Omega)$, then it holds $-\Delta_p^N \phi(x_0) \leq 0$. The viscosity solution u of (3) is required to satisfy the following conditions:

- The function u is continuous.

- For all $x_0 \in \Omega$ and for all test functions $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local maximum at x_0 and $D\phi(x_0) \neq 0$, one has $-\Delta_p^N \phi(x_0) \leq 0$.
- For all $x_0 \in \Omega$ and for all test functions $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a local minimum at x_0 and $D\varphi(x_0) \neq 0$, one has $-\Delta_p^N \varphi(x_0) \geq 0$.

This is not the whole definition. The case $D\phi(x_0) = 0$ requires special attention, because then $-\Delta_p^N \phi(x_0)$ is not well defined. For the precise definition, see for example [C, Definition 2.1]. It follows from the work of Juutinen, Lindqvist and Manfredi [JLM01] (see also [JJ12]) that viscosity solutions of (3) are distributional weak solutions of the standard p-Laplace equation $-\text{div}(|Du|^{p-2}Du) = 0$. Hence, viscosity solutions of (3) are *p-harmonic*, see also Section 3 below.

To give an idea of the connection between equations (2) and (3), we assume again that u is a smooth solution of (3) with $Du \neq 0$, and obtain heuristically a generalized mean value formula for u. Averaging the Taylor expansion

$$u(y) = u(x) + \langle Du(x), (y-x)\rangle + \frac{1}{2}\langle D^2u(x)(y-x), (y-x)\rangle + o(|y-x|^2)$$

over the ball $B_\varepsilon(x)$ and calculating

$$\fint_{B_\varepsilon(x)} \langle Du(x), (y-x)\rangle \, dy = 0$$

and

$$\fint_{B_\varepsilon(x)} \frac{1}{2}\langle D^2u(x)(y-x), (y-x)\rangle \, dy = \frac{\varepsilon^2}{2(n+2)}\Delta u(x),$$

we obtain

$$u(x) - \fint_{B_\varepsilon(x)} u(y) \, \mathrm{d}y = -\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + o(\varepsilon^2). \tag{4}$$

Moreover, by using the Taylor expansion for $u(x+h)$ with $h = \varepsilon\frac{Du(x)}{|Du(x)|}$ and $h = -\varepsilon\frac{Du(x)}{|Du(x)|}$, we get

$$\begin{aligned} u(x) - \frac{1}{2}\left(u(x + \varepsilon\frac{Du(x)}{|Du(x)|}) + u(x - \varepsilon\frac{Du(x)}{|Du(x)|})\right) \\ \approx -\frac{1}{2}\varepsilon^2\Delta_\infty^N u(x) + o(\varepsilon^2). \end{aligned}$$

From this we heuristically deduce

$$u(x) - \frac{1}{2}\left(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\right) = -\frac{1}{2}\varepsilon^2\Delta_\infty^N u(x) + o(\varepsilon^2). \tag{5}$$

By combining the expressions (4) and (5) and recalling that $\alpha = (p-2)/(n+p)$ and $\beta = (n+2)/(n+p)$, we get a generalized mean value formula for u,

$$u(x) = \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\right) + \beta \fint_{B_\varepsilon(x)} u(y)\,\mathrm{d}y + o(\varepsilon^2).$$

However, not all viscosity solutions of (3) are smooth in general, and the actual proof of the connection between value functions and viscosity solutions of (3) is more subtle, see [MPR10b].

2. Local regularity of tug-of-war games and articles [A, B]

In [A, B] we study the local regularity properties of two variants of tug-of-war games. The main goal is to understand the properties of these games on their own right. In [A] we study the game *tug-of-war with noise and running payoff*, where after each move Max gains a small payoff. In [B] we study the local regularity of time-dependent tug-of-war games where the probabilities α and β may vary. In the following we denote by $B_r \subset \mathbb{R}^n$ a ball centered at the origin with the radius r, and

$$\operatorname{osc}(u_\varepsilon, B_r) := \sup_{B_r} u_\varepsilon - \inf_{B_r} u_\varepsilon.$$

2.1. **Tug-of-war with noise and running payoff.** In [A] we study the following variant of tug-of-war games. We fix $\varepsilon > 0$ and denote by $\Omega \subset \mathbb{R}^n$ a bounded domain satisfying the exterior ball condition. The final payoff of the game is given by a bounded, measurable function $F : \Gamma_\varepsilon \to [-M, M]$. Moreover, we add a bounded, measurable *running payoff* $f : \Omega \to (0, M]$. The players play the tug-of-war with noise as explained in Section 1, but when the token moves from x_k to x_{k+1}, the amount $\varepsilon^2 f(x_k)$ is charged. First we show that this game has a unique value function u_ε satisfying the DPP

$$u_\varepsilon(x) = \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon\right) + \beta \fint_{B_\varepsilon(x)} u_\varepsilon(y)\,\mathrm{d}y + \varepsilon^2 f(x).$$

Moreover, u_ε is uniformly bounded with respect to ε, that is, there is a constant $C = C(p, n)$ for which

$$|u_\varepsilon| \le C(\sup_{\Gamma_\varepsilon} F + \sup_{\Omega} f).$$

We focus on the local regularity of the value function. Luiro, Parviainen and Saksman [LPS13] proved asymptotic Lipschitz regularity for value functions of tug-of-war with noise by using an idea they call cancellation strategy. They estimate the value function by setting sub-optimal strategies where players first try to cancel each others' moves and then pull the token in a certain fixed direction. Suppose that $B_{6R} \subset \Omega$, $x, y \in B_R$ with $|x - y| \ge \varepsilon$, and for $z \in B_{2R}$ it holds $|x - z| = |y - z| \le |x - y|$ and $\frac{3|x-z|}{\varepsilon} =: m \in \mathbb{N}$. The precise strategies used in [A] are the following. When the game starts from x, we fix the following strategy for Minnie: If Max has won more coin tosses than Minnie, then she cancels the earliest move of Max she has not yet cancelled. Otherwise, she moves the token to the direction of the vector $\frac{z-x}{m}$. The length of her move is always $\frac{\varepsilon}{3}$. We re-examine the situation (technically speaking, we define a new stopping time) if Minnie wins m coin tosses more than Max, or if Max wins at least $2R/\varepsilon$ times more than Minnie, or if the length of the sum of random vectors exceeds $2R$. Then the game stays in B_{6R} before the re-examination. For the game that starts from y, Max follows the cancellation strategy where he cancels the earliest move of Minnie he has not yet cancelled, or otherwise moves the token to the direction of the vector $\frac{z-y}{m}$.

The authors showed in [LPS13] that because of translation invariance and the fact that the game stops almost surely in a finite time, such strategies yield an asymptotic Lipschitz estimate. The most important difference between the game with a strictly positive running payoff f and the game with $f \equiv 0$ considered in [LPS13] is that for our purpose it is not enough to know that the game ends almost surely in a finite time. We usually need an estimate for the expected stopping time of the game. By using the cancellation strategies, we get an estimate

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq (1-P)\operatorname{osc}(u_\varepsilon, B_{6R}) + \varepsilon^2 \mathbb{E}[\tau^*]\operatorname{osc}(f, B_{6R}),$$

where P is the probability that the game, started from x, was re-examined because Minnie won m coin tosses more than Max. (By symmetry, P is also the probability that the game, started from y, was re-examined because Max won m coin tosses more than Minnie.) To obtain the Lipschitz estimate with a Lipschitz constant independent of ε in the proof of [A, Theorem 4.1], the main task is to estimate $\mathbb{E}[\tau^*]$, which is the expected number of steps before the re-examination.

In [A, Theorem 4.2] we use the Lipschitz estimate and an iteration in dyadic balls to prove Harnack's inequality for non-negative value functions. As in the proof of the Lipschitz estimate, the main task is to control the cumulative effect of the running payoff. The following theorem summarizes the main results of [A].

Theorem 1. [A, Theorems 4.1 and 4.2] *Suppose that for some $R > \varepsilon$ it holds $B_{30R} \subset \Omega$. Let $r \in (\varepsilon, R]$. Then there is a constant $C = C(p, n)$ for which the value function u_ε satisfies the asymptotic Lipschitz estimate*

$$\operatorname{osc}(u_\varepsilon, B_r) \leq C\frac{r}{R}\left[\operatorname{osc}(u_\varepsilon, B_{6R}) + \operatorname{osc}(f, B_{6R})\right].$$

Moreover, in the case $u \geq 0$ the value function satisfies Harnack's inequality

$$\sup_{B_R} u_\varepsilon \leq C(\inf_{B_R} u_\varepsilon + \sup_{B_{30R}} f).$$

2.2. **Time-dependent tug-of-war with varying probabilities.** In this section we consider the time-dependent variant of tug-of-war games which we study in [B]. For $T > 0$, we denote by $\Omega_T := \Omega \times (0, T)$ a parabolic cylinder with the parabolic boundary

$$\Gamma_T := \{\partial\Omega \times [0, T]\} \cup \{\Omega \times \{0\}\}$$

and the parabolic boundary strip of width $\varepsilon > 0$,

$$\Gamma_T^\varepsilon := \left(\Gamma_\varepsilon \times [-\frac{\varepsilon^2}{2}, T]\right) \cup \left(\Omega \times [-\frac{\varepsilon^2}{2}, 0]\right).$$

Here Γ_ε is the ε-boundary strip of Ω. For a measurable function $p : \Omega_T \to (2, \infty)$, we define the functions $\alpha : \Omega_T \to (0, 1)$ and $\beta : \Omega_T \to (0, 1)$,

$$\alpha(x, t) = \frac{p(x, t) - 2}{p(x, t) + n}, \quad \beta(x, t) = \frac{n + 2}{p(x, t) + n}.$$

The game starts from the point $(x_0, t_0) \in \Omega_T$. When the token is at a point (x_k, t_k), with the probability $\alpha(x_k, t_k)$, the players flip a fair coin, and the winner of the toss moves the token to a point

$$(x_{k+1}, t_{k+1}) \in B_\varepsilon(x_k) \times \{t_k - \frac{\varepsilon^2}{2}\},$$

according to his or her strategy. With the probability $\beta(x_k, t_k)$, the token is moved to (x_{k+1}, t_{k+1}) in a set $B_\varepsilon(x_k) \times \{t_k - \frac{\varepsilon^2}{2}\}$ according to the uniform probability density. We denote by $(x_\tau, t_\tau) \in \Gamma_T^\varepsilon$ the first point of the sequence on Γ_T^ε. Then Minnie pays Max the amount $F(x_\tau, t_\tau)$, where $F : \Gamma_T^\varepsilon \to [-M, M]$ is a given measurable payoff function.

It follows that the number of steps during the game is bounded, and the value function satisfies the DPP

$$\begin{aligned} u_\varepsilon(x, t) = &\frac{\alpha(x, t)}{2} \left(\sup_{y \in B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2}) + \inf_{y \in B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2}) \right) \\ &+ \beta(x, t) \fint_{B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2})\, dy. \end{aligned}$$

Manfredi, Parviainen and Rossi showed in [MPR10a] that in the case of constant probabilities this game has a connection to the parabolic normalized p-Laplace equation. For this variant with constant probabilities $\alpha(x, t) \equiv \alpha$ and $\beta(x, t) \equiv \beta$ we show asymptotic Lipschitz continuity by relying on a similar cancellation strategy that was first used in [LPS13], and later in [Hei] in the context of game regularity.

Theorem 2. [B, Theorem 3.3] *Suppose that $B_{6r} \subset \Omega$, where $0 < \varepsilon < r < \left(\frac{\alpha T}{6}\right)^{\frac{1}{2}}$. Suppose also that $x, y \in B_r$ and $\frac{6r^2}{\alpha} < t_0 < t_1 < T$ with $t_1 - t_0 \le r^2$. Then for the constant probabilities α and β, the value function satisfies the*

Lipschitz estimate

$$|u_\varepsilon(x,t_1)-u_\varepsilon(y,t_0)| \leq C(p,n)\frac{|x-y|+|t_1-t_0|^{1/2}}{r}\left\|u_\varepsilon\right\|_\infty + C'(p,n)\frac{\varepsilon^{1/2}}{r}\left\|u_\varepsilon\right\|_\infty .$$

For the general case with varying probabilities $\alpha(x,t)$ and $\beta(x,t)$ the cancellation strategy does not seem to work, mainly because the game lacks good symmetry properties and translation invariance. To study local Hölder regularity, we use a time-dependent version of the idea that was first introduced by Luiro and Parviainen in [LP]. This strategy uses a suitable comparison function defined in $\mathbb{R}^{2n+1}$ and having a certain favorable curvature in space.

Theorem 3. [B, Theorems 4.1, 4.2 and 4.7] *Suppose that for $r>0$ it holds $B_{2r}\subset\Omega$ and $r^2<t_0<t_1<T$. Then the value function satisfies the Hölder estimate for some $\delta=\delta(n)\in(0,1)$,*

$$|u_\varepsilon(x,t_1)-u_\varepsilon(y,t_0)| \leq C(n)\frac{|x-y|^\delta+|t_1-t_0|^{\delta/2}}{r^\delta}+C'(n)\frac{\varepsilon^{\delta/2}}{r^\delta}.$$

Moreover, in the case $u_\varepsilon\geq 0$ the value function satisfies Harnack's inequality

$$\sup_{x\in B_r} u_\varepsilon(x,t_0-r^2)\leq C(n)\inf_{x\in B_r} u_\varepsilon(x,t_0).$$

As an application, we show that when $\varepsilon\to 0$, the value functions u_ε approximate uniformly the viscosity solution of the normalized parabolic $p(x,t)$-Laplace equation

$$(n+p(x,t))u_t=\Delta u+(p(x,t)-2)\Delta_\infty^N u \tag{6}$$

with continuous boundary data F. As a consequence of Theorem 3, it follows that the viscosity solution u of (6) is locally Hölder continuous with a Hölder constant depending only on the dimension n, and if $u\geq 0$, it satisfies Harnack's inequality.

3. The normalized p-Poisson problem and articles [A, C]

In [PS08] Peres and Sheffield showed the connection between the tug-of-war with noise and running payoff and smooth solutions of the PDE

$$-\Delta_p^N u=f. \tag{7}$$

We call equation (7) *the normalized p-Poisson problem.* It is a nonlinear PDE in non-divergence form. For $1<p<\infty$, the normalized p-Laplacian Δ_p^N is a uniformly elliptic operator with ellipticity constants $\Lambda=\max(p-1,1)$ and $\lambda=\min(p-1,1)$. Krylov and Safonov [KS79, KS80] proved Harnack's inequality and Hölder continuity for solutions of linear uniformly elliptic equations with bounded and measurable coefficients. Caffarelli [Caf89] extended these results for viscosity solutions of fully nonlinear uniformly elliptic equations. For equations in non-divergence form, these results have a similar role as the De Giorgi−Nash−Moser theorem has for equations in divergence form. Caffarelli also proved $C^{1,\alpha}$ regularity under such regularity assumptions which the normalized p-Laplacian does not satisfy because of gradient dependence and discontinuity. Hence, only Hölder continuity for solutions of (7) follows from the general regularity theory for uniformly elliptic equations.

In [A] we show that for a given continuous boundary data and a continuous, strictly positive and bounded f, value functions of tug-of-war with noise and running payoff converge uniformly towards the unique viscosity solution of (7). From Theorem 1 we get a game-theoretic proof that in the case $p > 2$ the unique viscosity solution of (7) is locally Lipschitz continuous and satisfies Harnack's inequality. (Uniqueness was shown in [KMP12] in the case $f > 0$.) Earlier Charro, De Philippis, Di Castro and Máximo had obtained Lipschitz-type estimates in the case of $p > n$ in [CPCM13] by using PDE methods. In the case $p \geq 2$ Birindelli and Demengel [BD10] proved $C^{1,\alpha}$ regularity for a class of operators including the normalized p-Laplacian.

After [PS08], other normalized p-Laplace equations have received attention as well. For the normalized parabolic p-Laplace equation, Banerjee and Garofalo [BG15] studied the boundary behavior, Jin and Silvestre [JS] showed local $C^{1,\alpha}$ regularity, and Does [Doe11] studied applications to image processing.

In [C] we study higher regularity of viscosity solutions of (7). The main theorem is the following regularity result in the whole range $1 < p < \infty$.

Theorem 4. [C, Theorem 1.1] *Assume that* $p > 1$ *and* $f \in L^\infty(\Omega) \cap C(\Omega)$. *There exists* $\alpha = \alpha(p, n) > 0$ *such that any viscosity solution* u *of* (7) *is in* $C^{1,\alpha}_{\text{loc}}(\Omega)$, *and for any* $\Omega' \subset\subset \Omega$,

$$[u]_{C^{1,\alpha}(\Omega')} \leq C = C\left(p, n, d, d', ||u||_{L^\infty(\Omega)}, ||f||_{L^\infty(\Omega)}\right),$$

where $d = \text{diam}(\Omega)$ *and* $d' = \text{dist}(\Omega', \partial\Omega)$.

Heuristically, we show that if a solution u can be approximated by a plane in a small ball, then in a smaller ball there is a plane giving a better approximation. To get a $C^{1,\alpha}$ estimate, we have to show that the error in the approximation improves by a multiplicative factor $\rho < 1$. An inductive argument leads us to analyze the regularity of deviations of solutions from planes, and the required oscillation estimate for these deviations is called improvement of flatness. Recently, Imbert and Silvestre [IS13] used this method to show $C^{1,\alpha}$ regularity for viscosity solutions of $|Du|^\gamma F(D^2u) = f$, where F is uniformly elliptic.

In the case $p \geq 2$ we give another proof, which is essentially to show that viscosity solutions of (7) are distributional weak solutions of

$$-\mathrm{div}(|Du|^{p-2}Du) = |Du|^{p-2}f.$$

Then we can rely on the known regularity results for quasilinear PDEs. When using these techniques, it seems necessary that the $C^{1,\alpha}$ estimate depends on the L^∞ norm of f. Relaxing the dependence to $||f||_{L^q(\Omega)}$ for some $q < \infty$ requires a stricter assumption on p and a different method for the proof.

Theorem 5. [C, Theorem 1.2] *Assume that* $p > 2$, $q > \max(2, n, p/2)$, $f \in C(\Omega) \cap L^q(\Omega)$. *Then any viscosity solution* u *of* (7) *is in* $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha = \alpha(p, q, n)$. *Moreover, for any* $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, *with* Ω' *smooth enough, we have*

$$[u]_{C^{1,\alpha}(\Omega'')} \leq C = C\left(p, q, n, d, d'', ||u||_{L^\infty(\Omega)}, ||f||_{L^q(\Omega)}\right),$$

where $d = \mathrm{diam}\,(\Omega)$ *and* $d'' = \mathrm{dist}\,(\Omega'', \partial\Omega')$.

The proof is based on proving uniform gradient estimates for weak solutions v_ε of certain regularized equations in divergence form. We use the potential estimates of Duzaar and Mingione [DM10] together with the De Giorgi iteration. From the result of Lieberman [Lie93] we get a uniform estimate for $[v_\varepsilon]_{C^{1,\beta}(\Omega)}$ for some $\beta > 0$, and Theorem 5 follows from a compactness argument.

We also study the question of optimal regularity of solutions of (7). We recall that distributional weak solutions of the standard p-Laplace equation $-\mathrm{div}(|Du|^{p-2}Du) = 0$ are called *p-harmonic*, and they are of class $C^{1,\alpha_0}_{\mathrm{loc}}$ for some $\alpha_0 > 0$ depending only on p and the dimension n. Since the case $f \equiv 0$ is covered in Theorem 4 and viscosity solutions of the homogeneous normalized p-Laplace equation are p-harmonic, viscosity solutions of (7) should not be expected to be more regular than p-harmonic functions in general. Hence, α_0 is a natural upper bound for Hölder exponent of gradients of solutions of (7). By approximating the solutions of (7) by p-harmonic functions and using suitable rescaled function and iteration, for arbitrary $\varepsilon > 0$ we show local $C^{1,\alpha_0-\varepsilon}$ regularity for solutions of (7) in [C, Theorem 1.3]. The possible C^{1,α_0} regularity remains an open problem.

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Local regularity results for value functions of tug-of-war with noise and running payoff

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Research Article

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Local regularity results for value functions of tug-of-war with noise and running payoff

Abstract: We prove local Lipschitz continuity and Harnack's inequality for value functions of the stochastic game tug-of-war with noise and running payoff. As a consequence, we obtain game-theoretic proofs for the same regularity properties for viscosity solutions of the inhomogeneous p-Laplace equation when $p > 2$.

Keywords: Stochastic games, Lipschitz estimates, Harnack's inequality, p-Laplacian, viscosity solutions, tug-of-war

MSC 2010: 35B65, 91A15

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1 Introduction

Max and Minnie play a zero-sum stochastic game as follows. Fix $\epsilon > 0$. First a token is placed at $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. With probability $\alpha \in (0, 1)$ they flip a fair coin and the winner can move the token anywhere in an open ball $B_\epsilon(x_0)$. With probability $\beta = 1 - \alpha$ the token moves to a random point in $B_\epsilon(x_0)$. The game continues until the token hits the boundary of Ω for the first time in, say x_τ. Then Minnie pays Max a *total payoff*

$$F(x_\tau) + \epsilon^2 \sum_{j=0}^{\tau-1} f(x_j),$$

where F is a bounded *final payoff* on the boundary and f a positive, bounded *running payoff* in Ω. Since $\beta > 0$, the game ends almost surely in a finite time. Max tries to maximize the total payoff and Minnie tries to minimize it.

For given payoffs and $\epsilon > 0$, the game has a value u_ϵ, which is locally Lipschitz continuous up to the scale ϵ. To be more precise, we show in Theorem 4.1 that if $B_{6R}(a) \subset \Omega$ and $\epsilon < r \leq R$, then

$$\operatorname{osc}(u_\epsilon, B_r(a)) \leq C\frac{r}{R}\big[\operatorname{osc}(u_\epsilon, B_{6R}(a)) + \operatorname{osc}(f, B_{6R}(a))\big],$$

where $C > 0$ depends only on p and n. In Theorem 4.2 we show that if $B_{30r}(a) \subset \Omega$, the value function u_ϵ satisfies Harnack's inequality

$$\sup_{B_r(a)} u_\epsilon \leq K(\inf_{B_r(a)} u_\epsilon + \sup_{\Omega} f),$$

where $K = K(p, n) > 0$. In the proofs of Theorems 4.1 and 4.2 key ideas are related to controlling the expected cumulative effect of running payoff during the game under proper strategies.

According to Lemmas 2.1 and 2.2, the value functions satisfy

$$u_\epsilon(x) = \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x)} u_\epsilon + \inf_{B_\epsilon(x)} u_\epsilon\Big\} + \beta \fint_{B_\epsilon(x)} u_\epsilon dy + \epsilon^2 f(x) \tag{1.1}$$

for all $x \in \Omega$. Choosing the probabilities α and β properly, this *dynamic programming principle* (hereafter DPP) gives a connection to viscosity solutions of the inhomogeneous p-Laplace equation

$$-\frac{1}{p}|\nabla u|^{2-p} \operatorname{div}(|\nabla u|^{p-2}\nabla u) =: -\Delta_p^N u = f, \tag{1.2}$$

where $p > 2$. Let $f > 0$ be continuous and bounded. If u is a viscosity solution to (1.2) in Ω with some continuous and bounded boundary values, by Lemma 5.4 there is a sequence (u_ϵ) of value functions converging locally uniformly to u. By Theorems 4.1 and 4.2, the function u is locally Lipschitz continuous and satisfies Harnack's inequality. To the best of our knowledge, Lipschitz estimate is unavailable in the literature in the case $2 < p \leq n$. In the case $p > n \geq 2$ similar estimates were proven in [7] by using PDE methods. Harnack's inequality also follows by utilizing PDE methods described, e.g., in [6, 10]. For recent advances, see, e.g., [8].

Tug-of-war games were first introduced by Peres, Schramm, Sheffield and Wilson in [21] and by Peres and Sheffield in [22]. Various versions of the game have connections to the theory of non-linear PDEs. For example, value functions of the tug-of-war approximate infinity harmonic functions and value functions of the tug-of-war with noise approximate p-harmonic functions. Game-theoretic arguments have generated many new, intuitive proofs for uniqueness and regularity properties of infinity harmonic and p-harmonic functions. See, e.g., [1–5, 13, 15, 20]. For existence of viscosity solutions to certain parabolic equations, see, e.g., [9, 17]. For different versions of DPP, see, e.g., [11].

In Section 2 we define the game, show that it has a value which satisfies DPP (1.1), and give tools to estimate the value functions under different strategies and payoffs. In Section 3 we prove lemmas regarding expected stopping times under specific strategies, local comparison of value functions and control of infimum. In Section 4 we prove Theorems 4.1 and 4.2. In Section 5 we discuss the connection to the inhomogeneous p-Laplace equation.

2 Background of the game

Fix $\epsilon > 0$ and $p > 2$. The probabilities in the game are $\alpha = (p-2)/(n+p)$ and $\beta = (n+2)/(n+p)$. Define

$$\Gamma_\epsilon := \{x \in \mathbb{R}^n \setminus \Omega : \operatorname{dist}(x, \partial\Omega) \leq \epsilon\}$$

and

$$\Omega_\epsilon := \Omega \cup \Gamma_\epsilon.$$

Then $B_\epsilon(x) \subset \Omega_\epsilon$ for all $x \in \Omega$. The game ends when the token hits Γ_ϵ for the first time. In Sections 2, 3 and 4 the payoffs $F : \Gamma_\epsilon \to \mathbb{R}$ and $f : \Omega \to (0, \infty)$ are bounded and Borel measurable.

Let us briefly describe the stochastic terminology used in this paper. Strategies S_{I} for Max and S_{II} for Minnie are collections of Borel measurable functions that give the next game position given the history of the game. When we fix a certain strategy for a player, we usually write $S_{\mathrm{I}}^{\mathrm{Max}}$ for Max and $S_{\mathrm{II}}^{\mathrm{Min}}$ for Minnie. By a history of the game up to step k we mean a sequence

$$(x_0, (c_1, x_1), \dots, (c_k, x_k)),$$

where $x_0, \dots, x_k \in \Omega_\epsilon$ are game positions and $c_j \in \mathcal{C} := \{0, 1, 2\}$. Here $c_j = 0$ means that Max wins, $c_j = 1$ that Minnie wins and $c_j = 2$ that a random step occurs. Our probability space is the space of all game sequences

$$H^\infty := \{\omega : \omega \in x_0 \times (\mathcal{C}, \Omega_\epsilon) \times \cdots\}.$$

Put $\mathcal{F}_0 := \sigma(x_0)$ and

$$\mathcal{F}_k := \sigma(x_0, (c_1, x_1), \dots, (c_k, x_k))$$

for $k \geq 1$. Note that here (c_1, x_1) etc. are random variables. Then

$$\tau(\omega) := \inf\{k : x_k \in \Gamma_\epsilon,\ k = 0, 1, \dots\}$$

is a stopping time relative to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$.

The fixed starting point x_0 and the strategies S_{I} and S_{II} determine a unique probability measure $\mathbb{P}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}$ on the product σ-algebra, see, e.g., [15].

The expected total payoff, when starting from x_0 and using strategies S_{I} and S_{II}, is obtained as a sum of final payoff and running payoff

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F(x_\tau)+\epsilon^2\sum_{i=0}^{\tau-1}f(x_i)\Big]:=\int_{H^\infty}\Big(F(x_\tau(\omega))+\epsilon^2\sum_{i=0}^{\tau-1}f(x_i)\Big)d\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\omega).$$

The value of the game for Max in $x_0\in\Omega$ is given by

$$u^\epsilon_{\mathrm{I}}(x_0):=\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F(x_\tau)+\epsilon^2\sum_{i=0}^{\tau-1}f(x_i)\Big],$$

while *the value of the game for Minnie* is given by

$$u^\epsilon_{\mathrm{II}}(x_0):=\inf_{S_{\mathrm{II}}}\sup_{S_{\mathrm{I}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F(x_\tau)+\epsilon^2\sum_{i=0}^{\tau-1}f(x_i)\Big].$$

If a function u is defined in Ω_ϵ, $u=F$ on Γ_ϵ and

$$u=u^\epsilon_{\mathrm{I}}=u^\epsilon_{\mathrm{II}}$$

in Ω, then u is *the value of the game*.

The next two lemmas guarantee that the game has a value which satisfies DPP (1.1). For similar results without a running payoff, see [16, Theorems 2.1, 2.2, 3.2].

Lemma 2.1. *For given payoffs and $\epsilon>0$, there is a unique Borel measurable function $u_\epsilon:\Omega_\epsilon\to\mathbb{R}$, $u_\epsilon=F$ on Γ_ϵ, which satisfies DPP* (1.1) *for all $x\in\Omega$.*

Proof. First we show the existence. Let $(u_k)_{k=0}^\infty$ be a sequence of functions $\Omega_\epsilon\to\mathbb{R}$ such that $u_k=F$ on Γ_ϵ for all $k\in\mathbb{N}$, $u_0=\inf_{\Gamma_\epsilon}F$ in Ω and

$$u_{k+1}(x)=\frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x)}u_k+\inf_{B_\epsilon(x)}u_k\Big\}+\beta\fint_{B_\epsilon(x)}u_k\,dy+\epsilon^2f(x)$$

for all $k\in\mathbb{N}$ and $x\in\Omega$. If $k\geq1$ and $u_k\geq u_{k-1}$, then

$$u_{k+1}(x)\geq\frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x)}u_{k-1}+\inf_{B_\epsilon(x)}u_{k-1}\Big\}+\beta\fint_{B_\epsilon(x)}u_{k-1}\,dy+\epsilon^2f(x)=u_k(x)$$

for all $x\in\Omega$. Since $f>0$, we have $u_1\geq u_0$. By induction, the sequence (u_k) is increasing.

The sequence (u_k) is also bounded. Let $D=\operatorname{diam}(\Omega)$ and $N=\sup_\Omega f$. Note that for any point $y_0\in\Omega$ there is a sequence $(y_i)_{i=0}^{2D/\epsilon}$ for which $|y_{i+1}-y_i|\leq\frac{\epsilon}{2}$ and $y_{2D/\epsilon}\in\Gamma_\epsilon$. Choose an arbitrary $k_0\in\mathbb{N}$. We may assume

$$\sup_\Omega u_{k_0}\geq\sup_{\Gamma_\epsilon}F.$$

Choose a point $x_0\in\Omega$ such that

$$u_{k_0}(x_0)>\Big(1-\frac{1}{2}\Big(\frac{\alpha}{2}\Big)^{2D/\epsilon}\Big)\sup_\Omega u_{k_0}.$$

Let $(x_j)_{j=0}^J\subset\Omega_\epsilon$ be a sequence for which $|x_{j+1}-x_j|\leq\frac{\epsilon}{2}$, $x_J\in\Gamma_\epsilon$ and $J\leq\frac{2D}{\epsilon}$. By using the rough estimates

$$\sup_{\Omega_\epsilon}u_{k_0-1}\leq\sup_\Omega u_{k_0},$$
$$\inf_{B_\epsilon(x_j)}u_{k_0-1}\leq u_{k_0}(x_j)$$

and DPP we obtain

$$u_{k_0}(x_0)\leq\Big(\frac{\alpha}{2}+\beta\Big)\sup_\Omega u_{k_0}+\frac{\alpha}{2}u_{k_0}(x_1)+\epsilon^2N.$$

Repeating this estimate for the values $u_{k_0}(x_j)$, $j \in \{1, \dots, J\}$, we get

$$u_{k_0}(x_0) \le \left(\frac{\alpha}{2} + \beta\right) \sup_\Omega u_{k_0} \sum_{j=0}^{2D/\epsilon} \left(\frac{\alpha}{2}\right)^j + \epsilon^2 N \sum_{j=0}^{2D/\epsilon} \left(\frac{\alpha}{2}\right)^j$$
$$\le \left(1 - \left(\frac{\alpha}{2}\right)^{2D/\epsilon}\right) \sup_\Omega u_{k_0} + 2\epsilon^2 N.$$

Remembering how x_0 was chosen, we have

$$\sup_\Omega u_{k_0} \le 4\epsilon^2 N \left(\frac{2}{\alpha}\right)^{2D/\epsilon}.$$

Since k_0 was arbitrary and the right hand side does not depend on k_0, the sequence (u_k) is bounded. Hence it converges pointwise to a bounded, Borel measurable limit function u. We show that the convergence is uniform. Suppose not. Since a sequence $\sup_{\Omega_\epsilon}(u - u_k)$ is positive, decreasing and bounded, we have

$$M = \lim_k \sup_{\Omega_\epsilon}(u - u_k) > 0.$$

If $1 \le k_1 \le k_2$, we have

$$\max\left\{ \sup_{B_\epsilon(x)} u_{k_2} - \sup_{B_\epsilon(x)} u_{k_1}, \inf_{B_\epsilon(x)} u_{k_2} - \inf_{B_\epsilon(x)} u_{k_1} \right\} \le \sup_{B_\epsilon(x)} (u - u_{k_1})$$

for all $x \in \Omega$. Using DPP we estimate

$$u_{k_2+1}(x) - u_{k_1+1}(x) \le \alpha \sup_{B_\epsilon(x_0)} (u - u_{k_1}) + \beta \fint_{B_\epsilon(x)} (u - u_{k_2}) dz. \tag{2.1}$$

Fix $\delta > 0$. Select k_1 such that

$$\sup_{\Omega_\epsilon}(u - u_{k_1}) < M + \delta$$

and

$$\sup_{x\in\Omega} \beta \fint_{B_\epsilon(x)} (u - u_{k_1}) dz \le \delta.$$

Then pick $y \in \Omega$ such that $u(y) - u_{k_1}(y) > M - \delta$, and finally $k_2 \ge k_1$ such that $u(y) - u_{k_2}(y) < \delta$. Then

$$u_{k_2+1}(y) - u_{k_1+1}(y) > M - 2\delta,$$

and since $\alpha < 1$, the estimate (2.1) contradicts the assumption $M > 0$ when δ is small enough. Since the convergence is uniform, the limit function u satisfies DPP (1.1).

In the proof of uniqueness the running payoff plays a minor role, so we just explain the ideas and refer to the proof of [16, Theorem 2.2] for details. Assume that u and v are defined in Ω_ϵ, satisfy DPP in Ω and $u = F = v$ on Γ_ϵ. Assume that $u(y) > v(y)$ for some $y \in \Omega$. Since $u - v$ is bounded, we have

$$\sup_\Omega (u - v) =: M > 0.$$

Using DPP, we can estimate

$$u(x) - v(x) \le \frac{\alpha}{2}\Big(\sup_{B_\epsilon(x)} u(x) - \sup_{B_\epsilon(x)} v(x)\Big) + \frac{\alpha}{2}\Big(\inf_{B_\epsilon(x)} u(x) - \inf_{B_\epsilon(x)} v(x)\Big) + \beta \fint_{B_\epsilon(x)} (u - v) dz + f(x) - f(x)$$
$$\le \alpha M + \beta \fint_{B_\epsilon(x)} (u - v) dz.$$

Because of absolute continuity of the integral, a set

$$G := \{x : u(x) - v(x) = M\}$$

is non-empty, and if $x_0 \in G$, then $u - v = M$ almost everywhere in a ball $B_\epsilon(x_0)$. This contradicts the assumption that G is bounded. A similar contradiction follows if $v(y) > u(y)$ for some $y \in \Omega$. Hence $u = v$ in Ω_ϵ. □

Lemma 2.2. *Given the payoffs and $\epsilon > 0$, the tug-of-war with noise and running payoff has a unique value function $u_\epsilon := u_{\mathrm{I}}^\epsilon = u_{\mathrm{II}}^\epsilon$ which satisfies DPP (1.1) in Ω.*

Proof. By Lemma 2.1, there is a unique function u_ϵ, $u_\epsilon = F$ in Γ_ϵ, satisfying DPP (1.1). We show that

$$u_{\mathrm{II}}^\epsilon \le u_\epsilon \le u_{\mathrm{I}}^\epsilon \le u_{\mathrm{II}}^\epsilon.$$

Since

$$\sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f(x_i)\Big] \ge u_{\mathrm{I}}^\epsilon$$

for all strategies S_{II}, we get $u_{\mathrm{I}}^\epsilon \le u_{\mathrm{II}}^\epsilon$.

Next we show that $u_\epsilon \le u_{\mathrm{I}}^\epsilon$. Max follows a strategy $S_{\mathrm{I}}^{\mathrm{Max}}$ in which from x_{k-1} he steps to a point $x_k \in B_\epsilon(x_{k-1})$ so that for fixed $\eta > 0$

$$u_\epsilon(x_k) \ge \sup_{B_\epsilon(x_{k-1})} u_\epsilon - \eta 2^{-k}.$$

Minnie uses a strategy S_{II}. Using DPP for u_ϵ at a point x_{k-1}, we estimate

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}\Big[u_\epsilon(x_k) + \epsilon^2 \sum_{i=0}^{k-1} f(x_i) - \eta 2^{-k} | \mathcal{F}_{k-1}\Big] \\
&\quad \ge \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x_{k-1})} u_\epsilon - \eta 2^{-k} + \inf_{B_\epsilon(x_{k-1})} u_\epsilon\Big\} + \beta \fint_{B_\epsilon(x_{k-1})} u_\epsilon dy + \epsilon^2 f(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-2} f(x_i) - \eta 2^{-k} \\
&\quad = u_\epsilon(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-2} f(x_i) - \eta 2^{-k}\Big(1 + \frac{\alpha}{2}\Big) \\
&\quad \ge u_\epsilon(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-2} f(x_i) - \eta 2^{-(k-1)}.
\end{aligned}$$

Therefore the process

$$M_k := u_\epsilon(x_k) + \epsilon^2 \sum_{i=0}^{k-1} f(x_i) - \eta 2^{-k}$$

for $k \ge 1$, $M_0 = u_\epsilon(x_0) - \eta$, is a submartingale with respect to the strategies $S_{\mathrm{I}}^{\mathrm{Max}}$ and S_{II}. Using the Optional Stopping Theorem we obtain

$$\begin{aligned}
u_{\mathrm{I}}^\epsilon(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f(x_i)\Big] \\
&\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}\Big[F(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f(x_i) - \eta 2^{-\tau}\Big] \\
&= \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}\Big[u_\epsilon(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f(x_i) - \eta 2^{-\tau}\Big] \\
&\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}[u_\epsilon(x_0) - \eta] \\
&= u_\epsilon(x_0) - \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, we have $u_{\mathrm{I}}^\epsilon(x_0) \ge u_\epsilon(x_0)$. The inequality

$$u_{\mathrm{II}}^\epsilon(x_0) \le u_\epsilon(x_0)$$

follows from a symmetric argument, so

$$u_\epsilon = u_{\mathrm{I}}^\epsilon = u_{\mathrm{II}}^\epsilon$$

in Ω. Hence u_ϵ is the value of the game. □

The next two lemmas are useful tools in estimating the value function.

Lemma 2.3. *Let τ be the stopping time of the game and let $\tau^* \le \tau$ be a stopping time with respect to the filtration $\mathcal{F}_k$. Then*

$$u_\epsilon(y) \ge \inf_{S_{\mathrm{II}}} \mathbb{E}^y_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}} \Big[u_\epsilon(x_{\tau^*}) + \epsilon^2 \sum_{i=0}^{\tau^*-1} f(x_i) \Big]$$

for any fixed strategy $S_{\mathrm{I}}^{\mathrm{Max}}$, and

$$u_\epsilon(y) \le \sup_{S_{\mathrm{I}}} \mathbb{E}^y_{S_{\mathrm{I}}, S_{\mathrm{II}}^{\mathrm{Min}}} \Big[u_\epsilon(x_{\tau^*}) + \epsilon^2 \sum_{i=0}^{\tau^*-1} f(x_i) \Big]$$

for any fixed strategy $S_{\mathrm{II}}^{\mathrm{Min}}$.

Proof. We only prove the first inequality, since the second follows from a similar argument. Max has fixed a strategy $S_{\mathrm{I}}^{\mathrm{Max}}$. Let $\eta > 0$. Minnie follows a strategy $S_{\mathrm{II}}^{\mathrm{Min}}$ in which from $x_{k-1} \in \Omega$ she steps to a point $x_k \in B_\epsilon(x_{k-1})$ in which

$$u_\epsilon(x_k) \le \inf_{B_\epsilon(x_{k-1})} u_\epsilon + \eta 2^{-k}.$$

Let us first prove that

$$M_k := u_\epsilon(x_k) + \epsilon^2 \sum_{i=0}^{k-1} f(x_i) + \eta 2^{-k}$$

for $k \ge 1$, $M_0 = u_\epsilon(x_0) + \eta$, is a supermartingale under the strategies $S_{\mathrm{I}}^{\mathrm{Max}}$ and $S_{\mathrm{II}}^{\mathrm{Min}}$.

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}^{\mathrm{Min}}} [M_k | \mathcal{F}_{k-1}] &\le \frac{\alpha}{2} \Big\{ \sup_{B_\epsilon(x_{k-1})} u_\epsilon + \inf_{B_\epsilon(x_{k-1})} u_\epsilon + \eta 2^{-k} \Big\} + \beta \fint_{B_\epsilon(x_{k-1})} u_\epsilon \, dy + \epsilon^2 \sum_{i=0}^{k-1} f(x_i) + \eta 2^{-k} \\
&\le u_\epsilon(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-2} f(x_i) + \eta 2^{-(k-1)} = M_{k-1}.
\end{aligned}$$

Hence M_k is a supermartingale, and we get

$$\begin{aligned}
\inf_{S_{\mathrm{II}}} \mathbb{E}^y_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}} \Big[u_\epsilon(x_{\tau^*}) + \epsilon^2 \sum_{i=0}^{\tau^*-1} f(x_i) \Big] &\le \mathbb{E}^y_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}^{\mathrm{Min}}} \Big[u_\epsilon(x_{\tau^*}) + \epsilon^2 \sum_{i=0}^{\tau^*-1} f(x_i) + \eta 2^{-\tau^*} \Big] \\
&\le \mathbb{E}^y_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}^{\mathrm{Min}}} [M_0] = u_\epsilon(y) + \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, the result follows. □

Lemma 2.4. *If v_ϵ and u_ϵ are value functions with payoff functions f_v and F_v for v_ϵ, f_u and F_u for u_ϵ, and $f_v \ge f_u$, $F_v \ge F_u$, then $v_\epsilon \ge u_\epsilon$.*

Proof. Max plays with a strategy S_{I} and Minnie follows a strategy $S_{\mathrm{II}}^{\mathrm{Min}}$ in which from $x_{k-1} \in \Omega$ she steps to a point $x_k \in B_\epsilon(x_{k-1})$ in which

$$v(x_k) \le \inf_{B_\epsilon(x_{k-1})} v + \eta 2^{-k}$$

for some fixed $\eta > 0$. Then

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}^{\mathrm{Min}}} \Big[v(x_k) + \epsilon^2 \sum_{i=0}^{k-1} f_u(x_i) + \eta 2^{-k} | \mathcal{F}_{k-1} \Big] &\le \frac{\alpha}{2} \Big\{ \inf_{B_\epsilon(x_{k-1})} v + \eta 2^{-k} + \sup_{B_\epsilon(x_{k-1})} v \Big\} + \beta \fint_{B_\epsilon(x_{k-1})} v \, dy + \epsilon^2 \sum_{i=0}^{k-1} f_u(x_i) + \eta 2^{-k} \\
&\le v(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-1} f_u(x_i) - \epsilon^2 f_v(x_{k-1}) + \eta 2^{-(k-1)} \\
&\le v(x_{k-1}) + \epsilon^2 \sum_{i=0}^{k-2} f_u(x_i) + \eta 2^{-(k-1)},
\end{aligned}$$

since v is a value function and $f_v \ge f_u$. Thus

$$M_k = v(x_k) + \epsilon^2 \sum_{i=0}^{k-1} f_u(x_i) + \eta 2^{-k}$$

for $k \geq 1$, $M_0 = v_\epsilon(x_0) + \eta$, is a supermartingale. Since $F_v \geq F_u$ on Γ_ϵ, we deduce by the Optional Stopping Theorem that

$$\begin{aligned} u_\epsilon(x_0) &= \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[F_u(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f_u(x_i)\Big] \\ &\leq \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S^{\mathrm{Min}}_{\mathrm{II}}}\Big[F_v(x_\tau) + \epsilon^2 \sum_{i=0}^{\tau-1} f_u(x_i) + \eta 2^{-\tau}\Big] \\ &\leq \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S^{\mathrm{Min}}_{\mathrm{II}}}[M_0] \\ &= v(x_0) + \eta. \end{aligned}$$

Since η was arbitrary, this proves the claim. □

3 Stopping time estimates and regularity lemmas

Recall that since the running payoff is positive, the value function u_ϵ is bounded from below by $\inf_{\Gamma_\epsilon} F$. In the proof of Lemma 2.1 we saw that u_ϵ is bounded from above by

$$\max\Big\{\sup_{\Gamma_\epsilon} F,\ 4\epsilon^2\Big(\frac{2}{\alpha}\Big)^{2\,\mathrm{diam}(\Omega)/\epsilon} \sup_{\Omega} f\Big\}.$$

Unfortunately, this upper bound depends on ϵ. Using the lemmas of Section 2, we can now show that the value functions u_ϵ for different ϵ are uniformly bounded. The idea is to fix for Minnie a strategy in which she tries to push the token to a certain boundary point. No matter which strategy Max uses, the expected value of the stopping time can be estimated so that the total effect of the running payoff is under control.

Lemma 3.1. *For given payoffs F and f, there is a constant $C > 0$, independent of ϵ, such that*

$$u_\epsilon \leq C\Big(\sup_{\Gamma_\epsilon} F + \sup_{\Omega} f\Big).$$

Proof. Fix $\epsilon > 0$ and let $x_0 \in \Omega$. Choose $z \in \mathbb{R}^n \setminus \Omega_\epsilon$, then $r > 0$ such that $B_r(z) \subset \mathbb{R}^n \setminus \Omega_\epsilon$, and finally $R > 0$ such that $\Omega_\epsilon \subset B_{R/2}(z)$. Let v be a solution to the problem

$$\begin{cases} \Delta v = -2(n+2) & \text{in } B_{R+\epsilon} \setminus \overline{B}_r(z), \\ v = 0 & \text{on } \partial B_r(z), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial B_{R+\epsilon}(z), \end{cases}$$

where $\frac{\partial v}{\partial \nu}$ is the normal derivative. As discussed in the proof of [19, Lemma 4.5], the function v is concave in $r = |x - z|$, satisfies

$$v(x) = \fint_{B_\epsilon(x)} v\,dy + \epsilon^2 \tag{3.1}$$

and can be extended as a solution to the same equation to $\overline{B}_{r(z)} \setminus \overline{B}_{r-\epsilon(z)}$ so that equation (3.1) holds also near the boundary $\partial B_r(z)$.

The game starts from $x_0 \in \Omega$. Max plays with any strategy and Minnie plays with the strategy $S^{\mathrm{Min}}_{\mathrm{II}}$ in which from a point $x_{k-1} \in B_R(z)$ she moves to a point x_k for which

$$v(x_k) \leq \inf_{B_\epsilon(x_{k-1})} v + \frac{\beta}{\alpha}\epsilon^2.$$

Let τ be the smallest k for which $x_k \in B_R(z) \setminus \Omega$ and τ^* the smallest k for which x_k hits the complement of $B_R(z) \setminus B_r(z)$. Then $\tau \leq \tau^*$ for any game sequence (x_k). Let us estimate the expected value of τ^*. By radial concavity of v we get

$$\begin{aligned}\mathbb{E}^{x_0}_{S_I,S_{II}^{Min}}[v(x_k)|\mathcal{F}_{k-1}] &\le \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x_{k-1})} v + \inf_{B_\epsilon(x_{k-1})} v + \frac{\beta}{\alpha}\epsilon^2\Big\} + \beta \fint_{B_\epsilon(x_{k-1})} v\,dy \\ &\le \alpha v(x_{k-1}) + \frac{\beta}{2}\epsilon^2 + \beta(v(x_{k-1}) - \epsilon^2) \\ &= v(x_{k-1}) - \frac{\beta}{2}\epsilon^2.\end{aligned}$$

Hence $M_k := v(x_k) + k\frac{\beta}{2}\epsilon^2$ is a supermartingale. In particular, we have

$$\mathbb{E}^{x_0}_{S_I,S_{II}^{Min}}[M_{\tau^*}] \le v(x_0) \le C,$$

where C is independent of ϵ. On the other hand, since $v(x_{\tau^*}) = 0$, we have $\mathbb{E}[M_{\tau^*}] \ge \frac{\beta}{2}\epsilon^2\mathbb{E}\tau^*$. Hence

$$\mathbb{E}[\tau] \le \mathbb{E}[\tau^*] \le C\epsilon^{-2}.$$

Then

$$u_\epsilon(x_0) \le \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^{Min}}\Big[F(x_\tau) + \epsilon^2\sum_{i=0}^{\tau-1} f(x_i)\Big] \le C\Big(\sup_{\Gamma_\epsilon} F + \sup_{\Omega} f\Big).$$

Since $x_0 \in \Omega$ and $\epsilon > 0$ were arbitrary, the proof is complete. □

In the proof of Theorem 4.1 we need the following lemma, which is proven in the appendix of [15].

Put a token to the point $(0,t) \in B_{2r}(0) \times [0,2r] \subset \mathbb{R}^{n+1}$ and fix $r > 0$. From a point (x_j, t_j), with probability $\frac{\alpha}{2}$ the token moves to the point $(x_j, t_j - \epsilon)$, and with the same probability to $(x_j, t_j + \epsilon)$. With probability β the token moves to the point (x_{j+1}, t_j), where x_{j+1} is randomly chosen from the ball $B_\epsilon(x_j) \subset \mathbb{R}^n$.

Lemma 3.2. *The probability that the token does not escape the cylinder through its bottom is less than*

$$\frac{C(p,n)(t+\epsilon)}{r}$$

for all $\epsilon > 0$ small enough.

Also the next lemma is needed in the proof of Theorem 4.1, because it describes the expected total effect of the running payoff under the strategies used there.

Let $0 < \epsilon < t_0 < 1$ and start a random walk from t_0 as follows. From the point t_{j-1} we step with probability $\frac{\alpha}{2}$ to $t_j = t_{j-1} + \epsilon$, with the same probability to $t_j = t_{j-1} - \epsilon$, and with probability β we do not move, $t_j = t_{j-1}$. The random walk stops when $x_j \in \mathbb{R} \setminus (0,1)$ for the first time. Let $\overline{\tau}$ be the stopping time.

Lemma 3.3. *In the random walk described above,*

$$\mathbb{E}[\overline{\tau}] \le 5t_0\alpha^{-1}\epsilon^{-2}.$$

Proof. Since

$$\mathbb{E}[t_j^2|t_0,\dots,t_{j-1}] = \frac{\alpha}{2}(t_{j-1}+\epsilon)^2 + \frac{\alpha}{2}(t_{j-1}-\epsilon)^2 + \beta t_{j-1}^2 = t_{j-1}^2 + \alpha\epsilon^2,$$

we have

$$\mathbb{E}[(t_j^2 - j\alpha\epsilon^2)|t_0,\dots,t_{j-1}] = t_{j-1}^2 - (j-1)\alpha\epsilon^2.$$

Hence also $(t_j^2 - j\alpha\epsilon^2)$ is a martingale. Let $p = \mathbb{P}(x_{\overline{\tau}} \le 0)$. Then

$$t_0 = \mathbb{E}t_\tau \ge p(-\epsilon) + (1-p) = -p(\epsilon+1) + 1.$$

For the function $f : [0,\infty) \to \mathbb{R}$, $f(x) = (1-t_0)(1+x)^{-1} + x + t_0 - 1$ it holds that $f(0) = 0$ and $f' \ge 0$, so we have

$$p \ge (1-t_0)(1+\epsilon)^{-1} \ge 1 - t_0 - \epsilon.$$

Since $(t_j^2 - j\alpha\epsilon^2)$ is a martingale, we have

$$t_0^2 = t_0^2 - 0\cdot\alpha\epsilon^2 = \mathbb{E}(t_{\overline{\tau}}^2 - \overline{\tau}\alpha\epsilon^2) \le p\epsilon^2 + (1-p)(1+\epsilon)^2 - \alpha\epsilon^2\mathbb{E}\overline{\tau}.$$

We can estimate

$$\begin{aligned}
\mathbb{E}\overline{\tau} &\le \epsilon^{-2}(p\epsilon^2 + (1-p)(1+\epsilon)^2 - t_0^2) \\
&= \epsilon^{-2}\alpha^{-1}((1+\epsilon)^2 - t_0^2 + p\epsilon^2 - p(1+\epsilon)^2) \\
&\le \epsilon^{-2}\alpha^{-1}((1+\epsilon)^2 - t_0^2 - p) \\
&\le \epsilon^{-2}\alpha^{-1}((1+\epsilon)^2 - t_0^2 + t_0 + \epsilon - 1) \\
&= \epsilon^{-2}\alpha^{-1}(\epsilon^2 + 3\epsilon + t_0 - t_0^2) \\
&\le \epsilon^{-2}\alpha^{-1}(t_0 + 4\epsilon) \\
&\le 5t_0\alpha^{-1}\epsilon^{-2}.
\end{aligned}$$

□

The next two lemmas are needed in the proof of Harnack's inequality. The first is a simple local comparison estimate, and the second gives estimates for $\inf u_\epsilon$ in balls of radius $2\epsilon < r < 1$.

Lemma 3.4. *Let $u_\epsilon > 0$ be a value function and $x, y \in B_R(z) \subset \Omega$, $|x-y| \le 10\epsilon$. Then*

$$u_\epsilon(x) \ge \left(\frac{\alpha}{2}\right)^{20} u_\epsilon(y).$$

Proof. Start the game from x. Max uses a strategy $S_{\mathrm{I}}^{\mathrm{Max}}$ in which he takes $\frac{\epsilon}{2}$-step towards y, and jumps to y if possible. The game is stopped when the token reaches either y or $\Omega_\epsilon \setminus B_R(z)$. Let this stopping time be τ^*. Since the probability to stop at y is bigger than $(\frac{\alpha}{2})^{20}$, we obtain from Lemma 2.3

$$u_\epsilon(x) \ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}}, S_{\mathrm{II}}}[u_\epsilon(x_{\tau^*})] + 20\epsilon^2 \inf_\Omega f \ge \left(\frac{\alpha}{2}\right)^{20} u_\epsilon(y).$$

□

Lemma 3.5. *Let $u_\epsilon > 0$ be a value function and $B_{30R}(y) \subset \Omega$ for some $R > 0$. For $z \in B_{2R}(y)$ and $r \in (2\epsilon, R)$*

$$\inf_{B_r(z)} u_\epsilon \le Cr^{-n} u_\epsilon(y),$$

where $C = C(p, n)$

Proof. Without a loss of generality, we may assume that $y = 0$ and $R = 1$. Fix $\epsilon > 0$ and $r \in (2\epsilon, 1)$. Let $U = B_4(z) \setminus \overline{B}_r(z)$. There is no loss of generality in assuming that $0 \in U$.

Define

$$v(x) = \begin{cases} (|x-z|^{2-n} - 4^{2-n})(r^{2-n} - 4^{2-n})^{-1} & \text{if } n \ge 3, \\ \log(\frac{4}{|x-z|}) \log(\frac{4}{r})^{-1} & \text{if } n = 2. \end{cases}$$

Then v is harmonic in U with boundary values

$$\begin{cases} v = 1 & \text{on } \partial B_r(z), \\ v = 0 & \text{on } \partial B_4(z). \end{cases}$$

In both the cases there is a constant $c > 0$ such that

$$v(0) \ge cr^n.$$

The game starts from $x_0 = 0$. Minnie uses any strategy and Max uses the following strategy S_I^{Max}: In a ball $B_\epsilon(x_{k-1})$, he aims to a point x_k where

$$v(x_k) \ge \sup_{B_\epsilon(x_{k-1})} v - \eta r^n \epsilon^2,$$

where $\eta > 0$ is selected so that the stopping time estimation of the proof of Lemma 3.1 is at our disposal. The game is stopped at the ϵ-boundary Γ_ϵ of U and employing the boundary values $(u_\epsilon)_{|\Gamma_\epsilon}$. The corresponding stopping time is τ^*.

We want to estimate the probability of stopping at the inner boundary.

Since $y \mapsto v(x+y)$ is radially decreasing and the map $0 < t \mapsto v(x+ty_0)$ is convex for any fixed $y_0 \neq 0$, we obtain

$$\begin{aligned}\mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}[v(x_{k+1})|x_0,x_1,\ldots,x_k] &\geq \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x_k)} v - \eta r^n\epsilon^2 + \inf_{B_\epsilon(x_k)} v\Big\} + \beta \fint_{B_\epsilon(x_k)} v\,dy\\ &\geq \alpha v(x_k) + \beta v(x_k) - \eta r^n\epsilon^2\\ &= v(x_k) - \eta r^n\epsilon^2.\end{aligned}$$

Hence $M_k := v(x_k) + k\eta r^n\epsilon^2$ is a submartingale. Denote by P the probability of stopping at the inner boundary. The Optional Stopping Theorem gives

$$cr^n \leq v(0) = v(x_0) \leq \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}[v(x_{\tau^*}) + \eta r^n\epsilon^2\tau^*] \leq 2^{n+1}P + \eta C_1 r^n,$$

where $C_1 > 0$ is a constant such that $\mathbb{E}\tau^* \leq C_1\epsilon^{-2}$, and the term $2^{n+1}P$ comes from the fact that $v \leq 2^{n+1}$ in $B_r(z)\setminus B_{r-\epsilon}(z)$. We can select η so that $\eta C_1 < c$. Thus $P \geq c'r^n$, where $c' > 0$. Using Lemma 2.3 we obtain

$$\begin{aligned}u_\epsilon(0) = u_\epsilon(x_0) &\geq \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}^{\mathrm{Max}},S_{\mathrm{II}}}[u_\epsilon(x_{\tau^*}) + \epsilon^2\tau^* \inf_\Omega f]\\ &\geq P \inf_{B_r(z)} u_\epsilon\\ &\geq c'r^n \inf_{B_r(z)} u_\epsilon.\end{aligned}$$

We get

$$\inf_{B_r(z)} u_\epsilon \leq (c'r^n)^{-1}u_\epsilon(0) \leq Cr^{-n}u_\epsilon(0),$$

where $C = c'^{-1}$. □

4 Lipschitz and Harnack estimates

We are ready to prove the main results, Lipschitz continuity and Harnack's inequality. In the proof of the following theorem we use the cancellation strategy idea that was introduced in the proof of [15, Theorem 3.2]. Because of Lemma 3.3, the running payoff behaves well under this strategy.

Theorem 4.1. *Let $u_\epsilon > 0$ be a value function and $B_{6R}(a) \subset \Omega$, where $R > \epsilon$. When $\epsilon < r \leq R$, we have*

$$\operatorname{osc}(u_\epsilon, B_r(a)) \leq C\frac{r}{R}[\operatorname{osc}(u_\epsilon, B_{6R}(a)) + \operatorname{osc}(f, B_{6r}(a))], \tag{4.1}$$

where C is a constant depending only on p and n.

Proof. Take $x, y \in B_r(a)$, $|x-y| \geq \epsilon$, and then $z \in B_{2r}(a)$ such that

$$|x-z| = |y-z| = |x-y|.$$

When the game starts from x, Minnie plays according to the following cancellation strategy $S_{\mathrm{II}}^{\mathrm{Min}}$: If Max has won more coin tosses than Minnie, then she cancels one of the moves of Max. Otherwise, she moves towards z length $\frac{\epsilon}{2}$ or keeps the token in z. We stop the game if Minnie wins $\frac{3|x-z|}{\epsilon}$ coin tosses more than Max, or if Max wins at least $\frac{2R}{\epsilon}$ times more than Minnie, or if the length of the sum of random vectors exceeds $2R$. Then the game stays in B_{6R}. For the game that starts from y, Max follows the cancellation strategy $S_{\mathrm{I}}^{\mathrm{Max}}$, and we define stopping time τ^* as previously. For this stopping time $\tau^* \leq \tau$, where τ is the normal stopping time of the game. Hence Lemma 2.3 is at our disposal.

Notice that by putting

$$t_0 = \frac{3|x-y|}{3|x-y|+2R},$$

Lemma 3.3 gives

$$\mathbb{E}[\tau^*] \leq 5\alpha^{-1}\frac{3|x-y|}{3|x-y|+2R}\epsilon^{-2} \leq C_1\frac{|x-y|}{R}\epsilon^{-2}.$$

Let P be the probability that the game, started from x, ended because Minnie won more (hereafter Min w). By symmetry, P is also the probability that the game, started from y, ended because Max won more (hereafter Max w). Then, because of the cancellation effect, by using Lemma 2.3 we can estimate

$$\begin{aligned}|u_\epsilon(x) - u_\epsilon(y)| &\le \Big|P\Big(\mathbb{E}^x_{S_{\mathrm{I}},S^{\mathrm{Min}}_{\mathrm{II}}}[u_\epsilon(x_\tau) \mid \text{Min w}] - \mathbb{E}^y_{S^{\mathrm{Max}}_{\mathrm{I}},S_{\mathrm{II}}}[u_\epsilon(x_\tau) \mid \text{Max w}]\Big)\Big| \\ &\qquad + (1-P)\operatorname{osc}_{B_{6R}(z_0)} u_\epsilon + \epsilon^2\mathbb{E}[\tau^*]\operatorname{osc}_{B_{6R}(z_0)} f \\ &\le (1-P)\operatorname{osc}_{B_{6R}(z_0)} u_\epsilon + C_1|x-y|R^{-1}\operatorname{osc}_{B_{6R}(z_0)} f \\ &\le C\frac{r}{R}[\operatorname{osc}(u_\epsilon, B_{6R}(a)) + \operatorname{osc}(f, B_{6R}(a))]\end{aligned}$$

for $C = C(p, n)$, since by Lemma 3.2 we have

$$1 - P \le \frac{C_2|x-y|}{R},$$

where C_2 depends only on p and n.

If $x, y \in B_r$ and $|x - y| < \epsilon$, we can take a point $z \in B_r$ such that $|x - z| \ge \epsilon$ and $|z - y| \ge \epsilon$. By the triangle inequality the estimate follows from previous estimate. □

Next we prove Harnack's inequality. The idea is to show that if Harnack's inequality does not hold for a fixed, large constant, then by iteration argument the value functions are unbounded when ϵ is small. The cumulative effect of oscillations of the running payoff during iteration seems to cause trouble, but surprisingly, it is not even necessary to require the running payoff to be continuous.

Theorem 4.2. *Let $u_\epsilon > 0$ be a value function. Assuming $B_{30r}(a) \subset \Omega$, where $r > 0$, there exists a positive constant K, depending only on p and n, for which*

$$\sup_{B_r(a)} u_\epsilon \le K\big(\inf_{B_r(a)} u_\epsilon + \sup_\Omega f\big).$$

Proof. Without a loss in generality, we may assume that $r = 1$ and $a = 0$. For convenience of notation, let

$$N := \sup_\Omega f.$$

First we show that

$$\inf_{\overline{B}_1(0)} u_\epsilon > 0. \tag{4.2}$$

Suppose not. Then there is a converging sequence $(x_j) \subset \overline{B}_1(0)$, $x_j \to x_0$, such that $u_\epsilon(x_j) < \frac{1}{j}$. According to Lemma 3.4,

$$u_\epsilon(y) \ge \Big(\frac{\alpha}{2}\Big)^{20} u_\epsilon(x_0)$$

when $|y - x_0| < 10\epsilon$. This is a contradiction, so (4.2) holds.

Pick first a point $x_1 \in B_1(0)$ such that

$$u_\epsilon(x_1) < 2\inf_{B_1(0)} u_\epsilon,$$

and then a point $x_2 \in B_2(x_1)$ such that

$$M_1 := u_\epsilon(x_2) \ge \sup_{B_2(x_1)} u_\epsilon - N.$$

For $k \ge 2$, let $R_k = 2^{1-k}$ and pick $x_{k+1} \in B_{R_k}(x_k)$ such that

$$M_k := u_\epsilon(x_{k+1}) \ge \sup_{B_{R_k}(x_k)} u_\epsilon - N.$$

We are going to show that

$$M_1 < (2^{1+2n}C)^{1+2n} u_\epsilon(x_1) + 2N, \tag{4.3}$$

where $C = C(p, n)$ is a constant such that Lemma 3.5 and Theorem 4.1 are valid.

On the contrary, suppose that inequality (4.3) does not hold. Put $\delta := (2^{1+2n}C)^{-1}$. Let us show by induction that the counter assumption yields

$$M_k \ge 2C(\delta R_{k+1})^{-2n} u_\epsilon(x_1). \tag{4.4}$$

Notice first that from a straightforward calculation we get

$$2C(\delta R_{k+1})^{-2n} = (2C\delta)^{-k+1}(2^{1+2n}C)^{1+2n}. \tag{4.5}$$

By observation (4.5) the case $k = 1$ holds. Assume that (4.4) holds for $k \le j$. Let $k \in \{2, \dots, j+1\}$. Then

$$\inf_{B_{\delta R_k}(x_k)} u_\epsilon \le C(\delta R_k)^{-2n} u_\epsilon(x_1) \le \frac{M_{k-1}}{2} = \frac{u_\epsilon(x_k)}{2}, \tag{4.6}$$

where we used a weakened form of Lemma 3.5 and the induction assumption that (4.4) holds for $k \le j$.

By Theorem 4.1,

$$\operatorname{osc}(u_\epsilon, B_{\delta R_k}(x_k)) \le C\delta(\operatorname{osc}(u_\epsilon, B_{R_k}(x_k)) + \operatorname{osc}(f, B_{R_k}(x_k))),$$

or in other words,

$$\operatorname{osc}(u_\epsilon, B_{R_k}(x_k)) \ge (C\delta)^{-1} \operatorname{osc}(u_\epsilon, B_{\delta R_k}(x_k)) - \operatorname{osc}(f, B_{R_k}(x_k)). \tag{4.7}$$

By using first (4.7) and then (4.6) we obtain

$$\begin{aligned} M_k &\ge \operatorname{osc}(u_\epsilon, B_{R_k}(x_k)) - N \\ &\ge (C\delta)^{-1}\Big(\sup_{B_{\delta R_k}(x_k)} u_\epsilon - \inf_{B_{\delta R_k}(x_k)} u_\epsilon\Big) - \operatorname{osc}(f, B_{R_k}(x_k)) - N \\ &\ge (C\delta)^{-1}(u(x_k) - 2^{-1}u(x_k)) - 2N \\ &= (2C\delta)^{-1}M_{k-1} - 2N. \end{aligned}$$

Now we come to an important point, when we want an estimation between M_{j+1} and M_1. At first glance it seems that the cumulative effect of the oscillation of running payoff is an issue, but it turns out to be under control. We get

$$\begin{aligned} M_{j+1} &\ge (2C\delta)^{-1}M_j - 2N \\ &\ge (2C\delta)^{-1}[(2C\delta)^{-1}M_{j-1} - \operatorname{osc}(f, B_{R_j}(x_j))] - 2N \\ &\ge (2C\delta)^{-j}M_1 - 2N\sum_{i=1}^{j}(2C\delta)^{-i+1}. \end{aligned}$$

Remembering the counter assumption $M_1 \ge (2^{1+2n}C)^{1+2n}u_\epsilon(x_1) + 2N$ and noticing that $2C\delta = 2^{-2n} < \frac{1}{2}$, we obtain

$$M_{j+1} \ge (2C\delta)^{-j}(2^{1+2n}C)^{1+2n}u_\epsilon(x_1) + 2N\Big[(2C\delta)^{-j} - \sum_{i=1}^{j}(2C\delta)^{-i+1}\Big] \ge (2C\delta)^{-j}(2^{1+2n}C)^{1+2n}u_\epsilon(x_1).$$

Taking into account the observation (4.5), the induction is complete.

Take k_0 such that $\delta R_{k_0} \in (2\epsilon, 4\epsilon]$. By Lemma 3.4,

$$\inf_{B_{\delta R_{k_0}}(x_{k_0})} u_\epsilon \ge \Big(\frac{\alpha}{2}\Big)^{20} \sup_{B_{\delta R_{k_0}}(x_{k_0})} u_\epsilon.$$

By using Lemma 3.5 and inequality (4.4) we obtain

$$\begin{aligned} \Big(\frac{\alpha}{2}\Big)^{-20} &\ge \frac{\sup_{B_{\delta R_{k_0}}(x_{k_0})} u_\epsilon}{\inf_{B_{\delta R_{k_0}}(x_{k_0})} u_\epsilon} \\ &\ge \frac{u_\epsilon(x_{k_0})}{C(\delta R_{k_0})^{-n}u_\epsilon(x_1)} \\ &= \frac{M_{k_0-1}}{C(\delta R_{k_0})^{-n}u_\epsilon(x_1)} \\ &\ge \frac{(2C\delta)^{2-k_0}(2^{1+2n}C)^{1+2n}}{C(\delta 2^{1-k_0})^{-n}} \\ &\ge \widehat{C}(C\delta 2^{n+1})^{-k_0} \\ &= \widehat{C}2^{nk_0}, \end{aligned}$$

where $\widehat{C}$ is independent of k_0. This is a contradiction when k_0 is big enough, or in other words, when ϵ is small enough. Therefore inequality (4.3) holds and we get

$$\begin{aligned}\sup_{B_1(0)} u_\epsilon \le \sup_{B_2(x_1)} u_\epsilon &\le M_1 + N\\ &\le (2^{1+2n}C)^{1+2n} u_\epsilon(x_1) + 3N\\ &\le 2(2^{1+2n}C)^{1+2n} \inf_{B_1(0)} u_\epsilon + 3\sup_\Omega f\\ &\le K\big(\inf_{B_1(0)} u_\epsilon + \sup_\Omega f\big),\end{aligned}$$

where K depends only on p and n. □

5 Relation to PDE

In this section we study local regularity of viscosity solutions to the inhomogeneous p-Laplace equation

$$-\Delta_p^N u = f \tag{5.1}$$

in Ω. As before, $p > 2$. In the whole section $f > 0$ is continuous and bounded in $\overline{\Omega}$, and boundary values of viscosity solutions are required to be continuous and bounded. Recall that

$$\Delta_p^N u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u$$

is the normalized p-Laplacian. Here

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}((p-2)\Delta_\infty^N u + \Delta u),$$

where

$$\Delta_\infty^N u = |\nabla u|^{-2}\Delta_\infty u = |\nabla u|^{-2}\langle D^2 u\,\nabla u, \nabla u\rangle.$$

By [12, Proposition 3], we can define viscosity solutions to (5.1) as follows.

Definition 5.1. A continuous function u is a viscosity solution to (5.1) at $x \in \Omega$ if and only if every C^2-function ϕ, $\nabla\phi(x) \neq 0$ or $D^2\phi(x) = 0$, that touches u from below in $x \in \Omega$, satisfies

$$-\Delta_p^N \phi(x) \ge f(x),$$

and every C^2-function ϕ, $\nabla\phi(x) \neq 0$ or $D^2\phi(x) = 0$, that touches u from above in $x \in \Omega$, satisfies

$$-\Delta_p^N \phi(x) \le f(x).$$

Note that if a test function ϕ satisfies $\nabla\phi(x) = 0$ and $D^2\phi(x) = 0$ for some $x \in \Omega$, by the convergence argument explained in [12] we can set

$$\Delta_p^N \phi(x) = 0.$$

The idea for showing local regularity properties for viscosity solutions to (5.1) is to notice that viscosity solutions can be approximated uniformly by value functions of tug-of-war with noise and running payoff. We need the following Arzela–Ascoli type lemma, which is proven in [19, Lemma 4.2].

Lemma 5.2. *Let $\{u_\epsilon : \overline{\Omega} \to \mathbb{R},\ \epsilon > 0\}$ be a uniformly bounded set of functions such that given $\eta > 0$, there are constants r_0 and ϵ_0 such that for every $\epsilon < \epsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x-y| < r_0$ it holds that*

$$|u_\epsilon(x) - u_\epsilon(y)| < \eta.$$

Then there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by u_ϵ such that $u_\epsilon \to u$ uniformly in $\overline{\Omega}$ as $\epsilon \to 0$.

Let u be a viscosity solution to (5.1) in Ω. We may assume $0 \in \Omega$. Choose $R > 0$ such that $B_{2R}(0) \subset \Omega$. Let u_ϵ, $0 < \epsilon < R$, be the value function of tug-of-war with noise and running payoff in $B_R(0)$, where running payoff is f from equation (5.1), and final payoff is u on the boundary strip

$$\Gamma_\epsilon = \{x \in \Omega \setminus B_R(0) : \operatorname{dist}(x, \partial B_R(0)) \leq \epsilon\}.$$

Lemma 5.3. *The sequence (u_ϵ), defined in $B_R(0) \cup \Gamma_\epsilon$ as described above, satisfies the conditions of Lemma 5.2 in $\overline{B}_R(0)$.*

Proof. By Lemma 3.1, the sequence (u_ϵ) is uniformly bounded in $\overline{B}_R(0)$. Fix $\eta > 0$. Since u is uniformly continuous in $B_R(0) \cup \Gamma_\epsilon$, there is $r_1 > 0$ such that $x, y \in B_R(0) \cup \Gamma_\epsilon$, $|x - y| < r_1$, implies

$$|u(x) - u(y)| < \frac{\eta}{2}.$$

When $x, y \in \partial B_R(0)$, the same estimate holds between $u_\epsilon(x)$ and $u_\epsilon(y)$ for all $0 < \epsilon < R$, since $u_\epsilon = u$ on Γ_ϵ.

Let us next work out the case $x \in B_R(0)$, $y \in \partial B_R(0)$. Select $0 < s < S < r_1$ and $z \in \Gamma_\epsilon$ such that $y \in \partial B_s(z)$ and $B_{2S}(z) \subset B_R(0) \cup \Gamma_\epsilon$. Consider a function $v : \overline{B}_{2S}(z) \setminus B_s(z) \to \mathbb{R}$,

$$\begin{cases} \Delta v = -4(n+2) \sup_\Omega f & \text{in } B_{2S}(z) \setminus \overline{B}_s(z), \\ v = \sup_{B_R(z)} u & \text{on } \partial B_s(z), \\ v = \sup_{\Gamma_\epsilon} u & \text{on } \partial B_{2S}(z). \end{cases}$$

Note that this function satisfies

$$v(a) = \fint_{B_\epsilon(a)} v\,dy + 2\epsilon^2 \sup_\Omega f$$

when $B_\epsilon(a) \subset B_{2S}(z) \setminus \overline{B}_s(z)$.

Let $r_2 < S - s$ be so small that

$$\sup_{B_{r_2}(y)} v < \sup_{B_S(z)} u + \frac{\eta}{2}.$$

Pick $x \in B_{r_2}(y) \cap B_R(0)$. Since $|x - y| < S - s$, by the triangle inequality $x \in B_R(0) \cap B_S(z)$. Let $\epsilon < S$. We start a game from $x_0 = x$. Minnie plays with the following strategy $S_{\mathrm{II}}^{\mathrm{Min}}$: at x_{k-1}, she aims to a point x_k where

$$v(x_k) \leq \inf_{B_\epsilon(x_{k-1})} v + \frac{1}{2}\epsilon^2 \sup_\Omega f.$$

We stop the game when $x_k \in B_{2S}(z) \setminus (B_S(z) \cap B_R(0))$ for the first time. Let this stopping time be τ^*.

Max plays with a strategy S_{I}. From radial convexity of v we obtain

$$\begin{aligned} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}^{\mathrm{Min}}}(v(x_k)|\mathcal{F}_{k-1}) &\leq \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x_{k-1})} v + \inf_{B_\epsilon(x_{k-1})} v + \frac{1}{2}\epsilon^2 \sup_\Omega f\Big\} + \beta \fint_{B_\epsilon(x_{k-1})} v\,dy \\ &\leq \alpha v(x_{k-1}) + \frac{\alpha}{4}\epsilon^2 \sup_\Omega f + \beta v(x_{k-1}) - 2\epsilon^2 \sup_\Omega f \\ &\leq v(x_{k-1}) - \epsilon^2 \sup_\Omega f. \end{aligned}$$

Hence $M_k := v(x_k) + k\epsilon^2 \sup_\Omega f$ is a supermartingale, and we obtain

$$\begin{aligned} u_\epsilon(x_0) &\leq \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}^{\mathrm{Min}}}\Big[u_\epsilon(x_{\tau^*}) + \epsilon^2 \sum_{i=0}^{\tau^*-1} f(x_i)\Big] \\ &\leq \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}^{\mathrm{Min}}}\Big[v(x_{\tau^*}) + \tau^* \epsilon^2 \sup_\Omega f\Big] \\ &\leq v(x_0). \end{aligned}$$

We conclude that when $x \in B_R(0)$, $y \in \partial B_R(0)$, $|x-y| < r_2$ and $\epsilon < S$, we have

$$|u_\epsilon(x) - u_\epsilon(y)| \le |u_\epsilon(x_0) - \sup_{B_S(z)\cap\Gamma_\epsilon} u| + |\sup_{B_S(z)\cap\Gamma_\epsilon} u - u_\epsilon(y)| < \eta.$$

Finally, let us examine the case $x, y \in B_R(0)$. If $\operatorname{dist}(x, \partial B_R(0)) < \frac{r_2}{5}$ and $|x-y| < \frac{r_2}{5}$, there is $y_0 \in \partial B_R(0)$ such that $|x-y_0| < r_2$ and $|y-y_0| < r_2$. Then

$$|u_\epsilon(x) - u_\epsilon(y)| < 2\eta.$$

Hence we can assume that $x, y \in B_{R-r_2/3}(0)$, and the asymptotic estimate

$$|u_\epsilon(x) - u_\epsilon(y)| < \eta$$

follows straightforwardly from Theorem 4.1. □

The ideas in the proof of Lemma 5.4 are similar to those used in the proof of [19, Theorem 4.9], where the uniform limit of the value functions of tug-of-war with noise was shown to be a viscosity solution to the homogeneous p-Laplace equation.

Lemma 5.4. *Let u be a viscosity solution to* (5.1) *in Ω and $B_{2R}(0) \subset \Omega$. Then u can be approximated uniformly by value functions of tug-of-war with noise and running payoff in $B_R(0)$.*

Proof. Let (u_ϵ) be a sequence of value functions in $B_R(0)$ with final payoff u and running payoff

$$\overline{f} = \frac{p\beta}{2(n+2)} f.$$

By Lemma 5.2, it follows from Lemma 5.3 that there is a uniformly continuous function v in $\overline{B}_R(0)$, $v = u$ on $\partial B_R(0)$, such that there is a subsequence of (u_ϵ) converging uniformly to v in $\overline{B}_R(0)$ when $\epsilon \to 0$. For convenience of notation, we denote this subsequence (u_ϵ). We are going to show that the function v is a viscosity solution to (5.1) in $B_R(0)$. By comparison principle (see, e.g., [12, Theorem 5] and also [14]), we will conclude $v = u$ in $B_R(0)$.

Choose a point $x \in B_R(0)$. We only work out the supersolution part, since the subsolution part is similar. Let $\phi \in C^2(B)$, $\nabla\phi(x) \neq 0$ or $D^2\phi(x) = 0$, be defined in a neighborhood B of x, touching v from below in x. We need to show that

$$p(\Delta_p^N \phi(x) + f(x)) = (p-2)\Delta_\infty^N \phi + \Delta\phi + pf(x) \le 0. \tag{5.2}$$

If $\nabla\phi(x) = 0$ and $D^2\phi(x) = 0$, we have $\Delta_p^N\phi(x) = 0$ and inequality (5.2) cannot hold. Hence we can assume that $\nabla\phi(x) \neq 0$. From Taylor expansion results in [18] it follows that there is a point $\overline{x}^\epsilon \in B_\epsilon(x)$ in the direction of $\nabla\phi(x)$ such that

$$\begin{aligned}
&\frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x)} \phi + \inf_{B_\epsilon(x)} \phi\Big\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy - \phi(x) \\
&\quad \ge \frac{\beta\epsilon^2}{2(n+2)}\Big((p-2)\Big\langle D^2\phi(x)\Big(\frac{\overline{x}^\epsilon - x}{|\overline{x}^\epsilon - x|}\Big), \Big(\frac{\overline{x}^\epsilon - x}{|\overline{x}^\epsilon - x|}\Big)\Big\rangle + \Delta\phi(x)\Big) + o(\epsilon^2).
\end{aligned}$$

Since v is the uniform limit of the sequence (u_ϵ), there is a sequence $(x_\epsilon) \subset B$ converging to x so that

$$u_\epsilon(y) - \phi(y) \ge -\epsilon^3$$

when $y \in B_\epsilon(x_\epsilon)$. Using the DPP characterization of u_ϵ, we obtain

$$\epsilon^3 \ge -\phi(x_\epsilon) + \frac{\alpha}{2}\Big\{\sup_{B_\epsilon(x)} \phi + \inf_{B_\epsilon(x)} \phi\Big\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy + \epsilon^2 \overline{f}(x_\epsilon).$$

Hence

$$-\epsilon^3 \ge \frac{\beta\epsilon^2}{2(n+2)}\Big((p-2)\Big\langle D^2\phi(x)\Big(\frac{\overline{x}^\epsilon - x}{|\overline{x}^\epsilon - x|}\Big), \Big(\frac{\overline{x}^\epsilon - x}{|\overline{x}^\epsilon - x|}\Big)\Big\rangle + \Delta\phi(x)\Big) + \epsilon^2\overline{f}(x_\epsilon) + o(\epsilon^2).$$

Since $\overline{f}$ is continuous and $\nabla\phi(x) \neq 0$, dividing by ϵ^2 and then letting $\epsilon \to 0$ we get

$$0 \geq \frac{\beta}{2(n+2)}((p-2)\Delta_\infty^N \phi(x) + \Delta\phi(x)) + \overline{f}(x).$$

Remembering how the running payoff $\overline{f}$ was chosen, we have

$$0 \geq \frac{\beta}{2(n+2)}((p-2)\Delta_\infty^N \phi(x) + \Delta\phi(x) + pf(x)).$$

Hence v is a viscosity supersolution to (5.1) in $B_R(0)$. By similar argument v is also a viscosity subsolution, hence a viscosity solution. By the discussion in the beginning of the proof, the proof is complete. □

Theorem 5.5. *Nonnegative viscosity solutions of* (5.1) *are locally Lipschitz continuous and satisfy Harnack's inequality.*

Proof. By the previous lemma, each viscosity solution can be approximated locally uniformly by value functions. Hence, Harnack's inequality for viscosity solutions follows immediately from Theorem 4.2. By Theorem 4.1, value functions are locally Lipschitz continuous up to the scale ϵ with a Lipschitz constant depending only on p and n. Therefore viscosity solutions are locally Lipschitz continuous. □

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Local regularity for time-dependent tug-of-war games with varying probabilities

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Abstract

We study local regularity properties of value functions of time-dependent tug-of-war games. For games with constant probabilities we get local Lipschitz continuity. For more general games with probabilities depending on space and time we obtain Hölder and Harnack estimates. The games have a connection to the normalized $p(x,t)$-parabolic equation $u_t = \Delta u + (p(x,t) - 2)\Delta_\infty^N u$.

MSC: 35K20; 91A15

Keywords: Parabolic $p(x,t)$-Laplacian; Hölder continuity; Harnack inequality; Stochastic games; Tug-of-war

1. Introduction

In this paper we study local regularity properties of tug-of-war type games related to parabolic PDEs. First, we establish asymptotic Lipschitz continuity for value functions of the game with constant probabilities, and then continue analyzing the regularity of a more general game with space and time dependent probabilities that we call $p(x,t)$-game.

The value functions of this particular two player zero sum game satisfy the so called dynamic programming principle (hereafter DPP)

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$$u_\varepsilon(x,t) = \frac{\alpha(x,t)}{2}\Big(\sup_{B_\varepsilon(x)} u(y,t-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)} u(y,t-\frac{\varepsilon^2}{2})\Big)$$
$$+\beta(x,t) \fint_{B_\varepsilon(x)} u(y,t-\frac{\varepsilon^2}{2})\,dy,$$

which may arise for example from stochastic games or discretization schemes. In game terms, this equation can be heuristically interpreted as summing up the three different alternatives of the game round with the corresponding (x,t)-dependent probabilities while the step takes $\frac{\varepsilon^2}{2}$ time.

Lipschitz estimate for the game with constant probabilities is based on the good symmetry properties produced by utilizing cancellation strategies that allows us to directly obtain Lipschitz continuity. In the $p(x,t)$-case the symmetry properties and sharp cancellation effects break down. Moreover, global approaches to the problem are hampered by the of loss of translation invariance, which makes it hard to keep track of accumulated error.

Our proofs are of local nature. The idea of the proof for Hölder continuity of $p(x,t)$-game arises from the stochastic game theory: we start the game simultaneously at two points x and z and try to pull the points closer to each other, where 'closer' is in terms of a suitable comparison function. In particular, in the stochastic terminology the process is a supermartingale. To show that we may pull the points closer in this sense, we may consider the process in the higher dimensional space by setting $(x,z) \in \mathbb{R}^{2n}$ and then apply suitable strategies for such a game. There are several differences in the parabolic setting compared to the elliptic proofs in [17] and [16] related to controlling the dynamic effects. Indeed, in the Lipschitz proof we utilize estimates for probability distributions on different time instances whereas in the elliptic case it suffices to deal with the long time limit distribution. In the case of the $p(x,t)$-game the resulting DPP is in $\mathbb{R}^{2n+1}$, and the comparison function will have to take the time direction into account.

As an application, our results can be used to prove local Lipschitz continuity for the solutions to the normalized p-parabolic equation

$$u_t = |\nabla u|^{2-p}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Delta u + (p-2)\Delta_\infty^N u,$$

where $\Delta_\infty^N u = \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle$ is the normalized or game theoretic infinity Laplacian. This equation has been recently studied by Jin and Silvestre [14], Banerjee and Garofalo [2], Does [7], as well as Manfredi, Parviainen and Rossi [19]. In the $p(x,t)$-case, we show that under suitable assumptions the value functions of the game converge to the unique viscosity solution of the Dirichlet problem to the normalized $p(x,t)$-parabolic equation

$$u_t = \Delta u + (p(x,t)-2)\Delta_\infty^N u.$$

However, a priori our methods and results are not relying on the PDE techniques; rather they are quite different from those.

The connection between the infinity Laplacian and tug-of-war games was established by Peres, Schramm, Sheffield and Wilson in [22], for the p-Laplacian in [21] and for the normalized p-parabolic equation by Manfredi, Parviainen and Rossi in [19], see also [4].

This paper is organized as follows. In Section 2 we fix the notation and define the game. In Section 3 we assume that $p(x,t) \equiv p > 2$ is a constant and obtain local asymptotic Lipschitz continuity for value functions by using game-theoretic methods. In Section 4 we get Hölder and Harnack estimates for the $p(x,t)$-game. In Section 5 it is proved that value functions of the $p(x,t)$-game converge uniformly to a viscosity solution of the normalized $p(x,t)$-parabolic equation. In Section 6 we show that there is a unique viscosity solution to the $p(x,t)$-parabolic equation with given continuous boundary data.

2. Preliminaries

Throughout the paper $\Omega \subset \mathbb{R}^n$ is a bounded domain. If not mentioned otherwise, for $T > 0$, $\Omega_T := \Omega \times (0,T)$ is a parabolic cylinder with the parabolic boundary

$$\Gamma_T := \{\partial\Omega \times [0,T]\} \cup \{\Omega \times \{0\}\}.$$

For our game we also need the parabolic boundary strip of width $\varepsilon > 0$,

$$\Gamma_T^\varepsilon := \left(\Gamma_\varepsilon \times (-\frac{\varepsilon^2}{2}, T]\right) \cup \left(\Omega \times (-\frac{\varepsilon^2}{2}, 0]\right).$$

Here

$$\Gamma_\varepsilon := \{x \in \mathbb{R}^n \setminus \Omega \;:\; \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}$$

is the ε-boundary strip of Ω.

For a measurable function $p : \Omega_T \to (2,\infty)$, we define the functions $\alpha : \Omega_T \to (0,1)$ and $\beta : \Omega_T \to (0,1)$,

$$\alpha(x,t) = \frac{p(x,t)-2}{p(x,t)+n}, \quad \beta(x,t) = \frac{n+2}{p(x,t)+n}.$$

Notice that $\alpha(x,t) + \beta(x,t) = 1$ for all $(x,t) \in \Omega_T$.

Next we define a tug-of-war type game, which we call $p(x,t)$-*game* to emphasize the connection with $p(x,t)$-Laplacian, see Section 5. The game is a zero sum stochastic game between Player I and Player II in Ω_T. Fix $\varepsilon > 0$. First a token is placed at $(x_0, t_0) \in \Omega_T$. With probability $\alpha(x_0, t_0)$, the players flip a fair coin, and the winner of the toss moves the token to a point

$$(x_1, t_1) \in B_\varepsilon(x_0) \times \{t_0 - \frac{\varepsilon^2}{2}\},$$

according to his or her strategy. We use a notation $B_\varepsilon(x_0)$ for an open ball centered at x_0 with radius ε. With probability $\beta(x_0, t_0)$, the token moves according to the uniform probability to a random point (x_1, t_1) in a set $B_\varepsilon(x_0) \times \{t_0 - \frac{\varepsilon^2}{2}\}$, and in this paper we call such moves *random vectors* for short. From (x_1, t_1) the game continues according to the same rules, and the token moves to a point

$$(x_2, t_2) \in B_\varepsilon(x_1) \times \{t_1 - \frac{\varepsilon^2}{2}\}.$$

We denote by $(x_\tau, t_\tau) \in \Gamma_T^\varepsilon$ the first point of the sequence on Γ_T^ε. Then Player II pays Player I the payoff $F(x_\tau, t_\tau)$, where $F : \Gamma_T^\varepsilon \to [-M, M]$ is a given measurable payoff function. Naturally, Player I tries to maximize the payoff and Player II tries to minimize it. The number of steps during the game is bounded,

$$\tau \leq 2\varepsilon^{-2} t_0 + 1 \leq 2\varepsilon^{-2} T + 1.$$

The value function u_ε of the game is

$$u_\varepsilon(x_0, t_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}, S_{\mathrm{II}}} [F(x_\tau, t - \frac{\varepsilon^2}{2}\tau)],$$

where S_{I} and S_{II} are strategies of Player I and Player II. For further details on stochastic vocabulary regarding tug-of-war games, we refer to [22].

Since the number of steps during the game is bounded, adding a bounded running payoff to the game would not cause any new difficulties. In the case of unlimited number of steps the situation is different, see [23].

A crucial property of value functions of tug-of-war type games is DPP characterization. In the parabolic case this characterization is much easier to verify than in the elliptic case. Moreover, proving DPP characterization for value functions of our game does not differ from the case where the probabilities α and β are space independent. The following two lemmas can be proved by using the techniques of [18]. We use the notation

$$\fint_{B_r} u \,\mathrm{d}x := \frac{1}{|B_r|} \int_{B_r} u \,\mathrm{d}x$$

for the mean value of a function u in a ball B_r. Here $|B_r|$ denotes the Lebesgue measure of B_r.

Lemma 2.1. *For given $\varepsilon > 0$ and payoff function F on Γ^ε_T, there is a unique measurable function u equal to F on Γ^ε_T and satisfying the parabolic DPP*

$$\begin{aligned} u(x,t) = {} & \frac{\alpha(x,t)}{2} \Big(\sup_{B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2}) \Big) \\ & + \beta(x,t) \fint_{B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2}) \, dy \end{aligned}$$

for $(x,t) \in \Omega_T$.

Lemma 2.2. *Given $\varepsilon > 0$ and a bounded payoff function F on Γ^ε_T, the value function u_ε satisfies the parabolic DPP.*

A typical idea to estimate the value function u_ε is to fix a strategy for one of the players. We may also localize the situation by using a new stopping time $\tau^* \leq \tau$. The following lemma is a standard tool for fixed strategies. Again we omit the proof which is similar to [23, Lemma 2.3].

Lemma 2.3. *If the game starts from* $(x_0, t_0) \in \Omega_T$ *and* $\tau^* < 2t_0\varepsilon^{-2}$ *is a stopping time, then*

$$u_\varepsilon(x_0, t_0) \geq \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,t_0)}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}} u_\varepsilon(x_{\tau^*}, t_0 - \frac{\tau^*}{2}\varepsilon^2)$$

for any fixed strategy S^0_{I} *of Player I, and*

$$u_\varepsilon(x_0, t_0) \leq \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}, S^0_{\mathrm{II}}} u_\varepsilon(x_{\tau^*}, t_0 - \frac{\tau^*}{2}\varepsilon^2)$$

for any fixed strategy S^0_{II} *of Player II.*

3. Lipschitz estimate for *p*-game

In this section we study local regularity properties of $p(x,t)$-games when $p(x,t) \equiv p > 2$ is a constant. Then the probability functions are also constants, $\alpha(x,t) \equiv \alpha \in (0,1)$ and $\beta(x,t) \equiv \beta \in (0,1)$. This game was defined in [19].

We start with constant p for simplicity: these games have symmetry properties suitable for cancellation strategy idea, developed in [17], to get asymptotic Lipschitz continuity. In order to establish this, we use the following stochastic estimate, which combines well known Hoeffding's and Kolmogorov's inequalities.

Lemma 3.1. *Consider i.i.d. symmetric real-valued random variables* Y_m, $m = 1, \dots, N$, *for which* $|Y_m| \leq b$ *for some* $b > 0$. *Then for* $\lambda > 0$ *the following inequalities hold:*

$$\mathbb{P}(|Y_1 + \ldots + Y_N| \geq \lambda) \leq 2\exp\left(-\frac{\lambda^2}{2Nb^2}\right),$$

$$\mathbb{P}(\max_{1\leq m\leq N} |Y_1 + \ldots + Y_m| \geq \lambda) \leq 2\mathbb{P}(|Y_1 + \ldots + Y_N| \geq \lambda).$$

When the game starts from $(x_0, t_0) \in \Omega_T$ and Player I follows *a cancellation strategy with target z*, she tries to cancel the earliest move of Player I which she has not yet canceled. If there are no moves to cancel, she tries to pull the token to the direction of vector $(z - x_0) \in \Omega$. Notice that Player I pays no attention to random moves.

We want to use the cancellation strategy to prove asymptotic Lipschitz estimate for the $p(x,t)$-game with constant p. The two main difficulties are the possibility to reach the maximum number of steps too soon and the case of different time levels. We estimate the probability for reaching maximum number of steps in the proof of Theorem 3.2, and the problem of different time levels is solved in Theorem 3.3.

Theorem 3.2. *Suppose that* $B_{6r}(z_0) \subset \Omega$, *where* $0 < \varepsilon < r < \left(\frac{\alpha T}{6}\right)^{\frac{1}{2}}$. *Then, for points* $(x,t), (y,t) \in B_r(z_0) \times (\frac{6r^2}{\alpha}, T) \subset \Omega_T$ *and for sufficiently small* ε, *the value function* u_ε *satisfies the Lipschitz estimate*

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \leq C(p,n)\frac{|x-y|}{r}\|u_\varepsilon\|_\infty + C'(p,n)\frac{\varepsilon}{r}\|u_\varepsilon\|_\infty.$$

Proof. Because of the error term, we may suppose that $|x-y| \geq \varepsilon$. Let z be the midpoint of $[x, y] \subset \Omega$ and suppose first that

$$u_\varepsilon(y,t) \geq u_\varepsilon(x,t).$$

When the game starts from $(x, t) =: (x_0, t_0)$, Player II follows the cancellation strategy with a target z. Let us define the stopping time τ^*. There are four conditions to stop the game:

(1) Player II wins $\lceil |x-z|/\varepsilon \rceil$ fair coin tosses more than Player I.
(2) Player I wins $\lceil r/\varepsilon \rceil$ fair coin tosses more than Player II.
(3) The sum of random vectors has length larger than r.
(4) We reach the maximum number of steps.

When the game starts from $(y, t) =: (y_0, t_0)$, Player I follows the cancellation strategy with target z, and we define τ^* as before by changing the roles of the players. By using the cancellation effect and Lemma 2.3, we obtain

$$|u_\varepsilon(x_0,t_0) - u_\varepsilon(y_0,t_0)| \leq 2\,\|u_\varepsilon\|_\infty \sum_{j=1}^{k} \overline{P}_j + 2\delta\,\|u_\varepsilon\|_\infty\,,$$

where $\overline{P}_j$ is the probability that $\tau^* = j$ and the game ended because of Condition 2 or 3, and δ is the probability that the game ended because the maximum number of steps was reached. The number k is the maximum number of steps during the game, $k = \lceil 2\varepsilon^{-2} t_0 \rceil$.

We get an upper estimate for $\sum \overline{P}_j$ from [17, Lemma 3.1]. The lemma gives an upper bound $C(p,n)|x-y|/r$ for the probability P' that the tug-of-war with noise ends because of Condition 2 or 3. Since there is not Condition 4 in the elliptic case (there is not an upper bound for the number of steps during the game), we have

$$\sum_{j=1}^{k} \overline{P}_j \leq P' \leq C(p,n)\frac{|x-y|}{r}.$$

Hence, we get

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \leq C(p,n)\frac{|x-y|}{r}\,\|u_\varepsilon\|_\infty + 2\delta\,\|u_\varepsilon\|_\infty\,. \tag{3.1}$$

The previous inequality also holds if $u_\varepsilon(x,t) > u_\varepsilon(y,t)$, which can be seen by fixing a cancellation strategy for Player I when starting from (x,t) and for Player II when starting from (y,t).

The main part of this proof is to estimate the probability δ that the game ends when the maximum number $\lceil 2t_0/\varepsilon^2 \rceil$ of steps is reached. First we need a rough estimate for the number of fair coin tosses between the players during the game. Denote by Z_m the Bernoulli variables with $Z_m \in \{0,1\}$ and $\mathbb{P}(Z_m = 1) = \alpha$. Define

$$A := \left\{ \sum_{m=1}^{l} Z_m > \frac{\alpha}{2} l \text{ for all } l \geq \varepsilon^{-1} \right\}.$$

We estimate

$$\begin{aligned}\mathbb{P}(A^c) &= \mathbb{P}\left(\sum_{m=1}^{l} Z_m \le \frac{\alpha}{2} l \text{ for some } l \ge \varepsilon^{-1}\right)\\ &\le \sum_{l\ge\varepsilon^{-1}} \mathbb{P}\left(\sum_{m=1}^{l} Z_m \le \frac{\alpha}{2} l\right)\\ &\le \sum_{l\ge\varepsilon^{-1}} \mathbb{P}\left(\left|\sum_{m=1}^{l} Z_m - l\alpha\right| \ge \frac{\alpha}{2} l\right)\\ &= \sum_{l\ge\varepsilon^{-1}} \mathbb{P}\left(\left|\sum_{m=1}^{l} (Z_m - \alpha)\right| \ge \frac{\alpha}{2} l\right).\end{aligned}$$

Using Lemma 3.1 with $Y_m = Z_m - \alpha$, $\lambda = \frac{\alpha}{2} l$, $b = 1$ and $N = l$ gives

$$\sum_{l\ge\varepsilon^{-1}} \mathbb{P}\left(\left|\sum_{m=1}^{l} (Z_m - \alpha)\right| \ge \frac{\alpha}{2} l\right) \le \sum_{l\ge\varepsilon^{-1}} 2\exp(-\frac{\alpha^2}{8} l) \le O(\varepsilon).$$

Hence, for small enough ε there is a constant $C'(p,n) > 0$ such that

$$\mathbb{P}(A) \ge 1 - C'(p,n)\frac{\varepsilon}{r}. \tag{3.2}$$

Supposing that $\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil$ is an even number, we estimate combinatorially the probability $\widetilde{P}_0$ that after exactly $\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil$ fair coin flips there have been exactly the same number of heads and tails,

$$\begin{aligned}\widetilde{P}_0 &= \binom{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil}{\frac{1}{2}\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil}\left(\frac{1}{2}\right)^{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil}\\ &= \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil - 1}{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil}\\ &\le \left(\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil}{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil + 1}\right)^{\frac{1}{2}}\\ &= \left(\frac{1}{\lceil \frac{\alpha}{2}\varepsilon^{-2} t_0 \rceil + 1}\right)^{\frac{1}{2}}\\ &\le \frac{\varepsilon}{3r},\end{aligned}$$

where in the last inequality we used the requirement $t_0 > \frac{6r^2}{\alpha}$. For probability $\widetilde{P}_k$, $k \in \mathbb{Z}$, that after $\lceil \frac{\alpha}{2}\varepsilon^{-2}t_0 \rceil$ of fair coin flips there have been k heads more than tails, we have $\widetilde{P}_k \le \widetilde{P}_0$. (When k is negative, $\widetilde{P}_k$ means that there have been $-k$ tails more than heads.) We get the estimate

$$\sum_{k=-\lceil |x-y|/\varepsilon \rceil}^{\lceil |x-y|/\varepsilon \rceil} \widetilde{P}_k \le \left(\frac{2|x-y|}{\varepsilon} + 1 \right) \widetilde{P}_0 \le \frac{|x-y|}{r}.$$

Denote by D an event that the event A occurred and at the time $\lceil \frac{\alpha}{2}\varepsilon^{-2}t_0 \rceil$ of fair coin flips there have been at least $\lceil \frac{|x-y|}{\varepsilon} \rceil$ heads more than tails. Moreover, denote by E an event that the event A occurred and there have been at least $\lceil \frac{|x-y|}{\varepsilon} \rceil$ heads more than tails at some point before $\lceil \frac{\alpha}{2}\varepsilon^{-2}t_0 \rceil$ fair coin flips. By the previous estimate we have

$$\begin{aligned}\mathbb{P}(D) &\ge \frac{1}{2}\left(1 - \frac{|x-y|}{r}\right)(1 - C'(p,n)\frac{\varepsilon}{r}) \\ &\ge \frac{1}{2}\left(1 - 2C'(p,n)\frac{|x-y|}{r}\right).\end{aligned}$$

To estimate $\mathbb{P}(E)$, observe first that

$$\mathbb{P}(E \cap D) = \frac{1}{2}\mathbb{P}(E)$$

and

$$\mathbb{P}(E^c \cap D) \le \frac{\varepsilon}{3r}.$$

Since

$$\mathbb{P}(D) = \mathbb{P}(E \cap D) + \mathbb{P}(E^c \cap D),$$

we get

$$\mathbb{P}(E) \ge 1 - 2C'(p,n)\frac{|x-y|}{r} - \frac{\varepsilon}{3r}.$$

Since the probability that the game ends before step $\lceil 2t_0/\varepsilon^2 \rceil$ is greater than $\mathbb{P}(E)$, we get an estimate for δ,

$$\delta \le C(p,n)\frac{|x-y|}{r} + 3C'(p,n)\frac{\varepsilon}{r},$$

and recalling estimate (3.1), we have

$$\begin{aligned}|u_\varepsilon(x,t) - u_\varepsilon(y,t)| &\le C(p,n)\frac{|x-y|}{r}\|u_\varepsilon\|_\infty + 2\delta\|u_\varepsilon\|_\infty \\ &\le 2C(p,n)\frac{|x-y|}{r}\|u_\varepsilon\|_\infty + 6C'(p,n)\frac{\varepsilon}{r}\|u_\varepsilon\|_\infty. \qquad \square\end{aligned}$$

Theorem 3.3. *Let $x, y \in \Omega$ and $t =: t_0$ satisfy the conditions of Theorem 3.2 and $t_1 \in (t_0, T)$ satisfy $t_1 - t_0 \le r^2$. Then for $(x, t_1), (y, t_0) \in \Omega_T$ we have the Lipschitz estimate*

$$|u_\varepsilon(x, t_1) - u_\varepsilon(y, t_0)| \le C(p, n)\frac{|x - y| + |t_1 - t_0|^{\frac{1}{2}}}{r} \|u_\varepsilon\|_\infty + C'(p, n)\frac{\varepsilon^{\frac{1}{2}}}{r} \|u_\varepsilon\|_\infty .$$

Proof. We prove the case $x = y$, for otherwise we use triangle inequality and Theorem 3.2. Because of the error term, we may suppose that $t_1 \ge t_0 + \varepsilon^2$. Denote

$$s := \sqrt{t_1 - t_0} \ge \varepsilon.$$

Suppose first that $u_\varepsilon(y, t_1) \ge u_\varepsilon(y, t_0)$. The game starts from (y, t_1). Player II uses a strategy S^0_{II} in which he pulls towards y and stays there if possible. The game ends when the token leaves the cylinder $S := B_s(y) \times (t_0, t_1)$ for the first time. Let A be the event that the token hits the bottom of S. Then, regardless of the strategy of Player I,

$$P := \mathbb{P}(A) \ge \left(\frac{1}{10}\right)^{2(n+1)^2}.$$

This estimate follows from the proof of Lemma 4.6 below.

Denote

$$M := C(p, n)\frac{s}{r}||u_\varepsilon||_\infty,$$

where $C(p, n)$ is the constant from Theorem 3.2. By using Theorem 3.2 to estimate values of u_ε in the ball $B_s(y)$, we get

$$\begin{aligned} u_\varepsilon(y, t_1) - u_\varepsilon(y, t_0) &\le P(u_\varepsilon(y, t_0) + M) + (1 - P) \sup_{\partial B_s(y)\times[t_0,t_1]} u_\varepsilon - u_\varepsilon(y, t_0) \\ &= PM + (1 - P)\Big(\sup_{\partial B_r(y)\times[t_0,t_1]} u_\varepsilon - u_\varepsilon(y, t_0)\Big) \\ &\le PM + (1 - P)\Big(\sup_{t\in[t_0,t_1]} u_\varepsilon(y, t) + M - u_\varepsilon(y, t_0)\Big) \\ &= M + (1 - P)\Big(\sup_{t\in[t_0,t_1]} u_\varepsilon(y, t) - u_\varepsilon(y, t_0)\Big). \end{aligned}$$

Choose $k \in \mathbb{N}$ such that

$$(1 - P)^k \le C(p, n)\frac{s}{r}.$$

By continuing the previous estimation we get

$$\begin{aligned} u_\varepsilon(y,t_1)-u_\varepsilon(y,t_0) &\le M\sum_{j=0}^{k-1}(1-P)^j+(1-P)^k\big(\sup_{t\in[t_0,t_1]}u_\varepsilon(y,t)-u_\varepsilon(y,t_0)\big)\\ &\le \frac{1}{P}M+2C(p,n)\frac{s}{r}||u_\varepsilon||_\infty\\ &=\widetilde{C}(p,n)\frac{s}{r}||u_\varepsilon||_\infty. \end{aligned}$$

If $u_\varepsilon(y,t_1)<u_\varepsilon(y,t_0)$, we fix a strategy for Player I when starting from (y,t_1) and by symmetric argument we get

$$u_\varepsilon(y,t_0)-u_\varepsilon(y,t_1)\le \widetilde{C}(p,n)\frac{s}{r}||u_\varepsilon||_\infty.$$

Hence, we have

$$\begin{aligned} |u_\varepsilon(y,t_1)-u_\epsilon(y,t_1)| &\le \widetilde{C}(p,n)\frac{s}{r}||u_\varepsilon||_\infty\\ &=\widetilde{C}(p,n)\frac{|t_1-t_0|^{\frac{1}{2}}}{r}||u_\varepsilon||_\infty. \end{aligned}$$

The error term of the scale $\varepsilon^{1/2}$ has to be added when $t_1-t_0\le\varepsilon^2$. □

4. Hölder and Harnack estimates for $p(x,t)$-game

In this section we study regularity properties of the $p(x,t)$-game, which was defined in Section 2. We assume throughout the section that $u_\varepsilon>0$ is a value function of the game. In the first subsection we show asymptotic Hölder continuity for u_ε, and then continue with Harnack's inequality in the second subsection.

4.1. Asymptotic Hölder continuity

Since our location dependent parabolic game is not translation invariant, we cannot immediately use the cancellation strategy. Instead, we use a more general idea developed by Luiro and Parviainen for the elliptic case in [16]. The main idea is to start the game simultaneously at two points and try to pull them closer to each other by using a suitable comparison function f with a certain favorable curvature in space. The player trying to pull the two points closer, say Player I, has a certain flexibility in her strategy depending on what the opponent does. If Player II does not pull the points further away from each other, then Player I tries to pull them directly closer. Instead, if Player II tries to pull the points almost optimally further away, then Player I aims at the exactly opposite step.

As in the previous section concerning Lipschitz regularity, we break the proof of parabolic Hölder continuity into two parts. In the first part, Theorem 4.1, we consider the case where the points $x,y\in\Omega$ are at the same time level t in Ω_T. We use the strategy of [16], but add a time-dependent term $g(t)=|t|^{\delta/2}$ to the comparison function f. The purpose of the term g in our comparison function $F(x,t)=f(x)+g(t)$ is to get the right boundary values for F without allowing too large error in estimates.

In the other part of the proof of Hölder continuity we handle the time direction. This part is easier, and we could actually prove it by utilizing the technique we used in the proof of Theorem 3.3. However, we present another proof relying more on the DPP property of the value function u_ε.

Theorem 4.1. *Let $B_{2r}(0) \times [-2r^2, 0] \subset \Omega \times (-T, T)$. Then u_ε satisfies the Hölder estimate*

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \le C(n)\frac{|x-y|^\delta}{r^\delta}\,\|u_\varepsilon\|_\infty + C'(n)\frac{\varepsilon^\delta}{r^\delta}\,\|u_\varepsilon\|_\infty\,,$$

when $x, y \in B_r(0)$ and $t \in (-r^2, 0)$.

Proof. Denote

$$S_1 := B_r(0) \times (-r^2, 0),\;\; S_2 := B_{2r}(0) \times (-2r^2, 0).$$

To define a suitable comparison function, define the functions g, f_1 and f_2,

$$g(t) = |t|^{\delta/2},$$
$$f_1(x,z) = C(n)|x-z|^\delta + |x+z|^2,$$

as well as

$$f_2(x,z) = \begin{cases} C^{2(N-i)}\varepsilon^\delta & \text{if } (x,z) \in A_i, \\ 0 & \text{if } |x-z| > N\frac{\varepsilon}{10}. \end{cases}$$

Here

$$A_i := \{(x,z) \in \mathbb{R}^{2n} \;:\; (i-1)\frac{\varepsilon}{10} < |x-z| \le i\frac{\varepsilon}{10}\}$$

for $i = \{1, \ldots, N\}$. Finally, our comparison function is

$$F(x,z,t) = f(x,z) + g(t),$$

where

$$f(x,z) = f_1(x,z) - f_2(x,z).$$

We use this notation to emphasize the time dependent term g needed in the parabolic case.

By scaling, we may assume that

$$0 \le u_\varepsilon \le r^\delta \text{ in } S_2 \setminus S_1.$$

This implies

$$u_\varepsilon(x,t) - u_\varepsilon(z,t) - F(x,z,t) \le C^{2N}\varepsilon^\delta \text{ in } S_2 \setminus S_1,$$

and we want to show that the same inequality holds in S_1. Suppose not. Then

$$M := \sup_{(x',t'),(z',t')\in S_1} (u_\varepsilon(x',t') - u_\varepsilon(z',t') - F(x',z',t')) > C^{2N}\varepsilon^\delta.$$

Thriving for contradiction, let $\eta > 0$ and choose $(x,t), (z,t) \in S_1$ such that

$$u_\varepsilon(x,t) - u_\varepsilon(z,t) - F(x,z,t) \geq M - \eta. \tag{4.3}$$

Recall that DPP for u_ε reads as

$$u_\varepsilon(x,t) = \frac{\alpha(x,t)}{2}\left\{\sup_{y\in B_\varepsilon(x)} u_\varepsilon(y, t-\frac{\varepsilon^2}{2}) + \inf_{y\in B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2})\right\}$$
$$+ \beta(x,t) \fint_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2})dy.$$

By using the DPP characterization for the difference $u_\varepsilon(x,t) - u_\varepsilon(z,t)$, it is easy to see that

$$u_\varepsilon(x,t) - u_\varepsilon(z,t) = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{\alpha(z,t)}{2}(\sup_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - \inf_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2})$$
$$+ \inf_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - \sup_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2})),$$

$$I_2 = \beta(x,t)\left(\fint_{B_\varepsilon(x)} u(y,t-\frac{\varepsilon^2}{2})dy - \fint_{B_\varepsilon(z)} u(y,t-\frac{\varepsilon^2}{2})dy\right),$$

and

$$I_3 = \frac{\alpha(x,t)-\alpha(z,t)}{2}$$
$$\left(\sup_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - 2\fint_{B_\varepsilon(z)} u(y,t-\frac{\varepsilon^2}{2})dy\right).$$

This identity together with inequality (4.3) gives

$$M \leq I_1 + I_2 + I_3 - F(x,z,t) + \eta. \tag{4.4}$$

We are going to estimate the terms I_1, I_2 and I_3 to get a contradiction with (4.4). To be more precise, we are going to show the following inequalities,

$$\alpha(z,t)M > I_1 - \alpha(z,t)(F(x,z,t)-\eta),$$
$$\beta(x,t)M > I_2 - \beta(x,t)((F(x,z,t)-\eta),$$

as well as

$$(\alpha(x,t)-\alpha(z,t))M > I_3 - (\alpha(x,t)-\alpha(z,t))(F(x,z,t)-\eta).$$

To estimate I_1, first we prove the following inequalities

$$\sup_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - \inf_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) \le M + \sup_{B_\varepsilon(x)\times B_\varepsilon(z)} F(\overline{x},\overline{z},t-\frac{\varepsilon^2}{2}) + \eta$$

and

$$\inf_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - \sup_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) \le M + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} F(\overline{x},\overline{z},t-\frac{\varepsilon^2}{2}) + \eta.$$

The first inequality follows by picking $x' \in B_\varepsilon(x)$, $z' \in B_\varepsilon(z)$ such that $u_\varepsilon(x') \ge \sup_{B_\varepsilon(x)} u_\varepsilon - \eta/2$ and $u_\varepsilon(z') \le \inf_{B_\varepsilon(z)} u_\varepsilon - \eta/2$ and estimating

$$\begin{aligned}
&\sup_{y\in B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - \inf_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) \\
&\le u_\varepsilon(x',t-\frac{\varepsilon^2}{2}) - u_\varepsilon(z',t-\frac{\varepsilon^2}{2}) + \eta \\
&\le M + F(x',z',t-\frac{\varepsilon^2}{2}) + \eta \\
&\le M + \sup_{(y,y')\in B_\varepsilon(x)\times B_\varepsilon(z)} F(y,y',t-\frac{\varepsilon^2}{2}) + \eta.
\end{aligned}$$

The second inequality follows the same way, and we get an estimate for I_1,

$$\begin{aligned}
&I_1 - \frac{\alpha(z,t)}{2}\eta \\
&\le \alpha(z,t)\left(M + \frac{1}{2}\left(\sup_{B_\varepsilon(x)\times B_\varepsilon(z)} F(x,z,t-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} F(x,z,t-\frac{\varepsilon^2}{2})\right)\right).
\end{aligned}$$

Let us show that

$$\begin{aligned}
F(x,z,t) &> \frac{1}{2}\left(\sup_{B_\varepsilon(x)\times B_\varepsilon(z)} F(x',z',t-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} F(x',z',t-\frac{\varepsilon^2}{2})\right) + 2\eta \\
&= \frac{1}{2}\left(\sup_{B_\varepsilon(x)\times B_\varepsilon(z)} f(x',z') + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} f(x',z')\right) + \left|t-\frac{\varepsilon^2}{2}\right|^{\delta/2} + \eta.
\end{aligned}$$

Since

$$\left|t - \frac{\varepsilon^2}{2}\right|^{\delta/2} - |t|^{\delta/2} \le \left|\frac{\varepsilon^2}{2}\right|^{\delta/2} \le \varepsilon^\delta, \tag{4.5}$$

it suffices to show that

$$f(x,z) > \frac{1}{2}\left(\sup_{B_\varepsilon(x)\times B_\varepsilon(z)} f + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} f\right) + \varepsilon^\delta.$$

Throughout the proof the error caused by the term g is in the acceptable scale ε^δ.

During the rest of the argument we just write $\sup f$ and $\inf f$ meaning that sup and inf are taken over $B_\varepsilon(x) \times B_\varepsilon(z)$.

Suppose first that $|x-z| > N\frac{\varepsilon}{10}$. Then $f_2 = 0$. Choose $h_x, h_z \in B_\varepsilon(0)$ such that

$$\sup f_1 \le f_1(x+h_x, z+h_z) + \eta.$$

Let $\theta = \frac{1}{10}$ and assume first that

$$(h_x - h_z)_V^2 \ge (4-\theta)\varepsilon^2,$$

where V is the space spanned by $x-z$ and

$$(h_x - h_z)_V = (h_x - h_z) \cdot \frac{x-z}{|x-z|}.$$

To estimate $\sup f_1 + \inf f_2 - 2f_1$, it is useful to write Taylor's expansion for $f_1(x+h_x, z+h_z)$ as

$$\begin{aligned}
&f_1(x+h_x, z+h_z) \\
&\quad = f_1(x,z) + C\delta|x-z|^{\delta-1}(h_x-h_z)_V + 2(x+z)\cdot(h_x+h_z) \\
&\quad + \frac{C}{2}\delta|x-z|^{\delta-2}\left((\delta-1)(h_x-h_z)_V^2 + (h_x-h_z)_{V^\perp}^2\right) \\
&\quad + |h_x+h_z|^2 + \mathcal{E}_{x,z}(h_x,h_z).
\end{aligned}$$

Here $\mathcal{E}_{x,z}$ is an error term satisfying

$$\begin{aligned}
\mathcal{E}_{x,z}(h_x,h_z) &\le C|(h_x,h_z)|^3(|x-z|-2\varepsilon)^{\delta-3} \\
&\le 10\varepsilon^2|x-z|^{\delta-2}
\end{aligned}$$

when N is large enough, for example $N > 100C/\delta$.

By using the Taylor estimate and the estimate for the error term, we obtain

$$\begin{aligned}
&\sup f_1 + \inf f_2 - 2f_1 \\
&\le f_1(x+h_x, z+h_z) + f_1(x-h_x, z-h_z) - 2f_1(x,z) + \eta \\
&= \frac{C}{2}\delta|x-z|^{\delta-2}\Big(2(\delta-1)(h_x-h_z)_V^2 + 2(h_x-h_z)_{V^\perp}^2\Big) \\
&+ 2|h_x+h_z|^2 + \mathcal{E}_{x,z}(h_x,h_z) + \mathcal{E}_{x,z}(-h_x,-h_z) + \eta \\
&\le |x-z|^{\delta-2}(20 - C\delta)\varepsilon^2 + 8\varepsilon + \eta + \varepsilon^\delta \\
&\le -\widetilde{C}\varepsilon^\delta + 8\varepsilon^2 + \eta + \varepsilon^\delta < -2\varepsilon^\delta,
\end{aligned}$$

when $\widetilde{C} = C\delta - 20$ has been chosen large.

If

$$(h_x - h_z)_V^2 < (4-\theta)\varepsilon^2,$$

then

$$(h_x - h_z)_V \le (2 - \theta/4)\varepsilon,$$

and the second order term of the Taylor estimate, together with the error term, can be estimated by

$$\begin{aligned}
&\frac{C}{2}\delta|x-z|^{\delta-2}(2\varepsilon)^2 + (2\varepsilon)^2 + 10\varepsilon^2|x-z|^{\delta-2} \\
&\quad\le \frac{30C}{N}\delta|x-z|^{\delta-1}\varepsilon \\
&\quad< \delta^2|x-z|\delta - 1\varepsilon.
\end{aligned}$$

Now we get

$$\begin{aligned}
&\sup f_1 + \inf f_1 - 2f_1 \\
&\le f_1(x+h_x, z+h_z) + f_1(x - \varepsilon\frac{x-z}{|x-z|}, x + \varepsilon\frac{x-z}{|x-z|}) - 2f_1(x,z) + \eta \\
&\le C\delta|x-z|^{\delta-1}(-\theta\varepsilon/4) + 16\varepsilon + \delta^2|x-z|^{\delta-1}\varepsilon + \eta \\
&\le (\delta - \frac{\theta}{4}C)\delta|x-z|^{\delta-1}\varepsilon + 16\varepsilon + \eta + \varepsilon^\delta \\
&\le (\delta - \frac{\theta}{4}C)\delta\varepsilon^\delta + 16\varepsilon + \eta \\
&< -\varepsilon^\delta,
\end{aligned}$$

when C is large enough.

Suppose next that $|x-z| \le N\frac{\varepsilon}{10}$. Then a straightforward estimate gives

$$|f_1(x+h_x, z+h_z) - f_1(x,z)| \le 3C\varepsilon^\delta.$$

We also have

$$\inf(f_1 - f_2) \le \sup f_1 - 10C\varepsilon^\delta - 2f_2,$$

which implies

$$\begin{aligned}\sup f + \inf f &\le 2\sup f_1 - 10C\varepsilon^\delta - 2f_2(x,z)\\ &\le 2f_1 + 6C\varepsilon^\delta - 10C\varepsilon^\delta - 2f_2 + \varepsilon^\delta\\ &\le 2f - 2\varepsilon^\delta\end{aligned}$$

when C is large enough. It follows that

$$f(x,z) > \frac{1}{2}\left(\sup_{B_\varepsilon(x)\times B_\varepsilon(z)} f + \inf_{B_\varepsilon(x)\times B_\varepsilon(z)} f\right) + \varepsilon^\delta.$$

Hence, we have shown that

$$I_1 - \frac{\alpha(z,t)}{2}\delta < \alpha(z,t)(M + F(x,z,t)),$$

or equivalently,

$$\alpha(z,t)M > I_1 - \alpha(z,t)F(x,z,t) - \frac{\alpha(z,t)}{2}\delta.$$

Let us next estimate I_2. We want to show that

$$\beta(x,t)M > I_2 - \beta(x,t)((F(x,z,t)+\eta),$$

where

$$I_2 = \beta(x,t)\left(\fint_{B_\varepsilon(x)} u(y, t-\frac{\varepsilon^2}{2})dy - \fint_{B_\varepsilon(z)} u(y, t-\frac{\varepsilon^2}{2})dy\right).$$

Let $P_{x,z}(h)$ be a mirror point of h with respect to $\operatorname{span}(x-z)^\perp$. If $|x-z| \ge 2\varepsilon$, we get an estimate

$$\begin{aligned}I_2 = \frac{\beta(x,t)}{|B_\varepsilon|}\Bigg(&\int_{B_\varepsilon(0)} u_\varepsilon(x+h) - u_\varepsilon(z+P_{x,z}(h)) - F(x+h, z+P_{x,z}(h))dy\\ &+ \int_{B_\varepsilon(0)} F(x+h, z+P_{x,z}(h))dy\Bigg)\end{aligned}$$

$$\leq \beta(x,t)M + \frac{\beta(x,t)}{|B_\varepsilon|} \int\limits_{B_\varepsilon(0)} F(x+h, z+P_{x,z}(h))dy.$$

If $|x-z| \leq 2\varepsilon$, there is a perfect cancellation in the intersection $B_\varepsilon(x) \cap B_\varepsilon(z)$. We refer to [16] for details and just state that in this case we have an estimate

$$I_2 \leq \beta(x,t)M + \beta(x,t)J_1,$$

where

$$J_1 = \frac{1}{|B_\varepsilon|} \left(\int\limits_{B_\varepsilon(0)\setminus B_\varepsilon(x-z)} F(x+h, z+P_{x,z}(h))dh + \int\limits_{B_\varepsilon(x)\cap B_\varepsilon(z)} F(y,y)dy \right).$$

We want to show that

$$F(x,z,t) > J_1.$$

Notice that since

$$\begin{aligned} &\int\limits_{B_\varepsilon(0)\setminus B_\varepsilon(x-z)} F(x+h, z+P_{x,z}(h))dh + \int\limits_{B_\varepsilon(x)\cap B_\varepsilon(z)} F(y,y)dy \\ &\leq \int\limits_{B_\varepsilon(0)\setminus B_\varepsilon(x-z)} f(x+h, z+P_{x,z}(h))dh + \int\limits_{B_\varepsilon(x)\cap B_\varepsilon(z)} f(y,y)dy \\ &\quad + \frac{2}{|B_\varepsilon|} \int\limits_{B_\varepsilon} \left| t - \frac{\varepsilon^2}{2} \right|^{\delta/2} dy, \end{aligned}$$

it is sufficient to show that

$$\begin{aligned} &f(x,z,t) - 2\varepsilon^\delta \\ &> \frac{1}{|B_\varepsilon|} \left(\int\limits_{B_\varepsilon(0)\setminus B_\varepsilon(x-z)} f(x+h, z+P_{x,z}(h))dh + \int\limits_{B_\varepsilon(x)\cap B_\varepsilon(z)} f(y,y)dy \right). \end{aligned}$$

If $|x-z| > N\frac{\varepsilon}{10}$, the key estimate is

$$\varepsilon^2 |x-z|^{\delta-2}(10 - \frac{C\delta}{4(n+2)}) + 2\varepsilon^\delta < 0,$$

which holds when C is sufficiently large. In the same manner, if $|x-z| \leq N\frac{\epsilon}{10}$, the additional error term $2\varepsilon^\delta$ does not cause extra difficulty compared to the elliptic case. These estimates can be obtained by using similar Taylor expansion ideas than in the case I_1, see [16].

In the last case we need to show that

$$(\alpha(x,t)-\alpha(z,t))M > I_3 - (\alpha(x,t)-\alpha(z,t))(F(x,z,t)+\eta),$$

where

$$I_3 = \frac{\alpha(x,t)-\alpha(z,t)}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) - 2 \fint_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2}) dy \right).$$

Again the extra error term compared to the elliptic case is on the scale of ε^δ. By choosing a sequence (x_j) such that $u_\varepsilon(x_j) \to \sup_{B_\varepsilon(x)} u_\varepsilon$, we have

$$\begin{aligned}
&\sup_{B_\varepsilon(x)} u_\varepsilon - \fint_{B_\varepsilon(z)} u_\varepsilon(y,t-\frac{\varepsilon^2}{2})dy \\
&= \fint_{B_\varepsilon(z)} \lim_j (u_\varepsilon(x_j) - u_\varepsilon(y) - F(x_j,y,t-\frac{\varepsilon^2}{2}) + F(x_j,y,t-\frac{\varepsilon^2}{2}))dy \\
&\le M + \sup_{a\in B_\varepsilon(x)} \fint_{B_\varepsilon(z)} F(a,y,t-\frac{\varepsilon^2}{2})dy.
\end{aligned}$$

We also get

$$\inf_{B_\varepsilon(x)} u_\varepsilon - \fint_{B_\varepsilon(z)} u_\varepsilon(y)dy \le M + \fint_{B_\varepsilon(z)} \inf_{b\in B_\varepsilon(x)} F(b,y,t-\frac{\varepsilon^2}{2})dy,$$

and finally

$$I_3 \le \left(\frac{\alpha(x,t)-\alpha(z,t)}{2}\right) \left(2M + \sup_{a\in B_\varepsilon(x)} \fint_{B_\varepsilon(z)} F(a,y,t-\frac{\varepsilon^2}{2}) + \inf_{b\in B_\varepsilon(x)} F(b,y,t-\frac{\varepsilon^2}{2})dy \right).$$

Hence, it is sufficient to show that

$$f(x,z) > \frac{1}{2} \sup_{a\in B_\varepsilon(x)} \left[\fint_{B_\varepsilon(z)} f(a,y) + \inf_{b\in B_\varepsilon(x)} f(b,y)dy \right] + 2\varepsilon^\delta.$$

The arguments are analogous to those used before, and we refer to [16] for details. □

Next we consider the time direction. For the similar oscillation estimate in the PDE context, we refer to [14, Lemma 4.3] and [1].

Theorem 4.2. *Let* $B_{2r}(0) \times [-2r^2, 0] \subset \Omega \times (-T, T)$ *and* $-r^2 < t_0 < t_1 < 0$. *Then* u_ε *satisfies*

$$|u_\varepsilon(x, t_1) - u_\varepsilon(x, t_0)| \le C(n)\frac{|t_1 - t_0|^{\delta/2}}{r^\delta} + C'(n)\frac{\varepsilon^\delta}{r^\delta},$$

when $x \in B_r(0)$.

Proof. Define

$$Q_r := B_r(0) \times (-r^2, 0).$$

We want to show that the oscillation of u in Q_r is comparable with the oscillation of u on the bottom of Q_r by a constant depending only on the dimension n. The idea is to control the oscillation of u_ε by suitable comparison functions $\overline{v}$ and $\underline{v}$. We use the DPP together with suitable iteration to get estimates for u_ε and the comparison functions.

Denote

$$A := \operatorname{osc}_{B_r(0)\times\{-r^2\}} u_\varepsilon$$

and set the first comparison function $\overline{v}$ as

$$\overline{v}(x, t) = \overline{c} + 7r^{-2}At + 2r^{-2}A|x|^2,$$

where $\overline{c}$ is chosen so that $\overline{v}(x, -r^2) \ge u_\varepsilon(x, -r^2)$ for all $x \in B_r(0)$, and there is an equality for some $\overline{x} \in \overline{B}_r(0)$. Then actually $\overline{x} \in B_r(0)$, for otherwise

$$2A = \overline{v}(\overline{x}, -r^2) - \overline{v}(0, -r^2) \le u_\varepsilon(\overline{x}, -r^2) - u_\varepsilon(0, -r^2) \le A,$$

a contradiction. First we estimate

$$\begin{aligned}\beta(x,t)r^{-2}A \fint_{B_\varepsilon(0)} |x+h|^2 dh &\le \beta(x,t)r^{-2}A \fint_{B_\varepsilon(0)} |x|^2 + 2x\cdot h + h^2 dh \\ &\le \beta(x,t)r^{-2}A(|x|^2 + \varepsilon^2).\end{aligned}$$

Supposing that $|x| \ge \varepsilon$ and using the previous estimate together with a simple calculation

$$\sup_{B_\varepsilon(x)} |y|^2 + \inf_{B_\varepsilon(x)} |y|^2 = |x+\varepsilon|^2 + |x-\varepsilon|^2 = 2(|x|^2 + \varepsilon^2),$$

we obtain

$$
\begin{aligned}
&\frac{\alpha(x,t)}{2}\Big(\sup_{y\in B_\varepsilon(x)} \overline{v}(y,t-\frac{\varepsilon^2}{2})+\inf_{y\in B_\varepsilon(x)} \overline{v}(y,t-\frac{\varepsilon^2}{2})\Big)+\beta(x,t)\fint_{B_\varepsilon(x)} \overline{v}(y,t-\frac{\varepsilon^2}{2})dy \\
&\qquad = 2r^{-2}A\alpha(x,t)(|x|^2+\varepsilon^2)+2r^{-2}A\beta(x,t)(|x|^2+\varepsilon^2)+7r^{-2}A(t-\frac{\varepsilon^2}{2})+\overline{c} \\
&\qquad = \overline{c}+2r^{-2}A|x|^2+7r^{-2}At+\left(2r^{-2}A\alpha(x,t)+2r^{-2}A\beta(x,t)-\frac{7r^{-2}A}{2}\right)\varepsilon^2 \\
&\qquad < \overline{v}(x,t).
\end{aligned}
$$

One can easily see that the same inequality holds when $|x|<\varepsilon$.

We want to show that

$$
M := \sup_{Q_r}(u_\varepsilon - \overline{v}) \le 0.
$$

Suppose not, so that $M>0$. By using the DPP for u_ε we get

$$
\begin{aligned}
&u_\varepsilon(x,t)-\overline{v}(x,t) \\
&\le Tu_\varepsilon(x,t)-T\overline{v}(x,t) \\
&\le \alpha(x,t)\sup_{B_\varepsilon(x)}\Big(u_\varepsilon(y,t-\frac{\varepsilon^2}{2})-\overline{v}(y,t-\frac{\varepsilon^2}{2})\Big) \\
&+\beta(x,t)\fint_{B_\varepsilon(x)}\Big(u_\varepsilon(y,t-\frac{\varepsilon^2}{2})-\overline{v}(y,t-\frac{\varepsilon^2}{2})\Big)dy \\
&\le \alpha(x,t)M+\beta(x,t)\fint_{B_\varepsilon(x)}\Big(u_\varepsilon(y,t-\frac{\varepsilon^2}{2})-\overline{v}(y,t-\frac{\varepsilon^2}{2})\Big)dy.
\end{aligned}
$$

Since we can find a sequence $(x_j,t_j)\subset\Omega\times(-T,T)$ such that $(x_j,t_j)\to(x_0,t_0)$ and $(u_\varepsilon-\overline{v})(x_j,t_j)\to M$, by absolute continuity of integral we have

$$
\fint_{B_\varepsilon(x_0)}(u_\varepsilon-\overline{v})(y,t_0)dy=\lim_j\fint_{B_\varepsilon(x_j)}(u_\varepsilon-\overline{v})(y,t_j)dy=M.
$$

Hence the set

$$
G := \{(x,t): u_\varepsilon(x,t)-\overline{v}(x,t)=M\}
$$

is non-empty, and if $(x_0,t_0)\in G$, then $(u_\varepsilon-\overline{v})(y,t_0)=M$ for almost all $y\in B_\varepsilon(x_0)$. This contradicts the assumption that G is bounded. Hence $M\le 0$.

Similarly, we can show that for

$$
\underline{v}(x,t)=\underline{c}-7r^{-2}At-2r^{-2}A|x|^2
$$

we have $\underline{v} \le u$ in the cylinder Q_r. Hence

$$\overline{v}(\overline{x}, -r^2) - \underline{v}(\underline{x}, -r^2) \le \operatorname{osc}_{B_r(0)\times\{-r^2\}} u_\varepsilon,$$

so

$$\overline{c} - \underline{c} \le 3A.$$

Finally, we get

$$\operatorname{osc}_{Q_r} u \le \sup \overline{v} - \inf \underline{v} \le \overline{c} - \underline{c} + 4A \le CA,$$

so the oscillation in the cylinder Q_r is comparable with the oscillation on the bottom of the cylinder. □

Remark 4.3. Another way to prove the previous lemma is to use the same technique that was used in the proof of Theorem 3.3.

Combining the two previous theorems, we get local Hölder continuity for the $p(x, t)$-game.

Theorem 4.4. *Under the conditions of Theorem 4.2, u_ε satisfies the Hölder estimate*

$$|u_\varepsilon(x, t_1) - u_\epsilon(y, t_2)| \le C(n)\frac{|x - y|^\delta + |t_1 - t_2|^{\delta/2}}{R^\delta} + C'(n)\frac{\varepsilon^\delta}{R^\delta}.$$

4.2. Harnack's inequality

In this subsection we assume that $u_\varepsilon > 0$. We are going to prove Harnack's inequality for u_ε, Theorem 4.7, by using a well known iteration technique. Besides Hölder continuity, we need two lemmas to control the iteration process. We assume for function $p : \Omega_T \to (2, \infty)$ that

$$\inf p > 2,$$

which implies that $\inf \alpha > 0$. This requirement is not absolutely necessary, but makes the proof less technical.

Since Hölder continuity for u_ε breaks down at the ε-scale, we need a rough estimate to control the oscillation of the value function at this scale.

Lemma 4.5. *If $\frac{a}{2}\varepsilon^2 > t_2 - t_1 > 0$ for $a \in \mathbb{Z}_+$, and $|x - y| < 2(t_2 - t_1)/\varepsilon$, then*

$$u_\varepsilon(x, t_2) \ge \left(\frac{\inf \alpha}{2}\right)^a u_\varepsilon(y, t_1).$$

Proof. When the game starts from (x, t_2), Player I uses a strategy in which she takes $\frac{|x-y|}{a}$-steps towards y and steps to y if possible. We stop the game when the token hits the time level t_1, and denote the stopping time by τ^*. By simply estimating the probability that the first a moves are tug-of-war won by Player I and using Lemma 2.3, we obtain

$$
\begin{aligned}
u_\varepsilon(x,t_2) &\geq \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,t_2)}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[F(x_{\tau_{t^*}}, t_2 - \frac{\tau^*}{2}\varepsilon^2)] \\
&\geq \left(\frac{\inf \alpha}{2}\right)^a u_\varepsilon(y,t_1). \qquad \square
\end{aligned}
$$

Another lemma needed for Theorem 4.7 gives estimates for the infimum of u_ε. We use a comparison function which is often used in the literature to get Harnack estimates for parabolic equations.

Lemma 4.6. *When $x_0 \in B_{2R}(z) \subset \Omega$ for $R \leq 1$, $r \in [9\varepsilon, R)$ and $t_0 \geq 0$, then*

$$
\inf_{y \in B_r(z)} u_\varepsilon(y,t_0) \leq C(n) r^{-2(n+1)^2} u_\varepsilon(x_0, t_0 + R^2).
$$

Proof. Without a loss of generality, we may assume that $z = 0$ and $t_0 = 0$. Consider a comparison function

$$
\Psi(x,t) = \left(\frac{1}{9}\right)^3 \inf_{y \in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{(t + (\frac{1}{3}r)^2)^{(n+1)^2}} \left(9 - \frac{|x|^2}{t + (\frac{1}{3}r)^2}\right)^2_+
$$

in Ω_T. We have

$$
\max_{x \in B_r(0)} \Psi(x,0) = \frac{1}{9} \inf_{y \in B_r(0)} u_\varepsilon(y,0),
$$

and $\Psi(x,0) = 0$ when $|x - z| \geq r$.

When $x \in B_{2R}(0)$ and $R^2 \leq t \leq 2R^2$, we get

$$
\begin{aligned}
\Psi(x,t) &\geq \left(\frac{1}{9}\right)^3 \inf_{y \in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{(2R^2 + (\frac{1}{3}R)^2)^{(n+1)^2}} \left(9 - \frac{4R^2}{R^2}\right)^2_+ \\
&\geq \left(\frac{1}{9}\right)^3 3^{-3(n+1)^2} r^{2(n+1)^2} \inf_{y \in B_r(0)} u_\varepsilon(y,0).
\end{aligned}
$$

We use a martingale argument to show that

$$
u_\varepsilon(x,t) > \Psi(x,t),
$$

when $x \in \Omega$ and $t > 0$. Let us start the game from $(x_0, \widetilde{t})$, where $\widetilde{t} = R^2$. The fixed strategy S^0_I of Player I is to push towards $0 \in \Omega$ and stay there if possible. We show in Appendix that the function Ψ satisfies the following inequalities:

Case 1) If $x = 0$ and $t \geq \varepsilon^2/2$, then for $e \in \mathbb{R}^n$, $|e| = 1$,

$$
\frac{1}{2}[\Psi(0, t - \frac{\varepsilon^2}{2}) + \Psi(\varepsilon e, t - \frac{\varepsilon^2}{2})] \geq \Psi(0,t).
$$

Case 2) If $0 < |x| < \varepsilon$, then

$$\frac{1}{2}[\Psi(0, t-\frac{\varepsilon^2}{2}) + \Psi(x+\frac{x}{|x|}\varepsilon, t-\frac{\varepsilon^2}{2})] \geq \Psi(x,t).$$

Case 3) If $|x| \geq \varepsilon$, then

$$\frac{1}{2}[\Psi(x+\frac{x}{|x|}\varepsilon, t-\frac{\varepsilon^2}{2}) + \Psi(x-\frac{x}{|x|}\varepsilon, t-\frac{\varepsilon^2}{2})] \geq \Psi(x,t).$$

The previous three inequalities guarantee that Ψ satisfies

$$\Psi(x,t) \leq \frac{1}{2}(\sup_{y\in B_\varepsilon(x)} \Psi(y, t-\frac{\varepsilon^2}{2}) + \inf_{y\in B_\varepsilon(x)} \Psi(y, t-\frac{\varepsilon^2}{2})).$$

In the appendix we also show that Ψ is a subsolution to the scaled heat equation

$$(n+2)u_t(x,t) = \Delta u(x,t).$$

According to [19], this implies

$$\Psi(x,t) \leq \fint_{B_\varepsilon(x)} \Psi\left(y, t-\frac{\varepsilon^2}{2}\right) dy + o(\varepsilon^2),$$

when $x \in \Omega$ and $t > 0$. Denote $t_k := \widetilde{t} - k(\varepsilon^2/2)$. For arbitrary $\eta > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}_{S_I^0, S_{II}}[\Psi(x_{k+1}, t_{k+1})|(x_0, \widetilde{t}), \dots, (x_k, t_k)] \\
&\quad \geq \alpha(x)\Psi(x_k, t_k) + \beta(x) \fint_{B_\varepsilon(x_k)} \Psi(y, t_{k+1})\, dy - \frac{\eta}{2R^2}\varepsilon^2 k \\
&\quad \geq \Psi(x_k, t_k) - \frac{\eta}{2R^2}\varepsilon^2 k,
\end{aligned}$$

when ε is sufficiently small. According to Lemma 2.3, u_ε satisfies

$$\mathbb{E}_{S_I^0, S_{II}}[u_\varepsilon(x_{k+1}, t_{k+1})|(x_0, \widetilde{t}), \dots, (x_k, t_k)] \leq u_\varepsilon(x_k, t_k).$$

Hence $M_k := u_\varepsilon(x_k, t_k) - \Psi(x_k, t_k) - \frac{\eta}{2R^2}\varepsilon^2 k$ is a supermartingale. Let us stop the game when either $\Psi = 0$ or $t_k = 0$. Denote the stopping time by τ^*. We have

$$-\eta \leq \mathbb{E}_{S_I^0, S_{II}}[M_{\tau^*}|(x_0, \widetilde{t}), \dots, (x_{\tau^*-1}, \widetilde{t} - \frac{\tau^*-1}{2}\varepsilon^2] \leq M_0 = u_\varepsilon(x_0, \widetilde{t}) - \Psi(x_0, \widetilde{t}).$$

Since $\eta > 0$ was arbitrary, we obtain

$$u_\varepsilon(x_0, \widetilde{t}) - \Psi(x_0, \widetilde{t}) \geq 0.$$

Hence

$$\inf_{y\in B_r(z)} u_\varepsilon(y,t_0) \le C(n) r^{-2(n+1)^2} u_\varepsilon(x_0, t_0+R^2). \qquad \square$$

Using the Hölder estimate together with Lemmas 4.5 and 4.6, we get Harnack's inequality for u_ε.

Theorem 4.7. *If $B_{10r}(0)\times[t_0-r^2,t_0]\subset\Omega_T$, then for sufficiently small $\varepsilon>0$, u_ε satisfies Harnack's inequality*

$$\sup_{x\in B_r(0)} u_\varepsilon(x,t_0-r^2) \le C(n) \inf_{x\in B_r(0)} u_\varepsilon(x,t_0).$$

Proof. By scaling, we may assume that there is a point $x_1\in B_r(0)$ such that

$$1 = u_\varepsilon(x_1,t_0) < 2 \inf_{x\in B_r(0)} u_\varepsilon(x,t_0).$$

Let $R_k := 2^{1-k} r$ for all natural numbers $k\ge 2$, and pick $x_2, x_3, \ldots \in \Omega$ such that

$$M_1 := u_\varepsilon(x_2,t_0) = \sup_{x\in B_r(x_1)} u_\varepsilon(x,t_0),$$

and for $k\ge 2$

$$M_k := u_\varepsilon(x_{k+1}, t_0-r^2+R_{2k-1}^2) = \sup_{x\in B_{R_k}(x_k)} (x, t_0-r^2+R_{2k-1}^2).$$

Let $\eta = (2^{1+3(n+1)^2}C)^{-1}$, where $C=C(n)$ is a constant from the Hölder and infimum estimates. We are going to show that

$$M_1 < \eta^{-1-3(n+1)^2\delta^{-1}}, \tag{4.6}$$

where δ is a Hölder exponent for u_ε.

On the contrary, suppose that inequality (4.6) does not hold. Let us show by induction that the counter assumption yields

$$M_k \ge (2C\eta)^{-k+1}\eta^{-1-3(n+1)^2\delta^{-1}} = 2C(\eta^{1/\delta}R_{k+1})^{-3(n+1)^2}. \tag{4.7}$$

The case $k=1$ is clear, so assume that the inequality holds for M_{k-1}. Then

$$\begin{aligned}\inf_{B_{\eta^{1/\delta}R_k}(x_k)} u_\varepsilon(x,t_0-r^2+R_{2(k-1)}^2) &\le \frac{M_{k-1}}{2}\\ &= \frac{u_\varepsilon(x_k,t_0-r^2+R_{2(k-1)-1}^2)}{2},\end{aligned} \tag{4.8}$$

where we first used Lemma 4.6 and then the induction assumption.

Hölder estimate gives

$$\begin{aligned}
&\operatorname{osc}(u_\varepsilon, B_{\eta^{1/\delta}R_k}(x_k) \times \{t_0 - r^2 + R^2_{2(k-1)}\}) \\
&\qquad \le C\eta \operatorname{osc}(u_\varepsilon, B_{R_k}(x_k) \times \{t_0 - r^2 + R^2_{2k-1}\}),
\end{aligned}$$

so we get

$$\begin{aligned}
&\operatorname{osc}(u_\varepsilon, B_{R_k}(x_k) \times \{t_0 - r^2 + R^2_{2k-1}\}) \\
&\quad \ge (C\eta)^{-1} \operatorname{osc}(u_\varepsilon, B_{\eta^{1/\delta}R_k}(x_k) \times \{t_0 - r^2 + R^2_{2(k-1)}\}) \\
&\quad \ge (2C\eta)^{-1} M_{k-1} \\
&\quad \ge (2C\eta)^{-k+1} M_1,
\end{aligned}$$

and the induction is complete.

Take k_0 such that $\eta^{1/\delta} R_{k_0} \in (10\varepsilon, 20\varepsilon]$. Then

$$R^2_{2(k_0-1)} \le 100\eta^{-2/\delta}\varepsilon^2 \le (2^{8+3(n+1)^2}C)^{2/\delta}\varepsilon^2,$$

and we obtain

$$\begin{aligned}
\left(\frac{\inf\alpha}{2}\right)^{-2(2^{8+3(n+1)^2}C)^{2/\delta}} &\ge \frac{\sup_{B_{R_{k_0-1}}(x_{k_0-1})} u_\varepsilon(x, t_0 - r^2 + R^2_{k_0-1})}{\inf_{B_{\eta^{1/\delta}R_{k_0}}(x_{k_0})} u_\varepsilon(x, t_0 - r^2 + R^2_{k_0})} \\
&\ge \frac{u_\varepsilon(x_{k_0-1}, t_0 - r^2 + R^2_{k_0-1})}{C(\eta^{1/\delta}R_{k_0})^{-2(n+1)^2}} \\
&= \frac{M_{k_0-2}}{C(\eta^{1/\delta}R_{k_0})^{-2(n+1)^2}} \\
&\ge \frac{(2C\eta)^{3-k_0} M_1}{C(\eta^{1/\delta}2^{1-k_0})^{-2(n+1)^2}} \\
&\ge \widehat{C}(n) 2^{(n+1)^2 k_0},
\end{aligned}$$

which is a contradiction when k_0 is big enough, or in other words, when ε is small enough. Therefore inequality (4.6) holds and the proof is complete. □

5. Uniform convergence to viscosity solution

In Section 6 we will show that if the function p is Lipschitz continuous, there is a unique viscosity solution u to the boundary value problem

$$\begin{cases} (n + p(x,t))u_t = \Delta^N_{p(x,t)} u, & \text{for} \quad (x,t) \in \Omega_T, \\ u = F, & \text{for} \quad (x,t) \in \partial_p \Omega_T, \end{cases}$$

where F is continuous and bounded. Let (u_{ε_j}), $\varepsilon_j \to 0$, be a sequence of value functions of the $p(x,t)$-game with final payoff equal to F on the parabolic boundary strip Γ^ε_T. In this section we show that $u_{\varepsilon_j} \to u$ uniformly on $\overline{\Omega}_T$. The most notable difference is that now we don't have translation invariance at our disposal. Instead, we will make use of local Hölder continuity of functions u_{ε_j}, see Theorem 4.4. We assume during the rest of the paper that Ω satisfies exterior sphere condition.

First we need the following Arzela–Ascoli-type lemma. For the proof in the elliptic context, see [20, Lemma 4.2].

Lemma 5.1. *Let* $\{u_\varepsilon : \overline{\Omega}_T \to \mathbb{R},\ \varepsilon > 0\}$ *be a uniformly bounded set of functions such that given* $\eta > 0$, *there are constants* r_0 *and* ε_0 *such that for every* $\varepsilon < \varepsilon_0$ *and any* $(x,t),(y,s) \in \overline{\Omega}_T$ *with*

$$|x-y| + |t-s| < r_0$$

it holds that

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| < \eta.$$

Then there exists a uniformly continuous function $v : \overline{\Omega}_T \to \mathbb{R}$ *and a subsequence still denoted by* (u_ε) *such that* $u_\varepsilon \to v$ *uniformly in* $\overline{\Omega}_T$ *as* $\varepsilon \to 0$.

The plan is to first show that the sequence (u_{ε_j}) satisfies the conditions of Lemma 5.1, and then show that the uniform limit v is a viscosity solution to

$$(n + p(x,t))v_t = \Delta^N_{p(x,t)} v$$

with boundary data F. By using the uniqueness result of Section 6, we will conclude that $v = u$ on $\overline{\Omega}_T$. Our proofs yield that an arbitrary subsequence of (u_{ε_j}) has a uniformly convergent subsequence. Hence, by uniqueness of u, the sequence (u_{ε_j}) itself converges uniformly to u.

To show that the sequence (u_{ε_j}) satisfies the conditions of Lemma 5.1, we first need the following technical lemma, in which the function $p(x,t)$ does not cause extra difficulties compared to the case where $p > 2$ is a constant. The method for proof has been used before for different games, see [19, Lemma 4.9] and [20, Lemma 4.5].

Lemma 5.2. *For arbitrary* $\eta > 0$, *there are* $r_0 > 0$ *and* $\varepsilon_1 > 0$ *such that when* $(y,t) \in \partial_p\Omega_T$, $(x,s) \in \Omega_T$, $\varepsilon < \varepsilon_1$ *and* $|y-x| + |t-s| < r_0$, *we have*

$$|u_\varepsilon(y,t) - u_\varepsilon(x,s)| < \eta.$$

Proof. If (y,t) is on the bottom of the cylinder Ω_T, the result follows from Theorem 4.2. Assume next that $y \in \partial\Omega$. It is enough to verify the case $t = s =: t_0$, since otherwise triangle inequality gives

$$|u_{\varepsilon_j}(x,t) - u_{\varepsilon_j}(y,s)| \le |u_{\varepsilon_j}(x,t) - u_{\varepsilon_j}(y,t)| + |u_{\varepsilon_j}(y,t) - u_{\varepsilon_j}(y,s)|,$$

and the last term can be estimated by using uniform continuity of the boundary data.

Since Ω satisfies the exterior sphere condition, we have $y \in \partial B_\delta(z)$ for some $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$. Let us start the game from $(x,t) =: (x_0,t_0)$ and fix for Player I a strategy S^0_{I} of pulling towards z. Player II uses a strategy S_{II}. Then,

$$\begin{aligned}
&\mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}^0,S_{\mathrm{II}}}[|x_k-z||x_0,\dots,x_{k-1}] \\
&\le \frac{\alpha(x_{k-1},t_{k-1})}{2}\left(|x_{k-1}-z|+\varepsilon+|x_{k-1}-z|-\varepsilon\right) \\
&+\beta(x_{k-1},t_{k-1}) \fint_{B_\varepsilon(x_{k-1})} |x-z|\,dx \\
&\le |x_{k-1}-z|+C\varepsilon^2,
\end{aligned}$$

where C does not depend on ε. Therefore, $M_k=|x_k-z|-C\varepsilon^2 k$ is a supermartingale. Jensen's inequality gives

$$\mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}^0,S_{\mathrm{II}}}[|x_\tau-z|+|t_\tau-t_0|^{\frac12}]\le |x_0-z|+C\varepsilon\left(\mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}^0,S_{\mathrm{II}}}[\tau]\right)^{\frac12}.$$

Suppose that for the stopping time τ we have the estimate

$$\mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}^0,S_{\mathrm{II}}}[\tau]\le \frac{C(R/\delta)\operatorname{dist}(\partial B_\delta(z),x_0)+o(1)}{\varepsilon^2}, \tag{5.9}$$

where $R>0$ is chosen so that $\Omega\subset B_R(z)$, and $o(1)\to 0$ when $\varepsilon\to 0$. Then we have

$$\mathbb{E}^{(x_0,t_0)}_{S_{\mathrm{I}}^0,S_{\mathrm{II}}}[|x_\tau-z|+|t_\tau-t_0|^{\frac12}]\le |x_0-z|+C(R/\delta)|x_0-y|+o(1),$$

and the proof is complete by uniform continuity of the boundary function F.

It remains to justify estimate (5.9). In Ω, let v be a solution to the problem

$$\begin{cases}\Delta v=-2(n+2) & \text{in } B_{R+\varepsilon}\setminus \overline{B}_r(z),\\ v=0 & \text{on } \partial B_r(z),\\ \frac{\partial v}{\partial \nu}=0 & \text{on } \partial B_{R+\varepsilon}(z),\end{cases}$$

where $\frac{\partial v}{\partial \nu}$ is the normal derivative. The function v satisfies

$$v(x)=\fint_{B_\epsilon(x)} v\,dy+\varepsilon^2, \tag{5.10}$$

and it can be extended as a solution to the same equation in $\overline{B}_{r(z)}\setminus\overline{B}_{r-\varepsilon(z)}$ so that equation (5.10) holds also near the boundary $\partial B_r(z)$.

By concavity of v, it follows from (5.10) that $(v(x_k)+k\varepsilon^2)$ is a supermartingale. Define a new stopping time τ^*,

$$\tau^*=\inf\{k\ :\ x_k\in\overline{B}_\delta(z)\}.$$

Since

$$v(x_0)\le C(R/\delta)\operatorname{dist}(\partial B_\delta(z),x_0),$$

we have

$$\mathbb{E}^{x_0}[\tau^*] \le \frac{v(x_0) - \mathbb{E}[v(x_{\tau^*})]}{\varepsilon^2} \le \frac{C(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2}.$$

Since the function v is concave in $r = |x - z|$ and $\tau \le \tau^*$, we obtain estimate (5.9), and the proof is complete. □

Lemma 5.3. *The sequence (u_ε) of value functions satisfies the conditions of Lemma 5.1.*

Proof. Since $u_\varepsilon \le \max F$, the sequence (u_ε) is uniformly bounded. For asymptotic uniform continuity, fix η. Since u is uniformly continuous in $\Omega_T \times \Gamma_\varepsilon$, there is $r_1 > 0$ such that $(x,t), (y,s) \in \Omega_T \times \Gamma_\varepsilon$,

$$|x - y| + |t - s| < r_1,$$

implies

$$|u(x,t) - u(y,s)| < \eta/2.$$

When $x, y \in \partial B_R(0)$, the same estimate holds between $u_\varepsilon(x)$ and $u_\epsilon(y)$ for all $0 < \varepsilon < R$, since $u_\varepsilon = u$ on Γ_ε.

When $y \in \Gamma_\varepsilon$ and $x \in \Omega_T$, by the previous lemma there are $r_0 > 0$ and $\varepsilon_1 > 0$ such that when $|y - x| + |t - s| < r_0$, we have

$$|u_\varepsilon(y,t) - u_\varepsilon(x,s)| < \eta/2.$$

If $x, y \in \Omega_T$ and $\mathrm{dist}(\{x,y\}, \Gamma_\varepsilon) < r_0/2$, then by using the triangle inequality with a boundary point, we obtain $|u_\varepsilon(y) - u_\varepsilon(x)| < \eta$.

Finally, assume that $\mathrm{dist}(\{x,y\}, \Gamma_\varepsilon) \ge r_0/2$. By local Hölder continuity there is $\varepsilon_2 > 0$ such that when $\varepsilon < \varepsilon_1$, we have

$$|u_\varepsilon(y,t) - u_\varepsilon(x,s)| < \eta.$$

The proof is complete by taking $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. □

We have shown that the sequence (u_ε) converges uniformly towards a uniformly continuous limit function v, and next we show that the function is a viscosity solution to the normalized parabolic $p(x,t)$-equation.

Below we denote by $\lambda_{\max}((p(x,t) - 2)D^2\phi(x,t))$, and $\lambda_{\min}((p(x,t) - 2)D^2\phi(x,t))$ the largest, and the smallest of the eigenvalues to the symmetric matrix $(p(x,t) - 2)D^2\phi(x,t) \in \mathbb{R}^{n\times n}$ for a smooth test function.

Definition 5.4. A function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution to

$$u_t = \Delta u + (p(x,t) - 2)\Delta_\infty^N u,$$

if u is continuous and whenever $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ is such that

i) $u(x_0,t_0)=\phi(x_0,t_0)$,
ii) $\phi(x,t)>u(x,t)$ for $(x,t)\in\Omega_T$, $(x,t)\neq(x_0,t_0)$,

then we have at the point (x_0,t_0)

$$\begin{cases}(n+p(x,t))\phi_t\leq(p(x,t)-2)\Delta_\infty^N\phi+\Delta\phi, & \text{if}\quad \nabla\phi(x_0,t_0)\neq 0,\\ (n+p(x,t))\phi_t\leq\lambda_{\max}((p(x,t)-2)D^2\phi)+\Delta\phi, & \text{if}\quad \nabla\phi(x_0,t_0)=0.\end{cases}$$

Moreover, we require that when touching u with a test function from below all the inequalities are reversed and $\lambda_{\max}((p(x,t)-2)D^2\phi)$ is replaced by $\lambda_{\min}((p(x,t)-2)D^2\phi)$.

Lemma 5.5. *The limit function v is a viscosity solution to*

$$u_t=\Delta u+(p(x,t)-2)\Delta_\infty^N u,$$

with boundary data F.

Proof. We only show that the function v is a viscosity supersolution. (Showing that v is a subsolution is similar.) Choose $(x,t)\in Q_R$ and $\varphi\in C^2$ touching v from below at (x,t). We need to show that

$$\frac{\beta(x,t)}{2(n+2)}\Big((p(x,t)-2)\Delta_\infty^N\varphi(x,t)+\Delta\varphi(x,t)-(n+p(x,t))\varphi_t(x,t)\Big)\leq 0. \tag{5.11}$$

As a direct consequence of [19, Theorem 2.4], we have

$$\begin{aligned}&\frac{\alpha(x,t)}{2}\left\{\sup_{B_\varepsilon(x)}\varphi(y,t-\frac{\varepsilon^2}{2})+\inf_{B_\varepsilon(x)}\varphi(y,t-\frac{\varepsilon^2}{2})\right\}\\ &\quad+\beta(x,t)\fint_{B_\varepsilon(x)}\varphi(y,t-\frac{\varepsilon^2}{2})dy-\varphi(x,t)\\ &\geq\frac{\beta(x,t)\varepsilon^2}{2(n+2)}\Bigg((p(x,t)-2)\left\langle D^2\varphi(x,t)\left(\frac{\overline{x}^\varepsilon-x}{|\overline{x}^\varepsilon-x|}\right),\left(\frac{\overline{x}^\varepsilon-x}{|\overline{x}^\varepsilon-x|}\right)\right\rangle\\ &\qquad+\Delta\varphi(x,t)-(n+p(x,t))\varphi_t(x,t)\Bigg)+o(\varepsilon^2),\end{aligned}$$

where $\overline{x}^\varepsilon\in B_\varepsilon(x)$ is nearly to the direction of $\nabla\varphi(x)$.

By the uniform convergence, there is a sequence $(x_\varepsilon,t_\varepsilon)\to(x,t)$ such that when (y,s) is near $(x_\varepsilon,t_\varepsilon)$, we have

$$u_\varepsilon(y,s)-\varphi(y,s)\geq u_\varepsilon(x_\varepsilon,t_\varepsilon)-\varphi(x_\varepsilon,t_\varepsilon)-\eta_\varepsilon.$$

Setting $\widetilde{\varphi}=\varphi+u_\varepsilon(x_\varepsilon,t_\varepsilon)-\varphi(x_\varepsilon,t_\varepsilon)$ we have

$$u_\varepsilon(x_\varepsilon,t_\varepsilon)=\widetilde{\varphi}(x_\varepsilon,t_\varepsilon),\;\; u_\varepsilon(y,s)\geq\widetilde{\varphi}(y,s)-\eta_\varepsilon.$$

We get

$$\eta_\varepsilon \geq \frac{\alpha(x,t_\varepsilon)}{2}\left\{\sup_{B_\varepsilon(x)} \widetilde{\varphi}(y,t_\varepsilon-\frac{\varepsilon^2}{2}) + \inf_{B_\varepsilon(x)} \widetilde{\varphi}(y,t_\varepsilon-\frac{\varepsilon^2}{2})\right\}$$
$$+\beta(x,t_\varepsilon)\fint_{B_\varepsilon(x)} \widetilde{\varphi}(y,t_\varepsilon-\frac{\varepsilon^2}{2})dy - \widetilde{\varphi}(x_\varepsilon,t_\varepsilon)$$

Let us first assume that $\nabla\varphi(x,t)\neq 0$. Then, since we can choose $\eta_\varepsilon = o(\varepsilon^2)$, we obtain

$$0 \geq \frac{\beta(x_\varepsilon,t_\varepsilon)\varepsilon^2}{2(n+2)}\Bigg((p(x_\varepsilon,t_\varepsilon)-2)\left\langle D^2\varphi(x_\varepsilon,t_\varepsilon)\left(\frac{\overline{x}^\varepsilon - x_\varepsilon}{|\overline{x}^\varepsilon - x_\varepsilon|}\right),\left(\frac{\overline{x}^\varepsilon - x_\varepsilon}{|\overline{x}^\varepsilon - x_\varepsilon|}\right)\right\rangle$$
$$+\Delta\varphi(x_\varepsilon,t_\varepsilon) - (n+p(x_\varepsilon,t_\varepsilon))\varphi_t(x_\varepsilon,t_\varepsilon)\Bigg) + o(\varepsilon^2).$$

When $\varepsilon\to 0$, it follows that

$$\frac{\beta(x,t)}{2(n+2)}\Big((p(x,t)-2)\Delta_\infty^N\varphi(x) + \Delta\varphi(x) - (n+p(x,t))\varphi_t(x,t)\Big) \leq 0.$$

When $\nabla\varphi(x,t)=0$, also $D^2\varphi(x,t)=0$, and it is easy to verify the required inequality $\varphi_t(x,t)\geq 0$. □

6. Uniqueness for $p(x,t)$-equation

In this section we assume that the function p is Lipschitz continuous in Ω_T with Lipschitz constant C_1. We prove that there is a unique viscosity solution to

$$(n+p(x,t))u_t = \Delta^N_{p(x,t)}u \tag{6.12}$$

with classical Dirichlet boundary conditions. Existence is well known, and in fact the previous section provided a game-theoretic proof.

The technique for uniqueness is well known; $p(x,t)$ causes slight modifications. For the convenience of the reader, we give the details. For additional literature, see [15,10,2,3].

The parabolic equation (6.12) is discontinuous when the gradient vanishes. We recall the definition of viscosity solution based on semicontinuous extensions of the operator, and refer the reader to Chen–Giga–Goto [5], Evans–Spruck [8], and Giga's monograph [9].

The next lemma allows us reduce the test functions in the case $\nabla\phi(x_0,t_0)=0$ and only test by those having $D^2\phi(x_0,t_0)=0$.

Lemma 6.1. *A function $u:\Omega_T\to\mathbb{R}$ is a viscosity solution to (6.12) if u is continuous and whenever $(x_0,t_0)\in\Omega_T$ and $\phi\in C^2(\Omega_T)$ is such that*

i) $u(x_0,t_0)=\phi(x_0,t_0)$,
ii) $\phi(x,t)>u(x,t)$ *for* $(x,t)\in\Omega_T$, $(x,t)\neq(x_0,t_0)$,

then at the point (x_0, t_0) *we have*

$$\begin{cases} (n+p(x,t))\phi_t \le (p(x,t)-2)\Delta_\infty^N \phi + \Delta\phi, & \text{if } \nabla\phi(x_0,t_0) \neq 0, \\ \phi_t(x_0,t_0) \le 0, & \text{if } \nabla\phi(x_0,t_0)=0, \text{ and } D^2\phi(x_0,t_0)=0. \end{cases}$$

We also require that when testing from below all the inequalities are reversed.

Proof. The proof is by contradiction: We assume that u satisfies the conditions in the statement but still fails to be a viscosity solution in the sense of Definition 5.4. If this is the case, we must have $\phi \in C^2(\Omega_T)$, $(x_0, t_0) \in \Omega_T$ and $\eta > 0$ such that

i) $u(x_0,t_0) = \phi(x_0,t_0)$,
ii) $\phi(x,t) > u(x,t)$ for $(x,t) \in \Omega_T$, $(x,t) \neq (x_0,t_0)$,

for which $\nabla\phi(x_0,t_0) = 0$, $D^2\phi(x_0,t_0) \neq 0$ and

$$\begin{aligned} &(n + p(x_0,t_0))\phi_t(x_0,t_0) - \eta \\ &\quad > \lambda_{\min}((p(x_0,t_0)-2)D^2\phi(x_0,t_0)) + \Delta\phi(x_0,t_0), \end{aligned} \tag{6.13}$$

or the analogous inequality when testing from below (in this case the argument is symmetric and we omit it). Let

$$w_j(x,t,y,s) = u(x,t) - \phi(y,s) - \left(\frac{j^2}{4}|x-y|^4 + \frac{j}{2}|t-s|^2\right) \tag{6.14}$$

and denote by (x_j, t_j, y_j, s_j) the maximum point of w_j in $\overline{\Omega}_T \times \overline{\Omega}_T$. Since (x_0, t_0) is a local maximum for $u - \phi$, we may assume that

$$(x_j, t_j, y_j, s_j) \to (x_0, t_0, x_0, t_0) \quad \text{as} \quad j \to \infty,$$

and $(x_j, t_j), (y_j, s_j) \in \Omega_T$ for all large j, similarly to [11]. Since (x_0, t_0) is a local maximum of $u - \phi$, it follows from (6.14) that

$$\frac{j^2}{4}\left|x_j - y_j\right|^4 \to 0 \text{ and } \frac{j}{2}\left|t_j - s_j\right|^2 \to 0,$$

when $j \to \infty$. If not, there would be $\alpha > 0$ and subsequences $(x_j), \dots (s_j)$ such that

$$\frac{j^2}{4}|x-y|^4 + \frac{j}{2}|t-s|^2 > \alpha.$$

Let U_α be a neighborhood of (x_0, t_0) where oscillation of $(u - \phi)$ is less than α. Since the subsequences converge to (x_0, t_0), we get a contradiction.

We consider two cases: either $x_j = y_j$ infinitely often or $x_j \neq y_j$ for all j large enough. First, let $x_j = y_j$, and denote

$$\varphi(y,s)=\frac{j^2}{4}\left|x_j-y\right|^4+\frac{j}{2}(t_j-s)^2.$$

Then

$$\phi(y,s)+\varphi(y,s)$$

has a local minimum at (y_j,s_j). Since the function p is continuous, by (6.13) we have

$$(n+p(y_j,s_j))\phi_t(y_j,s_j)-\eta>\lambda_{\min}((p(y_j,s_j)-2)D^2\phi(y_j,s_j))+\Delta\phi(y_j,s_j)$$

for j large enough. As $\phi_t(y_j,s_j)=\varphi_t(y_j,s_j)$ and $-D^2\phi(y_j,s_j)\le D^2\varphi(y_j,s_j)$, we have by the previous inequality

$$\begin{aligned}\eta&<(n+p(y_j,s_j))\varphi_t(y_j,s_j)+\lambda_{\max}((p(y_j,s_j)-2)D^2\varphi(y_j,s_j))+\Delta\varphi(y_j,s_j)\\&=(n+p(x_j,s_j))j(t_j-s_j),\end{aligned}\tag{6.15}$$

where we also used the fact that $y_j=x_j$ and thus $D^2\varphi(y_j,s_j)=0$.

Next denote

$$\psi(x,t)=\frac{j^2}{4}\left|x-y_j\right|^4+\frac{j}{2}(t-s_j)^2.$$

Similarly,

$$u(x,t)-\psi(x,t)$$

has a local maximum at (x_j,t_j), and thus since $D^2\psi(x_j,t_j)=0$, our assumptions imply

$$0\ge(n+p(x_j,t_j))\psi_t(x_j,t_j)=(n+p(x_j,t_j))j(t_j-s_j),\tag{6.16}$$

for j large enough. This contradicts (6.15), because both t_j and s_j converge to t_0 and the function p is continuous.

Next we consider the case $y_j\neq x_j$. For the following notation, we refer to [6] and [12,13]. We also use the parabolic theorem of sums for w_j which implies that there exists symmetric matrices X_j,Y_j such that

$$\begin{aligned}&\left(j(t_j-s_j),\ j^2\left|x_j-y_j\right|^2(x_j-y_j),\ X_j\right)\in\overline{\mathcal{P}}^{2,+}u(x_j,t_j),\\&\left(j(t_j-s_j),\ j^2\left|x_j-y_j\right|^2(x_j-y_j),\ Y_j\right)\in\overline{\mathcal{P}}^{2,-}\phi(y_j,s_j),\end{aligned}$$

and

$$\begin{pmatrix}X_j&0\\0&-Y_j\end{pmatrix}\le D^2\Psi_j(x_j,y_j)+\frac{1}{j}[D^2\Psi_j(x_j,y_j)]^2$$

with $\Psi_j(x_j, y_j) = \frac{j^2}{4}\left|x_j - y_j\right|^4$. Here

$$D^2\Psi_j(x_j, y_j) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix},$$

where $M = j^2\left|x_j - y_j\right|^2\left(2\frac{x_j-y_j}{|x_j-y_j|} \otimes \frac{x_j-y_j}{|x_j-y_j|} + I\right)$, and

$$[D^2\Psi_j(x_j, y_j)]^2 = 2\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}.$$

Let $\xi := \frac{x_j-y_j}{|x_j-y_j|}$ and use $(\sqrt{p(x_j, t_j)}\xi, \sqrt{p(y_j, s_j)}\xi)$. The above implies

$$\begin{aligned} & p(x_j, t_j)\xi' X_j \cdot \xi - p(y_j, s_j)\xi' Y_j \cdot \xi \\ & \quad \le C\left(p(x_j, t_j) - p(y_j, s_j)\right)^2\left(\xi' M\xi + \frac{2}{j}\xi' M^2\xi\right), \end{aligned}$$

where we used a simple estimate

$$\left(\sqrt{p(x_j, t_j)} - \sqrt{p(y_j, s_j)}\right)^2 \le (p(x_j, t_j) - p(y_j, s_j))^2,$$

which holds since the function p is greater than 2.

We obtain

$$\begin{aligned} \eta &< (n + p(x_j, t_j))j(t_j - s_j) - (n + p(y_j, s_j))j(t_j - s_j) \\ &\quad - (p(x_j, t_j) - 2)\langle X_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle - \operatorname{tr}(X_j) \\ &\quad + (p(y_j, s_j) - 2)\langle Y_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle + \operatorname{tr}(Y_j) \\ &\le (n + p(x_j, t_j))j(t_j - s_j) - (n + p(y_j, s_j))j(t_j - s_j) \\ &\quad - p(x_j, t_j)\langle X_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle + p(y_j, s_j)\langle Y_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle. \end{aligned}$$

Since the function p is Lipschitz continuous, we have

$$\begin{aligned}
&|-(n+p(x_j,t_j))j(t_j-s_j)+(n+p(y_j,s_j))j(t_j-s_j)| \\
&\quad = |j(t_j-s_j)(p(y_j,s_j)-p(x_j,t_j))| \\
&\quad < C_1 j|t_j-s_j|(|x_j-y_j|^2+|t_j-s_j|^2)^{\frac{1}{2}} \\
&\quad \le C_1 j|t_j-s_j|\sqrt{2}(|x_j-y_j|+|t_j-s_j|) \\
&\quad = \sqrt{2}C_1\left((j|t_j-s_j|^2)^{\frac{1}{2}}(j^2|x_j-y_j|^4)^{\frac{1}{4}}+j|t_j-s_j|^2\right) \\
&\quad < \frac{\eta}{2}
\end{aligned}$$

when j is large enough. By theorem of sums, we get

$$\begin{aligned}
\frac{\eta}{2} &< -p(x_j,t_j)\langle X_j\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle + p(y_j,s_j)\langle Y_j\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle \\
&\le C\left(p(x_j,t_j)-p(y_j,s_j)\right)^2\left(\xi' M\xi+\frac{2}{j}\xi' M^2\xi\right) \\
&\le C\left(|x_j-y_j|^2+|t_j-s_j|^2\right)(j^2|x_j-y_j|^2+j^3|x_j-y_j|^4) \\
&< C\left(j^2|x_j-y_j|^4+(j^2|x_j-y_j|^4)^{3/2}\right)
\end{aligned}$$

when j is large. This is a contradiction, since $j^2|x_j-y_j|^4\to 0$ when $j\to\infty$. In the last two estimates we used Lipschitz continuity of p. □

By modifying the above proof we also get the uniqueness. For viscosity solutions we assume continuity on $\overline{\Omega}_T$.

Lemma 6.2. *Viscosity solutions to (6.12) are unique.*

Proof. The proof is by contradiction: We assume that u and v are viscosity solutions with the same boundary values and yet

$$u(x_0,t_0)-v(x_0,t_0)=\sup(u-v)>0.$$

Further, by considering

$$u-\frac{\eta}{T-t},$$

we may assume that

$$(n+p(x,t))u_t \le \Delta^N_{p(x,t)}u-\frac{\eta}{T}$$

in the viscosity sense when testing from above.

Let

$$w_j(x,t,y,s) = u(x,t) - v(y,s) - \Big(\frac{j^2}{4}|x-y|^4 + \frac{j}{2}|t-s|^2\Big)$$

and denote by (x_j, t_j, y_j, s_j) the maximum point of w_j in $\overline{\Omega}_T \times \overline{\Omega}_T$. Since (x_0, t_0) is a local maximum for $u - v$, we may assume that

$$(x_j, t_j, y_j, s_j) \to (x_0, t_0, x_0, t_0), \quad \text{as} \quad j \to \infty$$

and $(x_j, t_j), (y_j, s_j) \in \Omega_T$.

We consider two cases: either $x_j = y_j$ infinitely often or $x_j \neq y_j$ for all j large enough. First, denote

$$\varphi(x,t,y,s) = \frac{j^2}{4}|x-y|^4 + \frac{j}{2}(t-s)^2$$

and let $x_j = y_j$. Then $(y,s) \mapsto v(y,s) + \varphi(x_j, t_j, y, s)$, has a local minimum at (y_j, s_j), and $(x,t) \mapsto u(x,t) - \varphi(x,t,y_j,s_j)$ a local maximum at (x_j, t_j). From this we deduce (denote with abuse of notation $\varphi(y,s) = \varphi(x_j, t_j, y, s)$ in the next display)

$$(n + p(y_j, s_j))j(t_j - s_j) = (n + p(y_j, s_j))\varphi_s(y_j, s_j) \geq 0$$

and (denote with abuse of notation $\varphi(x,t) = \varphi(x,t,y_j,s_s)$ in the next display)

$$(n + p(x_j, t_j))(t_j - s_j) = (n + p(x_j, t_j))\varphi_t(x_j, t_j) \leq -\eta/T.$$

Thus

$$\frac{\eta}{T} \leq (n + p(x_j, t_j))(t_j - s_j) - (n + p(y_j, s_j))(t_j - s_j) = 0,$$

a contradiction.

Next we consider the case $y_j \neq x_j$. For the following notation, we refer to [6] and [12]. We also use the parabolic theorem of sums for w_j which implies that there exist symmetric matrices X_j, Y_j such that $Y_j - X_j$ is positive semidefinite and

$$\Big(j(t_j - s_j),\ j^2|x_j - y_j|^2(x_j - y_j),\ X_j\Big) \in \overline{\mathcal{P}}^{2,+}u(y_j, s_j)$$
$$\Big(j(t_j - s_j),\ j^2|x_j - y_j|^2(x_j - y_j),\ Y_j\Big) \in \overline{\mathcal{P}}^{2,-}v(x_j, t_j).$$

Using (6.13) and the assumptions on u, we get

$$\begin{aligned}\frac{\eta}{T} \leq &-(n + p(y_j, s_j))j(t_j - s_j) + (n + p(x_j, t_j))j(t_j - s_j)\\ &+ (p(x_j, t_j) - 2)\langle Y_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle + \operatorname{tr}(Y_j)\\ &- (p(y_j, s_j) - 2)\langle X_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\rangle - \operatorname{tr}(X_j).\end{aligned}$$

The right hand side can be estimated similarly as in the previous lemma to obtain a contradiction. □

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Appendix A

Let us show Cases 1, 2, and 3 from the proof of Lemma 4.6. Recall that the comparison function in that lemma was

$$\Psi(x,t) = \left(\frac{1}{9}\right)^3 \inf_{y\in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{(t+(\frac{1}{3}r)^2)^{(n+1)^2}} \left(9 - \frac{|x|^2}{t+(\frac{1}{3}r)^2}\right)_+^2$$

Starting from Case 1, we need to show that for $e \in \mathbb{R}^n$, $|e| = 1$,

$$\Psi(0,t) \le \frac{1}{2}[\Psi(0,t-\frac{\varepsilon^2}{2}) + \Psi(\varepsilon e, t-\frac{\varepsilon^2}{2})].$$

Since

$$\Psi(0,t) = \frac{1}{9} \inf_{y\in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t+(\frac{1}{3}r)^2]^{(n+1)^2}},$$

$$\Psi(0,t-\frac{\varepsilon^2}{2}) = \frac{1}{9} \inf_{y\in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^{(n+1)^2}},$$

and

$$\Psi(\varepsilon e, t-\frac{\varepsilon^2}{2}) = \left(\frac{1}{9}\right)^3 \inf_{y\in B_r(0)} u_\varepsilon(y,0) \frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^{(n+1)^2}} \left[9 - \frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right]^2,$$

we have to show that

$$1 \le \frac{1}{2}\left[\frac{t+(\frac{1}{3}r)^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right]^{(n+1)^2} \left[1 + \left(\frac{1}{9}\right)^2 \left(9 - \frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right] =: A_1.$$

Since

$$1 + \left(\frac{1}{9}\right)^2 \left(9 - \frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2 \ge 2 - \frac{2}{9}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right),$$

we get

$$
\begin{aligned}
A_1 &\geq \left[\frac{t+(\frac{1}{3}r)^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right]\left[1-\frac{1}{9}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\right] \\
&= \left[1+\frac{1}{2}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\right]\left[1-\frac{1}{9}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\right] \\
&\geq 1-\frac{1}{9}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)+\frac{1}{2}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)-\frac{1}{18}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right) \\
&\geq 1,
\end{aligned}
$$

and Case 1 is complete.

In Case 2, $|x| = \eta\varepsilon$ for some $0 < \eta < 1$, and we need to show that

$$
\Psi(x,t) \leq \frac{1}{2}[\Psi(0, t-\frac{\varepsilon^2}{2}) + \Psi(x+\frac{x}{|x|}\varepsilon, t-\frac{\varepsilon^2}{2})].
$$

Since

$$
\begin{aligned}
\Psi(x,t) &= \left(\frac{1}{9}\right)^3 \inf_{y\in B_r(0)} u_\varepsilon(y,0)\frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t+(\frac{1}{3}r)^2]^{(n+1)^2}}\left[9-\frac{|\eta\varepsilon|^2}{t+(\frac{1}{3}r)^2}\right]^2, \\
\Psi(0, t-\frac{\varepsilon^2}{2}) &= \frac{1}{9}\inf_{y\in B_r(0)} u_\varepsilon(y,0)\frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^{(n+1)^2}},
\end{aligned}
$$

and

$$
\Psi(x+\frac{x}{|x|}\varepsilon, t-\frac{\varepsilon^2}{2}) = \left(\frac{1}{9}\right)^3 \inf_{y\in B_r(0)} u_\varepsilon(y,0)\frac{(\frac{1}{3}r)^{2(n+1)^2}}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^{(n+1)^2}}\left[9-\frac{|(1+\eta)\varepsilon|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right]^2,
$$

it is sufficient to show that

$$
\left[9-\frac{|\eta\varepsilon|^2}{t+(\frac{r}{3})^2}\right]^2 \leq \frac{1}{2}\left(\frac{t+(\frac{1}{3}r)^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\left[9^2+\left(9-\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right].
$$

Notice that in Cases 1 and 2 we don't need to take the cut-off into account, since $\Psi(x,t) > 0$ when $|x| \leq 2\varepsilon$.

Recalling that $r \geq 9\varepsilon$, we have

$$
9^2+\left(9-\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2 \geq 144,
$$

from which it follows that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{t+(\frac{1}{3}r)^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\left[9^2+\left(9-\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right]\\
&=\frac{1}{2}\left(1+\frac{1}{2}\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\left[9^2+\left(9-\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right]\\
&\geq\frac{1}{2}\left[9^2+\left(9-\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right]+36\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}.
\end{aligned}$$

Hence, it is enough to show that

$$\begin{aligned}
&\left[9-\frac{|k\varepsilon|^2}{t+(\frac{r}{3})^2}\right]^2-\frac{1}{2}\left[9^2+\left(9-\frac{(1+k)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\right]\\
&\leq 36\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}.
\end{aligned}\tag{A.17}$$

The left hand side can be written as

$$-18\frac{|k\varepsilon|^2}{t+(\frac{r}{3})^2}+\left(\frac{|k\varepsilon|^2}{t+(\frac{r}{3})^2}\right)^2+9\left(\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)-\frac{1}{2}\left(\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2.$$

Since

$$9\left(\frac{(1+\eta)^2\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\leq 36\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}$$

and

$$\left(\frac{|k\varepsilon|^2}{t+(\frac{r}{3})^2}\right)^2\leq\frac{|k\varepsilon|^2}{t+(\frac{r}{3})^2},$$

inequality (A.17) holds, and Case 2 is proved.

In Case 3, we need to show that if $|x|\geq\varepsilon$, then

$$\frac{1}{2}[\Psi(x+\frac{x}{|x|}\varepsilon,t-\frac{\varepsilon^2}{2})+\Psi(x-\frac{x}{|x|}\varepsilon,t-\frac{\varepsilon^2}{2})]\geq\Psi(x,t).$$

Suppose first that

$$\frac{(|x|+\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}<9.$$

Then also

$$\frac{|x|^2}{t+(\frac{r}{3})^2} < 9 \text{ and } \frac{(|x|-\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2} < 9.$$

Since $\left(\frac{1}{9}\right)^3 \inf_{y\in B_r(0)} u_\varepsilon(y,0) \left(\frac{r}{3}\right)^{2(n+1)^2}$ cancels out, it is enough to show that

$$\left(\frac{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}{t+(\frac{r}{3})^2}\right)\left(9-\frac{|x|^2}{t+(\frac{r}{3})^2}\right)^2$$
$$\leq \frac{1}{2}\left[\left(9-\frac{(|x|+\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2+\left(9-\frac{(|x|-\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2\right],$$

or equivalently,

$$\left[9-\frac{|x|^2}{t+(\frac{r}{3})^2}\right]^2$$
$$\leq \frac{1}{2}\left(1+\frac{1}{2}\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)\left[\left(9-\frac{(|x|+\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2+\left(9-\frac{(|x|-\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2\right].$$

This is equivalent to showing that

$$18\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right) \leq \frac{1}{2}\left(\frac{(|x|+\varepsilon)^4+(|x|-\varepsilon)^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)-\frac{|x|^4}{[t+(\frac{1}{3}r)^2]^2}$$
$$+\frac{1}{4}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)I,$$

where

$$I=\left(9-\frac{(|x|+\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2+\left(9-\frac{(|x|-\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)^2.$$

Since

$$\frac{|x|^4}{[t+(\frac{1}{3}r)^2]^2} = \left(1-\frac{1}{2}\frac{\varepsilon^2}{t+(\frac{r}{3})^2}\right)^2\left(\frac{|x|^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)$$
$$\leq \left(1-\frac{1}{2}\frac{\varepsilon^2}{t+(\frac{r}{3})^2}\right)\left(\frac{|x|^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)$$

and

$$(|x|+\varepsilon)^4+(|x|-\varepsilon)^4 \leq 2|x|^4+12|x|^2\varepsilon^2,$$

we get an estimate

$$\begin{aligned}
&\frac{1}{2}\left(\frac{(|x|+\varepsilon)^4+(|x|-\varepsilon)^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)-\frac{|x|^4}{[t+(\frac{1}{3}r)^2]^2}\\
&\geq \frac{1}{2}\left(\frac{(|x|+\varepsilon)^4+(|x|-\varepsilon)^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)-\left(1-\frac{1}{2}\frac{\varepsilon^2}{t+(\frac{r}{3})^2}\right)\left(\frac{|x|^4}{[t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2]^2}\right)\\
&\geq 6\left(\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)+\frac{1}{2}\left(\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right).
\end{aligned}$$

Hence, it is sufficient to show that

$$\begin{aligned}
&18\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\\
&\leq 6\left(\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)+\frac{1}{2}\left(\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)^2\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)\\
&+\frac{1}{4}\left(\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\right)I.
\end{aligned}$$

If

$$\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\geq\frac{5}{2},$$

the previous inequality clearly holds. If

$$\frac{|x|^2}{t-\frac{\varepsilon^2}{2}+(\frac{1}{3}r)^2}\leq\frac{5}{2},$$

then $I\geq 72$ and the previous inequality holds again.

When

$$\frac{(|x|+\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\geq 9,$$

we need to show that

$$\left(9-\frac{|x|^2}{t+(\frac{r}{3})^2}\right)_+^2\leq\frac{1}{2}\left(1+\frac{1}{2}\frac{\varepsilon^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)\left(9-\frac{(|x|-\varepsilon)^2}{t-\frac{\varepsilon^2}{2}+(\frac{r}{3})^2}\right)_+^2,$$

and this follows by the previous estimates of Case 3.

Let us next show that Ψ is a viscosity subsolution to

$$(n+2)u_t(x,t) = \Delta u(x,t).$$

Denote

$$|z|^2 = \frac{|x|^2}{t+r^2}.$$

Then

$$\begin{aligned}
&(n+2)\Psi_t(x,t) - \Delta\Psi(x,t) \\
&= \frac{(\frac{r}{3})^{2(n+1)^2}}{(t+(\frac{r}{3})^2)^{(n+1)^2+1}}(9-|z|^2)_+ \\
&\quad \left[-(n+2)(n+1)^2(9-|z|^2)_+ + 2(n+2)|z|^2 + 4n - \frac{8|z|^2}{(9-|z|^2)_+}\right] \\
&=: \frac{(\frac{r}{3})^{2(n+1)^2}}{(t+(\frac{r}{3})^2)^{(n+1)^2+1}}(9-|z|^2)_+ A.
\end{aligned}$$

When $a := 9-|z|^2 > 0$, we get

$$\begin{aligned}
\frac{1}{a}A &= -(n+2)(n+1)^2a^2 + 2(n+2)(9-a)a + 4na - 8(9-a) \\
&= \left[-(n+2)(n+1)^2 - 2(n+2)\right]a^2 + 22(n+2)a - 72 < 0
\end{aligned}$$

when $a=8$, and the discriminant is

$$D = 22^2(n+2)^2 - 4\times 72(n+2)[(n+1)^2+2] < 0,$$

since $(n+1)^2+2 > 2(n+2)$. Hence $A<0$ when $0<a\le 9$.

That Ψ is a subsolution in Ω_T follows from the fact that the maximum of two subsolutions is a subsolution.

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$C^{1,\alpha}$ regularity for the normalized p-Poisson problem

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$C^{1,\alpha}$ regularity for the normalized p-Poisson problem

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Abstract

We consider the normalized p-Poisson problem

$$-\Delta_p^N u = f \qquad \text{in} \quad \Omega \subset \mathbb{R}^n.$$

The normalized p-Laplacian $\Delta_p^N u := |Du|^{2-p}\Delta_p u$ is in non-divergence form and arises for example from stochastic games. We prove $C^{1,\alpha}_{\text{loc}}$ regularity with nearly optimal α for viscosity solutions of this problem. In the case $f \in L^\infty \cap C$ and $p > 1$ we use methods both from viscosity and weak theory, whereas in the case $f \in L^q \cap C$, $q > \max(n, \frac{p}{2}, 2)$, and $p > 2$ we rely on the tools of nonlinear potential theory.

Résumé

On considère l'équation de Poisson pour le p-Laplacien normalisé

$$-\Delta_p^N u = f \qquad \text{dans} \quad \Omega \subset \mathbb{R}^n.$$

Le p-Laplacien normalisé est un opérateur sous forme non-divergence et il apparaît dans l'étude de certains jeux aléatoires. On démontre un résultat de régularité $C^{1,\alpha}_{\text{loc}}$ pour des solutions de viscosité de ce problème avec un exposent α quasi optimal. Dans le cas d'une function $f \in L^\infty \cap C$ et pour $p > 1$, on combine des arguments utilisés dans la théorie des solutions de viscosité avec des arguments provenant de la théorie des solutions distributionnelles. Dans le cas d'une fonction $f \in L^q \cap C$ où $q > \max(n, \frac{p}{2}, 2)$ et $p > 2$, on se base sur des outils de la théorie du potentiel non-linéaire.

Keywords: Normalized p-Laplacian, p-Poisson problem, viscosity solutions, local $C^{1,\alpha}$ regularity.
2010 MSC: 35J60, 35B65, 35J92

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1. Introduction

In this paper we study local regularity properties of the inhomogeneous normalized p-Laplace equation

$$-\Delta_p^N u = f \qquad \text{in} \quad \Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The normalized p-Laplacian is defined as

$$\Delta_p^N u := |Du|^{2-p}\Delta_p u = \Delta u + (p-2)\Delta_\infty^N u,$$

where $\Delta_\infty^N u := \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$ denotes the normalized infinity Laplacian. The motivation to study these types of normalized operators stems partially from their connections to stochastic games and their applications to image processing. Equation (1.1) is different from the standard p-Laplace equation due to the right hand side f. Indeed, it is in non-divergence form. The normalized p-Laplacian is gradient dependent and discontinuous, so we cannot directly rely on the existing general $C^{1,\alpha}$ regularity theory of viscosity solutions. Only Hölder continuity for solutions of (1.1) follows from the regularity theory for uniformly elliptic equations, see [7, 8].

Our aim is to show local Hölder continuity for gradients of viscosity solutions of (1.1) by relying on different methods depending on regularity assumptions of the source term f. Assuming first that $f \in L^\infty(\Omega) \cap C(\Omega)$, we show that solutions of (1.1) for $p > 1$ are of class $C^{1,\alpha}_{\text{loc}}$ for some $\alpha > 0$ depending on p and the dimension n.

Theorem 1.1. *Assume that $p > 1$ and $f \in L^\infty(\Omega) \cap C(\Omega)$. There exists $\alpha = \alpha(p, n) > 0$ such that any viscosity solution u of (1.1) is in $C^{1,\alpha}_{\text{loc}}(\Omega)$, and for any $\Omega' \subset\subset \Omega$,*

$$[u]_{C^{1,\alpha}(\Omega')} \le C = C\left(p, n, d, d', ||u||_{L^\infty(\Omega)}, ||f||_{L^\infty(\Omega)}\right),$$

where $d = \operatorname{diam}(\Omega)$ and $d' = \operatorname{dist}(\Omega', \partial\Omega)$.

By translation and scaling, we may prove the result at the origin and assume that $u(0) = 0$ and $\operatorname{osc}_{B_1} u \le 1$. It is sufficient to show that for some $\rho \in (0,1)$ and for all $k \in \mathbb{N}$, there exists $q_k \in \mathbb{R}^n$ for which

$$\underset{x \in B_{r_k}}{\operatorname{osc}} (u(x) - q_k \cdot x) \le r_k^{1+\alpha},$$

where $r_k := \rho^k$. Heuristically, we want to show that if a solution u can be approximated by a plane $q_k \cdot x$ in a small ball B_{r_k}, then in a smaller ball $B_{r_{k+1}}$ there is a slightly different plane $q_{k+1} \cdot x$ giving a better approximation. To get a $C^{1,\alpha}$ estimate, we have to show that the approximation improves by a sufficiently small multiplicative factor. An inductive argument leads us to analyze regularity of deviations of solutions from planes, $w(x) = u(x) - q \cdot x$ for different $q \in \mathbb{R}^n$. The required oscillation estimate for these deviations is called improvement of flatness:

$$\underset{x \in B_\rho}{\operatorname{osc}} (w(x) - q' \cdot x) \le \frac{1}{2}\rho$$

for some $q' \in \mathbb{R}^n$, under the assumption that the oscillation of f is sufficiently small. This is shown in Lemma 3.3 by using a compactness and contradiction argument. Recently, Imbert and Silvestre [18] used this method to show $C^{1,\alpha}$ regularity for viscosity solutions of $|Du|^\gamma F(D^2 u) = f$, where F is uniformly elliptic. In our case, the most technical part of the proof of Lemma 3.3 is to show a uniform $C^{1,\alpha}$ estimate for functions w under the assumption $f \equiv 0$. This is done in the proof of Lemma 3.2 by using the Ishii-Lions method.

Earlier, in the restricted case $p \ge 2$, a C^2 domain Ω and $f \in C(\overline{\Omega})$, Birindelli and Demengel [5, Proposition 3.5] proved global Hölder continuity for the gradient of viscosity solutions of (1.1) in connection to eigenvalue problems related to the p-Laplacian. In the case $p \geq 2$ we provide an alternative proof by showing first that viscosity solutions of (1.1) are weak solutions of

$$-\Delta_p u = |Du|^{p-2} f \qquad \text{in} \quad \Omega, \tag{1.2}$$

and then relying on the known regularity results for quasilinear PDEs to see that weak solutions of (1.2) are locally of class $C^{1,\alpha}$.

Restricting to the case $p > 2$, we can relax the estimate of Theorem 1.1 by providing a control on the Hölder estimate of the gradient that depends on a weaker norm of f.

Theorem 1.2. *Assume that $p > 2$, $q > \max(2, n, p/2)$, $f \in C(\Omega) \cap L^q(\Omega)$. Then any viscosity solution u of (1.1) is in $C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha = \alpha(p, q, n)$. Moreover, for any $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, with Ω' smooth enough, we have*

$$[u]_{C^{1,\alpha}(\Omega'')} \leq C = C\left(p, q, n, d, d'', ||u||_{L^\infty(\Omega)}, ||f||_{L^q(\Omega)}\right),$$

where $d = \text{diam}\,(\Omega)$ and $d'' = \text{dist}\,(\Omega'', \partial\Omega')$.

In the proof we first consider weak solutions u_ε of certain regularized equations, and by using the De Giorgi iteration and the potential estimates of Duzaar and Mingione [15] we obtain local uniform estimates for $||Du_\varepsilon||_{L^\infty(\Omega)}$. From the classical result of Lieberman we get a uniform estimate for $[Du_\varepsilon]_{C^\beta(\Omega)}$ for some $\beta > 0$, and Theorem 1.2 follows from a compactness argument. In the proof we also show that under the assumptions of Theorem 1.2, there exists a weak solution of equation (1.2) which is in $C^{1,\alpha}_{\text{loc}}(\Omega)$.

It is well known that p-harmonic functions are of class $C^{1,\alpha_0}_{\text{loc}}$ for some maximal exponent $0 < \alpha_0 < 1$ that depends only upon n and p. This was shown independently by Uraltseva [43] and Uhlenbeck [42] in the case $p > 2$, and later extended to the case $p > 1$, see [13, 29] and also [34, 20] for related research. The question of optimal regularity for p-Laplace equations in divergence form has attracted a lot of attention recently, see Section 5 for further references. Since the solutions of (1.1) should not be expected to be more regular than p-harmonic functions, the maximal exponent α_0 is a natural upper bound for $C^{1,\alpha}$ regularity for equation (1.1). In the following theorem we obtain nearly optimal regularity for solutions of (1.1).

Theorem 1.3. *Fix an arbitrary $\xi \in (0, \alpha_0)$, where α_0 is the optimal Hölder exponent for gradients of p-harmonic functions in terms of an a priori estimate.*

If $p > 1$ and $f \in L^\infty(\Omega) \cap C(\Omega)$, then viscosity solutions to (1.1) are in $C^{1,\alpha_0-\xi}_{\text{loc}}(\Omega)$.

If $p > 2$, $q > \max(2, n, p/2)$ and $f \in C(\Omega) \cap L^q(\Omega)$, then viscosity solutions to (1.1) are in $C^{1,\alpha_\xi}_{\text{loc}}(\Omega)$, where $\alpha_\xi := \min(\alpha_0 - \xi, 1 - n/q)$. Moreover the estimates given in the previous theorems hold for α_ξ.

When the gradient is sufficiently large, the result follows from the classical regularity results for uniformly elliptic equations. When the gradient is small, the first step is to use local $C^{1,\alpha}$ regularity of the solutions of (1.1), proved in Theorems 1.1 and 1.2, to show that the solutions can be approximated by p-harmonic functions in $C^{1,\alpha}$. The next step is to use suitable rescaled functions and iteration to obtain the required oscillation estimate.

Over the last decade, equation (1.1) and similar normalized equations have received growing attention, partly due to the stochastic zero-sum *tug-of-war* games defined by Peres, Schramm, Sheffield and Wilson in [37, 38]. In [37] Peres and Sheffield studied a connection between equation

(1.1) and the game tug-of-war with noise and running pay-off. The game-theoretic interpretation led to new regularity proofs in the case $f = 0$ in [32], and later in the case of bounded and positive f in [39], see also [9] for a PDE approach. Regularity studies were extended to the parabolic version $u_t = \Delta_p^N u$ in [35, 4, 21] and led to applications in image processing, see e.g. [14, 16].

This paper is organized as follows. In Section 2 we fix the notation and gather some definitions and tools which we need later. In Section 3 we give two proofs for Theorem 1.1, in Section 4 we prove Theorem 1.2, and in Section 5 Theorem 1.3.

2. Preliminaries

Throughout the paper $\Omega \subset \mathbb{R}^n$ is a bounded domain. We use the notation

$$\fint_A u \, \mathrm{d}x := \frac{1}{|A|} \int_A u \, \mathrm{d}x$$

for the mean value of a function u in a measurable set $A \subset \Omega$ with Lebesgue measure $|A| > 0$. The oscillation of a function u in a set A is denoted by

$$\operatorname*{osc}_A u := \sup_A u - \inf_A u.$$

For $p > 1$, we denote by Λ and λ the ellipticity constants of the normalized p-Laplacian Δ_p^N. Recalling the expression

$$\Delta_p^N u = \Delta u + (p-2)\Delta_\infty^N u = \operatorname{tr}\left((I + (p-2)\frac{Du \otimes Du}{|Du|^2})D^2 u \right)$$

and calculating for arbitrary $\eta \in \mathbb{R}^n$, $|\eta| = 1$,

$$\begin{aligned} \langle (I + (p-2)\frac{Du \otimes Du}{|Du|^2})\eta, \eta \rangle &= |\eta|^2 + (p-2)\frac{\langle \eta, Du \rangle^2}{|Du|^2} \\ &= 1 + (p-2)\frac{\langle \eta, Du \rangle^2}{|Du|^2}, \end{aligned}$$

we see that $\Lambda = \max(p-1, 1)$ and $\lambda = \min(p-1, 1)$.

We denote by S^n the set of symmetric $n \times n$ matrices. For $a, b \in \mathbb{R}^n$, we denote by $a \otimes b$ the $n \times n$-matrix for which $(a \otimes b)_{ij} = a_i b_j$.

We will use the *Pucci operators*

$$P^+(X) := \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} -\operatorname{tr}(AX)$$

and

$$P^-(X) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} -\operatorname{tr}(AX),$$

where $\mathcal{A}_{\lambda,\Lambda} \subset S^n$ is a set of symmetric $n \times n$ matrices whose eigenvalues belong to $[\lambda, \Lambda]$.

When studying Hölder and $C^{1,\alpha}$ regularity, for $\alpha \in (0, 1]$ and a ball $B_r \subset \mathbb{R}^n$ we use the notation

$$[u]_{C^{0,\alpha}(B_r)} := \sup_{x,y \in B_r, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

for Hölder continuous functions, and

$$[u]_{C^{1,\alpha}(B_r)} := [u]_{C^1(B_r)} + \sup_{x,y\in B_r, x\neq y} \frac{|Du(x)-Du(y)|}{|x-y|^\alpha}$$

for functions of class $C^{1,\alpha}$. Here $[u]_{C^1(B_r)} := \sup_{x\in B_r} |Du(x)|$.

Recall that weak solutions to $-\Delta_p u := -\operatorname{div}(|Du|^{p-2}\, Du) = 0$ are called p-harmonic functions. We will use the known $C^{1,\alpha_0}_{\text{loc}}$ a priori estimate in Sections 4 and 5. The existence of the optimal $\alpha_0 = \alpha_0(p,n)$ follows from the known regularity estimates for the homogeneous p-Laplace equation.

The normalized p-Laplacian is undefined when $Du = 0$, where it has a bounded discontinuity. This can be remediated adapting the notion of viscosity solution using the upper and lower semicontinuous envelopes (relaxations) of the operator, see [10].

Definition 2.1. *Let Ω be a bounded domain and $1 < p < \infty$. An upper semicontinuous function u is a viscosity subsolution of* (1.1) *if for all $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local maximum at x_0, one has*

$$\begin{cases} -\Delta_p^N \phi(x_0) \le f(x_0), & \text{if } \ D\phi(x_0) \neq 0, \\ -\Delta\phi(x_0) - (p-2)\lambda_{max}(D^2\phi(x_0)) \le f(x_0), & \text{if } D\phi(x_0) = 0 \text{ and } p \ge 2, \\ -\Delta\phi(x_0) - (p-2)\lambda_{min}(D^2\phi(x_0)) \le f(x_0), & \text{if } D\phi(x_0) = 0 \text{ and } 1 < p < 2. \end{cases}$$

A lower semicontinuous function u is a viscosity supersolution of (1.1) *if for all $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local minimum at x_0, one has*

$$\begin{cases} -\Delta_p^N \phi(x_0) \ge f(x_0), & \text{if } \ D\phi(x_0) \neq 0, \\ -\Delta\phi(x_0) - (p-2)\lambda_{min}(D^2\phi(x_0)) \ge f(x_0), & \text{if } D\phi(x_0) = 0 \text{ and } p \ge 2, \\ -\Delta\phi(x_0) - (p-2)\lambda_{max}(D^2\phi(x_0)) \ge f(x_0), & \text{if } D\phi(x_0) = 0 \text{ and} 1 < p < 2. \end{cases}$$

We say that u is a viscosity solution of (1.1) *in Ω if it is both a viscosity sub- and supersolution.*

We will make use of the equivalence between weak and viscosity solutions to the p-Laplace equation $\Delta_p u = 0$. This was first proved in [23] by using the full uniqueness machinery of the theory of viscosity solutions, and later in [22] without relying on the uniqueness. The techniques of the second paper are particularly important for us in Section 3.2, where we do not have uniqueness.

3. Two proofs for Theorem 1.1

In this section we give two proofs for Theorem 1.1. In the first subsection we use an iteration method often used to show $C^{1,\alpha}$ regularity for elliptic equations.

In Section 3.2 we give another proof for Theorem 1.1 in the case $p \ge 2$ by showing that a viscosity solution to (1.1) is also a weak solution to (1.2).

3.1. First proof by improvement of flatness and iteration

In this subsection we give a first proof for Theorem 1.1. We assume that $p > 1$ and $f \in L^\infty(\Omega) \cap C(\Omega)$, and we want to show that there exists $\alpha = \alpha(p,n) > 0$ such that any viscosity solution u of (1.1) is in $C^{1,\alpha}_{\text{loc}}(\Omega)$, and for any $\Omega' \subset\subset \Omega$,

$$[u]_{C^{1,\alpha}(\Omega')} \le C = C\left(p, n, d, d', ||u||_{L^\infty(\Omega)}, ||f||_{L^\infty(\Omega)}\right),$$

where $d = \text{diam}(\Omega)$ and $d' = \text{dist}(\Omega', \partial\Omega)$.

Since Hölder continuous functions can be characterized by the rate of their approximations by polynomials (see [25]), it is sufficient to prove that there exists some constant C such that for all $x \in \Omega$ and $r \in (0,1)$, there exists $q = q(r,x) \in \mathbb{R}^n$ for which

$$\operatorname*{osc}_{y \in B_r(x)} (u(y) - u(x) - q \cdot (x-y)) \leq Cr^{1+\alpha}.$$

If one also starts with a solution u such that $\operatorname{osc} u \leq 1$, then it is sufficient to choose a suitable $\rho \in (0,1)$ such that the previous inequality holds true for $r = r_k = \rho^k$, $q = q_k$ and $C = 1$ by proceeding by induction on $k \in \mathbb{N}$. The balls $B_r(x)$ for $x \in \Omega$ and $r < \text{dist}\,(x, \partial\Omega)$ covering the domain Ω, we may work on balls. By translation, it is enough to show that the solution is $C^{1,\alpha}$ at 0, and by considering

$$u_r(y) = r^{-2}u(x+ry),$$

we may work on the unit ball $B_1(0)$. Finally, considering $u - u(0)$ if necessary, we may suppose that $u(0) = 0$. We also reduce the problem by rescaling. Let $\kappa = (2||u||_{L^\infty(B_1)} + \varepsilon_0^{-1}||f||_{L^\infty(B_1)})^{-1}$. Setting $\tilde{u} = \kappa u$, then $\tilde{u}$ satisfies

$$-\Delta_p^N(\tilde{u}) = \tilde{f}$$

with $||\tilde{u}||_{L^\infty(B_1)} \leq \frac{1}{2}$ and $||\tilde{f}||_{L^\infty(B_1)} \leq \varepsilon_0$. Hence, without loss of generality we may assume in Theorem 1.1 that $||u||_{L^\infty(B_1)} \leq 1/2$ and $||f||_{L^\infty(B_1)} \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(p,n)$ is chosen later.

The idea of the proof is to first study the deviations of u from planes $w(x) = u(x) - q \cdot x$ which satisfy

$$-\Delta w - (p-2)\left\langle D^2 w \frac{Dw+q}{|Dw+q|}, \frac{Dw+q}{|Dw+q|} \right\rangle = f \quad \text{in} \quad B_1 \tag{3.1}$$

in the viscosity sense, and show equicontinuity for uniformly bounded solutions in Lemma 3.1. By the Arzelà-Ascoli theorem we get compactness, which, together with Lemma 3.2, we use to show improvement of flatness for solutions of (3.1) in Lemma 3.3. Finally, we prove $C^{1,\alpha}$ regularity for solutions of (1.1) in Lemma 3.4 by using Lemma 3.3 and iteration.

In order to prove Theorem 1.1, we will first need the following equicontinuity lemma for viscosity solutions to equation (3.1).

Lemma 3.1. *For all $r \in (0,1)$, there exist a constant $\beta = \beta(p,n) \in (0,1)$ and a positive constant $C = C(p,n,r,\operatorname*{osc}_{B_1}(w), ||f||_{L^n(B_1)})$ such that any viscosity solution w of (3.1) satisfies*

$$[w]_{C^{0,\beta}(B_r)} \leq C. \tag{3.2}$$

Proof. Equation (3.1) can be rewritten as

$$-\operatorname{tr}\left(\left(I + (p-2)\frac{Dw+q}{|Dw+q|} \otimes \frac{Dw+q}{|Dw+q|}\right) D^2 w\right) = f.$$

Recalling the definitions of the Pucci operators P^+ and P^- respectively, we have

$$\begin{cases} P^+(D^2w) + |f| \geq 0 \\ P^-(D^2w) - |f| \leq 0. \end{cases}$$

By the classical result of Caffarelli in [7, Proposition 4.10], there exists $\beta = \beta(p,n) \in (0,1)$ such that

$$[w]_{C^{0,\beta}(B_r)} \leq C = C\left(p,n,r,\operatorname*{osc}_{B_1}(w), ||f||_{L^n(B_1)}\right). \qquad \square$$

The next lemma is needed to prove the key Lemma 3.3, where we show improvement of flatness. For convenience, we postpone the technical proof of Lemma 3.2 and present it at the end of this section.

Lemma 3.2. *Assume that $f \equiv 0$ and let w be a viscosity solution to equation* (3.1) *with* $\operatorname*{osc}_{B_1} w \leq 1$. *For all $r \in (0, \frac{1}{2}]$, there exist constants $C_0 = C_0(p, n) > 0$ and $\beta_1 = \beta_1(p, n) > 0$ such that*

$$[w]_{C^{1,\beta_1}(B_r)} \leq C_0. \tag{3.3}$$

We are now in a position to show an improvement of flatness for solutions to equation (3.1) by using the previous lemmas together with known regularity results for elliptic PDEs. Intuitively, we show that graphs of the solutions get more flat when we look at them in smaller balls.

Lemma 3.3. *There exist $\varepsilon_0 \in (0, 1)$ and $\rho = \rho(p, n) \in (0, 1)$ such that, for any viscosity solution w of* (3.1), $\operatorname{osc}_{B_1}(w) \leq 1$ *and* $||f||_{L^\infty(B_1)} \leq \varepsilon_0$, *there exists $q' \in \mathbb{R}^n$ such that*

$$\operatorname*{osc}_{x \in B_\rho} (w(x) - q' \cdot x) \leq \frac{1}{2}\rho.$$

Proof. Thriving for a contradiction, assume that there exist a sequence of functions (f_j) with $||f_j||_{L^\infty(B_1)} \to 0$, a sequence of vectors (q_j) and a sequence of viscosity solutions (w_j) with $\operatorname{osc}_{B_1}(w_j) \leq 1$ to

$$-\Delta w_j - (p-2)\left\langle D^2 w_j \frac{Dw_j + q_j}{|Dw_j + q_j|}, \frac{Dw_j + q_j}{|Dw_j + q_j|}\right\rangle = f_j, \tag{3.4}$$

such that, for all $q' \in \mathbb{R}^n$ and any $\rho \in (0, 1)$

$$\operatorname*{osc}_{x \in B_\rho} (w_j(x) - q' \cdot x) > \frac{\rho}{2}. \tag{3.5}$$

Using the compactness result of Lemma 3.1, there exists a continuous function w_∞ such that $w_j \to w_\infty$ uniformly in B_ρ for any $\rho \in (0, 1)$. Passing to the limit in (3.5), we have that for any vector q',

$$\operatorname*{osc}_{x \in B_\rho} (w_\infty(x) - q' \cdot x) > \frac{\rho}{2}. \tag{3.6}$$

Suppose first that the sequence (q_j) is bounded. Using the result of Appendix A, we extract a subsequence (w_j) converging to a limit w_∞, which satisfies

$$-\operatorname{tr}\left(\left(I + (p-2)\frac{Dw_\infty + q_\infty}{|Dw_\infty + q_\infty|} \otimes \frac{Dw_\infty + q_\infty}{|Dw_\infty + q_\infty|}\right) D^2 w_\infty\right) = 0 \quad \text{in } B_1$$

in a viscosity sense. (Here $q_j \to q_\infty$ up to the same subsequence.) By the regularity result of Lemma 3.2, there exist $\beta_1 = \beta_1(p, n) > 0$ and $C_0 = C_0(p, n) > 0$ such that $||w_\infty||_{C^{1,\beta_1}(B_{1/2})} \leq C_0$.

If the sequence (q_j) is unbounded, we extract a converging subsequence from $e_j = \frac{q_j}{|q_j|}$, $e_j \to e_\infty$, and obtain (see Appendix A)

$$-\Delta w_\infty - (p-2)\left\langle D^2 w_\infty \, e_\infty, e_\infty\right\rangle = 0 \qquad \text{in} \quad B_1, \tag{3.7}$$

with $|e_\infty| = 1$. Noticing that equation (3.7) can be written as

$$-\operatorname{tr}\left((I + (p-2)e_\infty \otimes e_\infty) D^2 w_\infty\right) = 0,$$

we see that equation (3.7) is uniformly elliptic and depends only on $D^2 w_\infty$. By the regularity result of [8, Corollary 5.7], there is $\beta_2 = \beta_2(p,n) > 0$ so that $w_\infty \in C^{1,\beta_2}_{\text{loc}}$ and there exists $C_0 = C_0(p,n) > 0$ such that $||w_\infty||_{C^{1,\beta_1}(B_{1/2})} \le C_0$.

We have shown that $w_\infty \in C^{1,\beta}_{\text{loc}}$ for $\beta = \min(\beta_1, \beta_2) > 0$. Choose $\rho \in (0, 1/2)$ such that

$$C_0 \rho^\beta \le \frac{1}{4}. \tag{3.8}$$

By $C^{1,\beta}_{\text{loc}}$ regularity, there exists a vector k_ρ such that

$$\operatorname*{osc}_{x \in B_\rho} (w_\infty(x) - k_\rho \cdot x) \le C_0 \rho^{1+\beta} \le \frac{1}{4}\rho. \tag{3.9}$$

This contradicts (3.6) so the proof is complete. □

Proceeding by iteration, we obtain the following lemma.

Lemma 3.4. *Let ρ and $\varepsilon_0 \in (0,1)$ be as in Lemma 3.3 and let u be a viscosity solution of* (1.1) *with* $\operatorname{osc}_{B_1}(u) \le 1$ *and* $||f||_{L^\infty(B_1)} \le \varepsilon_0$. *Then, there exists $\alpha \in (0,1)$ such that for all $k \in \mathbb{N}$, there exists $q_k \in \mathbb{R}^n$ such that*

$$\operatorname*{osc}_{y \in B_{r_k}} (u(y) - q_k \cdot y) \le r_k^{1+\alpha}, \tag{3.10}$$

where $r_k := \rho^k$.

Proof. For $k = 0$, the estimate (3.10) follows from the assumption $\operatorname{osc}_{B_1}(u) \le 1$. Next we take $\alpha \in (0,1)$ such that $\rho^\alpha > 1/2$. We assume for $k \ge 0$ that we already constructed $q_k \in \mathbb{R}^n$ such that (3.10) holds true. To prove the inductive step $k \to k+1$, we rescale the solution considering for $x \in B_1$

$$w_k(x) = r_k^{-1-\alpha}\big(u(r_k x) - q_k \cdot (r_k x)\big).$$

By induction assumption, we have $\operatorname*{osc}_{B_1}(w_k) \le 1$, and w_k satisfies

$$-\Delta w_k - (p-2)\left\langle D^2 w_k \frac{Dw_k + (q_k/r_k^\alpha)}{|Dw_k + (q_k/r_k^\alpha)|}, \frac{Dw_k + (q_k/r_k^\alpha)}{|Dw_k + (q_k/r_k^\alpha)|} \right\rangle = f_k,$$

where $f_k(x) = r_k^{1-\alpha} f(r_k x)$ with $||f_k||_{L^\infty(B_1)} \le \varepsilon_0$ since $\alpha < 1$. Using the result of Lemma 3.3, there exists $l_{k+1} \in \mathbb{R}^n$ such that

$$\operatorname*{osc}_{x \in B_\rho} (w_k(x) - l_{k+1} \cdot x) \le \frac{1}{2}\rho.$$

Setting $q_{k+1} = q_k + l_{k+1} r_k^\alpha$, we get

$$\operatorname*{osc}_{B_{r_{k+1}}} (u(x) - q_{k+1} \cdot x) \le \frac{\rho}{2} r_k^{1+\alpha} \le r_{k+1}^{1+\alpha}.$$

□

Since the estimate (3.10) holds for every k, the proof of Theorem 1.1 is complete.

The rest of the section is devoted to the proof of Lemma 3.2. First we need the following technical lemma concerning Lipschitz regularity of solutions of equation (3.1) in the case $f \equiv 0$. For $n \times n$ matrices we use the matrix norm

$$||A|| := \sup_{|x| \le 1} \{|Ax|\}.$$

Lemma 3.5. *Assume that $f \equiv 0$ and let w be a viscosity solution to equation* (3.1) *with* $\operatorname{osc}_{B_1} w \leq 1$. *For all $r \in (0, \frac{3}{4})$, there exists a constant $Q = Q(p,n) > 0$ such that, if $|q| > Q$, then for all $x, y \in B_r$,*

$$|w(x) - w(y)| \leq \tilde{C}\, |x - y|\,, \tag{3.11}$$

where $\tilde{C} = \tilde{C}(p,n) > 0$.

Proof. We use the viscosity method introduced by Ishii and Lions in [19].

Step 1

It suffices to show that w is Lipschitz in $B_{3/4}$, because this will imply that w is Lipschitz in any smaller ball B_ρ for $\rho \in \left(0, \frac{3}{4}\right)$ with the same Lipschitz constant. Take $r = \frac{4}{5}$. First we fix $x_0, y_0 \in B_{\frac{15r}{16}}$, where now $\frac{15r}{16} = \frac{3}{4}$, and introduce the auxiliary function

$$\Phi(x,y) := w(x) - w(y) - L\phi(|x-y|) - \frac{M}{2}\,|x - x_0|^2 - \frac{M}{2}\,|y - y_0|^2\,,$$

where ϕ is defined below. Our aim is to show that $\Phi(x,y) \leq 0$ for $(x,y) \in B_r \times B_r$. For a proper choice of ϕ, this yields the desired regularity result. We take

$$\phi(t) = \begin{cases} t - t^\gamma \phi_0 & 0 \leq t \leq t_1 := (\frac{1}{\gamma\phi_0})^{1/(\gamma-1)} \\ \phi(t_1) & \text{otherwise,} \end{cases}$$

where $2 > \gamma > 1$ and $\phi_0 > 0$ is such that $t_1 \geq 2$ and $\gamma\phi_0 2^{\gamma-1} \leq 1/4$.

Then

$$\phi'(t) = \begin{cases} 1 - \gamma t^{\gamma-1}\phi_0 & 0 \leq t \leq t_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi''(t) = \begin{cases} -\gamma(\gamma-1)t^{\gamma-2}\phi_0 & 0 < t \leq t_1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\phi'(t) \in [\frac{3}{4}, 1]$ and $\phi''(t) < 0$ when $t \in [0,2]$.

Step 2

We argue by contradiction and assume that Φ has a positive maximum at some point $(x_1, y_1) \in \bar{B}_r \times \bar{B}_r$. Since w is continuous and its oscillation is bounded by 1, we get

$$\begin{aligned} M\,|x_1 - x_0|^2 &\leq 2\operatorname{osc}_{B_1} w \leq 2, \\ M\,|y_1 - y_0|^2 &\leq 2\operatorname{osc}_{B_1} w \leq 2. \end{aligned} \tag{3.12}$$

Notice that $x_1 \neq y_1$, otherwise the maximum of Φ would be non positive. Choosing $M \geq \left(\frac{32}{r}\right)^2$, we have that $|x_1 - x_0| < r/16$ and $|y_1 - y_0| < r/16$ so that x_1 and y_1 are in B_r.

We know that w is locally Hölder continuous and that there exists a constant $C_\beta > 0$ depending only on p, n, r such that

$$|w(x) - w(y)| \leq C_\beta |x-y|^\beta \quad \text{for}\, x, y \in B_r.$$

Using that w is Hölder continuous, it follows, adjusting the constants (by choosing $2M \leq C_\beta$), that

$$\begin{aligned} M\,|x_1 - x_0| &\leq C_\beta\,|x_1 - y_1|^{\beta/2}\,, \\ M\,|y_1 - y_0| &\leq C_\beta\,|x_1 - y_1|^{\beta/2}\,. \end{aligned} \tag{3.13}$$

By Jensen-Ishii's lemma (also known as theorem of sums, see [10, Theorem 3.2]), there exist

$$\begin{aligned} (\tilde{\zeta}_x, X) &\in \overline{\mathcal{J}}^{2,+}\left(w(x_1) - \frac{M}{2}\,|x_1 - x_0|^2\right), \\ (\tilde{\zeta}_y, Y) &\in \overline{\mathcal{J}}^{2,-}\left(w(y_1) + \frac{M}{2}\,|y_1 - y_0|^2\right), \end{aligned}$$

that is

$$\begin{aligned} (a, X + MI) &\in \overline{\mathcal{J}}^{2,+} w(x_1), \\ (b, Y - MI) &\in \overline{\mathcal{J}}^{2,-} w(y_1), \end{aligned}$$

where ($\tilde{\zeta}_x = \tilde{\zeta}_y$)

$$\begin{aligned} a &= L\phi'(|x_1 - y_1|)\frac{x_1 - y_1}{|x_1 - y_1|} + M(x_1 - x_0) = \tilde{\zeta}_x + M(x_1 - x_0), \\ b &= L\phi'(|x_1 - y_1|)\frac{x_1 - y_1}{|x_1 - y_1|} - M(y_1 - y_0) = \tilde{\zeta}_y - M(y_1 - y_0). \end{aligned}$$

If L is large enough (depending on the Hölder constant C_β), we have

$$|a|\,, |b| \geq L\phi'(|x_1 - y_1|) - C_\beta\,|x_1 - y_1|^{\beta/2} \geq \frac{L}{2}.$$

Moreover, by Jensen-Ishii's lemma, for any $\tau > 0$, we can take $X, Y \in \mathcal{S}^n$ such that

$$-\left[\tau + 2\,||B||\right]\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \tag{3.14}$$

and

$$\begin{aligned} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} &\leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \frac{2}{\tau}\begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix} \\ &= D^2\phi(x_1, y_1) + \frac{1}{\tau}\left(D^2\phi(x_1, y_1)\right)^2, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} B =& L\phi''(|x_1 - y_1|)\frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|} \\ &+ \frac{L\phi'(|x_1 - y_1|)}{|x_1 - y_1|}\left(I - \frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|}\right) \end{aligned}$$

and

$$B^2 = \frac{L^2(\phi'(|x_1-y_1|))^2}{|x_1-y_1|^2}\left(I - \frac{x_1-y_1}{|x_1-y_1|}\otimes\frac{x_1-y_1}{|x_1-y_1|}\right) + L^2(\phi''(|x_1-y_1|))^2\frac{x_1-y_1}{|x_1-y_1|}\otimes\frac{x_1-y_1}{|x_1-y_1|}.$$

Notice that $\phi''(t) + \dfrac{\phi'(t)}{t} \geq 0$, $\phi''(t) \leq 0$ for $t \in (0,2)$ and hence

$$||B|| \leq L\phi'(|x_1-y_1|), \tag{3.16}$$

$$\left|\left|B^2\right|\right| \leq L^2\left(|\phi''(|x_1-y_1|)| + \frac{\phi'(|x_1-y_1|)}{|x_1-y_1|}\right)^2. \tag{3.17}$$

Moreover, for $\xi = \frac{x_1-y_1}{|x_1-y_1|}$, we have

$$\langle B\xi,\xi\rangle = L\phi''(|x_1-y_1|) < 0, \qquad \langle B^2\xi,\xi\rangle = L^2(\phi''(|x_1-y_1|))^2.$$

Choosing $\tau = 4L\left(|\phi''(|x_1-y_1|)| + \dfrac{\phi'(|x_1-y_1|)}{|x_1-y_1|}\right)$, we have that for $\xi = \frac{x_1-y_1}{|x_1-y_1|}$,

$$\langle B\xi,\xi\rangle + \frac{2}{\tau}\langle B^2\xi,\xi\rangle = L\left(\phi''(|x_1-y_1|) + \frac{2}{\tau}L(\phi''(|x_1-y_1|))^2\right) \leq \frac{L}{2}\phi''(|x_1-y_1|) < 0. \tag{3.18}$$

In particular applying inequalities (3.14) and (3.15) to any vector (ξ,ξ) with $|\xi| = 1$, we have that $X - Y \leq 0$ and $||X||, ||Y|| \leq 2||B|| + \tau$. We refer the reader to [19, 10] for details.
Thus, setting $\eta_1 = a + q$, $\eta_2 = b + q$, we have for $|q|$ large enough (depending only on L)

$$|\eta_1| \geq |q| - |a| \geq \frac{|a|}{2} \geq \frac{L}{4},$$
$$|\eta_2| \geq |q| - |b| \geq \frac{|b|}{2} \geq \frac{L}{4}, \tag{3.19}$$

where L will be chosen later on and L will depend only on p, n, C_β. We write the viscosity inequalities

$$0 \leq \operatorname{tr}(X + MI) + (p-2)\frac{\langle (X+MI)(a+q),(a+q)\rangle}{|a+q|^2}$$
$$0 \geq \operatorname{tr}(Y - MI) + (p-2)\frac{\langle (Y-MI)(b+q),(b+q)\rangle}{|b+q|^2}.$$

In other words

$$0 \leq \operatorname{tr}(A(\eta_1)(X+MI))$$
$$0 \leq -\operatorname{tr}(A(\eta_2)(Y-MI))$$

where for $\eta \neq 0$ $\bar\eta = \dfrac{\eta}{|\eta|}$ and

$$A(\eta) := I + (p-2)\overline{\eta}\otimes\overline{\eta}.$$

Adding the two inequalities, we get

$$0 \le \mathrm{tr}(A(\eta_1)(X+MI)) - \mathrm{tr}(A(\eta_2)(Y-MI)).$$

It follows that

$$\begin{aligned} 0 \le & \underbrace{\mathrm{tr}(A(\eta_1)(X-Y))}_{(1)} + \underbrace{tr((A(\eta_1)-A(\eta_2))Y)}_{(2)} \\ & + \underbrace{M\big[\,\mathrm{tr}(A(\eta_1)) + \mathrm{tr}(A(\eta_2))\big]}_{(3)}. \end{aligned} \tag{3.20}$$

Estimate of (1). Notice that all the eigenvalues of $X-Y$ are non positive. Moreover, applying the previous matrix inequality (3.15) to the vector $(\xi, -\xi)$ where $\xi := \frac{x_1-y_1}{|x_1-y_1|}$ and using (3.18), we obtain

$$\begin{aligned} \langle (X-Y)\xi, \xi\rangle &\le 4\left(\langle B\xi,\xi\rangle + \frac{2}{\tau}\langle B^2\xi,\xi\rangle\right) \\ &\le 2L\phi''(|x_1-y_1|) < 0. \end{aligned} \tag{3.21}$$

Hence at least one of the eigenvalue of $X-Y$ that we denote by λ_{i_0} is negative and smaller than $2L\phi''(|x_1-y_1|)$. The eigenvalues of $A(\eta_1)$ belong to $[\min(1,p-1), \max(1,p-1)]$. Using (3.21), it follows by [40] that

$$\begin{aligned} \mathrm{tr}(A(\eta_1)(X-Y)) &\le \sum_i \lambda_i(A(\eta_1))\lambda_i(X-Y) \\ &\le \min(1,p-1)\lambda_{i_0}(X-Y) \\ &\le 2\min(1,p-1)L\phi''(|x_1-y_1|). \end{aligned}$$

Estimate of (2). We have

$$\begin{aligned} A(\eta_1) - A(\eta_2) &= (\overline{\eta}_1 \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes \overline{\eta}_2)(p-2) \\ &= [(\overline{\eta}_1 - \overline{\eta}_2 + \overline{\eta}_2) \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes (\overline{\eta}_2 - \overline{\eta}_1 + \overline{\eta}_1)](p-2) \\ &= [(\overline{\eta}_1 - \overline{\eta}_2) \otimes \overline{\eta}_1 + \overline{\eta}_2 \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes (\overline{\eta}_2 - \overline{\eta}_1) - \overline{\eta}_2 \otimes \overline{\eta}_1](p-2) \\ &= [(\overline{\eta}_1 - \overline{\eta}_2) \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes (\overline{\eta}_2 - \overline{\eta}_1)](p-2). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{tr}((A(\eta_1) - A(\eta_2))Y) &\le n\,||Y||\,||A(\eta_1) - A(\eta_2)|| \\ &\le n\,|p-2|\,||Y||\,|\overline{\eta}_1 - \overline{\eta}_2|\,(|\overline{\eta}_1| + |\overline{\eta}_2|) \\ &\le 2n\,|p-2|\,||Y||\,|\overline{\eta}_1 - \overline{\eta}_2|. \end{aligned}$$

On one hand we have

$$\begin{aligned} |\overline{\eta}_1 - \overline{\eta}_2| = \left|\frac{\eta_1}{|\eta_1|} - \frac{\eta_2}{|\eta_2|}\right| &\le \max\left(\frac{|\eta_2-\eta_1|}{|\eta_2|}, \frac{|\eta_2-\eta_1|}{|\eta_1|}\right) \\ \le \frac{8C_\beta}{L}\,|x_1-y_1|^{\beta/2}, \end{aligned}$$

where we used (3.19) and (3.13).

On the other hand, by (3.14)–(3.17),

$$||Y|| = \max_{\overline{\xi}} |\langle Y\overline{\xi}, \overline{\xi}\rangle| \le 2|\langle B\overline{\xi}, \overline{\xi}\rangle| + \frac{4}{\tau}|\langle B^2\overline{\xi}, \overline{\xi}\rangle|$$
$$\le 4L\left(\frac{\phi'(|x_1-y_1|)}{|x_1-y_1|} + |\phi''(|x_1-y_1|)|\right).$$

Hence, remembering that $|x_1 - y_1| \le 2$, we end up with

$$\mathrm{tr}((A(\eta_1) - A(\eta_2))Y) \le 128n\,|p-2|\,C_\beta \phi'(|x_1-y_1|)\,|x_1-y_1|^{-1+\beta/2}$$
$$+ 128n\,|p-2|\,C_\beta|\phi''(|x_1-y_1|)|.$$

Estimate of (3). Finally, we have

$$M(\mathrm{tr}(A(\eta_1)) + \mathrm{tr}(A(\eta_2))) \le 2Mn\max(1, p-1).$$

Step 3

Gathering the previous estimates with (3.20) and recalling the definition of ϕ, we get

$$0 \le 128n\,|p-2|\,C_\beta\left(\phi'(|x_1-y_1|)\,|x_1-y_1|^{\beta/2-1} + |\phi''(|x_1-y_1|)|\right)$$
$$+ 2\min(1,p-1)L\phi''(|x_1-y_1|) + +2Mn\max(1,p-1)$$
$$\le 128n\,|p-2|\,C_\beta\,|x_1-y_1|^{\beta/2-1} + 2nM\max(1,p-1)$$
$$+ 128n\,|p-2|\,C_\beta\gamma(\gamma-1)\phi_0\,|x_1-y_1|^{\gamma-2}$$
$$- 2\min(1,p-1)\gamma(\gamma-1)\phi_0 L\,|x_1-y_1|^{\gamma-2}.$$

Taking $\gamma = 1 + \beta/2 > 1$ and choosing L large enough depending on p, n, C_β, we get

$$0 \le \frac{-\min(1,p-1)\gamma(\gamma-1)\phi_0}{200} L\,|x_1-y_1|^{\gamma-2} < 0,$$

which is a contradiction. Hence, by choosing first L such that

$$0 > 128n\,|p-2|\,C_\beta\left(\phi'(|x_1-y_1|)\,|x_1-y_1|^{\beta/2-1} + |\phi''(|x_1-y_1|)|\right)$$
$$+ \min(1,p-1)L\phi''(|x_1-y_1|) + 2nM\max(1,p-1)$$

and then taking $|q|$ large enough (depending on L, it suffices that $|q| > 6L > \frac{3}{2}|a|$ see (3.19)), we reach a contradiction and hence $\Phi(x,y) \le 0$ for $(x,y) \in B_r \times B_r$. The desired result follows since for $x_0, y_0 \in B_{\frac{15r}{16}}$, we have $\Phi(x_0, y_0) \le 0$, we get

$$|w(x_0) - w(y_0)| \le L\phi(|x_0-y_0|) \le L|x_0-y_0|. \qquad \square$$

Remembering that $\frac{15r}{16} = \frac{15\cdot 4}{16\cdot 5} = \frac{3}{4}$, we get that w is Lipschitz in $B_{\frac{3}{4}}$.

Finally, once we have a control on the Lipschitz norm of w, we can prove Lemma 3.2.

Proof of Lemma 3.2. Introducing the function $v(x) := w(x) + q \cdot x$, we notice that v is a viscosity solution to

$$-\Delta_p^N v = 0 \qquad \text{in } B_1,$$

and thus also a viscosity solution to the homogeneous p-Laplace equation $\Delta_p v = 0$, see [23]. By the equivalence result first proved by [23], v is a weak solution to the homogeneous p-Laplace equation. By the classical regularity result, there is $\beta_1 = \beta_1(p, n) > 0$ so that $v \in C^{1,\beta_1}_{\text{loc}}(B_1)$ and hence also $w \in C^{1,\beta_1}_{\text{loc}}(B_1)$. The main difficulty is to provide C^{1,β_1} estimates which are uniform with respect to q.

We notice that if $|q|$ is large enough, then the equation satisfied by w is uniformly elliptic and the operator is not discontinuous. Taking Q from Lemma 3.5 and assuming that $|q| > Q$, we know from Lemma 3.5 that $|Dw(x)|$ is controlled by some constant $\tilde{C}$ depending only on p, n and independent of $|q|$ for any $x \in B_{3/4}$. It follows that, if q satisfies

$$|q| \geq \theta_0 := \max(Q, 2\tilde{C}) \geq 2\, ||Dw||_{L^\infty(B_{3/4})}\,,$$

then denoting $\nu := \frac{1}{|q|}$ and $e := \frac{q}{|q|}$, we have

$$\frac{1}{2} \leq |e| - |\nu Dw| \leq |\nu Dw + e| \leq |e| + |\nu Dw| \leq \frac{3}{2}.$$

Defining

$$\Sigma(x) := (p-2)\frac{Dw(x) + q}{|Dw(x) + q|} \otimes \frac{Dw(x) + q}{|Dw(x) + q|},$$

we note that (3.1) can be rewritten as

$$-\operatorname{tr}(F(D^2 w, x)) = 0,$$

where $F : \mathcal{S}^n \times B_{3/4} \to \mathbb{R}$,

$$F(M, x) = -\operatorname{tr}((I + \Sigma(x))M),$$

is continuous.

Since Dw is Hölder continuous, we can see this equation as a linear elliptic equation with C^α coefficients. The standard Calderón-Zygmund theory provides local $C^{2,\alpha}$ regularity on w (bootstrapping the argument gives even C^∞ regularity on w).

Moreover, since v is a weak solution to the usual p-Laplacian, it follows that w is a weak solution to

$$-\operatorname{div}\big(|Dw + q|^{p-2}(Dw + q)\big) = 0 \quad \text{in} \quad B_{3/4}. \tag{3.22}$$

Writing the weak formulation, we have that for any test function $\varphi \in C_0^\infty(B_{3/4})$

$$\int_{B_{3/4}} |Dw + q|^{p-2}(Dw + q) \cdot D\varphi \, dx = 0. \tag{3.23}$$

Fixing k, $1 \leq k \leq n$, taking $\varphi_k = \dfrac{\partial \varphi}{\partial x_k}$ instead of φ as a test function and integrating by parts, we obtain

$$\int_{B_{3/4}} (|Dw + q|^{p-2}(I + \Sigma(x))\, Dw_k) \cdot D\varphi \, dx = 0.$$

Dividing by $|q|^{p-2}$, we have for any function $\varphi \in C_0^\infty(B_{3/4})$

$$\int_{B_{3/4}} \left(|\nu Dw + e|^{p-2}(I + \Sigma(x))Dw_k\right) \cdot D\varphi \, dx = 0.$$

We conclude that $h := w_k$ is a weak solution to the linear uniformly elliptic equation

$$-\operatorname{div}(\mathbf{A}(x)\, Dh) = 0,$$

where $\mathbf{A}(x) := |\nu Dw(x) + e|^{p-2}(I + \Sigma(x)) \in \mathcal{S}^n$ satisfies

$$\mathbf{A}(x) \geq \min(1, p-1) \min\left(\left(\frac{3}{2}\right)^{p-2}, \left(\frac{1}{2}\right)^{p-2}\right) I,$$
$$\mathbf{A}(x) \leq \max(1, p-1) \max\left(\left(\frac{3}{2}\right)^{p-2}, \left(\frac{1}{2}\right)^{p-2}\right) I.$$

Using the classical result of De Giorgi [12] for uniformly elliptic equations with bounded coefficients (see also [36], [17, Theorems 8.24, 12.1]) we get that h is locally Hölder continuous and

$$[h]_{C^{0,\beta_1}(B_{1/2})} \leq C(p,n)\, ||h||_{L^2(B_{3/4})} \tag{3.24}$$

for some $\beta_1 = \beta_1(p,n) > 0$.
We conclude that if $|q| > \theta_0 = \theta_0(p,n)$, then there exist $C = C(p, n, ||w||_{L^\infty(B_1)}, ||Dw||_{L^\infty(B_{3/4})}) = C_0(p,n) > 0$ (see Lemma 3.5) and $\beta_1 = \beta_1(n,p) > 0$ such that

$$[w]_{C^{1,\beta_1}(B_{1/2})} \leq C_0.$$

If $|q| \leq \theta_0$, we have

$$\operatorname*{osc}_{B_1} v \leq \operatorname*{osc}_{B_1} w + 2|q| \leq 1 + 2\theta_0.$$

It follows that

$$[w]_{C^{1,\beta_1}(B_{1/2})} \leq [v]_{C^{1,\beta_1}(B_{1/2})} + 2|q| \leq C(p,n) \operatorname*{osc}_{B_1} v + 2\theta_0 \leq C_0(p,n).$$

□

3.2. Second proof by using distributional weak theory

In this part we establish a second method to show that viscosity solutions to (1.1) are in $C^{1,\alpha}_{\text{loc}}(\Omega)$, when $f \in L^\infty(\Omega) \cap C(\Omega)$ and $p \geq 2$. Recall that equation (1.2) reads as

$$-\Delta_p u = |Du|^{p-2} f \qquad \text{in} \quad \Omega.$$

Since the exponent of the nonlinear gradient term is less than p and $f \in L^\infty(\Omega)$, locally Hölder continuous weak solutions of (1.2) are known to be of class $C^{1,\alpha}_{\text{loc}}$ for some $\alpha \in (0,1)$, see [41]. More precisely, if u is a weak solution to (1.2) in B_{2r}, then

$$[u]_{C^{1,\alpha}(B_r)} \leq C = C\left(p, n, r, ||u||_{L^\infty(B_{2r})}, ||f||_{L^\infty(B_{2r})}\right).$$

We know that in the case $p \geq 2$ viscosity solutions of (1.1) are viscosity solutions to (1.2), and our aim is to show that they are also weak solutions to (1.2). The next theorem holds for the more general case $f \in L^q(\Omega) \cap C(\Omega)$, where $q > \max(n, p/2)$, and will be useful not only in this subsection, but in Section 4 and Section 5 as well. Our proof cannot rely on uniqueness, see Example 3.7 below. Instead, we use a technique developed by Julin and Juutinen in [22]. We point out that the uniqueness of viscosity solutions is known only when f is either 0 or has constant sign (see [24]). A detailed discussion can be found in [3, 38] for the case of the normalized infinity Laplacian.

Theorem 3.6. *Assume that $p \geq 2$, $\max(n, p/2) < q \leq \infty$, and $f \in L^q(\Omega) \cap C(\Omega)$. Let u be a bounded viscosity solution to* (1.1)*. Then u is a weak solution to* (1.2)*.*

Proof. We will prove that a viscosity supersolution u to (1.2) is also a weak supersolution to (1.2) (the proof adapts to the case of subsolutions with obvious changes). We need to show that

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx \geq \int_\Omega |Du|^{p-2} f \varphi \, dx,$$

where $\varphi \in C_0^\infty(\Omega)$.

Step 1: regularization.

Let us start by showing that the inf-convolution u_ε of u,

$$u_\varepsilon(x) := \inf_{y \in \Omega} \left(u(y) + \frac{|x-y|^2}{2\varepsilon} \right), \tag{3.25}$$

is a weak supersolution to

$$-\Delta_p u_\varepsilon \geq |Du_\varepsilon|^{p-2} f_\varepsilon \qquad \text{in} \quad \Omega_{r(\varepsilon)}, \tag{3.26}$$

where

$$f_\varepsilon(x) = \inf_{|y-x| \leq 2\sqrt{\varepsilon \operatorname{osc}_\Omega u}} f(y)$$

and

$$\Omega_{r(\varepsilon)} = \{x \ : \ \operatorname{dist}(x, \partial\Omega) > 2\sqrt{\varepsilon \operatorname{osc}_\Omega u}\}.$$

We recall some properties of inf-convolutions. For more general discussion and proofs, see the appendix of [22]. First we mention that u_ε is a semi-concave viscosity supersolution to (3.26). Moreover, $u_\varepsilon \in W^{1,\infty}_{\text{loc}}(\Omega_{r(\varepsilon)})$ is twice differentiable a.e and satisfies

$$\begin{aligned} -\Delta_p u_\varepsilon &= -|Du_\varepsilon|^{p-2} \left(\Delta u_\varepsilon + (p-2) D^2 u_\varepsilon \frac{Du_\varepsilon}{|Du_\varepsilon|} \cdot \frac{Du_\varepsilon}{|Du_\varepsilon|} \right) \\ &\geq |Du_\varepsilon|^{p-2} f_\varepsilon \end{aligned} \tag{3.27}$$

a.e. in $\Omega_{r(\varepsilon)}$. Finally we mention that $u_\varepsilon \to u$ locally uniformly and $||u_\varepsilon||_{L^\infty(\Omega_{r(\varepsilon)})} \leq ||u||_{L^\infty(\Omega)}$, see [10].

Since the function $\phi(x) := u_\varepsilon(x) - \frac{1}{2\varepsilon}|x|^2$ is concave in $\Omega_{r(\varepsilon)}$, we can approximate it by a sequence (ϕ_j) of smooth concave functions by using standard mollification. Denoting $u_{\varepsilon,j} := \phi_j + \frac{1}{2\varepsilon}|x|^2$, we can integrate by parts to obtain

$$\int_{\Omega_{r(\varepsilon)}} |Du_{\varepsilon,j}|^{p-2} Du_{\varepsilon,j} \cdot D\varphi \, dx = \int_{\Omega_{r(\varepsilon)}} (-\Delta_p u_{\varepsilon,j}) \varphi \, dx. \tag{3.28}$$

Since Du_ε is locally bounded, the dominated convergence theorem implies

$$\lim_{j\to\infty}\int_{\Omega_{r(\varepsilon)}}|Du_{\varepsilon,j}|^{p-2}Du_{\varepsilon,j}\cdot D\varphi\,dx=\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot D\varphi\,dx. \tag{3.29}$$

Next, using the concavity of $u_{\varepsilon,j}$ (we have $D^2u_{\varepsilon,j}\le\frac{1}{\varepsilon}I$) and the local boundedness of $Du_{\varepsilon,j}$, we get

$$-\Delta_p u_{\varepsilon,j}\ge-\frac{C^{p-2}(n+p-2)}{\varepsilon}$$

locally in $\Omega_{r(\varepsilon)}$. Applying Fatou's lemma, we obtain

$$\liminf_{j\to\infty}\int_{\Omega_{r(\varepsilon)}}(-\Delta_p u_{\varepsilon,j})\varphi\,dx\ge\int_{\Omega_{r(\varepsilon)}}\liminf_{j\to\infty}(-\Delta_p u_{\varepsilon,j})\varphi\,dx. \tag{3.30}$$

Since

$$\liminf_{j\to\infty}(-\Delta_p u_{\varepsilon,j}(x))=-\Delta_p u_\varepsilon(x)$$

almost everywhere, by using (3.28), (3.29) and (3.30) we obtain

$$\begin{aligned}\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot D\varphi\,dx&\ge\int_{\Omega_{r(\varepsilon)}}(-\Delta_p u_\varepsilon)\varphi\,dx\\&\ge\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}f_\varepsilon\varphi\,dx.\end{aligned}$$

Hence, we have shown that $u_\varepsilon\in W^{1,p}_{\mathrm{loc}}(\Omega_{r_\varepsilon})$ is a weak supersolution to (3.26).

Step 2: passing to the limit in the regularization.

Take an arbitrary test function $\varphi\in C_0^\infty(\Omega)$. We finish the proof by showing that

$$\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot D\varphi\,dx\to\int_\Omega|Du|^{p-2}Du\cdot D\varphi\,dx \tag{3.31}$$

and

$$\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}f_\varepsilon\varphi\,dx\to\int_\Omega|Du|^{p-2}f\varphi\,dx. \tag{3.32}$$

Let Ω'' be the support of φ and ε so small that $\Omega''\subset\Omega'\subset\subset\Omega_{r(\varepsilon)}$. We start by showing that Du_ε is uniformly bounded in $L^p(\Omega')$. Take a compactly supported smooth cut-off function $\xi:\Omega_{r(\varepsilon)}\to[0,1]$ such that $\xi\equiv1$ on Ω'' and such that the support of ξ is included in Ω'. Choose the test function $(2L-u_\varepsilon)\xi^p$ in the weak formulation, where $L=\sup_{\Omega'}|u_\varepsilon|$. By using Hölder's inequality we obtain

$$\begin{aligned}\int_{\Omega_{r(\varepsilon)}}\xi^p|Du_\varepsilon|^p\,dx\quad&\le\quad\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p-2}(2L-u_\varepsilon)\xi^2\xi^{p-2}|f_\varepsilon|\,dx\\&\qquad+p\int_{\Omega_{r(\varepsilon)}}\xi^{p-1}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot D\xi(2L-u_\varepsilon)\,dx\\&\le\quad 1/4\int_{\Omega_{r(\varepsilon)}}\xi^p|Du_\varepsilon|^p\,dx+C(p)L^{p/2}\int_{\Omega_{r(\varepsilon)}}|f_\varepsilon|^{p/2}\xi^p\,dx\\&\qquad+C(p)\int_{\Omega_{r(\varepsilon)}}L^p|D\xi|^p dx+1/4\int_{\Omega_{r(\varepsilon)}}\xi^p|Du_\varepsilon|^p\,dx.\end{aligned}$$

It follows that

$$\int_{\Omega_{r(\varepsilon)}} \xi^p |Du_\varepsilon|^p\, dx \le C(p)L^{p/2} \int_{\Omega_{r(\varepsilon)}} |f_\varepsilon|^{p/2} \xi^p\, dx + \ C(p) \int_{\Omega_{r(\varepsilon)}} L^p |D\xi|^p dx$$
$$\le C = C\left(p, n, ||u||_{L^\infty(\Omega)}, ||f||_{L^q(\Omega)}\right). \tag{3.33}$$

Hence, Du_ε is uniformly bounded with respect to ε in $L^p(\Omega')$. It follows that there exists a subsequence such that $Du_\varepsilon \to Du$ weakly in $L^p(\Omega')$, and we can also show that $Du_\varepsilon \to Du$ strongly in $L^p(\Omega')$. Indeed, taking this time the test function $(u - u_\varepsilon)\xi^p$, we estimate

$$-\int_{\Omega_{r(\varepsilon)}} \xi^p |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot (Du - Du_\varepsilon)\, dx \le \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p-2} (u - u_\varepsilon) \xi^p |f_\varepsilon|\, dx$$
$$+ p \int_{\Omega_{r(\varepsilon)}} \xi^{p-1} |Du_\varepsilon|^{p-1} |D\xi| (u - u_\varepsilon)\, dx.$$

Adding $\int_{\Omega_{r(\varepsilon)}} |Du|^{p-2} Du \cdot (Du - Du_\varepsilon)\xi^p\, dx$ to this inequality and recalling that for $p > 2$

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \ge C(p)|a - b|^p,$$

we get

$$C(p) \int_{\Omega_{r(\varepsilon)}} |Du - Du_\varepsilon|^p \xi^p\, dx \le ||u - u_\varepsilon||_{L^\infty(\Omega')} ||Du_\varepsilon \xi||^{p-2}_{L^p(\Omega')} ||f_\varepsilon \xi||_{L^{p/2}(\Omega')}$$
$$+ \ p\, ||u - u_\varepsilon||_{L^\infty(\Omega')} ||Du_\varepsilon \xi||^{p-1}_{L^p(\Omega')} ||D\xi||_{L^p(\Omega')}$$
$$+ \int_{\Omega_{r(\varepsilon)}} |Du|^{p-2} Du \cdot (Du - Du_\varepsilon)\xi^p\, dx.$$

By using the local uniform convergence of u_ε to u, the facts that $||f_\varepsilon||_{L^q(\Omega')} \le C(q, \Omega)\, ||f||_{L^q(\Omega)}$, $Du \in L^p(\Omega')$ and the weak convergence of Du_ε in $L^p(\Omega')$, we obtain

$$\int_{\Omega_{r(\varepsilon)}} |Du - Du_\varepsilon|^p \xi^p\, dx \to 0,$$

so $Du_\varepsilon \to Du$ strongly in $L^p(\Omega')$.

Finally, we are ready to show that (3.31) and (3.32) hold. First we use the triangle inequality to obtain

$$\left| \int_{\Omega'} |Du_\varepsilon|^{p-2} f_\varepsilon \varphi\, dx - \int_{\Omega'} |Du|^{p-2} f \varphi\, dx \right|$$
$$\le \left| \int_{\Omega'} |Du_\varepsilon|^{p-2} (f_\varepsilon - f)\varphi\, dx \right| + \left| \int_{\Omega'} (|Du_\varepsilon|^{p-2} - |Du|^{p-2}) f \varphi\, dx \right|$$
$$=: I_1 + I_2.$$

Using the generalized Hölder's inequality, we get

$$I_1 \le ||Du_\varepsilon||_{L^p(\Omega')} ||f_\varepsilon - f||_{L^q(\Omega)} ||\varphi||_{L^\infty(\Omega')} \le C||f_\varepsilon - f||_{L^{p/2}(\Omega')} \to 0.$$

To estimate I_2, notice first that since f and φ are continuous in Ω', $f\varphi$ is bounded in Ω'. By using Hölder's inequality and the convexity of $\frac{p}{p-2}$ power function, we obtain

$$\begin{aligned}
I_2 &\le C \left|\left||Du_\varepsilon|^{p-2} - |Du|^{p-2}\right|\right|_{L^{\frac{p}{p-2}}(\Omega')} \\
&\le C \left| ||Du_\varepsilon||^p_{L^p(\Omega')} - ||Du||^p_{L^p(\Omega')} \right|^{\frac{p-2}{p}} \to 0,
\end{aligned}$$

since $Du_\varepsilon \to Du$ in $L^p(\Omega')$. Hence, (3.32) holds, and by using the same argument, also (3.31) holds. The proof is complete. □

Finally, we give an example to show why we deliberately avoided using the uniqueness machinery. For similar counterexamples in the case of the infinity Laplacian, see [11].

Example 3.7. *We give an example to show that for given continuous boundary data, there can be several weak solutions to equation* (1.2)*. Let* $f = (p-1)$*. Consider the 1-dimensional situation, where for* $R \in [0,1]$ *we define a function*

$$u(x) = \begin{cases} C - C(\frac{x+R}{-1+R})^2 & x \in (-1, -R) \\ C & [-R, R] \\ C - C(\frac{x-R}{1-R})^2 & x \in (R, 1). \end{cases}$$

Solving C *from*

$$-(p-1)u'' = (p-1)$$

gives

$$\frac{2C}{(-1+R)^2} = 1 \text{ i.e. } C = \frac{1}{2}(-1+R)^2.$$

This gives different weak solutions for the whole range of R*. Indeed, assuming that* $u \in W^{1,p}((-1,1))$*, for any test function* $\varphi \in C_0^\infty((-1,1))$

$$\begin{aligned}
\int_{-1}^{1} |u'|^{p-2}u'\varphi' \, dx &= -\int_{-1}^{-R} (x+R)^{p-1}\varphi'(x)\, dx + \int_R^1 (x-R)^{p-1}\varphi'(x)\, dx \\
&= (p-1)\Big(\int_{-1}^{-R} (x+R)^{p-2}\varphi(x)\, dx \\
&\quad + \int_R^1 (R-x)^{p-2}\varphi(x)\, dx\Big) \\
&= \int_{-1}^{1} |u'|^{p-2}\varphi f \; dx.
\end{aligned}$$

Only the largest i.e. $R = 0$ *is a solution to the original* $-\Delta_p^N u = (p-1)$*.*
This counterexample also shows that in general weak solutions to (1.2) *are not necessarily viscosity solutions to* (1.1)*.*

4. Uniform gradient estimates when $f \in C(\Omega) \cap L^q(\Omega)$

In this section we assume that $p > 2$, $f \in C(\Omega) \cap L^q(\Omega)$ for some $q > \max\left(n, \frac{p}{2}, 2\right)$. Our aim is to prove Theorem 1.2, which states that viscosity solutions of (1.1) are of class $C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha = \alpha(p, q, n)$, and for any $\Omega'' \subset \Omega' \subset\subset \Omega$,

$$[u]_{C^{1,\beta}(\Omega'')} \leq C = C\left(p, q, n, d, d'', ||u||_{L^\infty(\Omega)}, ||f||_{L^q(\Omega)}\right),$$

where $d = \text{diam}\,(\Omega)$ and $d'' = \text{dist}\,(\Omega'', \partial\Omega')$.

Let u be a viscosity solution of (1.1). From Lemma 3.1, we know that u is locally of class $C^{0,\beta}$ for some $\beta = \beta(p, n)$. From Section 3, we know that u is a weak solution to (1.2) and passing to the limit in (3.33), we know that for any $\Omega' \subset\subset \Omega$,

$$||Du||_{L^p(\Omega')} \leq C(p, n, ||u||_{L^\infty(\Omega)}, ||f||_{L^q(\Omega)}). \tag{4.1}$$

Moreover, for any $\lambda > 0$ the function u is a bounded viscosity solution to the following equation

$$-\Delta_p^N v(x) + \lambda v(x) = h(x) := f(x) + \lambda u(x), \quad x \in \Omega. \tag{4.2}$$

Let $\Omega' \subset\subset \Omega$ with Ω' smooth enough so that weak solutions to (1.2) satisfy the boundary condition in a classical sense. In the sequel we fix small enough $\lambda = \lambda(p, n, \Omega') > 0$ and a viscosity solution u of (1.1). We take Hölder continuous functions $f_\varepsilon \in C(\Omega) \cap L^q(\Omega)$ such that f_ε converges uniformly to f in Ω' and f_ε converges to f in $L^q(\Omega')$. The idea for the proof of Theorem 1.2 is to obtain uniform estimates for solutions v_ε to the following regularized problems

$$\begin{cases} -\text{div}\left(\left(|Dv_\varepsilon|^2 + \varepsilon^2\right)^{(p-2)/2} Dv_\varepsilon\right) = (|Dv_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}}(h_\varepsilon - \lambda v_\varepsilon), & x \in \Omega', \\ v_\varepsilon = u & x \in \partial\Omega', \end{cases} \tag{4.3}$$

where $h_\varepsilon = f_\varepsilon + \lambda u$. Notice that the right-hand side of the equation has a growth of power less than p with respect to the gradient, and h_ε is bounded. Since the regularized equations are uniformly elliptic with smooth coefficients, in Step 1 we notice that $v_\varepsilon \in C^{1,\beta(\varepsilon)}_{\text{loc}}(\Omega') \cap W^{2,2}_{\text{loc}}(\Omega')$. In the next two steps we obtain uniform estimate for the norm $||Dv_\varepsilon||_{L^p(\Omega')}$ and local Lipschitz estimate for v_ε. Once we know that v_ε and $|Dv_\varepsilon|^{p-2}$ are locally uniformly bounded, in Step 4 we use the regularity result of Lieberman [30] to get a local uniform Hölder estimate for the gradient Dv_ε. By using the equicontinuity of (Dv_ε), we obtain a subsequence (v_ε) converging to a viscosity solution v of equation (4.2) in $C^{1,\alpha}_{\text{loc}}(\Omega')$ when $\varepsilon \to 0$.

For $\lambda > 0$ and a given continuous boundary data, uniqueness for viscosity solutions of (4.2) is easy to prove. By using uniqueness, we conclude in Step 5 that the function v is the unique viscosity solution to (4.2) with given boundary data u. Since u is a solution to (4.2), we get that $u = v$. This gives a proof for Theorem 1.2.

Step 1: Local $C^{1,\beta}$ regularity for v_ε. Let $v_\varepsilon \in W^{1,p}(\Omega')$ be a weak solution of the regularized problem (4.3). Since $p - 2 < p$ and $h_\varepsilon \in L^q(\Omega')$ with $q > n/2$, regularity theory implies that the solutions v_ε are bounded and locally Hölder continuous. This follows from the Sobolev embedding for $p > n$ and from [28, Theorems 7.1,7.2, Chapter 4 p. 286–290] for $p \leq n$. Since $h_\varepsilon \in C(\overline{\Omega'})$ is bounded and the exponent on the gradient in the left term is less than p, we also have $v_\varepsilon \in C^{1,\alpha(\varepsilon)}_{\text{loc}}(\Omega') \cap W^{2,2}_{\text{loc}}(\Omega')$ (see [28, Theorem 8.7, chapter 4, p. 311], and also [13, 41] for more general regularity results.) This observation is useful, since we will derive estimates for Dv_ε by using test functions involving the derivatives of v_ε.

Step 2: Uniform boundedness of $||Dv_\varepsilon||_{L^p(\Omega')}$ and $||v_\varepsilon||_{L^\infty(\Omega')}$. First we derive a uniform bound for $||Dv_\varepsilon||_{L^p(\Omega')}$. Considering the weak formulation and taking $\varphi = v_\varepsilon - u$ as a test function, we have

$$\begin{aligned}\int_{\Omega'} \left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}} |Dv_\varepsilon|^2\,dx \le & \int_{\Omega'} (|Dv_\varepsilon|+\varepsilon)^{p-2}|v_\varepsilon-u||h_\varepsilon|\,dx \\ & + \int_{\Omega'} \left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}} |Dv_\varepsilon\cdot Du|\,dx \\ & + \lambda\int_{\Omega'} (|Dv_\varepsilon|+\varepsilon)^{p-2}|v_\varepsilon||v_\varepsilon-u|\,dx \\ \le & \int_{\Omega'} (|Dv_\varepsilon|+\varepsilon)^{p-2}|v_\varepsilon-u||h_\varepsilon|\,dx \\ & + \int_{\Omega'} \left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}} |Dv_\varepsilon||Du|\,dx \\ & + \lambda\int_{\Omega'} (|Dv_\varepsilon|+\varepsilon)^{p-2}|v_\varepsilon-u|^2\,dx \\ & + \lambda\int_{\Omega'} (|Dv_\varepsilon|+\varepsilon)^{p-2}|v_\varepsilon-u||u|\,dx.\end{aligned}$$

Using the inequality

$$\int_{\Omega'} |Dv_\varepsilon|^p\,dx \le \int_{\Omega'} \left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}} |Dv_\varepsilon|^2\,dx$$

together with Young's inequality and the previous estimate, we obtain

$$\begin{aligned}\int_{\Omega'} |Dv_\varepsilon|^p\,dx \le & \ \delta_0\int_{\Omega'} |Dv_\varepsilon|^p\,dx + C(p)\varepsilon^p|\Omega'| + \int_{\Omega'} |Du|^p\,dx \\ & + C(p)\int_{\Omega} |v_\varepsilon-u|^{p/2}|h_\varepsilon|^{p/2}\,dx \\ & + \lambda C(p)\int_{\Omega'} |v_\varepsilon-u|^p\,dx + C(p)\lambda\int_{\Omega'} |u|^p\,dx. \end{aligned} \tag{4.4}$$

If $\lambda = \lambda(p,n,\Omega') > 0$ is small enough, then using the Sobolev embedding, we get

$$\begin{aligned}\int_{\Omega'} |Dv_\varepsilon|^p\,dx \quad \le \quad & \delta_1\int_{\Omega'} |Dv_\varepsilon|^p\,dx + C(p)\int_{\Omega'} |v_\varepsilon-u|^{p/2}|h_\varepsilon|^{p/2}\,dx \\ & +\delta_2\int_{\Omega'} |Dv_\varepsilon|^p\,dx + C(p,n)\int_{\Omega'} |Du|^p\,dx \qquad (4.5) \\ & +C(p)\lambda\int_{\Omega'} |u|^p\,dx + C(p)\varepsilon^p|\Omega'|. \qquad (4.6)\end{aligned}$$

Now we have to estimate $\int_{\Omega'} |v_\varepsilon-u|^{p/2}|h_\varepsilon|^{p/2}\,dx$. We deal separately with the cases $p<n$, $p=n$ and $p>n$.

Case $p < n$

We denote by $p^* = \dfrac{np}{n-p}$ the Sobolev's conjugate exponent of p. Using Sobolev's and Hölder's inequalities and noticing that $\frac{p^* p}{2p^* - p} = \frac{np}{n+p}$, we get

$$\begin{aligned}
\int_{\Omega'} |v_\varepsilon - u|^{\frac{p}{2}} |h_\varepsilon|^{\frac{p}{2}}\, dx \quad &\leq \quad ||v_\varepsilon - u||^{\frac{p}{2}}_{L^{p^*}(\Omega')} \left(\int_{\Omega'} |h_\varepsilon|^{\frac{p^* p}{2p^* - p}}\, dx \right)^{\frac{2p^* - p}{2p^*}} \\
&\leq \quad C(p, n, |\Omega'|)\, ||Dv_\varepsilon - Du||^{\frac{p}{2}}_{L^p(\Omega')}\, ||h_\varepsilon||^{p/2}_{L^{\frac{np}{n+p}}(\Omega')} \\
&\leq \quad \delta_3 \int_{\Omega'} |Dv_\varepsilon - Du|^p\, dx + C(p, n, |\Omega'|)\, ||h_\varepsilon||^p_{L^{\frac{np}{n+p}}(\Omega')} \\
&\leq \quad \delta_4 \int_{\Omega'} |Dv_\varepsilon|^p\, dx + C(p, n, |\Omega'|)\, ||Du||^p_{L^p(\Omega')} \\
&\qquad + C(p, n, |\Omega'|)\, ||h_\varepsilon||^p_{L^{\frac{np}{n+p}}(\Omega')}\, .
\end{aligned} \tag{4.7}$$

Combining (4.4) and (4.7) and choosing $\delta_1 + \delta_2 + C(p)\delta_4 = 1/2$ in order to absorb terms, we obtain remembering the definition of the function h_ε

$$\begin{aligned}
||Dv_\varepsilon||^p_{L^p(\Omega')} &\leq C(p, n, |\Omega'|) \left(||h_\varepsilon||^p_{L^{\frac{np}{n+p}}(\Omega')} + \int_{\Omega'} (|Du| + 1 + |u|)^p\, dx \right) \\
&\leq C(p, n, |\Omega'|) \left(||f||^p_{L^{\frac{np}{n+p}}(\Omega')} + |\Omega'|^{1+p/n}\, ||u||^p_{L^\infty(\Omega)} \right) \\
&\quad + C(p, n, |\Omega'|) \int_{\Omega'} (|Du| + 1 + |u|)^p\, dx.
\end{aligned} \tag{4.8}$$

Case $p = n$

First we calculate

$$\begin{aligned}
\int_{\Omega'} |v_\varepsilon - u|^{p/2} |h_\varepsilon|^{p/2}\, dx &\leq \delta_5\, ||v_\varepsilon - u||^p_{L^p(\Omega')} + C(p)\, ||h_\varepsilon||^p_{L^p(\Omega')} \\
&\leq \delta_5 C(p, n, |\Omega'|)\, ||Dv_\varepsilon - Du||^p_{L^p(\Omega')} + C(p)\, ||h_\varepsilon||^p_{L^p(\Omega')} \\
&\leq \delta_6\, ||Dv_\varepsilon||^p_{L^p(\Omega')} + C(n, p, |\Omega'|)\, ||Du||^p_{L^p(\Omega')} \\
&\quad + C(p)\, ||h_\varepsilon||^p_{L^p(\Omega')}\, .
\end{aligned} \tag{4.9}$$

Combing (4.4) and (4.9) and choosing $\delta_1 + \delta_2 + C(p)\delta_6 = 1/2$, we obtain

$$\begin{aligned}
||Dv_\varepsilon||^p_{L^p(\Omega')} &\leq C(p, n, |\Omega'|)\, ||h_\varepsilon||^p_{L^p(\Omega')} + C(p, n, \Omega') \int_{\Omega'} (|Du| + |u| + 1)^p\, dx \\
&\leq C(p, n, |\Omega'|) \left(||f||^p_{L^n(\Omega')} + |\Omega'|\, ||u||^p_{L^\infty(\Omega')} \right) \\
&\quad + C(p, n, \Omega') \int_{\Omega'} (|Du| + |u| + 1)^p\, dx.
\end{aligned} \tag{4.10}$$

Case $p > n$

First we calculate

$$\begin{aligned}\int_{\Omega'} |v_\varepsilon - u|^{p/2} |h_\varepsilon|^{p/2} \, dx &\le \delta_7 \, ||v_\varepsilon - u||^p_{L^\infty(\Omega')} + C(p,n) \, ||h_\varepsilon||^p_{L^{\frac{p}{2}}(\Omega')} \\ &\le \delta_7 C(p,n,|\Omega'|) \, ||Dv_\varepsilon - Du||^p_{L^p(\Omega')} \\ &\quad + C(p,n) \, ||h_\varepsilon||^p_{L^{\frac{p}{2}}(\Omega')} \\ &\le \delta_8 \, ||Dv_\varepsilon||^p_{L^p(\Omega')} + C(p,n) \, ||h_\varepsilon||^p_{L^{\frac{p}{2}}(\Omega')} \\ &\quad + C(p,n,|\Omega'|) \, ||Du||^p_{L^p(\Omega')} \, . \end{aligned} \tag{4.11}$$

Combing (4.4) and (4.11) and choosing $\delta_1 + \delta_2 + C(p)\delta_8 = 1/2$, we get

$$\begin{aligned} ||Dv_\varepsilon||^p_{L^p(\Omega')} &\le C(p,n,|\Omega'|) \, ||h_\varepsilon||^p_{L^{\frac{p}{2}}(\Omega')} + C(p,n,|\Omega'|) \int_{\Omega'} (|Du| + 1 + |u|)^p \, dx \\ &\le C(p,n,|\Omega'|) \left(||f||^p_{L^{\frac{p}{2}}(\Omega')} + |\Omega'|^2 \, ||u||^p_{L^\infty(\Omega')} \right) \\ &\quad + C(p,n,|\Omega'|) \int_{\Omega'} (|Du| + 1 + |u|)^p \, dx. \end{aligned} \tag{4.12}$$

Once the boundedness of $||Dv_\varepsilon||_{L^p(\Omega')}$ is proved, we can derive a uniform bound for $||v_\varepsilon||_{L^\infty(\Omega')}$. Using the Sobolev embedding, in the case $p > n$ we get

$$\begin{aligned} ||v_\varepsilon||_{L^\infty(\Omega')} &\le ||v_\varepsilon - u||_{L^\infty(\Omega')} + ||u||_{L^\infty(\Omega')} \\ &\le C(n,\Omega',p) \, ||Dv_\varepsilon - Du||_{L^p(\Omega')} + ||u||_{L^\infty(\Omega')} \\ &\le C(p,n,|\Omega'|) \left(||f||_{L^q(\Omega')} + ||u||_{W^{1,p}(\Omega')} + ||u||_{L^\infty(\Omega')} + 1 \right). \end{aligned}$$

For $p \le n$, since $h_\varepsilon \in L^q(\Omega)$ for $q > \frac{n}{2}$, we can apply [28, Theorem 7.1, Chapter 4] giving an estimate for $||v_\varepsilon||_{L^\infty(\Omega')}$ when combined with the previous estimates of $||Dv_\varepsilon||_{L^p(\Omega')}$. We get

$$\begin{aligned} ||v_\varepsilon||_{L^\infty(\Omega')} &\le C \left(||u||_{L^\infty(\Omega)} , p, n, |\Omega'|, ||h_\varepsilon||_{L^q(\Omega')} , ||v_\varepsilon||_{L^{p^*}(\Omega')} \right) \\ &\le C \left(||u||_{L^\infty(\Omega')} , p, n, q, |\Omega'|, ||f||_{L^q(\Omega')} , ||u||_{W^{1,p}(\Omega')} \right), \end{aligned}$$

where we also used the estimate

$$\begin{aligned} ||v_\varepsilon||_{L^{p^*}(\Omega')} &\le ||v_\varepsilon - u||_{L^{p^*}(\Omega')} + ||u||_{L^{p^*}(\Omega')} \\ &\le C(p,n,\Omega')(||u||_{L^\infty(\Omega')} + ||u||_{W^{1,p}(\Omega')} + ||v_\varepsilon||_{W^{1,p}(\Omega')}). \end{aligned}$$

In both cases $p \le n$ and $p > n$, by using the estimate (4.1) we get

$$||v_\varepsilon||_{L^\infty(\Omega')} \le C \left(||u||_{L^\infty(\Omega')} , p, n, q, |\Omega'|, ||f||_{L^q(\Omega')} \right). \tag{4.13}$$

Step 3: Local uniform Lipschitz estimate for v_ε. In this subsection we derive a uniform local gradient estimate for v_ε by combining [15, Theorem 1.5] with the previous estimates (4.8)-(4.13).

We follow the main steps of Duzaar and Mingione [15]. For the sake of completeness, we give some details of these steps. We denote by $V(x) := h_\varepsilon(x) - \lambda v_\varepsilon(x)$. Then v_ε solves the equation

$$\begin{cases} -\mathrm{div}\left(\left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{(p-2)/2} Dv_\varepsilon\right) = (|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p-2}{2}} V, & x \in \Omega', \\ v_\varepsilon = u & x \in \partial\Omega'. \end{cases}$$

The Duzaar-Mingione gradient estimate relies on the use of a nonlinear potential of the function $|V|^2$ defined by

$$\mathcal{P}^V(x,R) := \int_0^R \left(\frac{|V|^2(B(x,\rho))}{\rho^{n-2}}\right)^{\frac{1}{2}} \frac{d\rho}{\rho}, \tag{4.14}$$

where

$$|V|^2(B(x,\rho)) := \int_{B(x,\rho)} |V(y)|^2\,dy.$$

Let us recall the main ingredients of the proof of the result of [15]. A key step is to derive a Caccioppoli type estimate for the function $(|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p}{2}}$ with a suitable remainder involving $|V|^2$. Relying on the regularity result of Step 1, this can be done by taking

$$\varphi_{ij}(x) := \frac{\partial}{\partial x_j}\left(\eta(x)^2\left((|Dv_\varepsilon(x)|^2+\varepsilon^2)^{\frac{p}{2}} - k\right)_+ \frac{\partial v_\varepsilon(x)}{\partial x_i}\right)$$

as test functions in the weak formulation, where η is a non negative cut-off function. Next, a modification of the De Giorgi techniques allowed them to get pointwise estimate of $|Dv_\varepsilon|^p$ in terms of the L^{2p} norm of Dv_ε and the nonlinear potential $\mathcal{P}^V$. Finally, using interpolation, they improved the estimate in terms of the natural L^p norm of the gradient and the L^∞ norm of the nonlinear potential.

Our approximation is slightly different, but the Caccioppoli type estimate of [15, Lemma 3.1] (adapted for the new right hand side) holds for $2 < p \le n$ and also for $p > n$. Indeed, by using the weak formulation with the test function φ_{ij} and integration by parts, there exists a constant $C = C(p,n)$ such that for any ball $B_R := B(x,R) \subset \Omega'$,

$$\begin{aligned}\int_{B_{\frac{R}{2}}} \left| D\left((|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p}{2}} - k\right)_+\right|^2 dy \le \frac{C}{R^2}\int_{B_R}\left((|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p}{2}} - k\right)_+^2 dy \\ +C\int_{B_R}\left|\left(\varepsilon^2 + ||Dv_\varepsilon||^2_{L^\infty(B_R)}\right)^{(p-1)/2} V\right|^2 dy.\end{aligned}$$

It follows that the oscillation improvement estimate [15, Lemma 3.2] holds. Once we have such control on the level sets of $|Dv_\varepsilon|^p$, a standard modification of the De Giorgi iteration argument implies the following potential estimate (see for example [15, Lemma 3.3])

$$\begin{aligned}(|Dv_\varepsilon(x)|^2+\varepsilon^2)^{\frac{p}{2}} &\le C\left(\fint_{B_R}(|Dv_\varepsilon|^2+\varepsilon^2)^p\,dy\right)^{1/2} \\ &+ C\left(\varepsilon^2 + ||Dv_\varepsilon||^2_{L^\infty(B_R)}\right)^{\frac{p-1}{2}} \mathcal{P}^V(x,R),\end{aligned}$$

where $C = C(p,n)$. Proceeding as in [15] we get for $R/2 < \rho < r < R$,

$$\begin{aligned}\left(||Dv_\varepsilon||^2_{L^\infty(B_\rho)} + \varepsilon^2\right)^{\frac{p}{2}} &\leq C\frac{\left(\varepsilon^2 + ||Dv_\varepsilon||^2_{L^\infty(B_r)}\right)^{\frac{p}{4}}}{(r-\rho)^{n/2}}\left(\int_{B_r}\left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{p/2}\,dy\right)^{1/2} \\ &\quad + C\left(\varepsilon^2 + ||Dv_\varepsilon||^2_{L^\infty(B_r)}\right)^{\frac{p-1}{2}}\left|\left|\mathcal{P}^V(\cdot,R)\right|\right|_{L^\infty(B_R)} \\ &\leq \frac{1}{2}\left(\varepsilon^2 + ||Dv_\varepsilon||^2_{L^\infty(B_r)}\right)^{\frac{p}{2}} + C\left|\left|\mathcal{P}^V(\cdot,R)\right|\right|^p_{L^\infty(B_R)} \\ &\quad + \frac{C}{(r-\rho)^n}\int_{B_r}\left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{p/2}\,dy,\end{aligned}$$

where $C = C(p,n)$. Now the standard iteration lemma (see for example [15, Lemma 2.1]) implies that

$$\begin{aligned}\left(||Dv_\varepsilon||^2_{L^\infty(B_{R/2})} + \varepsilon^2\right)^{\frac{p}{2}} \;\; &\leq C \fint_{B_R}\left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{p/2}\,dy \\ &+C\left|\left|\mathcal{P}^V(\cdot,R)\right|\right|^p_{L^\infty(B_R)}, \end{aligned} \tag{4.15}$$

where $C = C(p,n)$. Consequently, combining (4.8), (4.10), (4.12) and (4.15) we get

$$||Dv_\varepsilon||_{L^\infty(B_{R/2})} \;\; \leq \;\; C\left(R^{-n/p}\,||Dv_\varepsilon||_{L^p(B_R)} + \left|\left|\mathcal{P}^V(\cdot,R)\right|\right|_{L^\infty(B_R)} + 1\right),$$

for all R such that $B_R \subset \Omega'$ and where $C = C(p,n)$. Since v_ε is uniformly bounded in $L^\infty(\Omega')$ and h_ε is uniformly bounded in $L^q(\Omega')$, we have $V \in L^q(\Omega')$. We obtain

$$\int_{B(x,\rho)}|V(y)|^2\,dy \leq ||V||^2_{L^q(\Omega')}\,|B(x,\rho)|^{\frac{q-2}{q}} \leq C\,||V||^2_{L^q(\Omega')}\,\rho^{\frac{n(q-2)}{q}},$$

where $C = C(n)$, and

$$\mathcal{P}^V(x,R) \leq ||V||_{L^q(\Omega')}\int_0^R \rho^{\frac{n(q-2)}{2q}-\frac{n}{2}}\,d\rho \leq CR^{\frac{q-n}{q}},$$

where $C = C(q,n)\,||V||_{L^q(\Omega')}$. It follows that

$$\sup_{B(x,R)}\mathcal{P}^V(\cdot,R) \leq C\sup_{B(x,R)}R^{\frac{q-n}{q}} < \infty, \tag{4.16}$$

where $C = C(n,q,||V||_{L^q(\Omega')})$. Recalling that $V = h_\varepsilon - \lambda v_\varepsilon$, and using the bound (4.13) for $||v_\varepsilon||_{L^\infty(\Omega')}$, we get

$$||V||_{L^q(\Omega')} \leq C\left(p,n,q,|\Omega'|,||f||_{L^q(\Omega')},||u||_{L^\infty(\Omega')}\right). \tag{4.17}$$

Hence,

$$||Dv_\varepsilon||_{L^\infty(B_{R/2})} \leq \tilde{C}\left(p,n,\Omega,q,||f||_{L^q(\Omega')},||u||_{L^\infty(\Omega')},R\right).$$

Step 4: Local uniform $C^{1,\beta}$ estimate for u_ε. Since Dv_ε is locally uniformly bounded in L^∞ with respect to ε, the function

$$\mu_\varepsilon := (|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p-2}{2}}V$$

is also locally bounded in L^q with $q > n$ and satisfies

$$\begin{aligned}\int_{B_r(x)} |\mu_\varepsilon| dy &\leq C(p)\left(||Dv_\varepsilon||^{p-2}_{L^\infty(B_r(x))}+1\right)\int_{B_r(x)}|V(y)|\,dy\\ &\leq C(p,n)\left(||Dv_\varepsilon||^{p-2}_{L^\infty(B_r(x))}+1\right)||V||_{L^q(\Omega')}\,r^{\frac{n(q-1)}{q}}\\ &\leq \tilde{C}\left(q,n,p,\Omega',||f||_{L^q(\Omega')},||u||_{L^\infty(\Omega')}\right)r^{n-p+\delta},\end{aligned}$$

where $\delta = \frac{qp-n}{q}$, $\delta \in (p-1,p)$. Applying the result of Lieberman [30, Theorem 5.3] ((v_ε) being also bounded in L^∞), we get that v_ε are locally of class $C^{1,\beta}$ for some $\beta = \beta(p,q,n)$ and for any $\Omega'' \subset\subset \Omega'$

$$[v_\varepsilon]_{C^{1,\beta}(\Omega'')} \leq C = C\left(p,q,n,|\Omega'|,||u||_{L^\infty(\Omega')},d'',||f||_{L^q(\Omega')}\right), \tag{4.18}$$

where $d'' = \text{dist}(\Omega'',\partial\Omega')$.

Step 5: Convergence in the weak and viscosity sense and conclusion. We get from (4.18) and the Arzelà-Ascoli theorem that (u_ε) converges (up to a subsequence) to a function v in $C^{1,\alpha}_{\text{loc}}(\Omega')$ for some $\alpha = \alpha(q,p,n) < \beta$. Passing to the limit within the weak formulation, v is a weak solution to

$$-\Delta_p v = |Dv|^{p-2}(h-\lambda v), \tag{4.19}$$

see Appendix B for details. Passing to the limit in (4.18), we get that for any $\Omega'' \subset\subset \Omega'$, we have the estimate

$$||v||_{C^{1,\alpha}(\Omega'')} \leq C\left(p,n,q,d'',|\Omega'|,||u||_{L^\infty(\Omega')},||f||_{L^q(\Omega')}\right).$$

From the boundedness of v_ε, it follows that v is a bounded weak solution of the Dirichlet problem associated to (4.19). Since $(v_\varepsilon - u)$ is uniformly bounded in $W^{1,p}_0(\Omega')$, we have $(v-u) \in W^{1,p}_0(\Omega')$. Assuming sufficient regularity for the boundary $\partial\Omega'$, we have $v \in C(\overline{\Omega'})$ and for any $x_0 \in \partial\Omega'$ $\lim\limits_{x\to x_0} v(x) = u(x_0)$. The reader can find further discussion of the boundary regularity problem for elliptic equations in the monograph of Malý and Ziemer [33]. On the other hand, the local Hölder continuity of Dv_ε and the Hölder continuity of h_ε imply, by the classical elliptic regularity theory, that v_ε is also a classical solution to

$$-\text{div}\left(\left(|Dv_\varepsilon|^2+\varepsilon^2\right)^{(p-2)/2} Dv_\varepsilon\right) = (|Dv_\varepsilon|^2+\varepsilon^2)^{\frac{p-2}{2}}(h_\varepsilon - \lambda v_\varepsilon) \quad \text{in } \Omega'.$$

This implies that v_ε solves in the classical sense

$$-\Delta v_\varepsilon - (p-2)\frac{D^2v_\varepsilon Dv_\varepsilon \cdot Dv_\varepsilon}{|Dv_\varepsilon|^2+\varepsilon^2} = h_\varepsilon - \lambda v_\varepsilon \qquad \text{in } \Omega'. \tag{4.20}$$

Hence v_ε is a continuous viscosity solution of the Dirichlet problem associated to equation (4.20) with continuous boundary data u. Passing to the limit in (4.20), we get that the limit function v is also a continuous viscosity solution of (4.2) with boundary data equals u, see Appendix C. The viscosity solution to (4.2) is understood in the sense of Definition C.1. It is easy to see that the fixed viscosity solution u of (1.1) is a viscosity solution to (4.2) with the weaker Definition C.1 (η is then taken as an eigen-vector of $D^2\phi(x_0)$). It follows (see the Appendix D for details) that, for a given boundary data, the Dirichlet problem associated to (4.2) admits a unique viscosity solution. By uniqueness, we conclude that the limit function v is the unique viscosity solution of (4.2) and since u is a viscosity solution to this problem, we conclude that $u = v$ in Ω'. It follows that u is of class $C^{1,\alpha}_{\text{loc}}$ for some $\alpha = \alpha(p,q,n)$ and the estimate of Theorem 1.2 holds.

5. Nearly optimal Hölder exponent for gradients

In this section we prove Theorem 1.3. Assume that $f \in L^q(\Omega) \cap C(\Omega)$ and fix arbitrary $\xi > 0$. We will prove that the viscosity solutions to (1.1) are of class $C^{1,\alpha_\xi}_{\text{loc}}$, where

$$\alpha_\xi = \begin{cases} \alpha_0 - \xi & \text{when } q = \infty, \\ \min(\alpha_0 - \xi, 1 - \frac{n}{q}) & \text{when } \max(n, \frac{p}{2}, 2) < q < \infty, \end{cases}$$

and α_0 is the optimal Hölder exponent in an a priori estimate for gradients of p-harmonic functions. In the case $q = \infty$ we only assume that $p > 1$, whereas in the case $q < \infty$ we require $p > 2$.

The question of optimal regularity for inhomogeneous p-Laplacian in divergence form has received attention as well, see [31, 27, 1, 2]. An alternative approach to study optimal regularity questions for p-Poisson problem in divergence form could be based on [26, equation (1.38)]. In our paper we do not try to quantify the explicit optimal value of α in $C^{1,\alpha}$ estimate to the homogenous case.

Remark 5.1. *If $p \geq 2$ and f is a continuous and bounded function, in the case that Ω is either a ball or an annulus, radial viscosity solutions to* (1.1) *have a better regularity and they are in* $C^{1,1}(\Omega)$ *(see [6, Theorem 1.1]).*

5.1. The case $q = \infty$

In this subsection we prove Theorem 1.3 when $f \in L^\infty(\Omega) \cap C(\Omega)$. Since our results are local, by translation and rescaling we can restrict our study in the unit ball $B_1 \subset \Omega$ and show the regularity at $0 \in B_1 \subset \Omega$. Like previously, it is useful to do suitable rescaling to get an Arzelà-Ascoli type compactness lemma. During the rest of this section, for $\delta_0 > 0$ to be determined later, we assume that $||u||_{L^\infty(B_1)} \leq 1$ and $||f||_{L^\infty(B_1)} \leq \delta_0$ without loss of generality. This can be seen like before: Let $\kappa = (||u||_{L^\infty(B_1)} + \delta_0^{-1}||f||_{L^q(B_1)})^{-1}$. Setting $\tilde{u} = \kappa u$, then $\tilde{u}$ satisfies

$$-\Delta_p^N(\tilde{u}) = \tilde{f}$$

with $||\tilde{u}||_{L^\infty(B_1)} \leq 1$ and $||\tilde{f}||_{L^q(B_1)} \leq \delta_0$.

For convenience, in this subsection we denote by C different constants depending only on p and n. First we use our regularity result from Section 3 to show that the solutions to (1.1) can be approximated by p-harmonic functions in $C^{1,\alpha}_{\text{loc}}$ for some small $\alpha > 0$.

Lemma 5.2. *Let $u \in C(B_1)$ be a viscosity solution to equation* (1.1)*. For given $\varepsilon > 0$, there exists $\delta_0 = \delta_0(p, n, \varepsilon)$ such that for $||u||_{L^\infty(B_1)} \leq 1$, $||f||_{L^\infty(B_1)} \leq \delta_0$, there exists a p-harmonic function h in $B_{3/4}$ satisfying*

$$||u - h||_{L^\infty(B_{1/2})} < \varepsilon \quad \textit{and} \quad ||Du - Dh||_{L^\infty(B_{1/2})} < \varepsilon.$$

Proof. Suppose that the lemma is not true. Then, for some $\varepsilon_0 > 0$ there is a uniformly bounded sequence of continuous functions (u_j) and a sequence $(f_j) \subset C(\Omega) \cap L^\infty(\Omega)$, $||f_j||_{L^\infty(B_1)} \to 0$, such that

$$-\Delta_p^N u_j = f_j,$$

but for all p-harmonic functions h defined in $B_{3/4}$ we have either $||u_j - h||_{L^\infty(B_{1/2})} \geq \varepsilon_0$ or $||Du_j - Dh||_{L^\infty(B_{1/2})} \geq \varepsilon_0$.

By Theorem 1.1, $(u_j) \subset C^{1,\alpha}(B_{3/4})$ for some $\alpha > 0$, so by the Arzelà-Ascoli theorem there is a subsequence, still denoted by (u_j), which converges to some function h in $C^{1,\alpha}(B_{1/2})$. Then the limit function h satisfies $\Delta_p^N h = 0$ in the viscosity sense, so it also satisfies $\Delta_p h = 0$ in the weak sense. By $C^{1,\alpha}$ convergence, there is $j_0 \in \mathbb{N}$ such that $||u_{j_0} - h||_{L^\infty(B_{1/2})} < \varepsilon_0$ and $||Du_{j_0} - Dh||_{L^\infty(B_{1/2})} < \varepsilon_0$. We have reached a contradiction. □

By using the approximation with p-harmonic functions, in the next lemma we obtain an oscillation estimate for solutions u to (1.1) near the critical set $\{x : Du(x) = 0\}$.

Lemma 5.3. *There exist $\lambda_0 = \lambda_0(p, n) \in (0, \frac{1}{2})$ and $\delta_0 > 0$ such that if $||f||_{L^\infty(B_1)} \le \delta_0$ and $u \in C^{1,\alpha}(B_1)$ is a viscosity solution to* (1.1) *in B_1 with $||u||_{L^\infty(B_1)} \le 1$, then*

$$\sup_{x \in B_{\lambda_0}} |u(x) - u(0)| \le \lambda_0^{1+\alpha_\xi} + |Du(0)|\lambda_0.$$

Proof. Take the approximating p-harmonic function h from the previous lemma. By the a priori estimate for p-harmonic functions, there exist $\lambda_0 = \lambda_0(p, n) \in (0, \frac{1}{2})$ such that

$$\sup_{x \in B_{\lambda_0}} |h(x) - [h(0) + Dh(0) \cdot x]| \le C\lambda_0^{1+\alpha_0},$$

and $C\lambda_0^{1+\alpha_0} \le \frac{1}{2}\lambda_0^{1+\alpha_\xi}$. Now we choose $\varepsilon > 0$ satisfying $\varepsilon < \frac{1}{6}\lambda_0^{1+\alpha_\xi}$. This ε determines δ_0 through the previous lemma. We get for all $x \in B_{\lambda_0}$,

$$\begin{aligned}
|u(x) - [u(0) + Du(0) \cdot x]| &\le |h(x) - [h(0) + Dh(0) \cdot x]| \\
&\quad + |(u - h)(x)| + |(u - h)(0)| + |D(u - h)(0) \cdot x| \\
&\le C\lambda_0^{1+\alpha_0} + 3\varepsilon \\
&\le \lambda_0^{1+\alpha_\xi}.
\end{aligned}$$

The result follows by the triangle inequality. □

Next we iterate the previous estimate to control the oscillation of the solutions in dyadic balls.

Theorem 5.4. *Under the assumptions of the previous lemma, there exists a constant C such that*

$$\sup_{x \in B_r} |u(x) - u(0)| \le Cr^{1+\alpha_\xi}\left(1 + |Du(0)|r^{-\alpha_\xi}\right)$$

for all sufficiently small $r \in (0, 1)$.

Proof. For $k \in \mathbb{N}$, consider the rescaled function defined in B_1,

$$v_k(x) = \frac{u(\lambda_0^k x) - u(0)}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}}.$$

We have $v_k(0) = 0$,

$$Dv_k(0) = \frac{\lambda_0^k}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}} Du(0),$$

and

$$-\Delta_p^N v_k(x) = \frac{\lambda_0^{2k}}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}} f(\lambda_0^k x) \le |\lambda_0^{k(1-\alpha_\xi)} f(\lambda_0^k x)|,$$

where $|\lambda_0^{k(1-\alpha_\xi)} f(\lambda_0^k x)| \le \delta_0$, since $\lambda_0^{k(1-\alpha_\xi)} \le 1$.

Let us show by induction that $||v_k||_{L^\infty(B_1)} \le 1$. By the previous lemma, this holds for $k = 1$, so assume that $||v_j||_{L^\infty(B_1)} \le 1$ for $j \le k$. As shown above, the function v_k satisfies the conditions of the previous lemma, so we have

$$\sup_{x \in B_{\lambda_0}} |v_k(x) - v_k(0)| \le \lambda_0^{1+\alpha_\xi} + |Dv_k(0)|\lambda_0.$$

Hence,

$$\begin{aligned} &\sup_{x\in B_1} \frac{|u(\lambda_0^{k+1}x) - u(0)|}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}} \\ &\qquad \le \lambda_0^{1+\alpha_\xi} + \frac{\lambda_0^{k+1}}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}} |Du(0)|, \end{aligned}$$

which reads

$$\sup_{x\in B_1} |u(\lambda_0^{k+1}x) - u(0)| \le \lambda_0^{(k+1)(1+\alpha_\xi)} + \sum_{j=0}^{k} |Du(0)|\lambda_0^{k+j\alpha_\xi+1}.$$

This is equivalent to $||v_{k+1}||_{L^\infty(B_1)} \le 1$, so induction is complete.

We obtain for arbitrary k,

$$\begin{aligned} \sup_{x\in B_{\lambda_0^{k+1}}} \frac{|u(x) - u(0)|}{\lambda_0^{(k+1)(1+\alpha_\xi)}} &\le 1 + \frac{\sum_{j=0}^{k} |Du(0)|\lambda_0^{k+j\alpha_\xi+1}}{\lambda_0^{(k+1)(1+\alpha_\xi)}} \\ &\le 1 + |Du(0)|\lambda_0^{-(k+1)\alpha_\xi} \sum_{j=0}^{k} \lambda_0^{j\alpha_\xi} \\ &\le \left(1 + \frac{1}{1-\lambda_0^{\alpha_\xi}}\right)\left(1 + |Du(0)|\lambda_0^{-(k+1)\alpha_\xi}\right) \\ &= C\left(1 + |Du(0)|\lambda_0^{-(k+1)\alpha_\xi}\right). \end{aligned}$$

Since this holds for all $k \in \mathbb{N}$, we obtain for all sufficiently small $r > 0$,

$$\sup_{x\in B_r} |u(x) - u(0)| \le Cr^{1+\alpha_\xi}\left(1 + |Du(0)|r^{-\alpha_\xi}\right). \qquad \square$$

We are ready to show $C^{1,\alpha_\xi}_{\text{loc}}$ regularity for solutions to equation (1.1). If the gradient $Du(0)$ is very small, we obtain the result from the previous theorem. In the other case the result follows from a more classical reasoning using the regularity theory of uniformly elliptic equations.

Theorem 5.5. *Under the assumptions of Lemma 5.3, we have for all sufficiently small $r \in (0,1)$,*

$$\sup_{x\in B_r} |u(x) - [u(0) + Du(0)\cdot x]| \le Cr^{1+\alpha_\xi}.$$

Proof. When $|Du(0)| \leq r^{\alpha_\xi}$, Theorem 5.4 gives

$$\begin{aligned}\sup_{x\in B_r} |u(x) - [u(0) + Du(0)\cdot x]| &\leq \sup_{x\in B_r} |u(x) - u(0)| + |Du(0)|r \\ &\leq Cr^{1+\alpha_\xi}.\end{aligned}$$

When $|Du(0)| > r^{\alpha_\xi}$, define $\mu := \min\left(\frac{3}{4}, |Du(0)|^{1/\alpha_\xi}\right)$ and use the rescaled function

$$w(x) = \frac{u(\mu x) - u(0)}{\mu^{1+\alpha_\xi}}.$$

We have $w(0) = 0$, $|Dw(0)| \geq 1$, and

$$-\Delta_p^N w(x) = \frac{\mu^2 f(\mu x)}{\mu^{1+\alpha_\xi}} = \mu^{1-\alpha_\xi} f(\mu x),$$

where $||\mu^{1-\alpha_\xi} f||_{L^\infty(B_1)} \leq \delta_0$. From Theorem 5.4 we obtain

$$\sup_{x\in B_1} |w(x)| = \sup_{x\in B_\mu} \frac{|u(x) - u(0)|}{\mu^{1+\alpha_\xi}} \leq C\left(1 + |Du(0)|\mu^{-\alpha_\xi}\right) = C.$$

Since $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ for some $\alpha > 0$, there exists $\gamma \in (0, \frac{1}{2})$ such that

$$|Dw(x)| \geq \frac{1}{2} \qquad \text{in } B_\gamma.$$

For all $p > 1$ w is a viscosity solution to $-\Delta_p w = |Dw|^{p-2}\mu^{1-\alpha_\xi} f(\mu x) =: g \in C(B_\gamma)$ in B_γ, so by [22] it is a weak solution to the same equation, which also satisfies the conditions of [28, Theorem 5.2, p. 277], see also [41]. Hence, $w \in W^{2,2}(B_\gamma)$, so by the local version of [17, Lemma 9.16, p 241], for arbitrary $\varepsilon > 0$ it holds $w \in C^{1,1-\varepsilon}(B_\gamma)$. In particular, $w \in C^{1,\alpha_\xi}(B_\gamma)$. Hence, for all $s \in (0, \frac{\gamma}{2})$, we have

$$\sup_{x\in B_s} |w(x) - Dw(0)\cdot x| \leq Cs^{1+\alpha_\xi},$$

or equivalently,

$$\sup_{x\in B_s} \left| \frac{u(\mu x) - u(0)}{\mu^{1+\alpha_\xi}} - \mu^{-\alpha_\xi} Du(0)\cdot x \right| \leq Cs^{1+\alpha_\xi},$$

and we get

$$\sup_{x\in B_s} |u(\mu x) - [u(0) + Du(0)\cdot(\mu x)]| \leq C(\mu s)^{1+\alpha_\xi}.$$

If $r < \frac{\mu\gamma}{2}$, then the previous estimate gives

$$\sup_{x\in B_r} |u(x) - [u(0) + Du(0)\cdot x]| \leq Cr^{1+\alpha_\xi}.$$

If $r \geq \frac{\mu\gamma}{2}$, noticing that $r < \mu$ and $|Du(0)| \leq C\mu^{\alpha_\xi}$ we obtain

$$\begin{aligned}\sup_{x\in B_r} |u(x) - [u(0) + Du(0)\cdot x]| &\leq \sup_{x\in B_\mu} |u(x) - u(0)| + |Du(0)|\mu \\ &\leq C\mu^{1+\alpha_\xi} \\ &\leq C\left(\frac{2}{\gamma}\right)^{1+\alpha_\xi} r^{1+\alpha_\xi} \\ &\leq Cr^{1+\alpha_\xi}.\end{aligned}$$

□

This theorem completes the proof of Theorem 1.3 when $f \in C(\Omega) \cap L^\infty(\Omega)$.

5.2. The case $f \in C \cap L^q$

In this subsection we assume that $p > 2$ and $f \in C(B_1) \cap L^q(B_1)$, and use Theorem 1.2 to show that the solutions to equation (1.1) are of class $C^{1,\alpha_\xi}_{\text{loc}}$. As previously, for $\delta_0 > 0$ to be determined later, we take the assumptions $||u||_{L^\infty(B_1)} \le 1$ and $||f||_{L^q(B_1)} \le \delta_0$ without loss of generality. We also denote by C different constants depending only on p and n.

We follow the reasoning of the first subsection. First we show that the solutions to equation (1.1) can be approximated by p-harmonic functions in $C^{1,\alpha}_{\text{loc}}$.

Lemma 5.6. *Let $u \in C(B_1)$, $||u||_{L^\infty(B_1)} \le 1$, be a viscosity solution to equation* (1.1)*. Given $\varepsilon > 0$, there is $\delta_0 = \delta_0(p, n, \varepsilon)$ such that if $||f||_{L^q(B_1)} \le \delta_0$, there is a p-harmonic function h in $B_{3/4}$ satisfying*

$$||u - h||_{L^\infty(B_{1/2})} < \varepsilon \quad \textit{and} \quad ||Du - Dh||_{L^\infty(B_{1/2})} < \varepsilon.$$

Proof. Thriving for contradiction, assume that there exists $\varepsilon_0 > 0$ such that there are sequences (u_j) and (f_j) satisfying $||u_j||_{L^\infty(B_1)} \le 1$, $f_j \in C(B_1) \cap L^q(B_1)$, $||f_j||_{L^q(B_1)} \to 0$, and

$$-\Delta_p^N u_j = f_j,$$

but for all p-harmonic functions h in $B_{3/4}$

$$||u_j - h||_{L^\infty(B_{1/2})} > \varepsilon_0 \quad \text{or} \quad ||Du_j - Dh||_{L^\infty(B_{1/2})} > \varepsilon_0.$$

Recall from Theorem 3.6 that u_j is a weak solution to

$$-\Delta_p u_j = |Du_j|^{p-2} f_j \qquad \text{in} \quad B_1.$$

From Theorem 1.2 we know that $(u_j) \subset C^{1,\alpha}(B_{3/4})$ for some $\alpha > 0$, so by the Arzelà-Ascoli theorem, there is a subsequence, still denoted by (u_j), converging in $C^{1,\alpha}(B_{3/4})$ to a function h. By Appendix B, h is a p-harmonic function. We have reached a contradiction. □

The next lemma follows from the previous approximation result as in the first subsection.

Lemma 5.7. *There exists $\lambda_0 = \lambda_0(p, n) \in (0, \frac{1}{2})$ and $\delta_0 > 0$ such that if $||f||_{L^q(B_1)} \le \delta_0$ and $u \in C^{1,\alpha}_{loc}(B_1)$ is a viscosity solution to* (1.1) *in B_1 with $||u||_{L^\infty(B_1)} \le 1$, then*

$$\sup_{x \in B_{\lambda_0}} |u(x) - u(0)| \le \lambda_0^{1+\alpha_\xi} + |Du(0)|\lambda_0.$$

Theorem 5.8. *Under the assumptions of the previous lemma, we have*

$$\sup_{x \in B_r} |u(x) - u(0)| \le C r^{1+\alpha_\xi} \left(1 + |Du(0)| r^{-\alpha_\xi}\right)$$

for all sufficiently small $r > 0$.

Proof. The proof is similar to the proof of Theorem 5.4. Again we consider the rescaled function

$$v_k(x) = \frac{u(\lambda_0^k x) - u(0)}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}},$$

and see that $v_k(0) = 0$,

$$Dv_k(0) = \frac{\lambda_0^k}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)|\lambda_0^{k+j\alpha_\xi}} Du(0),$$

and

$$-\Delta_p^N v_k(x) = \frac{\lambda_0^{2k}}{\lambda_0^{k(1+\alpha_\xi)} + \sum_{j=0}^{k-1} |Du(0)\lambda_0^{k+j\alpha_\xi}} f(\lambda_0^k x) =: f_k(x).$$

Since $q(1-\alpha_\xi) - n > 0$, we estimate

$$\begin{aligned}
\int_{B_1} |f_k(x)|^q dx &\leq \int_{B_1} \left(\lambda_0^{k(1-\alpha_\xi)} |f(\lambda_0^k x)|\right)^q dx \\
&= \int_{B_{\lambda_0^k}} \left(\lambda_0^{k(1-\alpha_\xi)} |f(y|)\right)^q \lambda_0^{-nk} dy \\
&= \int_{B_{\lambda_0^k}} \lambda_0^{kq(1-\alpha_\xi)-nk} |f(y)|^q dy \\
&\leq \int_{B_{\lambda_0^k}} |f(y)|^q dy.
\end{aligned}$$

Hence, we have $||f_k||_{L^q(B_1)} \leq \delta_0$. By continuing as in the proof of Theorem 5.4, we get the result. □

Theorem 5.9. *Under the assumptions of Lemma 5.7, we have*

$$\sup_{x \in B_r} |u(x) - [u(0) + Du(0) \cdot x]| \leq Cr^{1+\alpha_\xi}$$

for all sufficiently small $r \in (0,1)$.

Proof. We follow the ideas of the proof of Theorem 5.5. We get the result from Theorem 5.8 when $|Du(0)| \leq r^{\alpha_\xi}$. In the case $|Du(0)| > r^{\alpha_\xi}$, define the rescaled function $w(x) = (u(\mu x) - u(0))/\mu^{1+\alpha_\xi})$, for which $w(0) = 0$, $|Dw(0)| \geq 1$, and

$$-\Delta_p^N w(x) = \frac{\mu^2 f(\mu x)}{\mu^{1+\alpha_\xi}} = \mu^{1-\alpha_\xi} f(\mu x) =: f_\mu(x),$$

where $||f_\mu||_{L^q(B_1)} \leq \delta_0$. From Theorem 5.8 we get

$$\sup_{x \in B_1} |w(x)| = \sup_{x \in B_\mu} \frac{|u(x) - u(0)|}{\mu^{1+\alpha_\xi}} \leq C\left(1 + |Du(0)|\mu^{-\alpha_\xi}\right) = C.$$

Since $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ for some $\alpha > 0$, there exists $\gamma \in (0, 1/2)$ such that

$$|Dw(x)| \geq \frac{1}{2} \qquad \text{in } B_\gamma.$$

As explained in the proof of Theorem 5.5, we know that $w \in C^{1,1-n/q}(B_\gamma)$. Since $\alpha_\xi \leq 1 - n/q$, we have $w \in C^{1,\alpha_\xi}(B_\gamma)$. Hence, for all $s \in (0, \frac{\gamma}{2})$, we have

$$\sup_{x \in B_s} |w(x) - Dw(0) \cdot x| \leq C s^{1+\alpha_\xi},$$

and the rest of the argument follows as in the proof of Theorem 5.5. □

The proof of Theorem 1.3 is complete.

Appendix

A. The limit equation in Lemma 3.3

We prove two convergence results needed in the proof of Lemma 3.3. Assume that there exist a sequence of continuous functions (f_j) with $||f_j||_{L^\infty(B_1)} \to 0$, a sequence of vectors (q_j) and a sequence of viscosity solutions (w_j) with $\operatorname{osc}_{B_1} w_j \leq 1$ to

$$-\Delta w_j - (p-2) \left\langle D^2 w_j \frac{Dw_j + q_j}{|Dw_j + q_j|}, \frac{Dw_j + q_j}{|Dw_j + q_j|} \right\rangle = f_j.$$

Case 1: (q_j) is bounded

First we show that if (q_j) is bounded, there is a subsequence (w_j) converging to a limit w_∞, which satisfies

$$-\operatorname{tr}\left(\left(I + (p-2)\frac{Dw_\infty + q_\infty}{|Dw_\infty + q_\infty|} \otimes \frac{Dw_\infty + q_\infty}{|Dw_\infty + q_\infty|}\right) D^2 w_\infty\right) = 0 \quad \text{in } B_1 \tag{A.1}$$

in a viscosity sense. Here $q_j \to q_\infty$ up to the same subsequence. We show that w_∞ is a subsolution of (A.1) (the case of supersolution being similar). We fix $\phi \in C^2(\Omega)$ such that $w_\infty - \phi$ has a strict maximum at x_0. As w_∞ is the uniform limit of the subsequence (w_j) and x_0 is a strict maximum point, there exists a sequence of points $x_j \to x_0$ such that $(w_j - \phi)$ has a local maximum at x_j.

Suppose first that $-D\phi(x_0) \neq q_\infty$. Then $-D\phi(x_j) \neq q_j$ when j is large, and at those points we have

$$-\Delta \phi_j - (p-2) \left\langle D^2 \phi_j \frac{D\phi_j + q_j}{|D\phi_j + q_j|}, \frac{D\phi_j + q_j}{|D\phi_j + q_j|} \right\rangle \leq f_j.$$

Passing to the limit, we get the desired result.

Suppose next that $-D\phi(x_0) = q_\infty$. We have to consider two cases. Assuming that there exists a subsequence still indexed by j such that $|D\phi(x_j) + q_j| > 0$ for all j in the subsequence, then

$$-\Delta \phi_j - (p-2) \left\langle D^2 \phi_j \frac{D\phi_j + q_j}{|D\phi_j + q_j|}, \frac{D\phi_j + q_j}{|D\phi_j + q_j|} \right\rangle \leq f_j,$$

and we conclude by passing to the limit. If such a subsequence does not exist, then we have

$$-\Delta \phi(x_j) - (p-2)\lambda_{max}(D^2\phi(x_j)) \leq f_j(x_j)$$

for j large enough. Passing to the limit we get

$$-\Delta \phi(x_0) - (p-2)\lambda_{max}(D^2\phi(x_0)) \leq 0.$$

We have shown the desired result.

Case 2: (q_j) is unbounded

When (q_j) is unbounded, take a subsequence, still denoted by (q_j), for which $|q_j| \to \infty$, and then a converging subsequence from $e_j = \frac{q_j}{|q_j|}$, $e_j \to e_\infty$. We have

$$-\Delta w_j - (p-2)\left\langle D^2 w_j \frac{Dw_j|q_j|^{-1} + e_j}{|Dw_j|q_j|^{-1} + e_j|}, \frac{Dw_j|q_j|^{-1} + e_j}{|Dw_j|q_j|^{-1} + e_j|}\right\rangle = f_j.$$

We show that the uniform limit w_∞ (up to a subsequence) satisfies in the viscosity sense

$$-\Delta w_\infty - (p-2)\left\langle D^2 w_\infty\, e_\infty, e_\infty\right\rangle = 0 \qquad \text{in} \quad B_1, \tag{A.2}$$

with $|e_\infty| = 1$.

We only show that w_∞ is a subsolution of (A.2) (the case of supersolution is similar). We fix $\phi \in C^2(\Omega)$ such that $w_\infty - \phi$ has a strict maximum at x_0. By the uniform convergence of w_j to w_∞, there are points x_j such that $w_j - \phi$ has a maximum at x_j and $x_j \to x_0$. Since $D\phi(x_j) \to D\phi(x_0)$ and $|q_j| \to \infty$, we know that

$$\frac{D\phi(x_j)}{|q_j|} \neq -e_j$$

for j large. Denoting $A_j := D\phi(x_j)|q_j|^{-1}$ for short, we get at those points

$$-\Delta\phi(x_j) - (p-2)\left\langle D^2\phi(x_j)\frac{A_j + e_j}{|A_j + e_j|}, \frac{A_j + e_j}{|A_j + e_j|}\right\rangle \leq f_j(x_j).$$

Since $A_j \to 0$, we get the desired result.

B. Convergence in the weak formulation

Assume that $p > 2$, $q > \max(2, n, p/2)$, $f_\varepsilon, f \in C(\Omega) \cap L^q(\Omega)$ and $f_\varepsilon \to f$ in $L^q(\Omega)$. We show that if u_ε is a weak solution to

$$-\Delta_p u_\varepsilon = |Du_\varepsilon|^{p-2} f_\varepsilon,$$

and if $u_\varepsilon \to u$ in $C^{1,\alpha}(K)$ for any $K \subset\subset \Omega$, then u is a weak solution to

$$-\Delta_p u = |Du|^{p-2} f.$$

For any test function $\phi \in C_0^\infty(\Omega)$, u_ε satisfies

$$\int_\Omega |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot D\phi \, dx = \int_\Omega |Du_\varepsilon|^{p-2} f_\varepsilon \phi \, dx.$$

Since $Du_\varepsilon \to Du$ locally uniformly, we have for all sufficiently small ε,

$$|Du_\varepsilon|^{p-2}|Du_\varepsilon \cdot D\phi| \leq (||Du||_{L^\infty(\operatorname{supp}\phi)} + 1)^{p-1}|D\phi| \in L^1(\Omega),$$

so by the dominated convergence theorem,

$$\int_\Omega |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot D\phi \, dx \to \int_\Omega |Du|^{p-2} Du \cdot D\phi \, dx.$$

It remains to show that

$$\int_\Omega |Du_\varepsilon|^{p-2} f_\varepsilon \phi \, dx \to \int_\Omega |Du|^{p-2} f \phi \, dx. \tag{B.1}$$

Notice that

$$|Du_\varepsilon|^{p-2} f_\varepsilon \phi = |Du_\varepsilon|^{p-2}(f_\varepsilon - f)\phi + |Du_\varepsilon|^{p-2} f \phi. \tag{B.2}$$

Since $Du_\varepsilon \in L^\infty_{\mathrm{loc}}(\Omega)$, by the dominated convergence and identity (B.2), (B.1) holds.

C. Convergence in the viscosity sense

Assume that $h_\varepsilon \in C(\Omega)$ and let v_ε be a viscosity solution to

$$-\Delta v_\varepsilon - (p-2)\frac{D^2 v_\varepsilon D v_\varepsilon \cdot D v_\varepsilon}{|Dv_\varepsilon|^2+\varepsilon^2} + \lambda v_\varepsilon = h_\varepsilon \quad \text{in } \Omega', \tag{C.1}$$

and assume that $v_\varepsilon \to v$ locally uniformly in Ω' and $h_\varepsilon \to h$ locally uniformly. We prove that the limit v is a viscosity solution of (4.2). Viscosity solutions to (4.2) are understood in the following sense

Definition C.1. Let Ω' be a bounded domain and $2 < p < \infty$. An upper semicontinuous function v is a viscosity subsolution of (4.2) if, for all $x_0 \in \Omega'$ and $\phi \in C^2(\Omega')$ such that $v - \phi$ attains a local maximum at x_0 and $v(x_0) = \phi(x_0)$, one has either

$$-\Delta_p^N \phi(x_0) + \lambda v(x_0) \le h(x_0) \qquad \text{if } D\phi(x_0) \neq 0,$$

or there exists a vector $\eta \in \mathbb{R}^n$ with $|\eta| \le 1$ such that

$$-\Delta\phi(x_0) - (p-2)\langle D^2\phi(x_0)\eta, \eta\rangle + \lambda v(x_0) \le h(x_0) \quad \text{if } D\phi(x_0) = 0.$$

The notion of viscosity supersolution is defined similarly and a function v is a viscosity solution to (4.2) if and only if it is a sub- and supersolution.

We only show that v is a viscosity subsolution to (4.2). To show that v is a viscosity supersolution, one proceeds similarly. Let $\phi \in C^2$ be such that $v - \phi$ has a local strict maximum at x_0 and $v(x_0) = \phi(x_0)$. Since $v_\varepsilon \to v$ locally uniformly, there exists a sequence $x_\varepsilon \to x_0$ such that $v_\varepsilon - \phi$ has a local maximum at x_ε. Since v_ε is a viscosity solution of (4.20), it follows that

$$-\Delta\phi(x_\varepsilon) - (p-2)\frac{D^2\phi(x_\varepsilon)D\phi(x_\varepsilon)\cdot D\phi(x_\varepsilon)}{|D\phi(x_\varepsilon)|^2+\varepsilon^2} + \lambda v_\varepsilon(x_\varepsilon) \le h_\varepsilon(x_\varepsilon). \tag{C.2}$$

First suppose that $D\phi(x_0) \neq 0$, then $D\phi(x_\varepsilon) \neq 0$ for ε small enough. Since h_ε converges to h locally uniformly and v_ε converges to v locally uniformly, passing to the limit in (C.2), we get that

$$-\Delta\phi(x_0) - (p-2)\frac{D^2\phi(x_0)D\phi(x_0)\cdot D\phi(x_0)}{|D\phi(x_0)|^2} + \lambda v(x_0) \le h(x_0).$$

Next suppose that $D\phi(x_0) = 0$. Noticing that $\left|\dfrac{D\phi(x_\varepsilon)}{\sqrt{|D\phi(x_\varepsilon)|^2+\varepsilon^2}}\right| \le 1$, it follows that (up to a subsequence) the sequence $\dfrac{D\phi(x_\varepsilon)}{\sqrt{|D\phi(x_\varepsilon)|^2+\varepsilon^2}}$ converges to a vector $\eta \in \mathbb{R}^n$ with $|\eta| \le 1$. Passing to the limit in (C.2), we get that, there exists a vector η such that

$$-\Delta\phi(x_0) - (p-2)\langle D^2\phi(x_0)\eta, \eta\rangle + \lambda v(x_0) \le h(x_0).$$

D. Uniqueness of viscosity solutions to (4.2)

In this section we prove the uniqueness of viscosity solutions to (4.2), where viscosity solutions of (4.2) are understood in the sense of Definition C.1 and $\lambda > 0$. Notice that, for $\lambda > 0$, the operator

$$F(X, \xi, r, x) := -\operatorname{tr}(A(\xi)X) + \lambda r - h(x)$$

where

$$A(\xi) := \begin{cases} I + (p-2)\overline{\xi} \otimes \overline{\xi} & \text{if} \quad \xi \neq 0 \\ I + (p-2)\eta \otimes \eta & \text{for a certain } \eta, |\eta| \leq 1 \quad \text{if } \xi = 0 \end{cases}$$

with $\overline{\xi} := \dfrac{\xi}{|\xi|}$ is proper, that is

$$F(X, \xi, s, x) \leq F(Y, \xi, r, x) \quad \text{for} \quad Y \leq X, \quad s \leq r.$$

Now, let v_1 and v_2 be two continuous viscosity solutions to (4.2) in Ω' and such that $v_1 = v_2$ on $\partial\Omega'$. We want to show that $v_1 = v_2$. We argue by contradiction. Without loss of generality, we assume that $v_1 - v_2$ reaches a positive maximum at an interior point $x_0 \in \Omega'$. For $\varepsilon > 0$, the function

$$\Phi(x, y) := v_1(x) - v_2(y) - \frac{|x-y|^4}{4\varepsilon},$$

reaches a maximum in $\overline{\Omega'} \times \overline{\Omega'}$ at $(x_\varepsilon, y_\varepsilon)$. By classical arguments we have that $x_\varepsilon \in \Omega'$, $y_\varepsilon \in \Omega'$ for $\varepsilon > 0$ small enough and $x_\varepsilon \to x_0$, $y_\varepsilon \to x_0$ when $\varepsilon \to 0$. We also observe that the function $x \mapsto v_1(x) - \left(v_2(y_\varepsilon) + \dfrac{|x-y_\varepsilon|^4}{4\varepsilon}\right) = v_1(x) - \phi_1(x)$ reaches a maximum at x_ε and $y \mapsto v_2(y) - \left(v_1(x_\varepsilon) - \dfrac{|x_\varepsilon - y|^4}{4\varepsilon}\right) = v_2(y) - \phi_2(y)$ reaches a minimum at y_ε. From the definition of viscosity sub- and supersolution we obtain the following. If $x_\varepsilon = y_\varepsilon$ then $D^2\phi_1(x_\varepsilon) = D^2\phi_2(y_\varepsilon) = 0$ and writing the viscosity inequalities we get that

$$\lambda v_1(x_\varepsilon) \leq h(x_\varepsilon), \qquad \lambda v_2(x_\varepsilon) \geq h(x_\varepsilon).$$

It follows that $\lambda(v_1(x_\varepsilon) - v_2(x_\varepsilon)) \leq 0$ and passing to the limit we get that $\lambda(v_1(x_0) - v_2(x_0)) \leq 0$, which is a contradiction since $\lambda > 0$ and $v_1(x_0) - v_2(x_0) > 0$.
If $x_\varepsilon \neq y_\varepsilon$, then by the theorem of sums [10, Theorem 3.2] there are

$$(\xi_x, X) \in \overline{\mathcal{J}}^{2,+}(v_1(x_\varepsilon)), \quad (\xi_y, Y) \in \overline{\mathcal{J}}^{2,-}(v_2(y_\varepsilon))$$

with $X \leq Y$ and $\xi_x = \xi_y = D\phi_1(x_\varepsilon) = D\phi_2(y_\varepsilon) \neq 0$. Writing the viscosity inequalities, we have

$$\begin{aligned} &-\operatorname{tr}(A(\xi_x)X) + \lambda v_1(x_\varepsilon) \leq h(x_\varepsilon) \\ &-\operatorname{tr}(A(\xi_x)Y) + \lambda v_2(y_\varepsilon) \geq h(y_\varepsilon). \end{aligned}$$

Since $A(\xi_x) = I + (p-2)\overline{\xi}_x \otimes \overline{\xi}_x \geq 0$ and $X - Y \leq 0$, subtracting the previous two inequalities, we get that

$$\lambda(v_1(x_\varepsilon) - v_2(y_\varepsilon)) \leq h(x_\varepsilon) - h(y_\varepsilon)$$

and passing to the limit we get a contradiction.

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1. Final comments

During the preparation of this dissertation we noticed the following errors and typos in the published articles [A,B]:

[A] In the statement of Theorem 4.1, instead of $\operatorname{osc}(f, B_{6r}(a))$ should read $\operatorname{osc}(f, B_{6R}(a))$. In the proof of the same theorem, p.10, l.-10, instead of 'Otherwise, she moves towards z length $\frac{\varepsilon}{2}$ or keeps the token at z.' should read: 'Otherwise, she moves the token towards vector $\frac{z-x}{m}$, where $m = \frac{3|x-z|}{\varepsilon}$.'

In the proof of Lemma 5.3, in the case $x \in B_R(0)$ and $y \in \partial B_R(0)$, we choose $\varepsilon < \frac{s}{2}$, so we do not require that $B_{2S}(z) \subset B_R(0) \cup \Gamma_\varepsilon$. The boundary values of a radially concave v should read $v = \sup_{B_s(z)} u$ on $\partial B_s(z)$ and $v = \sup_\Omega u$ on $\partial B_{2S}(z)$. Then the argument of the proof gives $u_\varepsilon(x) - u_\varepsilon(y) < \eta$, and the other inequality $u_\varepsilon(y) - u_\varepsilon(x) < \eta$ follows from the symmetric argument by considering a radially convex function v' with boundary values $v' = \inf_{B_s(z)} u$ on $\partial B_s(z)$ and $v' = \inf_\Omega u$ on $\partial B_{2S}(z)$.

[B] The citation [19] ([MPR10b] in the introduction of this thesis) was meant to be [MPR10a] by the same authors.

In p.1389, l.5, the correct choice is to use

$$\left(\sqrt{p(x_j,t_j)-1}\,\xi, \sqrt{p(y_j,s_j)-1}\,\xi\right).$$

Then the argument, starting from line 11, reads as follows:

We have

$$\begin{aligned}\eta < &-(n+p(x_j,t_j))j(t_j-s_j) + (n+p(y_j,s_j))j(t_j-s_j)\\ &+ (p(x_j,t_j)-2)\langle X_j\xi,\, \xi\rangle + \operatorname{tr}(X_j) - (p(y_j,s_j)-2)\langle Y_j\xi,\, \xi\rangle - \operatorname{tr}(Y_j).\end{aligned}$$

Since the function p is Lipschitz continuous, we have

$$|-(n+p(x_j,t_j))j(t_j-s_j) + (n+p(y_j,s_j))j(t_j-s_j)| < \frac{\eta}{2}$$

when j is large enough. Hence, we get

$$\begin{aligned}\frac{\eta}{2} &< (p(x_j,t_j)-2)\langle X_j\xi,\, \xi\rangle + \operatorname{tr}(X_j) - (p(y_j,s_j)-2)\langle Y_j\xi,\, \xi\rangle - \operatorname{tr}(Y_j)\\ &\le \langle (X_j-Y_j)\xi, \xi\rangle + (p(x_j,t_j)-2)\langle X_j\xi,\, \xi\rangle - (p(y_j,s_j)-2)\langle Y_j\xi,\, \xi\rangle\\ &\le \langle X_j(\sqrt{p(x_j,t_j)-1}\,\xi), (\sqrt{p(x_j,t_j)-1}\,\xi)\rangle\\ &\quad - \langle Y_j(\sqrt{p(y_j,s_j)-1}\,\xi), (\sqrt{p(y_j,s_j)-1}\,\xi)\rangle\\ &\le C\,(p(x_j,t_j)-p(y_j,s_j))^2\left(\xi' M\xi + \frac{2}{j}\xi' M^2\xi\right)\\ &\le C\left(|x_j-y_j|^2 + |t_j-s_j|^2\right)(j^2|x_j-y_j|^2 + j^3|x_j-y_j|^4)\\ &< C\left(j^2|x_j-y_j|^4 + (j^2|x_j-y_j|^4)^{3/2}\right)\end{aligned}$$

when j is large. This is a contradiction, since $j^2|x_j - y_j|^4 \to 0$ when $j \to \infty$. In the last two estimates we used Lipschitz continuity of p.

All occurrences of the parabolic normalized $p(x,t)$-Laplace equations should be $(n + p(x,t))u_t = \Delta u + (p(x,t) - 2)\Delta_\infty^N u$.

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