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PLANAR SOBOLEV EXTENSION DOMAINS

YI ZHANG

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Abstract

This doctoral thesis deals with geometric characterizations of bounded planar simply connected Sobolev extension domains. It consists of three papers. In the first and third papers we give full geometric characterizations of $W^{1,p}$-extension domains for $1 < p < 2$ and $p = 1$, respectively. The second paper establishes a density result for Sobolev functions on planar domains, necessary for the solution for the case $p = 1$. Combining with the known results, we obtain a full geometric characterization of $W^{1,p}$-extension domains for every $1 \leq p \leq \infty$.

The author had an active role in the research and preparation of each of the three papers.

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Yi Zhang

Introduction

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1 Sobolev Extension domains

Let $\Omega \subset \mathbb{R}^n$ be a domain, that is, an open connected subset of $\mathbb{R}^n$.

Definition 1.1. *Assume that $f \in L^1_{\mathrm{loc}}(\Omega)$, and let $1 \le i \le n$. We say that $g_i \in L^1_{\mathrm{loc}}(\Omega)$ is the weak partial derivative (or distributional partial derivative) of f with respect to x_i if*

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi\, dx$$

for all $\varphi \in C^1_c(\Omega)$.

The *Sobolev space* $W^{1,p}(\Omega)$ for $p \in [1, \infty]$ is the collection of all functions $u \in L^1_{\mathrm{loc}}(\Omega)$ with the (semi)norm

$$\|u\|_{W^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} < \infty.$$

Here $\nabla u = (g_1, \ldots, g_n)$ is the *distributional gradient* of u, where g_i is the weak partial derivative of u with respect to x_i. Also recall that every Sobolev function f has a *precise representative* with

$$f(x) = \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y)\, dy$$

almost everywhere (with respect to the p-capacity; see e.g. [5, Page 160]). Here $|B(x, r)|$ denotes the Lebesgue measure of $B(x, r)$. In what follows we identify every Sobolev function with its precise representative.

A domain Ω is called a $W^{1,p}$*-extension domain* if there exists a bounded (linear) operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ such that $Eu|_\Omega = u$ for all $u \in W^{1,p}(\Omega)$. Regarding the issue of linearity in our definition for $1 < p < \infty$ we refer the reader to [12].

Our definition of $W^{1,p}$ and the corresponding norm only deal with ∇u. This is called the homogeneous norm. The usual Sobolev norm for u is

$$\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The extension problem for the usual norm is equivalent to our definition when Ω is bounded [13].

Regarding extendability of functions in Euclidean spaces, it is clear that every function in $L^p(A)$ can be zero-extended to $\mathbb{R}^n$ for every set $A \subset \mathbb{R}^n$. Also any L-Lipschitz function $f : A \to \mathbb{R}$ can be extended from any set $A \subset \mathbb{R}^n$ to a global Lipschitz function. Indeed, define

$$Ef(x) = \inf_{z \in A} \{f(z) + L|z - x|\}.$$

One can easily check that this really gives us a Lipschitz extension (even with the same Lipschitz constant).

For Sobolev spaces, extension problems become more complicated. When $n = 1$, namely on the real line, we can extend Sobolev functions easily for any open interval by reflection and suitable cut-off functions.

Calderón [4] and Stein [24] dealt with the case where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. They showed that, for each of these domains, it is possible to construct an extension operator for every $1 \le p \le \infty$. Later in [9, 10, 11, 14] it was proven that a bounded simply connected planar domain is a $W^{1,2}$-extension domain if and only if it is a quasidisk (equivalently, a uniform domain). Moreover Jones showed in his seminal paper [14] that every uniform domain of $\mathbb{R}^n$ is a $W^{1,p}$-extension domain for all $p \in [1, \infty)$. Let us recall the definition.

Definition 1.2. A domain Ω is called *uniform* if there exists a positive constant ϵ_0 such that for any two different points $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x, y and satisfying

$$\ell(\gamma) \le \frac{1}{\epsilon_0}|x - y| \quad \text{and} \quad \mathrm{dist}\,(z, \partial\Omega) \ge \epsilon_0 \min\{\ell(\gamma_{xz}), \ell(\gamma_{zy})\} \text{ for all } z \in \gamma, \tag{1.1}$$

where $\ell(\gamma_{xz})$ is the length of the part of γ joining from x to z, and γ_{zy} corresponds to z and y.

Jones' idea was to construct an extension operator via a reflection technique $Q_j \to Q_j^*$, motivated by "quasiconformal reflections"; it was well-known that a bounded simply connected planar domain is uniform if and only if it is the image of the unit disk under some quasiconformal mapping on $\mathbb{R}^2$; see [7]. Recall here that a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, is *quasiconformal* if $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and there is a constant $K \ge 1$ so that

$$|Df(x)|^2 \le KJ_f(x), \quad a.e.\ x \in \mathbb{R}^2.$$

In the case $p > 2$, the correct geometric condition is $\frac{p-2}{p-1}$-subhyperbolicity: each such domain is a $W^{1,p}$-extension domain and a simply connected planar domain is a $W^{1,p}$-extension domain if and only if it is a $\frac{p-2}{p-1}$-subhyperbolic domain; see [23] and also [17]. This class of domains is strictly larger than the class of uniform domains.

In this thesis we deal with the planar case where $1 \le p < 2$. Then even a simply connected planar $W^{1,p}$-extension domain is not necessarily uniform (see [21] and related examples in [16, 19]). Before going further to our results, let us recall more details about classical results together with the ideas of their proofs in the following subsections, case by case. Then we introduce the main ideas of our proofs. All the discussions below are always in the plane, even though some of them can be generalized into the higher dimensional cases.

1.1 Case $p = \infty$: locally Lipschitz functions

A (path)connected set $A \subset \mathbb{R}^2$ is called C-*quasiconvex* if there exists a constant $C > 0$ such that for any $x, y \in A$ there exists a rectifiable curve γ joining them with

$$\ell(\gamma) \leq C|x - y|,$$

where $\ell(\gamma)$ denotes the length of γ. Recall that the *inner distance* $\operatorname{dist}_\Omega(x, y)$ between x and y in a domain Ω is defined as

$$\operatorname{dist}_\Omega(x, y) = \inf \ell(\gamma_{x,y})$$

where the infimum is taken over all rectifiable curves $\gamma_{x,y} \subset \Omega$ joining x and y. Also recall that (the precise representatives of) $W^{1,\infty}$-functions are locally Lipschitz [5, Section 4.2.3].

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then it is a $W^{1,\infty}$-extension domain if and only if it is C-quasiconvex, where the norm of the extension operator and C depend only on each other.*

Sketch of the proof. To show the necessity of C-quasiconvexity, given $z_1, z_2 \in \Omega$ let

$$u(x) = \operatorname{dist}_\Omega(x, z_1).$$

Then $u \in W^{1,\infty}(\Omega)$ and the L^∞-norm of $|\nabla u|$ is no more than 1. Moreover since Ω is a $W^{1,\infty}$-extension domain, then

$$\operatorname{dist}_\Omega(z_1, z_2) = |Eu(z_1) - Eu(z_2)| \leq \|\nabla Eu\|_{L^\infty(\mathbb{R}^2)}|z_1 - z_2| \leq \|E\||z_1 - z_2|,$$

where E is the extension operator. Thus there exists a curve γ joining z_1 and z_2 whose length is no more than $2\|E\||z_1 - z_2|$.

The sufficiency of quasiconvexity is essentially trivial since functions in $W^{1,\infty}(\Omega)$ are (globally) $C\|\nabla f\|_{L^\infty(\Omega)}$-Lipschitz when Ω is quasiconvex. Then by applying the Lipschitz extension operator we conclude the sufficiency. □

1.2 Case $2 < p < \infty$: Hölder continuity

We denote by $C^{0,\alpha}(A)$ the space of all Hölder continuous functions on $A \subset \mathbb{R}^2$ with power α, equipped with the (semi-)norm

$$\|u\|_{C^{0,\alpha}(A)} = \sup_{x,y \in A, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}. \tag{1.2}$$

Similarly we denote by $Lip_\alpha(\Omega)$ the space of all continuous functions on a domain $\Omega \subset \mathbb{R}^2$ such that for every $u \in Lip_\alpha(\Omega)$ the quantity on the right-hand side of (1.2) is finite with the supremum taken over all disks $B \subset \Omega$ and all $x, y \in B$. A domain $\Omega \subset \mathbb{R}^2$ is called a *Lip_α-extension domain* if it admits a bounded extension operator from $Lip_\alpha(\Omega)$ to $Lip_\alpha(\mathbb{R}^2) = C^{0,\alpha}(\mathbb{R}^2)$.

It is well-known that $W^{1,p}$-functions are Hölder continuous [5, Section 4.5.3]. More precisely for each $2 < p < \infty$ and any disk $B \subset \Omega$ there exists a constant C depending only on p such that

$$|u(x) - u(y)| \le C|x-y|^{1-\frac{2}{p}} \|\nabla u\|_{L^p(B)} \tag{1.3}$$

for all $u \in W^{1,p}(\Omega)$ and $x, y \in B \subset \Omega$; recall that we have identified u with its precise representative. Especially $u \in Lip_\alpha(\Omega)$.

It follows that a $W^{1,p}$-extension domain is necessarily quasiconvex for $2 < p < \infty$. Indeed for fixed $z_1, z_2 \in \Omega$ one defines

$$u(x) = \min\{\operatorname{dist}_\Omega(x, z_1), 2\operatorname{dist}_\Omega(z_1, z_2)\}.$$

Then a straightforward calculation shows that

$$\int_\Omega |\nabla u|^p \, dx \lesssim |B(z_1, 2\operatorname{dist}_\Omega(z_1, z_2))| \lesssim \operatorname{dist}_\Omega(z_1, z_2)^2.$$

On the other hand, by letting E be the extension operator associated to Ω, we conclude from the uniform Hölder continuity that

$$\begin{aligned}\operatorname{dist}_\Omega(z_1, z_2) = |Eu(z_1) - Eu(z_2)| &\le C(p)\|\nabla Eu\|_{L^p(\mathbb{R}^2)}|z_1 - z_2|^{1-\frac{2}{p}} \\ \le C(p, \|E\|)|z_1 - z_2|^{1-\frac{2}{p}} \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} &\le C(p, \|E\|)|x-y|^{1-\frac{2}{p}} \operatorname{dist}_\Omega(z_1, z_2)^{\frac{2}{p}},\end{aligned}$$

which with the assumption $p > 2$ implies

$$\operatorname{dist}_\Omega(z_1, z_2) \lesssim |x - y|.$$

Here the constant depends only on p and the norm of the extension operator.

However quasiconvexity is not sufficient for extendability of $W^{1,p}$-functions with $2 < p < \infty$. There are quasiconvex domains which do not admit a $W^{1,p}$-extension operator for any $2 < p < \infty$; see e.g. [12] for more necessary conditions.

Nevertheless, a heuristic calculation leads to a correct condition. Given a sequence of disks $B_i = B_i(x_i, r_i) \subset \Omega$ joining $z_1, z_2 \in \Omega$, by (1.3) we have via Hölder's inequality

$$\begin{aligned}|u(z_1) - u(z_2)| &\le C\sum_i r_i^{1-\frac{2}{p}} \|\nabla u\|_{L^p(B_i)} \le C\left(\sum_i r_i^{\frac{2-p}{1-p}}\right)^{\frac{p-1}{p}} \left(\sum_i \|\nabla u\|^p_{L^p(B_i)}\right)^{\frac{1}{p}} \\ &\le C\left(\sum_i r_i^{\frac{2-p}{1-p}}\right)^{\frac{p-1}{p}} \|\nabla u\|^p_{L^p(\Omega)}.\end{aligned} \tag{1.4}$$

We would like to obtain global Hölder continuity, which is necessary for the extension, but by concavity of $x \mapsto x^{\frac{2-p}{1-p}}$ we only have

$$\left(\sum_i r_i\right)^{\frac{2-p}{1-p}} \leq \sum_i r_i^{\frac{2-p}{1-p}}$$

If the reversed inequality

$$\sum_i r_i^{\frac{2-p}{1-p}} \leq C' \left(\sum_i r_i\right)^{\frac{2-p}{1-p}} \tag{1.5}$$

were to hold, then by the quasiconvexity (with a little bit more effort) we would obtain from (1.4)

$$|u(z_1) - u(z_2)| \leq C' \left(\sum_i r_i\right)^{\frac{p-2}{p}} \|\nabla u\|^p_{L^p(\Omega)} \leq C''|z_1 - z_2|^{1-\frac{2}{p}} \|\nabla u\|^p_{L^p(\Omega)}$$

as desired. Indeed (1.5) is essentially the required geometric condition.

Theorem 1.4. *Let Ω be a bounded simply connected domain in the plane. Then for $2 < p < \infty$ the following are equivalent.*

1) *Ω is a $W^{1,p}$-extension domain.*

2) *Ω is a $Lip_{1-2/p}$-extension domain.*

3) *$W^{1,p}(\Omega)$ can be continuously embedded into $C^{0,1-2/p}(\overline{\Omega})$.*

4) *there exist a constant C such that, for any two points $x, y \in \Omega$, there is a rectifiable curve $\gamma \subset \Omega$ joining them such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-p}}\, dz \leq C|x-y|^{\frac{2-p}{1-p}}. \tag{1.6}$$

The curve condition in (4), to my best knowledge, appeared first in [7] where it is proven that (2) is equivalent to (4). Motivated by this, later Koskela [17] obtained a weaker version of above theorem. The missing part was completed by Shvartsman [23] via a key observation that (1.6) essentially has self-improvement, namely a version of it holds if we replace p by some $q = q(p) < p$, up to a multiplicative constant depending only on p and the original constant C; a heuristic reason for this comes from the reversed inequality (1.5).

1.3 Case $p = 2$: Quasidisks

The case where $p = 2$ has attracted considerable attention since it has very deep relations with the theory of quasiconformal mappings. Here we list a sequence of related results. For further results and information, we refer to e.g. [6].

Theorem 1.5. *Let Ω be a bounded simply connected domain in the plane. The following are equivalent:*

(1) *Ω is a uniform domain;*

(2) *$\widetilde{\Omega} = \mathbb{R}^2 \setminus \overline{\Omega}$ is a uniform domain;*

(3) *for all $z_1,\, z_2 \in \Omega$, there is a curve $\gamma \subset \Omega$ joining z_1 and z_2 and satisfying*

$$\int_\gamma \operatorname{dist}(z,\, \partial\Omega)^{-1}\, ds(z) \le C \log\left(1 + \frac{|z_1 - z_2|}{\min\{\operatorname{dist}(z_1,\, \partial\Omega),\ \operatorname{dist}(z_2,\, \partial\Omega)\}}\right), \tag{1.7}$$

(4) *there exists a quasiconformal (or even biLipschitz with respect to the spherical metric) reflection between Ω and $\widetilde{\Omega}$;*

(5) *Ω is a $W^{1,2}$-extension domain.*

(6) *Ω is the image of $\mathbb{D}$ under a quasiconformal mapping $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

Towards introducing our method later, we sketch a proof of the implication of (1) to (5) here following [14], under the assumption that Ω is Jordan.

Sketch of the proof: (1) $\Rightarrow$ (5). Apply the Whitney decomposition [24] to Ω and $\widetilde{\Omega}$ respectively. Fix $u \in W^{1,2}(\Omega) \cap C^\infty(\overline{\Omega})$. For each Whitney square $\widetilde{Q} \subset \widetilde{\Omega}$, by the uniformity of the domain Ω we can choose a Whitney square $Q \subset \Omega$ such that

$$\ell(\widetilde{Q}) \sim \ell(Q) \sim \operatorname{dist}(\widetilde{Q},\, Q), \tag{1.8}$$

where the constants depend only on the uniformity constant ϵ of Ω. Then we define

$$Eu(x) = \sum_i a_{Q_i} \varphi_i,$$

where the index i goes over all the Whitney squares in some large disk B containing Ω, a_{Q_i} is the average of u over the Whitney square Q_i associated to $\widetilde{Q}_i$, and $\{\varphi_i\}$ is a standard partition of unity associated to $\{\widetilde{Q}_i\}$ such that each φ_i is compactly supported in $\frac{11}{10}\widetilde{Q}_i$, and

$$|\nabla \varphi_i| \lesssim \ell(\widetilde{Q}_i)^{-1}.$$

Notice that by (1.8) and the geometry of the plane, there are at most $C(\epsilon)$ Whitney squares in $\widetilde{\Omega}$ related to any given $Q \subset \Omega$. Moreover, for any two neighboring Whitney squares $\widetilde{Q},\, \widetilde{Q}' \subset \widetilde{\Omega}$, the corresponding Whitney squares in Ω are within at most $C(\epsilon)\ell(\widetilde{Q})$ (inner) distance. These properties allow us to control $\|\nabla Eu\|_{L^2(B)}$ by $\|\nabla u\|_{L^2(\Omega)}$ via the Poincaré inequality, up to a constant depending only on ϵ. Also $\|Eu\|_{L^2(B)}$ is bounded from above by $C(\epsilon)\|u\|_{L^2(\Omega)}$. It is not difficult to check that we obtain a function of the Sobolev class in B, or see [15] for a general theorem on this. Since B is an extension domain, we conclude (5) by the fact that $W^{1,2}(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$ for any uniform domain Ω, which is also shown in [14]. □

1.4 Case $1 < p < 2$: A generalized version of Jones' reflection

The uniformity condition (1.1) is sufficient but not necessary for a simply connected planar domain to be a $W^{1,p}$-extension domain when $1 < p < 2$; see e.g. [16]. In the first paper [KRZ16] of this thesis we established the following result.

Theorem 1.6. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Ω is a $W^{1,p}$-extension domain if and only if for all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining z_1 and z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \le C(\Omega, p)|z_1 - z_2|^{2-p}. \tag{1.9}$$

The theorem above with Theorem 1.4 implies (with some effort) the following corollary.

Corollary 1.7. *Let $1 < p, q < \infty$ be Hölder dual exponents and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω is a $W^{1,p}$-extension domain if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is a $W^{1,q}$-extension domain.*

Let us now sketch the proof. We first consider the case where Ω is a Jordan domain in the plane, and the general case then follows from an approximation argument with the weak compactness of $W^{1,p}(\Omega)$.

In this case, the idea to show the sufficiency of (1.9) is to modify Jones' reflection. The key difference is that, for a Jordan domain $\widetilde{\Omega}$, after applying Whitney decomposition we do not associate squares as in (1.8), but via the shadows of squares on $\partial\Omega$.

To be precise, let us define the shadow $S(A)$ of a set $A \subset \Omega$. Since Ω is Jordan, a conformal map $\varphi\colon \mathbb{D} \to \Omega$ can be extended homeomorphically up to the boundary by the Caratheodory-Osgood theorem. The images of radial line segments in $\mathbb{D}$ under φ are called hyperbolic rays. Then we define the shadow $S(A)$ of A as the subset of $\partial\Omega$ where the hyperbolic rays crossing A end. It follows from the definition and the homeomorphity of φ that $S(A)$ is connected whenever A is. Indeed S can be regarded as a projection map from Ω to its boundary. In a similar manner we define the shadow of $\widetilde{A} \subset \widetilde{\Omega}$; here the hyperbolic rays are images of radial rays towards ∞ in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ under $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$.

Then mimicking (1.8) for each Whitney square $\widetilde{Q} \subset \widetilde{\Omega}$ we choose a Whitney square $Q \subset \Omega$ such that

$$\operatorname{diam}(S(Q)) \sim \operatorname{diam}(S(\widetilde{Q})) \sim \operatorname{diam}(S(Q) \cap S(\widetilde{Q})) \tag{1.10}$$

with the constant depending only on the constant in (1.9). In the case where Ω is uniform, we in fact obtain the condition (1.8) of Jones' since

$$\operatorname{diam}(S(Q)) \sim \ell(Q) \quad \text{and} \quad \operatorname{diam}(S(\widetilde{Q})) \sim \ell(\widetilde{Q}).$$

However in our case only the former inequality holds, i.e. only for $Q \subset \Omega$, since Ω is a John domain with the John constant depending only on the constant in (1.9) by [22]. This might not hold for Whitney squares in $\widetilde{\Omega}$; see [KRZ16, Lemma 4.3]. Thus we cannot expect either the uniform finiteness of Whitney squares corresponding to a square $Q \subset \Omega$ or the comparable sizes of Q and $\widetilde{Q}$.

Instead, a motivational lemma [KRZ16, Lemma 4.5] indicates that we may control the sizes of all (possible) Whitney squares $\widetilde{Q}_j$ corresponding to Q in the following sense:

$$\sum_{\widetilde{Q}_j} \ell(\widetilde{Q}_j)^{2-p} \lesssim \ell(Q)^{2-p}, \tag{1.11}$$

which essentially comes from (1.9) with considerable effort. Roughly speaking this gives the control for change of variable with respect to the “reflection”, and hints us that, after defining the extension operator E via a suitable partition of unity, we can control the gradient of the extended function by the gradient of the original function, for each function in $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$. If so, then by a density result [20] for bounded Jordan domains in the plane, we obtain the desired extension operator as in the case of Jones’ in the previous subsection.

However (1.11) is not enough for our purpose. The reason is that, for a given pair of Whitney squares $\widetilde{Q}_1$, $\widetilde{Q}_2 \subset \widetilde{\Omega}$ with $\widetilde{Q}_1 \cap \widetilde{Q}_2 \neq \emptyset$, the corresponding Whitney squares Q_1, Q_2 might not be of comparable sizes. Assume that $\ell(Q_1) << \ell(Q_2)$ for our illustration later.

To estimate the difference of the averages, we need lots of intermediate Whitney squares of geometric-type sizes joining Q_1 to Q_2 in Ω, namely their side-lengths form a geometric-type sequence. However neither $\widetilde{Q}_1$ nor $\widetilde{Q}_2$ are associated to these intermediate Whitney squares. Thus (1.11) is too weak; some extra terms should appear in the left-hand side of (1.11), and for the usage of the Hölder inequality later we also need a slightly smaller power than $2 - p$ in (1.11).

To this end, we introduce “fake squares” $\widetilde{F}_j$ inside $\widetilde{\Omega}$ as suitable (in the sense that each $S(\widetilde{F}_j)$ satisfies a property similar to (1.10)) sets corresponding to the intermediate Whitney squares, such that $\widetilde{F}_j \subset \widetilde{Q}_1 \cup \widetilde{Q}_2$ and $\widetilde{Q}_1 \subset \widetilde{F}_j$. These “fake squares” by definition have diameters comparable to their distances to $\partial\Omega$ with a uniform constant. Then by an argument similar to the one for (1.11), with an important observation of Shvartsman [23] we obtain the following key estimate

$$\sum_{\widetilde{F}_j} \operatorname{diam}(\widetilde{F}_j)^{2-s} + \sum_{\widetilde{Q}_j} \ell(\widetilde{Q}_j)^{2-s} \lesssim \ell(Q)^{2-s}, \tag{1.12}$$

where $\widetilde{F}_j$’s and $\widetilde{Q}_j$’s are sets corresponding to Q, and $s = s(p) > p$. This gives us the correct estimate on change of variable with respect to the “reflection”, and the rest follows similarly as in the Jones’ case.

To show the necessity of the condition (1.9), we again first consider the case where Ω is Jordan. First observe that when $1 < p < 2$ the function $\frac{1}{\pi} Arg(z)$ is a $W^{1,p}$-function of the upper half disk. Motivated by this, and the fact that simply connected $W^{1,p}$-extension domains in the plane are necessarily John for $1 < p < 2$ (see e.g. [16, Theorem 6.4], [10, Theorem 3.4], [22, Theorem 4.5] and references therein), we construct an “arctangent-like” function inside Ω; recall that every point on the boundary of a John domain admits a “twisted cone” inside the domain by the definition of John domain. See [KRZ16, Lemma 3.3] for the construction. Then by the extension property of Ω with this test function, we conclude the curve condition (1.9). The case where Ω is simply connected is handled via an approximation argument, in which we apply a version of Jones' "reflection" inside John domains to show an inner $W^{1,p}$-extension property; see [KRZ16, Theorem 3.6] and recall that simply connected John domains are inner uniform [2], [25]; namely uniformity here is with respect to the inner distance, comparing to (1.1).

1.5 Case $p = 1$: Hyperbolic triangulation

This is the endpoint of the whole story, but the most difficult one. To begin with, a result of Burago and Maz'ya [3] implies that a domain, whose complement is quasiconvex, is a BV-extension domain. Later in [18] it was shown that quasiconvexity is also necessary. Then the fact that the complement of a bounded simply connected $W^{1,1}$-extension domain in the plane is necessarily quasiconvex follows as a corollary [18, Corollary 1.2].

Quasiconvexity of the complement does not imply $W^{1,1}$-extendability in general. A typical example is given by a slit disk

$$\Omega = \mathbb{D} \setminus \{(x, 0) \mid x \geq 0\},$$

which has a quasiconvex complement but fails to be a $W^{1,1}$-extension domain. In the third paper [KRZ17] we show that in order to obtain a characterization of $W^{1,1}$-extendability, in addition to the quasiconvexity of the complement, one needs to control the size of the set of self-intersections of the boundary $\partial\Omega$.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Ω is a $W^{1,1}$-extension domain if and only if there exists a constant $C < \infty$ such that for each pair $x, y \in \Omega^c$ of points there exists a curve $\gamma \subset \Omega^c$ connecting x and y with*

$$\int_\gamma \frac{1}{\chi_{\mathbb{R}^2 \setminus \partial\Omega}(z)}\, dz \leq C|x-y|. \tag{1.13}$$

In other words, (1.13) requires that $\ell(\gamma) \leq C|x-y|$ and that $\mathcal{H}^1(\gamma \cap \partial\Omega) = 0$, where $\mathcal{H}^1$ denotes the 1-Hausdorff measure.

This time we cannot construct the extension operator via the standard Whitney decomposition. One of the reasons is that, to deal with overlaps of Whitney squares in the reflection procedure, one usually uses the maximal function operator which preserves the L^p-norm; however it maps L^1-functions to $L^{1,\infty}$-functions.

To deal with this problem, we introduce a hyperbolic triangulation of $\Omega^c = \mathbb{R}^2 \setminus \Omega$. First assume that Ω is Jordan. We apply a dyadic decomposition to $\partial\mathbb{D}$ and then via the (extended) conformal map $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ decompose $\partial\Omega$. Then based on the points on $\partial\Omega$ we decompose $\widetilde{\Omega} = \mathbb{R}^2 \setminus \overline{\Omega}$ into hyperbolic triangles (that is, a closed set whose boundary consists of three hyperbolic geodesics) by joining neighboring points of the same dyadic level on $\partial\Omega$ via hyperbolic geodesics in $\widetilde{\Omega}$. In order to control the length of these geodesics via the Gehring-Hayman theorem inside an unbounded domain, we start our decomposition with a sufficiently high level, and apply the quasiconvexity of $\widetilde{\Omega}$ (or, equivalently, the quasiconvexity of Ω^c). Also we apply a Whitney-type decomposition for Ω via the images of a Whitney-type decomposition of $\mathbb{D}$ under φ.

This time our association is not given with respect to the Whitney-type sets, but between the Whitney-type sets in Ω and the hyperbolic geodesics (which are the boundaries of hyperbolic triangles) in $\widetilde{\Omega}$. Precisely we associate to each hyperbolic geodesic the integral average of a function $u \in C^{\infty}(\overline{\Omega}) \cap W^{1,1}(\Omega)$ on the Whitney-type set in Ω such that the shadows of the sets coincide with each other on $\partial\Omega$. We construct a function in $W^{1,1}$ for each hyperbolic triangle with suitably controlled norm, such that it takes the corresponding value continuously (except the three vertices) on each of the edges; note that the vertices of these hyperbolic triangles are on $\partial\Omega$. Moreover, we extend the function as constant towards infinity; in order to do this some technical treatments are needed. This gives us a function Eu which is continuous except for a countable number of points on $\partial\Omega$, and it can be checked that we obtain a global Sobolev function. Finally a density result in the second paper [KZ16] is applied to define Eu for all $u \in W^{1,1}(\Omega)$.

For the simply connected case, we can neither use the approximation argument since $W^{1,1}(B)$ is not weakly compact for any disk $B \subset \mathbb{R}^2$, nor apply the method above directly since there might be lots of complementary domains. Nevertheless, we can take the limit of hyperbolic geodesics by the Arzelá-Ascoli lemma because of the (uniform) quasiconvexity of $\mathbb{R}^2 \setminus \varphi(\overline{B(0, 1-2^{-n})})$. This motivates us to introduce the concept of "piecewise hyperbolic geodesic", and then similar ideas as above can be applied except for an important issue: The limits of hyperbolic geodesics might overlap with each other, and because of this the limits of hyperbolic triangles might degenerate. To deal with this problem, we study the cases of degeneracy, and check how the overlaps of piecewise hyperbolic geodesics might happen; see [KRZ17, Lemma 5.1, Lemma 5.2, Lemma 5.3]. Finally we use a result in the second paper [KZ16] about the density of $C(\Omega) \cap W^{1,\infty}(\Omega)$ in $W^{1,1}(\Omega)$ when Ω is a bounded simply connected planar domain; see [KRZ17, Section 5.3]. This, with some other techniques, allows us to check that Eu is in the Sobolev class when $u \in C(\Omega) \cap W^{1,\infty}(\Omega)$.

To show the necessity of (1.13), we first use a result in [18] saying that Ω is necessarily quasiconvex. This allows us to define piecewise hyperbolic geodesics between points relatively close to each other. Then via suitable test functions, we show that for each piecewise hyperbolic geodesic $\gamma \subset \Omega^c$

$$\mathcal{H}^1(\gamma \cap \partial\Omega) = 0.$$

In the end the injectivity of piecewise hyperbolic geodesics implies (1.13); see [KRZ17, Lemma 4.1].

2 Density of regular functions in Sobolev spaces

As mentioned in the previous section, density of regular functions is needed for extendability. The second paper [KZ16] was motivated exactly by this.

Let us recall some earlier results. First smooth functions are dense in $W^{1,p}(\Omega)$ for any domain $\Omega \subset \mathbb{R}^2$ with $1 \le p < \infty$. Consequently, if Ω is a $W^{1,p}$-extension domain, then global smooth functions can approximate functions in $W^{1,p}(\Omega)$ with respect to $W^{1,p}(\Omega)$-norm. Indeed we extend $u \in W^{1,p}(\Omega)$ to $Eu \in W^{1,p}(\mathbb{R}^2)$, pick a sequence $v_j \in C^{\infty}(\mathbb{R}^2)$ approximating Eu in $W^{1,p}(\mathbb{R}^2)$-norm and then restrict these v_j to $\overline{\Omega}$.

Recall that Lipschitz domains are extension domains. However, if we would like to approximate Sobolev functions by functions that are smooth up to the boundary, then the Lipschitz condition can be relaxed. For instance, if Ω satisfies a cone condition or the weaker segment condition, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. On the other hand, it is easy to construct domains Ω for which $C^\infty(\overline{\Omega})$ fails to be dense. For example, take Ω to be a slit disk: the unit disk minus a radius. See e.g. [1] for more information.

Lewis gave another condition for density of global smooth functions in [20]. He proved that $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for every $1 < p < \infty$ provided Ω is a planar Jordan domain: the bounded component of $\mathbb{R}^2 \setminus \gamma$, where γ is a Jordan curve. Note that only a topological property is assumed.

More recently, Giacomini and Trebeschi established in [8] density results that especially yield the density of $W^{1,2}(\Omega)$ in $W^{1,p}(\Omega)$ for all $1 \le p < 2$ when Ω is bounded and simply connected. The Helmholtz decomposition of $L^2(\Omega, \mathbb{R}^2)$ was applied to characterize the orthonormal subspaces of certain Sobolev spaces. Thus only the density of $W^{1,2}(\Omega)$ can be obtained by this technique.

We established the following theorem in the second paper.

Theorem 2.1. *If $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain, then $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for any $1 \le p < \infty$. Moreover if Ω is a planar Jordan domain, then $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for any $1 \le p < \infty$.*

The idea of the proof is quite straightforward. Firstly fix a bounded simply connected domain in the plane, and let $\Omega_n = \varphi(\overline{B(0,\, 1-2^{-n})})$, where $\varphi\colon \mathbb{D} \to \Omega$ is conformal. Applying a (radial) dyadic Whitney decomposition to $\mathbb{D}$ up to the n-th level, we obtain a decomposition of $\overline{B(0,\, 1-2^{-n})}$, and by φ we also decompose Ω_n into Whitney-type sets of Ω. We call $\varphi(\overline{B(0,\, 1-2^{-n})} \setminus \overline{B(0,\, 1-2^{-n+1})})$, $n \ge 2$, the boundary layer of Ω_n, consisting of Whitney-type sets R_j.

We next decompose $\Omega \setminus \Omega_n$. The main idea is to regard $\Omega \setminus \Omega_n$ as a "copy" of the boundary layer of Ω_n. To be precise, $\Omega \setminus \Omega_n$ is decomposed into sets S_j such that

$$\mathcal{H}^1(\partial S_j \cap S_{j-1}) \sim \operatorname{diam}(R_j) \sim \mathcal{H}^1(\partial S_j \cap R_j) \tag{2.1}$$

where S_j and R_j are neighboring sets. This can be done because of the geometry of simply connected domains in the plane; see the proof of inequalities (3.4) and (3.5) in [KZ16].

Now we start to construct the approximating sequence. Pick $u \in W^{1,p}(\Omega)$. We may assume that u is smooth and bounded by the classical results; the general case then follows from a diagonal argument. Define v_n on Ω_n as the restriction of u to Ω_n, and then extend v_n to $\Omega \setminus \Omega_n$. Towards this, for each S_j, we associate to it the integral average of u on R_j, and use a partition of unity, coming from the inner distance, to "glue" these values together. It is not difficult to see that $v_n \in W^{1,\infty}(\Omega)$. Additionally, by (2.1) and the Poincaré inequality, one sees that the $W^{1,p}$-norm of v_n is bounded from above by a multiple of the $W^{1,p}$-norm of u on the boundary layer of Ω_n.

Regarding the case where Ω is Jordan, we use a sequence of Lipschitz domains to approximate it from outside with respect to the Hausdorff distance, and apply the method above to the Lipschitz domains (sufficiently close to Ω_n) with a function u defined in Ω.

Then we obtain a sequence of $W^{1,\infty}$-functions inside these Lipschitz domains. Since Lipschitz domains are $W^{1,\infty}$-extension domains, we then obtain a sequence of Lipschitz functions approaching u in $W^{1,p}(\Omega)$. Via standard mollifiers, another diagonal argument gives a desired sequence of smooth functions.

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A GEOMETRIC CHARACTERIZATION OF PLANAR SOBOLEV EXTENSION DOMAINS (LONG VERSION)

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Abstract. We characterize bounded simply connected planar $W^{1,p}$-extension domains for $1 < p < 2$ as those bounded domains $\Omega \subset \mathbb{R}^2$ for which any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected with a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}z \lesssim |z_1 - z_2|^{2-p}.$$

Combined with Shvartsman's characterization of $W^{1,p}$-extension domains for $2 < p < \infty$, we obtain the following duality result: a Jordan domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,p}$-extension domain, $1 < p < \infty$, if and only if the complementary domain $\mathbb{R}^2 \setminus \overline{\Omega}$ is a $W^{1,p/(p-1)}$-extension domain.

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1. Introduction

In this paper we study those planar domains $\Omega \subset \mathbb{R}^2$ for which there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. Here the Sobolev space $W^{1,p}$, $1 \le p \le \infty$, is

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \ : \ \nabla u \in L^p(\Omega, \mathbb{R}^2) \right\},$$

where ∇u denotes the distributional gradient of u. The usual norm in $W^{1,p}(\Omega)$ is $||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}$. More precisely, $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ is an extension operator if there exists a constant $C \ge 1$ so that for every $u \in W^{1,p}(\Omega)$ we have

$$||Eu||_{W^{1,p}(\mathbb{R}^2)} \le C||u||_{W^{1,p}(\Omega)}$$

and $Eu|_\Omega = u$. Notice that we are not assuming the operator E to be linear. However, for $p > 1$ there always exists also a linear extension operator provided that there exists an extension operator, see [15] and also [34]. Finally, a domain $\Omega \subset \mathbb{R}^2$ is called a $W^{1,p}$-extension domain if there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. For example, each Lipschitz domain is a $W^{1,p}$-extension domain for each $1 \le p \le \infty$ by the results of Calderón [6] and Stein [36].

In this paper we prefer to use the homogeneous seminorm $||u||_{L^{1,p}(\Omega)} = ||\nabla u||_{L^p(\Omega)}$. This makes no difference because we only consider domains Ω with bounded (and hence compact) boundary; for such domains one has a bounded (linear) extension operator for the homogeneous seminorms if and only if there is one for the non-homogeneous ones; see [17]. In what follows, the norm in question is always the homogeneous one, even if we happen to refer to it by $||u||_{W^{1,p}(\Omega)}$.

The main result of our paper is the following geometric characterization of simply connected bounded planar $W^{1,p}$-extension domains when $1 < p < 2$.

Theorem 1.1. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Ω is a $W^{1,p}$-extension domain if and only if for all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining z_1 and z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C(\Omega, p)|z_1 - z_2|^{2-p}. \tag{1.1}$$

Both the necessity and sufficiency in Theorem 1.1 are new. Notice that the curve γ above is allowed to touch the boundary of Ω even if the points in question lie outside the closure of Ω. This is crucial: there exist bounded simply connected $W^{1,p}$-extension domains for which $\mathbb{R}^2 \setminus \overline{\Omega}$ has multiple components, see e.g. [20], [7].

When combined with earlier results, Theorem 1.1 essentially completes the search for a geometric characterization of bounded simply connected planar $W^{1,p}$-extension domains. The unbounded case requires extra technical work and it will be discussed elsewhere.

The condition (1.1) on the complement in Theorem 1.1 appears also in the characterization of $W^{1,q}$-extension domains when $2 < q < \infty$. For such domains a characterization using condition (1.1) in the domain itself with the Hölder dual exponent $q/(q-1)$ of q was proved in [35, Theorem 1.2], see also earlier results [5, 21].

Theorem 1.2 (Shvartsman)**.** *Let $2 < q < \infty$ and let Ω be a bounded simply connected planar domain. Then Ω is a $W^{1,q}$ -extension domain if and only if for all $z_1, z_2 \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining z_1 to z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-q}}\, \mathrm{d}s(z) \le C(\Omega, q)|z_1 - z_2|^{\frac{q-2}{q-1}}. \tag{1.2}$$

The above two theorems leave out the case $p = 2$. This is settled by earlier results [12, 13, 14, 18], according to which a bounded simply connected domain is a $W^{1,2}$-extension domain if and only if it is a quasidisk (equivalently, a uniform domain). Since the complementary domain of a Jordan uniform domain is also uniform, one rather easily concludes that a Jordan domain is a $W^{1,2}$-extension domain if and only if the complementary domain is.

Combining our characterization in Theorem 1.1 (also see the remark after Theorem 3.1 and Proposition 4.1 with Lemma 2.11 for the case where $\partial\Omega$ is Jordan) with Shvartsman's characterization stated in Theorem 1.2 (see also Lemma 2.1 for the passage between bounded domains and unbounded domains with bounded boundary, Theorem 3.1, Lemma 4.1 and Lemma 4.12)) we get the following duality result between the extendability of Sobolev functions from a Jordan domain and from its complementary domain.

Corollary 1.3. *Let $1 < p, q < \infty$ be Hölder dual exponents and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω is a $W^{1,p}$-extension domain if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is a $W^{1,q}$-extension domain.*

Corollary 1.3 was hinted by the example in [23] (see also [28, 33]) that exhibits such duality.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $W^{1,p}$-extension domain, where $1 < p \leq 2$. Then there is $q > p$ so that Ω is a $W^{1,s}$-extension domain for all $1 < s < q$.*

The case $1 < p < 2$ follows from Theorem 1.1 together with the fact that (1.1) implies the analogous inequality for all $1 < s < p + \epsilon$. The case of smaller s is essentially just Hölder's inequality, see [26], while the improvement to larger exponents follows from the proof of Proposition 2.6 in [35]; see Lemma 2.11 below. Again, the case $p = 2$ of Corollary 1.4 was already known to hold: one then has extendability for all $1 < s < \infty$.

Combining Corollary 1.4 with results from [21] and [35] we obtain an open-ended property.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $W^{1,p}$-extension domain, where $1 < p < \infty$. Then the set of all $1 < s < \infty$ for which Ω is a $W^{1,s}$-extension domain is an open interval.*

Actually, the open interval above can only be one of $1 < s < \infty$, $1 < s < q$ with $q \leq 2$, or $q < s < \infty$ with $q \geq 2$.

Let us finally comment on some earlier partial results related to Theorem 1.1. First of all, bounded simply connected $W^{1,p}$-extension domains are John domains when $1 \leq p < 2$; see e.g. [20, Theorem 6.4], [13, Theorem 3.4], [29, Theorem 4.5] and references therein. The definition of a John domain is given in Definition 2.12 below. However, there exist John domains that fail to be extension domains and, even after Theorem 1.1 there is no interior geometric characterization available for this range of exponents. Secondly, in [22] it was shown that the complement of a bounded simply connected $W^{1,1}$-extension domain is quasiconvex. This was obtained as a corollary to a characterization of bounded simply connected BV-extension domains. Recall that a set $E \subset \mathbb{R}^2$ is called *quasiconvex* if there exists a constant $C \geq 1$ such that any pair of points $z_1, z_2 \in E$ can be connected to each other with a rectifiable curve $\gamma \subset E$ whose length satisfies $\ell(\gamma) \leq C|z_1 - z_2|$. In [22] it was conjectured that quasiconvexity of the complement holds for every $W^{1,p}$-extension domain when $1 < p \leq 2$. This conjecture follows from our Theorem 1.1, but again, quasiconvexity is a weaker condition than our geometric characterization.

Before going into the proof of Theorem 1.1 in Sections 3 and 4, we fix some notation and recall basic results in Section 2. The necessity of (1.1) is proved in Section 3 by first verifying this condition under the additional assumption that the domain in question is a Jordan domain. The general case is then handled via an approximation argument, for which we need to approximate Ω from inside by Jordan $W^{1,p}$-extension domains. For sufficiency, we again first deal with the Jordan case, and then use a compactness argument to pass to a limit. This is done in Section 4. The crucial point in the proof is the construction of a new version of the Whitney extension technique.

2. Preliminaries

Let us start this section with the following lemma stating that we can always swap an unbounded domain to a bounded domain (and vice versa) with the same extendability and curve properties. This is the minor observation needed to conclude Corollary 1.3 from Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Take $x \in \Omega$ and define an unbounded domain $\hat{\Omega} = i_x(\Omega)$ using the inversion*

$$i_x \colon \mathbb{R}^2 \setminus \{x\} \to \mathbb{R}^2 \setminus \{x\} \colon y \mapsto x + \frac{y-x}{|y-x|^2}.$$

Then

(1) For any $1 \le p \le \infty$ the domain Ω is a $W^{1,p}$-extension domain if and only if $\hat{\Omega}$ is a $W^{1,p}$-extension domain.

(2) For any $1 < p < 2$ the domain Ω has the curves satisfying (1.1) if and only if $\hat{\Omega}$ has them for some constant $C(\hat{\Omega}, p)$.

Proof. Let $R = 2\,\mathrm{diam}\,(\Omega)$ and $2r = \mathrm{dist}\,(x, \partial\Omega)$. Then $\partial\Omega \subset A(x, r, R) := B(x, R) \setminus \overline{B(x, r)}$.

Notice that i_x is a biLipschitz map when restricted to $A(x, r, R)$, with the biLipschitz constant only depending on r and R. Hence for any function $u \in W^{1,p}(\hat{\Omega})$, the pull-backed function $u \circ i_x|_{A(x,r,R)\cap\Omega} \in W^{1,p}(\Omega \setminus B(x, r))$. Since the annulus $A(x, r, 2r) \subset A(x, r, R) \cap \Omega$ is a $W^{1,p}$-extension domains for any $1 \le p \le \infty$, we can extend $u \circ i_x|_{A(x,r,R)\cap\Omega}$ to a function $v \in W^{1,p}(\Omega)$, and then apply the operator E to extend v as a global $W^{1,p}$ function Ev, whose norm is less than the norm of u up to a multiplicative constant.

Next we use the diffeomorphism i_x to push the global function Ev forward to the image side and restrict it on the set $i_x(A(x, r, R))$, namely let $w = Ev \circ i_x^{-1}|_{i_x(A(x,r,R))}$. Again by the biLipschitz property of i_x on $A(x, r, R)$, we know $w \in W^{1,p}(i_x(A(x, r, R)))$, $\|w\|_{W^{1,p}(i_x(A(x,r,R)))}$ is less than the norm of u up to a multiplicative constant, and by definition $w|_{\hat{\Omega}\cap i_x(A(x,r,R))} = u$. Therefore we can additionally define $w(z) = u(z)$ for all $z \in \hat{\Omega}$.

Since complementary domains of disks are also $W^{1,p}$-extension domains for any $1 \le p \le \infty$, we can extend the function w globally to $\mathbb{R}^2$. Note that w coincides with u on $\hat{\Omega}$. Hence finally we construct an extension operator for u with the norm depending only on the norm of E, p, R and r. The other direction follows from a similar argument.

Additionally, the fact that i_x is biLipschitz when restricted to $A(x, r, R)$ and the fact that outside $A(x, r, R)$ and its image one can always connect using curves satisfying (1.1), imply claim (2). Indeed, if Ω has curves satisfying (1.1), then let x_1, x_2 be any two points in $\hat{\Omega}$, and let $z_1 = i_x^{-1}(x_1)$ and $z_2 = i_x^{-1}(x_2)$. Then z_1, $z_2 \in \Omega$. If the curve $\gamma \subset \Omega$ connecting them lies in $A(x, r, R)$, then the bi-Lipschitz property of i_x directly gives the desired inequality for the curve $i_x \circ \gamma$ up to a multiplicative constant depending only on p, r and R.

Next if $z_1,\, z_2 \in A(x,\, r,\, R)$ but the corresponding curve is not contained in $\overline{A(x,\, r,\, R)}$, since $r = \frac{\operatorname{dist}(x, \partial\Omega)}{2}$, then we can replace the part of curve inside $B(x,\, r)$ by the shorter subarc of the circle $S^1(x,\, r)$ connecting the corresponding points on the circle $S^1(x,\, r)$. The new curve that we still denote by γ satisfies inequality (1.1) with a constant that only depends on the original constant and p. The desired inequality for the curve $i_x \circ \gamma$ follows by the argument in the previous case.

The case where $z_1,\, z_2 \in B(x,\, r)$ is trivial, since then $x_1,\, x_2$ are contained in the complement of a disk, and this complement is contained in $\hat{\Omega}$ and far away from $\partial\hat{\Omega}$. The case $z_1 \in B(x,\, r)$ while $z_2 \in A(x,\, r,\, R)$ follows easily from the combination of previous cases, and by symmetry we finish the proof of one direction. The other direction is similar. □

Let us fix some notation. When we make estimates, we often write the constants as positive real numbers $C(\cdot)$ with the paranthesis including all the parameters on which the constant depends. The constant $C(\cdot)$ may vary between appearances, even within a chain of inequalities. By $a \sim b$ we mean that $b/C \le a \le Cb$ for some constant $C \ge 2$. If we need to make the dependence of the constant on the parameters $(\cdot)$ explicit, we write $a \sim_{(\cdot)} b$. The Euclidean distance between two sets $A,\, B \subset \mathbb{R}^2$ is denoted by $\operatorname{dist}(A,\, B)$. By $\mathbb{D}$ we always mean the open unit disk in $\mathbb{R}^2$ and by $\mathbb{S}^1$ its boundary. We call a *dyadic square* in $\mathbb{R}^2$ any set

$$[m_i 2^{-k},\, (m_i+1)2^{-k}] \times [m_j 2^{-k},\, (m_j+1)2^{-k}],$$

where $m_i,\, m_j,\, k \in \mathbb{Z}$. We denote by $\ell(Q)$ the side length of the square Q.

Recall that any open set in $\mathbb{R}^n$ admits a Whitney decomposition; see e.g. [36, Chapter VI].

Lemma 2.2 (Whitney decomposition)**.** *For any $U \subset \mathbb{R}^2$ open there exists a collection $W = \{Q_j\}_{j\in\mathbb{N}}$ of countably many closed dyadic squares such that*
(i) $U = \cup_{j\in\mathbb{N}} Q_j$ and $(Q_k)^\circ \cap (Q_j)^\circ = \emptyset$ for all $j,\, k \in \mathbb{N}$ with $j \ne k$;
(ii) $\ell(Q_k) \le \operatorname{dist}(Q_k,\, \partial\Omega) \le 4\sqrt{2}\ell(Q_k)$;
(iii) $\frac{1}{4}\ell(Q_k) \le \ell(Q_j) \le 4\ell(Q_k)$ whenever $Q_k \cap Q_j \ne \emptyset$.

Definition 2.3. *A bounded connected set $A \subset \Omega \subset \mathbb{R}^2$ is called a λ-Whitney-type set in Ω with some constant $\lambda \ge 1$ if the following holds.*
(i) There exists a disk with radius $\frac{1}{\lambda}\operatorname{diam}(A)$ contained in A;
(ii) $\frac{1}{\lambda}\operatorname{diam}(A) \le \operatorname{dist}(A,\, \partial\Omega) \le \lambda \operatorname{diam}(A)$.

For example, the Whitney squares in Lemma 2.2 are $4\sqrt{2}$-Whitney-type sets. Observe that for a λ-Whitney-type set A in Ω and any $x \in A$, by the triangle inequality we have

$$\operatorname{dist}(A,\, \partial\Omega) \le \operatorname{dist}(x,\, \partial\Omega) \le (1+\lambda)\operatorname{dist}(A,\, \partial\Omega). \tag{2.1}$$

Thus if two λ-Whitney-type set $A_1,\, A_2$ have non-empty intersection, then

$$\operatorname{diam}(A_1) \sim \operatorname{diam}(A_2) \tag{2.2}$$

with the constant depending only on λ. Now let us recall some terminology and results from complex analysis that will be needed in what follows.

Recall that for $z_1,\, z_2 \in \mathbb{D}$, the hyperbolic distance between them is defined to be

$$\operatorname{dist}_h(z_1,\, z_2) = \inf_\gamma \int_\gamma \frac{2}{1-|z|^2}\,|dz|,$$

where the infimum is over all rectifiable curves γ joining z_1 to z_2 in $\mathbb{D}$. Notice that the density above is comparable to $\frac{1}{1-|z|} = \operatorname{dist}(z,\, \partial\mathbb{D})^{-1}$. The hyperbolic geodesics in $\mathbb{D}$ are arcs of (generalized) circles that intersect the unit circle orthogonally, and both the hyperbolic metric and hyperbolic geodesics are preserved under conformal maps. To be precise, if $\varphi\colon \mathbb{D} \to \Omega$ is conformal, then for $x,\, y \in \Omega$

$$\operatorname{dist}_h(x,\, y) = \operatorname{dist}_h(\varphi^{-1}(x),\, \varphi^{-1}(y))$$

by definition. This is independent of the choice of φ since φ is unique modulo a Möbius transformation that maps $\mathbb{D}$ to $\mathbb{D}$, and the hyperbolic distance in $\mathbb{D}$ is invariant under such transformation. The hyperbolic metric in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ is defined via the Möbius transformation $\frac{1}{z}$, and the hyperbolic geodesics in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ are arcs of (generalized) circles that intersect the unit circle orthogonally. Then the associated density is still controlled from above by an absolute constant multiple of $\frac{1}{|z|-1} = \operatorname{dist}(z,\, \partial\mathbb{D})^{-1}$ (and also from below when $z \in B(0,\, 10)$). By the Koebe distortion theorem [1, Theorem 2.10.6], up to a multiplicative constant these density estimates also hold for domains conformally equivalent to $\mathbb{D}$ or $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. One may refer to [1, Chapter 2] for more information.

Observe that for any λ-Whitney-type set A contained in $\mathbb{D}$ or $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, the hyperbolic diameter of A satisfies

$$\operatorname{diam}_h(A) \le C(\lambda), \tag{2.3}$$

as A is connected. Indeed if $A \subset \mathbb{D}$ is a λ-Whitney-type set, then for any two points $z_1,\, z_2 \in A$, the line segment L connecting them is also contained in $\mathbb{D}$ by the convexity of $\mathbb{D}$. Moreover the distance from L to $\partial\mathbb{D}$ is not less than $\operatorname{dist}(A,\, \partial\mathbb{D})$. Then by Definition 2.3 one easily gets that

$$\operatorname{dist}_h(z_1,\, z_2) \le C(\lambda),$$

and hence we get (2.3) for the unit disk. For the complement of the unit disk we just apply the standard Möbius transformation $\frac{1}{z}$.

Moreover (2.3) even holds for any domain Ω conformally equivalent to $\mathbb{D}$ or $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. To see this, for every point x contained in the λ-Whitney-type set A, by definition $B_x = B(x,\, \frac{1}{10\lambda} \operatorname{diam}(A)) \subset\subset \Omega$, and then by applying the $5r$-covering theorem to the collection $\{B_x\}_{x\in A}$, there is a set $I_A \subset A$, consisting of at most countable many points, such that

$$A \subset \bigcup_{y\in I_A} 5B_y \subset\subset \Omega$$

with $\{B_y\}_{y\in I_A}$ pairwise disjoint; indeed this disjointness implies that the cardinality of I_A is controlled by $C(\lambda)$. Moreover this covering together with the connectedness of A further implies that for any points $z,\, w \in A$ there is a curve $\gamma \subset \Omega$ joining them and satisfying

$$\operatorname{dist}(\gamma,\, \partial\Omega) \ge \frac{1}{3\lambda} \operatorname{diam}(A)$$

with

$$\ell(\gamma) \le C(\lambda) \operatorname{diam}(A).$$

Hence the desired control follows.

Further recall that a Jordan curve divides the plane into two domains, the boundary of each of which equals to this curve; we refer to the bounded one as a Jordan domain. Given a Jordan domain Ω and a conformal map $\varphi : \mathbb{D} \to \Omega$ or $f : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$, our map φ extends homeomorphically up to the boundary by the Caratheodory-Osgood theorem [30]. Then the *hyperbolic ray in* Ω, ending at $z \in \partial\Omega$, is the image under φ of the radial ray from the origin to $\varphi^{-1}(z)$, or in $\mathbb{R}^2 \setminus \overline{\Omega}$ the image under f of the radial half-line starting from $\varphi^{-1}(z)$. We sometimes also use the hyperbolic metric in $\widetilde{\Omega} = \mathbb{R}^2 \setminus \overline{\Omega}$ when Ω is Jordan.

Next let us recall the definition of conformal capacity. For a given pair of continua $E,\, F \subset \overline{\Omega} \subset \mathbb{R}^2$, define the *conformal capacity between* E *and* F *in* Ω as

$$\mathrm{Cap}(E,\, F,\, \Omega) = \inf\{\|\nabla u\|^2_{L^2(\Omega)} :\ u \in \Delta(E,\, F)\},$$

where $\Delta(E,\, F)$ denotes the class of all $u \in W^{1,\,2}(\Omega)$ that are continuous in $\Omega \cup E \cup F$ and satisfy $u = 1$ on E, and $u = 0$ on F. By definition, we see that the conformal capacity is increasing with respect to Ω.

For two continua $E, F \subset \overline{\mathbb{D}}$,

$$\frac{\min\{\,\mathrm{diam}\,(E),\ \mathrm{diam}\,(F)\}}{\mathrm{dist}\,(E,\, F)} \geq \delta > 0 \quad \Longrightarrow \quad \mathrm{Cap}(E,\, F,\, \mathbb{D}) \geq C(\delta) > 0, \tag{2.4}$$

and the analogous estimate holds in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. Moreover, for a continuum E and a Jordan domain Ω_1 satisfying

$$E \subset\subset \Omega_1 \subset \Omega,$$

by letting $\delta = \frac{\mathrm{diam}\,(E)}{\mathrm{dist}\,(E,\, \partial\Omega_1)}$ we have

$$C_1(\delta) \leq \mathrm{Cap}(E,\, \partial\Omega_1,\, \Omega) \leq C_2(\delta), \tag{2.5}$$

where $C_1(\delta)$, $C_2(\delta)$ are continuous increasing functions with respect to δ such that

$$\lim_{\delta \to 0^+} C_i(\delta) = 0 \qquad \lim_{\delta \to \infty} C_i(\delta) = \infty,$$

for both $i = 1,\, 2$. The lower bound of (2.5) follows from [37, Theorem 11.7, Theorem 11.9]. For the upper bound, if $0 < \delta < \frac{1}{2}$ we have $E \subset B \subset 2B \subset \Omega_1$ for some suitable ball B, and then by the monotonicity of capacity and [37, Example 7.5] one obtains the upper bound $C(\log(1/\delta))^{-1}$ for some absolute constant $C > 0$. When $\frac{1}{2} \leq \delta < \infty$ one just applies the test function

$$u(x) = \min\left\{1,\, \max\left\{0,\, 1 - \frac{\mathrm{dist}\,(x,\, E)}{\mathrm{dist}\,(E,\, \partial\Omega_1)}\right\}\right\}.$$

A simple calculation shows that

$$\|\nabla u\|^2_{L^2(\Omega)} \lesssim \left(\frac{\mathrm{diam}\,(E) + \mathrm{dist}\,(E,\, \partial\Omega_1)}{\mathrm{dist}\,(E,\, \partial\Omega_1)}\right)^2 \sim (1+\delta)^2.$$

We remark that, while using (2.5), we may not directly apply it to the two continua in question, but to some suitable related sets. For example, when applying (2.5) to the case where Ω_1 is just simply connected, we in fact apply it to a sequence of Jordan domains contained in Ω_1 and approximating Ω_1 (in Hausdorff distance). Since (2.5) depends only on δ, then the desired inequality still holds for E and $\partial\Omega_1$. To conclude, the essence of (2.5) tells us that the capacity of two continua in the plane is comparable to 1 if and only if the corresponding ratio δ is also comparable to 1, with the constants depending on each other.

Moreover recall that conformal capacities are conformally invariant. Here the conformal invariance is in the following sense: For $f\colon \Omega' \to \Omega$ conformal, for two continua $E,\, F \subset \Omega$ and any $u \in \Delta(E,\, F)$ we have $u \circ f \in \Delta(f(E),\, f(F))$, and hence the chain rule and the change of variable give us

$$\mathrm{Cap}(E,\, F,\, \Omega) = \mathrm{Cap}(f(E),\, f(F),\, \Omega).$$

When $E,\, F \subset \overline{\Omega}$, if f can be extended as a homeomorphism to $\overline{\Omega'}$, or $F = \partial\Omega$ (and then $f(F) = f(\partial\Omega)$), we can also deduce a similar result via a truncation argument. Indeed for any $0<\epsilon<\frac{1}{3}$ and a function $u\in W^{1,2}(\Omega)$ that is continuous in $\Omega\cup E\cup F$ and satisfies $u=1$ on E, and $u=0$ on F, we define

$$u_\epsilon(x)=\frac{u-\epsilon}{1-2\epsilon}.$$

Then the two sets $U_\epsilon=\{u_\epsilon>1\}$ and $V_\epsilon=\{u_\epsilon<0\}$ are relatively open with $E\subset U_\epsilon$, $F\subset V_\epsilon$. Moreover $u_\epsilon\to u$ in $W^{1,2}(\Omega)$ as $\epsilon\to 0$. Therefore via a suitable truncation for each u_ϵ, we obtain the claim as in the previous case. See [37, Chapter 1] for more properties. Actually, [37] states these results for "modulus", but "modulus" is equivalent with conformal capacity in our setting below (see e.g. [32, Proposition 10.2, Page 54]).

Define the *inner distance with respect to* Ω between $x,\,y\in\Omega$ by

$$\operatorname{dist}_\Omega(x,\,y)=\inf_{\gamma\subset\Omega}\ell(\gamma),$$

where the infimum runs over all curves joining x and y in Ω. The inner diameter $\operatorname{diam}_\Omega(E)$ of a set $E\subset\Omega$ is then defined in the usual way. We record the following estimate, which states a converse version of (2.4).

Lemma 2.4. *Let $E,\,F\subset\Omega$ be a pair of continua. Then if $\operatorname{Cap}(E,\,F,\,\Omega)\geq c_0$, we have*

$$\min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\}\gtrsim\operatorname{dist}_\Omega(E,\,F),$$

where the constant only depends on c_0. Especially

$$\min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\}\gtrsim\operatorname{dist}(E,\,F),$$

and if $\Omega=\mathbb{R}^2$

$$\min\{\operatorname{diam}(E),\ \operatorname{diam}(F)\}\gtrsim\operatorname{dist}(E,\,F).$$

Proof. We may assume that $\operatorname{diam}_\Omega(E)\leq\operatorname{diam}_\Omega(F)$ and $2\operatorname{diam}_\Omega(E)\leq\operatorname{dist}_\Omega(E,\,F)$. Let $z\in E$, and $\frac{\operatorname{dist}_\Omega(E,F)}{\operatorname{diam}_\Omega(E)}=\delta$. We define

$$f(x)=\begin{cases}1, & \text{if } \operatorname{dist}_\Omega(x,\,z)\leq\operatorname{diam}_\Omega(E)\\ 0, & \text{if} \operatorname{dist}_\Omega(x,\,z)\geq\operatorname{dist}_\Omega(E,\,F)\ .\\ \frac{\log(\operatorname{dist}_\Omega(E,F))-\log(\operatorname{dist}_\Omega(x,z))}{\log(\operatorname{dist}_\Omega(E,F))-\log(\operatorname{diam}_\Omega(E))}, & \text{otherwise}\end{cases}$$

Then a direct calculation via a dyadic annular decomposition with respect to the inner distance gives

$$c_0\leq\int_\Omega|\nabla f|^2\,dx\lesssim(\log\delta)^{-1}.$$

Hence $\delta\leq C(c_0)$, which means that $\operatorname{dist}_\Omega(E,\,F)\lesssim\operatorname{diam}_\Omega(E)$. □

We remark that, even though this lemma is stated in the case $E,\,F\subset\Omega$, it is also true if $E,\,F$ are on the boundary of Ω when Ω is Jordan and $\operatorname{dist}_\Omega(E,\,F)<\infty$. Indeed let $\varphi\colon\overline{\mathbb{D}}\to\overline{\Omega}$ be a homeomorphism given by the Caratheodory-Osgood theorem. Then for $\varphi^{-1}(E)$, $\varphi^{-1}(F)$ we can find a sequence of continua $\varphi^{-1}(E_j)$, $\varphi^{-1}(F_j)\subset\mathbb{D}$ approximating them (in Hausdorff distance). Note that for every two points $z_1,\,z_2\in E$, the hyperbolic geodesic Γ' joining them satisfies (see Lemma 2.7 below with the remark afterward)

$$\ell(\Gamma')\lesssim\operatorname{dist}_\Omega(z_1,\,z_2).$$

For any $w_1,\,w_2\in E_j$, denoting by Γ the hyperbolic geodesic connecting them (and extended to the boundary), when j is large enough we have $\Gamma\cap\partial\Omega\subset E$. Therefore

$$\operatorname{dist}_\Omega(w_1,\,w_2)\leq\ell(\Gamma)\lesssim\operatorname{diam}_\Omega(E).$$

By the arbitrariness of w_1, w_2, we conclude that

$$\operatorname{diam}_\Omega(E_j) \lesssim \operatorname{diam}_\Omega(E)$$

for large enough j. Hence by applying a similar argument to F_j and F, we obtain that

$$\min\{\operatorname{diam}_\Omega(E_j),\ \operatorname{diam}_\Omega(F_j)\} \lesssim \min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\} \tag{2.6}$$

when j is large enough.

Additionally when j is big we also have

$$\operatorname{Cap}(E_j,\ F_j,\ \Omega) \geq \frac{1}{2}c_0,$$

and hence applying Lemma 2.4 and (2.6) to each E_j, F_j we obtain that there is a rectifiable curve γ_j connecting E_j and F_j such that

$$\ell(\gamma_j) \lesssim \min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\},$$

with the constant independent of j. Then by parameterizing γ_j with arc length and applying the Arzelá-Ascoli lemma, we get a curve $\gamma \subset \overline{\Omega}$ joining E, F with the length bounded by $\min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\}$ up to a multiplicative constant as desired.

The following lemma states a distortion property of conformal maps.

Lemma 2.5 ([1], Theorem 2.10.8)**.** *Suppose φ is conformal in U, where U is the unit disk $\mathbb{D}$ or $U = \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and $z,\ w \in U$. Then*

$$\exp\left(-3\operatorname{dist}_h(z,\ w)\right)|\varphi'(w)| \leq |\varphi'(z)| \leq \exp\left(3\operatorname{dist}_h(z,\ w)\right)|\varphi'(w)|.$$

Given a λ-Whitney-type set $A \subset \mathbb{D}$, one has $\operatorname{dist}_h(z,\ w) \leq C(\lambda)$ for all $z,\ w \in A$ by (2.3). Hence Lemma 2.5 implies $|\varphi'(z)| \sim |\varphi'(w)|$ with a constant depending only on λ.

By this (applied to suitable disks), condition (2.4) and the capacity estimate (2.5), one obtains the following well-known property. Also see [8, Theorem 11] for a more general statement.

Lemma 2.6. *Suppose $\varphi\colon \Omega \to \Omega'$ is conformal, where $\Omega, \Omega' \subset \mathbb{R}^2$ are domains so that at least one of them is the unit disk or its complementary domain and $Q \subset \Omega$ is a λ_1-Whitney-type set. Then $\varphi(Q) \subset \Omega'$ is a λ_2-Whitney-type set with $\lambda_2 = \lambda_2(\lambda_1)$.*

Proof. Since Q is a λ_1-Whitney-type set, for every point $x \in Q$ by definition we have

$$\frac{1}{\lambda_1}\operatorname{diam}(Q) \leq \operatorname{dist}(x,\ \partial\Omega) \leq (1+\lambda_1)\operatorname{diam}(Q),$$

and for some $x_0 \in Q$, the ball $B(x_0,\ \frac{1}{\lambda_1}\operatorname{diam}(Q))$ is contained in Q. Therefore for any $x \in Q$, by (2.3) we have

$$\operatorname{dist}_h(x_0,\ x) \leq C(\lambda_1).$$

Then via Lemma 2.5 and the fact that Q is a λ_1-Whitney-type set, using the change of variable we obtain that there exists a ball $B(\varphi(x_0),\ \frac{1}{\lambda_2}\operatorname{diam}(f(Q))) \subset \varphi(Q)$ with $\lambda_2 = \lambda_2(\lambda_1)$. Hence we get (i) in Definition 2.3.

To check the second property, we study the cases separately. If Ω or Ω' is the unit disk, then by (2.5) (with the remark afterward) and the conformal invariance of capacity,

$$1 \sim_{\lambda_1} \operatorname{Cap}(\overline{Q},\ \partial\Omega,\ \Omega) = \operatorname{Cap}(\varphi(\overline{Q}),\ \partial\Omega',\ \Omega'),$$

and therefore $\operatorname{diam}(\varphi(Q)) \sim_{\lambda_1} \operatorname{dist}(\varphi(Q),\ \partial\Omega')$ by (2.5) (with the remark afterward) again. Thus $\varphi(Q)$ is a λ_2-Whitney-type set with $\lambda_2 = \lambda_2(\lambda_1)$.

Next let us discuss the case where Ω or Ω' is the complementary domain of the unit disk; we first assume that $\Omega = \mathbb{R}^2 \setminus \overline{\mathbb{D}}$. If $\operatorname{diam}(Q) \le 4\lambda_1$, then by (2.5) (with the remark afterward) and a similar argument as above we conclude that $\varphi(Q)$ is also a Whitney-type set. Hence we only consider the case where $\operatorname{diam}(Q) > 4\lambda_1$.

By scaling, we then may assume that $\operatorname{diam}(\partial\Omega') = 1$. Then by [25, Pages 254–256] and $\varphi(\infty) = \infty$ we know that $\frac{1}{4} \le |\varphi'(\infty)| \le \frac{1}{2}$. Observe that for any $z, w \in \Omega$ with $|z|, |w| > 4$ we have $\operatorname{dist}_h(z, w) \le 1$. Hence Lemma 2.5 implies that for any $|w| > 4$

$$|\varphi'(w)| \sim 1 \tag{2.7}$$

for some absolute constant. Notice that by Definition 2.3 and the assumption $\operatorname{diam}(Q) \ge 4\lambda_1$, one has $\operatorname{dist}(Q, \partial\mathbb{D}) \ge 4$. Hence (2.7) gives

$$\operatorname{diam}(\varphi(Q)) \sim \operatorname{diam}(Q) \sim \operatorname{dist}(Q, \partial\Omega). \tag{2.8}$$

Then for any λ_1-Whitney-type set $Q \subset \Omega$ with $\operatorname{diam}(Q) \ge 4\lambda_1$, let $\gamma \subset \Omega$ be a line segment connecting Q to a nearest point on $\partial\Omega$. By applying the Jordan curve theorem to $B(0, 4)$, we find a λ_1-Whitney-type set $Q' \subset \Omega$ intersecting γ such that $\operatorname{diam}(Q') = 4\lambda_1$, and then by the conclusion of the paragraph before we have

$$\operatorname{dist}(\partial\Omega', \varphi(Q')) \sim \operatorname{diam}(\varphi(Q')).$$

Hence via change of variable, (2.7) and (2.8) we conclude that

$$\begin{aligned}\operatorname{dist}(\varphi(Q), \partial\Omega') &\le \operatorname{dist}(Q', \partial\Omega) + \operatorname{dist}(Q, Q')\\ &\lesssim 1 + \int_{\gamma\cap\{z \mid |z|\ge 4\}} |\varphi'(w)|\, dw \lesssim \operatorname{dist}(Q, \partial\Omega) \lesssim \operatorname{diam}(\varphi(Q)).\end{aligned}$$

In a similar manner we obtain the other direction of the inequality, and the case where $\Omega' = \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ also follows from a similar argument. Consequently the lemma follows. □

We remark that in the proof of the case where $\Omega = \mathbb{R}^2 \setminus \overline{\mathbb{D}}$, we essentially use the fact that when $\varphi'(\infty) = 1$, the mapping φ behaves almost like the identity near ∞. This philosophy will be used again in what follows.

Sometimes we omit the constant λ when we are dealing with the Whitney squares from Lemma 2.2. We record the following estimates, often called the Gehring-Hayman inequalities.

Lemma 2.7 ([9],[31])**.** *Let $\varphi : \mathbb{D} \to \Omega$ be a conformal map. Given a pair of points $x, y \in \mathbb{D}$, denoting the corresponding hyperbolic geodesic in $\mathbb{D}$ by $\Gamma_{x,y}$, and by $\gamma_{x,y}$ any arc connecting x and y in $\mathbb{D}$, we have*

$$\ell(\varphi(\Gamma_{x,y})) \le C\ell(\varphi(\gamma_{x,y}))$$

and

$$\operatorname{diam}(\varphi(\Gamma_{x,y})) \le C \operatorname{diam}(\varphi(\gamma_{x,y})),$$

where C is an absolute constant.

When Ω is Jordan, Lemma 2.7 also holds for points on the boundary, as one just extends the hyperbolic geodesic to the boundary and argues by approximation. Actually, we also need a version of a step in the proof of Lemma 2.7.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain, and let a homeomorphism $\varphi\colon \mathbb{R}^2 \setminus \mathbb{D} \to \mathbb{R}^2 \setminus \Omega$ be conformal in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. For $z_1 \in \partial\Omega$, define*

$$A(z_1, k) := \{x \in \mathbb{R}^2 \setminus \overline{\mathbb{D}} \mid 2^{k-1} < |x - \varphi^{-1}(z_1)| \le 2^k\},$$

for $k \in \mathbb{Z}$. Furthermore, let $\Gamma \subset \mathbb{R}^2 \setminus \overline{\Omega}$ be the hyperbolic ray corresponding to z_1, let $z_2 \in \Gamma$, and let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be a curve connecting z_1 and z_2. Set

$$\Gamma_k := \varphi(A(z_1, k)) \cap \Gamma$$

when $2^k \le |\varphi^{-1}(z_1) - \varphi^{-1}(z_2)|$ and let γ_k be any subcurve of γ in $\varphi(A(z_1, k))$ joining the inner and outer boundaries of $\varphi(A(z_1, k))$. (Here the inner and outer boundaries of $\varphi(A(z_1, k))$ are the images under φ of the inner and outer boundaries of $A(z_1, k)$.) Then

$$\ell(\Gamma_k) \sim \operatorname{dist}(\Gamma_k, \partial\Omega)$$

and

$$\ell(\gamma_k) \gtrsim \ell(\Gamma_k) \sim \operatorname{diam}(\Gamma_k).$$

Here all the constants are independent of Ω and the choice of $\varphi, z_1, \gamma, z_2, k$.

Proof. The fact that $\ell(\Gamma_k) \sim \operatorname{dist}(\Gamma_k, \partial\Omega) \sim \operatorname{diam}(\Gamma_k)$ follows immediately from Lemma 2.5 and Lemma 2.6, since by definition $\varphi^{-1}(\Gamma_k)$ is contained in a Whitney-type set in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$.

Hence we only need to prove that $\ell(\gamma_k) \gtrsim \ell(\Gamma_k)$. Observe that, since γ_k by definition joins the inner and outer boundaries of $\varphi(A(z_1, k))$, then

$$\ell(\varphi^{-1}(\gamma_k)) \gtrsim \operatorname{diam}(\varphi^{-1}(\Gamma_k)) \sim \operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D}). \tag{2.9}$$

We next argue by case study.
Case 1: $\operatorname{dist}(\varphi^{-1}(\gamma_k), \varphi^{-1}(\Gamma_k)) \ge \frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D})$. By Lemma 2.5, the assumption and the fact that $\varphi^{-1}(\Gamma_k)$ is contained in a Whitney-type set, we know that for any curve γ' joining γ_k and Γ_k, its length satisfies

$$\ell(\gamma') \gtrsim \operatorname{diam}(\Gamma_k),$$

and hence

$$\operatorname{dist}(\gamma_k, \Gamma_k) \gtrsim \operatorname{diam}(\Gamma_k). \tag{2.10}$$

Moreover by (2.4) for the exterior of the unit disk, (2.9) and the monotonicity of the capacity we obtain

$$1 \lesssim \operatorname{Cap}(\varphi^{-1}(\overline{\gamma}_k), \varphi^{-1}(\overline{\Gamma}_k), \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \operatorname{Cap}(\overline{\gamma}_k, \overline{\Gamma}_k, \mathbb{R}^2 \setminus \overline{\Omega}) \le \operatorname{Cap}(\overline{\gamma}_k, \overline{\Gamma}_k, \mathbb{R}^2).$$

Hence by (2.10) and Lemma 2.4 we know that

$$\ell(\gamma_k) \ge \operatorname{diam}(\gamma_k) \gtrsim \operatorname{diam}(\Gamma_k) \sim \ell(\Gamma_k).$$

Case 2: $\operatorname{dist}(\varphi^{-1}(\gamma_k), \partial\mathbb{D}) \ge \frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D})$. This assumption implies that the set $\varphi^{-1}(\gamma_k) \cup \varphi^{-1}(\Gamma_k)$ is contained in a Whitney-type set. Then γ_k is also contained in a Whitney-type set by Lemma 2.6, and then the desired estimate follows directly from Lemma 2.5 and (2.9).
Case 3:

$$\operatorname{dist}(\varphi^{-1}(\gamma_k), \varphi^{-1}(\Gamma_k)) < \frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D})$$

and

$$\operatorname{dist}(\varphi^{-1}(\gamma_k), \partial\mathbb{D}) < \frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D}).$$

In this case, by assumption there is a subcurve $\widetilde{\gamma}_k \subset \gamma_k$ such that $\ell(\varphi^{-1}(\widetilde{\gamma}_k)) \gtrsim \ell(\varphi^{-1}(\Gamma_k))$ and $\operatorname{dist}(\varphi^{-1}(\widetilde{\gamma}_k), \partial\mathbb{D}) \gtrsim \operatorname{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D})$, as γ_k is a (connected) curve. Then we are reduced to a case similar to the second one, and it follows that

$$\ell(\gamma_k) \gtrsim \ell(\widetilde{\gamma}_k) \ge \operatorname{diam}(\widetilde{\gamma}_k) \gtrsim \operatorname{diam}(\Gamma_k) \sim \ell(\Gamma_k).$$

Consequently we obtain the desired estimate. □

One can also apply Lemma 2.8 to prove a version of Lemma 2.7 for hyperbolic rays in the exterior of the unit disk. We record another similar result, see [31, Corollary 4.18] and [4, Proof of Theorem 3.1, Page 645].

Lemma 2.9. *Let $\varphi : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal map, where Ω is a Jordan domain. Let $z_0 \in \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and let I be an arc of $\partial\mathbb{D}$ with*

$$\ell(I) \geq \delta(|z_0| - 1)$$

and

$$\mathrm{dist}\,(I, z_0) \leq \frac{|z_0| - 1}{\delta}.$$

Then there is a curve $\alpha \subset \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ joining z_0 to I so that

$$\ell(\varphi(\alpha)) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0),\, \partial\Omega).$$

Moreover, given a Whitney square $Q \subset \mathbb{R}^2 \setminus \overline{\Omega}$ with

$$\ell(Q) \leq \frac{\mathrm{diam}\,(\Omega)}{\delta},$$

there is a hyperbolic ray (i.e. radial half-line) Γ_w starting at some $w \in \partial\mathbb{D}$ so that $\Gamma_w \cap \varphi^{-1}(Q) \neq \emptyset$ and

$$\ell(\varphi([w, y])) \leq C(\delta)\,\mathrm{diam}\,(Q)$$

for the arc $[w, y]$ of Γ_w between w and the last point y where Γ_w intersects $\varphi^{-1}(Q)$. Above $C(\delta)$ is independent of φ, z_0, Ω, Q.

Proof. We begin with the first claim. We may assume that $\ell(I) \leq (|z_0| - 1)$ and that I is closed. Let w be the midpoint of I and set $z = |z_0|w$. Then $|z - z_0| \leq (1 + \frac{1}{\delta})(|z_0| - 1)$ and $|z| - 1 = |z_0| - 1$. It follows from this and the geometry of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ that we can join z to z_0 via a chain of no more than $M(\delta)$ Whitney squares. By Lemma 2.5 we conclude that $|\varphi'(z)| \sim |\varphi'(z_0)|$ with a constant only depending on $M(\delta)$ and that this estimate also holds in the union of the squares in our chain. Noticing that the diameter of each of the above Whitney squares is no more than $C(1 - |z_0|)$ with C only depending on $M(\delta)$, we conclude that there is a curve β that joins $\varphi(z)$ to $\varphi(z_0)$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ and satisfies $\ell(\beta) \leq C(\delta)|\varphi'(z)|(1 - |z_0|)$. By the Koebe distortion theorem [1, Theorem 2.10.6] we conclude that $\ell(\beta) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0), \partial\Omega)$.

We proceed to show that we may join z to I with a suitable curve. Towards this end, define $I_t = \{t\xi :\ \xi \in I\}$ when $1 < t \leq |z|$. From the argument in the first paragraph we see that $\ell(\varphi(I_{|z|})) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0), \partial\Omega)$. Write $G = \mathbb{R}^2 \setminus \overline{\Omega}$. According to (2.4) (for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$),

$$C_0(\delta) \leq \mathrm{Cap}(I_t,\, I_{|z|},\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \mathrm{Cap}(\varphi(I_t),\, \varphi(I_{|z|}),\, G).$$

Then by Lemma 2.4 and the above estimate on the length of $\varphi(I_{|z|})$, we conclude that

$$\mathrm{dist}_G(\varphi(I_t),\, \varphi(I_{|z|})) \lesssim \ell(\varphi(I_{|z|}) \sim \mathrm{dist}\,(\varphi(z_0),\, \partial\Omega).$$

Hence, for every $1 < t < |z|$, we obtain a curve β_t joining $I_{|z|}$ to I_t so that $\ell(\varphi(\beta_t)) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0),\, \partial\Omega)$. Since $\ell(\varphi(I_{|z|})) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0),\, \partial\Omega)$, by the triangle inequality we may assume that β_t contains points $w_t \in I_t$ and $z_t \in I_{|z|}$ that lie on the same hyperbolic ray. Lemma 2.8 (together with its proof) now shows that

$$\ell(\varphi(\Gamma_{|z|,t})) \leq C(\delta)\,\mathrm{dist}\,(\varphi(z_0),\, \partial\Omega),$$

where $\Gamma_{|z|,t}$ is the corresponding (hyperbolic) segment. The Arzelá-Ascoli theorem now gives us the desired curve α.

For the second claim, notice first that

$$C_0(\delta) \leq \mathrm{Cap}(Q,\, \partial\Omega,\, G) = \mathrm{Cap}(\varphi^{-1}(Q),\, \partial\mathbb{D},\, \mathbb{R}^2 \setminus \overline{\mathbb{D}})$$

by (2.5). Hence Lemma 2.4 shows that $\mathrm{dist}\,(\varphi^{-1}(Q), \partial\mathbb{D})$ is bounded from above by a constant depending on δ. Let $I \subset \partial\mathbb{D}$ consist of those w for which the corresponding hyperbolic ray intersects $\varphi^{-1}(Q)$. Then I is a closed arc and since $\varphi^{-1}(Q)$ is a Whitney-type set and of diameter absolutely bounded from above in terms of δ, it follows that the diameter of I is comparable (with constants only depending on δ) to both the diameter of $\varphi^{-1}(Q)$ and to the distance between these two sets. Hence we may repeat the argument at the end of the proof of the first claim to obtain the second claim. $\square$

We record a consequence of (1.1) from [26], also see the proof of [10, Theorem 2.15].

Lemma 2.10. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Suppose that $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ and $\gamma \subset \mathbb{R}^2 \setminus \Omega$ is a curve joining z_1, z_2 with*

$$\int_\gamma \mathrm{dist}\,(z, \partial\Omega)^{1-p}\,\mathrm{d}s(z) \leq C|z_1 - z_2|^{2-p}.$$

Then $\ell(\gamma) \leq C'|z_1 - z_2|$, where C' depends only on p, C.

The following self-improving property of (1.1) can be established by the proof of [35, Proposition 2.6].

Lemma 2.11. *Inequality* (1.1) *implies the analogous inequality for all $1 < s < p + \epsilon$ with $\epsilon > 0$ that only depends on p and the constant $C(\Omega, p)$ in* (1.1)*. Namely, for all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \widetilde{\Omega}$ joining z_1 and z_2 such that*

$$\int_\gamma \mathrm{dist}\,(z, \partial\Omega)^{1-s}\,\mathrm{d}s(z) \leq C|z_1 - z_2|^{2-s}.$$

Proof. Fix $z_1,\, z_2 \in \mathbb{R}^2 \setminus \Omega$ and pick a sequence of curves γ_j joining z_1, z_2 in $\mathbb{R}^2 \setminus \Omega$ that minimizes the integral in (1.1). Then Lemma 2.10 permits us to use the Arzela-Ascoli lemma to conclude the existence of a minimizer for (1.1); the limiting argument is a special case of the reasoning in Section 3.3 below. This observation allows one to employ the argument of the proof of [35, Proposition 2.6]. Indeed, the essential condition for the proof of [35, Proposition 2.6] is that any subcurve of a minimizer also satisfies (1.2) for its end points with a uniform constant. If a minimizer $\gamma \subset \mathbb{R}^2 \setminus \Omega$ exists for any given two points z_1, z_2 (which will be shown this soon), then for any $w_1,\, w_2 \in \gamma$, the subcurve of γ between them is also a minimizer for them; otherwise by changing the subcurve to a corresponding minimizer for w_1, w_2 we obtain a new curve γ' joining z_1, z_2 with a smaller totally integral because of the linearity of the integral. This contradicts the minimality assumption on γ.

Then let us show the existence of a minimizer. Let γ_j be a sequence of curves joining z_1 and z_2, whose corresponding constants c_j in (1.1) converge to the minimal constant c. Then this condition ensures that

$$\ell(\gamma_j) \leq C|z_1 - z_2| =: M$$

by Lemma 2.10. Parametrize each γ_j by arc length, $\gamma_j \colon [0,\, \ell(\gamma_j)] \to \mathbb{R}^2 \setminus \Omega$, starting from z_1, and extend γ_n to $[\ell(\gamma_j),\, M]$ as $\gamma_j(t) = z_2$. Notice that $\gamma_j \subset \overline{B}(z_1,\, M)$, and therefore by the Arzelá-Ascoli lemma we obtain a 1-Lipschitz parametrized curve $\gamma\colon [0,\, M] \to \overline{B}(z_1,\, M) \setminus \Omega$

such that a subsequence of $\{\gamma_j\}$ converges to γ uniformly. Then γ is a curve connecting z_1 and z_2.

Fix $m \in \mathbb{N}$ and $\epsilon > 0$. For $z \in \mathbb{R}^2$ set

$$\omega^{(m)}(z) = \min\{m,\ \mathrm{dist}\,(z,\,\Omega)^{1-p}\}.$$

Then $\omega^{(m)}(z)$ is continuous and bounded.

Let us now show that

$$\int_\gamma \mathrm{dist}\,(z,\,\partial\Omega)^{1-p}\,ds \le c|z_1 - z_2|^{2-p}. \tag{2.11}$$

To this end, since $M < +\infty$, up to choosing a subsequence and redefining γ, we may assume that $\ell(\gamma_j)$ converges to M as $n \to \infty$. Therefore for j large enough, by Fatou's lemma we have

$$\begin{aligned}
&\int_0^{M-\epsilon} \omega^{(m)} \circ \gamma(t)|\gamma'(t)|\,dt \le \int_0^{M-\epsilon} \omega^{(m)} \circ \gamma(t)\,dt \\
\le &\int_0^{M-\epsilon} \lim_{j\to\infty} \omega^{(m)} \circ \gamma_j(t)\,dt \le \liminf_{j\to\infty} \int_0^{M-\epsilon} \omega^{(m)} \circ \gamma_j(t)\,dt \\
\le &\liminf_{j\to\infty} \int_0^{\ell(\gamma_j)} \omega^{(m)} \circ \gamma_j(t)\,dt \le \liminf_{j\to\infty} \int_{\gamma_j} \omega^{(m)}(z)\,dz \\
\le &\liminf_{j\to\infty} \int_{\gamma_j} \mathrm{dist}\,(z,\,\Omega)^{1-p}\,dt \le c|z_1 - z_2|^{2-p},
\end{aligned}$$

where we used the fact that

$$\lim_{j\to\infty} \omega^{(m)} \circ \gamma_j(t) = \omega^{(m)} \circ \gamma(t)$$

when $t \in [0,\, M - \epsilon]$. Letting $\epsilon \to 0$ we obtain

$$\int_\gamma \omega^{(m)}(z)\,dz \le c|z_1 - z_2|^{2-p},$$

and by the monotone convergence theorem we finally get (2.11). □

Finally, let us recall a few things about John domains.

Definition 2.12 (John domain). *An open bounded subset $\Omega \subset \mathbb{R}^2$ is called a John domain provided it satisfies the following condition: There exist a distinguished point $x_0 \in \Omega$ and a constant $J > 0$ such that, for every $x \in \Omega$, there is a curve $\gamma : [0,\, l(\gamma)] \to \Omega$ parameterized by arc length, such that $\gamma(0) = x$, $\gamma(l(\gamma)) = x_0$ and*

$$\mathrm{dist}\,(\gamma(t),\, \mathbb{R}^2 \setminus \Omega) \ge Jt.$$

The curve γ is called a John curve.

We further need the following results from [29], see [29, Theorem 2.18, Theorem 4.5]; see Lemma 2.10 for the comment regarding (1.1).

Lemma 2.13. *A bounded simply connected planar domain Ω whose complement is quasiconvex, especially if the complement satisfies* (1.1), *is a John domain, where the John constant J only depends on the constant in quasiconvexity or the constant in* (1.1). *Moreover, each* *simply connected John domain* Ω *is finitely connected along its boundary, we may use hyperbolic geodesics to the base point as John curves, and any given pair of points* z_1, z_2 *in* $\mathbb{R}^2 \setminus \Omega$ *can be joined by a curve* $\gamma \subset \mathbb{R}^2 \setminus \Omega$ *with* $\operatorname{diam}(\gamma) \leq C(J)|z_1 - z_2|$.

We remark that, for a simply connected planar John domain Ω with the base point x_0 and a point $y \in \partial\Omega$, the hyperbolic geodesic γ connecting x_0 and y is also a John curve by Lemma 2.13 and the definition of a geodesic.

We say that a homeomorphism $\varphi\colon \mathbb{D} \to \Omega$ is *quasisymmetric with respect to the inner distance* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$|z - x| \leq t|y - x| \text{ implies } \operatorname{dist}_\Omega(\varphi(z), \varphi(x)) \leq \eta(t) \operatorname{dist}_\Omega(\varphi(y), \varphi(x))$$

for each triple z, x, y of points in $\mathbb{D}$. It is clear from the definition that the inverse of a quasisymmetric map is also quasisymmetric. Roughly speaking the homeomorphism φ maps round objects to round objects (with respect to the inner distance).

Lemma 2.14 ([16], Theorem 3.1)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected domain, and* $\varphi\colon \mathbb{D} \to \Omega$ *be a conformal map. Then* Ω *is John if and only if* φ *is quasisymmetric with respect to the inner distance. This statement is quantitative in the sense that the John constant and the function* η *in quasisymmetry depend only on each other and* $\operatorname{diam}(\Omega)/\operatorname{dist}(\varphi(0), \partial\Omega)$. *Especially, if* Ω *is John with constant* J *and* $\varphi(0) = x_0$*, where* x_0 *is the distinguished point, then, for any disk* $B \subset \mathbb{D}$*,* $f(B)$ *is a John domain with the John constant only depending on* J.

To be precise, the definition of quasisymmetry with respect to the inner distance in [16] is based on another version of the inner distance, where $\ell(\gamma)$ is replaced by $\operatorname{diam}(\gamma)$. By Lemma 2.7 one may replace the arcs γ in these definitions by hyperbolic geodesics. If our simply connected domain Ω is John with constant J it then follows from [11, Theorem 5.14] that these two distances are comparable modulo a multiplicative constant that only depends on J. Conversely, if φ is quasisymmetric in our sense, then it easily follows from the definitions and Lemma 2.7 that Ω is John with a constant J that only depends on the quasisymmetry function η. Hence the two distances are again comparable. Thus the statement of Lemma 2.14 holds also under our definition of quasisymmetry.

Let us give an example of the usage of quasisymmetry. Suppose that $\varphi\colon \mathbb{D} \to \Omega$ is a conformal map from the unit disk onto a J-John domain with $\varphi(0) = x_0$ the John center of Ω, and $Q \subset \mathbb{D}$ is a simply connected J'-John domain with John center z_0. We claim that $Q' = \varphi(Q)$ is a John domain with constant depending only on J, J'.

Fix $w \in Q'$ and let $\varphi^{-1}(w) = x \in Q$. Then by Lemma 2.13 the hyperbolic geodesic (with respect to Q) $\Gamma \subset Q$ connecting z_0 to x satisfies that, for every $y \in \Gamma$

$$\operatorname{dist}(y, \partial Q) \geq C(J')|x - y|.$$

Now for $\varphi(y) \in \varphi(\Gamma)$, take $z \in \partial Q'$ such that

$$\operatorname{dist}(\varphi(y), \partial Q') = |z - \varphi(y)|.$$

Then

$$C(J')|x - y| \leq \operatorname{dist}(y, \partial Q) \leq |\varphi^{-1}(z) - y|.$$

As φ is quasisymmetric with respect to the inner distance, we have

$$\operatorname{dist}(\varphi(y), \partial Q') = |z - \varphi(y)| = \operatorname{dist}_\Omega(\varphi(y), z) \geq (\eta(1/C(J')))^{-1} \operatorname{dist}_\Omega(w, \varphi(y)).$$

Notice that $\varphi(\Gamma)$ is a hyperbolic geodesic of Q' since φ is conformal. Then by Lemma 2.7 the length of $\varphi(\Gamma)$ between w and $\varphi(y)$ is comparable to $\operatorname{dist}_{\Omega}(w, \varphi(y))$ with an absolute constant. Hence our claim follows.

3. Proof of necessity

In this section we prove that a bounded simply connected planar $W^{1,p}$-extension domain necessarily has the property that any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected with a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying

$$\int_{\gamma} \operatorname{dist}(z, \partial\Omega)^{1-p}\, dz \le C(||E||, p)|z_1 - z_2|^{2-p}.$$

We will first consider the case where Ω is additionally assumed to be Jordan. Under this assumption, we usually denote by $\widetilde{\Omega}$ its complementary domain.

Theorem 3.1. *Let $1 < p < 2$ and let Ω be a Jordan domain. Suppose that there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. Then, given z_1, $z_2 \in \widetilde{\Omega} \cup \partial\Omega$, there is a curve $\gamma \subset \widetilde{\Omega} \cup \partial\Omega$ so that*

$$\int_{\gamma} \operatorname{dist}(z, \partial\Omega)^{1-p}\, dz \le C(||E||, p)|z_1 - z_2|^{2-p}. \tag{3.1}$$

where $C(||E||, p)$ depends only on p and the norm of the extension operator.

After this, based on inner uniformity (see Definition 3.7 below), we prove that, if Ω is a bounded simply connected $W^{1,p}$-extension domain, then, for $n \ge 2$, the Jordan domains $\Omega_n = \varphi(B(0, 1 - \frac{1}{n}))$ are also $W^{1,p}$-extension domains with extension operator norms only depending on p and the norm of the extension operator for Ω. Here $\varphi\colon \mathbb{D} \to \Omega$ is a conformal map. Finally by approximation and a limiting argument we obtain the result for the general case.

We remark that, actually when z_1, $z_2 \in \widetilde{\Omega}$ one can require that the curve γ in Theorem 3.1 is contained in $\widetilde{\Omega}$. For this see the comment after the proof of Theorem 3.1.

3.1. Necessity in the Jordan case.

In this section we will prove Theorem 3.1. Recall that the existence of our extension operator guarantees that Ω is a John domain with a constant J only depending on p and the norm of E. In what follows, J refers to this constant. For notational simplicity, we first consider the case z_1, $z_2 \in \partial\widetilde{\Omega} = \partial\Omega$.

Since Ω is Jordan, the two points z_1, z_2 separate the boundary into two open curves P_1 and P_2. Without loss of generality we assume that $\operatorname{diam}(P_1) \le \operatorname{diam}(P_2)$. For the following four lemmas let Ω, z_1, z_2, P_1 and P_2 be fixed.

We need the following general lower bound on the Sobolev norm.

Lemma 3.2. *Let Q be a square, $1 \le p < 2$ and let $u \in W^{1,1}(Q)$ be absolutely continuous on almost all lines parallel to the coordinate axes. Write*

$$A_0 = \{x \in Q \mid u(x) \le 0\} \quad \text{and} \quad A_1 = \{x \in Q \mid u(x) \ge 1\}.$$

Suppose further that

$$\max\{\mathscr{H}^1(\pi_1(A_0)), \mathscr{H}^1(\pi_2(A_0))\} \ge \delta\ell(Q)$$

and

$$\max\{\mathscr{H}^1(\pi_1(A_1)), \mathscr{H}^1(\pi_2(A_1))\} \ge \delta\ell(Q)$$

for some $\delta > 0$, where the notation $\mathscr{H}^1$ means the 1-dimensional Hausdorff measure, and π_i stands for the projection to the x_i-axis for each $i = 1, 2$. Then

$$\int_Q |\nabla u|^p \, dx \geq C(\delta, p)\ell(Q)^{2-p}.$$

Proof. Assume first that $\mathscr{H}^1(\pi_1(A_0)) \geq \delta\ell(Q)$ and $\mathscr{H}^1(\pi_1(A_1)) \geq \delta\ell(Q)$. If for $\mathscr{H}^1$−almost every $x_1 \in \pi_1(A_0)$, there exists some $x_2 \in \pi_2(Q)$ such that $u(x_1, x_2) \geq \frac{1}{3}$, then

$$\frac{1}{3} \leq \int_{\pi_2(Q)} |\nabla u(x_1, t)| \, dt \leq \ell(Q)^{\frac{p-1}{p}} \left(\int_{\pi_2(Q)} |\nabla u(x_1, t)|^p \, dt \right)^{\frac{1}{p}}$$

for $\mathscr{H}^1$−almost every $x_1 \in \pi_1(A_0)$, and our claim follows by Fubini's theorem. Similarly, the claim holds if for $\mathscr{H}^1$−almost every $x_1 \in \pi_1(A_1)$, there exists $x_2 \in \pi_2(Q)$ such that $u(x_1, x_2) \leq \frac{2}{3}$. If both of the above two conditions fail, we find $x_1 \in \pi_1(A_0)$ and $\hat{x}_1 \in \pi_1(A_1)$ such that for all $x_2 \in \pi_2(Q)$, $u(x_1, x_2) \leq \frac{1}{3}$ and $u(\hat{x}_1, x_2) \geq \frac{2}{3}$. Then the claim again follows by using the fundamental theorem of calculus, Hölder's inequality and the Fubini's theorem.

If $\mathscr{H}^1(\pi_2(A_0)) \geq \delta\ell(Q)$ and $\mathscr{H}^1(\pi_2(A_1)) \geq \delta\ell(Q)$, the argument from the previous paragraph applies with obvious modifications. We are left with the cases where

$$\mathscr{H}^1(\pi_1(A_0)) \geq \delta\ell(Q) \quad \text{and} \quad \mathscr{H}^1(\pi_2(A_1)) \geq \delta\ell(Q)$$

and

$$\mathscr{H}^1(\pi_2(A_0)) \geq \delta\ell(Q) \quad \text{and} \quad \mathscr{H}^1(\pi_1(A_1)) \geq \delta\ell(Q).$$

By symmetry, it suffices to consider the first one. As above, we get reduced to the case in which there exist $x_1 \in \pi_1(A_0)$ and $x_2 \in \pi_2(A_1)$ such that for all $t \in \pi_2(Q)$ and $s \in \pi_1(Q)$, $u(x_1, t) \leq \frac{1}{3}$ and $u(s, x_2) \geq \frac{2}{3}$. Since u is absolutely continuous along these two line segments, this is impossible as these segments intersect. □

Now we are ready to state the existence of a suitable test function. We remark again that the two curves P_1 and P_2 are open.

Lemma 3.3. *Let $c_1 \geq 1$. With the above notation, there exists a function $\Phi \in W^{1,p}(\Omega)$ such that for any $0 < \epsilon < \frac{1}{9}$, we have $\Phi \geq 1 - \epsilon$ in some neighborhood of $P_1 \cap B(z_1, c_1|z_2 - z_1|)$, $\Phi \leq \epsilon$ in some neighborhood of $P_2 \cap B(z_1, c_1|z_2 - z_1|)$, and*

$$\|\nabla\Phi\|^p_{L^p(\Omega)} \leq C(p, c_1, J)|z_1 - z_2|^{2-p}.$$

Here the neighborhoods are defined with respect to the topology of $\overline{\Omega}$.

Proof. Recall that a bounded $W^{1,p}$-extension domain Ω is a John domain with a constant only depending on p and the norm of the extension operator when $1 < p < 2$. Let J be the John constant and x_0 the distinguished point as in Definition 2.12. Denote by γ_1 a John curve connecting x_0 and z_1. By Lemma 2.13 we may assume that γ_1 is a hyperbolic geodesic. Similarly we define γ_2 for x_0 and z_2, and let $\gamma_0 = \gamma_1 \cup \gamma_2$. The existence of John curves is actually only guaranteed by the definition for points inside the domain, but the general case follows easily from this; see Lemma 2.13 and the remark after it for our setting. Observe that P_1 and γ_0 give a Jordan subdomain $\Omega_1 \subset \Omega$. Indeed let $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism which is conformal inside and $\varphi(0) = x_0$. Then it is clear that the preimages of γ_1 and γ_2 under φ are radial line segments, and $\varphi^{-1}(P_1 \cup \gamma_0)$ is a Jordan curve. Hence $P_1 \cup \gamma_0$ is also Jordan as φ is a homeomorphism.

Define a function $\phi\colon \Omega \to \mathbb{R}$ by setting

$$\phi(x) = \max\left\{ \inf_{\gamma(x,\,P_2)} \int_{\gamma(x,\,P_2)} \frac{1}{|z-z_1|}\,dz,\ \inf_{\gamma(x,\,P_2)} \int_{\gamma(x,\,P_2)} \frac{1}{|z-z_2|}\,dz \right\},$$

for $x \in \Omega$, where the infima are taken over all the rectifiable curves $\gamma(x,\,P_2) \subset \Omega$ from x to some point of P_2. We may define $\phi(x) = 0$ for $x \in P_2$.

Since Ω is a Jordan domain, γ_0, P_1 and P_2 are disjoint. From the John condition we have for every $w \in \gamma_1$,

$$\mathrm{dist}\,(w,\,\partial\Omega) \geq J|w - z_1|.$$

Therefore for $w \in \gamma_1$, we get

$$\phi(w) \geq \inf_{\gamma(w,\,P_2)} \int_{\gamma(w,\,P_2)} \frac{1}{|z_1 - z|}\,dz \geq \frac{\mathrm{dist}\,(w,\,\partial\Omega)}{\mathrm{dist}\,(w,\,\partial\Omega) + |w - z_1|} \geq \frac{J}{J+1} =: c_0,$$

where we have used the triangle inequality, and the fact that $\gamma(w,\,P_2)$ necessarily exits $B(w,\,\mathrm{dist}\,(w,\,\partial\Omega))$. The same estimate follows similarly for $w \in \gamma_2$. Hence for any point $w \in \Omega_1$, we have $\phi(w) \geq c_0$ as Ω_1 is Jordan and P_2 is outside Ω_1; any curve $\gamma(x,\,P_2) \subset \Omega$ must cross γ_0 by the Jordan curve theorem.

Fix $0 < \epsilon < \frac{1}{9}$. We claim that we have $\phi \leq \epsilon$ in some neighborhood of P_2. Indeed for any $x \in P_2$ there is a radius $r_x > 0$ such that $B(x,\,3r_x) \cap P_1 = \emptyset$. Then for any $y \in B(x,\,r_x) \cap \Omega$ there is a point $z \in P_2 \cup \{z_1,\,z_2\}$ such that

$$|y - z| = \mathrm{dist}\,(y,\,P_2) = \mathrm{dist}\,(y,\,\partial\Omega) \leq r_x < 2r_x \leq \mathrm{dist}\,(y,\,P_1)$$

via the triangle inequality. Moreover the definition of ϕ implies $\phi(y) \leq \epsilon$ if r_x is sufficiently small (compared to $\min\{|x - z_1|,\,|x - z_2|\}$). Hence by taking the union of $B(x,\,r_x) \cap \overline{\Omega}$ over $x \in P_2$ we obtain a neighborhood of P_2 in which $\phi \leq \epsilon$.

Let $c_1 \geq 1$. We define a cut-off function by setting

$$\alpha(z) = \begin{cases} 1, & \text{if } |z - z_1| < c_1|z_1 - z_2| \\ \log_2 \frac{2c_1|z_1 - z_2|}{|z - z_1|}, & \text{if } c_1|z_1 - z_2| \leq |z - z_1| \leq 2c_1|z_1 - z_2| \\ 0, & \text{otherwise} \end{cases}$$

for $z \in \Omega$. Using this cut-off function we define

$$\Phi(z) = \alpha(z) \min\left\{ \frac{1}{c_0}\phi(z),\, 1 \right\}$$

when $z \in \Omega$. Then by the properties of ϕ we know that, for any $0 < \epsilon < \frac{1}{9}$, $\Phi \geq 1 - \epsilon$ in some neighborhood of the set $P_1 \cap B(z_1,\,c_1|z_1 - z_2|)$, and $\Phi \leq \epsilon$ in some neighborhood of $P_2 \cap B(z_1,\,c_1|z_1 - z_2|)$. We may also define $\Phi(x) = 1$ for $x \in P_1 \cap B(z_1,\,c_1|z_1 - z_2|)$, and $\Phi(x) = 0$ when $x \in P_2$.

Moreover, we claim that ϕ is locally Lipschitz in Ω with

$$|\nabla\phi(z)| \leq \frac{3}{2} \max\left\{ |z - z_1|^{-1},\, |z - z_2|^{-1} \right\}$$

almost everywhere. Indeed, for any $y \in B(z, 3^{-1} \operatorname{dist}(z, \partial\Omega))$, we have, by the definition of ϕ and the fact $z_1, z_2 \in \partial\Omega$,

$$|\phi(y) - \phi(z)| \le \max\left\{ \int_{[y,\, z]} |z - z_1|^{-1}\, dz,\ \int_{[y,\, z]} |z - z_2|^{-1}\, dz \right\}$$
$$\le \frac{3}{2} \max\left\{ |z - z_1|^{-1},\ |z - z_2|^{-1} \right\} |y - z|,$$

where $[y, z]$ is the line segment joining y and z. Thus our claim follows. Furthermore applying the Leibniz's rule we obtain

$$\begin{aligned} &\|\nabla\Phi\|^p_{L^p(\Omega)} \lesssim \|\nabla\alpha\|^p_{L^p(\Omega)} + \|\nabla\phi\|^p_{L^p(\Omega\cap B(z_1,\, 2c_1|z_1 - z_2|))} \\ &\lesssim \int_{B(z_1,\, 2c_1|z_1-z_2|)\setminus B(z_1,\, |z_1-z_2|)} |z - z_1|^{-p}\, dz + \int_{B(z_1,\, 2c_1|z_1-z_2|)} |z - z_1|^{-p} + |z - z_2|^{-p}\, dz \\ &\lesssim |z_1 - z_2|^{2-p}, \end{aligned}$$

by calculating in polar coordinates with $1 < p < 2$. Thus $\Phi \in W^{1,p}(\Omega)$ with the desired properties since $\|\Phi\|_{L^\infty(\Omega)} \le 1$ and Ω is bounded. □

Let $\widetilde{\varphi} : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal map. Since Ω is Jordan, $\widetilde{\varphi}$ extends homeomorphically up to the boundary. We refer to this extension also by $\widetilde{\varphi}$. Given $z_1, z_2 \in \partial\widetilde{\Omega}$, let Γ_j be the hyperbolic ray starting at $\widetilde{\varphi}^{-1}(z_j)$, where $j = 1, 2$. Pick $y_j \in \Gamma_j$ with

$$|\widetilde{\varphi}^{-1}(z_j) - y_j| = |\widetilde{\varphi}^{-1}(z_2) - \widetilde{\varphi}^{-1}(z_1)|,$$

and let α be the curve obtained from the arcs $[\widetilde{\varphi}^{-1}(z_j), y_j]$ together with the shorter one of the two circular arcs between y_1, y_2. Set $\gamma = \widetilde{\varphi}(\alpha)$. See Figure 1.

Let W be a Whitney decomposition of $\widetilde{\Omega}$ and set

$$W_\gamma = \{Q_i \in W \mid Q_i \cap \gamma \neq \emptyset\}.$$

We index the squares in W_γ according to side length: $Q_{i1}, \cdots, Q_{in_i}$ are those with side length 2^i when $i \in \mathbb{Z}$, if there are such squares. Notice that each n_i is necessarily finite. Moreover observe that by applying Lemma 2.6 to Whitney squares, there are at most uniformly finitely many $\widetilde{\varphi}^{-1}(Q_{ij})$ intersecting the circular part of α.

Lemma 3.4. *For the curve γ defined above, we have*

$$\operatorname{diam}(\gamma) \le C|z_1 - z_2|, \tag{3.2}$$

where $C = C(J)$ is independent of $z_1, z_2, \widetilde{\varphi}$.

Proof. Since Ω is John, Lemma 2.13 gives us a (closed) curve $\beta \subset \mathbb{R}^2 \setminus \Omega$ that joins z_1, z_2 and so that $\operatorname{diam}(\beta) \le C(J)|z_1 - z_2|$. Then $\beta \subset \overline{B}(z_1, C(J)|z_1 - z_2|)$. Let $z \in \gamma$. We claim that $z \in B(z_1, NC(J)|z_1 - z_2|)$ for some absolute constant N. Let $Q \in W_\gamma$ be a Whitney square containing z. If $Q \cap \beta \neq \emptyset$, the desired conclusion follows. Otherwise, notice that $\widetilde{\varphi}^{-1}(Q)$ is a Whitney-type set by Lemma 2.6 and hence the definition of γ together with the lower bound on the capacity obtained via the version of (2.4) for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and the conformal invariance of the capacity show that the capacity of Q and β in $\tilde{\Omega}$ is bounded away from zero by an absolute constant. In fact as $\widetilde{\varphi}$ is a homeomorphism, we have

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\beta)) \ge |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)|.$$

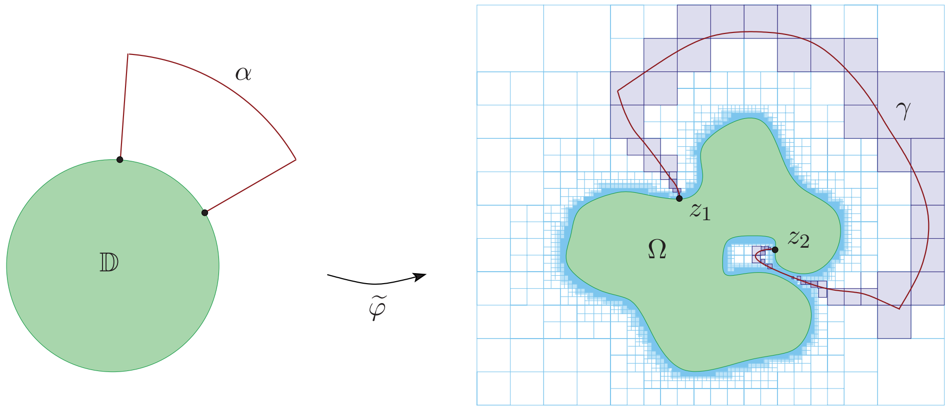

Figure 1. The curve γ satisfying inequality (3.2) is obtained as the image of the curve α under the conformal map $\widetilde{\varphi}\colon \mathbb{R}^2\setminus\overline{\mathbb{D}} \to \mathbb{R}^2\setminus\overline{\Omega}$. In the illustration the Whitney squares W_γ are highlighted.

Since $\widetilde{\varphi}^{-1}(Q)\cap\alpha\neq\emptyset$, by (2.1) and the fact $\widetilde{\varphi}^{-1}(Q)$ is a Whitney-type set by Lemma 2.6, we further have

$$\begin{aligned}\operatorname{dist}(\widetilde{\varphi}^{-1}(Q),\,\varphi^{-1}(\beta)) &\le \min\{\operatorname{dist}(\widetilde{\varphi}^{-1}(Q),\,\widetilde{\varphi}^{-1}(z_1)),\ \operatorname{dist}(\widetilde{\varphi}^{-1}(Q),\,\widetilde{\varphi}^{-1}(z_2))\}\\ &\lesssim \operatorname{diam}(\widetilde{\varphi}^{-1}(Q)) \lesssim |\widetilde{\varphi}^{-1}(z_1)-\widetilde{\varphi}^{-1}(z_2)|.\end{aligned}$$

In conclusion we have

$$\min\{\operatorname{diam}(\widetilde{\varphi}^{-1}(Q)),\ \operatorname{diam}(\widetilde{\varphi}^{-1}(\beta))\}\ge C(J)\operatorname{dist}(\widetilde{\varphi}^{-1}(Q),\,\varphi^{-1}(\beta)).$$

Then the version of (2.4) for $\mathbb{R}^2\setminus\overline{\mathbb{D}}$ and conformal invariance of capacity give

$$1\lesssim \operatorname{Cap}(\widetilde{\varphi}^{-1}(Q),\,\widetilde{\varphi}^{-1}(\beta),\,\mathbb{R}^2\setminus\overline{\mathbb{D}}) = \operatorname{Cap}(Q,\,\beta,\,\widetilde{\Omega})\le \operatorname{Cap}(Q,\,\beta,\,\mathbb{R}^2),$$

where in the last inequality we use the monotonicity of capacity.

Hence Lemma 2.4 shows that $\operatorname{dist}(Q,\,\beta)\lesssim \operatorname{diam}(\beta)$, and by the definition of β we conclude that Q must intersect $B(z_1, NC(J)|z_1-z_2|)$, where N is an absolute constant. Since Q is a Whitney square, the side length of Q is no more than $\operatorname{dist}(Q,\partial\Omega)$; especially no more than $\operatorname{dist}(z_1,\,Q)$ as $z_1\in\partial\Omega$. The asserted inequality then follows. □

Lemma 3.5. *For $Q_{i,j}$ defined via W_γ above, we have*

$$\sum_i n_i 2^{-i(2-p)}\le C(||E||,\,p)|z_1-z_2|^{2-p}.$$

Proof. Let Φ be defined as in Lemma 3.3 for the choice $c_1=cC$, where C is the constant in Lemma 3.4, and c will be determined momentarily.

Since Ω is a $W^{1,p}-$extension domain, we have $E\Phi\in W^{1,p}(\mathbb{R}^2)$, where E is the corresponding extension operator. Therefore, by denoting by M the Hardy-Littlewood maximal operator and by fixing $c_2 \geq 1$ to be determined momentarily and $1 < s < p$, we get

$$\begin{aligned}
&\sum_i \sum_{j=1}^{n_i} |Q_{ij}|^{1-\frac{p}{s}} \left(\int_{2c_2 Q_{ij}} |\nabla E\Phi(x)|^s \, dx \right)^{\frac{p}{s}} \\
\lesssim\ & \sum_i \sum_{j=1}^{n_i} |Q_{ij}| \left(\fint_{2c_2 Q_{ij}} |\nabla E\Phi(x)|^s \, dx \right)^{\frac{p}{s}} \\
\lesssim\ & \sum_i \sum_{j=1}^{n_i} \int_{Q_{ij}} |M((\nabla E\Phi)^s)(x)|^{\frac{p}{s}} \, dx \\
\lesssim\ & \int_{\widetilde{\Omega}} |M((\nabla E\Phi)^s)(x)|^{\frac{p}{s}} \, dx \\
\lesssim\ & \|E\Phi\|^p_{W^{1,p}(\mathbb{R}^2)} \lesssim |z_1 - z_2|^{2-p},
\end{aligned}$$

where $|Q_{ij}|$ denotes the Lebesgue measure of Q_{ij}.

By Lemma 2.9, there is a constant c_2 such that, for every $Q_{ij} \in W_\gamma$,

$$c_2 Q_{ij} \cap P_1 \neq \emptyset \neq c_2 Q_{ij} \cap P_2. \tag{3.3}$$

Indeed for any $z_0 \in [\widetilde{\varphi}^{-1}(z_j), y_j] \subset \alpha$ with $j = 1, 2$, Lemma 2.9 gives a curve α' connecting P_1 and P_2 and passing through z_0 such that $\ell(\varphi(\alpha')) \lesssim \operatorname{dist}(\varphi(z_0), \partial\widetilde{\Omega})$. Suppose $\varphi(z_0) \in Q'$ with $Q' \in W_\gamma$. Then there is a constant c_2' such that $\varphi(\alpha') \subset c_2' Q'$, and by the definition of α' we conclude (3.3) for such Q'. For the circular part of α, as there are only uniformly finitely many $\widetilde{\varphi}^{-1}(Q_{ij})$ intersecting α, there exists a constant c_2'' such that

$$\ell(Q') \leq c_2'' \ell(Q_{ij}) \quad \text{and} \quad \operatorname{dist}(Q_{ij}, Q') \leq c_2'' \ell(Q_{ij})$$

for each such Q_{ij} and some Q' as above. By setting $c_2 = c_2' c_2''$ we obtain (3.3). This determines the value of c_2 in the above estimate.

We now choose a constant c depending on c_2 so that $2c_2 Q_{ij} \subset B(z_1, cC|z_1 - z_2|)$ for each Q_{ij}; this determines the constant c in the definition of c_1 in the beginning of our proof. Notice that for any Q_{ij},

$$\operatorname{diam}(\gamma_1) \sim \ell(Q_{ij}) \sim \operatorname{diam}(\gamma_2)$$

for subcurves $\gamma_1 \subset 2c_2 Q_{ij}$ of P_1 and $\gamma_2 \subset 2c_2 Q_{ij}$ of P_2 by Lemma 3.4 and the definition of c_2. Then, by Lemma 3.2 (with $p = s$ there) applied to a representative of $E\Phi$ that is absolutely continuous on almost every line segment parallel to the coordinate axes, relying on the values of Φ on P_1, P_2 from Lemma 3.3 we have

$$\sum_i \sum_{j=1}^{n_i} |2c_2 Q_{ij}|^{1-\frac{p}{s}} \left(\int_{2c_2 Q_{ij}} |\nabla E\Phi(x)|^s \, dx \right)^{\frac{p}{s}} \gtrsim \sum_i n_i 2^{-i(2-p)}.$$

Therefore we have established the asserted inequality. □

Proof of Theorem 3.1. For each Q_{ij}, its diameter is comparable to $\operatorname{dist}(Q_{ij}, \partial\Omega)$, which means for the points $w \in \gamma \cap Q_{ij}$ that

$$\operatorname{dist}(w, \partial\Omega) \sim \operatorname{diam}(Q_{ij}). \tag{3.4}$$

Moreover we claim that

$$\mathcal{H}^1(Q_{ij} \cap \gamma) \lesssim \ell(Q_{ij}). \tag{3.5}$$

Indeed, first of all by Lemma 2.6 we have that $\varphi^{-1}(Q_{ij})$ is a λ-Whitney-type set with an absolute constant λ. Then it can be covered by $C = C(\lambda)$ Whitney squares of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$; see (2.2). According to the geometry of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and the definition of α, we have

$$\mathcal{H}^1(S \cap \alpha) \lesssim \ell(S)$$

for each Whitney square S of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ with an absolute constant. Here $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Then by applying Lemma 2.5 to these C Whitney squares which cover $\varphi^{-1}(Q_{ij})$, we obtain (3.5) by a change of variable. Therefore the claim of Lemma 3.5 with (3.4) and (3.5) gives

$$\begin{aligned}\int_\gamma \operatorname{dist}(z,\,\partial\Omega)^{1-p}\,ds &\le \sum_{Q_{ij}} \int_{\gamma\cap Q_{ij}} \operatorname{dist}(z,\,\partial\Omega)^{1-p}\,ds\\ &\lesssim \sum_{Q_{ij}} \operatorname{dist}(Q_{ij},\,\partial\Omega)^{2-p} \le C(||E||,\,p)|z_1-z_2|^{2-p}.\end{aligned}$$

In fact the inequality in Lemma 3.5 is equivalent to (1.2) for γ. One direction is shown above. For the other direction, first we note that each Whitney square has at most 20 neighboring squares, which tells us that we can divide the squares in W_γ into at most 21 subcollections $\{W_k\}_{k=1}^{21}$ such that in each of the subcollections the squares are pairwise disjoint. Then for any two distinct $Q_i,\,Q_j \in W_k$, by Lemma 2.2 we have

$$1.1Q_i \cap 1.1Q_j = \emptyset.$$

Notice that for each $Q_{ij} \in W_\gamma$, by definition we have

$$\mathcal{H}^1(1.1Q_{ij} \cap \gamma) \ge 0.1\ell(Q_{ij}).$$

Thus by applying

$$\ell(Q_{ij}) \sim \operatorname{dist}(Q_{ij},\,\partial\Omega),$$

we have

$$\begin{aligned}\sum_{Q_{ij}} \operatorname{dist}(Q_{ij},\,\partial\Omega)^{2-p} &\lesssim \sum_{k=1}^{21}\sum_{Q_{ij}\in W_k} \int_{\gamma\cap Q_{ij}} \operatorname{dist}(z,\,\partial\Omega)^{1-p}\,ds\\ &\lesssim \int_\gamma \operatorname{dist}(z,\,\partial\Omega)^{1-p}\,ds \le C(||E||,\,p)|z_1-z_2|^{2-p},\end{aligned}$$

which gives the other direction. Hence we have proven the existence of the desired curve when $z_1, z_2 \in \partial\Omega$.

Suppose now that $z_1 \in \partial\Omega$ and z_2 lies on the hyperbolic ray Γ starting at z_1. If $|\widetilde{\varphi}^{-1}(z_2)| \le 2$, then

$$1 \lesssim \operatorname{Cap}(\widetilde{\varphi}^{-1}(Q),\,\partial\mathbb{D},\,\mathbb{R}^2\setminus\overline{\mathbb{D}}) = \operatorname{Cap}(Q,\,\partial\Omega,\,\widetilde{\Omega}) \le \operatorname{Cap}(Q,\,\partial\Omega,\,\mathbb{R}^2)$$

for every Whitney square $Q \in W$ with $Q \cap [z_1, z_2] \neq \emptyset$, where $[z_1, z_2]$ is the part of the hyperbolic ray Γ between z_1 and z_2. Now Lemma 2.4 shows that $\operatorname{dist}(Q,\,\partial\Omega) \lesssim \operatorname{diam}(\Omega)$, or equivalently $\operatorname{diam}(Q) \lesssim \operatorname{diam}(\Omega)$. Especially, $\operatorname{dist}(z_2, \partial\Omega) \lesssim \operatorname{diam}(\Omega)$ and

$$|z_2 - z_1| \lesssim \operatorname{diam}(\Omega). \tag{3.6}$$

Moreover, the proof of Lemma 3.4 applies with γ replaced by $[z_1,\,z_2]$ to show that

$$\operatorname{diam}([z_1, z_2]) \lesssim |z_2 - z_1|. \tag{3.7}$$

Indeed since Ω is John, Lemma 2.13 gives us a closed curve $\beta \subset \mathbb{R}^2 \setminus \Omega$ that joins z_1, z_2 and so that $\operatorname{diam}(\beta) \le C(J)|z_1 - z_2|$. Then $\beta \subset \overline{B}(z_1, C(J)|z_1 - z_2|)$. Let $z \in [z_1, z_2]$. We claim that $z \in B(z_1, NC(J)|z_1 - z_2|)$ for some absolute constant N. Let $Q \in W$ be a Whitney square containing z. If $Q \cap \beta \neq \emptyset$, the desired conclusion follows. Otherwise, notice that $\widetilde{\varphi}^{-1}(Q)$ is a Whitney-type set by Lemma 2.6. Then as $\widetilde{\varphi}$ is a homeomorphism, we have

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\beta)) \ge |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)|.$$

Since $\widetilde{\varphi}^{-1}(Q) \cap \alpha \neq \emptyset$ and $\widetilde{\varphi}^{-1}(Q)$ is a Whitney-type set by Lemma 2.6, by (2.1) we further have

$$\begin{aligned}\operatorname{dist}(\widetilde{\varphi}^{-1}(Q), \widetilde{\varphi}^{-1}(\beta)) &\le \operatorname{dist}(\widetilde{\varphi}^{-1}(Q), \widetilde{\varphi}^{-1}(z_1)) \\ &\lesssim \operatorname{diam}(\widetilde{\varphi}^{-1}(Q)) \lesssim |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)|,\end{aligned}$$

where we applied the geometry of $\mathbb{R}^2 \setminus \mathbb{D}$. In conclusion we have

$$\min\{\operatorname{diam}(\widetilde{\varphi}^{-1}(Q)),\ \operatorname{diam}(\widetilde{\varphi}^{-1}(\beta))\} \ge C(J)\operatorname{dist}(\widetilde{\varphi}^{-1}(Q), \varphi^{-1}(\beta)).$$

Then the version of (2.4) for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and conformal invariance of capacity give

$$1 \lesssim \operatorname{Cap}(\widetilde{\varphi}^{-1}(Q), \widetilde{\varphi}^{-1}(\beta), \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \operatorname{Cap}(Q, \beta, \widetilde{\Omega}) \le \operatorname{Cap}(Q, \beta, \mathbb{R}^2),$$

where in the last inequality we use the monotonicity of capacity. Hence Lemma 2.4 shows that $\operatorname{dist}(Q, \beta) \lesssim \operatorname{diam}(\beta)$, and by the definition of β we conclude that Q must intersect $B(z_1, NC(J)|z_1 - z_2|)$, where N is an absolute constant. Since Q is a Whitney square, the side length of Q is no more than $\operatorname{dist}(Q, \partial\Omega)$; especially no more than $\operatorname{dist}(z, Q)$ as $z_1 \in \partial\Omega$. The asserted inequality then follows.

With the property (3.6) above, we can find an auxiliary point $z_2' \in \partial\Omega$ with

$$|z_2' - z_1| \sim |z_2 - z_1|. \tag{3.8}$$

Then we apply Lemma 3.3 for the pair z_1, z_2' to obtain a suitable test function Φ such that, for any $0 < \epsilon < \frac{1}{3}$, $\Phi \ge 1 - \epsilon$ in some neighborhood of $P_1 \cap B(z_1, c_1|z_2' - z_1|)$, $\Phi \le \epsilon$ in some neighborhood of $P_2 \cap B(z_1, c_1|z_2' - z_1|)$, where $P_1, P_2 \subset \partial\Omega$ are the two open curves separated by z_1 and z_2', and the constant c_1 is again to be determined. Moreover we also have

$$\|\nabla\Phi\|^p_{L^p(\Omega)} \le C(p, c_1, J)|z_1 - z_2'|^{2-p} \le C(p, c_1, J)|z_1 - z_2|^{2-p}.$$

by (3.8). This allows one to apply the argument of the proof of Lemma 3.5 with (the first paragraph in this proof) to $[z_1, z_2]$, determine the constant c_1 above via (3.7) and (3.8), and obtain the desired inequality for the integral over $[z_1, z_2]$. When $|\widetilde{\varphi}^{-1}(z_2)| > 2$, let $z_3 \in [z_1, z_2]$ be the last point such that $|\widetilde{\varphi}^{-1}(z_3)| = 2$. Then the estimate on $[z_1, z_3]$ follows from the previous case, and the estimate on $[z_3, z_2]$ follows directly from Lemma 2.5. This gives us the desired curve.

Suppose now that $z_1, z_2 \in \widetilde{\Omega}$. If $|z_1 - z_2| \le \operatorname{dist}(z_1, \partial\Omega)$ or $|z_1 - z_2| \le \operatorname{dist}(z_2, \partial\Omega)$, we may choose γ to be a line segment between z_1, z_2. Otherwise, Lemma 2.9 allows us to pick hyperbolic rays Γ_j starting at some w_j, $j = 1, 2$ so that Γ_j intersects $B(z_j, \frac{1}{2}\operatorname{dist}(z_j, \partial\Omega))$ at some y_j and the length of the arc $[w_j, y_j]$ of Γ_j is no more than $C\operatorname{dist}(z_j, \partial\Omega)$. Now one obtains the curve γ by joining w_1, w_2 by the first part of our proof, applying the beginning of this paragraph to the arcs $[w_j, y_j]$ and using additional line segments inside $B(z_j, \frac{1}{2}\operatorname{dist}(z_j, \partial\Omega))$ if necessary. The case when only one of the points is in $\widetilde{\Omega}$ is similar.

□

We remark that, even though for $z_1,\, z_2 \in \widetilde{\Omega}$ the curve which we construct in the proof above may touch the boundary $\partial\Omega$, it can be modified to be contained in $\widetilde{\Omega}$. Indeed for $z_i \in \widetilde{\Omega}$ with $i = 1,\, 2$, in the proof of Theorem 3.1 we first go along a suitable hyperbolic ray near z_i to the boundary point w_i, and then go along the curve γ constructed for boundary points (before Lemma 3.4). However γ goes from the boundary to the interior $\widetilde{\Omega}$ along the same hyperbolic rays again. Hence if we cut off the overlap in the union of the original curve in the proof of Theorem 3.1, we obtain a curve inside $\widetilde{\Omega}$ with the desired bound.

3.2. **Inner extension.** We prove the following inner extension theorem in this subsection.

Theorem 3.6. *Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map, where $\Omega \subset \mathbb{R}^2$ is a simply connected John domain with John constant J. Suppose that $\varphi(0)$ is the distinguished point in the definition of a John domain. Set $\Omega_\epsilon = \varphi(B(0,\, 1-\epsilon))$ for $0 < \epsilon < \frac{1}{2}$. Then there exists an extension operator $E_\epsilon\colon W^{1,p}(\Omega_\epsilon) \to W^{1,p}(\Omega)$ such that $\|E_\epsilon\| \le C(p,\, J)$ for $1 < p < \infty$.*

Fix ϵ, and notice that Ω_ϵ is a Jordan domain. Let $\Omega'_\epsilon = \mathbb{R}^2 \setminus \overline{\Omega}_\epsilon$, and $\widetilde{\Omega}_\epsilon = \Omega'_\epsilon \cap \Omega$. Since a John domain is finitely connected along its boundary, see Lemma 2.13, by [29, Theorem 2.18] we may extend φ continuously to the boundary $\partial\mathbb{D}$; we denote the extended map still by φ. We are going to modify the method of P.W. Jones from [18] to prove Theorem 3.6.

First, recall a concept introduced in [38], also see [2].

Definition 3.7 (Inner uniform domain)**.** *A domain Ω is called inner uniform if there exists a positive constant ϵ_0 such that for any pair of points $x,\, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x, y and satisfying*

$$\ell(\gamma) \le \frac{1}{\epsilon_0} \operatorname{dist}_\Omega(x,\, y) \quad \text{and} \quad \operatorname{dist}(z,\, \partial\Omega) \ge \epsilon_0 \min\{\ell(\gamma_{xz}),\, \ell(\gamma_{zy})\} \text{ for all } z \in \gamma, \tag{3.9}$$

where γ_{xz} is the part of γ joining from x to z, and γ_{zy} correspondingly for z and y.

By [2], [38] we know that each simply connected J-John domain Ω is an inner uniform domain, with ϵ_0 depending only on J. Moreover, one can require γ to be the hyperbolic geodesic between x and y.

Proof of Theorem 3.6. We wish to construct a suitable cover for $\widetilde{\Omega}_\epsilon$ inside Ω and an associated partition of unity. Towards this, recall that Ω is John and that, by Lemma 2.14, so is Ω_ϵ, with a constant only depending on J. From the discussion after Definition 3.7 we may further assume that Ω_ϵ is inner uniform, and that we may use hyperbolic geodesics of Ω_ϵ as curves referred to in the definition, with constant ϵ_0 only depending on J.

Fix $2^{-k_0-1} < \epsilon \le 2^{-k_0}$ for some $k_0 \in \mathbb{N}$. We begin by constructing a decomposition of the preimage $A = \mathbb{D} \setminus B(0,\, 1-\epsilon)$, of $\widetilde{\Omega}_\epsilon$ under φ, and then obtain a decomposition of $\widetilde{\Omega}_\epsilon$ with the help of the map φ.

For $k \in \mathbb{N}$ let

$$A_k = B(0,\, 1-\epsilon + 2^{-k}\epsilon) \setminus B(0,\, 1-\epsilon+2^{-k-1}\epsilon).$$

For each $k \ge 0$, the collection of the 2^{k+k_0} radial rays obtained by dividing the polar angle 2π evenly and by starting with the zero angle subdivides A_k into Whitney-type (with respect to $\mathbb{R}^2 \setminus B(0, 1-\epsilon)$) sets. Run this process for all $k \in \mathbb{N}$. We denote these Whitney-type sets by $\widetilde{Q}_i$. They satisfy (i) in the Definition 2.3 with

$$\frac{1}{\lambda} \operatorname{diam}(\widetilde{Q}_i) \le \operatorname{dist}(\widetilde{Q}_i,\, \partial\,(B(0, 1-\epsilon))) \le \lambda \operatorname{diam}(\widetilde{Q}_i)$$

for some absolute constant λ. Also note that they are John domains with an absolute constant.

According to Lemma 2.14, φ is quasisymmetric with respect to the inner metrics, and it follows from this and the geometry of the sets $\widetilde{Q}_i$ that each $\widetilde{S}_i = \varphi(\widetilde{Q}_i)$ is a John domain with the John constant only depending on J; see the argument after Lemma 2.14. Set $\widetilde{W} = \{\widetilde{S}_i\}$. Then according to the quasisymmetry of φ, the set $\widetilde{S}_i$ is a λ-Whitney-type set with respect to distance to $\partial\Omega_\epsilon$ in the inner distance of Ω : replace the diameter by the inner diameter and $\operatorname{dist}(A, \partial\Omega)$ with $\operatorname{dist}_\Omega(A, \partial\Omega_\epsilon)$ in Definition 2.3. Here the constant λ depends only on J.

Indeed, we will show that there exists a constant $0 < c = c(J) < 1$ such that

$$B_\Omega(\varphi(z_0),\, c\operatorname{diam}_\Omega(\widetilde{S}_i)) \subset \widetilde{S}_i \tag{3.10}$$

and

$$c\operatorname{diam}_\Omega(\widetilde{S}_i) \leq \operatorname{dist}_\Omega(\widetilde{S}_i,\, \partial\Omega_\epsilon) \leq \frac{1}{c}\operatorname{diam}_\Omega(\widetilde{S}_i). \tag{3.11}$$

Here $B_\Omega(x,\, r)$ denotes the open ball centered at x with radius r with respect to the inner distance. Fixing $\widetilde{S}_i$ with the corresponding set $\widetilde{Q}_i$, by the geometry of $\widetilde{Q}_i$ there is a ball $B(z_0,\, c_0\operatorname{diam}(\widetilde{Q}_i))$ contained in $\widetilde{Q}_i$ for some absolute constant $c_0 \leq 1$. Let z_1 be an arbitrary point on the boundary of $B(z_0,\, c_0\operatorname{diam}(\widetilde{Q}_i))$ and $z_2 \in \partial\widetilde{Q}_i$ such that

$$\operatorname{dist}_\Omega(\varphi(z_2),\, \varphi(z_0)) \geq \frac{1}{3}\operatorname{diam}_\Omega(\widetilde{S}_i);$$

the existence of such a point follows from the triangle inequality. Then by quasisymmetry we have

$$\operatorname{diam}_\Omega(\widetilde{S}_i) \leq 3\operatorname{dist}_\Omega(\varphi(z_2),\, \varphi(z_0)) \leq 3\eta(c_0^{-1})\operatorname{dist}_\Omega(\varphi(z_1),\, \varphi(z_0)).$$

By the arbitrariness of z_1 and the fact that φ is a homeomorphism, we conclude (3.10) for some constant $c = c(J)$.

To show (3.11), first choose points $z_3 \in \partial B(1-\epsilon)$ and $z_4 \in \partial\widetilde{Q}_i$ such that

$$\operatorname{dist}_\Omega(\widetilde{S}_i,\, \partial\Omega_\epsilon) = \operatorname{dist}_\Omega(\varphi(z_4),\, \varphi(z_3)). \tag{3.12}$$

Notice that by the geometry of $\widetilde{Q}_i$

$$|z_4 - z_0| \sim \operatorname{diam}(\widetilde{Q}_i) \sim \operatorname{dist}(\widetilde{Q}_i,\, B(1-\epsilon)) \lesssim |z_3 - z_4|$$

with absolute constants. Then the quasisymmetry of φ implies

$$\operatorname{dist}_\Omega(\varphi(z_4),\, \varphi(z_0)) \leq C(J)\operatorname{dist}_\Omega(\varphi(z_3),\, \varphi(z_4)) = C(J)\operatorname{dist}_\Omega(\widetilde{S}_i,\, \partial\Omega_\epsilon). \tag{3.13}$$

By (3.10) above, we have that

$$\operatorname{dist}_\Omega(\varphi(z_4),\, \varphi(z_0)) \sim \operatorname{diam}_\Omega(\widetilde{S}_i),$$

and hence obtain the lower bound of the distance in (3.11) by combining (3.12) with (3.13). For the upper bound, pick points $z_5 \in \partial B(1-\epsilon)$ and $z_6 \in \partial\widetilde{Q}_i$ such that

$$|z_5 - z_6| = \operatorname{dist}(\widetilde{Q}_i,\, \partial B(1-\epsilon)).$$

Again by the geometry of $\widetilde{Q}_i$ we get

$$|z_5 - z_6| \lesssim |z_0 - z_6|,$$

and by quasisymmetry

$$\operatorname{dist}_\Omega(\widetilde{S}_i,\, \partial\Omega_\epsilon) \leq \operatorname{dist}_\Omega(\varphi(z_5),\, \varphi(z_6)) \leq C(J)\operatorname{dist}_\Omega(\varphi(z_6),\, \varphi(z_0)) \sim \operatorname{diam}_\Omega(\widetilde{S}_i),$$

where we apply (3.10) with $\varphi(z_6) \in \partial \widetilde{S}_i$ in the last estimate. All in all we have shown (3.10) and (3.11).

We then claim that $\operatorname{diam}_\Omega(\widetilde{S}_i) \sim \operatorname{diam}_\Omega(\widetilde{S}_j)$ if $\overline{\widetilde{S}_i} \cap \overline{\widetilde{S}_j} \cap \Omega \neq \emptyset$. This follows from the facts that

$$\operatorname{diam}(\widetilde{Q}_i) \sim \operatorname{diam}(\widetilde{Q}_j) \tag{3.14}$$

if $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$, and that φ is quasisymmetric with respect to the inner metric of Ω; see Lemma 2.14.

Now for each $\widetilde{S}_i \in \widetilde{W}$, define

$$\widetilde{U}_i := \{x \in \Omega \mid \operatorname{dist}_\Omega(x, \widetilde{S}_i) < \frac{1}{c} \operatorname{diam}_\Omega(\widetilde{S}_i)\}.$$

We claim that, we can choose $c > 1$ depending only on J such that these sets $\widetilde{U}_i$ have uniformly finite overlaps. Namely for every $x \in \widetilde{\Omega}_\epsilon$

$$1 \leq \sum_i \chi_{\widetilde{U}_i}(x) \leq C \tag{3.15}$$

for some absolute constant C, where $\chi_{\widetilde{U}_i}$ is the characteristic function of $\widetilde{U}_i$. This easily follows from the quasisymmetry of φ with respect to the inner metric and the fact that $\widetilde{W}$ forms a cover of $\widetilde{\Omega}_\epsilon$. Indeed the lower bound is trivial. For the upper bound, first observe that $\widetilde{Q}_i \cap \widetilde{Q}_j = \emptyset$ gives

$$\operatorname{dist}(\widetilde{Q}_i, \widetilde{Q}_j) \gtrsim \max\{\operatorname{diam}(\widetilde{Q}_i), \operatorname{diam}(\widetilde{Q}_j)\},$$

and then by quasisymmetry, $\widetilde{S}_i \cap \widetilde{S}_j = \emptyset$ implies

$$\operatorname{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j) \gtrsim \max\{\operatorname{diam}_\Omega(S_i), \operatorname{diam}_\Omega(S_j)\}, \tag{3.16}$$

where the constant depends only on the John constant; also see [24, Formula (3.5)] for a version of this. Then by the fact that each $\widetilde{S}_i$ has at most 9 neighboring sets, we conclude the upper bound in (3.15) by choosing the constant c according to (3.16).

Given $\widetilde{S}_i \in \widetilde{W}$, we construct a locally Lipschitz function ϕ_i whose support is bounded and relatively closed in Ω, and contained in $\widetilde{U}_i$, such that $|\nabla \phi_i| \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)^{-1}$ and $\phi_i(x) = 1$ for any $x \in \widetilde{S}_i$. Indeed, simply set

$$\phi_i(x) = \max\{1 - 2c \operatorname{diam}_\Omega(\widetilde{S}_i)^{-1} \operatorname{dist}_\Omega(x, \widetilde{S}_i), 0\}$$

for $x \in \Omega$.

Since we have uniformly finite overlaps for $\widetilde{U}_i$, our collection of the functions ϕ_i give rise to a partition of unity, still denoted by $\{\phi_i\}$, such that $\sum \phi_i(x) = 1$ for every $x \in \widetilde{\Omega}_\epsilon$. By (3.15) and our construction of ϕ_i, we know that $|\nabla \phi_i| \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)^{-1}$ still holds up to a multiplicative constant. Indeed by the definition of $\widetilde{U}_i$ and the fact that for each $\widetilde{S}_i$ there are at most 9 neighboring sets $\widetilde{S}_j$, we have for each $x \in \widetilde{U}_i$ with the original ϕ_i's

$$\left| \nabla \left(\frac{\phi_j(x)}{\sum_i \phi_i(x)} \right) \right| \leq |\nabla \phi_i(x)| + \sum_{\widetilde{S}_j \cap \widetilde{S}_i \neq \emptyset} |\nabla \phi_j(x)| \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)^{-1}.$$

We are now ready to construct the extension operator. First, let us associate to each $\widetilde{S}_i \in \widetilde{W}$ a square $S_i \in W$ such that

$$\operatorname{diam}(S_i) = \operatorname{diam}_\Omega(S_i) \sim_J \operatorname{dist}_\Omega(\widetilde{S}_i, S_i) \sim_J \operatorname{diam}_\Omega(\widetilde{S}_i), \tag{3.17}$$

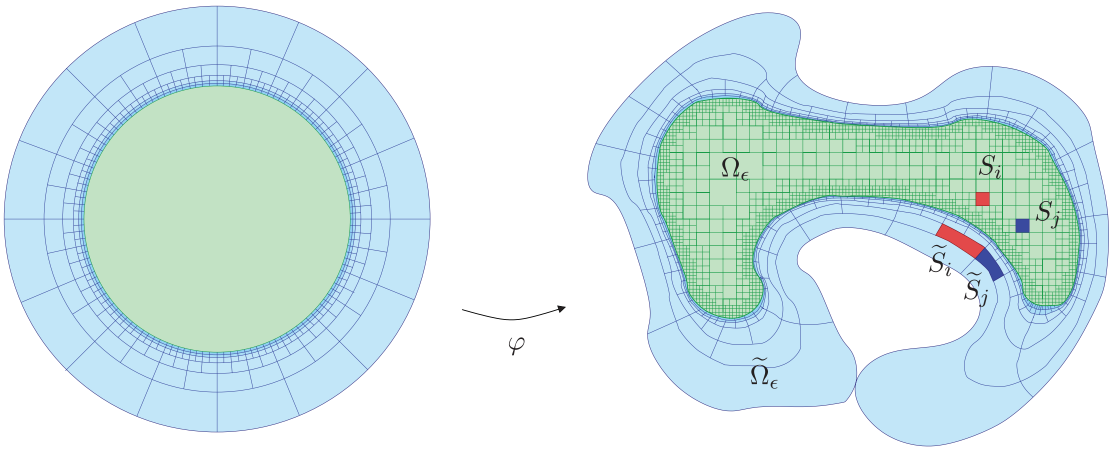

FIGURE 2. In the inner extension the annular region $\widetilde{\Omega}_\epsilon$ is divided into Whitney-type sets that are obtained by mapping a Whitney-type decomposition of the annulus inside the disk conformally.For the inner part Ω_ϵ we use a standard Whitney decomposition. Two pairs of sets $(\widetilde{S}_i, S_i)$ and $(\widetilde{S}_j, S_j)$ are highlighted.

where W is the corresponding Whitney decomposition of Ω_ϵ; see Figure 2. To see that a Whitney square of desired size can be chosen, trace back towards $\varphi(0)$ along any hyperbolic ray of Ω that intersects $\widetilde{S}_i$ and let S_i be a first Whitney square of Ω_ϵ intersecting that hyperbolic ray such that

$$\operatorname{dist}(\varphi^{-1}(S_i),\, \varphi^{-1}(\widetilde{S}_i)) \geq \frac{1}{9\lambda} \operatorname{diam}(\varphi^{-1}(\widetilde{S}_i)), \tag{3.18}$$

where λ is an absolute constant such that $\varphi^{-1}(S_i)$ is of λ-Whitney-type for Ω_ϵ by Lemma 2.6. The existence of such a square follows from Definition 2.3 and the assumption that $0 < \epsilon < \frac{1}{2}$. Indeed if there is no such a set, then

$$\frac{\operatorname{dist}(\varphi^{-1}(S_i),\, \varphi^{-1}(\widetilde{S}_i))}{\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i))} \leq \frac{1}{9\lambda}$$

for all the S_i intersecting the hyperbolic ray. However, the diameter of $\varphi^{-1}(\widetilde{S}_i) = \widetilde{Q}_i$ is at most 2, while a λ-Whitney-type set in $B(0,\, 1-\epsilon)$ containing the origin has distance to $\partial B(0,\, 1-\epsilon)$ at least $\frac{1}{4\lambda}$ since $\epsilon < \frac{1}{2}$ and $\lambda \geq 1$. Therefore we have

$$\frac{1}{8\lambda} \leq \frac{\operatorname{dist}(\varphi^{-1}(S_i),\, \varphi^{-1}(\widetilde{S}_i))}{\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i))} \leq \frac{1}{9\lambda},$$

which leads to a contradiction. Then by the fact that S_i is a first square satisfying (3.18), (2.2) implies

$$\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i)) \sim \operatorname{dist}(\varphi^{-1}(S_i),\, \varphi^{-1}(\widetilde{S}_i)) \sim \operatorname{diam}(\varphi^{-1}(S_i)).$$

Recalling that φ is quasisymmetric with respect to the inner distance, we conclude (3.17).

By (3.17), we know that the inner distance between $\widetilde{S}_i$ and S_i with respect to Ω is no more than a constant times $\operatorname{diam}_\Omega(\widetilde{S}_i)$. By the triangle inequality it follows that

$$\operatorname{dist}_\Omega(S_i,\, S_j) \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)$$

if $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$. Given such $\widetilde{S}_i, \widetilde{S}_j$ and corresponding S_i, S_j, consider the hyperbolic geodesic in Ω between the centers of S_i, S_j. From Lemma 2.7 we conclude that the Euclidean length of this geodesic is no more than constant times $\operatorname{diam}_\Omega(\widetilde{S}_i)$. Since $\Omega_\epsilon = \varphi(B(0,\, 1-\epsilon))$, it follows that this geodesic is contained in Ω_ϵ. We use Lemma 2.7 a second time to conclude that the Euclidean length of the hyperbolic geodesic Γ with respect to Ω_ϵ is also bounded from above by a constant times $\operatorname{diam}_\Omega(\widetilde{S}_i)$. Let us define $G(\widetilde{S}_i, \widetilde{S}_j)$ to be the union of all Whitney squares of Ω_ϵ that intersect this geodesic. By the inner uniformity of Ω_ϵ and the comment after Definition 3.7, one concludes that there are uniformly finitely many Whitney squares in every $G(\widetilde{S}_i, \widetilde{S}_j)$ with $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$, Namely

$$\#\left\{S_k \in W \mid S_k \in G(\widetilde{S}_i, \widetilde{S}_j),\, \widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset\right\} \leq C(J), \tag{3.19}$$

where # denotes the counting measure; this is a counterpart of [18, Lemma 2.8] with a similar proof. Indeed since

$$\ell(\Gamma) \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)$$

and by (3.17) with (3.14)

$$\operatorname{diam}_\Omega(S_i) \sim \operatorname{diam}_\Omega(\widetilde{S}_i) \sim \operatorname{diam}_\Omega(\widetilde{S}_j) \sim \operatorname{diam}_\Omega(S_j), \tag{3.20}$$

the diameters of Whitney squares of Ω_ϵ intersecting Γ are bounded from above by a multiple of $\operatorname{diam}_\Omega(\widetilde{S}_i)$. On the other hand, the second condition of (3.9) with (3.20) tells us that

$$\operatorname{dist}(Q,\, \partial\Omega_\epsilon) \gtrsim \operatorname{diam}_\Omega(\widetilde{S}_i)$$

if $Q \cap \Gamma \neq \emptyset$ while $Q \cap S_i = \emptyset = Q \cap S_j$; for those Q with $Q \cap S_i \neq \emptyset$ or $Q \cap S_j \neq \emptyset$, by the Whitney decomposition with (3.20) it trivially follows that $\operatorname{diam}(Q) \sim \operatorname{diam}_\Omega(\widetilde{S}_i)$. Thus we obtain an upper and a lower bound for the diameters of Q with $Q \cap \Gamma \neq \emptyset$, and hence there are finitely many Whitney squares in every $G(\widetilde{S}_i, \widetilde{S}_j)$ as $\ell(\Gamma) \lesssim \operatorname{diam}_\Omega(\widetilde{S}_i)$.

Define

$$E_\epsilon u(x) = \sum_i a_i \phi_i(x)$$

for a given Lipschitz (with respect to the Euclidean metric) function $u \in W^{1,p}(\Omega_\epsilon)$. Here

$$a_i = \fint_{S_i} u(x)\, dx = \frac{1}{|S_i|}\int_{S_i} u(x)\, dx,$$

where $S_i \in W$ is the square associated to $\widetilde{S}_i \in \widetilde{W}$. Therefore for each $\widetilde{S} \in \widetilde{W}$, by letting

$$a = \fint_S u(x)\, dx,$$

we obtain by applying the Poincaré inequality (see e.g. [18, Lemma 3.1]) to the chain of cubes $G(S,\, S_k)$

$$\begin{aligned}
\|\nabla(E_\epsilon u(x))\|^p_{L^p(\widetilde{S})} &\lesssim \int_{\widetilde{S}} \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} |a_k-a|^p|\nabla\phi_k(x)|^p\,dx\\
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} |a_k-a|^p(\operatorname{diam}_\Omega(S))^{2-p}\\
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} (\operatorname{diam}_\Omega(S))^{2-p}(\operatorname{diam}_\Omega(S))^{p-2}\int_{G(S,\,S_k)}|\nabla u(x)|^p\,dx\\
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} \int_{G(S,\,S_k)}|\nabla u(x)|^p\,dx,
\end{aligned}$$

where in the first inequality we used the identity

$$\nabla u(x) = \nabla(u(x)-a) = \nabla\left(\sum_i \phi_i(x)(a_i-a)\right),$$

and $G(S,\, S_k)$ is the union of squares along a suitable hyperbolic geodesic connecting S and S_k, as defined in the previous paragraph, with

$$\operatorname{diam}_\Omega(G(S,\, S_k)) \lesssim \operatorname{diam}_\Omega(\widetilde{S}).$$

Note that (3.17) implies that, for a fixed $S_i\in W$, all the sets in $\widetilde{W}$ associated to it are contained in a disk with (inner) radius controlled by a multiple of $\operatorname{diam}_\Omega(S_i)$. By (3.10) we conclude that there are uniformly finitely many such sets, where the constant depending only on J. Indeed note that for any $x\in\Omega$ and $0<r<\operatorname{diam}(\Omega)$, the hyperbolic geodesic Γ of Ω joining x to a point $y\in B_\Omega(x,\, r)$ satisfies

$$\frac{1}{2}r = \operatorname{dist}_\Omega(x,\, y) \le \ell(B_\Omega(x,\, r)\cap\Gamma).$$

Then since hyperbolic geodesics of Ω satiesfy (3.9) with a constant $0<c=c(J)<1$, we have

$$B\left(z,\, \frac{1}{16}cr\right)\subset B_\Omega(x,\, r)$$

by the triangle inequality, where z is the middle point (with respect to the length) on Γ. Thus

$$C(J)r^2 \le |B_\Omega(x,\, r)| \le \pi r^2, \tag{3.21}$$

where the upper bound comes from

$$B_\Omega(x,\, r)\subset B(x,\, r).$$

By (3.21) with (3.10) we conclude the uniform finiteness of the number of the sets. Therefore since $\widetilde{S}_i$ has uniformly finitely many neighbors, with (3.17), (3.21) and (3.19) we conclude

$$\sum_{\widetilde{S}_i\in\widetilde{W}}\sum_{\widetilde{S}_i\cap\widetilde{S}_k\neq\emptyset}\chi_{G(S_k,\,S_i)}(x)\lesssim 1, \tag{3.22}$$

for all x; notice that (3.19) is the counterpart of [18, Lemma 2.8], which with (3.17) and (3.21) implies that each Whitney square is contained in at most uniformly finitely many chains. One may also see [18, Page 80, Formula (3.2)] for a similar version. Hence we finally obtain

$$\begin{aligned}\|\nabla(E_\epsilon u)\|^p_{L^p(\widetilde{\Omega}_\epsilon)} &\lesssim \sum_{\widetilde{S}_i\in\widetilde{W}}\sum_{\widetilde{S}_k\cap\widetilde{S}_i\neq\emptyset}\int_{G(S_i,\,S_k)}|\nabla u(x)|^p\,dx\\ &\lesssim \|\nabla u\|^p_{L^p(\Omega_\epsilon)},\end{aligned}$$

with the constant only depending on p and J.

Since u is Lipschitz in $\overline{\Omega}_\epsilon$, the above procedure gives us an extension in $W^{1,\,p}(\Omega)$ of u to the entire Ω with the desired norm bound. Indeed we only need to check that $E_\epsilon u$ is absolutely continuous along almost every line segment parallel to the coordinate axes. We claim that $E_\epsilon u$ is even locally Lipschitz.

According to our construction, $E_\epsilon u$ is smooth in $\widetilde{\Omega}_\epsilon$. Hence to show the local Lipschitz continuity, we only need to consider the case where $z_1\in\overline{\Omega}_\epsilon$ and $z_2\in\widetilde{\Omega}_\epsilon$ with

$$B(z_2,\,2|z_1-z_2|)\subset\Omega.$$

Suppose that $z_2\in\widetilde{S}$ for some $\widetilde{S}\in\widetilde{W}$. Then by (3.17) and the Lipschitz continuity of u we have

$$\begin{aligned}|E_\epsilon u(z_2)-u(z_1)| &\le \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset}\phi_k(z_2)|a_k-u(z_1)|\\ &\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset}\phi_k(z_2)(\,\mathrm{dist}\,(z_1,\,S_k)+\,\mathrm{diam}\,(S_k))\\ &\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset}\phi_k(z_2)(|z_1-z_2|+\,\mathrm{diam}_{\,\Omega}(\widetilde{S}_k))\lesssim|z_1-z_2|,\end{aligned}$$

where in the last inequality we apply the facts that for $\widetilde{S}_k\cap\widetilde{S}\neq\emptyset$ it holds that

$$\mathrm{diam}_{\,\Omega}(\widetilde{S}_k)\sim\,\mathrm{dist}_{\,\Omega}(S,\,\Omega_\epsilon)\sim\,\mathrm{dist}_{\,\Omega}(z_2,\,\Omega_\epsilon)\le|z_1-z_2|.$$

Therefore we obtain the local Lipschitz continuity of E_ϵ.

It also follows from the construction (especially (3.17) and (3.22)) that also $\|E_\epsilon u\|_{L^p(\Omega)}\lesssim\|u\|_{L^p(\Omega_\epsilon)}$. Since E_ϵ is linear, then the general case of $u\in W^{1,\,p}(\Omega_\epsilon)$ follows by density of Lipschitz functions in this class: even $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,\,p}(G)$ for $1<p<\infty$ if G is a planar Jordan domain [27]. □

3.3. **Proof of the general case.** In this subsection, we prove the necessity in the more general case, where Ω is a bounded simply connected $W^{1,p}$-extension domain.

Fix $z_1,\,z_2\in\mathbb{R}^2\setminus\Omega$. Let $\Omega_n=\varphi(B(0,\,1-\frac{1}{n}))$ for $n\ge4$, where $\varphi\colon\mathbb{D}\to\Omega$ is a conformal map with $\varphi(0)$ the John center of Ω. By Theorem 3.6 we know that each Ω_n is also a $W^{1,p}$-extension domain with the norm of the operator only depending on p, the John constant of Ω, and the norm of the extension operator for Ω. Denoting by $\widetilde{\Omega}_n$ the complementary domain of Ω_n, we know that

$$\bigcap_{n=4}^{\infty}\widetilde{\Omega}_n=\mathbb{R}^2\setminus\Omega.$$

Moreover, by Theorem 3.1, there is a curve $\gamma_n \subset \widetilde{\Omega}_n \cup \partial\Omega_n$ connecting z_1 and z_2 so that

$$\int_{\gamma_n} \mathrm{dist}\,(z,\, \partial\Omega_n)^{1-p}\, ds \leq C(J,\, \|E\|,\, p)|z_1 - z_2|^{2-p}.$$

We proceed to find a curve γ such that (1.1) holds.

Notice that the condition above ensures that $\ell(\gamma_n) \leq C(J,\, \|E\|,\, p)|z_1 - z_2| := M$ by Lemma 2.10. Parametrize each γ_n by arc length, $\gamma_n\colon [0,\, \ell(\gamma_n)] \to \widetilde{\Omega}_n$, starting from z_1, and extend γ_n to $[\ell(\gamma_n),\, M]$ as $\gamma_n(t) = z_2$. Notice that $\gamma_n \subset \overline{B}(z_1,\, M)$, and therefore by the Arzelá-Ascoli lemma we obtain a 1-Lipschitz parametrized curve $\gamma\colon [0,\, M] \to \overline{B}(z_1,\, M) \setminus \Omega$ such that a subsequence of (γ_n) converges to γ uniformly. Then γ is a curve connecting z_1 and z_2.

Fix $m \in \mathbb{N}$ and $\epsilon > 0$. For $z \in \mathbb{R}^2$ and $n \in \mathbb{N}$ set

$$\omega_n^{(m)}(z) = \min\{m,\ \mathrm{dist}\,(z,\, \Omega_n)^{1-p}\}.$$

Then $\omega_n^{(m)}(z)$ is continuous and

$$\lim_{n\to\infty} \omega_n^{(m)}(z) = \min\{m,\ \mathrm{dist}\,(z,\, \Omega)^{1-p}\} := \omega^{(m)}(z).$$

Let us now show that

$$\int_{\gamma} \mathrm{dist}\,(z,\, \partial\Omega)^{1-p}\, ds \lesssim |z_1 - z_2|^{2-p}. \tag{3.23}$$

To this end, since $M < +\infty$, up to choosing a subsequence and redefining γ, we may assume that $\ell(\gamma_n)$ converges to M as $n \to \infty$. Therefore for n large enough, by Fatou's lemma we have

$$\begin{aligned}
&\int_0^{M-\epsilon} \omega^{(m)} \circ \gamma(t)|\gamma'(t)|\, dt \leq \int_0^{M-\epsilon} \omega^{(m)} \circ \gamma(t)\, dt \\
&\leq \int_0^{M-\epsilon} \lim_{n\to\infty} \omega_n^{(m)} \circ \gamma_n(t)\, dt \leq \liminf_{n\to\infty} \int_0^{M-\epsilon} \omega_n^{(m)} \circ \gamma_n(t)\, dt \\
&\leq \liminf_{n\to\infty} \int_0^{\ell(\gamma_n)} \omega_n^{(m)} \circ \gamma_n(t)\, dt \leq \liminf_{n\to\infty} \int_{\gamma_n} \omega_n^{(m)}(z)\, dz \\
&\leq \liminf_{n\to\infty} \int_{\gamma_n} \mathrm{dist}\,(z,\, \Omega_n)^{1-p}\, dt \leq C(J,\, \|E\|,\, p)|z_1 - z_2|^{2-p},
\end{aligned}$$

where we used the fact that since $\Omega_n \to \Omega$ and $\gamma_n \to \gamma$ uniformly, for fixed $m \in \mathbb{N}$

$$\lim_{n\to\infty} \omega_n^{(m)} \circ \gamma_n(t) = \omega^{(m)} \circ \gamma(t)$$

when $t \in [0,\, M - \epsilon]$. Letting $\epsilon \to 0$ we obtain

$$\int_{\gamma} \omega^{(m)}(z)\, dz \leq C(J,\, \|E\|,\, p)|z_1 - z_2|^{2-p},$$

and by the monotone convergence theorem we finally get (3.23).

4. Proof of sufficiency

In this section we prove the sufficiency of the condition (1.1) in Theorem 1.1, but begin with an auxiliary version. Namely, let $1 < p < s < 2$ and suppose that Ω is a bounded Jordan domain with the property that there exists a constant C such that for every pair of points $z_1,\, z_2 \in \mathbb{R}^2 \setminus \overline{\Omega}$ one can find a curve $\gamma \subset \mathbb{R}^2 \setminus \overline{\Omega}$ joining them with

$$\int_\gamma \operatorname{dist}(z,\, \partial\Omega)^{1-s}\, dz \le C|z_1 - z_2|^{2-s}. \tag{4.1}$$

We claim that Ω is a $W^{1,p}$-extension domain. Write $\widetilde{\Omega}$ for the complementary domain of Ω.

Proposition 4.1. *Let $1 < p < s < 2$ and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Suppose that for all $z_1, z_2 \in \widetilde{\Omega}$ there exists a curve $\gamma \subset \widetilde{\Omega}$ joining z_1 and z_2 such that* (4.1) *holds. Then Ω is a $W^{1,p}$-extension domain and the norm of the extension operator only depends on p, s and the constant C in* (4.1).

The proof of Proposition 4.1 is given in three steps. In the first step, in the following subsection, we show that (4.1) also holds for initial arcs of hyperbolic rays $\Gamma \subset \widetilde{\Omega}$, up to a multiplicative constant. In the second subsection we then assign a Whitney square of the domain Ω to each such Whitney square $\widetilde{Q}$ of its complementary domain $\widetilde{\Omega}$ that satisfies $\ell(\widetilde{Q}) \le 3\operatorname{diam}(\Omega)$. In the third subsection we use the relation between the Whitney squares to construct our extension operator.

Eventually in the final subsection of this section we prove Theorem 1.1 via Proposition 4.1 and an approximation argument. For this, it is crucial that the norm of the extension operator in Proposition 4.1 only depends on s, p and C in inequality (4.1) and that (4.1) for some $s > p$ follows from (1.1) by Lemma 2.11.

4.1. **Transferring the condition to hyperbolic rays.**

According to the Riemann mapping theorem there is a conformal map $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$. Since $\widetilde{\Omega}$ is a Jordan domain, we can extend $\widetilde{\varphi}$ continuously to the boundary as a homeomorphism. We denote the extension still by $\widetilde{\varphi}$. Recall the definition of a hyperbolic ray from Section 2.

Lemma 4.2. *Assume that* (4.1) *holds and that Ω is Jordan. Let $z_1 \in \partial\Omega$ and $[z_2, z_3]$ be an arc of the hyperbolic ray $\Gamma \subset \overline{\widetilde{\Omega}}$ corresponding to z_1. Then*

$$\int_{[z_2,z_3]} \operatorname{dist}(z,\, \partial\Omega)^{1-s}\, dz \le C'|z_2 - z_3|^{2-s}, \tag{4.2}$$

where C' depends only on s and the constant in (4.1).

Proof. Let γ be a curve from Proposition 4.1 for the pair z_2, z_3. By symmetry we may assume that z_3 is after z_2 on Γ when one moves towards infinity. Suppose first that $z_1 \neq z_2$. We use the notation from Lemma 2.8; especially, we let γ_k be a subcurve of γ that joins the inner and outer boundaries of $\widetilde{\varphi}(A(z_1,\, k))$, provided that $[z_2, z_3]$ hits at least three such annuli. If $[z_2, z_3]$ is contained in the union of two of these annuli, we claim that (4.2) follows from Lemma 2.5. Indeed (4.2) holds trivially for the hyperbolic rays of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. By the geometry of hyperbolic rays in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, $\widetilde{\varphi}^{-1}([z_2,\, z_3])$ is contained in a λ-Whitney-type set for some absolute constant λ; see also Lemma 2.8. Then a change of variable with Lemma 2.5 tells us that (4.2) holds for $[z_2,\, z_3]$.

For each $k \in \mathbb{Z}$ with

$$|\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)| \le 2^{k-1} \le 2^k \le |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_3)|,$$

let

$$Z_k = \widetilde{\varphi}(S_k^1) \cap \Gamma_k,$$

where S_k^1 is the circle centered at $\widetilde{\varphi}^{-1}(z_1)$ and with radius $3 \times 2^{k-2}$.

Fix $k \le 2$ as above. According to Lemma 2.8,

$$\operatorname{dist}(\Gamma_k,\, \partial\Omega) \sim \operatorname{dist}(Z_k,\, \partial\Omega) \tag{4.3}$$

and

$$\ell(\Gamma_k) \sim \operatorname{dist}(\Gamma_k,\, \partial\Omega). \tag{4.4}$$

Hence

$$\int_{\Gamma_k} \operatorname{dist}(z, \partial\Omega)^{1-s}\, dz \lesssim \operatorname{dist}(Z_k,\, \partial\Omega)^{2-s}. \tag{4.5}$$

Next we claim that

$$\operatorname{dist}(Z_k,\, \partial\Omega) \gtrsim \operatorname{dist}(\gamma_k,\, \partial\Omega). \tag{4.6}$$

Indeed let $B_k = B(Z_k,\, \frac{1}{4}\operatorname{dist}(Z_k,\, \partial\Omega))$. If $\gamma_k \cap B_k \neq \emptyset$, then by the triangle inequality we obtain the claim. For the other case, notice that B_k is a 4-Whitney-type set, and then by Lemma 2.6, $\widetilde{\varphi}^{-1}(B_k)$ is of λ-Whitney-type for some absolute constant λ. Hence

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(Z_k),\, \mathbb{S}^1) \sim \operatorname{diam}(\widetilde{\varphi}^{-1}(B_k)). \tag{4.7}$$

By the geometry of $A(z_1,\, k)$ in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, we have

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(\Gamma_k),\, \widetilde{\varphi}^{-1}(\gamma_k)) \lesssim \operatorname{dist}(\widetilde{\varphi}^{-1}(Z_k),\, \mathbb{S}^1)$$

and

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\gamma_k)) \gtrsim \operatorname{dist}(\widetilde{\varphi}^{-1}(Z_k),\, \mathbb{S}^1)$$

for some absolute constants. Hence with (2.4) and (4.7) we conclude that

$$\operatorname{Cap}(\widetilde{\varphi}^{-1}(B_k),\, \widetilde{\varphi}^{-1}(\gamma_k),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \gtrsim 1.$$

The conformal invariance of capacity gives

$$\operatorname{Cap}(B_k,\, \gamma_k,\, \widetilde{\Omega}) \gtrsim 1.$$

Applying Lemma 2.4 we have

$$\operatorname{dist}(B_k,\, \gamma_k) \lesssim \operatorname{diam}(B_k).$$

We then conclude the claim in this case by the definition of B_k and the triangle inequality. Thus we have shown (4.6).

By Lemma 2.8

$$\ell(\gamma_k) \gtrsim \ell(\Gamma_k).$$

Then by (4.6) (4.3) and (4.4), this gives that there is an subcurve $\gamma_k' \subset \gamma_k$ such that

$$\operatorname{dist}(Z_k,\, \partial\Omega) \gtrsim \operatorname{dist}(\gamma_k',\, \partial\Omega)$$

and

$$\ell(\gamma_k') \sim \ell(\Gamma_k).$$

Combining this with (4.3) and (4.4) again, we have

$$\int_{\gamma_k} \operatorname{dist}(z, \partial\Omega)^{1-s}\, dz \gtrsim \int_{\gamma_k'} \operatorname{dist}(z, \partial\Omega)^{1-s}\, dz \gtrsim \operatorname{dist}(Z_k, \partial\Omega)^{2-s}. \tag{4.8}$$

We are left to consider the remaining values of k. If $k \geq 2$, then $A(z_1, k)$ is a full annulus and the analogs of (4.5) and (4.8) easily follow from Lemma 2.5. The only remaining values of k to consider are those potential k with

$$2^{k-1} \leq |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_3)| \leq 2^k$$

and

$$2^{k-1} \leq |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)| \leq 2^k.$$

For such k, (4.5) still holds and Lemma 2.5 shows that $\operatorname{dist}(Z_k, \partial\Omega) \sim \operatorname{dist}(Z_{k-1}, \partial\Omega)$. By our assumption $[z_2, z_3]$ is not contained in the union of two of our annuli, and hence these additional terms are controlled by the other terms. Consequently the claim follows by summing over k.

Finally if $z_1 = z_2$ we conclude (4.2) by picking $w_j \in [z_1, z_3] \cap \widetilde{\Omega}$ with $w_j \to z_1$ and applying the conclusion from the proof above (to $[w_j, z_3]$) and the monotone convergence theorem. □

4.2. **Assigning Whitney squares for reflection.** Let Ω be a Jordan domain. We will assign "reflected" squares in the Whitney decomposion $W = \{Q_i\}$ of Ω to squares $\widetilde{Q}_i$ in the Whitney decomposition $\widetilde{W} = \{\widetilde{Q}_i\}$ of the complementary domain $\widetilde{\Omega}$. This will only be done for those $\widetilde{Q}_i$ for which $\ell(\widetilde{Q}_i) \leq 3 \operatorname{diam}(\Omega)$. The construction of our extension operator will then rely on these squares. We continue under the asumption that Ω satisfies (4.1). In what follows we usually use the notation $\widetilde{A}$ to remind the readers that the set in question is contained in $\widetilde{\Omega}$.

Given a set $\widetilde{A} \subset \widetilde{\Omega}$, we consider all the hyperbolic rays in $\widetilde{\Omega}$ starting from ∞ and passing through $\widetilde{A}$, and define the *shadow* $S(\widetilde{A})$ as the set of all points where these rays hit the boundary $\partial\Omega$.

Similarly, we define $S(A)$ for $A \subset \Omega$, with the difference that the hyperbolic rays are now starting from $\varphi(0)$, where $\varphi\colon \mathbb{D} \to \Omega$ is a conformal map. If Ω happens to be John, we require that $\varphi(0)$ is the distinguished point of Ω and otherwise the center of one of the largest Whitney squares in Ω. Notice that the shadow of a connected set is connected. Moreover, for Whitney squares we have the following properties.

Lemma 4.3. *For each $Q_i \in W$, we have that $S(Q_i)$ is connected and $\operatorname{diam}(S(Q_i)) \gtrsim \ell(Q_i)$ for some absolute constant. Analogous properties hold for each $\widetilde{Q}_i \in \widetilde{W}$ that satisfies*

$$\ell(\widetilde{Q}_i) \leq M \operatorname{diam}(\Omega),$$

with the constant further depending only on M. Moreover, if Ω is John, then

$$\operatorname{diam}_{\Omega}(S(Q_i)) \sim_J \operatorname{diam}(S(Q_i)) \sim_J \ell(Q_i),$$

where the constant J here is the John constant.

Proof. Consider a conformal map $\varphi\colon \mathbb{D} \to \Omega$ and extend it continuously to the boundary as a homeomorphism. Then $\varphi^{-1}(Q_i)$ is connected. Therefore, by the fact that φ maps hyperbolic geodesics to hyperbolic geodesics, $\varphi^{-1}(S(Q_i))$ is connected, and so is $S(Q_i)$.

Additionally, by Lemma 2.6, $\varphi^{-1}(Q_i)$ is a Whitney-type set and hence the conformal capacity between $\varphi^{-1}(S(Q_i))$ and $\varphi^{-1}(Q_i)$ in $\mathbb{D}$ is bounded from below by a positive absolute constant; see (2.4). Since φ preserves conformal capacity, we obtain

$$\mathrm{Cap}(S(Q_i),\, Q_i,\, \Omega) \gtrsim 1,$$

and hence $\operatorname{diam}(S(Q_i)) \gtrsim \ell(Q_i)$ by Lemma 2.4. Indeed by the monotonicity of capacity we have

$$1 \lesssim \mathrm{Cap}(S(Q_i),\, Q_i,\, \Omega) \leq \mathrm{Cap}(S(Q_i),\, Q_i,\, \mathbb{R}^2),$$

which with Lemma 2.4 shows that

$$\operatorname{dist}(Q_i,\, S(Q_i)) \lesssim \operatorname{diam}(S(Q_i)). \tag{4.9}$$

Hence by the definition of Whitney squares

$$\ell(Q_i) \lesssim \operatorname{dist}(Q_i,\, \partial\Omega) \lesssim \operatorname{dist}(Q_i,\, S(Q_i)) \lesssim \operatorname{diam}(S(Q_i)).$$

The connectivity of $S(\widetilde{Q}_i)$ and the analogous estimate for $\widetilde{Q}_i$ follow similarly; notice that by the assumption $\ell(\widetilde{Q}_i) \leq 3\operatorname{diam}(\Omega)$, we have by(2.5)

$$\mathrm{Cap}(\widetilde{Q}_i, \partial\Omega, \widetilde{\Omega}) \geq C(M)$$

and hence by (2.5) again

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}_i), \mathbb{D}) \sim \operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_i)) \lesssim 1,$$

where the constants depend only on M. This implies that

$$1 \lesssim \mathrm{Cap}(\widetilde{\varphi}^{-1}(\widetilde{Q}_i),\, \widetilde{\varphi}^{-1}(S(\widetilde{Q}_i)),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \mathrm{Cap}(\widetilde{Q}_i,\, S(\widetilde{Q}_i),\, \widetilde{\Omega}).$$

Hence we may argue as earlier.

If Ω is John, then we know by Lemma 2.13 that hyperbolic rays are in fact John curves. Then by the definition of John curves and the triangle inequality we have

$$S(Q_i) \subset C(J)Q_i.$$

Especially

$$\operatorname{diam}(S(Q_i)) \lesssim \ell(Q_i),$$

and hence we can find a constant $C(J) > 2$ such that

$$\frac{1}{C(J)}\ell(Q_i) \leq \operatorname{diam}(S(Q_i)) \leq C(J)\ell(Q_i).$$

Finally we show that

$$\operatorname{diam}_{\Omega}(S(Q_i)) \sim \operatorname{diam}(S(Q_i)). \tag{4.10}$$

It suffices to prove $\operatorname{diam}_{\Omega}(S(Q_i)) \lesssim \operatorname{diam}(S(Q_i))$ since the other direction is trivial. First pick $x,\, y \in S(Q_i)$ such that

$$\operatorname{diam}_{\Omega}(S(Q_i)) \leq 3\operatorname{dist}_{\Omega}(x,\, y).$$

By the definition of inner distance, the hyperbolic geodesic Γ joining x, y satisfies

$$\operatorname{dist}_{\Omega}(x,\, y) \lesssim \ell(\Gamma).$$

Let z be the middle point (in the sense of length) of Γ. Then by the inner uniformity (3.9) of Ω we conclude that

$$\ell(\Gamma) \lesssim \operatorname{dist}(z,\, \partial\Omega).$$

Let Q' be a λ'-Whitney-type set such that $\varphi^{-1}(Q')$ is a closed ball of radius $\frac{1}{8}|1-\varphi^{-1}(z)|$ tangent to the circular arc $\varphi^{-1}(\Gamma)$ at z and contained in the Jordan domain enclosed by $\varphi^{-1}(\Gamma)$ and $\varphi^{-1}(\gamma)$; recall Lemma 2.6 and that $\varphi^{-1}(\Gamma)$ is a hyperbolic geodesic in $\mathbb{D}$. Here λ' is absolute. By the geometry of the unit disk we have $\varphi^{-1}(S(Q')) \subset \varphi^{-1}(S(Q_i))$ and then

$$\begin{aligned}\operatorname{diam}(\varphi^{-1}(S(Q_i))) &\geq \operatorname{diam}(\varphi^{-1}(S(Q'))) \sim \operatorname{diam}(\varphi^{-1}(Q')) \\ &\sim \operatorname{dist}(\varphi^{-1}(Q'),\, \varphi^{-1}(S(Q'))) \geq \operatorname{dist}(\varphi^{-1}(Q'),\, \varphi^{-1}(S(Q_i))).\end{aligned}$$

Hence by (2.4)

$$\operatorname{Cap}(\varphi^{-1}(Q'),\, \varphi^{-1}(S(Q_i)),\, \mathbb{D}) \gtrsim 1.$$

By the conformal invariance of capacity and the monotonicity,

$$1 \lesssim \operatorname{Cap}(Q',\, S(Q_i);\, \Omega) \leq \operatorname{Cap}(Q',\, S(Q_i);\, \mathbb{R}^2),$$

which with Lemma 2.4 implies

$$\operatorname{dist}(Q',\, S(Q_i)) \lesssim \operatorname{diam}(S(Q_i)).$$

Since Q' is a Whitney square and $z \in Q'$ we conclude that

$$\operatorname{dist}(z,\, \partial\Omega) \sim \operatorname{diam}(Q') \lesssim \operatorname{dist}(Q',\, S(Q_i)) \lesssim \operatorname{diam}(S(Q_i)).$$

To conclude, we have

$$\operatorname{diam}_{\Omega}(S(Q_i)) \lesssim \ell(\Gamma) \lesssim \operatorname{dist}(z,\, \partial\Omega) \lesssim \operatorname{diam}(S(Q_i)),$$

and accomplish our proof. □

Note that if we change the Whitney squares into λ-Whitney-type sets for fixed $\lambda \geq 1$ in the proof above, the conclusions still hold with constants further depend on λ; we only used the capacity estimates, Lemma 2.6, the connectivity of Whitney squares and the estimate

$$\operatorname{diam}(Q) \sim \operatorname{dist}(Q,\, \partial\Omega) \tag{4.11}$$

for Whitney squares. A λ-Whitney-type set is also connected and satisfies (4.11). Moreover, for a Jordan J-John domain Ω and a (closed) subarc $\gamma \subset \partial\Omega$, we always have

$$\operatorname{diam}_{\Omega}(\gamma) \sim \operatorname{diam}(\gamma)$$

with the constant depending only on J. This follows directly from the proof of (4.10) with notational changes; one just replaces $S(Q_i)$ by γ.

We need to associate a square $Q_i \in W$ to each square $\widetilde{Q}_i \in \widetilde{W}$ that satisfies

$$\ell(\widetilde{Q}_i) \leq 3\operatorname{diam}(\Omega).$$

Lemma 4.4. *Let Ω be a Jordan John domain with constant J. For each (closed) subarc $\gamma \subset \partial\Omega$, there exists a Whitney square $Q_i \in W$ satisfying*

$$\operatorname{diam}(S(Q_i)) \leq C(J)\operatorname{diam}(\gamma), \tag{4.12}$$

and

$$\operatorname{diam}(\gamma) \leq C(J)\operatorname{diam}(S(Q_i) \cap \gamma), \tag{4.13}$$

where $C(J)$ depends only on J.

Proof. Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map with $\varphi(0) = x_0$, where x_0 is the distinguished point in the John condition. Extend φ to the boundary as a homeomorphism. Given γ, let $\alpha = \varphi^{-1}(\gamma)$. We only need to show the existence for those γ that satisfy

$$\ell(\alpha) \le \frac{1}{2}.$$

Indeed suppose that $\ell(\alpha) > \frac{1}{2}$. Recall that Lemma 2.14 shows that φ is quasisymmetric with respect to the inner distance of Ω with η only depending on J. Then by taking $z_1,\, z_2 \in \alpha$ such that

$$\operatorname{dist}_\Omega(\varphi(z_1),\, \varphi(0)) = \operatorname{dist}_\Omega(\varphi(0),\, \gamma)$$

and

$$|z_1 - z_2| = \frac{1}{4},$$

quasisymmetry implies

$$\operatorname{dist}_\Omega(\varphi(0),\, \gamma) \le \eta(4) \operatorname{dist}_\Omega(\varphi(z_1),\, \varphi(z_2)) \le \eta(4) \operatorname{diam}_\Omega(\gamma).$$

By the John property and the triangle inequality we have

$$\operatorname{dist}_\Omega(\varphi(0),\, \partial\Omega) \gtrsim \operatorname{diam}(\Omega). \tag{4.14}$$

Moreover Lemma 4.3 implies that

$$\operatorname{diam}(\gamma) \sim \operatorname{diam}_\Omega(\gamma).$$

Thus from the inequalities above we conclude that

$$\operatorname{diam}(\gamma) \ge \frac{1}{C(J)} \operatorname{diam}(\partial\Omega).$$

Therefore in this case if one chooses the Whitney square containing $\varphi(0)$, then by (4.14) the desired conclusion follows.

When $\ell(\alpha) \le \frac{1}{2}$, denote the midpoint of α by w, let

$$r = \frac{\sin\left(\frac{\ell(\alpha)}{2}\right)}{1 + 2\sin\left(\frac{\ell(\alpha)}{2}\right)},\ z = (1-2r)w$$

and set $B = \overline{B(z,r)}$. Observe that by the assumption $\ell(\alpha) \le \frac{1}{2}$, the set B satisfies

$$2\operatorname{dist}(B,\, \partial\mathbb{D}) = 2r = \operatorname{diam}(B),$$

and is of 2-Whitney-type, and the shadow of B is exactly α.

Consider the collection of all Whitney squares that intersect $\varphi(B)$. Since $\varphi(B)$ is a λ-Whitney-type set by Lemma 2.6 for some absolute constant λ, this collection has no more than N elements for some universal N; see (2.2). Since φ can be extended to the boundary as a homeomorphism, the shadow of $\varphi(B)$ is exactly $\varphi(\alpha) = \gamma$. Thus the shadow of one of the N Whitney-squares, call it Q_i, satisfies

$$\operatorname{diam}(S(Q_i) \cap \gamma) \ge \operatorname{diam}(\gamma)/N$$

by the triangle inequality, the fact that γ is connected and the fact that the closed sets $S(Q_i)$ cover γ. This gives (4.13).

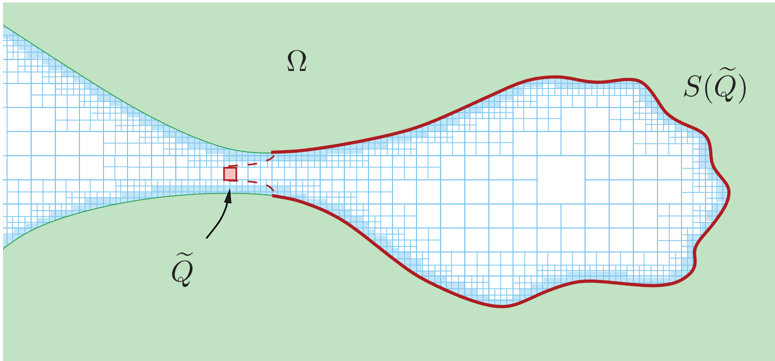

FIGURE 3. The shadow $S(\widetilde{Q})$ of a Whitney square $\widetilde{Q}$ of the complementary domain $\widetilde{\Omega}$ may have much larger diameter than the square in question.

For the remaining part, first notice that $\varphi^{-1}(Q_i) \cap B \neq \emptyset$ and that $\varphi^{-1}(Q_i)$ is a λ-Whitney-type set by Lemma 2.6 with a universal constant λ. Since B is of 2-Whitney-type, we conclude that

$$\operatorname{diam}(\varphi^{-1}(Q_i)) \lesssim \operatorname{diam}(B)$$

with an absolute constant. Then by Lemma 2.5

$$\operatorname{diam}(Q_i) \lesssim \operatorname{diam}(\varphi(B)). \tag{4.15}$$

By Lemma 4.3 (and the comment after it), we further have

$$\operatorname{diam}(S(Q_i)) \sim \operatorname{diam}(Q_i)$$

and

$$\operatorname{diam}(\varphi(B)) \sim \operatorname{diam}(\gamma)$$

since

$$S(\varphi(B)) = \gamma.$$

Combining these with (4.15) we conclude that

$$\operatorname{diam}(S(Q_i)) \lesssim \operatorname{diam}(\gamma).$$

Hence (4.12) follows.

$\square$

Lemma 4.4 allows us to associate a Whitney square in Ω to each Whitney square $\widetilde{Q}_i \in \widetilde{W}$. Indeed, simply choose $\gamma = S(\widetilde{Q}_i)$ in the lemma; observe that Lemma 4.3 ensures that $S(\widetilde{Q}_i)$ is a subarc of $\partial\Omega$.

Notice that there may be many possible candidates of Q_i for a given $\widetilde{Q}_i$, namely satisfying (4.12) and (4.13), and we just choose one of them. Since Ω is John, the Euclidean distance between any two of these candidates is no more than $C \operatorname{diam}(S(\widetilde{Q}_i))$, where C depends only on $C(J)$. However, a single Q_i may well be chosen for many distinct $\widetilde{Q}_j$, of different sizes: $S(\widetilde{Q}_i)$ can be much larger in size than $\widetilde{Q}_i$; see Figure 3. Even though the previous lemma does not require that $\ell(\widetilde{Q}_i) \leq 3 \operatorname{diam}(\Omega)$, the first estimate from Lemma 4.3 does require that $\ell(\widetilde{Q}_i) \leq M \operatorname{diam}(\Omega)$. Because of this, we only consider squares of the above type.

Let us relabel those $\widetilde{Q}_i \in \widetilde{W}$ with $\ell(\widetilde{Q}_i) \le 3\,\mathrm{diam}\,(\Omega)$ that have the same associated square via $Q_i \in W$. To be more specific, for each Q_i that is an associated square for some square in $\widetilde{W}$, consider all those squares $\widetilde{Q}_{ij}$ with $\ell(\widetilde{Q}_{ij}) \le 3\,\mathrm{diam}\,(\Omega)$ from $\widetilde{W}$ whose associated square is Q_i. Since $\widetilde{\varphi}$ is homeomorphic up to the boundary, $\mathrm{diam}\,(S(\widetilde{Q}_{ij}))$ tends to zero when $\mathrm{diam}\,(\widetilde{Q}_{ij})$ goes to zero, and hence there are only finitely many $\widetilde{Q}_{ij}$ corresponding to a fixed Q_i; say $\widetilde{Q}_{i1}, \dots, \widetilde{Q}_{il}$ where l may depend on i. We order them so that for every $1 \le j \le k \le l$, we have

$$\mathrm{diam}\,(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_{ij}))) \ge \mathrm{diam}\,(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_{ik}))),$$

where $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \mathbb{D} \to \widetilde{\Omega}$ is our fixed conformal map.

Given $Q_i \in W$ set $\widetilde{Q}_i^* := \{\widetilde{Q}_{i1}, \dots, \widetilde{Q}_{il}\}$, where $\widetilde{Q}_{ij}$ are as above. Observe that $\widetilde{Q}_i^*$ and $\widetilde{Q}_j^*$ have no common element when $i \neq j$. Next we prove an important estimate related to $\widetilde{Q}_i^*$.

Lemma 4.5. *For each $i \in \mathbb{N}$, we have*

$$\sum_{\widetilde{Q}_{ij} \in \widetilde{Q}_i^*} \ell(\widetilde{Q}_{ij})^{2-s} \lesssim \ell(Q_i)^{2-s}.$$

In order to prove this, we need two auxiliary lemmas.

Lemma 4.6. *Given C, there are no more than $N = N(C, J)$ pairwise disjoint (closed) subarcs γ_k of $S(Q_i)$ such that*

$$\mathrm{diam}\,(S(Q_i)) \le C\,\mathrm{diam}\,(\gamma_k).$$

The bound N depends only on C and the John constant of Ω.

Proof. Let $\gamma_1, \dots, \gamma_n$ be pairwise disjoint as in the statement. In order to bound n it suffices to associate to each γ_j a disk B_j of radius $r \ge \mathrm{diam}\,(S(Q_i))/C'$ so that these disks are pairwise disjoint and all have distance to $S(Q_i)$ no more than $C'\,\mathrm{diam}\,(S(Q_i))$, for a constant C' only depending on C, J.

Given k, let $x_k^{(1)}$ and $x_k^{(2)}$ be the two end points of γ_k.

First of all, observe that we may assume that $|x_k^{(1)} - x_k^{(2)}| \ge \frac{1}{3C}\,\mathrm{diam}\,(S(Q_i))$. Indeed, if $|x_k^{(1)} - x_k^{(2)}| < \frac{1}{3C}\,\mathrm{diam}\,(S(Q_i))$, then there exists a point $x_k^{(3)} \in \gamma_k$ such that $|x_k^{(1)} - x_k^{(3)}| \ge \frac{1}{3C}\,\mathrm{diam}\,(S(Q_i))$, and we replace $x_k^{(2)}$ with $x_k^{(3)}$. The existence of $x_k^{(3)} \in \gamma_k$ comes from the triangle inequality. Namely if there is no such a point, then for all $x, y \in \gamma_k$, we have

$$|x - y| \le |x - x_k^{(1)}| + |x_k^{(1)} - y| < \frac{2}{3C}\,\mathrm{diam}\,(S(Q_i)),$$

contradicting the condition that $\mathrm{diam}\,(S(Q_i)) \le C\,\mathrm{diam}\,(\gamma_k)$.

Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map with $\varphi(0) = x_0$, the distinguished point in the John condition. Let w_k be the midpoint of $\varphi^{-1}(\gamma_k)$, set

$$r_k = \frac{\sin\left(\frac{\ell(\varphi^{-1}(\gamma_k))}{2}\right)}{1 + 2\sin\left(\frac{\ell(\varphi^{-1}(\gamma_k))}{2}\right)}, \; z_k = (1 - 2r_k)w_k$$

and set $B_k = \overline{B\,(z_k, r_k)}$, such that $S(B_k) = \varphi^{-1}(\gamma_k)$. Since the arcs γ_k are pairwise disjoint, so are also $\varphi^{-1}(\gamma_k)$ and consequently also the sets B_k. Then the sets $\varphi(B_k)$ are also pairwise disjoint and of uniform Whitney-type by Lemma 2.6. From the proof of Lemma 4.3 (see the comment after Lemma 4.3) it follows that

$$\operatorname{diam}(\varphi(B_k)) \geq C(J) \operatorname{diam}(\gamma_k)$$

and (by (4.9) for λ-Whitney-type set)

$$\operatorname{dist}(\varphi(B_k), \gamma_k) \leq C(J) \operatorname{diam}(\gamma_k).$$

Then the claim follows by recalling that a λ-Whitney-type set A contains a disk of radius $\frac{1}{\lambda} \operatorname{diam}(A)$ and that $C \operatorname{diam}(\gamma_k) \geq \operatorname{diam}(S(Q_i))$; the sets $\varphi(B_k)$ are λ-Whitney-type for an absolute λ.

□

For a Whitney square $\widetilde{Q} \subset \widetilde{\Omega}$ and a hyperbolic ray Γ with $\Gamma \cap \widetilde{Q} \neq \emptyset$, corresponding to a point $z \in \partial\Omega$, we define the *tail* of Γ with respect to $\widetilde{Q}$ to be the arc of Γ between z and $\widetilde{Q}$. Denote this set by $T(\Gamma, \widetilde{Q})$. Then the tail of $\widetilde{Q}$ is

$$T(\widetilde{Q}) = \{y \in \widetilde{\Omega} \mid y \in T(\Gamma, \widetilde{Q}) \text{ for some } \Gamma\}.$$

Lemma 4.7. *Let $\widetilde{Q}_0 \subset \widetilde{\Omega}$ be a Whitney square with $\ell(\widetilde{Q}_0) \leq 3 \operatorname{diam}(\Omega)$. Then for any Whitney square $\widetilde{Q} \subset \widetilde{\Omega}$ satisfying $\widetilde{Q} \cap T(\widetilde{Q}_0) \neq \emptyset$, we have*

$$\ell(\widetilde{Q}) \lesssim \operatorname{diam}(S(\widetilde{Q}_0)).$$

The constant here is absolute.

Proof. Let $\widetilde{\varphi}$ be a conformal map $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$. First of all, we claim that

$$\operatorname{diam}(\widetilde{Q}) \lesssim \operatorname{diam}(\Omega), \tag{4.16}$$

with an absolute constant. In fact, since $\ell(\widetilde{Q}_0) \leq 3 \operatorname{diam}(\Omega)$, by (2.5) we have

$$\operatorname{Cap}(\widetilde{Q}_0,\, \partial\Omega,\, \widetilde{\Omega}) \gtrsim 1.$$

By Lemma 2.6, $\varphi^{-1}(\widetilde{Q}_0)$ is of λ-Whitney-type with an absolute constant λ. Moreover as φ^{-1} preserves conformal capacity,

$$\operatorname{Cap}(\varphi^{-1}(\widetilde{Q}_0),\, \partial\mathbb{D},\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \gtrsim 1.$$

Hence

$$\operatorname{dist}(\varphi^{-1}(\widetilde{Q}_0),\, \mathbb{D}) \lesssim 1.$$

By the assumption that $\widetilde{Q} \cap T(\widetilde{Q}_0) \neq \emptyset$ and the geometry of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, we also have

$$\operatorname{diam}(\varphi^{-1}(\widetilde{Q})) \sim \operatorname{dist}(\varphi^{-1}(\widetilde{Q}),\, \mathbb{D}) \lesssim 1,$$

since $\varphi^{-1}(\widetilde{Q})$ is also a λ-Whitney-type set by Lemma 2.6. Then by (2.4) and the conformal invariance of the capacity

$$\operatorname{Cap}(\widetilde{Q},\, \partial\Omega,\, \widetilde{\Omega}) = \operatorname{Cap}(\varphi^{-1}(\widetilde{Q}),\, \partial\mathbb{D},\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \gtrsim 1.$$

Now by the fact that $\widetilde{Q}$ is a Whitney square, we conclude (4.16) by Lemma 2.4; notice that since $\widetilde{\Omega}$ is the exterior domain of a Jordan domain Ω, then

$$\operatorname{diam}_{\widetilde{\Omega}}(\partial\Omega) \leq 3 \operatorname{diam}(\partial\Omega).$$

Recall that by Lemma 2.6 the preimage of each Whitney square of $\widetilde{\Omega}$ is of λ-Whitney-type with an absolute constant λ. If $\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})) \geq c_1 \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_0)))$ with

$$c_1 = \min\left\{\frac{1}{9}, \frac{1}{6\lambda}, \frac{1}{8\lambda^2}\right\}, \tag{4.17}$$

then by the fact that $\widetilde{\varphi}^{-1}(\widetilde{Q})$ and $\widetilde{\varphi}^{-1}(\widetilde{Q}_0)$ are of λ-Whitney-type with an absolute constant λ, and the assumption $\widetilde{\varphi}^{-1}(\widetilde{Q}) \cap \widetilde{\varphi}^{-1}(T(\widetilde{Q}_0)) \neq \emptyset$, we know that

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\, \varphi^{-1}(\widetilde{Q}_0)) \lesssim \operatorname{diam}(\varphi^{-1}(\widetilde{Q}_0)) \sim \operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}))$$

according to the geometry of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. Hence there are at most C Whitney squares of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ between $\widetilde{\varphi}^{-1}(\widetilde{Q}_0)$ and $\widetilde{\varphi}^{-1}(\widetilde{Q})$ for some absolute constant C. Then by Lemma 2.6 and (2.2), there are at most C' Whitney squares of $\widetilde{\Omega}$ between $\widetilde{Q}_0$ and $\widetilde{Q}$ with another universal constant C'. Therefore we have $\ell(\widetilde{Q}) \lesssim \ell(\widetilde{Q}_0)$ with an absolute constant. By Lemma 4.3 and the assumption that

$$\ell(\widetilde{Q}_0) \leq 3 \operatorname{diam}(\Omega),$$

we conclude that $\ell(\widetilde{Q}) \lesssim \operatorname{diam}(S(\widetilde{Q}_0))$.

Now let us consider the case where

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})) < c_1 \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_0))). \tag{4.18}$$

If $\widetilde{Q} \subset T(\widetilde{Q}_0)$ then by Lemma 4.3 with (4.16) again we have

$$\ell(\widetilde{Q}) \lesssim \operatorname{diam}(S(\widetilde{Q})) \lesssim \operatorname{diam}(S(\widetilde{Q}_0)).$$

If not, let $d = \operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}))$. By (4.18) and (4.17), we have that

$$6\lambda d \leq \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_0))). \tag{4.19}$$

Thus by the geometry of $\varphi^{-1}(T(\widetilde{Q}_0))$ and the definition of d, we know that $\widetilde{\varphi}^{-1}(\widetilde{Q})$ only intersects one of the two hyperbolic rays in $\mathbb{R}^2 \setminus \mathbb{D}$ which have (non-constant) subarcs contained in the boundary of $\widetilde{\varphi}^{-1}(T(\widetilde{Q}_0))$; let Γ be this hyperbolic ray. Also let Γ' be the hyperbolic ray in $\mathbb{R}^2 \setminus \mathbb{D}$ which intersects $\widetilde{\varphi}^{-1}(T(\widetilde{Q}_0))$ and

$$\operatorname{dist}(\Gamma,\, \Gamma') = 2\lambda d, \tag{4.20}$$

and let z be the point on Γ' with $|z| = 1 + d$. Denote by $\widetilde{Q}'$ be the preimage under $\widetilde{\varphi}$ of some Whitney square so that $z \in \widetilde{Q}'$. Then since $\widetilde{Q}'$ is of λ-Whitney-type, by Definition 2.3, (4.17) and (4.18) we conclude that

$$\operatorname{diam}(\widetilde{Q}') + \operatorname{dist}(\widetilde{Q}',\, \partial\mathbb{D}) \leq \lambda d + d < \frac{1}{4\lambda} \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_0))).$$

Moreover the geometry of the exterior of the unit disk implies

$$\frac{1}{4\lambda} \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_0))) < \frac{1}{\lambda} \operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_0)) \leq \operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}_0),\, \partial\mathbb{D}).$$

To conclude, for any point $x \in Q'$

$$\operatorname{dist}(x,\, \partial\mathbb{D}) \leq \operatorname{diam}(\widetilde{Q}') + \operatorname{dist}(\widetilde{Q}',\, \partial\mathbb{D}) < \operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}_0),\, \partial\mathbb{D}); \tag{4.21}$$

especially

$$\widetilde{Q}' \cap \widetilde{\varphi}^{-1}(\widetilde{Q}_0) = \emptyset.$$

Furthermore, since
$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}')) \leq \lambda d,$$
by (4.19) and (4.20) we know that $\widetilde{Q}'$ does not intersect either of the two hyperbolic rays in $\mathbb{R}^2 \setminus \mathbb{D}$ which have (non-constant) subarcs contained in the boundary of $\widetilde{\varphi}^{-1}(T(\widetilde{Q}_0))$. This implies that
$$S(\widetilde{Q}') \subset \widetilde{\varphi}^{-1}(S(Q_0)).$$
Hence by the geometry of $\widetilde{\varphi}^{-1}(T(\widetilde{Q}_0))$, with (4.21) we have $\widetilde{Q}' \subset \widetilde{\varphi}^{-1}(T(\widetilde{Q}_0))$, or equivalently
$$\widetilde{\varphi}(\widetilde{Q}') \subset T(\widetilde{Q}_0).$$
Moreover by the definition of z and that $\widetilde{Q}'$ is of λ-Whitney-type, we have
$$\operatorname{dist}(\widetilde{Q}', \mathbb{D}) \sim d.$$
This with (4.20), the definition of d and the assumption that $\Gamma \cap \widetilde{\varphi}^{-1}(\widetilde{Q}) \neq \emptyset$ gives that
$$\operatorname{dist}(\widetilde{Q}', \widetilde{\varphi}^{-1}(\widetilde{Q})) \lesssim d = \operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})).$$
Thus by the geometry of the exterior of the disk there are at most C Whitney squares of $\mathbb{D}$ between $\widetilde{Q}'$ and $\widetilde{\varphi}^{-1}(\widetilde{Q})$ for some absolute constant C. Then by Lemma 2.6 and (2.2), there are at most C' Whitney squares of $\widetilde{\Omega}$ between $\widetilde{\varphi}(\widetilde{Q}')$ and $\widetilde{Q}$ with another universal constant C'. Therefore
$$\operatorname{diam}(\widetilde{\varphi}(\widetilde{Q}')) \sim \operatorname{diam}(\widetilde{Q}).$$
Since $\widetilde{\varphi}(\widetilde{Q}') \subset T(\widetilde{Q}_0)$ and similarly as the previous case we obtain the desired estimate by Lemma 4.3. □

Note that this lemma also holds for λ-Whitney-type sets with the constant depending on λ; see Lemma 4.10 below for the modification.

Proof of Lemma 4.5. First of all let us consider a conformal map $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$. Let ρ_0 be a hyperbolic ray that intersects $\widetilde{Q}_i^*$. Denote by ρ the tail of ρ_0 with respect to the square in
$$\{\widetilde{Q}_{ij} \in \widetilde{Q}_i^* \mid \rho_0 \cap \widetilde{Q}_{ij} \neq \emptyset\}$$
whose preimage under $\widetilde{\varphi}$ is furthest away from the origin, that is, the last square of these that ρ_0 hits towards infinity. Let $\widetilde{Q}_0$ be this square.

We claim that $\ell(\rho) \lesssim \ell(Q_i)$. First we prove that, for every square $\widetilde{Q} \in \widetilde{W}$ intersecting ρ, we have
$$\ell(\widetilde{Q}) \lesssim \operatorname{diam}(S(Q_i)) \sim \ell(Q_i).$$
By Lemma 4.3 and (4.13), we know that
$$\ell(\widetilde{Q}_{ij}) \lesssim \operatorname{diam}(S(\widetilde{Q}_{ij})) \lesssim \operatorname{diam}(S(Q_i)) \sim \ell(Q_i), \tag{4.22}$$
for all j; especially this holds for $\widetilde{Q}_0$. For the rest of the squares $\widetilde{Q}$ satisfying $\widetilde{Q} \cap \rho \neq \emptyset$, we have $\widetilde{Q} \cap T(\widetilde{Q}_0) \neq \emptyset$. By Lemma 4.7, (4.13) and (4.22) for $\widetilde{Q}_0$ we also get
$$\ell(\widetilde{Q}) \lesssim \operatorname{diam}(S(\widetilde{Q}_0)) \lesssim \operatorname{diam}(S(Q_i)) \sim \ell(Q_i),$$
as desired. Therefore, by Lemma 4.2 we have
$$\ell(Q_i)^{1-s}\ell(\rho) \lesssim \int_\rho \operatorname{dist}(z, \partial\Omega)^{1-s}\, dz \lesssim \ell(\rho)^{2-s}, \tag{4.23}$$

and finally

$$\ell(\rho) \lesssim \ell(Q_i). \tag{4.24}$$

Now, Lemma 4.2 and (4.24) give

$$\int_\rho \operatorname{dist}(z, \partial\Omega)^{1-s}\, dz \lesssim \ell(Q_i)^{2-s}. \tag{4.25}$$

Set

$$\widetilde{Q}^*_{i,\rho} = \left\{ \widetilde{Q}_{ij} \in Q^*_i \mid \widetilde{Q}_{ij} \cap \rho \neq \emptyset \right\}.$$

Since $\widetilde{Q}_{ij}$ are Whitney squares and $\ell(\widetilde{Q}_{ij}) \lesssim \ell(Q_i)$ for each $\widetilde{Q}_{ij} \in \widetilde{Q}^*_{i,\rho}$, then (4.25) easily gives

$$\sum_{\widetilde{Q}_{ij} \in \widetilde{Q}^*_{i,\rho}} \ell(\widetilde{Q}_{ij})^{2-s}\, dz \lesssim \ell(Q_i)^{2-s}. \tag{4.26}$$

Indeed first we note that each Whitney square has at most 20 neighboring squares, which tells us that we can divide the squares in $\widetilde{Q}^*_{i,\rho}$ into at most 21 subcollections $\{\widetilde{W}_k\}_{k=1}^{21}$ such that in each of the subcollections the squares are pairwise disjoint. Then for any two distinct $\widetilde{Q}_i, \widetilde{Q}_j \in \widetilde{W}_k$, by Lemma 2.2 we have

$$1.1\widetilde{Q}_i \cap 1.1\widetilde{Q}_j = \emptyset.$$

Notice that for each $\widetilde{Q}_{ij} \in \widetilde{Q}^*_{i,\rho_0}$, by definition we have

$$\mathcal{H}^1(1.1\widetilde{Q}_{ij} \cap \rho) \geq 0.1\ell(\widetilde{Q}_{ij}),$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Thus by the fact that

$$\ell(\widetilde{Q}_{ij}) \sim \operatorname{dist}(\widetilde{Q}_{ij}, \partial\Omega)$$

and (4.25), we have

$$\begin{aligned}\sum_{\widetilde{Q}_{ij} \in \widetilde{Q}^*_{i,\rho}} \ell(\widetilde{Q}_{ij})^{2-s}\, dz &\lesssim \sum_{k=1}^{21} \sum_{\widetilde{Q}_{ij} \in \widetilde{W}^*_k} \int_{\rho \cap \widetilde{Q}_{ij}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds \\ &\lesssim \int_\rho \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds \lesssim \ell(Q_i)^{2-s};\end{aligned}$$

which shows (4.26).

Recall that our finite collection of the squares $\widetilde{Q}_{ij} \in Q^*_i$ is ordered with respect to j, say $1 \leq j \leq k$, so that the diameters of $\widetilde{\varphi}^{-1}(S(\widetilde{Q}_{ij}))$ decrease when j increases. We choose $\widetilde{Q}^1_i = \widetilde{Q}_{ik}$. If $S(\widetilde{Q}_{ij}) \cap S(\widetilde{Q}^1_i) = \emptyset$ for $j = k-1$ we set $\widetilde{Q}^2_i = \widetilde{Q}_{ij}$. Otherwise we consider $\widetilde{Q}_{ij}$ with $j = k-2$ as a candidate for $\widetilde{Q}^2_i$ and continue inductively. Namely we choose $\widetilde{Q}^2_i$ to be $\widetilde{Q}_{ij}$ such that j is the largest integer smaller than k such that

$$S(\widetilde{Q}_{ij}) \cap S(\widetilde{Q}^1_i) = \emptyset.$$

Then choose $\widetilde{Q}^3_i$ to be $\widetilde{Q}_{ij}$ with the largest j such that its shadow does not intersect either $S(\widetilde{Q}^1_i)$ or $S(\widetilde{Q}^2_i)$, and continue this process. This gives us $\widetilde{Q}^1_i, \cdots, \widetilde{Q}^n_i$ with pairwise disjoint shadows. Since these squares come from $\widetilde{Q}^*_i$, Lemma 4.6 gives us a universal bound on n in terms of $C(J)$; see (4.12) and (4.13).

Let $\widetilde{Q}_{ij} \in \widetilde{Q}_i^*$. By the construction in the previous paragraph, there is an index l so that $S(\widetilde{Q}_{ij}) \cap S(\widetilde{Q}_i^l) \neq \emptyset$. Suppose that $\widetilde{Q}_{ij}$ is not one of the chosen squares $\widetilde{Q}_i^l$. Since shadows are connected, at least one of the end points of $S(\widetilde{Q}_i^l)$ is contained in $S(\widetilde{Q}_{ij})$; otherwise $S(\widetilde{Q}_{ij})$ is strictly contained in $S(\widetilde{Q}_i^l)$ since these shadows are closed and connected, which means

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_i^l))) > \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_{ij}))),$$

contradicting our selection of the squares $\widetilde{Q}_i^l$. Therefore by assigning two hyperbolic rays to each $\widetilde{Q}_i^l$ we obtain a collection of $2n$ hyperbolic rays that intersects all of our squares $\widetilde{Q}_{ij}$ in $\widetilde{Q}_i^*$.

Our claim follows by combining the estimate from the previous paragraph on the number of hyperbolic rays necessary to catch our squares $\widetilde{Q}_{ij}$ with (4.26).

□

4.3. **The extension operator in the Jordan case.** Let

$$B_\Omega = B(x_0, 2\operatorname{diam}(\Omega))$$

be a disk compactly containing Ω, where φ is a conformal map such that $\varphi(0) = x_0$ with x_0 the John center of Ω. Observe that by Lemma 2.2, we have

$$\ell(\widetilde{Q}) \leq \operatorname{dist}(\widetilde{Q}, \partial\Omega)$$

for each $\widetilde{Q} \in \widetilde{W}$. Then if $\widetilde{Q} \cap B_\Omega \neq \emptyset$, by definition we get

$$\ell(\widetilde{Q}) \leq \operatorname{dist}(\widetilde{Q}, \partial\Omega) \leq 2\operatorname{diam}(\Omega).$$

Hence

$$B_\Omega \subset \overline{\Omega} \cup \bigcup_{\substack{\widetilde{Q} \in \widetilde{W} \\ \ell(\widetilde{Q}) \leq 3\operatorname{diam}(\Omega)}} \widetilde{Q}.$$

We define

$$Eu(x) = \sum_j a_{Q_j} \phi_j(x)$$

for given $u \in W^{1,p}(\Omega)$ and $x \in B_\Omega \setminus \overline{\Omega}$. Here

$$a_{Q_j} = \fint_{Q_j} u(z)\, dz,$$

where Q_j is the square associated to $\widetilde{Q}_j$ with $\ell(\widetilde{Q}_j) \leq 3\operatorname{diam}(\Omega)$ and $\phi_j \in C^\infty(\widetilde{\Omega})$ is compactly supported in $\frac{11}{10}\widetilde{Q}_j$, $|\nabla \phi_j| \lesssim \ell(\widetilde{Q}_j)^{-1}$, and

$$\sum_j \phi_j(x) = 1,$$

for all $x \in \widetilde{\Omega}$ contained in a Whitney square of side length no more than $3\operatorname{diam}(\Omega)$. Notice that the support of ϕ_i and that of ϕ_j have no intersection unless $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$. See [18] for the existence of such a partition of unity $\{\phi_j\}$. Especially, Eu is defined in $B_\Omega \setminus \overline{\Omega}$. We will prove that $\|Eu\|_{W^{1,p}(B_\Omega \setminus \overline{\Omega})} \lesssim \|u\|_{W^{1,p}(\Omega)}$.

Let us first estimate the norm of the extension inside such a square $\widetilde{Q} \in \widetilde{W}$. Denote by $\widehat{|\nabla u|}$ the zero extension of $|\nabla u|$, and by M the maximal function operator. Before going to the general case, we first establish the estimate in a special case.

Lemma 4.8. *Given distinct Whitney squares $Q_1,\, Q_2 \subset \Omega$ such that*

$$\operatorname{dist}_{\Omega}(S(Q_1),\, S(Q_2)) \lesssim \ell(Q_1) \sim \ell(Q_2), \tag{4.27}$$

we have

$$|a_{Q_1} - a_{Q_2}| \le C_0 \ell(Q_1)^{-1} \int_{Q_1} M(\widehat{|\nabla u|})(z)\, dz.$$

Here C_0 only depends on s and C in (4.1) *and the constants in* (4.27).

Proof. Let $\varphi : \mathbb{D} \to \Omega$ be a conformal map. Recall that it extends homeomorphically up to the boundary. We further assume that $\varphi(0) = x_0$, the distinguished point in the definition of a John domain; recall that Ω is John with constant only depending on s and the constant in (4.1). Additionally, φ is quasisymmetric with respect to the inner distance by Lemma 2.14.

Notice that by the geometry of the unit disk and Lemma 2.6, for $k = 1, 2$

$$\operatorname{dist}(\varphi^{-1}(Q_k),\, \varphi^{-1}(S(Q_k))) \lesssim \operatorname{diam}(\varphi^{-1}(Q_k)),$$

and then the quasisymmetry of φ (Lemma 2.14) gives

$$\operatorname{dist}_{\Omega}(Q_k,\, S(Q_k)) \lesssim \operatorname{diam}_{\Omega}(Q_k) \sim \ell(Q_k).$$

Then it follows from the triangle inequality, Lemma 4.3 and (4.27) that

$$\begin{aligned} \operatorname{dist}_{\Omega}(Q_1,\, Q_2) \lesssim & \operatorname{dist}_{\Omega}(Q_1,\, S(Q_1)) + \operatorname{diam}_{\Omega}(S(Q_1)) + \operatorname{dist}_{\Omega}(S(Q_1),\, S(Q_2)) \\ & + \operatorname{diam}_{\Omega}(S(Q_2)) + \operatorname{dist}_{\Omega}(Q_2,\, S(Q_2)) \\ \lesssim & \ell(Q_1). \end{aligned}$$

By Lemma 2.7 we deduce that the length of the hyperbolic geodesic between the centers of Q_1 and Q_2 is comparable to $\ell(Q_1)$. Moreover, since a simply connected John domain is inner uniform (see Definition 3.7) by [2],[38] it follows that this hyperbolic geodesic provides us with a John subdomain $\Omega_{Q_1,\,Q_2} \subset \Omega \cap CQ_1$ of diameter no more than $C\ell(Q_1)$ containing both Q_1 and Q_2, where C only depends on the John constant J. For example, take

$$\Omega_{Q_1,\,Q_2} = Q_1 \cup Q_2 \cup \bigcup_{z \in \Gamma} B\left(z,\, 3^{-1} \operatorname{dist}(z,\, \partial\Omega)\right),$$

where Γ is the hyperbolic geodesic joining the centers of Q_1 and Q_2. By the fact that Γ is a John curve in Ω it is easy to see that $\Omega_{Q_1,\,Q_2}$ is also a John domain with the John constant depending only on J, and

$$\operatorname{diam}(\Omega_{Q_1,\,Q_2}) \lesssim \ell(Q_1);$$

Indeed for any point $z \in \Omega_{Q_1,\,Q_2}$, the following curve γ is the John curve of $\Omega_{Q_1,\,Q_2}$: first the curve goes from z to the $z_1 \in \Gamma$, where $z \in B\left(z,\, 3^{-1} \operatorname{dist}(z_1,\, \partial\Omega)\right)$, or z_1 is the center of Q_k if $z \in Q_k$ for some $k = 1, 2$, and then it coincides with $\Gamma[z_0,\, z_1]$, where z_0 is the middle point (in the sense of length) of Γ (and also the John center of $\Omega_{Q_1,\,Q_2}$); one checks via the inner uniformity (3.9) that the curve defined above is a John curve with some uniform constant.

By letting

$$a = \fint_{\Omega_{Q_1,\,Q_2}} u\, dz,$$

the Poincaré inequality on Ω_{Q_1, Q_2} from [3] (with the constant depending only on J) and (4.27) imply

$$\begin{aligned}|a_{Q_1} - a_{Q_2}| \le |a_{Q_1} - a| + |a_{Q_2} - a| &\lesssim \fint_{Q_1} |u - a| \, dz + \fint_{Q_2} |u - a| \, dz \\ &\lesssim \ell(Q_1)^{-1} \int_{\Omega_{Q_1, Q_2}} |\nabla u(z)| \, dz \lesssim \ell(Q_1) \fint_{CQ_1} \widehat{|\nabla u|}(z) \, dz \\ &\lesssim \ell(Q_1) \fint_{Q_1} M(\widehat{|\nabla u|})(z) \, dz \lesssim \ell(Q_1)^{-1} \int_{Q_1} M(\widehat{|\nabla u|})(z) \, dz.\end{aligned}$$

□

If for a fixed Whitney square $\widetilde{Q}$ and any of its neighboring Whitney squares $\widetilde{Q}_k$ we have that their reflected squares Q, Q_k satisfy (4.27), then by Lemma 4.8 and Hölder's inequality

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(\widetilde{Q})} &\lesssim \int_{\widetilde{Q}} \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} |a_{Q_k} - a_Q|^p |\nabla \phi_k(x)|^p \, dx \\ &\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} \ell(\widetilde{Q}_k)^{2-p} \ell(Q)^{p-2} \int_Q (M(\widehat{|\nabla u|})(z))^p \, dz,\end{aligned}$$

where in the first inequality we used

$$\nabla u(x) = \nabla(u(x) - a) = \nabla \left(\sum_j \phi_j(x)(a_{Q_j} - a) \right).$$

If this estimate could be used for all pairs, then it together with Hölder's inequality, the definition of our extension and changing the order of summation would give

$$\|\nabla Eu\|^p_{L^p(B_\Omega \setminus \Omega)} \lesssim \sum_i \sum_{\widetilde{Q}_{ij} \in \widetilde{Q}_i^*} \ell(\widetilde{Q}_{ij})^{2-p} \ell(Q_i)^{p-2} \int_{Q_i} (M(\widehat{|\nabla u|})(z))^p \, dz.$$

Then by Lemma 4.5 for $s = p$ (by Lemma 2.10 and Hölder's inequality, our (4.1) with $s > p$ gives (4.1) for $s = p$ and hence also Lemma 4.5 for $s = p$), we would conclude that

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(B_\Omega \setminus \Omega)} &\lesssim \sum_i \int_{Q_i} (M(\widehat{|\nabla u|})(z))^p \, dz \\ &\lesssim \int_\Omega \widehat{|\nabla u|}^p (z) \, dz = \|u\|^p_{W^{1,p}(\Omega)}.\end{aligned}$$

Especially we obtain the desired control for the case of those squares that satisfy (uniformly) the assumption of our lemma.

Unfortunately, the reflected squares of neighboring Whitney square $\widetilde{Q}_1$ and $\widetilde{Q}_2$ need not have comparable size (see Figure 4), and hence we cannot directly rely on Lemma 4.8. To fix this problem, we need to find a chain of suitable squares connecting Q_1 and Q_2 inside Ω to be able to use our estimate.

Notice that if

$$\frac{1}{8} \operatorname{diam}(S(\widetilde{Q}_2)) \le \operatorname{diam}(S(\widetilde{Q}_1)) \le 8 \operatorname{diam}(S(\widetilde{Q}_2)), \tag{4.28}$$

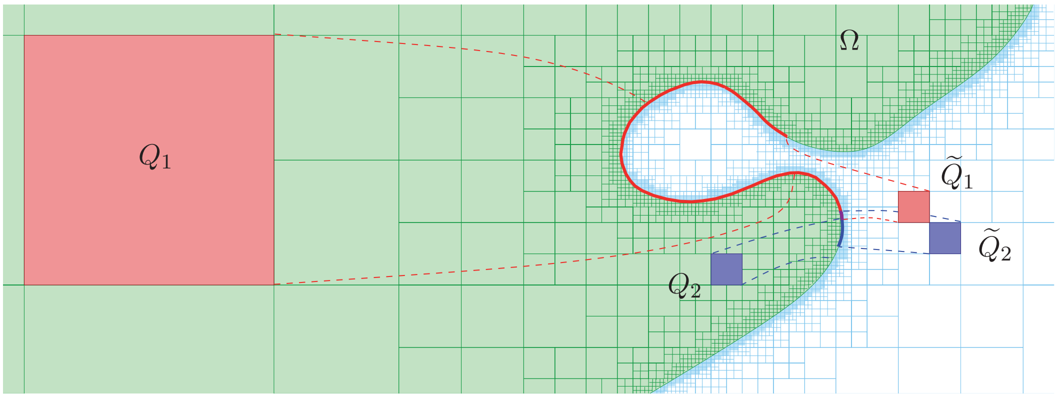

FIGURE 4. The shadows of neighboring squares $\widetilde{Q}_1$ and $\widetilde{Q}_2$ can differ significantly in size from each other. Consequently the reflected squares Q_1 and Q_2 may be of very different size.

then by Lemma 4.3, Lemma 4.4 and the fact that $\widetilde{Q}_1 \cap \widetilde{Q}_2 \neq \emptyset$, we have that (4.27) holds for Q_1, Q_2 with the constants depending only on J. This has been already considered by Lemma 4.8. Hence without loss of generality, we may assume that

$$8\,\mathrm{diam}\,(S(\widetilde{Q}_1)) \leq \mathrm{diam}\,(S(\widetilde{Q}_2)),$$

and hence $\ell(Q_1) \lesssim \ell(Q_2)$. As usual we only consider the squares $\widetilde{Q}$ satisfying $\ell(\widetilde{Q}) \leq 3\,\mathrm{diam}\,(\Omega)$.

Take a connected closed set $\widetilde{F}^1$ (a fake square) such that

$$\widetilde{Q}_1 \subset \widetilde{F}^1 \subset \widetilde{Q}_1 \cup \widetilde{Q}_2,\ S(\widetilde{Q}_1) \subset S(\widetilde{F}^1)$$

and

$$2\,\mathrm{diam}\,(S(\widetilde{F}^1)) = \mathrm{diam}\,(S(\widetilde{Q}_1 \cup \widetilde{Q}_2)). \tag{4.29}$$

The existence of $\widetilde{F}^1$ is clear since $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \mathbb{D} \to \overline{\widetilde{\Omega}}$ is a homeomorphism and conformal outside $\overline{\mathbb{D}}$. For example, we can construct $\widetilde{F}^1$ in the following way. Since $\widetilde{\varphi}$ is a homeomorphism, we know that both $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_1)$ and $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_2)$ are two Jordan curves, and they intersect each other. Suppose that $z \in \partial\widetilde{Q}_1 \cap \partial\widetilde{Q}_2$. Then parameterizing $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_2)$ via $\gamma\colon [0,\,1] \to \widetilde{\varphi}^{-1}(\partial\widetilde{Q}_2)$ with $\gamma(0) = \gamma(1) = z$, by continuity there is $0 < t < 1$ such that, by letting $\widetilde{F}^1 = \varphi(\gamma[0,\,t] \cup \widetilde{Q}_1)$, we have that (4.29) holds; notice that the preimage under $\widetilde{\varphi}$ of hyperbolic rays are radial rays, and then $\widetilde{\varphi}^{-1}(S(\partial\widetilde{Q}_2)) = \widetilde{\varphi}^{-1}(S(\widetilde{Q}_2))$. Then by our construction it is clear that $\widetilde{Q}_1 \subset \widetilde{F}^1 \subset \widetilde{Q}_1 \cup \widetilde{Q}_2$, and thus it is a desired set.

Notice that $\widetilde{F}^1$ is a Whitney-type set since $\ell(\widetilde{Q}_1) \sim \ell(\widetilde{Q}_2) \sim \mathrm{diam}\,(\widetilde{F}^1)$ and $\widetilde{Q}_1 \subset \widetilde{F}^1$. By Lemma 4.4, there is a Whitney square $Q^1 \subset \Omega$ such that

$$\mathrm{diam}\,(S(Q^1)) \leq C(J)\,\mathrm{diam}\,(S(\widetilde{F}^1)),$$

and

$$\mathrm{diam}\,(S(\widetilde{F}^1)) \leq C(J)\,\mathrm{diam}\,(S(Q^1) \cap S(\widetilde{F}^1)),$$

where $C(J)$ depends only on J; see the comment after Lemma 4.3.

Next we pick a connected closed set $\widetilde{F}^2$ such that $\widetilde{Q}_1 \subset \widetilde{F}^2 \subset \widetilde{F}^1 \subset \widetilde{Q}_1 \cup \widetilde{Q}_2$, $S(\widetilde{Q}_1) \subset S(\widetilde{F}^2)$ and

$$4\,\mathrm{diam}\,(S(\widetilde{F}^2)) = \mathrm{diam}\,(S(\widetilde{Q}_1 \cup \widetilde{Q}_2)),$$

and select a Whitney square $Q^2 \subset \Omega$ such that

$$\mathrm{diam}\,(S(Q^2)) \leq C(J)\,\mathrm{diam}\,(S(\widetilde{F}^2)),$$

and

$$\mathrm{diam}\,(S(\widetilde{F}^2)) \leq C(J)\,\mathrm{diam}\,(S(Q^2) \cap S(\widetilde{F}^2)),$$

where $C(J)$ depends only on J. We continue this process until we have

$$\frac{1}{2}\,\mathrm{diam}\,(S(\widetilde{F}^l)) \leq \mathrm{diam}\,(S(\widetilde{Q}_1)) \leq \mathrm{diam}\,(S(\widetilde{F}^l))$$

for some $l \in \mathbb{N}$.

Denote by $G(\widetilde{Q}_1, \widetilde{Q}_2)$ the collection of the Whitney squares $\{Q^l\}$ defined above together with Q_1, Q_2. We set $G(\widetilde{Q}_1, \widetilde{Q}_2) = G(\widetilde{Q}_2, \widetilde{Q}_1)$, and when (4.28) holds, let

$$G(\widetilde{Q}_1, \widetilde{Q}_2) = G(\widetilde{Q}_2, \widetilde{Q}_1) = \{Q_1, Q_2\}.$$

If $Q_m \in W$ is the Whitney square associated to $\widetilde{F}^l$, we denote $\widetilde{F}^l$ by $\widetilde{F}_m^{1,2}$. Here the upper indices 1, 2 are used to remind that $\widetilde{Q}_1 \subset \widetilde{F}^l = \widetilde{F}_m^{1,2} \subset \widetilde{Q}_1 \cup \widetilde{Q}_2$. Notice that all the fake squares $\widetilde{F}^l$ are of $8\sqrt{2}$-Whitney-type. See Figure 5 for an illustration.

By symmetry we also construct the fake squares and find their corresponding Whitney squares in W in the case where

$$8\,\mathrm{diam}\,(S(\widetilde{Q}_2)) \leq \mathrm{diam}\,(S(\widetilde{Q}_1)).$$

Accordingly we define sets $\widetilde{F}_m^{2,1}$ and the chain $G(\widetilde{Q}_2, \widetilde{Q}_1)$ and set $G(\widetilde{Q}_1, \widetilde{Q}_2) = G(\widetilde{Q}_2, \widetilde{Q}_1)$. Notice that either $\widetilde{F}_m^{1,2}$ or $\widetilde{F}_m^{2,1}$ exists.

Define the index set $I(m)$ as

$$\begin{aligned} I(m) = \{i \in \mathbb{N} \mid Q_m \in W, Q_m \in G(\widetilde{Q}_i, \widetilde{Q}_j) \\ \text{for some } \widetilde{Q}_i, \widetilde{Q}_j \in \widetilde{W} \text{ with } \widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset,\ \ell(\widetilde{Q}_i) \leq 3\,\mathrm{diam}\,(\Omega)\}. \end{aligned}$$

Now we need a stronger version of Lemma 4.5.

Lemma 4.9. *For each $m \in \mathbb{N}$, we have*

$$\sum_{i \in I(m)} \ell(\widetilde{Q}_i)^{2-s} \lesssim \ell(Q_m)^{2-s}.$$

The reader may expect that in Lemma 4.9 we have sum over all the sets which are associated to Q_m, namely all the (real) Whitney squares in $\widetilde{Q}_m^*$ and all the fake squares $\widetilde{F}_m^{i,j}$. However in the summation of Lemma 4.9 for the fake squares, each i is only considered once by the definition of $I(m)$. This is not a problem since for each fixed i with the corresponding Whitney square $\widetilde{Q}_i$, there are at most 21 Whitney squares $\widetilde{Q}_j$ satisfying $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$, and hence there are at most 21 fake squares $\widetilde{F}_m^{i,j}$ with fixed i and m; similarly there are also at most 21 fake squares $\widetilde{F}_m^{j,i}$ with fixed i and m. Therefore if we sum over all the sets associated to Q_m, each $i \in I(m)$ is referred at most uniformly finitely many times. Hence this lemma is enough for us after interchanging the order of summation later in the Proof of Proposition 4.1.

The proof of Lemma 4.9 is a modification to the proof of Lemma 4.5. For the sake of completeness we state and prove the necessary analog of Lemma 4.7.

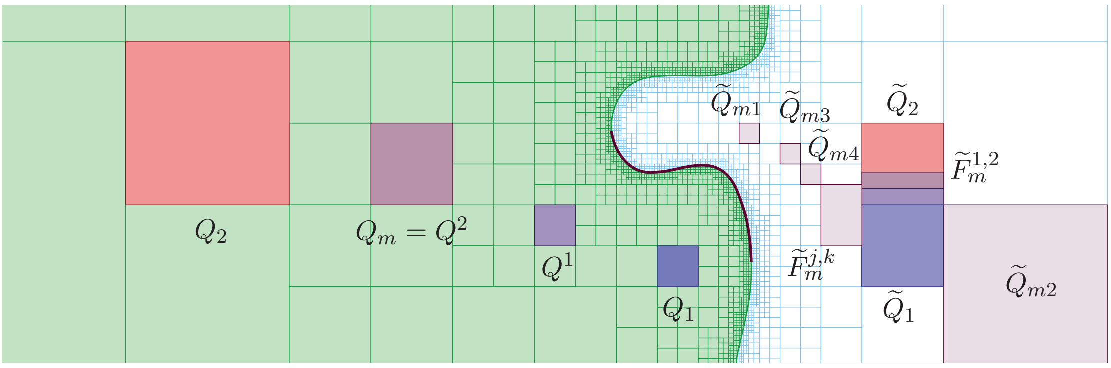

FIGURE 5. A square $Q_m \in W$ might be associated to several squares $\widetilde{Q}_{mk}$ as well as to fake squares $\widetilde{F}_m^{j,k}$. In the illustration the squares $\widetilde{Q}_1$ and $\widetilde{Q}_2$ give rise to two fake squares, one of which is associated with Q_m. Another fake square as well as four (real) squares that are associated with Q_m are exhibited. Also the shadow of Q_m is shown.

Lemma 4.10. *Let $\widetilde{Q}_0 \subset \widetilde{\Omega}$ be a λ-Whitney-type set with $\ell(\widetilde{Q}_0) \leq 3\,\mathrm{diam}\,(\Omega)$. Then for any λ-Whitney-type set $\widetilde{Q} \subset \widetilde{\Omega}$ satisfying $\widetilde{Q} \cap T(\widetilde{Q}_0) \neq \emptyset$, we have*

$$\ell(\widetilde{Q}) \lesssim \mathrm{diam}\,(S(\widetilde{Q}_0)),$$

where the constant depends only on λ.

Proof. In the proof of Lemma 4.7 the only things we use for the Whitney squares are (ii) of Lemma 2.2, Lemma 2.6 and Lemma 4.3. They also hold for the λ-Whitney-type sets in question, with constants which further depend on λ. Hence the conclusion follows similarly to the proof of Lemma 4.7. $\square$

Proof of Lemma 4.9. There are two kinds of Whitney-type squares taken into account in the summation: the Whitney squares in $\widetilde{\Omega}$ that are associated to Q_m, and the squares $\widetilde{Q}_i$ contained $\widetilde{F}_m^{i,j}$ or the squares $\widetilde{Q}_i$ with its neighbor $\widetilde{Q}_j$ such that $\widetilde{F}_m^{j,i} \subset \widetilde{Q}_i \cup \widetilde{Q}_j$ for some $i \in I(m)$ and $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$; recall that

$$\mathrm{diam}\,(\widetilde{F}_m^{i,j}) \sim \ell(\widetilde{Q}_i) \sim \ell(\widetilde{Q}_j) \tag{4.30}$$

if $\widetilde{F}_m^{i,j}$ exists, and an analogy to $\widetilde{F}_m^{j,i}$.

We only need to discuss the sets in the latter case, because Lemma 4.5 gives the estimate for the squares belonging to the first one. It suffices to consider $\widetilde{F}_m^{i,j}$ with $i \in I(m)$ since the by (4.30) the case for $\widetilde{F}_m^{j,i}$ follows from a similar argument.

Since each $\widetilde{Q}_i$ only has a uniformly bounded number of neighbors, $\widetilde{F}_m^{i,j}$ are Whitney-type sets with uniformly finite overlaps (for a fixed m). Moreover, these sets are chosen such that

$$\mathrm{diam}\,(S(Q_m)) \leq C(J)\,\mathrm{diam}\,(S(\widetilde{F}_m^{i,j})),$$

and

$$\mathrm{diam}\,(S(\widetilde{F}_m^{i,j})) \leq C(J)\,\mathrm{diam}\,(S(Q_m) \cap S(\widetilde{F}_m^{i,j})).$$

We next follow the argument for Lemma 4.5. First of all let us consider the conformal map $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$. Let ρ_0 be a hyperbolic ray that intersects at least one of the Whitney-type sets $\{\widetilde{F}_m^{i,j}\}_{i\in I(m)}$. Denote by ρ the tail of ρ_0 with respect to a Whitney-type set in

$$\{\widetilde{F}_m^{i,j} \mid i \in I(m),\, \rho_0 \cap \widetilde{F}_m^{i,j} \neq \emptyset\}$$

whose preimage under $\widetilde{\varphi}$ is furthest away from the origin, that is, the a Whitney-type set of these that ρ_0 hits towards infinity. Let $\widetilde{F}_0$ be this set.

We have already shown that

$$\ell(\rho) \lesssim \ell(Q_m), \tag{4.31}$$

in the proof of Lemma 4.5; even though at this time ρ ends at $\widetilde{F}_0$, however by the definition of $\widetilde{F}_0$ and Lemma 4.3 we know that

$$\mathrm{diam}\,(\widetilde{F}_0) \lesssim \mathrm{diam}\,(S(Q_m)) \sim \ell(Q_m),$$

and hence 4.31 follows up to a multiplicative constant by Lemma 4.10. Now, Lemma 4.2 and (4.31) give

$$\int_\rho \mathrm{dist}\,(z,\, \partial\Omega)^{1-s}\, dz \lesssim \ell(Q_m)^{2-s}. \tag{4.32}$$

Set

$$I_\rho(m) = \left\{ i \in I(m) \mid \widetilde{F}_m^{i,j} \cap \rho \neq \emptyset \right\}.$$

Recall that each fake square is contained in the union of a neighboring pair of Whitney squares. Notice that if two fake squares intersect each other, then their corresponding neighboring pairs of Whitney squares also intersect. For a fixed pair of Whitney squares, there are at most uniformly finitely many Whitney squares intersecting them, and thus there are at most uniformly finitely many (unions of) neighboring pairs of Whitney squares intersecting them. Hence by the definition of Whitney-type sets and the relations between $F_m^{i,j}$ and $\widetilde{Q}_i \in \widetilde{W}$, namely

$$\frac{1}{8}\,\mathrm{diam}\,(\widetilde{Q}_i) \leq \mathrm{diam}\,(F_m^{i,j}) \leq 8\,\mathrm{diam}\,(\widetilde{Q}_i),$$

(4.32) gives

$$\sum_{i\in I_\rho(m)} \ell(\widetilde{Q}_i)^{2-s}\, dz \lesssim \ell(Q_i)^{2-s}; \tag{4.33}$$

see the corresponding part in the proof of Lemma 4.5 for a similar argument.

Relabel all the Whitney-type sets $\widetilde{F}_m^{i,j}$ by $\widetilde{F}_n$ with respect to n, say $1 \leq n \leq k$ so that the diameters of $\widetilde{\varphi}^{-1}(S(\widetilde{F}_n))$ decrease when n increases. We choose $\widetilde{F}_m^1 = \widetilde{F}_n$. If $S(\widetilde{F}_{n-1}) \cap S(\widetilde{F}_m^1) = \emptyset$ we set $\widetilde{F}_m^2 = \widetilde{F}_{n-1}$. Otherwise we consider $\widetilde{F}_{n-2}$ as a candidate for $\widetilde{F}_m^2$ and continue inductively. Namely we choose $\widetilde{F}_m^2$ to be $\widetilde{F}_n$ such that n is the largest integer smaller than k such that

$$S(\widetilde{F}_n) \cap S(\widetilde{F}_m^1) = \emptyset.$$

Then choose $\widetilde{F}_m^3$ to be $\widetilde{F}_n$ with the largest n such that its shadow does not intersect either $S(\widetilde{F}_m^1)$ or $S(\widetilde{F}_m^2)$, and continue this process. This gives us $\widetilde{F}_m^1, \cdots, \widetilde{F}_m^{n_0}$ with pairwise disjoint shadows. By the construction of these sets, Lemma 4.6 gives us a universal bound on n_0 in terms of $C(J)$; see (4.12) and (4.13).

Let $\widetilde{F}_i$ be some set which is not chosen. By the construction in the previous paragraph, there is an index l so that $S(\widetilde{F}_i) \cap S(\widetilde{F}_m^l) \neq \emptyset$. Notice that $\widetilde{F}_i$ is not one of the chosen sets $\widetilde{F}_m^l$. Since shadows are connected, at least one of the end points of $S(\widetilde{F}_m^l)$ is contained in $S(\widetilde{F}_i)$; otherwise $S(\widetilde{F}_i)$ is strictly contained in $S(\widetilde{F}_m^l)$ since these shadows are closed and connected, which means

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{F}_m^l))) > \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{F}_i))),$$

contradicting our selection of the squares $\widetilde{F}_m^l$. Therefore by assigning two hyperbolic rays to each $\widetilde{F}_m^l$ we obtain a collection of $2n_0$ hyperbolic rays that intersects all of our sets $\widetilde{F}_m^{i,j}$ for $i \in I(m)$.

Our claim follows by combining the estimate from the previous paragraph on the number of hyperbolic rays necessary to catch our sets $\widetilde{F}_m^{i,j}$ with (4.33). $\square$

Now we can estimate the norm of the gradient of our extension over each square $\widetilde{Q} \in \widetilde{W}$.

Lemma 4.11. *For all $\widetilde{Q} \in \widetilde{W}$ with $\ell(\widetilde{Q}) \leq 3\operatorname{diam}(\Omega)$, we have*

$$\|\nabla Eu\|^p_{L^p(\widetilde{Q})} \lesssim \sum_k \sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(\widetilde{Q})^{2-s} \ell(Q_m)^{s-2} \int_{Q_m} M(|\widehat{\nabla u}|)(z))^p \, dz,$$

where the sum is over all the k's for which $\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset$.

Proof. Let $\widetilde{\varphi} : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$ and $\varphi : \mathbb{D} \to \Omega$ be conformal maps. Recall that both of them extend homeomorphically up to the boundary. We further assume that $\varphi(0) = x_0$, the distinguished point in the definition of a John domain; recall that Ω is John with constant only depending on s and the constant in (4.1).

Fix $\widetilde{Q}_k$ with $\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset$. Notice that

$$\operatorname{dist}(S(\widetilde{Q}_k),\, S(\widetilde{Q})) = 0 \tag{4.34}$$

since

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(S(\widetilde{Q}_k)),\, \widetilde{\varphi}^{-1}(S(\widetilde{Q}))) = 0$$

and $\widetilde{\varphi}$ is a homeomorphism. Next, consider the corresponding squares $Q, Q_k \subset \Omega$. From the definition, we know that

$$\operatorname{diam}(S(Q)) \leq C(J) \operatorname{diam}(S(\widetilde{Q}))$$

and

$$\operatorname{diam}(S(\widetilde{Q})) \leq C(J) \operatorname{diam}(S(Q) \cap S(\widetilde{Q})) \leq C(J) \operatorname{diam}(S(Q)). \tag{4.35}$$

Moreover, the corresponding inequalities hold for the pair $\widetilde{Q}_k, Q_k$. Thus Lemma 4.3 and (4.35) give that

$$\begin{aligned} \operatorname{dist}_\Omega(S(Q),\, S(Q_k)) &\lesssim \operatorname{diam}_\Omega(S(Q)) + \operatorname{diam}_\Omega(S(\widetilde{Q})) + \operatorname{diam}_\Omega(S(\widetilde{Q}_k)) + \operatorname{diam}_\Omega(S(Q_k)) \\ &\lesssim \max\{\operatorname{diam}_\Omega(S(Q)),\, \operatorname{diam}_\Omega(S(Q_k))\}. \end{aligned}$$

In conclusion, by Lemma 4.3 again we have that

$$\operatorname{dist}_\Omega(S(Q_k), S(Q)) \lesssim \max\{\ell(Q_k), \ell(Q)\}.$$

Similarly, for any pair of consecutive squares $Q^l,\, Q^{l+1} \in G(\widetilde{Q}, \widetilde{Q}_k)$ we have that

$$\operatorname{dist}_\Omega(S(Q^l), S(Q^{l+1})) \lesssim \max\{\ell(Q^l), \ell(Q^{l+1})\}. \tag{4.36}$$

Moreover, $\ell(Q^l) \sim \ell(Q^{l+1})$ by our construction, and hence (4.27) holds for each pair of consecutive squares in $G(\widetilde{Q}, \widetilde{Q}_k)$. Further notice that the side lengths of these squares form a geometric-type sequence by Lemma 4.3 and the rules which we applied to choose these squares. Here a positive sequence $\{a_j\}$ is called *a geometric-type sequence* if there exists a constant $c_1 > 0$ such that

$$\#\{a_j \colon a_j \in (2^{k-1}, 2^k]\} \le c_1 \quad \text{and} \quad \frac{1}{c_1} \le \frac{a_j}{a_{j+1}} \le c_1$$

for any $k \in \mathbb{Z}$, where $\#$ denotes the cardinality of the set. If $\{a_j\}$ is a geometric-type sequence and $a_j \le c_2$ for all j, then

$$\sum_j a_j \le 4c_1c_2. \tag{4.37}$$

For $\{\ell(Q^l)\}$ our constant c_1 is independent of $\widetilde{Q}$, $\widetilde{Q}_k$ and the choices of the squares Q^l.

Since $\{\phi_k\}$ is a partition of unity, we know that

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(\widetilde{Q})} &\lesssim \int_{\widetilde{Q}} \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} |a_{Q_k} - a_Q|^p |\nabla \phi_k(x)|^p \, dx \\ &\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} |a_{Q_k} - a_Q|^p \ell(\widetilde{Q})^{-p} |\widetilde{Q}| \lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} |a_{Q_k} - a_Q|^p \ell(\widetilde{Q})^{2-p}.\end{aligned}$$

Let $\epsilon = \frac{s-p}{p} > 0$. We apply Lemma 4.8 via (4.36), Hölder's inequality and the fact that the side lengths of the squares in our sequence form a geometric-type sequence with (4.37) to get

$$\begin{aligned}|a_{Q_k} - a_Q|^p &\lesssim \left(\sum_{Q^l \in G(\widetilde{Q}, \widetilde{Q}_k)} |a_{Q^l} - a_{Q^{l+1}}| \right)^p \\ &\lesssim \left(\sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(Q_m) ⨍_{Q_m} M(|\widehat{\nabla u}|)(z) \, dz \right)^p \\ &\lesssim \left[\sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(Q_m)^{1+\epsilon-\epsilon} \left(⨍_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz \right)^{\frac{1}{p}} \right]^p \\ &\lesssim \left(\sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(Q_m)^{p+p\epsilon} ⨍_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz \right) \left(\sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(Q_m)^{-\frac{\epsilon p}{p-1}} \right)^{p-1} \\ &\lesssim \min\{\ell(Q), \ell(Q_k)\}^{-\epsilon p} \sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(Q_m)^{p+p\epsilon-2} \int_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz \\ &\lesssim \sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(\widetilde{Q})^{-\epsilon p} \ell(Q_m)^{p+p\epsilon-2} \int_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz,\end{aligned}$$

where the last step comes from Lemma 4.3, (4.12) and (4.13). By recalling $\epsilon p = s - p$, we get

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(\widetilde{Q})} &\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} |a_{Q_k} - a_Q|^p \ell(\widetilde{Q})^{2-p} \\
&\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q} \neq \emptyset} \sum_{Q_m \in G(\widetilde{Q}, \widetilde{Q}_k)} \ell(\widetilde{Q})^{2-s} \ell(Q_m)^{s-2} \int_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz,
\end{aligned}$$

which gives the claim. □

Proof of Proposition 4.1. By interchanging the order of summation with respect to Q_m, and by the fact that each Whitney square has uniformly finitely many neighbors, we obtain from Lemmas 4.11 and 4.9 (with the comment after it) the estimate

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(B_\Omega \setminus \Omega)} &\lesssim \sum_m \sum_{i \in I(m)} \ell(\widetilde{Q}_i)^{2-s} \ell(Q_m)^{s-2} \int_{Q_m} (M(|\widehat{\nabla u}|)(z))^p \, dz \\
&\lesssim \sum_m \int_{Q_i} (M(|\widehat{\nabla u}|)(z))^p \, dz \\
&\lesssim \int_\Omega |\widehat{\nabla u}|^p(z) \, dz \leq \|u\|^p_{W^{1,p}(\Omega)}.
\end{aligned}$$

It remains to check that E generates a Sobolev function. Indeed, it then follows that we have extended u to B_Ω with a norm bound, and extendability to the entire plane follows from the fact that disks are extension domains (with operator norms independent of the radius for the homogeneous norm).

Towards this, notice that by the definition of E and Hölder's inequality

$$\begin{aligned}
\|Eu\|^p_{L^p(B_\Omega \setminus \Omega)} &\lesssim \sum_i \sum_{\widetilde{Q}_{ij} \in \widetilde{Q}_i^*} \ell(\widetilde{Q}_{ij})^2 \left(\fint_{Q_i} |u| \, dx \right)^p \\
&\lesssim \sum_i \sum_{\widetilde{Q}_{ij} \in \widetilde{Q}_i^*} \ell(\widetilde{Q}_{ij})^2 \ell(Q_i)^{-2} \int_{Q_i} |u|^p \, dx \\
&\lesssim \sum_i \int_{Q_i} |u|^p \, dx \lesssim \int_\Omega |u|^p \, dx, \qquad (4.38)
\end{aligned}$$

where we use the fact that

$$\sum_j \ell(\widetilde{Q}_{ij})^2 \lesssim \ell(Q_i)^2$$

since for every $\widetilde{Q}_{ij} \in \widetilde{Q}_i^*$ we have $\widetilde{Q}_{ij} \subset CQ_i$ by Lemma 4.3 and Lemma 4.5 (with Lemma 2.10). Recall that our operator E is linear by definition and $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for $1 < p < \infty$ if Ω is a planar Jordan domain, see [27]. By our norm estimates above, it thus suffices to show that, for $u \in W^{1,p}(\Omega) \cap C^\infty(\mathbb{R}^2)$, by defining the extension in $B_\Omega \setminus \overline{\Omega}$ as above and setting $Eu(x) = u(x)$ when $x \in \overline{\Omega}$, we obtain a function in $W^{1,p}(B_\Omega)$. For this, it suffices to show that our extended function is continuous at every point of B_Ω. Indeed, since Ω is a John domain, [19, Theorem 4] then guarantees that the above definition gives a Sobolev function. Notice that Eu is clearly continuous (even smooth) in $B_\Omega \setminus \overline{\Omega}$ and smooth in Ω. Hence we are reduced to show continuity at every $x \in \partial\Omega$.

Recall that Ω is Jordan. This implies that $\operatorname{diam}(S(\widetilde{Q}))$ tends to zero uniformly when $\ell(\widetilde{Q})$ tends to zero. Given $x \in \partial\Omega$ and points x_k converging to x from within $\widetilde{\Omega}$, pick Whitney squares $\widetilde{Q}_k$ containing x_k. Then by the fact that $\{\phi_j\}$ forms a partition of unity, we have

$$\begin{aligned}|Eu(x_k) - u(x)| &= \left| \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} a_j \phi_j(x_k) - \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} \phi_j(x_k) u(x) \right| \\ &\leq \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} \phi_j(x_k) |a_j - u(x)|.\end{aligned}$$

Since $\widetilde{Q}_k$ tends to x, then its neighboring squares also tend to x, and so do their shadows; one sees this via the conformal map φ in the complement of the disk. Furthermore by Lemma 4.3 and Lemma 4.4, the associated squares Q_k also converge to x; analogy as statement also hold for the squares associated to the neighbors of $\widetilde{Q}_k$. Thus we have

$$u(x_k) \to u(x)$$

by the assumption that u is the restriction of a smooth (especially continuous) function to Ω. □

4.4. **Proof of the general case.** We complete the proof for the general case of a bounded simply connected domain Ω by approximation.

Recall that we are claiming the existence of a bounded extension operator under the condition (1.1) for a given bounded simply connected domain Ω. We have already verified a weaker version of this if Ω is Jordan.

In order to be able to prove the general case by using the result for the Jordan case, we need a sequence of approximating Jordan domains to have extension operators with uniform norm bounds. For this purpose we have stated the dependence of the norm of the extension operator in Proposition 4.1 explicitly.

From now on, Ω is a bounded simply connected domain that satisfies (1.1) and $\varphi\colon \mathbb{D} \to \Omega$ is a conformal map. Towards the existence of a suitable approximating sequence, recall that (1.1) guarantees that Ω is John and finitely connected along its boundary, see Lemma 2.13. Thus we can extend φ continuously up to the boundary, see [30, Theorem 4.7]. We still denote the extended map by φ. Let $B_n = B(0, 1 - \frac{1}{n})$ for $n \geq 2$. Then $\Omega_n = \varphi(B_n)$ are Jordan John domains (with constant independent of n) contained in Ω by Lemma 2.14, and converge to Ω uniformly because of the uniform continuity of φ up to the boundary. Actually, φ is even uniformly Hölder continuous [10], [31].

We divide the proof into two steps. First we prove that the complementary domain of Ω_n satisfies condition (4.1) with a constant that is independent of n. In the second step, we apply Proposition 4.1 to Ω_n and complete the proof by a compactness argument.

Lemma 4.12. *Each of the complementary domains $\widetilde{\Omega}_n$ of Ω_n satisfies condition* (4.1) *with curves $\gamma \subset \widetilde{\Omega}_n$ for fixed $s > p$ and a constant independent of n.*

Proof. Fix $n \geq 2$. First we notice that, if z_1 and z_2 are both outside Ω, then condition (4.1) follows immediately from (1.1) and the self-improving property from Lemma 2.11, since $\operatorname{dist}(z, \partial\Omega) \leq \operatorname{dist}(z, \partial\Omega_n)$ for $z \in \mathbb{R}^2 \setminus \Omega$. Hence we may assume that $z_1 \in \Omega \setminus \Omega_n$.

Suppose first that $\varphi^{-1}(z_2) \in B(\varphi^{-1}(z_1), (1 - |\varphi^{-1}(z_1)|)/2) := B$. Then the existence of the desired curve easily follows from Lemma 2.5. Indeed, because of the geometry of $B \setminus \overline{B_n}$,

$\varphi^{-1}(z_2)$ and $\varphi^{-1}(z_1)$ can be joined in $B \setminus \overline{B_n}$ by a curve for which the analog of (4.1) holds with a universal constant. Then by Lemma 2.5 and the fact that B is of 2-Whitney-type, via changing of variable one shows that the image of this curve satisfies (4.1) with a multiplicative constant. The desired conclusion also follows if the roles of z_1, z_2 above are reversed. Applying Lemma 2.6 and the definition of Whitney-type set, we know that there exists an absolute constant C such that if

$$C|z_1 - z_2| \leq \max\{\operatorname{dist}(z_1, \partial\Omega),\ \operatorname{dist}(z_2, \partial\Omega)\},$$

then we are in one of the above cases. Thus we may assume that

$$C|z_1 - z_2| \geq \max\{\operatorname{dist}(z_1, \partial\Omega),\ \operatorname{dist}(z_2, \partial\Omega)\}.$$

provided also z_2 is contained in $\Omega \setminus \Omega_n$.

Recall from Lemma 2.14 that φ is quasisymmetric with respect to the inner distance. We now employ Lemma 2.14 and simple geometry to find an open disk U contained in $B(0,1)\setminus\overline{B_n}$ so that $z_1 \in \varphi(\overline{U})$, $\varphi(\overline{U}) \cap \partial\Omega \neq \emptyset$, $\operatorname{diam}(\varphi(U)) \leq C \operatorname{dist}(z_1, \partial\Omega)$, and $\varphi(U)$ is C-John with a constant only depending on the John constant of Ω. By connecting z_1 to the John center of $\varphi(U)$ and then the John center to the boundary, we obtain a curve $\Gamma_1 \subset U$ joining z_1 to $\partial\Omega$ so that

$$\int_{\Gamma_1} \operatorname{dist}(z,\, \partial\Omega_n)^{1-s}\, dz \leq \int_{\Gamma_1} \operatorname{dist}(z,\, \partial U)^{1-s}\, dz \lesssim \operatorname{dist}(z_1, \partial\Omega)^{2-s}.$$

Analogously, if $z_2 \in \Omega \setminus \Omega_n$, we find a corresponding curve for z_2. In this case, it remains to join the two endpoints of Γ_1 and Γ_2 in $\partial\Omega$ by a curve Γ_3 outside Ω guaranteed by our assumption. It is easy to check that the curve composed from Γ_1, Γ_2 and Γ_3 satisfies our requirements.

Finally, if $z_2 \notin \Omega$ above, we simply use Γ_1 and a curve Γ_3 joining z_2 and the endpoint of Γ_1 in $\partial\Omega$ as above. □

Proof of Theorem 1.1. By Section 3, we only need to prove the sufficiency of (1.1).

Pick a the conformal map $\varphi\colon \mathbb{D} \to \Omega$ with $\varphi(0) = x_0$ the John center of Ω. By Lemma 2.14 and the comment after it, the domains

$$\Omega_n = \varphi(B_n)$$

are John domains with John center x_0, where the John constants depend only on J.

Let $B = B(x_0,\, 1.5 \operatorname{diam}(\Omega))$. Then $B \subset B_{\Omega_n}$ when n is large enough; recall that

$$B_{\Omega_n} = B(x_0,\, 2 \operatorname{diam}(\Omega_n))$$

and Ω_n converges to Ω (in the Hausdorff distance). Then by Lemma 4.12 and Proposition 4.1, there exists an extension operator

$$E_n\colon W^{1,p}(\Omega_n) \to W^{1,p}(B),$$

for n large enough as $B \subset B_{\Omega_n}$.

Fix $u \in W^{1,p}(\Omega)$, and let $u_n = u|_{\Omega_n}$ for $n \geq 2$. Since the norms of the extension operators E_n depend only on p and the constant C in condition (4.1), $\|\nabla E_n u_n\|_{L^p(B)} + \|E_n u_n\|_{L^p(B)}$ is bounded independently of n; note that $\|\nabla E_n u_n\|_{L^p(B)}$ does not depend on $\operatorname{diam}(\Omega)$, while $\|E_n u_n\|_{L^p(B)}$ does. Hence by the assumption $p > 1$, there exists a subsequence weakly converging to some $v \in W^{1,p}(B)$. Define $Eu := v$. Observe that the sequence $\{E_n u_n\}$ converges to u pointwise a.e. on Ω. Hence we know that Eu is an extension of u, and the desired norm bound over B follows from the uniform bound on the extension operators E_n and the lower semicontinuity of the norm. Since B is a $W^{1,p}$-extension domain, this completes the proof of Theorem 1.1. $\square$

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PLANAR $W^{1,1}$-EXTENSION DOMAINS

PEKKA KOSKELA, TAPIO RAJALA, AND YI RUYA ZHANG

Abstract. We show that a bounded planar simply connected domain Ω is a $W^{1,1}$-extension domain if and only if for every pair x, y of points in Ω^c there exists a curve $\gamma \subset \Omega^c$ connecting x and y with

$$\int_\gamma \frac{1}{\chi_{\mathbb{R}^2 \setminus \partial\Omega}(z)}\, dz \le C|x-y|.$$

Consequently, a planar Jordan domain Ω is a $W^{1,1}$-extension domain if and only if it is a BV-extension domain, and if and only if its complementary domain $\widetilde{\Omega}$ is a $W^{1,\infty}$-extension domain.

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1. Introduction

Given a domain $\Omega \subset \mathbb{R}^2$, define for $1 \le p \le \infty$ the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \nabla u \in L^p(\Omega, \mathbb{R}^2) \right\},$$

where ∇u denotes the distributional gradient of u. One usually employs $W^{1,p}(\Omega)$ with the non-homogeneous norm $\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega,\mathbb{R}^2)}$ or with the homogeneous norm $\|\nabla u\|_{L^p(\Omega,\mathbb{R}^2)}$.

We call $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ an extension operator for Ω if there exists a constant $C \ge 1$ so that for every $u \in W^{1,p}(\Omega)$ we have

$$\|Eu\|_{W^{1,p}(\mathbb{R}^2)} \le C\|u\|_{W^{1,p}(\Omega)}$$

and $Eu|_\Omega = u$. A domain $\Omega \subset \mathbb{R}^2$ is called a $W^{1,p}$-extension domain if there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. By [15], a bounded domain Ω admits a bounded extension operator for the non-homogeneous norm if and only if it admits one for the homogeneous norm, when $1 \le p < \infty$.

Geometric properties of simply connected planar $W^{1,p}$-extension domains are by now rather well understood, see [11], [16], [24] and [18]. In particular, a full geometric characterization for bounded simply connected planar $W^{1,p}$-extension domains is available for $1 < p \le \infty$. For the range $1 < p < 2$ bounded simply connected planar $W^{1,p}$-extension domains were characterized in [18] as those bounded domains $\Omega \subset \mathbb{R}^2$ for which any two points $x, y \in \mathbb{R}^2 \setminus \Omega$ can be connected with a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, dz \lesssim |x-y|^{2-p}. \tag{1.1}$$

Regarding $p = 1$, in [17, Corollary 1.2] it was shown that the complement of a bounded simply connected $W^{1,1}$-extension domain is quasiconvex. This was obtained as a corollary to a characterization of bounded simply connected BV-extension domains. Recall that a set $E \subset \mathbb{R}^2$ is called C-*quasiconvex* if there exists a constant $C \ge 1$ such that any pair of points $z_1, z_2 \in E$ can be connected by a rectifiable curve $\gamma \subset E$ whose length $\ell(\gamma)$ satisfies $\ell(\gamma) \le C|z_1 - z_2|$.

Quasiconvexity of the complement does not imply $W^{1,1}$-extendability in general. Indeed, a slit disk

$$\Omega = \mathbb{D} \setminus \{(x, 0) \mid x \ge 0\}$$

has a quasiconvex complement but Ω fails to be a $W^{1,1}$-extension domain. In this paper we show that in order to obtain a characterization of $W^{1,1}$-extendability in the spirit of the curve estimate (1.1), in addition to the quasiconvexity of the complement, we have to take into account the size of the set of self-intersections of the boundary $\partial\Omega$. Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Ω is a $W^{1,1}$-extension domain if and only if there exists a constant $C < \infty$ such that for each pair $x, y \in \Omega^c$ of points there exists a curve $\gamma \subset \Omega^c$ connecting x and y with*

$$\int_\gamma \frac{1}{\chi_{\mathbb{R}^2 \setminus \partial\Omega}(z)}\, dz \le C|x-y|. \tag{1.2}$$

In other words, (1.2) requires that $\ell(\gamma) \le C|x-y|$ and that $\mathcal{H}^1(\gamma \cap \partial\Omega) = 0$, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

Both the necessity and sufficiency in Theorem 1.1 are new. Our construction of an extension operator actually gives a linear operator.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. If Ω is a $W^{1,1}$-extension domain, then Ω admits a bounded linear extension operator for $W^{1,1}$.*

Corollary 1.2 gives the first linearity result for Sobolev extension operators for $p = 1$ beyond concrete cases like Lipschitz domains. Up to now it has not been clear if linearity could be expected; recall that there is no bounded linear extension operator from the trace space $L^1(\mathbb{R})$ of $W^{1,1}(\mathbb{R}^2)$ to $W^{1,1}(\mathbb{R}^2)$. The existence of a bounded linear extension operator in the case of $p > 1$ was established in [13] for $W^{1,p}$-extension domains via a method that cannot be applied for $p = 1$.

In the case of a Jordan domain we do not have boundary self-intersections and hence our characterization easily reduces to quasiconvexity of the complementary domain. From Theorem 1.1 and (the proof of) Lemma 2.5 we obtain the following corollary.

Corollary 1.3. *Let Ω be a planar Jordan domain. Then Ω is a $W^{1,1}$-extension domain if and only if its complementary domain is quasiconvex.*

Corollary 1.3 together with earlier results (see e. g. [3], [26], [17]) yields the following somewhat surprising corollary.

Corollary 1.4. *Let Ω be a planar Jordan domain and $\widetilde{\Omega}$ be its complementary domain. Then the following are equivalent:*

(1) Ω is a $W^{1,1}$-extension domain.
(2) Ω is a BV-extension domain.
(3) $\widetilde{\Omega}$ is quasiconvex.
(4) $\widetilde{\Omega}$ is a $W^{1,\infty}$-extension domain.

This corollary together with Lemma 2.1 below also extends the duality result from [18, Corollary 1.3] to the case of all $1 \le p \le \infty$.

Corollary 1.5. *Let $1 \le p, q \le \infty$ be Hölder dual exponents and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω is a $W^{1,p}$-extension domain if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is a $W^{1,q}$-extension domain.*

One further obtains a monotonicity property via Theorem 1.1 and [18]: if Ω is a bounded simply connected $W^{1,p}$-extension domain with $1 < p < 2$, then it is also a $W^{1,q}$-extension domain for all $1 \le q \le p$. However, a $W^{1,1}$-extension domain need not be a $W^{1,p}$-extension domain for any $1 < p < 2$. To see this consider the Jordan domain

$$\Omega = \mathbb{D} \setminus \{(x,\, y) \mid y \le x^2\}.$$

The complement of Ω is clearly quasiconvex, but for any curve γ connecting the origin and the point $(t,\, 0)$ with $0 < t < 1$, we have

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, dz \gtrsim \int_0^t x^{2-2p}\, dx \gtrsim t^{3-2p}.$$

Since there is no constant $C > 0$ such that $t^{3-2p} < Ct^{2-p}$ for all $0 < t < 1$ when $1 < p < 2$, (1.1) fails, and thus Ω is not an extension domain for $W^{1,\,p}(\Omega)$ when $1 < p < 2$.

Resulting from [18, Corollary 1.5] and the argument above, we conclude that the set of all $1 \le s \le \infty$ for which a bounded simply connected planar domain Ω is a $W^{1,\,s}$-extension domain, can only be one of: $\emptyset$, $\{1\}$, $[1,\,q)$ with $q < 2$, $[1,\,\infty]$, $(q,\,\infty]$ with $q > 2$ or $\{\infty\}$.

2. Preliminaries

In this section we recall some definitions and properties. The notation in our paper is quite standard. When we make estimates, we sometimes write the constants as positive real numbers $C(\cdot)$ with the parenthesis including all the parameters on which the constant depends. The constant $C(\cdot)$ may vary between appearances, even within a chain of inequalities. By $a \lesssim b$ we mean $a \le Cb$, and $a \sim b$ means that $b/C \le a \le Cb$ for some constant $C \ge 2$. The Euclidean distance between two sets $A,\,B \subset \mathbb{R}^2$ is denoted by $\operatorname{dist}(A,\,B)$. By $\mathbb{D}$ we always mean the open unit disk in $\mathbb{R}^2$ and by $\partial\mathbb{D}$ its boundary. We denote by $\ell(\gamma)$ the length of a curve γ. For a set $A \subset \mathbb{R}^2$, we denote by A^o its interior, ∂A its boundary, $A^c = \mathbb{R}^2 \setminus A$ its complement, and $\overline{A}$ its closure respectively with respect to the Euclidean topology, unless another specific explanation is given. For an injective curve $\gamma\colon [0,\,1] \to \mathbb{R}^2$ we use γ^o to denote $\gamma((0,\,1))$. Notation $[x,\,y]$ means the line segment connecting x and y, and for an injective curve γ with $x,\,y \in \gamma$, the subarc of γ joining x and y is denoted by $\gamma[x,\,y]$. We denote the 1-dimensional Hausdorff measure by $\mathcal{H}^1$.

We have the following swapping lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Take $x \in \Omega$ and define an unbounded domain $\widetilde{\Omega} = i_x(\Omega)$ using the inversion mapping*

$$i_x\colon \mathbb{R}^2 \setminus \{x\} \to \mathbb{R}^2 \setminus \{x\}\colon y \mapsto x + \frac{y-x}{|y-x|^2}.$$

Then

(1) The domain Ω is a $W^{1,1}$-extension domain if and only if $\widetilde{\Omega}$ is such a domain.
(2) The domain Ω is quasiconvex if and only if $\widetilde{\Omega}$ is such a domain.

This lemma basically comes from the proof of [18, Lemma 2.1].

Definition 2.2. *A connected set $A \subset \Omega \subset \mathbb{R}^2$ is called a λ-Whitney-type set in Ω with $\lambda \ge 1$ if the following holds.*

(1) There exists a disk with radius $\frac{1}{\lambda}\operatorname{diam}(A)$ contained in A;
(2) $\frac{1}{\lambda}\operatorname{diam}(A) \le \operatorname{dist}(A,\,\partial\Omega) \le \lambda \operatorname{diam}(A)$.

Conformal mappings preserve Whitney-type sets in the following sense, see e. g. [18, Lemma 2.5, Lemma 2.6]

Lemma 2.3. *Suppose $\varphi\colon \Omega \to \Omega'$ is conformal, where $\Omega, \Omega' \subset \mathbb{R}^2$ are domains and $Q \subset \Omega$ is a λ_1-Whitney-type set. Then $\varphi(Q) \subset \Omega'$ is a λ_2-Whitney-type set with $\lambda_2 = \lambda_2(\lambda_1)$. Moreover for all $z,\,w \in Q$, one has*

$$|\varphi'(z)| \sim |\varphi'(w)|,$$

where the constant depends only on λ_1.

Sometimes we omit the constant λ when we are dealing with a Whitney-type set whose constant is clear from the context.

Recall that the hyperbolic geodesics in $\mathbb{D}$ and $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ are defined as the arcs of (generalized) circles that intersect the unit circle orthogonally. Moreover, hyperbolic geodesics are preserved under conformal maps. According to Lemma 7.2, hyperbolic geodesics in a (bounded) simply connected domain are essentially the shortest possible curves joining given endpoints. For the exterior domain case, one needs to rule out the case of geodesics with large diameters; see Lemma 7.3.

We need the following lemma that follows from [22, Corollary 4.18]. Also see [4] and [1, Theorem 0.1].

Lemma 2.4. *Let $\varphi : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to G$ be a conformal map, and x, $y \in G$ be a pair of points and Γ a hyperbolic geodesic joining them. If B is a disk with $B \cap \Gamma \neq \emptyset$ and of λ-Whitney type, then any curve γ connecting x, y in G has non-empty intersection with cB, where $c = c(\lambda)$.*

Lemma 2.5. *Let $\widetilde{\Omega}$ be the complementary domain of a Jordan domain. Then it is c_1-quasiconvex if and only if its closure is c_2-quasiconvex, where c_1 and c_2 depend only on each other.*

Proof. We first show the sufficiency of c_2-quasiconvexity of the closure. Let x, $y \in \widetilde{\Omega}$. At the beginning consider the special case where

$$|x-y| \leq \delta \operatorname{diam}(\partial\widetilde{\Omega}), \ \ \operatorname{dist}(x, \partial\widetilde{\Omega}) \leq \delta \operatorname{diam}(\partial\widetilde{\Omega}) \ \text{ and } \ \operatorname{dist}(y, \partial\widetilde{\Omega}) \leq \delta \operatorname{diam}(\partial\widetilde{\Omega})$$

with the constant δ in Lemma 7.6. Then by applying Lemma 7.5 and Lemma 7.6 to the hyperbolic geodesic connecting x and y we get the desired control on the length of the geodesic.

Let us now prove the general case. Let $x, y \in \widetilde{\Omega}$ be two distinct points. Since $\overline{\widetilde{\Omega}}$ is c_2-quasiconvex, there exists a curve $\gamma \subset \overline{\widetilde{\Omega}}$ connecting x to y with $\ell(\gamma) \leq c_2|x-y|$. Let $\{x_i\}_{i=0}^k \subset \gamma$ be such that $x_0 = x$, $x_k = y$,

$$0 < |x_i - x_{i-1}| \leq \frac{1}{6}\delta \operatorname{diam}(\partial\widetilde{\Omega}) \qquad \text{for all } i \in \{1, \ldots, k\}. \tag{2.1}$$

The existence of these points follows from the finiteness of the length of γ. By the definition of the length of a curve, we further have

$$\sum_{i=1}^{k} |x_i - x_{i-1}| \leq \ell(\gamma) \leq c_2|x-y|. \tag{2.2}$$

For each $i \in \{1, \ldots, k-1\}$ take a point

$$x_i' \in \widetilde{\Omega} \cap B(x_i, \min\{|x_i - x_{i-1}|, |x_i - x_{i+1}|\})$$

and define $x_0' = x$ and $x_k' = y$. By the choice of x_i' and (2.1) we have for every $i \in \{1, \ldots, k\}$ the estimate

$$|x_i' - x_{i-1}'| \leq |x_i' - x_i| + |x_i - x_{i-1}| + |x_{i-1} - x_{i-1}'| \leq 3|x_i - x_{i-1}| \leq \frac{1}{2}\delta \operatorname{diam}(\partial\widetilde{\Omega}). \tag{2.3}$$

Combining this with (2.2) gives

$$\sum_{i=1}^{k} |x_i' - x_{i-1}'| \le 3c_2|x-y|. \tag{2.4}$$

Now for each $i \in \{1, \dots, k\}$ we define a curve $\gamma_i \subset \widetilde{\Omega}$ connecting x_{i-1}' and x_i'. There are two cases. If

$$|x_i' - x_{i-1}'| \le \delta \operatorname{diam}(\partial\widetilde{\Omega}), \quad \operatorname{dist}(x_i', \partial\widetilde{\Omega}) \le \delta \operatorname{diam}(\partial\widetilde{\Omega}) \quad \text{and} \quad \operatorname{dist}(x_{i-1}', \partial\widetilde{\Omega}) \le \delta \operatorname{diam}(\partial\widetilde{\Omega}), \tag{2.5}$$

then by the special case we find $\gamma_i \subset \widetilde{\Omega}$ connecting x_{i-1}' and x_i' with $\ell(\gamma_i) \lesssim |x_i' - x_{i-1}'|$. If (2.5) fails, then by (2.3)

$$\operatorname{dist}(x_i', \partial\widetilde{\Omega}) > \delta \operatorname{diam}(\partial\widetilde{\Omega}) \quad \text{or} \quad \operatorname{dist}(x_{i-1}', \partial\widetilde{\Omega}) > \delta \operatorname{diam}(\partial\widetilde{\Omega}).$$

Therefore, again by (2.3) we can define $\gamma_i := [x_i', x_{i-1}'] \subset \widetilde{\Omega}$. By concatenating all the γ_i we get a curve γ' in $\widetilde{\Omega}$ connecting x to y with

$$\ell(\gamma') \le \sum_{i=1}^{k} \ell(\gamma_i) \lesssim \sum_{i=1}^{k} |x_i' - x_{i-1}'| \lesssim |x-y|,$$

as required.

For the necessity, we choose a sequence $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in \widetilde{\Omega}$. Then for every large $n \in \mathbb{N}$, there exists a curve γ_n of length uniformly bounded from above by a multiple of $|x-y|$ which joins x_n and y_n, according to the quasiconvexity of $\widetilde{\Omega}$. Therefore up to reparametrizing γ_n's, Arzelà-Ascoli lemma tells us that (a subsequence of) γ_n converge to a curve $\gamma \subset \overline{\widetilde{\Omega}}$ joining x and y, whose length is bounded from above by a multiple of $|x-y|$. □

Lemma 2.6. *Let Ω be a bounded simply connected domain so that Ω^c is C_1-quasiconvex and let $\varphi : \mathbb{D} \to \Omega$ be conformal. Set $B_n = B(0, 1-2^{-n})$ and $\Omega_n = \varphi(B_n)$ for $n \ge 1$. Then Ω_n^c is $C(C_1)$-quasiconvex for each n.*

Proof. Fix a pair of points $x, y \in \Omega_n^c$. If $x, y \in \Omega^c$, then we are done by the quasiconvexity of Ω^c. Hence, by symmetry we may assume that $x \in \Omega \setminus \Omega_n$.

Let $x_0 \in \partial\Omega$ be a point such that the line segment $[\varphi^{-1}(x), \varphi^{-1}(x_0)]$ joining $\varphi^{-1}(x)$ and $\varphi^{-1}(x_0)$ lies in the line segment $[0, \varphi^{-1}(x_0)]$. Notice that $\gamma_1 = \varphi([\varphi^{-1}(x), \varphi^{-1}(x_0)])$ is a hyperbolic geodesic.

Suppose first that $y \in \Omega^c$, then we can join x_0 and y by a curve γ_2 in Ω^c of length at most $C_1|x_0 - y|$ because Ω^c is quasiconvex. Note that Lemma 2.8 below implies

$$\ell(\gamma_1) \lesssim \operatorname{dist}(x, \partial\Omega)$$

with the constant depending only on C_1. Then by the fact that

$$\operatorname{dist}(x, \partial\Omega) \le |x-y|,$$

we have

$$\ell(\gamma_1) + \ell(\gamma_2) \lesssim |x-y| + |y-x_0| \lesssim |x-y|,$$

where we apply the triangle inequality to get

$$|x_0 - y| \le |x - y| + |x - x_0| \le 2|x - y|.$$

Hence $\gamma_1 \cup \gamma_2$ is the desired curve.

Next consider the case where $y \in \Omega \setminus \Omega_n$. Here we have two subcases depending whether (2.6) below holds.

Firstly suppose $\varphi^{-1}(y) \in B(\varphi^{-1}(x),\, (1 - |\varphi^{-1}(x)|)/2) =: B$. Then the existence of the desired curve follows directly from Lemma 2.3; note that we can join $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ inside $B \setminus \overline{B_n}$ by a curve with length uniformly bounded by a multiple of $|\varphi^{-1}(x) - \varphi^{-1}(y)|$ for some absolute constant, and then by applying change of variable we obtain the desired estimate from Lemma 2.3. A similar conclusion also holds if the roles of x, y are reversed. By Lemma 2.3 we know that $\varphi(B(z,\, (1 - |z|)/2))$ is of Whitney-type for any $z \in \mathbb{D}$, and hence there exists an absolute constant C such that if

$$C|x - y| \le \max\{\operatorname{dist}(x, \partial\Omega),\ \operatorname{dist}(y, \partial\Omega)\},$$

then these two points fall into one of the cases above.

Hence we only need to consider that $x, y \in \Omega \setminus \Omega_n$ and

$$C|x - y| \ge \max\{\operatorname{dist}(x, \partial\Omega),\ \operatorname{dist}(y, \partial\Omega)\}. \tag{2.6}$$

Recall the definition of x_0 above, and similarly define $y_0 \in \partial\Omega$. We can connect x_0 to y_0 by a curve γ_2 in Ω^c with length controlled from above by $C_1|x_0 - y_0|$. Also join y_0 to y by the hyperbolic geodesic γ_3, similar to γ_1 joining x_0 to x. The desired curve is obtained by concatenating γ_1 with γ_2 and γ_3, respectively. Indeed by (2.6) and Lemma 2.8

$$\ell(\gamma_1) + \ell(\gamma_2) + \ell(\gamma_3) \lesssim \operatorname{dist}(x, \partial\Omega) + \operatorname{dist}(y, \partial\Omega) + |x_0 - y_0| \lesssim |x - y|,$$

where again we apply the triangle inequality with (2.6) to have

$$|x_0 - y_0| \le |x - y| + \operatorname{dist}(x, \partial\Omega) + \operatorname{dist}(y, \partial\Omega) \lesssim |x - y|.$$

$\square$

Recall the definition of John domains.

Definition 2.7 (John domain)**.** *An open subset $\Omega \subset \mathbb{R}^2$ is called a John domain provided it satisfies the following condition: There exist a distinguished point $x_0 \in \Omega$ and a constant $J > 0$ such that, for every $x \in \Omega$, there is a curve $\gamma : [0,\, \ell(\gamma)] \to \Omega$ parameterized by the arclength, such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$ and*

$$\operatorname{dist}(\gamma(t),\, \mathbb{R}^2 \setminus \Omega) \ge Jt.$$

The curve γ is called a John curve, and J is called a John constant.

We collect a number of results related to John domains in the following lemma.

Lemma 2.8 ([20], [21])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. If Ω^c is C-quasiconvex, then Ω is a J-John domain with $J = J(C)$. Moreover, given a simply connected (bounded) J-John domain $\Omega \subset \mathbb{R}^2$ together with a conformal map $\varphi\colon \mathbb{D} \to \Omega$ with $\varphi(0) = x_0$, where x_0 is the distinguish point of Ω, we can extend φ continuously up to the boundary and the hyperbolic geodesic connecting $x \in \Omega$ and x_0 in Ω is a J'-John curve, where $J' = J'(J)$.*

Recall that the extended complex plane consists of the complex plane plus a point at infinity. As a complex manifold it can be described by two charts via the stereographic projections with $\frac{1}{z}$ as the transition map. Combining the lemma above with results in [6], we have the following conclusion.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain whose complement is quasiconvex. Then Ω^c is locally connected and each bounded component of its interior is a Jordan domain. (In fact, every component of the interior of Ω^c is Jordan on the Riemann sphere.)*

Proof. By Lemma 2.8 we know that a Riemann map $\varphi\colon \mathbb{D} \to \Omega$ can be extended continuously to the boundary. Hence [6, Page 14, Corollary] shows that Ω^c is locally connected.

To show the second part of the lemma, we may assume that $0 \in \Omega$. Then by letting $g(z) = \frac{z}{|z|^2}$ we have that $g(\Omega^c)$ is compact, locally connected and its complement is connected (on the Riemann sphere). Therefore by applying [6, Page 17, Proposition 2.3] to $g(\Omega^c)$ we conclude the second part of the lemma; note that g is a homeomorphism on the Riemann sphere. □

3. Sufficiency for Jordan domains

Let Ω be a Jordan domain and suppose that the complementary domain $\widetilde{\Omega}$ is C_1-quasiconvex; note that (1.2) implies quasiconvexity of $\widetilde{\Omega}$ by Lemma 2.5. We will prove that Ω is a $W^{1,1}$-extension domain.

First we decompose a part of $\widetilde{\Omega}$ into hyperbolic triangles (defined below), and then define auxiliary functions on them. After this we construct an extension operator based on these functions.

Recall first that Ω is a John domain by Lemma 2.8. Let $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism which is conformal inside $\mathbb{D}$ and satisfies $\varphi(0) = x_0$, where x_0 is the distinguished point of Ω. Such a homeomorphism exists by the Riemann mapping theorem and the Carathéodory-Osgood theorem [21] since Ω is Jordan. Moreover we denote by $\widetilde{\varphi}\colon \overline{\widetilde{\Omega}} \to \mathbb{R}^2 \setminus \mathbb{D}$ a homeomorphism which is conformal in $\widetilde{\Omega}$. The map $\widetilde{\varphi}$ is obtained also from an extension of a Riemann mapping.

3.1. Decomposition of the domain and the complementary domain.

Fix $k_0 \in \mathbb{N}$ to be determined later. Let $A_1 = B(0, \frac{1}{2})$ and

$$A_k = B(0, 1 - 2^{-k}) \setminus B(0, 1 - 2^{-k+1})$$

for $k \geq 2$. For each $k \geq 1$, the radial rays passing through the points $x_k^{(j)} = e^{j2^{-k-k_0}\pi i}$ with $1 \leq j \leq 2^{k+k_0+1}$ divide A_k into 2^{k+k_0+1} subsets. We denote them by $D_k^{(j)}$ with $1 \leq j \leq 2^{k+k_0+1}$. Here the upper indices are labeled anticlockwise from the positive real axis. Define $Q_k^{(j)} = \varphi(D_k^{(j)})$, and $z_k^{(j)} = \varphi(x_k^{(j)})$ with $x_k^{(0)} = x_k^{(2^{k+k_0+1})}$. Since $D_k^{(j)}$ is a Whitney-type set with the constant depending only on k_0, we obtain that $Q_k^{(j)}$ is also of $\lambda(k_0)$-Whitney-type by Lemma 2.3.

Moreover we also have

$$\operatorname{dist}_{\Omega}(z_k^{(j-1)}, z_k^{(j)}) \lesssim \operatorname{diam}(Q_k^{(j)}) \tag{3.1}$$

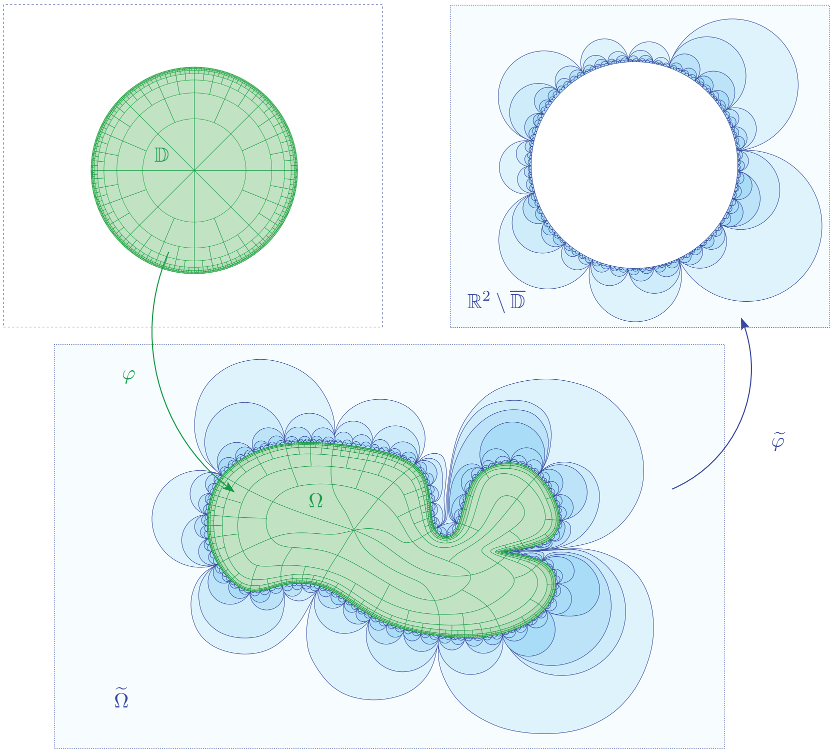

FIGURE 1. The hyperbolic triangulation of the complementary domain $\widetilde{\Omega}$ of Ω via conformal maps $\varphi\colon \mathbb{D} \to \Omega$ and $\widetilde{\varphi}\colon \widetilde{\Omega} \to \mathbb{R}^2 \setminus \overline{\mathbb{D}}$.

with the constant depending only on k_0. Indeed let x_1 be a point in $\Gamma_1 \cap Q_k^{(j)}$, where Γ_1 is the hyperbolic geodesic joining $z_k^{(j-1)}$ and the origin. Note that Γ_1 is a $C(C_1)$-John curve by Lemma 2.8. Thus we have

$$\operatorname{dist}(x_1, \partial\Omega) \gtrsim \ell(\Gamma_1[x_1, z_k^{(j-1)}]) \gtrsim \operatorname{dist}_\Omega(x_1, z_k^{(j)})$$

where $\Gamma_1[x_1, z_k^{(j-1)}]$ denotes the subarc of Γ_1 joining x_1 and $z_k^{(j-1)}$. Likewise we define Γ_2 as the hyperbolic geodesic joining $z_k^{(j)}$ and the origin, x_2 to be a point in $\Gamma_2 \cap Q_k^{(j)}$, and have the estimate

$$\operatorname{dist}(x_2, \partial\Omega) \gtrsim \ell(\Gamma_2[x_2, z_k^{(j)}]) \gtrsim \operatorname{dist}_\Omega(x_2, z_k^{(j)}).$$

Since $Q_k^{(j)}$ is a connected set of $\lambda(k_0)$-Whitney-type, we have

$$\operatorname{dist}_\Omega(x_1, x_2) \lesssim \operatorname{dist}(x_1, \partial\Omega) \sim \operatorname{dist}(x_2, \partial\Omega).$$

Therefore by the triangle inequality we conclude that

$$\begin{aligned}\operatorname{dist}_\Omega(z_k^{(j-1)}, z_k^{(j)}) &\le \operatorname{dist}_\Omega(z_k^{(j-1)}, x_1) + \operatorname{dist}_\Omega(z_k^{(j)}, x_2) + \operatorname{dist}_\Omega(x_1, x_2)\\ &\le \operatorname{dist}(x_1, \partial\Omega) \sim \operatorname{diam}(Q_k^{(j)})\end{aligned}$$

with the constant depending only on k_0 and C_1.

Denote by $\gamma_k^{(j)}$ the hyperbolic geodesic of $\widetilde{\Omega}$ connecting $z_k^{(j)}$ and $z_k^{(j-1)}$. We can choose $k_0 = k_0(C_1) \in \mathbb{N}$ large enough such that

$$|z_k^{(j-1)} - z_k^{(j)}| \le \delta \operatorname{diam}(\Omega)$$

for any $k,\, j \in \mathbb{N}$, where δ is the constant in Lemma 7.6. Indeed this follows from the uniform Hölder continuity of φ as Ω is $C(C_1)$-John by Lemma 2.8; see [22, Page 100, Corollary 5.3]. This allows us to apply Lemma 7.5 with (3.1) and conclude that

$$\ell(\gamma_k^{(j)}) \lesssim |z_k^{(j-1)} - z_k^{(j)}| \lesssim \operatorname{diam}(Q_k^{(j)}) \tag{3.2}$$

with the constant depending only on C_1.

A set is called a *hyperbolic triangle* if it is enclosed by three hyperbolic geodesics, meeting at the three vertices. For $k \in \mathbb{N}$ and $1 \le j \le 2^{k_0+k+1}$, we denote by $R_k^{(j)}$ the relatively closed set (with respect to the topology of $\widetilde{\Omega}$) enclosed by $\gamma_k^{(j)}$, $\gamma_{k+1}^{(2j-1)}$ and $\gamma_{k+1}^{(2j)}$. It is a closed hyperbolic triangle (with respect to the topology of $\widetilde{\Omega}$); see Figure 1.

Moreover, notice that for $1 \le j \le 2^{k_0+2}$ the image of each $\gamma_1^{(j)}$ under $\widetilde{\varphi}$ is a circular arc; recall that $\widetilde{\varphi}\colon \overline{\widetilde{\Omega}} \to \mathbb{R}^2 \setminus \mathbb{D}$ is a homeomorphism which is conformal in $\widetilde{\Omega}$. Denote the preimage under $\widetilde{\varphi}$ of the midpoint of each arc $\widetilde{\varphi}(\gamma_1^{(j)})$ by $z_0^{(j)}$, and the corresponding hyperbolic geodesic connecting $z_0^{(j)}$ and $z_0^{(j+1)}$ by $\gamma_0^{(j)}$; here $z_0^{(2^{k_0+2}+1)} = z_0^{(1)}$. Then with the help of $\gamma_1^{(j)}$ we obtain 2^{k_0+1} closed hyperbolic triangles $R_0^{(j)}$ such that every $R_0^{(j)}$ is enclosed by $\gamma_0^{(j)}$, $\gamma_1^{(j)}$ and $\gamma_1^{(j+1)}$; here $\gamma_1^{(2^{k_0+1}+1)} = \gamma_1^{(1)}$. Moreover $\widetilde{\varphi}(\gamma_0^{(j)}) \subset B(0, 10)$ for any $1 \le j \le 2^{k_0+2}$ since we defined the constant δ as in Lemma 7.6. Therefore it is legitimate to apply Lemma 7.5 with (3.2) to conclude

$$\begin{aligned}\ell(\gamma_0^{(j)}) &\lesssim \operatorname{dist}(z_0^{(j+1)}, z_0^{(j)}) \le \operatorname{diam}(\gamma_0^{(j+1)} \cup \gamma_0^{(j)}) \le \ell(\gamma_0^{(j+1)}) + \ell(\gamma_0^{(j)})\\ &\lesssim |z_1^{(j-1)} - z_1^{(j)}| + |z_1^{(j+1)} - z_1^{(j)}| \lesssim \operatorname{diam}(Q_1^{(j)}),\end{aligned} \tag{3.3}$$

where in the last inequality we used the fact that

$$\operatorname{diam}(Q_1^{(j+1)}) \sim \operatorname{diam}(Q_1^{(j)})$$

coming from Lemma 2.3 with a change of variable.

3.2. **Definition of the extension.** We define an auxiliary function as follows. Our construction is linear in the parameters a_1, a_2, a_3.

Lemma 3.1. *Let R be a closed hyperbolic triangle enclosed by three Jordan arcs γ_1, γ_2, γ_3 and with vertices z_1, z_2, z_3. Then for a_1, a_2, $a_3 \in \mathbb{R}$, there exists a function ϕ locally Lipschitz in $R \setminus \{z_1, z_2, z_3\}$ with the following properties:*

$$\min\{a_1, a_2, a_3\} \le \phi \le \max\{a_1, a_2, a_3\}, \tag{3.4}$$

and for $i = 1, 2, 3$, *we have* $\phi(x) = a_i$ *when* $x \in \gamma_i$. *Moreover,*

$$||\nabla\phi||_{L^1(R)} \lesssim \sum_{1\le i,\, j\le 3} |a_i - a_j| \min\{\ell(\gamma_i),\, \ell(\gamma_j)\} \tag{3.5}$$

with constants only depending on C_1.

Proof. Without loss of generality we may assume that γ_3 has the longest length among these three curves. We first consider the case where $\max\{a_1, a_2\} \le a_3$. Define

$$\phi(x) = \min\left\{a_1 + \inf_{\theta_1} \int_{\theta_1} \frac{20(a_3 - a_1)}{\operatorname{dist}(z,\, \partial R \setminus \gamma_1)}\, dz,\ a_2 + \inf_{\theta_2} \int_{\theta_2} \frac{20(a_3 - a_2)}{\operatorname{dist}(z,\, \partial R \setminus \gamma_2)}\, dz,\ a_3\right\},$$

where the infima are taken over all curves $\theta_i \subset R$ connecting x to γ_i. Observe that ϕ is bounded by its definition and locally Lipschitz in the interior of R by the triangle inequality; indeed

$$|\phi(x) - \phi(y)| \lesssim |x - y| \max_{i=1,\,2} (a_3 - a_i) \operatorname{dist}(z,\, \partial R \setminus \gamma_i)^{-1}$$

for $y \in B(x,\, 1/3 \operatorname{dist}(x,\, \partial R))$. Moreover ϕ takes the correct boundary value by its definition; note that both of the integrals in the definition of ϕ have logarithmic growth towards the boundary.

Define

$$A = \bigcup_{i=1}^{2} \bigcup_{y\in\gamma_i} B\left(y,\, \frac{1}{10}\operatorname{dist}(y,\, \partial R \setminus \gamma_i)\right).$$

Note that for $x \in R \setminus A$ we have

$$\phi(x) = a_3. \tag{3.6}$$

By the $5r$-covering theorem, we know that there exists a countable collection of pairwise disjoint disks

$$\left\{B\left(y_{ij},\, \frac{1}{10}\operatorname{dist}(y_{ij},\, \partial R \setminus \gamma_i)\right)\right\}_{y_{ij}\in\gamma_i}$$

such that

$$A \subset \bigcup_{i=1}^{2} \bigcup_{y_{ij}\in\gamma_i} B\left(y_{ij},\, \frac{1}{2}\operatorname{dist}(y_{ij},\, \partial R \setminus \gamma_i)\right) =: \bigcup_{i=1}^{2} \bigcup_{y_{ij}\in\gamma_i} B_{ij}.$$

Let us estimate the gradient inside B_{ij}. Define

$$A_i = \bigcup_{y_{ij}\in\gamma_i} B_{ij}$$

for $i = 1, 2$. Notice that $A_1 \cap A_2 = \emptyset$ by the triangle inequality; B_{ij} are open disks. Then for every $x \in A_i$, $i = 1, 2$, we have

$$\phi(x) = \min\left\{a_i + \inf_{\theta_i} \int_{\theta_i} \frac{20(a_3 - a_i)}{\operatorname{dist}(z,\, \partial R \setminus \gamma_i)}\, dz,\ a_3\right\},$$

and hence for $x \in A_i$, $i = 1, 2$,

$$|\nabla\phi(x)| \lesssim \frac{a_3 - a_i}{\operatorname{dist}(z,\, \partial R \setminus \gamma_i)}. \tag{3.7}$$

By the definition of A_i, (3.7) together with (3.6) gives us the local Lipschitz property of ϕ in $R \setminus \{z_1, z_2, z_3\}$.

Moreover by (3.7) we further have

$$\begin{aligned}\|\nabla\phi\|_{L^1(R)} &\lesssim \sum_{i=1,2}\sum_j \int_{B_{ij}} \frac{|a_3-a_i|}{\operatorname{dist}(z,\partial R\setminus\gamma_i)}\,dz\\ &\lesssim \sum_{i=1,2}\sum_j \int_{B_{ij}} \frac{|a_3-a_i|}{\operatorname{dist}(y_{ij},\partial R\setminus\gamma_i)}\,dz\\ &\lesssim \sum_{i=1,2}\sum_j |a_3-a_i|\operatorname{dist}(y_{ij},\partial R\setminus\gamma_i) \lesssim |a_3-a_1|\ell(\gamma_1)+|a_3-a_2|\ell(\gamma_2),\end{aligned}$$

where the second inequality follows from the fact that for any $z \in B_{ij}$, we have

$$\operatorname{dist}(z,\partial\Omega\setminus\gamma_i) \gtrsim \operatorname{dist}(y_{ij},\partial\Omega\setminus\gamma_i)$$

by the triangle inequality, and the last inequality follows from the fact that B_{ij} are centered on γ_i and pairwise disjoint.

For the other cases, we use suitable modifications to ϕ; the idea is to increase or decrease the value of $\phi(x)$ towards a longest curve according to the values on the rest two curves. Namely if $a_3 \le \min\{a_1, a_2\}$, we then define

$$\phi(x) = \max\left\{a_2 - \inf_{\theta_2}\int_{\theta_2} \frac{20(a_2-a_3)}{\operatorname{dist}(z,\partial R\setminus\gamma_1)}\,dz,\ a_1 - \inf_{\theta_1}\int_{\theta_1}\frac{20(a_1-a_3)}{\operatorname{dist}(z,\partial R\setminus\gamma_2)}\,dz,\ a_3\right\};$$

For the remaining case, by symmetry we consider the case where $a_1 \le a_3 \le a_2$, and then define

$$\phi(x) = \max\left\{a_2 - \inf_{\theta_2}\int_{\theta_2} \frac{20(a_2-a_3)}{\operatorname{dist}(z,\partial R\setminus\gamma_1)}\,dz,\ \min\left\{a_1 + \inf_{\theta_1}\int_{\theta_1}\frac{20(a_3-a_1)}{\operatorname{dist}(z,\partial R\setminus\gamma_2)}\,dz,\ a_3\right\}\right\}.$$

One may apply an argument similar to the one above to estimate the L^1-norm of the gradient. □

With the help of Lemma 3.1, we are ready to construct our extension operator.

Fix a function $u \in W^{1,1}(\Omega) \cap C^\infty(\mathbb{R}^2)$. Notice that Ω is J-John by Lemma 2.8. For $k \ge 1$ and $1 \le j \le 2^{k_0+k+1}$ we associate to $\gamma_k^{(j)}$ a constant

$$a_k^{(j)} = \fint_{Q_k^{(j)}} u(x)\,dx,$$

where $Q_k^{(j)} \in W$ is the Whitney-type set defined in Section 3.1 before.

Let

$$R = \bigcup R_k^{(j)},$$

where the indices run over all the $k, j \in \mathbb{N}$ such that $R_k^{(j)}$ is defined. First, by Lemma 3.1 we associate to each $R_k^{(j)}$ a function $\phi_k^{(j)} \in W^{1,1}(R_k^{(j)})$ for every $k \ge 1$ and $1 \le j \le 2^{k_0+k+1}$ so

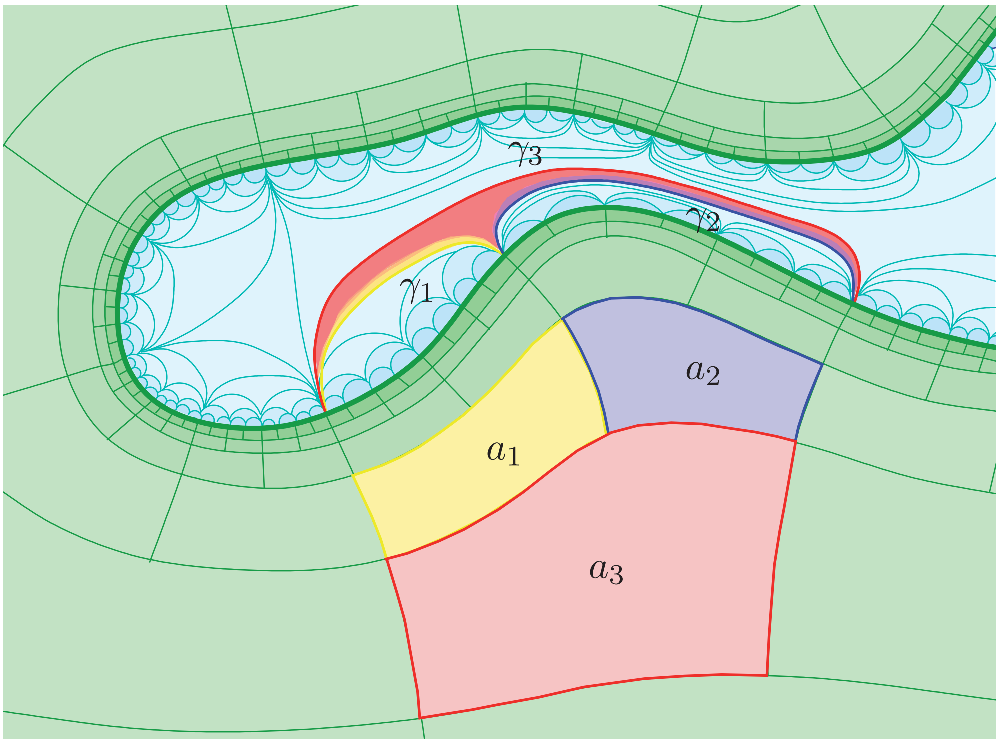

FIGURE 2. A hyperbolic triangle is shown with the three Whitney-type sets corresponding to its three edges, respectively. The colors of the edges and the Whitney-type sets show how the boundary values are taken by the function in the hyperbolic triangle.

that

$$\phi_k^{(j)}(x) = \begin{cases} a_k^{(j)} & x \in \gamma_k^{(j)} \cap R_k^{(j)} \\ a_{k+1}^{(2j-1)} & x \in \gamma_{k+1}^{(2j-1)} \cap R_k^{(j)} \\ a_{k+1}^{(2j)} & x \in \gamma_{k+1}^{(2j)} \cap R_k^{(j)} \end{cases} .$$

When $k = 0$ again by Lemma 3.1 for each $1 \leq j \leq 2^{k_0+1}$ we have a function $\phi_0^{(j)} \in W^{1,1}(R_0^{(j)})$ satisfying

$$\phi_0^{(j)}(x) = \begin{cases} a & x \in \gamma_0^{(j)} \setminus (\gamma_1^{(j)} \cup \gamma_1^{(j+1)}) \\ a_1^{(j)} & x \in \gamma_1^{(j)} \cap R_0^{(j)} \\ a_1^{(j+1)} & x \in \gamma_1^{(j+1)} \cap R_0^{(j)} \end{cases} ,$$

where $a_1^{(2^{k_0+1}+1)} = a_1^{(1)}$ and

$$a = \fint_\Omega u\,dx = \frac{1}{|\Omega|}\int_\Omega u\,dx;$$

see Figure 2.

Finally set

$$Eu(x) = \begin{cases} u(x), & \text{if } x \in \overline{\Omega} \\ \phi_k^{(j)}(x), & \text{if } x \in R_k^{(j)} \cap \widetilde{\Omega} \\ a, & \text{otherwise} \end{cases}.$$

Now let us estimate the homogeneous Sobolev norm of Eu. It is clear from the construction that $Eu \in W^{1,1}_{\mathrm{loc}}(\widetilde{\Omega})$ since Eu is locally Lipschitz in $\widetilde{\Omega}$ by Lemma 3.1. Notice that $Q_k^{(j)}$'s are $C(C_1)$-John domains by Lemma 2.3 since the conformal map φ is uniformly $C(\lambda)$-bi-Lipschitz on λ-Whitney-type sets up to a dilation factor, and so are, for example, $Q_k^{(j)} \cup Q_k^{(j+1)}$ and $Q_k^{(j)} \cup Q_{k+1}^{(2j+1)}$. Recall the following (1, 1)-Poincaré inequality for John domains: For any J-John domain $G \subset \mathbb{R}^2$ and $v \in W^{1,1}(G)$ we have

$$\fint_G |v - v_G|\,dx \le C(J)\,\mathrm{diam}\,(G) \fint_G |\nabla v|\,dx,$$

where v_G is the integral average of v in G; see e. g. [5] and the references therein. Also note that by the definition of John domain, the facts that every $Q_1^{(j)}$ is of $\lambda(C_1)$-Whitney-type and that the John center of Ω is contained in every $Q_1^{(j)}$, we have

$$\mathrm{diam}\,(Q_1^{(j)}) \sim \mathrm{diam}\,(\Omega) \qquad \text{and then} \qquad |Q_1^{(j)}| \sim |\Omega| \sim \mathrm{diam}\,(\Omega)^2 \tag{3.8}$$

for each $1 \le j \le 2^{k_0+1}$, where the constants depend only on C_1.

Now we can estimate the homogeneous Sobolev norm of Eu. By (3.3), we apply Lemma 3.1 and Poincaré inequality above to get

$$
\begin{aligned}
&\|\nabla Eu(x)\|_{L^1(\widetilde{\Omega})} \\
&\quad\lesssim \sum_{\substack{k\geq 1\\ 1\leq j\leq 2^{k_0+k+1}}} \int_{R_k^{(j)}} |\nabla \phi_k^{(j)}(x)|\,dx + \sum_{1\leq j\leq 2^{k_0+2}} \int_{R_0^{(j)}} |\nabla \phi_k^{(j)}(x)|\,dx \\
&\quad\lesssim \sum_{\substack{k\geq 1\\ 1\leq j\leq 2^{k_0+k+1}}} \left(|a_k^{(j)} - a_{k+1}^{(2j-1)}| + |a_k^{(j)} - a_{k+1}^{(2j)}| + |a_{k+1}^{(2j-1)} - a_{k+1}^{(2j)}|\right) \operatorname{diam}(Q_k^{(j)}) \\
&\qquad + \sum_{j=1}^{2^{k_0+1}} \left(|a_1^{(j)} - a_1^{(j+1)}| + |a_1^{(j)} - a| + |a_1^{(j+1)} - a|\right) \operatorname{diam}(Q_1^{(j)}) \\
&\quad\lesssim \sum_{\substack{k\in\mathbb{N}\\ 1\leq j\leq 2^{k_0+k+1}}} \int_{Q_k^{(j)}\cup Q_{k+1}^{(2j-1)}} |\nabla u|\,dx + \int_{Q_k^{(j)}\cup Q_{k+1}^{(2j)}} |\nabla u|\,dx \\
&\qquad + \int_{Q_{k+1}^{(2j-1)}\cup Q_{k+1}^{(2j)}} |\nabla u|\,dx + \int_{\Omega} |\nabla u|\,dx.
\end{aligned}
$$

Additionally, by our construction, we only have uniformly finite overlaps for the Whitney-type sets in the summation. Therefore we have

$$
\|\nabla Eu\|_{L^1(\widetilde{\Omega})} \lesssim \int_{\Omega} |\nabla u|\,dx. \tag{3.9}
$$

Let B be a ball of radius $4\operatorname{diam}(\Omega)$ such that Ω is properly contained in $\frac{1}{2}B$. Then additionally we have that

$$
\begin{aligned}
\|Eu\|_{L^1(B)} &\lesssim \sum_{\substack{k\geq 1\\ 1\leq j\leq 2^{k_0+k+1}}} \int_{R_k^{(j)}} |Eu(x)|\,dx + \int_{\Omega} |u(x)|\,dx \\
&\lesssim \sum_{\substack{k\geq 1\\ 1\leq j\leq 2^{k_0+k+1}}} \int_{Q_k^{(j)}} |u(x)|\,dx + \int_{\Omega} |u(x)|\,dx \lesssim \int_{\Omega} |u(x)|\,dx
\end{aligned} \tag{3.10}
$$

where we used (3.4) with the fact that

$$
|R_k^{(j)}| \lesssim |Q_k^{(j)}| \sim |Q_{k+1}^{(2j-1)}| \sim |Q_{k+1}^{(2j)}|
$$

coming from (3.2) and the fact that $Q_k^{(j)}$ is of Whitney-type.

3.3. **Absolute continuity.** If we now can prove that $Eu \in W^{1,1}(\mathbb{R}^n)$ for every $u \in W^{1,1}(\Omega) \cap C^\infty(\mathbb{R}^2)$, then by the fact that $C^\infty(\mathbb{R}^2) \cap W^{1,1}(\Omega)$ is dense in $W^{1,1}(\Omega)$ provided that Ω is Jordan domain [19], linearity and boundedness of E allow us to extend E to the entire $W^{1,1}(\Omega)$; notice that (3.9) and (3.10) permit us to use the density, and B is a $W^{1,1}$-extension domain.

Based on the previous argument, the only thing we need to check is that Eu is absolutely continuous along almost every line segment parallel to the coordinate axes. Notice that u is already smooth in $\overline{\Omega}$. Denoting the collection of the countable number of points $z_k^{(j)}$ by Z, we first claim that the restriction of Eu to $\mathbb{R}^2 \setminus Z$ is continuous.

Indeed since u is Lipschitz in $\overline{\Omega}$ and Eu is continuous in $\widetilde{\Omega}$ by our construction, we just need to verify continuity of Eu at points of $\partial\Omega \setminus Z$. Given a sequence of points $\{y_i\}_{i=1}^{\infty}$ in $\widetilde{\Omega}$ converging to $y \in \partial\Omega \setminus Z$, we claim that there is a sequence of hyperbolic triangles $\{R_i\}_{i=1}^{\infty}$ among $R_k^{(j)}$ such that $y_i \in R_i$ and R_i converge to y. If so, then by (3.4) we know that $Eu(y_i)$ is bounded from above and below respectively by one of the integral averages of u over the corresponding Whitney-type sets Q_i or two neighbors of Q_i. Let us show that now also Q_i converge to y (inside Ω). Indeed since R_i converge to y, especially the three vertices of R_i also converge to y as $i \to \infty$, by the continuity of φ and $\widetilde{\varphi}$ on the boundary, we know that $\operatorname{diam}(Q_i) \to 0$. Therefore Q_i also converge to y. Since u is Lipschitz in $\overline{\Omega}$, we get the continuity of $Eu(x)$ in $\mathbb{R}^2 \setminus Z$.

Now let us show the existence of the asserted sequence $\{R_i\}_{i=1}^{\infty}$. Fix $y \in \partial\Omega \setminus Z$, and for any $s \in \mathbb{N}$ define

$$\Gamma_s = \overline{\bigcup_{0 \le k \le s} \bigcup_j \gamma_k^{(j)}}, \tag{3.11}$$

where the second union in j is over all the indices j such that $\gamma_k^{(j)}$ is defined. Notice that Γ_s is compact, and $y \notin \Gamma_s$. Therefore there exists $\delta_s > 0$ such that $\operatorname{dist}(y, \Gamma_s) \ge \delta_s$. Thus if $\{y_i\}_{i=1}^{\infty}$ converges to y, then the lower indices of our hyperbolic triangles $R_{k(i)}^{(j)} = R_{k(i)}$ containing y_i tend to ∞ when $i \to \infty$. Since $\widetilde{\varphi} \colon \overline{\widetilde{\Omega}} \to \mathbb{R}^2 \setminus \mathbb{D}$ is a homeomorphism, and $\widetilde{\varphi}(R_{k(i)})$ converges to a point since $k(i) \to \infty$, we have $R_{k(i)} \to y$ as desired since $y_i \to y$ and $y_i \in R_{k(i)}$.

Since Z is countable, almost every line parallel to the x_1-axis contains no point of Z. Next for almost every such line S,

$$\int_{S \cap \widetilde{\Omega}} |\nabla Eu| \, dx < \infty \tag{3.12}$$

by Fubini's theorem. Since Eu is locally Lipschitz in $\widetilde{\Omega}$, $Eu|_S$ is absolutely continuous for $\mathcal{H}^1$-almost every each closed segment $I \subset S$ parallel to the coordinate axes with $I \subset \widetilde{\Omega}$. Fix S with this property.

Now for any line segment $I \subset S$, we first observe that $Eu|_I$ is certainly continuous. Next we show that $Eu|_I$ is absolutely continuous. Let $z, w \in I$.

If $z,\, w \in \overline{\Omega}$, then the L-Lipschitz continuity of u in $\overline{\Omega}$ gives

$$|Eu(z) - Eu(w)| \le L|z - w|. \tag{3.13}$$

If $[z,\, w] \cap \overline{\Omega} = \emptyset$, then since Eu is absolutely continuous on line segments in S, we obtain

$$|Eu(z) - Eu(w)| \le \int_{[z,\, w]} |\nabla Eu| \, dx. \tag{3.14}$$

For what is left, by symmetry we may assume that $z \in \overline{\Omega}$ while $w \in \widetilde{\Omega}$. Then let z_0 be the closest point of $\overline{\Omega} \cap I$ to w between z and w. Then

$$|Eu(z)-Eu(w)| \le |Eu(z)-Eu(z_0)|+|Eu(z_0)-Eu(w)| \le L|z-z_0|+\int_{[z_0,\,w]} |\nabla Eu|\,dx. \quad (3.15)$$

For the last inequality, we used the facts that Eu is continuous on S and absolutely continuous on each closed subinterval $I \subset S$ when $I \subset \widetilde{\Omega}$, together with (3.12).

Suppose that $\epsilon > 0$ and we are given $z_1, \ldots, z_{2n} \in I$ so that the one-dimensional open intervals (z_{2i-1}, z_{2i}), $1 \le i \le n$, are pairwise disjoint. By applying the relevant one of (3.13),(3.14) or (3.15) to each pair z_{2i-1}, z_{2i}, our assumption (3.12) together with the absolute continuity of integrals gives the existence of $\delta > 0$ so that $\sum_{i=1}^{n} |Eu(z_{2i-1}) - Eu(z_{2i})| \le \epsilon$ provided $\sum_{i=1}^{n} |z_{2i-1} - z_{2i}| < \delta$. This implies the desired absolute continuity property. The case of lines parallel to the x_2-axis is handled analogously.

4. Piecewise hyperbolic geodesics and cut-points

Recall that we relied on a hyperbolic triangulation of the complementary domain in the construction of our extension operator in the Jordan case. In order to obtain a suitable variant of this for the (strongly quasiconvex) exterior of our simply connected domain, we need a counterpart of exterior hyperbolic geodesics.

Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map. Our assumption that Ω^c is quasiconvex, say with constant C_1, together with Lemma 2.8 allows us to extend φ continuously to entire $\overline{\mathbb{D}}$. As usual, the extension is also denoted by φ. We set $\widetilde{\Omega} := \mathbb{R}^2 \setminus \overline{\Omega}$ and let $\{\widetilde{\Omega}_i\}_{i=1}^{N}$ be an enumeration of the connected components of $\widetilde{\Omega}$ with $N \in \mathbb{N} \cup \{+\infty\}$, so that $\widetilde{\Omega}_1$ is the unique unbounded component. Then each $\widetilde{\Omega}_i$ is Jordan for $i \ge 2$ by Lemma 2.9.

Because the boundary of Ω may have self-intersections, we base our construction on labeling via $\partial\mathbb{D}$. Towards this end, let $x, y \in \partial\mathbb{D}$ be distinct points of distance at most δ along the boundary of the unit circle, where $\delta > 0$ will be fixed later. When $\delta < \pi/4$, this guarantees that the hyperbolic geodesic between x, y in the exterior of the disk is contained in $B(0, 10)$. We will later need this also for certain additional points associated to x, y and hence will eventually take a smaller δ. A piecewise hyperbolic geodesic (in Ω^c) associated to x, y via φ between $\varphi(x), \varphi(y) \in \partial\Omega$ in the complement of Ω is any curve obtained via the following construction.

Define $x_n = (1 - 2^{-n})x$, $y_n = (1 - 2^{-n})y$. Then $\varphi(x_n), \varphi(y_n) \in \partial D_n$, where

$$D_n = \varphi(B(0,\, 1 - 2^{-n})),$$

for $n \ge 2$, is a Jordan domain. Denote by $\widetilde{D}_n$ the complementary domain of D_n. Since φ is continuous up to the boundary, the points $\varphi(x_n) \in \partial\widetilde{D}_n$, $\varphi(y_n) \in \partial\widetilde{D}_n$ converge to $\varphi(x), \varphi(y)$, respectively. Also observe that $\Omega^c \subset \widetilde{D}_{n+1} \subset \widetilde{D}_n$, and $\Omega^c = \bigcap_n \widetilde{D}_n$. Pick a hyperbolic geodesic $\gamma_n \subset \widetilde{D}_n$ of $\widetilde{D}_n$ connecting $\varphi(x_n)$ and $\varphi(y_n)$. Since Ω^c is C_1-quasiconvex, Lemma 2.6 shows that $\widetilde{D}_n$ is c_1-quasiconvex with $c_1 = c_1(C_1)$. We now fix $\delta_1 = \delta_1(c_1) < \frac{1}{4c_1}$ as in Lemma 7.6. Then Lemma 7.6 guarantees that $|\widetilde{\varphi}_n(z) - \widetilde{\varphi}_n(w)| \le \frac{1}{4}$ for any conformal map $\widetilde{\varphi}_n : \widetilde{D}_n \to \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and $z, w \in \partial D_n$ with $|z - w| < \delta_1 \operatorname{diam}(\Omega)$; recall here that such a $\widetilde{\varphi}_n$ extends continuously to the boundary and hence we may apply Lemma 7.6 even for points on the boundary. By the uniform Hölder continuity of φ on $\mathbb{D}$ [22, Page 100, Corollary 5.3], we may choose $\delta = \delta(\delta_1, C_1)$ so that

$$|\varphi(x_n) - \varphi(y_n)| < \delta_1 \operatorname{diam}(\Omega)$$

whenever $|x - y| < \delta$; notice that $|x_n - y_n| < |x - y|$. This determines the value of δ. Under this choice, $|\widetilde{\varphi}_n(\varphi(x_n)) - \widetilde{\varphi}_n(\varphi(y_n))| < \frac{1}{4}$ and hence the c_1-quasiconvexity of $\widetilde{D}_n$ together with Lemma 7.5 guarantees that

$$\ell(\gamma_n) \leq c_2|\varphi(x_n) - \varphi(y_n)|,$$

where $c_2 = c_2(c_1)$. Recalling that $\varphi(x_n) \to \varphi(x)$ and $\varphi(y_n) \to \varphi(y)$ when n tends to infinity, we conclude that $\gamma_n \subset B(\varphi(x), M)$ and $\ell(\gamma_n) \leq M$ for some M independently of n. Parametrize each γ_n by arc length. Then, by the Arzelà-Ascoli lemma, we obtain via uniform convergence a curve γ joining $\varphi(x)$ and $\varphi(y)$ in Ω^c. We call it a *piecewise hyperbolic geodesic joining* $\varphi(x)$ *and* $\varphi(y)$.

Similarly we also define piecewise hyperbolic geodesics joining sufficiently close points near $\partial\Omega$, for example, the points on the piecewise hyperbolic geodesics above; this is the reason why we chose $\delta_1 < \frac{1}{4c_1}$ when we applied Lemma 7.6 above. More precisely, the argument above applies whenever $x, y \in \Omega^c$ satisfy

$$|x - y| \leq \delta \operatorname{diam}(\Omega),\ \operatorname{dist}(x, \partial\Omega) \leq \delta \operatorname{diam}(\Omega),\ \text{ and } \operatorname{dist}(y, \partial\Omega) \leq \delta \operatorname{diam}(\Omega)$$

with our choice of δ. Indeed, regarding Lemma 7.6 with our approach above, it suffices to show that when n is large enough we have

$$|x - y| \leq \delta \operatorname{diam}(D_n),\ \operatorname{dist}(x, \partial D_n) \leq \delta \operatorname{diam}(D_n),\ \text{ and } \operatorname{dist}(y, \partial D_n) \leq \delta \operatorname{diam}(D_n);$$

notice that $x, y \in \widetilde{D}_n$ for any $n \in \mathbb{N}$. This follows from the uniform Hölder continuity of φ.

The following lemma justifies our terminology.

Lemma 4.1. *Let Ω be a bounded simply connected domain whose complement is C_1-quasiconvex. Then there exist constants $C(C_1)$ and $\delta = \delta(C_1) > 0$ such that, for every pair of distinct $x, y \in \Omega^c$ with*

$$|x - y| \leq \delta \operatorname{diam}(\Omega), \quad \operatorname{dist}(x, \partial\Omega) \leq \delta \operatorname{diam}(\Omega) \quad \text{and} \quad \operatorname{dist}(y, \partial\Omega) \leq \delta \operatorname{diam}(\Omega), \quad (4.1)$$

any piecewise hyperbolic geodesic γ joining x to y in Ω^c satisfies

$$\ell(\gamma[z, w]) \leq C(C_1)|z - w|,$$

whenever $z, w \in \gamma$ and $\gamma[z, w]$ is any subcurve of γ joining z and w. In particular, γ is an injective curve (when parametrized by arc length).

Additionally, for any $1 \leq i \leq N$ with $\gamma \cap \widetilde{\Omega}_i \neq \emptyset$, we have that $\gamma \cap \widetilde{\Omega}_i$ is a hyperbolic geodesic in $\widetilde{\Omega}_i$.

Proof. Let $\gamma[z, w]$ be a subcurve of γ joining z and w. Then by our construction of the piecewise hyperbolic geodesics, there are $t_n < s_n$ such that the restrictions of γ_n to $[t_n, s_n]$ converge to $\gamma[z, w]$. We may assume that $z_n = \gamma_n(t_n)$ converges to z and that $w_n = \gamma_n(s_n)$ converges to w. Hence by the lower semi-continuity of length (as a functional on curves), Lemma 7.5 and Lemma 2.6 one has

$$\ell(\gamma[z, w]) \lesssim \liminf_{n\to\infty} \ell(\gamma_n[z_n, w_n]) \lesssim \liminf_{n\to\infty} |z_n - w_n| \lesssim |z - w|.$$

Hence our first claim follows.

For the final claim, let $\widetilde{\varphi}_n \colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{D}_n$ to be conformal for each $n \in \mathbb{N}$, and extend it to $\overline{\mathbb{R}^2} \setminus \overline{\mathbb{D}}$ by setting $\widetilde{\varphi}_n(\infty) = \infty$. Also when $i \geq 2$ and $\gamma \cap \widetilde{\Omega}_i \neq \emptyset$, let $x_0 \in \widetilde{\Omega}_i$ be a point of maximal distance to $\partial\widetilde{\Omega}_i$; for $\widetilde{\Omega}_1$ we set $x_0 = \infty$.

Denote by $\phi \colon \overline{\mathbb{R}^2} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{R}^2} \setminus \overline{\mathbb{D}}$ a Möbius transformation such that $\phi(\infty) = \widetilde{\varphi}_n^{-1}(x_0)$. Then the compositions $\widetilde{\varphi}_n \circ \phi \colon \overline{\mathbb{R}^2} \setminus \overline{\mathbb{D}} \to \widetilde{D}_n \cup \{\infty\}$ form a normal family by [25, Theorem 19.2]. Hence there is a subsequence that converges locally uniformly to a conformal map $\widetilde{\varphi} \colon \overline{\mathbb{R}^2} \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}_i$; see [25, Theorem 21.1]. Notice that hyperbolic geodesics are invariant under conformal maps. Therefore γ_n can be regarded as a hyperbolic geodesic induced by $\widetilde{\varphi}_n \circ \phi$ on the Riemann sphere by the uniqueness of the limit, and hence also γ as induced by $\widetilde{\varphi} \circ \phi$. Thus the part of γ in $\widetilde{\Omega}_i$ is a hyperbolic geodesic; it easily follows from the above argument that $\gamma \cap \widetilde{\Omega}_i$ is connected. □

We need the following further properties of piecewise hyperbolic geodesics.

Lemma 4.2. *Let Ω and δ be as in Lemma 4.1, and $x, y \in \Omega^c$ satisfy* (4.1)*. Given any piecewise hyperbolic geodesic $\Gamma \subset \Omega^c$ associated to the pair x, y one has*

$$\Gamma \cap \partial\Omega \subset \gamma \cap \partial\Omega,$$

for any curve $\gamma \subset \Omega^c$ connecting x and y. Moreover, if the complement of Ω satisfies the curve condition (1.2)*, then*

$$\int_\Gamma \frac{1}{\chi_{\mathbb{R}^2 \setminus \partial\Omega}(z)}\, dz \leq C|x - y|.$$

Proof. Let $\gamma \subset \Omega^c$ be a curve joining $x, y \in \Omega^c$. Denote the corresponding hyperbolic geodesics approximating Γ by $\Gamma_n \subset \widetilde{D}_n$ with endpoints x_n, y_n, according to the definition of Γ.

Suppose $z \in \Gamma \cap \partial\Omega$. We claim that $z \in \gamma \cap \partial\Omega$. We may assume that $z \neq x$, $z \neq y$. Let $\{z_n\}$, $z_n \in \Gamma_n$ be a sequence of points converging to z, and let for every $n \geq 2$,

$$B_n = B\left(z_n,\, \frac{1}{2}\operatorname{dist}(z_n,\, \partial D_n)\right).$$

Since each B_n is a Whitney-type set in $\widetilde{D}_n$, then Lemma 2.4 gives a constant c independent of n such that $\gamma' \cap cB_n \neq \emptyset$ for any $\gamma' \subset \widetilde{D}_n$ joining x_n and y_n. Since $x_n \to x$ and $y_n \to y$, then when n is large enough, the uniform quasiconvexity of $\widetilde{D}_n$ implies that there exist curves in $\widetilde{D}_n$ joining x to x_n and y to y_n, respectively, such that these two curves do not intersect cB_n; see Lemma 2.6. By concatenating them with γ, we apply Lemma 2.4 to conclude that

$$\gamma \cap cB_n \neq \emptyset \tag{4.2}$$

when n is large enough; recall that $\gamma \subset \Omega^c \subset \widetilde{D}_n$.

Observe that by $z \in \Gamma \cap \partial\Omega$, we have that

$$\lim_{n\to\infty} \operatorname{dist}(z_n,\, \partial D_n) = 0.$$

Then by the assumption $z_n \to z$ when $n \to \infty$, we conclude that $cB_n \to z$, and hence $z \in \gamma$ by (4.2). Thus $z \in \gamma \cap \partial\Omega$. This proves the claim, and the first part of the lemma follows.

The second part follows from Lemma 4.1 together with the first part applied to a curve γ satisfying (1.2); note that Γ is an injective curve. □

For further reference we record the following consequences of the above lemmas.

Lemma 4.3. *Given Ω as in Lemma 4.1, and a piecewise hyperbolic geodesic Γ, a subcurve $\Gamma[x, y]$ of Γ and a curve γ joining x to y in Ω^c, we have*

$$\Gamma[x, y] \cap \partial\Omega \subset \gamma \cap \partial\Omega.$$

Moreover $\Gamma \cap \partial\widetilde{\Omega}_i$ consists of at most two points for each $1 \leq i \leq N$, and it is a doubleton if and only if $\Gamma \cap \widetilde{\Omega}_i$ is a hyperbolic geodesic joining boundary points.

Proof. Let x_0, y_0 be the end points of Γ. Then the concatenation of three curves, the subcurve $\Gamma[x,\, x_0]$ of Γ, γ, and the subcurve $\Gamma[y,\, y_0]$ of Γ, is a curve joining x_0 to y_0. Then the first conclusion follows from Lemma 4.2.

If $\Gamma \cap \partial\widetilde{\Omega}_i$ has more than two points, then consider the first and the last points of Γ intersecting $\partial\widetilde{\Omega}_i$ according to its parametrization, and join them by a hyperbolic geodesic α inside $\widetilde{\Omega}_i$; recall that $\widetilde{\Omega}_i$ is Jordan by Lemma 2.9. Then we obtain a new curve γ by rerouting the subarc of Γ via α. This contradicts Lemma 4.2 for γ since we have assumed that $\Gamma \cap \partial\widetilde{\Omega}_i$ has more than two points. Thus $\Gamma \cap \partial\widetilde{\Omega}_i$ has at most two points.

Let us show the last part of the lemma. We may assume that $\widetilde{\Omega}_i$ is bounded; otherwise we just apply suitable Möbius transformations on the Riemann sphere. If $\Gamma \cap \widetilde{\Omega}_i$ is a hyperbolic geodesic joining boundary points, then $\Gamma \cap \partial\widetilde{\Omega}_i$ consists of two points since $\widetilde{\Omega}_i$ is Jordan. For the other direction, by Lemma 4.1 it suffices to show that $\Gamma \cap \widetilde{\Omega}_i$ is non-empty. On the contrary let us assume that $\Gamma \cap \widetilde{\Omega}_i = \emptyset$. Then by joining the given two points $x,\, y \in \Gamma \cap \partial\widetilde{\Omega}_i$ with the hyperbolic geodesic $\alpha \subset \widetilde{\Omega}_i$, we obtain a Jordan curve γ' by concatenating $\Gamma[x,\, y]$ with α; see Lemma 4.1 and note that the open subarc $\Gamma(x,\, y)$ is contained in the exterior of $\widetilde{\Omega}$ by our assumption and the conclusion of the previous paragraph. However by our construction, $\partial\Omega$ intersects both the interior and the exterior of the Jordan domain given by γ'. This implies that there are points belonging to Ω on both sides of γ'. This contradicts the Jordan curve theorem since Ω is connected while $\gamma' \subset \Omega^c$. Therefore we conclude that $\Gamma \cap \widetilde{\Omega}_i \neq \emptyset$ as expected. □

The following corollary is a by-product of the lemmas above. A similar argument will also be applied for Lemma 5.1.

Lemma 4.4. *Given domain Ω with two points x, y as in Lemma 4.1, there exists a unique piecewise hyperbolic geodesic joining them, up to a reparametrization.*

Proof. The existence follows directly from Lemma 4.1, and we only need to show the uniqueness.

Suppose that there are two piecewise hyperbolic geodesics γ_1, γ_2 connecting x, y. Then by Lemma 4.2 we have

$$\gamma_1 \cap \partial\Omega = \gamma_2 \cap \partial\Omega. \tag{4.3}$$

Since $\Omega^c = \partial(\Omega^c) \cup \widetilde{\Omega} = \partial\Omega \cup \widetilde{\Omega}$, we then only need to show that γ_1 coincides with γ_2 componentwise. This follows from (4.3), Lemma 4.1 and Lemma 4.3 according to the uniqueness of hyperbolic geodesics in Jordan domains. □

In order to deal with self-intersections of the boundary, it is convenient to classify points on the boundary in terms of their preimages.

Definition 4.5. *Let $\varphi\colon \mathbb{D} \to \Omega$ be conformal and assume that it extends as a continuous map (still denoted by φ) to entire $\overline{\mathbb{D}}$. A point $x \in \partial\Omega$ is called a* cut-point *if $\varphi^{-1}(x)$ is not a singleton. A point x that is not a cut-point is called* one-sided.

We warn the reader that our terminology above is not standard. See e. g. [23, Chapter 14] for a discussion on the relation with the usual topological definition of cut-points. As stated, the definition appears to depend on the choice of the continuous conformal map. This is not the case since any other (continuous) conformal map differs from the chosen one only by a precomposition via a Möbius self-map of the (closed) disk.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected John domain. Then the set T of cut-points for a conformal map $\mathbb{D} \to \Omega$ can be characterized as*

$$T = \bigcup_{\gamma} \gamma^o \cap \partial\Omega, \tag{4.4}$$

where the union is over all piecewise hyperbolic geodesics γ associated to pairs of (suitable) points in $\partial\mathbb{D}$ and γ^o denotes the curve γ without its endpoints. Moreover, the union can equivalently be taken over a countable set of curves.

Proof. First of all, by Lemma 2.8 we have a continuous map $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ which is conformal in $\mathbb{D}$.

Let $\{x_i\}_{i=1}^{\infty} \subset \partial\mathbb{D}$ be dense. We first claim that

$$T \subset \bigcup \gamma_{i,j}^o \cap \partial\Omega. \tag{4.5}$$

where the union is taken over all the piecewise hyperbolic geodesics $\gamma_{i,j}$ joining $\varphi(x_i)$ to $\varphi(x_j)$ with

$$|\varphi(x_i) - \varphi(x_i)| \le \delta \operatorname{diam}(\Omega).$$

Recall that $\gamma_{i,j}$ is obtained via a subsequence of the domains $\widetilde{D}_n$. Since we only have countably many curves, by a usual diagonal argument, we may assume that each of them is obtained via the same subsequence. Observe that by Lemma 4.1, these curves are all injective.

Fix $z \in T$. By the definition of cut-points there exist $z_1, z_2 \in \partial\mathbb{D}$, $z_1 \neq z_2$, such that $\varphi(z_1) = \varphi(z_2) = z$. Recall here that the image of any nontrivial arc of $\partial\mathbb{D}$ under our conformal map that is continuous up to the boundary is connected and not a singleton. By the density of $\{x_i\}_{i=1}^{\infty}$ there exist $i, j \in \mathbb{N}$ such that

$$\varphi(x_i) \neq z \neq \varphi(x_j),\ |\varphi(x_i) - \varphi(x_j)| \le \delta \operatorname{diam}(\Omega)$$

with $\varphi(x_i) \neq \varphi(x_j)$ and x_i, x_j divide $\partial\mathbb{D}$ into two connected components; one contains z_1 and the other contains z_2. The curve $\varphi([0, z_1] \cup [0, z_2])$ is Jordan since $z_1 \neq z_2$ and φ is injective in $\mathbb{D}$, where $[0, z_i]$ is the (radial) line segment connecting the origin and z_i for $i = 1, 2$. Because

φ is continuous up to the boundary and Ω is simply connected, the points $\varphi(x_i)$ and $\varphi(x_j)$ belong to different connected components of $\mathbb{R}^2 \setminus \varphi([z_1, 0] \cup [0, z_2])$. Indeed, $\varphi([z_1, 0] \cup [0, z_2])$ divides Ω into two components, coming from the two components of $\mathbb{D} \setminus ([z_1, 0] \cup [0, z_2])$. Recall that x_i and x_j are in different connected components of $\partial\mathbb{D} \setminus \{z_1, z_2\}$. Then by taking two sequences of points inside the components of $\mathbb{D} \setminus ([z_1, 0] \cup [0, z_2])$ approaching x_i and x_j, respectively, we conclude that $\varphi(x_i)$ and $\varphi(x_j)$ are not in the same component of $\mathbb{R}^2 \setminus \varphi([z_1, 0] \cup [0, z_2])$ since φ is continuous up to the boundary and neither $\varphi(x_i)$ nor $\varphi(x_j)$ belong to $\varphi([z_1, 0] \cup [0, z_2])$. Consequently, $\gamma_{i,j}$ has to intersect the Jordan curve $\varphi([z_1, 0] \cup [0, z_2])$. Since $\varphi([z_1, 0] \cup [0, z_2]) \cap \Omega^c = \{z\}$ and $\gamma_{i,j} \subset \Omega^c$, we know that $\gamma_{i,j} \cap \varphi([z_1, 0] \cup [0, z_2]) = \{z\}$ and thus the claim is proved since $\varphi(x_i)$, z, and $\varphi(x_j)$ are distinct. Therefore we obtain (5.2).

Let us then show the other inclusion. Let γ be a piecewise hyperbolic geodesic joining two different points in $\partial\Omega^c$ and let $z \in \gamma^o \cap \partial\Omega$. Let $y_1, y_2 \in \partial\mathbb{D}$ be such that $\varphi(y_1)$ and $\varphi(y_2)$ are the endpoints of γ. Now the curve $\gamma' := \varphi([y_1, 0] \cup [0, y_2]) \cup \gamma$ is Jordan by Lemma 4.1 and it divides Ω into two connected components Ω_1 (the exterior component) and Ω_2 (the interior component). Let Ω' be the Jordan domain given by γ'.

Let us assume contrary to the claim that $z \notin T$. Then there exists $r > 0$ such that $B(z, r)$ intersects only one of the components, Ω_1 or Ω_2; otherwise we have two sequences of points converging to z in different components, which contradicts the assumption that $z \notin T$ but $z \in \partial\Omega$ by the continuity of φ.

Assume that $B(z, r) \cap \Omega_1 = \emptyset$. Since $z \in \gamma \cap \partial\Omega$, by the first part of Lemma 4.2 we know that there is no path connecting $\varphi(y_1)$ and $\varphi(y_2)$ in $\Omega^c \setminus \{z\}$. Towards a contradiction, we apply the fact that $\gamma \subset \partial\Omega'$ and the Jordan-Schoenflies theorem to Ω'.

Indeed, by Jordan-Schoenflies theorem there is a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\Omega') = \mathbb{D}$, and then $\phi(\gamma) \subset \partial\mathbb{D}$. Notice that $\varphi(y_1) \neq z \neq \varphi(y_2)$ since $z \in \gamma^o \cap \partial\Omega$. Choose $0 < \epsilon < 1$ such that

$$\min\{|\varphi(y_1) - z|, |\varphi(y_2) - z|\} > \epsilon r.$$

Then $\varphi(y_1), \varphi(y_2) \notin B(z, \epsilon r)$, and $\phi(B(z, \epsilon r))$ is also a Jordan domain. Certainly $\phi(z)$ is an interior point of $\phi(B(z, \epsilon r))$, and hence there exists $\delta > 0$ such that $B(\phi(z), \delta) \subset \phi(B(z, \epsilon r))$. Denote by σ the subarc of $\partial(\mathbb{D} \cup B(\phi(z), \delta))$ which joins $\phi(\varphi(y_1))$ and $\phi(\varphi(y_2))$ and reroutes $\phi(\gamma)$, and let $\Gamma = \phi^{-1}(\sigma)$.

Observe first that $\Gamma \subset \Omega^c$ by our construction and the assumptions that $\Omega_2 \subset \Omega'$ and $B(z, \epsilon r) \cap \Omega_1 = \emptyset$. Also Γ joins $\varphi(y_1)$ and $\varphi(y_2)$ by not passing through z. Therefore we obtain the desired curve, which leads to a contradiction. A similar argument can be applied to the case where $B(z, r) \cap \Omega_2 = \emptyset$.

The rest of the lemma follows from the proof above directly. □

An immediate consequence of Lemma 4.2 and Lemma 4.6 is the following.

Corollary 4.7. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded John domain whose complement satisfies the curve condition* (1.2) *of Theorem 1.1. Then the set of cut-points of $\partial\Omega$ has $\mathcal{H}^1$-measure zero.*

Let us record the following consequence of the proof of Lemma 4.6.

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded John domain whose complement is quasiconvex and $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map. Also let γ be a piecewise hyperbolic geodesic* *joining two distinct points* $\varphi(y_1)$, $\varphi(y_2) \in \partial\Omega^c$, *and let* $z \in \gamma^o \cap \partial\Omega$. *Then* z *is a cut-point, and every open disk centered at* z *has non-trivial intersection with* Ω_1 *and* Ω_2, *where* Ω_1 *and* Ω_2 *are the two components of* $\Omega \setminus (\varphi([y_1, 0] \cup [0, y_2]) \cup \gamma) = \Omega \setminus (\varphi([y_1, 0] \cup [0, y_2]))$.

5. Sufficiency for simply connected domains

We construct the desired extension via a modification to our procedure for the Jordan case. The first obstacle is that we cannot anymore use hyperbolic triangles in the complementary domain since there need not be a complementary domain to work with. To overcome this, we use piecewise hyperbolic geodesics to obtain a desired decomposition in each of the components of the interior of the complement of Ω. This allows us to mimic the Jordan case, but we have to work harder to verify the regularity of our extension.

5.1. **Decomposition of the complement.** Recall that $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ is conformal and continous up to the boundary. The decomposition of Ω is the same as in Section 3.1 for the Jordan case. The Whitney-type sets in Ω are denoted by $Q_k^{(j)}$ and $z_k^{(j)} = \varphi(x_k^{(j)})$ stand for the images of the endpoints on $\partial\mathbb{D}$ of the radial rays used in the decomposition of $\mathbb{D}$. Again our decomposition starts with suitable k_0 according to the constant δ in the previous section such that

$$|z_k^{(j)} - z_k^{(j-1)}| \le \delta \operatorname{diam}(\Omega).$$

This time the complement Ω^c will be simultaneously triangulated inside each connected component of $\widetilde{\Omega}$ using hyperbolic geodesics. For this purpose we consider all pairs $(z_k^{(j-1)}, z_k^{(j)})$ and pick for each of them a piecewise hyperbolic geodesic. Since this is a countable collection and our piecewise hyperbolic geodesics are obtained via the Arzelà-Ascoli theorem, we may assume that each of them is obtained via the same subsequence of the conformal maps $\widetilde{\varphi}_n$. Recall from Lemma 4.1 that a piecewise hyperbolic geodesic is a shortest curve, up to a multiplicative constant, such that its restriction to any of the connected components of $\widetilde{\Omega}$ is a hyperbolic geodesic. Moreover by Lemma 4.1 and (3.2) we again have

$$\ell(\gamma_k^{(j)}) \lesssim |z_k^{j-1} - z_k^j| \lesssim \operatorname{diam}(Q_k^{(j)}), \tag{5.1}$$

with the constant depending only on C_1.

Next we study the possible cases of degenerated hyperbolic triangles; also see Figure 3.

Lemma 5.1. *For* $k = 1, 2, 3$, *let* γ_k *be a piecewise hyperbolic geodesics with two different endpoints* z_k, $z_{k+1} \in \partial\Omega$ *respectively and assume that* $z_4 = z_1$. *Then there are two possibilities:*

1) *each of the curves is contained in the union of the other two, and the common set of these three curves is a single point.*
2) *For some* $1 \le j \le N$, *the intersection of each* γ_k *with* $\widetilde{\Omega}_j$ *is a hyperbolic geodesic and these three geodesics form a hyperbolic triangle of* $\widetilde{\Omega}_j$. *For any other* $1 \le j \le N$, *if two of the curves* γ_k *intersect* $\widetilde{\Omega}_j$, *then the respective intersections overlap with each other.*

Proof. First of all if $y \in \gamma_1 \cap \partial\Omega$, then γ_2 or γ_3 must contain y by Lemma 4.2. Namely

$$\gamma_1 \cap \partial\Omega \subset (\gamma_2 \cup \gamma_3) \cap \partial\Omega. \tag{5.2}$$

The analogous statements also hold for γ_2 and γ_3.

Let y_0 be the last point along γ_1 from z_2 towards z_1 such that $y_0 \in \gamma_1 \cap \gamma_2$. We claim that $y_0 \in \partial\Omega$.

Suppose on the contrary that $y_0 \notin \partial\Omega$. Then $y_0 \in \widetilde{\Omega}_j$ for some $1 \leq j \leq N$, and $\widetilde{\Omega}_j$ is Jordan by Lemma 2.9. Let y_1 be the first point on $\gamma_1 \cap \partial\widetilde{\Omega}_j$ when we trace (along γ_1) towards z_2 from y_0. Consider the subcurves $\gamma_{1,0}$ and $\gamma_{2,0}$ of γ_1, γ_2, respectively, between y_0 and z_2. By Lemma 4.3 we have that $y_1 \in \gamma_{2,0}$. It follows from Lemma 4.1 that the intersections of both γ_1 and γ_2 with $\widetilde{\Omega}_j$ are hyperbolic geodesics that contain both y_1 and y_0. By mapping $\widetilde{\Omega}_j$ conformally to $\mathbb{D}$ (or to the exterior of the closed unit disk if $\widetilde{\Omega}_j$ is unbounded), the uniqueness of hyperbolic geodesics yields that the intersection of γ_1 with $\widetilde{\Omega}_j$ equals to the intersection of γ_2 with $\widetilde{\Omega}_j$. This contradicts the definition of y_0.

Consider again the subcurves $\gamma_{1,0}$ and $\gamma_{2,0}$ from the previous paragraph. We conclude by Lemma 4.3 that

$$\gamma_{1,0} \cap \partial\Omega = \gamma_{2,0} \cap \partial\Omega. \tag{5.3}$$

Moreover, if $\gamma_{1,0}$ intersects $\widetilde{\Omega}_j$, then this intersection β is by Lemma 4.1 a hyperbolic geodesic, say with endpoints w_1, w_2. Then $w_1, w_2 \in \partial\Omega$ and it follows that $w_1, w_2 \in \gamma_{2,0}$. Consider the corresponding subcurve α of $\gamma_{2,0}$. Since $\beta \subset \widetilde{\Omega}_j$, it follows from Lemma 4.3 that $\alpha \subset \widetilde{\Omega}_i$ is a hyperbolic geodesic. As in the previous paragraph, we conclude that $\alpha = \beta$. Hence $\gamma_{1,0} \subset \gamma_{2,0}$. By reversing the roles of $\gamma_{1,0}$ and $\gamma_{2,0}$ we deduce that

$$\gamma_{1,0} = \gamma_{2,0}. \tag{5.4}$$

Suppose that $y_0 \in \gamma_3$. In this case, we repeat the argument from the previous paragraph first for the subcurve $\gamma_{1,2}$ of γ_1 between z_1 and y_0 and the subcurve $\gamma_{3,4}$ of γ_3 between $z_1 = z_4$ and y_0 and after that for the subcurve $\gamma_{3,3}$ of γ_3 between z_3 and y_0 and the subcurve $\gamma_{2,2}$ of γ_2 between z_3 and y_0. This gives us $\gamma_{1,2} = \gamma_{3,4}$ and $\gamma_{3,3} = \gamma_{2,2}$ and we conclude that $\gamma_3 \subset \gamma_1 \cup \gamma_2$.

We are left with the case $y_0 \notin \gamma_3$. Since y_0 is the last common point of γ_1 and γ_2 and $y_0 \in \partial\Omega$, we have that $y_0 \in \overline{\widetilde{\Omega}}_j$ for some j such that $\gamma_1 \cap \widetilde{\Omega}_j$ and $\gamma_2 \cap \widetilde{\Omega}_j$ do not intersect. Indeed by the definition of y_0 and (5.2) we have that

$$(\gamma_1 \setminus \gamma_{1,0}) \cap \gamma_2 = \emptyset$$

and

$$(\gamma_1 \setminus \gamma_{1,0}) \cap \partial\Omega \subset \gamma_3 \cap \partial\Omega.$$

Note that $\gamma_3 \cap \partial\Omega$ is a closed set, and $y_0 \in \partial\Omega$ while $y_0 \notin \gamma_3$. This implies that there is no sequence of points in $(\gamma_1 \setminus \gamma_{1,0}) \cap \partial\Omega$ converging to y_0. Let y_0' be the last point in $(\gamma_1 \setminus \gamma_{1,0}) \cap \partial\Omega$ towards y_0. Then the open subcurve $\gamma_1(y_0, y_0')$ is contained in the interior of Ω^c. By applying Lemma 4.1 to γ_1 we conclude the claim.

Then by Lemma 4.1, (5.2) and the definition of y_0, the endpoint w_1 on $\partial\widetilde{\Omega}_j$ of the part of γ_1 in $\widetilde{\Omega}_j$ has to be contained in γ_3. Arguing as above, we conclude that the remaining part of γ_1 coincides with a subcurve of γ_3. Similarly, the endpoint $w_2 \in \partial\widetilde{\Omega}$ of the part of γ_2 in $\widetilde{\Omega}_j$ must be contained in γ_3, and we conclude that the remaining part of γ_2 coincides with a subcurve of γ_3. Finally, let $\gamma_{3,1,2}$ be the subcurve of γ_3 joining w_1, w_2. Since the union of the (closures) of the subcurves of γ_1, γ_2 in $\widetilde{\Omega}_j$ also joins w_1 to w_2 and only intersects $\partial\Omega$ in the set

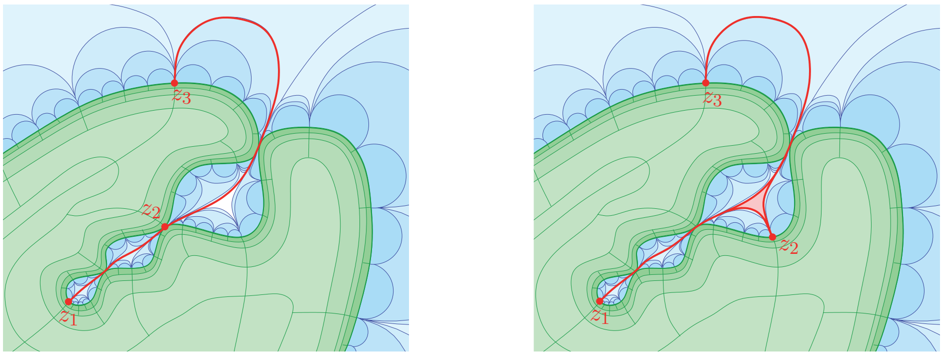

FIGURE 3. Here are the two possible cases of degenerated hyperbolic triangles.

$\{y_0, w_1, w_2\}$ and $y_0 \notin \gamma_3$, Lemma 4.3 implies that $\gamma_{3,1,2}$ joins w_1 to w_2 in $\widetilde{\Omega}_j$. By Lemma 4.1 all of $\gamma_{3,1,2}$ and the subcurves of γ_1, γ_2 in $\widetilde{\Omega}_j$ are hyperbolic geodesics and it follows that they form a hyperbolic triangle.

□

By Lemma 5.1, we can define the (degenerated) hyperbolic triangles $R_k^{(j)}$ similarly as in the Jordan case, namely the bounded (relatively) closed set enclosed by the union of $\gamma_k^{(j)}$, $\gamma_{k+1}^{(2j-1)}$ and $\gamma_{k+1}^{(2j)}$, and there is at most one connected component $\widetilde{\Omega}_i$ of $\widetilde{\Omega}$ such that $R_k^{(j)} \cap \widetilde{\Omega}_i$ is a (non-degenerated) hyperbolic triangle, denoted by $R_{k,i}^{(j)}$.

For every $k \geq 1$, $1 \leq j \leq 2^{k_0+k+1}$ and $1 \leq i \leq N$, set

$$\gamma_{k,i}^{(j)} = \gamma_k^{(j)} \cap \widetilde{\Omega}_i.$$

Notice that $\gamma_{k,i}^{(j)}$ may well be empty for a choice of k, j, i. Nevertheless

$$\bigcup_{i=1}^{N} \gamma_{k,i}^{(j)} = \gamma_k^{(j)} \cap \widetilde{\Omega};$$

see Lemma 4.6. We call $\gamma_{k,i}^{(j)}$ the i-th component of $\gamma_k^{(j)}$, call $\gamma_{k,i}^{(j)}$ the *mother* of $\gamma_{k+1,i}^{(2j-1)}$ and $\gamma_{k+1,i}^{(2j)}$, and refer to $\gamma_{k+1,i}^{(2j-1)}$ and $\gamma_{k+1,i}^{(2j)}$ (possibly empty) as *children* of $\gamma_{k,i}^{(j)}$. By construction each mother has two children, and every child has a mother; see Lemma 5.1. Moreover, $\gamma_{k,i}^{(j_1)}$ is called a *sibling* of $\gamma_{k,i}^{(j_2)}$ if they have the same mother. We say that $\gamma_{k,i}^{(j)}$ is in the first generation if $k = 1$, that two curves $\gamma_{k_1,i_1}^{(j_1)}, \gamma_{k_2,i_2}^{(j_2)}$ are in the same generation if $k_1 = k_2$, and that the generation of $\gamma_{k_1,i_1}^{(j_1)}$ is higher than that of $\gamma_{k_2,i_2}^{(j_2)}$ if $k_1 < k_2$.

Notice that potentially $\gamma_{k,i}^{(j)}$ may coincide with the i-th component of some other piecewise hyperbolic geodesic. The following lemma describes the potential overlaps.

Lemma 5.2. *If $k_1 < k_2$, then $\gamma_{k_1,i}^{(j_1)}$ and $\gamma_{k_2,i}^{(j_2)}$ are not siblings. Moreover if we have that $\gamma_{k_1,i}^{(j_1)}$ coincides with $\gamma_{k_2,i}^{(j_2)}$ for some i, j_1 and j_2, then the mother of $\gamma_{k_2,i}^{(j_2)}$ also coincides with $\gamma_{k_1,i}^{(j_1)}$.*

Similarly if $\gamma_{k,i}^{(j_1)}$ and $\gamma_{k,i}^{(j_2)}$ coincide with each other but they are not siblings, the mother of $\gamma_{k,i}^{(j_2)}$ also coincides with $\gamma_{k,i}^{(j_1)}$.

Proof. It is clear that $\gamma_{k_1,i}^{(j_1)}$ and $\gamma_{k_2,i}^{(j_2)}$ are not in the same generation since $k_1 < k_2$. Thus they are not siblings. Recall that piecewise hyperbolic geodesics are obtained by taking limits of hyperbolic geodesics in Jordan domains $\widetilde{D}_n$. Suppose that the end points of $\gamma_{k_1,i}^{(j_1)}$ are x and y. Then for each $l = 1, 2$ there exists an approaching sequence of $\gamma_{n,l} \subset \widetilde{D}_n$ for $\gamma_{k_l}^{(j_l)}$ such that $x_{n,l}, y_{n,l} \in \gamma_{n,l}$ approximate x and y, respectively. Let $\alpha_n \subset \widetilde{D}_n$ be the hyperbolic geodesic joining $x_{n,1}$ and $x_{n,2}$. Likewise $\beta_n \subset \widetilde{D}_n$ is the hyperbolic geodesic connecting $y_{n,1}$ and $y_{n,2}$.

Recall that by our choice of δ_1 in Section 4, according to Lemma 7.5

$$\ell(\alpha_n) \lesssim |x_{n,1} - x_{n,2}|, \qquad \ell(\beta_n) \lesssim |y_{n,1} - y_{n,2}| \tag{5.5}$$

for sufficiently large n, where the constants depend only on C_1. This allows to take the limit of α_n and β_n by Arzelà-Ascoli lemma, up to relabeling the sequences. Since $x_{n,l} \to x$ and $y_{n,l} \to y$ as $n \to \infty$ for $l = 1, 2$, we conclude from (5.5) that $\alpha_n \to x$ and $\beta_n \to y$.

On the other hand, by the geometry of hyperbolic geodesics in the exterior of the unit disk, via conformal mappings, we conclude that α_n and β_n have non-trivial intersections with $\gamma'_{n,2}$, where the sequence $\{\gamma'_{n,2}\}$ approaches the mother of $\gamma_{k_2,i}^{(j_2)}$. Thus the mother of $\gamma_{k_2,i}^{(j_2)}$ also goes through x and y, and our conclusion follows from Lemma 4.3.

The second part of the lemma follows from a similar argument with notational changes. □

Let $[\gamma_{k,i}^{(j)}]$ be the collection of all the i-th components of our piecewise hyperbolic geodesics which coincide with a non-empty curve $\gamma_{k,i}^{(j)}$ in $\widetilde{\Omega}_i$ (see Lemma 4.1) and order the elements in it as follows: $\gamma_{k_1,i}^{(j_1)}$ is older than $\gamma_{k_2,i}^{(j_2)}$ if and only if either $k_1 < k_2$ or $k_1 = k_2$ but $j_1 < j_2$. We denote the oldest one in $[\gamma_{k,i}^{(j)}]$ by $\mathcal{A}[\gamma_{k,i}^{(j)}]$, and call it *the ancestor of* $\gamma_{k,i}^{(j)}$. So as to define an extension operator conveniently later, we force $\gamma_{1,i}^{(1)} \in [\gamma_{k,i}^{(j)}]$ if $\gamma_{1,i}^{(j)} \in [\gamma_{k,i}^{(j)}]$ for some $2 \le j \le 2^{k_0+k+1}$; it is just an artificial curve which is only used in the next subsection, and we suppress the abuse of notation here.

To clarify the definition of an ancestor, consider the curves $\gamma_{k,i}^{(j)}$, $\gamma_{k+1,i}^{(2j-1)}$ and $\gamma_{k+1,i}^{(2j)}$ enclosing a non-degenerated hyperbolic triangle $R_{k,i}^{(j)}$. Then Lemma 5.2 shows that

$$\mathcal{A}[\gamma_{k+1,i}^{(2j-1)}] = \gamma_{k+1,i}^{(2j-1)} \quad \text{and} \quad \mathcal{A}[\gamma_{k+1,i}^{(2j)}] = \gamma_{k+1,i}^{(2j)}. \tag{5.6}$$

We need the concept of a *family chain of* $\gamma_{k,i}^{(j)}$. This is the finite ordered set $\mathcal{F}(\gamma_{k,i}^{(j)}) = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$, $l = l(k,j)$ consisting of elements in $[\gamma_{k,i}^{(j)}]$ and with the property that γ_n is the mother or sibling of γ_{n+1} for $1 \le n \le l-1$, obtained via the following procedure. Define $\hat{\gamma}_1 = \gamma_{k,i}^{(j)}$. Set $\hat{\gamma}_2$ to be the mother of $\hat{\gamma}_1$ and continue inductively; see Lemma 5.2. This procedure stops after a finite number of steps, when we reach $\hat{\gamma}_{l-1} = \gamma^{j_0}_{k_0,i}$ such that the mother of $\gamma^{j_0}_{k_0,i}$ is not in $[\gamma^{(j)}_{k,i}]$, or $k_0 = 1$; namely $\gamma^{j_0}_{k_0,i}$ is in the highest generation of $[\gamma^{(j)}_{k,i}]$ (by Lemma 5.2). Then define $\hat{\gamma}_l = \mathcal{A}[\gamma^{(j)}_{k,i}]$; note that it is possible that $\hat{\gamma}_l = \hat{\gamma}_{l-1}$. We now set $\gamma_n = \hat{\gamma}_{l-n+1}$, $1 \le n \le l$. Observe that for $n \ge 3$, γ_n is a child of γ_{n-1} by our construction. See Figure 4.

Lemma 5.3. *Each generation in $[\gamma^{(j)}_{k,i}]$ contains at most two distinct elements (expect the first generation which has at most three elements). Moreover if the highest (but not the first) generation of $[\gamma^{(j)}_{k,i}]$ contains two distinct elements, then they are siblings.*

Proof. We prove the first claim by contradiction. Suppose that $[\gamma^{(j)}_{k,i}]$ contains three distinct elements γ_l, $l = 1, 2, 3$ from the same generation, and that they coincide in the component $\widetilde{\Omega}_i$. By joining each of the two end points of γ_l via a hyperbolic geodesic in Ω starting from the image of the origin under the conformal map $\varphi\colon \mathbb{D} \to \Omega$, we obtain a Jordan domain Ω_l via concatenating the two hyperbolic geodesics together with γ_l, for every $l = 1, 2, 3$ correspondingly; recall here that γ_l is injective by Lemma 4.1 for each $l = 1, 2, 3$.

We claim that these three domains are pairwise disjoint. Indeed recall that

$$D_n = \varphi(B(0,\, 1 - 2^{-n}))$$

with $\varphi\colon \mathbb{D} \to \Omega$ conformal, and that each γ_l is the limit of a uniformly converging sequence, consisting of hyperbolic geodesics $\gamma_{l,n} \subset \widetilde{D}_n$, where $\widetilde{D}_n$ is the exterior of D_n. Then the end points $x_{l,n}$, $y_{l,n}$ of $\gamma_{l,n}$ converge to the end points x_l, y_l of γ_l, respectively. By the uniform continuity of φ the hyperbolic geodesics $\varphi([0,\, \varphi^{-1}(x_{l,n})])$ joining $\varphi(0)$ to $x_{l,n}$ also converge uniformly to the hyperbolic geodesics joining $\varphi(0)$ to x_l (in D_n); an analogous statement holds for $\varphi([0,\, \varphi^{-1}(y_{l,n})])$. Therefore

$$\alpha_{l,n} := \varphi([0,\, \varphi^{-1}(x_{l,n})]) \cup \varphi([0,\, \varphi^{-1}(y_{l,n})]) \cup \gamma_{l,n},\ n \in \mathbb{N}$$

is a sequence of Jordan curves converging to a Jordan curve

$$\alpha_l = \varphi([0,\, \varphi^{-1}(x_{l,n})]) \cup \varphi([0,\, \varphi^{-1}(y_{l,n})]) \cup \gamma_l$$

uniformly as $n \to \infty$ for $l = 1, 2, 3$.

Note that for each $n \in \mathbb{N}$, the Jordan domains enclosed by $\alpha_{l,n}$'s are pairwise disjoint since γ_l are in the same generation. Then the claim follows from the uniform convergence of $\alpha_{l,n}$. Indeed if two of Ω_1, Ω_2, Ω_3 were to intersect, then there would exist a disk $B(z, 3r)$ contained in the intersection, say, of Ω_1 and Ω_2. However for each $l = 1, 2$, when n is large enough, every point in $\alpha_{l,n}$ has at most distance r to α_l. Since α_l and $\alpha_{l,n}$ are Jordan, we conclude by the triangle inequality that $B(z, r)$ is in the intersection of the Jordan domains given by $\alpha_{1,n}$ and $\alpha_{2,n}$. This is a contradiction.

We then show that, since α_l's coincide in $\widetilde{\Omega}_i$ for each $i = 1, 2, 3$, at least two of the corresponding Jordan domains have non-trivial intersection; this gives the contradiction. In fact, by the Jordan-Schoenflies theorem there is a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the curve $\gamma_l \cap \widetilde{\Omega}_i$ is mapped to the interval $[-1, 1]$ on the real line. Notice that Ω_l are also

(a)

(b)

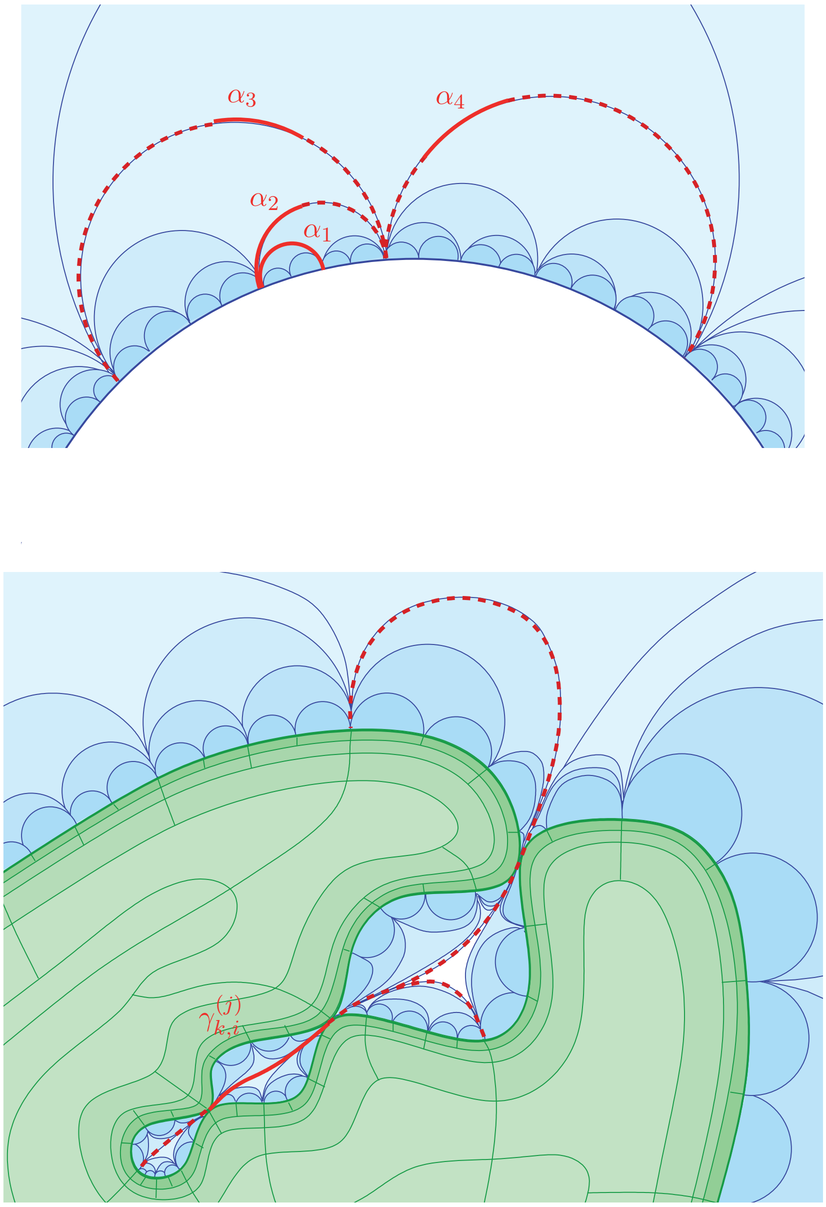

Figure 4. A collection of hyperbolic geodesics (in the approximation to the piecewise hyperbolic geodesics in the picture (b)) related to the elements in $[\gamma_{k,i}^{(j)}]$ is illustrated in the picture (a). Here the collection $[\gamma_{k,i}^{(j)}]$ gives rise to two different family chains: One corresponds to (the limit of) $\{\alpha_3, \alpha_3, \alpha_2, \alpha_1\}$ and the other corresponds to (the limit of) $\{\alpha_3, \alpha_4\}$.

mapped to Jordan domains. Then there exists a constant $0 < r < 1$ such that $B(0, r) \cap \phi(\Omega_l)$ is a half disk for each $l = 1, 2, 3$; indeed choose

$$r = \min_{l=1,2,3} \operatorname{dist}\left(0, \phi(\partial\Omega_l \setminus (\gamma_l \cap \widetilde{\Omega}_i))\right).$$

However there are only two possible disjointed half disks with the same line segment as a part of the common boundary. Thus we obtain the claim via the homeomorphism ϕ since we have three Jordan domains. All in all we have show the first part of the lemma.

The second part of the lemma follows from Lemma 5.2. □

Lemma 5.3 with Lemma 5.1 and Lemma 5.2 tells that, from each $[\gamma_{k,i}^{(j)}]$, one can extract at most two different family chains from the highest generation to the lowest one. Now we are ready to construct our extension operator.

5.2. **Definition of the extension.** Let $u \in W^{1,\infty}(\Omega)$. As in the Jordan case, we define

$$a_k^{(j)} = \fint_{Q_k^{(j)}} u(x)\, dx.$$

We still associate this value to $\gamma_k^{(j)}$ and write it as $a_{\gamma_k^{(j)}}$; we associate this value also to all the subarcs $\gamma_{k,i}^{(j)}$ of $\gamma_k^{(j)}$. However one cannot define our extension directly via them and Lemma 3.1 because of the degeneracy.

To overcome this, we use the terminology defined in the previous subsection. For each $x \in \gamma_{k,i}^{(j)}$, we define $Eu(x) = a_{\mathcal{A}[\gamma_{k,i}^{(j)}]}$, where $a_{\mathcal{A}[\gamma_{k,i}^{(j)}]}$ is the value associated to $\mathcal{A}[\gamma_{k,i}^{(j)}]$. Then Eu is well-defined on these hyperbolic geodesics by the definition of $[\gamma_{k,i}^{(j)}]$.

Next we associate to each non-degenerate $R_{k,i}^{(j)}$ in $\widetilde{\Omega}_i$ a function $\phi_k^{(j)} \in W^{1,1}(R_k^{(j)})$ so that (recall again that there is at most one possible i such that $R_{k,i}^{(j)}$ is non-degenerate by Lemma 5.1.)

$$\phi_k^{(j)}(x) = \begin{cases} a_{\mathcal{A}[\gamma_{k,i}^{(j)}]}, & x \in \gamma_k^{(j)} \cap R_k^{(j)} \\ a_{\mathcal{A}[\gamma_{k+1,i}^{(2j-1)}]}, & x \in \gamma_{k+1}^{(2j-1)} \cap R_k^{(j)} \\ a_{\mathcal{A}[\gamma_{k+1,i}^{(2j)}]}, & x \in \gamma_{k+1}^{(2j)} \cap R_k^{(j)} \end{cases}.$$

We choose $\phi_k^{(j)}$ to be the function ϕ from Lemma 3.1 with the given boundary values. Then we have the gradient estimate (3.5) that will be employed later.

Since $R_{k,i}^{(j)}$ is non-degenerate, we have (5.6). On the rest of $\widetilde{\Omega}$ we define $Eu = a_{\gamma_1^{(1)}}$. This is consistent because we force $\gamma_{1,i}^{(1)} \in [\gamma_{k,i}^{(j)}]$ if $\gamma_{1,i}^{(j)} \in [\gamma_{k,i}^{(j)}]$ for $2 \leq j \leq 2^{k_0+k+1}$. To conclude, the operator E defined by

$$Eu(x) = \begin{cases} u(x), & x \in \Omega \\ a_{\mathcal{A}[\gamma^{(j)}_{k,\,i}]}, & x \in \gamma^{(j)}_{k,\,i} \\ \phi^{(j)}_k(x), & x \text{ is in the interior of the non-degenerate } R^j_{k,\,i} \\ a_{\gamma^{(1)}_1}, & \text{otherwise} \end{cases}.$$

Observe that $Eu \in W^{1,1}_{loc}(\widetilde{\Omega})$. Indeed by Lemma 3.1 and Lemma 5.1, Eu is locally Lipschitz continuous inside $\widetilde{\Omega}$. Thus by Lemma 3.1 again we have $Eu \in W^{1,1}_{loc}(\widetilde{\Omega})$.

This time we estimate the Sobolev-norm of the extension inside a non-degenerate hyperbolic triangle $R^{(j)}_{k,\,i} \subset \widetilde{\Omega}_i$ by inserting the intermediate values given to the corresponding curves in the family chain. Namely by Lemma 3.1, (5.1) and (5.6),

$$\begin{aligned} \int_{R^{(j)}_{k,\,i}} |\nabla \phi^{(j)}_k(x)|\,dx &\lesssim \Big(|a_{\mathcal{A}[\gamma^{(j)}_{k,\,i}]} - a^{(2j-1)}_{k+1}|\ell(\gamma^{(j)}_{k,i}) + |a_{\mathcal{A}[\gamma^{(j)}_{k,\,i}]} - a^{(2j)}_{k+1}|\ell(\gamma^{(j)}_{k,i}) \\ &\qquad + |a^{(2j-1)}_{k+1} - a^{(2j)}_{k+1}| \min\{\ell(\gamma^{(2j-1)}_{k+1,i}), \ell(\gamma^{(2j)}_{k+1,i})\} \Big) \\ &\lesssim \Big(|a^{(j)}_k - a^{(2j-1)}_{k+1}| + |a^{(j)}_k - a^{(2j)}_{k+1}| + |a^{(2j-1)}_{k+1} - a^{(2j)}_{k+1}| \Big) \operatorname{diam}(Q^{(j)}_k) \\ &\qquad + \sum_{\gamma_l \in \mathcal{F}(\gamma^{(j)}_{k,\,i})} |a_{\gamma_l} - a_{\gamma_{l+1}}| \ell(\mathcal{A}[\gamma^{(j)}_{k,i}]) \\ &= S^{(j)}_k(1) + S^{(j)}_k(2), \end{aligned}$$

where we used the fact $\ell(\mathcal{A}[\gamma^{(j)}_{k,i}]) = \ell(\gamma^{(j)}_{k,i})$.

The first term $S^{(j)}_k(1)$ is the same as in the Jordan case, and so we know that, after summing over all the hyperbolic triangles, it can be controlled by $\|\nabla u\|_{L^1(\Omega)}$ up to a multiplicative constant.

Observe that

$$\sum_{i=1}^{N} \ell(\gamma^{(j)}_{k,i}) = \sum_{i=1}^{N} \ell(\mathcal{A}[\gamma^{(j)}_{k,i}]) \lesssim \operatorname{diam}(Q^{(j)}_k), \tag{5.7}$$

by (5.1).

Fix i and two distinct non-degenerate triangles $R^{(j)}_{k,\,i}$, $R^{(j')}_{k',\,i}$. Then $\mathcal{F}(\gamma^{(j)}_{k,\,i})$ is disjoint from $\mathcal{F}(\gamma^{(j')}_{k',\,i})$ unless $[\gamma^{(j)}_{k,\,i}] = [\gamma^{(j')}_{k',\,i}]$, or equivalently $R^{(j)}_{k,\,i}$ and $R^{(j')}_{k',\,i}$ share the same boundary. Recall that in each $[\gamma^{(j)}_{k,\,i}]$ there are at most two different family chains starting from the highest generation to the lowest one, and hence the ancestor in each $[\gamma^{(j)}_{k,\,i}]$ is counted at most twice if $k \neq 1$; if $k = 1$ the multiplicity of $\gamma^{(1)}_{1,\,i}$ is at most 2^{k_0+3}. As a result of changing the order of summation with (5.7) and (3.8) the following estimate holds:

$$\sum_{R_{k,i}^{(j)}} S_k^{(j)}(2) = \sum_{R_{k,i}^{(j)}} \sum_{\gamma_l \in \mathcal{F}(\gamma_{k,i}^{(j)})} |a_{\gamma_l} - a_{\gamma_{l+1}}| \ell(\mathcal{A}[\gamma_{k,i}^{(j)}])$$

$$\lesssim \sum_{k,j} (|a_k^{(j)} - a_{k+1}^{(2j-1)}| + |a_k^{(j)} - a_{k+1}^{(2j-1)}| + |a_k^{(j)} - a_k^{(j-1)}| + |a_k^{(j)} - a_k^{(j+1)}|) \sum_{i=1}^{N} \ell(\mathcal{A}[\gamma_{k,i}^{(j)}])$$

$$\lesssim \sum_{k,j} (|a_k^{(j)} - a_{k+1}^{(2j-1)}| + |a_k^{(j)} - a_{k+1}^{(2j-1)}| + |a_k^{(j)} - a_k^{(j-1)}| + |a_k^{(j)} - a_k^{(j+1)}|) \operatorname{diam}(Q_k^{(j)})$$

$$\lesssim \sum_{k,j} S_k^{(j)}(1).$$

Therefore the estimate for $\sum_{k,j} S_k^{(j)}(1)$ gives

$$\|\nabla Eu\|_{L^1(\widetilde{\Omega})} \lesssim \|\nabla u\|_{L^1(\Omega)}.$$

Finally let B be a disk of radius $4 \operatorname{diam}(\Omega)$ such that Ω is properly contained in $\frac{1}{2}B$. Similarly as in the Jordan case, we have that

$$\begin{aligned} \|Eu\|_{L^1(B)} &\lesssim \sum_{R_{k,i}^{(j)}} \int_{R_{k,i}^{(j)}} |Eu(x)|\, dx + \int_\Omega |u(x)|\, dx \\ &\lesssim \sum_{\substack{k \geq 1 \\ 1 \leq j \leq 2^{k_0+k+1}}} \int_{Q_k^{(j)}} |u(x)|\, dx + \int_\Omega |u(x)|\, dx \lesssim \int_\Omega |u(x)|\, dx \end{aligned} \tag{5.8}$$

where we used (3.4), (3.8) and the fact that

$$|R_k^{(j)}| \lesssim |Q_k^{(j)}|$$

coming from (3.2) and the fact that $Q_k^{(j)}$ is of Whitney-type.

5.3. **Continuity at one-sided points.** By our assumption and Corollary 4.7, the set $T \subset \partial\Omega$ of cut-points for the mapping φ has $\mathcal{H}^1$-measure zero. Therefore if we can extend u continuously to the one-sided points, then we have defined Eu everywhere except for a set of $\mathcal{H}^1$-measure zero.

Let us first show that we may assume u to be continuous at the one-sided points in the following sense:

Lemma 5.4. *Let Ω be a bounded simply connected John domain, and suppose that $u \in W^{1,\infty}(\Omega) \cap C(\Omega)$. Then we can extend u (uniquely) to $\widetilde{u}$ so that $\widetilde{u} \in C(\overline{\Omega} \setminus T) \cap W^{1,\infty}(\Omega)$, where T is the collection of cut-points of $\partial\Omega$.*

Proof. Let $z \in \partial\Omega \setminus T$. Since $u \in W^{1,\infty}(\Omega) \cap C(\Omega)$, it suffices to show that $\lim_{i\to\infty} u(z_i)$ exists whenever $z_i \in \Omega$ satisfies $|z_i - z| \to 0$ as $i \to \infty$. Let $\varphi \colon \overline{\mathbb{D}} \to \overline{\Omega}$ be continuous and conformal inside $\mathbb{D}$; see Lemma 2.8.

By [14], φ is still continuous when the Euclidean distance in Ω is replaced by $\operatorname{dist}_\Omega$. To be precise, the definition of continuity with respect to the inner distance in [14] is based on another version of the inner distance, where $\ell(\gamma)$ is replaced by $\operatorname{diam}(\gamma)$. By [8] one may replace the arcs γ in these definitions by hyperbolic geodesics. If our simply connected domain Ω is John with constant J it then follows from [9, Theorem 5.14] that these two distances are comparable modulo a multiplicative constant that only depends on J. Thus it is legitimate to apply [14] here.

Since $z \notin T$, we have that $\varphi^{-1}(z)$ is a singleton; say $\varphi^{-1}(z) = w$. Our claim follows if $|\varphi^{-1}(z_i) - w| \to 0$ when $i \to \infty$; indeed since φ is continuous with respect to the $\operatorname{dist}_\Omega$-metric of Ω, then by the assumption that $u \in W^{1,\infty}(\Omega)$ we conclude that $\lim_{i\to\infty} u(z_i)$ exists. This is necessarily the case as, otherwise a subsequence converges to some $w' \neq w$ and then $\varphi(w') = z$ by the continuity of φ at w', contradicting the fact that $\varphi^{-1}(z)$ is a singleton. □

Now the continuity of Eu with $u \in W^{1,\infty}(\Omega) \cap C(\Omega)$ restricted to $\mathbb{R}^2 \setminus H$ with $H := T \cup Z$ follows exactly as in the Jordan case, where Z is also defined like the previous one: the collection of the countably many points $z_k^{(j)}$ on the boundary. This time the fact that $y \notin \Gamma_s$ (defined in (3.11)) comes from Lemma 4.6, where we observed that, except for the endpoints, piecewise hyperbolic geodesics can meet the boundary only at cut-points. Therefore, by further defining $Eu(x) = \widetilde{u}(x)$ for $x \in \partial\Omega$, we have that $Eu(x)$ is continuous in $\mathbb{R}^2 \setminus H$.

5.4. **Absolute continuity.** We show the absolute continuity of Eu along almost every horizontal or vertical line under the assumption that $\widetilde{u} \in C(\Omega) \cap W^{1,\infty}(\Omega)$. By Lemma 5.4 we may assume that

$$u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega} \setminus T).$$

We begin by applying the Jordan case in the following way.

Since $\mathcal{H}^1(T) = 0$ by Corollary 4.7, for any $\epsilon > 0$, we find

$$U_\epsilon := \Omega \cap \bigcup_{z\in T} B(z, r_z)$$

with the property that the $\mathcal{H}^1$-measure of its vertical and horizontal projections are less than ϵ.

Now pull U_ϵ back to $\overline{\mathbb{D}}$ via the conformal map φ. Since φ is a homeomorphism in $\mathbb{D}$, the set $\varphi^{-1}(U_\epsilon)$ is open. For each $z \in T$ and every $w \in \varphi^{-1}(\{z\})$, by the continuity of φ we pick a small enough $\delta_w \in (0, \frac{1}{2})$ such that the set

$$V_w := \{v \in \mathbb{D} \colon \langle w, v\rangle > (1 - \delta_w)\}$$

satisfies

$$\operatorname{dist}(\varphi(V_w),\, z) \leq \frac{1}{2} r_z. \tag{5.9}$$

where $\langle w, v\rangle$ means the inner product of w and v. See Figure 5.

Finally we define

$$\Omega_\epsilon := \varphi(\mathbb{D} \setminus \overline{\bigcup_{w\in\varphi^{-1}(T)} V_w}).$$

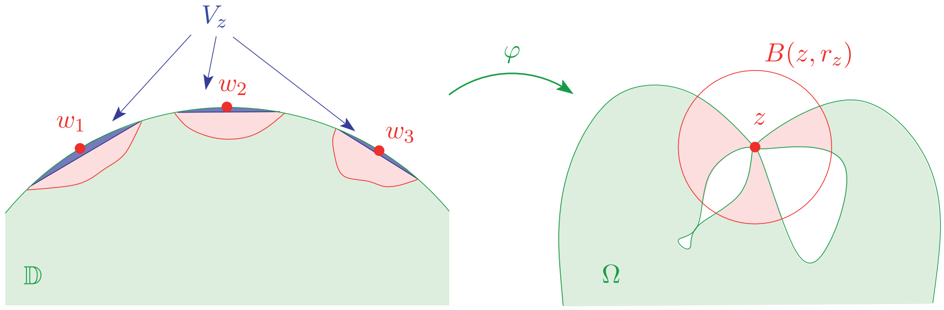

FIGURE 5. Here we show in red the ball $B(z, r_z)$ and its preimage under φ. The set V_z, whose boundaries are line segments inside the unit disk, is shown with blue color.

We claim that $\Omega \setminus \Omega_\epsilon \subset U_\epsilon$. If this held, then the $\mathcal{H}^1$-measure of the projection of $\Omega \setminus \Omega_\epsilon$ on each coordinate axis would be less than ϵ, as we desired.

Let us show the claim. Since φ is a homeomorphism in $\mathbb{D}$, we have

$$\Omega \setminus \Omega_\epsilon = \varphi(\overline{\bigcup_{w \in \varphi^{-1}(T)} V_w}) = \overline{\bigcup_{w \in \varphi^{-1}(T)} \varphi(V_w)}.$$

Pick $y \in \Omega \setminus \Omega_\epsilon$ and let $y_n \in \bigcup_{w \in \varphi^{-1}(T)} \varphi(V_w)$ be a sequence approximating y. Let $\delta := \operatorname{dist}(y, \partial\Omega) > 0$ For n large enough, we have $|y_n - y| \le \delta/4$, and hence

$$\operatorname{dist}(y_n, \partial\Omega) \ge \operatorname{dist}(y, \partial\Omega) - |y_n - y| \ge \frac{3}{4}\delta.$$

Moreover by (5.9) there exists $z_n \in T$ such that $|z_n - y_n| \le r_{z_n}/2$, and then we obtain

$$\delta \le \frac{2}{3} r_{z_n}.$$

Therefore

$$|y - z_n| \le |y_n - y| + |y_n - z_n| \le \frac{1}{4}\delta + \frac{1}{2} r_{z_n} \le \frac{2}{3} r_{z_n},$$

and then $y \in B(z_n, r_{z_n}) \cap \Omega \subset U_\epsilon$. Thus the claim follows.

Since we took out neighbourhoods of all the cut-points and φ is a conformal map and continuous up to the boundary, then Ω_ϵ is Jordan [23, Theorem 14.19]. As a consequence, by the result on the $W^{1,1}$-extension in the Jordan case (in Section 3) we know that there exists an extension $u_\epsilon \in W^{1,1}(\mathbb{R}^2)$ of $u|_{\Omega_\epsilon}$, provided $\mathbb{R}^2 \setminus \overline{\Omega}_\epsilon$ is quasiconvex, and we claim it is; the proof is similar to the one of Lemma 2.6.

Indeed if $x, y \in \Omega^c$, then (1.2) gives us a curve. If $x \in \Omega^c$ while $y \in \Omega \setminus \Omega_\epsilon$, then $|x - y| \ge \operatorname{dist}(y, \partial\Omega)$. Hence by connecting y to $y_0 \in \partial\Omega$ such that the line segment $[\varphi^{-1}(y), \varphi^{-1}(y_0)]$ joining $\varphi^{-1}(y)$ and $\varphi^{-1}(y_0)$ lies in the line segment $[0, \varphi^{-1}(y_0)]$, (1.2) and Lemma 2.8 give us the desired curve; note that $[\varphi^{-1}(y), \varphi^{-1}(y_0)]$ is contained in the union of V_z's by definition. Thus by symmetry we may assume that $x, y \in \Omega \setminus \Omega_\epsilon$. Moreover if x, y are relatively far from each other compared to their distances to the boundary, then we can also define $x_0 \in \partial\Omega$ in a similar way, connect x_0 with x via a hyperbolic geodesic in Ω and join x_0 and y_0 in Ω^c by a curve with length controlled by $C(C_1)|x_0-y_0|$. This gives us a desired curve by (1.2) and Lemma 2.8 again. Thus we only need to consider the case where $|x-y|\le C(J)\min\{\operatorname{dist}(x,\partial\Omega),\ \operatorname{dist}(y,\partial\Omega)\}$. Here $C(J)$ is a suitable constant such that

$$|\varphi^{-1}(x)-\varphi^{-1}(y)|\le\frac{1}{16}\min\{\operatorname{dist}(\varphi^{-1}(x),\partial\Omega),\ \operatorname{dist}(\varphi^{-1}(y),\partial\Omega)\};\tag{5.10}$$

see Lemma 2.3.

Let $B=B(\varphi^{-1}(x),4|\varphi^{-1}(x)-\varphi^{-1}(y)|)$. Then it is a Whitney-type set because of (5.10). We claim that there exists a curve $\gamma\subset B\setminus\varphi^{-1}(\Omega_\epsilon)$ joining $\varphi^{-1}(x)$, $\varphi^{-1}(y)$ such that its length is controlled by $|\varphi^{-1}(x)-\varphi^{-1}(y)|$ up to a multiplicative constant. If this holds, by Lemma 2.3 the curve $\varphi(\gamma)$ joining x, y in $\varphi(B)\cap(\Omega\setminus\Omega_\epsilon)$ has length smaller than $|x-y|$ up to a multiplicative constant. Consequently $\mathbb{R}^2\setminus\Omega_\epsilon$ is quasiconvex with the constant independent of ϵ.

Now let us show the existence of γ under the assumption (5.10). Observe first that by definition

$$\varphi^{-1}(\Omega_\epsilon)=\mathbb{D}\setminus\overline{\bigcup_{z\in T}V_z}=\left(\bigcap_{z\in T}(\mathbb{D}\setminus V_z)\right)^o$$

is a convex domain since it is the interior of the intersection of convex sets and the connectedness follows direction from the definition. Since we always require that $\delta_w\in(0,\frac{1}{2})$, then $B(0,\frac{1}{2})\subset\varphi^{-1}(\Omega_\epsilon)$, which combining with convexity yields that for every $z\in\varphi^{-1}(\overline{\Omega}_\epsilon)$, locally there is a Euclidean (open) cone contained in $\varphi^{-1}(\Omega_\epsilon)$ with angle at least $\frac{\pi}{3}$ centered at z towards the origin. Thus if we cannot join $\varphi^{-1}(x)$, $\varphi^{-1}(y)$ in $\frac{3}{4}B\setminus\varphi^{-1}(\Omega_\epsilon)$, then $\frac{3}{4}B\setminus\varphi^{-1}(\Omega_\epsilon)$ has at least two components. However this is impossible since $\varphi^{-1}(x)$, $\varphi^{-1}(y)\in\frac{1}{4}B$ but both $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ are not in the convex set $\varphi^{-1}(\Omega_\epsilon)$; planar geometry tells that such a cone with angle at least $\frac{\pi}{3}$ cannot exist. Therefore we can join $\varphi^{-1}(x)$, $\varphi^{-1}(y)$ in $\frac{3}{4}B\setminus\varphi^{-1}(\Omega_\epsilon)$, and the length of the curve is at most $(8+\frac{3}{2}\pi)|\varphi^{-1}(x)-\varphi^{-1}(y)|$. Therefore we proved the quasiconvexity of $\mathbb{R}^2\setminus\overline{\Omega}_\epsilon$.

Notice that Eu is absolutely continuous along almost every closed line segment parallel to the coordinate axes and contained in $\widetilde{\Omega}$ since Eu is locally Lipschitz in $\widetilde{\Omega}$ by our construction. Recall that by Lemma 5.4 we may assume that $u\in C(\overline{\Omega}\setminus T)\cap W^{1,\infty}(\Omega)$. Also take the representative of $u_{1/n}$, the extension of the restriction of u to the Jordan domain $\Omega_{1/n}$, so that it is absolutely continuous (and hence continuous) along almost every line segment parallel to the coordinate axes.

By the Fubini's theorem,

$$\int_L|\nabla Eu|\,dx<\infty,\qquad\int_L|\nabla u_{1/n}|\,dx<\infty\tag{5.11}$$

for $\mathcal{H}^1$-almost every line segment L parallel to the coordinate axes. It follows that for almost every horizontal (or vertical) line S, (5.11) holds, each $u_{1/n}$ is absolutely continuous on every closed line segment of S, and so is also Eu on those closed line sements of S.

Define $Z_0=\cap_{n\ge1}\Omega\setminus\Omega_{1/n}$. Since the vertical and horizontal projections of $\Omega\setminus\Omega_{1/n}$ have $\mathcal{H}^1$-measure no more than $1/n$, we conclude that almost every line S parallel to the coordinate axes is disjoint from Z_0. Equivalently, $S\cap\Omega\subset\Omega_{1/n}$ for some $n\in\mathbb{N}$. Furthermore, recalling that $H = T \cup Z$ has $\mathcal{H}^1$-measure zero, we have that almost every S parallel to the coordinate axes is disjoint from H. By Lemma 5.4, we may assume that u is continuous on $S \cap \overline{\Omega}$ for each such S.

Fix a horizontal line segment S that satisfies the conclusions of the preceding two paragraphs. Let us verify that $Eu|_I$ is absolutely continuous, where $I \subset S$ is a closed line segment. By the definition of S, there exists $M \in \mathbb{N}$ such that $Eu|_{\overline{\Omega}\cap S} = u_{1/M}|_{\overline{\Omega}\cap S}$, and $Eu_{1/M}$ is continuous on S; notice that by Lemma 5.4 and our assumption, $Eu(x) = u_{1/M}(x)$ for $x \in \partial\Omega \cap S$. Let $z_1, \dots, z_{2n} \in I$ be points whose first components are strictly increasing.

If $z_1, z_2 \in \overline{\Omega}$, then by the assumption that $u \in C(\overline{\Omega} \setminus T) \cap W^{1,\infty}(\Omega)$, we get

$$|Eu(z_1) - Eu(z_2)| = |u_{1/M}(z_1) - u_{1/M}(z_2)| \le \int_{[z_1,\, z_2]} |\nabla u_{1/M}|\, dx.$$

If $[z_1, z_2] \subset \widetilde{\Omega}$, then since Eu is absolutely continuous on closed line segments in S, we obtain

$$|Eu(z_1) - Eu(z_2)| \le \int_{[z_1,\, z_2]} |\nabla Eu|\, dx.$$

For the remaining case, by symmetry we may assume that $z_1 \in \overline{\Omega}$ while $z_2 \in \widetilde{\Omega}$. Then let z be the nearest point to z_2 on $\overline{\Omega} \cap I$ between z_1 and z_2, and observe by the same absolute continuity properties as above that

$$\begin{aligned} |Eu(z_1) - Eu(z_2)| &\le |Eu(z_1) - Eu(z)| + |Eu(z) - Eu(z_2)| \\ &= |u_{1/M}(z_1) - u_{1/M}(z)| + |Eu(z) - Eu(z_2)| \\ &\le \int_{[z_1,\, z]} |\nabla u_{1/M}|\, dx + \int_{[z,\, z_2]} |\nabla Eu|\, dx, \end{aligned}$$

where in the last inequality we applied the continuity of Eu on S.

Using a similar argument for the other pairs, we get

$$\begin{aligned} \sum_{i=1}^{n} |Eu(z_{2i-1}) - Eu(z_{2i})| &\le \sum_{i=1}^{n} 2 \int_{[z_{2i-1},\, z_{2i}]} (|\nabla u_{1/M}| + |\nabla Eu|)\, dx \\ &= 2 \int_{\cup_i [z_{2i-1},\, z_{2i}]} (|\nabla u_{1/M}| + |\nabla Eu|)\, dx \end{aligned}$$

from which we get the desired absolute continuity of Eu on I by the absolute continuity of the integral. The case of vertical lines is dealt with analogously, we conclude the absolute continuity of Eu along almost every vertical and horizontal line.

5.5. **Conclusion of the proof.** Combining the discussions in all the subsections above, we have shown that there exists a linear extension operator

$$E\colon W^{1,1}(\Omega) \cap C(\Omega) \cap W^{1,\infty}(\Omega) \to W^{1,1}(B)$$

such that

$$\|\nabla Eu\|_{L^1(B)} \lesssim \|\nabla u\|_{L^1(\Omega)}$$

and

$$\|Eu\|_{L^1(B)} \lesssim \|u\|_{L^1(\Omega)}.$$

Here B is a ball of radius $4\,\mathrm{diam}\,(\Omega)$ such that Ω is properly contained in $\frac{1}{2}B$. By the fact that B is an $W^{1,1}$-extension domain, and the fact that $C(\Omega)\cap W^{1,\infty}(\Omega)$ is dense in $W^{1,1}(\Omega)$ [19], we conclude the sufficiency of the curve condition (1.2).

6. Necessity for simply connected domains

In this section we prove the necessity of the curve condition (1.2) for bounded simply connected $W^{1,1}$-extension domains. We use the following result stated in [17, Corollary 1.2] and proved via results in [3].

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected $W^{1,1}$-extension domain. Then Ω^c is C_1-quasiconvex with C_1 depending only on the norm of the extension operator (with respect to the homogeneous norm).*

Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map. By Theorem 6.1 the complement of Ω is quasiconvex for some constant C_1. Therefore Ω is $J(C_1)$-John by Lemma 2.8 and also resulting from Lemma 2.8 the map φ extends as a continuous map (still denoted by φ) to the boundary $\partial\mathbb{D}$. As before, let us denote $\widetilde{\Omega} := \mathbb{R}^2 \setminus \overline{\Omega}$ and let $\{\widetilde{\Omega}_i\}_{i=1}^N$ be an enumeration of the connected components of $\widetilde{\Omega}$.

We show that (1.2) is necessary. Fix $x,\, y \in \Omega^c$. If

$$|x-y| \le \frac{1}{2}\max\{\,\mathrm{dist}\,(x,\,\partial\Omega),\ \mathrm{dist}\,(y,\,\partial\Omega)\},$$

then we may take γ to be the line segment joining x and y, and then (1.2) holds for this γ. Thus we may assume that

$$|x-y| \ge \frac{1}{2}\max\{\,\mathrm{dist}\,(x,\,\partial\Omega),\ \mathrm{dist}\,(y,\,\partial\Omega)\}.$$

By Theorem 6.1 there exists a curve $\gamma' \subset \Omega^c$ joining $x,\, y$ with length controlled by $C_1|x-y|$. If γ' does not touch the boundary, then we are done. Otherwise starting from $z_0 = x$, we choose z_1 such that

$$\ell(\gamma'[z_0,\, z_1]) = \delta\,\mathrm{diam}\,(\Omega)$$

for the subcurve $\gamma'[z_0,\, z_1]$ of γ' joining z_0 to z_1, with δ in Lemma 4.1; if there is no such a point, we let $z_1 = y$ and define γ_1 to be the piecewise hyperbolic geodesic joining z_0 and z_1. If $\gamma'[z_0,\, z_1]$ touches the boundary at most once, then we define $\gamma_1 = \gamma'[z_0,\, z_1]$. If not, choose the first and last points $x_1,\, y_1$ of $\gamma'[z_0,\, z_1] \cap \partial\Omega$ according to the parametrization of γ', and reroute $\gamma'[z_0,\, z_1]$ via the piecewise hyperbolic geodesic joining x_1 and y_1 to obtain a new curve γ_1; the existence follows from Lemma 4.1. Also according to Lemma 4.1,

$$\ell(\gamma_1) \lesssim \ell(\gamma'[z_0,\, z_1])$$

with the constant depending only on C_1. Continue this procedure via replacing z_0 by z_1 to find z_2 and to define γ_2, and iterate until we finally get the point y. Observe that this process stops after finitely many steps since γ' has finite length, and it produces a finite sequence of curves γ_i, $1 \le i \le N$ such that

$$\sum_{i=1}^{N} \ell(\gamma_i) \lesssim \sum_{i=1}^{N} \ell(\gamma'[z_{i-1},\, z_i]) \lesssim \ell(\gamma') \lesssim |x-y|.$$

We claim that $\gamma = \cup_{i=1}^{N} \gamma_i$ satisfies (6.1). By our construction and the injectivity of piecewise hyperbolic geodesics from Lemma 4.1, it suffices to show that for $x, y \in \partial\Omega$ with $|x-y| \le \delta\, \mathrm{diam}\,(\Omega)$ we have

$$\mathcal{H}^1(I) = 0$$

where I is the intersection between $\partial\Omega$ and the piecewise hyperbolic geodesic Γ joining x and y.

Let Γ be a piecewise hyperbolic geodesic joining $x, y \in \partial\Omega$ in Ω^c with

$$|x-y| \le \delta\, \mathrm{diam}\,(\Omega).$$

By Lemma 4.8 any $z \in I \setminus \{x, y\}$ is a cut-point, and every neighborhood of z intersects both Ω_0 and Ω_1, where Ω_1 and Ω_2 are the two components of $\Omega \setminus (\varphi([v_1, 0] \cup [0, v_2]) \cup \Gamma)$ with $x = \varphi(v_1)$ and $y = \varphi(v_2)$; write $\alpha = [v_1, 0] \cup [0, v_2]$.

Let us next construct a test function that will give us the contradiction. Pick $z_0 \in I \setminus \{x, y\}$ with

$$\mathcal{H}^1(I \cap B(z_0, r)) > 0 \qquad \text{for all } r > 0;$$

this is possible by the assumption that $\mathcal{H}^1(I) > 0$; see [7, Theorem 2, Page 72]. Since $\varphi(\alpha)$ is a compact set,

$$r_0 := \frac{1}{8}\, \mathrm{dist}\,(z, \varphi(\alpha)) > 0.$$

We define ψ by setting

$$\psi(z) := \max\left\{0, 1 - \frac{1}{4r_0}\, \mathrm{dist}\,(z, B(z_0, r_0))\right\} \chi_{\Omega_1}(z).$$

By our definition of r_0, we have $\psi \in W^{1,1}(\Omega)$; notice that $\partial\Omega_1 \cap \Omega \subset \varphi(\alpha)$.

Suppose that there exists an extension $E\psi \in W^{1,1}(\mathbb{R}^2)$ of ψ. By Lemma 4.6 and the John property of Ω, for every $z \in K := I \cap B(z_0, r_0)$ and every $0 < r_z < \delta < r_0$ with any $\delta > 0$, for each $i = 0, 1$ there is a point $z^{(i)} \in B(z, r_z)$ such that $B(z^{(i)}, cr_z) \subset B(z, r_z) \subset B(z_0, 2r_0) \cap \Omega_i \subset \Omega_i$, where the constant c depends only on the John constant J.

Therefore, by the Poincaré inequality,

$$1 \lesssim |E\psi_{B(z^{(0)}, cr_z)} - E\psi_{B(z^{(1)}, cr_z)}| \lesssim r_z \fint_{B(z, r_z)} |\nabla E\psi(w)|\, dw \tag{6.1}$$

for every $z \in K$, where $E\psi_B$ denotes the integral average of $E\psi$ on B.

Notice that $K \subset \bigcup_{z \in K} B(z, r_z)$. By the $5r$-covering theorem, there exists a collection of countably many pairwise disjoint disks

$$\{B(z_i, r_i)\} \subset \{B(z, r) : z \in K, 0 < r < \delta)\}$$

such that

$$K \subset \bigcup_i B(z_i, 5r_i).$$

Moreover since $\mathcal{H}^2(K) = 0$, for any $\epsilon > 0$, we can choose $\delta_0 > 0$ such that

$$\sum_i r_i^2 < \epsilon \tag{6.2}$$

whenever $0 < \delta < \delta_0$.

Now by (6.1) we have

$$\int_{\bigcup_i B(z_i,r_i)} |\nabla E\psi(w)|\, dw \gtrsim \sum_i r_i \gtrsim \mathcal{H}^1_\delta(K) \gtrsim \mathcal{H}^1(K).$$

for $\delta > 0$ small enough. However, since $|\nabla E\psi| \in L^1(\mathbb{R}^2)$, the absolute continuity of the integral and (6.2), gives that

$$\int_{\bigcup_i B(z_i,r_i)} |\nabla E\psi(w)|\, dw$$

can be made arbitrarily small by choosing a sufficiently small ϵ. Thus we have obtained a contradiction and hence the necessity of (6.1) for $W^{1,1}$-extension domains follows.

7. Appendix

7.1. **Conformal capacity.** The *conformal capacity* of a given pair of continua $E,\, F \subset \overline{\Omega} \subset \mathbb{R}^2$ in Ω is defined by

$$\mathrm{Cap}(E,\, F,\, \Omega) = \inf\{\|\nabla u\|^2_{L^2(\Omega)} :\ u \in \Delta(E,\, F)\},$$

where $\Delta(E,\, F)$ is the class of all $u \in W^{1,2}(\Omega)$ that are continuous in $\Omega \cup E \cup F$ and satisfy $u = 1$ on E, and $u = 0$ on F.

We have the following estimates for conformal capacities. First of all, if $E, F \subset \overline{\mathbb{D}}$, then

$$\frac{\min\{\operatorname{diam}(E),\ \operatorname{diam}(F)\}}{\operatorname{dist}(E,\, F)} \geq \delta > 0 \quad \Longrightarrow \quad \mathrm{Cap}(E,\, F,\, \mathbb{D}) \geq C(\delta) > 0, \tag{7.1}$$

and the analogous estimate holds in $\mathbb{R}^2 \setminus \mathbb{D}$. Moreover, suppose that E is a continuum and Ω_1 is a Jordan domain such that

$$E \subset\subset \Omega_1 \subset \Omega.$$

Then by defining $\delta = \frac{\operatorname{diam}(E)}{\operatorname{dist}(E,\partial\Omega_1)}$, we obtain that

$$\eta_1(\delta) \leq \mathrm{Cap}(E,\, \partial\Omega_1,\, \Omega) \leq \eta_2(\delta), \tag{7.2}$$

where $\eta_1(\delta)$, $\eta_2(\delta)$ are two continuous and increasing functions with respect to δ such that

$$\lim_{\delta\to 0^+} \eta_i(\delta) = 0 \qquad \lim_{\delta\to\infty} \eta_i(\delta) = \infty,$$

for both $i = 1,\, 2$. Note that (7.2) also holds if Ω_1 is the exterior of some Jordan domain; one may use a suitable (sense reversing) conformal map to see this.

We define the *inner distance with respect to* Ω between $x,\, y \in \Omega$ as

$$\operatorname{dist}_\Omega(x,\, y) = \inf_{\gamma\subset\Omega} \ell(\gamma),$$

where the infimum is taken over all (rectifiable) curves joining x and y in Ω. The inner diameter $\operatorname{diam}_\Omega(E)$ of a set $E \subset \Omega$ can be defined in a similar manner. We record the following estimate stating a converse version of (7.1); see e. g. [18, Lemma 2.4].

Lemma 7.1. *Let $E,\, F \subset \Omega$ be a pair of continua. Then if* $\mathrm{Cap}(E,\, F,\, \Omega) \geq c_0$*, we have*

$$\min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\} \gtrsim \operatorname{dist}_\Omega(E,\, F),$$

where the constant only depends on c_0. Especially

$$\min\{\operatorname{diam}_\Omega(E),\ \operatorname{diam}_\Omega(F)\} \gtrsim \operatorname{dist}(E,\, F),$$

and if $\Omega = \mathbb{R}^2$

$$\min\{\operatorname{diam}(E),\ \operatorname{diam}(F)\} \gtrsim \operatorname{dist}(E,\, F).$$

7.2. **Gehring-Hayman theorems.** We record the following theorem of Gehring and Hayman [8].

Theorem 7.2. *Let Ω be a bounded simply connected domain, and $\varphi : \mathbb{D} \to \Omega$ be a conformal map. Given a pair of points $x,\, y \in \mathbb{D}$, denoting the corresponding hyperbolic geodesic in $\mathbb{D}$ by $\Gamma_{x,y}$, and by $\gamma_{x,y}$ any arc connecting x and y in $\mathbb{D}$, we have*

$$\ell(\varphi(\Gamma_{x,y})) \leq C\ell(\varphi(\gamma_{x,y})).$$

We establish the following version of the Gehring-Hayman theorem.

Theorem 7.3. *Let $\widetilde{\Omega}$ be the exterior of a Jordan domain. If $\widetilde{\varphi}\colon \widetilde{\Omega} \to \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ is conformal and Γ is a hyperbolic geodesic in $\widetilde{\Omega}$ connecting x and y satisfying $\widetilde{\varphi}(\Gamma) \subset B(0,\, 100)$, and γ is any curve joining x and y, then we have*

$$\ell(\Gamma) \leq C\ell(\gamma)$$

for some absolute constant C.

To show this, we need the following lemma on conformal annuli.

Lemma 7.4. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain, and let a homeomorphism $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \Omega \to \mathbb{R}^2 \setminus \mathbb{D}$ be conformal in $\mathbb{R}^2 \setminus \overline{\Omega}$. For $z_1 \in \partial\Omega$, define*

$$A(z_1,\, k) := \{x \in \mathbb{R}^2 \setminus \overline{\mathbb{D}} \mid 2^{k-1} < |x - \widetilde{\varphi}(z_1)| \leq 2^k\},$$

for $k \in \mathbb{Z}$. Furthermore, let $\Gamma \subset \mathbb{R}^2 \setminus \overline{\Omega}$ be the hyperbolic geodesic joining z_1 and $z_2 \in \partial\Omega$ such that $\widetilde{\varphi}(\Gamma) \subset B(0,\, 100)$. Also let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be a curve connecting z_1 and z_2 inside Ω. Set

$$\Gamma_k := \widetilde{\varphi}^{-1}(A(z_1,\, k)) \cap \Gamma$$

and let γ_k be any subcurve of γ in $\widetilde{\varphi}^{-1}(A(z_1,\, k))$ joining the inner and outer boundaries of $\widetilde{\varphi}^{-1}(A(z_1,\, k))$, if such a subcurve exists. (Here the inner and outer boundaries of $\widetilde{\varphi}^{-1}(A(z_1,\, k))$ are the preimages under $\widetilde{\varphi}$ of the inner and outer boundaries of $A(z_1,\, k)$.) Then

$$\ell(\Gamma_k) \sim \operatorname{dist}(\Gamma_k,\, \partial\Omega) \tag{7.3}$$

and

$$\ell(\gamma_k) \gtrsim \ell(\Gamma_k) \sim \operatorname{diam}(\Gamma_k).$$

Here all the constants are independent of Ω and the choice of $\widetilde{\varphi}, z_1, \gamma, z_2, k$.

Proof. The fact that $\ell(\Gamma_k) \sim \operatorname{dist}(\Gamma_k, \partial\Omega) \sim \operatorname{diam}(\Gamma_k)$ follows immediately from Lemma 2.3, since by definition $\widetilde{\varphi}^{-1}(\Gamma_k)$ is contained in a Whitney-type set in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$.

Hence we only need to prove that $\ell(\gamma_k) \gtrsim \ell(\Gamma_k)$. Observe that, since γ_k by definition joins the inner and outer boundaries of $\widetilde{\varphi}(A(z_1, k))$, then

$$\ell(\widetilde{\varphi}(\gamma_k)) \gtrsim \operatorname{diam}(\widetilde{\varphi}(\Gamma_k)) \sim \operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D}). \tag{7.4}$$

We next argue by case study.
Case 1: $\operatorname{dist}(\widetilde{\varphi}(\gamma_k), \widetilde{\varphi}(\Gamma_k)) \geq \frac{1}{3}\operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D})$. By Lemma 2.3, the assumption and the fact that $\widetilde{\varphi}(\Gamma_k)$ is contained in a Whitney-type set, we know that for any curve γ' joining γ_k and Γ_k, its length satisfies

$$\ell(\gamma') \gtrsim \operatorname{diam}(\Gamma_k),$$

and hence

$$\operatorname{dist}(\gamma_k, \Gamma_k) \gtrsim \operatorname{diam}(\Gamma_k). \tag{7.5}$$

Moreover by (7.1) for the exterior of the unit disk, (7.4) and the monotonicity of the capacity we obtain

$$1 \lesssim \operatorname{Cap}(\widetilde{\varphi}(\overline{\gamma}_k), \widetilde{\varphi}(\overline{\Gamma}_k), \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \operatorname{Cap}(\overline{\gamma}_k, \overline{\Gamma}_k, \mathbb{R}^2 \setminus \overline{\Omega}) \leq \operatorname{Cap}(\overline{\gamma}_k, \overline{\Gamma}_k, \mathbb{R}^2).$$

Hence by (7.5) and Lemma 7.1 we know that

$$\ell(\gamma_k) \geq \operatorname{diam}(\gamma_k) \gtrsim \operatorname{diam}(\Gamma_k) \sim \ell(\Gamma_k).$$

Case 2: $\operatorname{dist}(\widetilde{\varphi}(\gamma_k), \partial\mathbb{D}) \geq \frac{1}{3}\operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D})$. This assumption implies that the set $\widetilde{\varphi}(\gamma_k) \cup \widetilde{\varphi}(\Gamma_k)$ is contained in a Whitney-type set. Then γ_k is also contained in a Whitney-type set by Lemma 2.3, and then the desired estimate follows directly from Lemma 2.3 and (7.4).
Case 3:

$$\operatorname{dist}(\widetilde{\varphi}(\gamma_k), \widetilde{\varphi}(\Gamma_k)) < \frac{1}{3}\operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D})$$

and

$$\operatorname{dist}(\widetilde{\varphi}(\gamma_k), \partial\mathbb{D}) < \frac{1}{3}\operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D}).$$

In this case, by assumption there is a subcurve $\widetilde{\gamma}_k \subset \gamma_k$ such that $\ell(\widetilde{\varphi}(\widetilde{\gamma}_k)) \gtrsim \ell(\widetilde{\varphi}(\Gamma_k))$ and $\operatorname{dist}(\widetilde{\varphi}(\widetilde{\gamma}_k), \partial\mathbb{D}) \gtrsim \operatorname{dist}(\widetilde{\varphi}(\Gamma_k), \partial\mathbb{D})$, as γ_k is a (connected) curve. Then we are reduced to a case similar to the second one, and it follows that

$$\ell(\gamma_k) \gtrsim \ell(\widetilde{\gamma}_k) \geq \operatorname{diam}(\widetilde{\gamma}_k) \gtrsim \operatorname{diam}(\Gamma_k) \sim \ell(\Gamma_k).$$

Consequently we obtain the desired estimate. □

We usually call the set $\widetilde{\varphi}^{-1}(A(z, k))$ defined above a *conformal annulus* centered at z. Now let us prove Theorem 7.3.

Proof of Theorem 7.3. Extend the hyperbolic geodesic joining x and y to the boundary of $\widetilde{\Omega}$ at z_1 and z_2, and construct the conformal annuli as in Lemma 7.4

We first consider the case where x, y are in the same conformal annulus (centered at z_1). Then the theorem follows directly from (7.3) and Lemma 2.3 as x, y are in some Whitney-type set.

Then we show the theorem in the case where x and y are in different annuli. We may assume that they lie on the boundary of different conformal annuli, since the general case follows from the triangle inequality with the conclusion of the first case.

We employ the notation in Lemma 7.4. If $|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| \leq 1$, then by the Jordan curve theorem, any curve joining x and y should go across $A(x, k)$ whenever Γ_k is non-empty. Therefore by Lemma 7.4 we have

$$\ell(\Gamma) \leq \sum_k \ell(\Gamma_k) \lesssim \sum_k \ell(\gamma_k) \lesssim \ell(\gamma).$$

Therefore we obtain the desired conclusion.

Suppose $|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| > 1$. If $\Gamma_k \neq \emptyset$ but $\gamma_k = \emptyset$, then Jordan curve theorem implies that $k \geq 1$. By $|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| > 1$ and the fact that $\widetilde{\varphi}(\Gamma) \subset B(0, 100)$, according to (7.1) we have

$$\mathrm{Cap}(\Gamma_k, \gamma, \widetilde{\Omega}) = \mathrm{Cap}(\widetilde{\varphi}(\Gamma_k), \widetilde{\varphi}(\gamma), \mathbb{R}^2 \setminus \mathbb{D}) \gtrsim 1.$$

Thus by Lemma 7.1 and Lemma 7.4 we have

$$\mathrm{diam}\,(\Gamma_k) \lesssim \mathrm{diam}\,(\gamma) \lesssim \ell(\gamma).$$

Therefore by Lemma 7.4 again we conclude that

$$\ell(\Gamma) \leq \sum_{k<0} \ell(\Gamma_k) + \sum_{0\leq k\leq 7} \ell(\Gamma_k) \lesssim \sum_{k<0} \ell(\gamma_k) + \sum_{0\leq k\leq 7} \ell(\Gamma_k) \lesssim \ell(\gamma).$$

Hence we conclude the theorem. □

7.3. **Some lemmas on quasiconvex sets.**

Lemma 7.5. *Let $\widetilde{\Omega}$ be a complementary domain of some Jordan domain and be C_1-quasiconvex. If $\widetilde{\varphi}\colon \widetilde{\Omega} \to \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ is conformal and Γ is a hyperbolic geodesic in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ connecting x and y with $\gamma \subset B(0, 100)$, then*

$$\ell(\widetilde{\varphi}^{-1}(\Gamma)) \leq C(C_1)|\widetilde{\varphi}^{-1}(x) - \widetilde{\varphi}^{-1}(y)|.$$

Proof. According to the definition of quasiconvexity, there exists a curve $\gamma \subset \widetilde{\Omega}$ joining x and y whose length is bounded from above by $C_1|x-y|$. Then by Theorem 7.3, we have

$$\ell(\widetilde{\varphi}^{-1}(\Gamma)) \lesssim \ell(\gamma) \lesssim |\widetilde{\varphi}^{-1}(x) - \widetilde{\varphi}^{-1}(y)|$$

with the constant depending only on C_1. □

Lemma 7.6. *Given $C_1 > 0$ and an unbounded C_1-quasiconvex domain $\widetilde{\Omega}$ whose boundary is Jordan, there is a constant $\delta = \delta(C_1) > 0$ so that the following holds: If $\widetilde{\varphi}\colon \widetilde{\Omega} \to \mathbb{R}^2 \setminus \overline{\mathbb{D}}$ is conformal and $z_1, z_2 \in \overline{\widetilde{\Omega}}$ satisfy $|z_1 - z_2| < \delta\,\mathrm{diam}\,(\partial\widetilde{\Omega})$ and $\mathrm{dist}\,(z_i, \partial\widetilde{\Omega}) \leq \delta\,\mathrm{diam}\,(\partial\widetilde{\Omega})$ for $i = 1, 2$, then $|\widetilde{\varphi}(z_1) - \widetilde{\varphi}(z_2)| < \frac{1}{4}$ and $\mathrm{dist}\,(\varphi(z_i), \partial\mathbb{D}) < \frac{1}{4}$ for $i = 1, 2$.*

Proof. We may assume that $0 \notin \widetilde{\Omega}$. Let $S = \partial B(0, 10\,\mathrm{diam}\,(\partial\widetilde{\Omega}))$, and Q be a 2-Whitney square of $\widetilde{\Omega}$ intersecting S. Then since $\mathrm{diam}\,(Q)/\,\mathrm{dist}\,(Q, \partial\widetilde{\Omega})$ is bounded from below, then by (7.2) we have

$$\mathrm{Cap}(\partial\widetilde{\Omega}, Q, \widetilde{\Omega}) \gtrsim 1. \tag{7.6}$$

By the conformal invariance of capacity, we conclude from (7.6) that

$$\mathrm{Cap}(\partial\mathbb{D},\, \widetilde{\varphi}(Q),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \gtrsim 1.$$

Notice that by Lemma 2.3, $\widetilde{\varphi}(Q)$ is of λ-Whitney-type for some absolute constant λ. Hence

$$\mathrm{diam}\,(\widetilde{\varphi}(Q)) \sim \mathrm{dist}\,(\partial\mathbb{D},\, \widetilde{\varphi}(Q)) \lesssim 1$$

according to Lemma 7.1. By the triangle inequality and the arbitrariness of Q we have

$$\mathrm{dist}\,(\widetilde{\varphi}(S),\, \partial\mathbb{D}) \le C_2^{-1} \;\text{ and }\; 1 \le \mathrm{diam}\,(\widetilde{\varphi}(S)) \le C_2^{-1}. \tag{7.7}$$

where $C_2 < 1$ is an absolute constant.

Since $\widetilde{\Omega}$ is C_1-quasiconvex, then for the points z_1, z_2 in the statement of the lemma, there exists a curve $\gamma \subset \widetilde{\Omega}$ joining z_1 and z_2 such that

$$\ell(\gamma) \le \delta C_1\,\mathrm{diam}\,(\partial\widetilde{\Omega}).$$

By letting $\delta < C_1^{-1} \le 1$, with the definition of z_1, z_2, we have $\gamma \subset B(0,\, 4\,\mathrm{diam}\,(\partial\widetilde{\Omega}))$ by the triangle inequality. Then by (7.2) again we obtain

$$\mathrm{Cap}(\gamma,\, S,\, \widetilde{\Omega}) \lesssim \eta_2(\delta C_1/6). \tag{7.8}$$

By conformal invariance we have

$$\mathrm{Cap}(\widetilde{\varphi}(\gamma),\, \widetilde{\varphi}(S),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \lesssim \eta_2(\delta C_1/6). \tag{7.9}$$

Notice that since $\gamma \subset B(0,\, 4\,\mathrm{diam}\,(\partial\widetilde{\Omega}))$, then $\widetilde{\varphi}(\gamma)$ is contained in the Jordan domain enclosed by $\widetilde{\varphi}(S)$. Hence if

$$|\widetilde{\varphi}(z_1) - \widetilde{\varphi}(z_2)| \ge \frac{1}{4},$$

by (7.7), (7.2) and (7.9) we further have

$$\eta_1(C_3/4) \le \mathrm{Cap}(\widetilde{\varphi}(\gamma),\, \widetilde{\varphi}(S),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \lesssim \eta_2(\delta C_1/6). \tag{7.10}$$

Thus by choosing δ small enough such that $\delta < C_1^{-1}$ but (7.10) fails, we obtain the first part of the lemma.

Towards the second part, by symmetry we only need to prove the inequality for z_1. If $z_1 \in \partial\widetilde{\Omega}$, then by the Caratheodory-Osgood theorem [21] we know that $\widetilde{\varphi}(z_1) \in \partial\mathbb{D}$, and the desired claim follows. If z_1 is in the interior of $\widetilde{\Omega}$, let Γ be the hyperbolic geodesic joining z_1 to ∞. Then since $\mathrm{dist}\,(z_i,\, \partial\widetilde{\Omega}) \le \delta\,\mathrm{diam}\,(\partial\widetilde{\Omega})$, then by (7.2) we have

$$\mathrm{Cap}(\partial\widetilde{\Omega},\, \Gamma,\, \widetilde{\Omega}) \gtrsim \eta_1(\delta^{-1}) \tag{7.11}$$

for some absolute constant. By the conformal invariance of capacity, we conclude from (7.11) that

$$\mathrm{Cap}(\partial\mathbb{D},\, \widetilde{\varphi}(\Gamma),\, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \gtrsim \eta_1(\delta^{-1}).$$

Then the desired estimate follows from Lemma 7.1 by choosing δ suitably. $\square$

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