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SHAPE OPTIMIZATION IN CONTACT PROBLEMS WITH FRICTION

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ABSTRACT

The optimal shape design of a two-dimensional elastic body on a rigid foundation is analyzed. The influence of the friction between the body and the support will be taken into account by applying the model with a given friction. The aim is to redesign the contact surface of the body in order to achieve such a shape that the total potential energy of the system in the equilibrium state will be minimized.

The solvability of the problem is proved. Discretization by finite elements as well as the sensitivity analysis is given which is necessary for solving numerically the problem in question.

1. Introduction

It is the aim of this paper to extend the analysis of [5, 6], where the optimal shape design of a two-dimensional elastic body on a rigid frictionless foundation was analyzed. Here the influence of the friction between the body and the support will be taken into account. Even in the simplest model with a given friction the situation is more involved than in the frictionless case.

In our design problem the contact boundary of the body with unilateral boundary conditions must be redesigned in such a way that the total potential energy of the system in the equilibrium state will be minimized. In chapter 2 we shall prove the existence of a solution to this contour design problem. In chapter 3 a finite element discretization with linear triangular elements is presented. We shall prove that the solution of the discrete state problem (governed by variational inequality) is directionally differentiable with respect to shape control of the body (Theorem 3.1). The main result of chapter 3 is Theorem 3.2 which contains formula for the directional derivative for the criteria functional. We present two ways to obtain the directional derivative: one based on variational inequality approach and second based on min-max principle (saddle-point approach). The gradient formulae are necessary for the use of efficient nonlinear programming algorithms in solving numerically the problem in question.

For contributions to shape optimization in problem governed by equations we refer to [8, 9, 10, 11, 13] and to bibliography therein.

2. Existence result

Let us assume an elastic body, represented by a bounded plane domain $\Omega(\alpha) \subset \mathbb{R}^2$ (see Fig. 1), where

$$\Omega(\alpha) \equiv \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 < b, \ \alpha(x_1) < x_2 < \gamma\}, \quad \gamma > 0$$

and

$$\begin{aligned} \alpha \in \mathcal{U}_{ad} \equiv \{\alpha \in C^{1,1}([a, b]) \mid 0 \le \alpha(x_1) \le C_0 < \gamma, \\ |\alpha'(x_1)| \le C_1, \ |\alpha''(x_1)| \le C_2, \\ \text{meas } (\Omega(\alpha)) = C_3\}. \end{aligned}$$

We suppose that positive constants C_0, C_1, C_2, C_3 are chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. Let the boundary $\partial\Omega(\alpha)$ of $\Omega(\alpha)$ be decomposed as follows (for the meaning see (2.2) - (2.6)):

$$\partial\Omega(\alpha) = \overline{\Gamma}_D \cup \overline{\Gamma}_P \cup \overline{\Gamma}_C(\alpha) , \quad \Gamma_D \neq \emptyset$$

and

$$\Gamma_C(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \alpha(x_1) , \ x_1 \in (a, b)\}$$

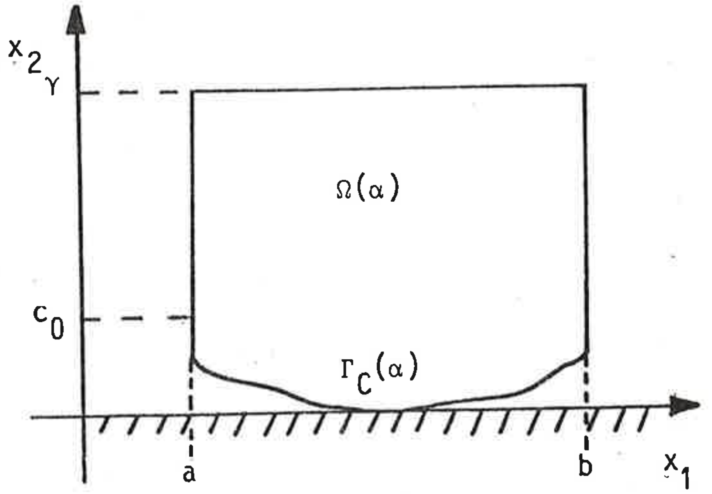

Fig. 1. $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$.

Let the elastic body be unilaterally supported by a rigid foundation (the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$) and subjected to a body force $F = (F_1, F_2)$ and to a surface traction $P = (P_1, P_2)$ on the portion Γ_P. Moreover, the influence of friction between the body and the support will be taken into account. Here we shall discuss the simplest model involving friction, namely the so called model with a *given friction*. Classical formulation of a contact problem with a given friction can be stated as follows (with $\alpha \in \mathcal{U}_{ad}$ being fixed): we look for a displacement field $u \equiv u(\alpha) = (u_1(\alpha), u_2(\alpha))$, satisfying the equilibrium equations

$$(2.1)\qquad \frac{\partial \tau_{ij}}{\partial x_j}(u) + F_i = 0 \quad \text{in } \Omega(\alpha)\,, \quad i = 1, 2 \qquad)^1$$

where the stress tensor $\tau(u) = (\tau_{ij}(u))^2_{i,j=1}$ is related to the strain (linearized) tensor $\varepsilon(u) = (\varepsilon_{ij}(u))^2_{i,j=1}$, with $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ by means of a linear Hooke's law:

$$\tau_{ij}(u) = c_{ijkl}\,\varepsilon_{kl}(u)$$

Elasticity coefficients c_{ijkl} are supposed to be bounded and measurable in $\hat{\Omega} \equiv (a, b) \times (0, \gamma)$, i.e. $c_{ijkl} \in L^\infty(\hat{\Omega})$, satisfying the symmetry conditions

$$c_{ijkl}(x) = c_{jikl}(x) = c_{klij}(x) \qquad \text{a.e. in } \hat{\Omega}$$

as well as the ellipticity condition

$$\exists\, q_0 > 0:\ c_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq q_0\,\xi_{ij}\,\xi_{ij} \qquad \forall \xi_{ij} = \xi_{ji} \qquad \text{a.e. in } \hat{\Omega}$$

Further, the displacement field u has to satisfy the following system of boundary conditions:

$$(2.2)\qquad u_i = 0 \qquad \text{on } \Gamma_D\,, \quad i = 1, 2\,;$$

$$(2.3)\qquad \tau_{ij}(u)\,n_j = P_i \quad \text{on } \Gamma_P\,, \quad i = 1, 2\,;$$

($n = (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$)

$$(2.4)\qquad u_2(x_1, \alpha(x_1)) \geq -\alpha(x_1) \qquad \forall x_1 \in (a, b)\,;$$

$$(2.5)\qquad T_2(u) \equiv \tau_{2j}(u)\,n_j \geq 0\,, \quad (u_2 + \alpha)\,T_2(u) = 0 \qquad \text{on } \Gamma_C(\alpha);$$

)[1] Throughout the paper, the summation convention is used

$$(2.6)\quad \begin{cases} |T_1(u)| \le g \quad \text{on} \quad \Gamma_C(\alpha)\,; \quad T_1(u) \equiv \tau_{1j}(u)\,n_j \\ |T_1(u)(x)| < g \Rightarrow u_1(x) = 0\,; \\ |T_2(u)(x)| = g \Rightarrow \exists\, \lambda(x) \ge 0: \quad u_1(x) = -\lambda(x)\, T_1(u)(x)\,. \end{cases}$$

Above (2.4) and (2.5) are conditions of the unilateral contact along $\Gamma_C(\alpha)$, (2.6) is the mathematical formulation of our model of friction. In order to give the variational formulation, we introduce some notations: Let

$$V(\alpha) \equiv V(\Omega(\alpha)) = \{v \in (H^1(\Omega(\alpha)))^2 \mid v_i = 0 \text{ on } \Gamma_D\,,\ i = 1\,,2\}$$

be the set of virtual displacements and

$$K(\alpha) \equiv K(\Omega(\alpha)) = \{v \in V(\alpha) \mid v_2(x_1\,, \alpha(x_1)) \ge -\alpha(x_1) \quad \forall x_1 \in (a\,,b)\}$$

be its closed, convex subset. By a *variational solution* of a contact problem with given friction we call a function $u(\alpha) \in K(\alpha)$ such that

$$(\mathcal{P}(\alpha))\qquad (\tau(u)\,, \varepsilon(v - u))_{\Omega(\alpha)} + j_\alpha(v) - j_\alpha(u) \ge \langle L\,, v - u\rangle_\alpha \qquad \forall v \in K(\alpha)\,,$$

where

$$(\tau(u), \varepsilon(v))_{\Omega(\alpha)} \equiv \int_{\Omega(\alpha)} \tau_{ij}(u)\, \varepsilon_{ij}(v)\, dx$$

$$j_\alpha(v) \equiv g \int_{\Gamma_C(\alpha)} |v_1|\, ds$$

$$\langle L, v\rangle_\alpha \equiv \int_{\Omega(\alpha)} F_i v_i\, dx + \int_{\Gamma_P} P_i v_i\, ds \equiv (F\,, v)_{\Omega(\alpha)} + (P\,, v)_{\Gamma_P}$$

with $F \in (L^2(\hat{\Omega}))^2$, $P \in (L^2(\Gamma_P))^2$, $g \in \mathbb{R}^1$, $g > 0$. It is well-known that for any $\alpha \in \mathcal{U}_{ad}$, the problem $(\mathcal{P}(\alpha))$ has a unique solution u (see [4]). Up to now, a function $\alpha \in \mathcal{U}_{ad}$ has been fixed. Our aim will be to design the contact surface $\Gamma_C(\alpha)$ to achieve a shape, the total potential energy of which is minimal. More precisely, we look for $\alpha^* \in \mathcal{U}_{ad}$ such that ([7])

$$(\mathbb{P}) \qquad E(\alpha^*) \leq E(\alpha) \qquad \forall\, \alpha \in \mathcal{U}_{ad} ,$$

where

$$E(\alpha) \equiv E(u(\alpha), \alpha) = \frac{1}{2} (\tau_{ij}(u(\alpha)), \varepsilon_{ij}(u(\alpha)))_{\Omega(\alpha)}$$

$$+ j_\alpha(u(\alpha)) - \langle L , u(\alpha)\rangle_\alpha$$

with $u(\alpha) \in K(\alpha)$ being the solution of $(\mathcal{P}(\alpha))$.

The main result of the paper is

Theorem 2.1. There exists at least one solution of $(\mathbb{P})$.

Before we prove the result, we shall need some auxiliary results.

Lemma 2.1. Let $\alpha_j \rightrightarrows \alpha$ (uniformly) in $[a, b]$ and let $\varphi \in K(\alpha)$, $\varphi = (\varphi_1, \varphi_2)$ be given. Then there exists $\varphi_j \in (H^1(\hat{\Omega}))^2$ and a subsequence $\{\alpha_{n(j)}\} \subset \{\alpha_n\}$ such that $\varphi_j|_{\Omega(\alpha_{n(j)})} \in K(\alpha_{n(j)})$ and $\varphi_j \to \hat{\varphi}$ in $(H^1(\hat{\Omega}))^2$, where $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)$ denotes the Calderon extension of φ from $\Omega(\alpha)$ on $\hat{\Omega}$.

Proof. Define a function

$$\psi_2(x_1, x_2) = \max\{\varphi_2(x_1, x_2), -x_2\}, \quad (x_1, x_2) \in \hat{\Omega}$$

We find that $\psi_2 \in H^1(\hat{\Omega})$, $\psi_2|_{\Gamma_u} = 0$ and

$$\psi_2(x_1, \alpha(x_1)) = \varphi_2(x_1, \alpha(x_1)) \geq -\alpha(x_1),$$

i.e. $\psi|_{\Omega(\alpha)} \in K(\alpha)$, where $\psi = (\psi_1, \psi_2)$ with $\psi_1 = \hat{\varphi}_1$. Let us split $\hat{\varphi}$ as follows: $\hat{\varphi} = \psi + \Phi$. From the construction of ψ

we see that

$$\Phi_1\big|_{\Gamma_u} = \Phi_2\big|_{\Gamma_u} = \Phi_2\big|_{\Gamma_C(\alpha)} = 0 \quad , \qquad \Phi = (\Phi_1 , \Phi_2) .$$

Applying the classical density result, one can find a sequence $\{\Phi_j\}$, $\Phi_j = (\Phi_{1j} , \Phi_{2j}) \in (C^\infty(\bar{\hat{\Omega}}))^2$ with Φ_{1j} , Φ_{2j} vanishing in a neighbourhood of $\bar{\Gamma}_u$, $\overline{\Gamma_u \cup \Gamma_C(\alpha)}$, respectively and such that

$$\Phi_j \to \Phi \qquad \text{in} \quad (H^1(\hat{\Omega}))^2 .$$

Let us define now:

$$\varphi_j = \psi + \Phi_j .$$

Then

$$\|\hat{\varphi} - \varphi_j\|_{1,\hat{\Omega}} = \|\Phi - \Phi_j\|_{1,\hat{\Omega}} \to 0 , \quad j \to \infty .$$

Let us denote $d_j \equiv \text{dist}\,\{\text{supp}\ \Phi_{2j} , \overline{\Gamma_u \cup \Gamma_C(\alpha)}\}$ and let j_0 be fixed. As $\alpha_j \rightrightarrows \alpha$ in $[a , b]$, then exists $n_0 = n(j_0)$ such that the graph of the function α_{n_0} doesn't intersect $\text{supp}\ \Phi_{2j_0}$. As

$$\varphi_{2j_0} = \psi_2 \qquad \text{on} \quad \hat{\Omega} \setminus \text{supp}\ \Phi_{2j_0}$$

and

$$\psi_2(x_1 , x_2) \geq - x_2 \qquad \forall (x_1 , x_2) \in \hat{\Omega}$$

(as follows from the definition of ψ_2) we immediately get

$$\psi_2(x_1 , \alpha_{n_0}(x_1)) \geq -\alpha_{n_0}(x_1) .$$

Hence

$$\varphi_{j_0}\Big|_{\Omega(\alpha_{n_0})} \in K(\alpha_{n_0}) . \qquad \square$$

Lemma 2.2. Let $\alpha_n \to \alpha$ in $C^1([a, b])$-topology and let $u_n \equiv u(\alpha_n) \in K_n \equiv K(\alpha_n)$ be solutions of $(P(\alpha_n))$. Then there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ such that

$$u_n \rightharpoonup u \quad \text{(weakly)} \quad \text{in} \quad (H^1(G_m(\alpha)))^2$$

for any m, where

$$(2.7) \qquad G_m(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a, b), \ \alpha(x_1) + \tfrac{1}{m} < x_2 < \gamma\}$$

and $u = u(\alpha) \in K(\alpha)$ solves $(P(\alpha))$.

Proof. There exists a constant $c > 0$ independent on $\alpha \in \mathcal{U}_{ad}$ and such that

$$(2.8) \qquad \|u_n\|_{1,\Omega_n} \le c \qquad (\Omega_n := \Omega(\alpha_n)) .$$

Indeed,

$$C\,\|u_n\|^2_{1,\Omega_n} \le (\tau(u_n), \varepsilon(u_n))_{\Omega_n} \le (\tau(u_n), \varepsilon(u_n))_{\Omega_n} + j_n(u_n)$$
$$\le (\tau(u_n), \varepsilon(v))_{\Omega_n} + \langle L, v - u_n \rangle_{\alpha_n} + j_n(v)$$

holds for any $v \in K(\alpha_n)$ $(j_n \equiv j_{\alpha_n})$. Setting $v = (0, 0) \in K_n$ $\forall n$ in the last inequality, we arrive at (2.8). Here we used the fact that the constant C, appearing in the Korn's inequality can be chosen independently on n (for the proof see Appendix in [6]). Let $\hat{u}_n \in (H^1(\hat{\Omega}))^2$ denote the Calderon extension of u_n from Ω_n on $\hat{\Omega}$. As $\{\Omega(\alpha)\}$, $\alpha \in \mathcal{U}_{ad}$ posesses the so called "uniform extension property" (see [2]), the norm of $\hat{u}_n$ in $(H^1(\hat{\Omega}))^2$ can be estimated independently on n , i.e.

(2.9) $\|\hat{u}_n\|_{1,\hat{\Omega}} \leq c$)[2]

Therefore there exists a subsequence of $\{\hat{u}_n\}$ (denoted by $\{\hat{u}_n\}$ again) and an element $\hat{u} \in (H^1(\hat{\Omega}))^2$ such that

(2.10) $\hat{u}_n \rightharpoonup \hat{u}$ in $(H^1(\hat{\Omega}))^2$.

Our goal is to show that $u \equiv \hat{u}|_{\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$.

First we prove that $u \in K(\alpha)$. It is clear that $u = 0$ on Γ_D . Let us denote by P the penalty mapping associated with the geometrical constraint $u(\alpha) \in K(\alpha)$:

$$P(z(\alpha))(x_1) \equiv (z(\alpha) + \alpha)^- = (z(x_1 , \alpha(x_1)) + \alpha(x_1))^- ,$$

$$x_1 \in (a , b) ,$$

$(\bar{a} \equiv (|a| - a)/2)$ and let

$$(I_\alpha(u_2) , \xi) \equiv \int_a^b P(u_2(\alpha))(x_1)\, \xi(x_1 , \alpha(x_1))\, dx_1 , \quad \xi \in \mathcal{D}(\mathbb{R}^2) .$$

Because of (2.10) and the fact that $\alpha_n \rightrightarrows \alpha$ in $[a , b]$, we have

(2.11) $(I_{\alpha_n}(u_{2n}) , \xi) \to (I_\alpha(u_2) , \xi)$

(for the proof see [5], Lemma 2.2). As $u_n \in K_n$ implies $(I_{\alpha_n}(u_{2n}) , \xi) = 0 \quad \forall \xi \in \mathcal{D}(\mathbb{R}^2)$ it holds by virtue of (2.11) that $(I_\alpha(u_2) , \xi) = 0$. This means that $u \in K(\alpha)$.

From (2.10) it follows that

(2.12) $u_n \rightharpoonup u$ in $(H^1(G_m(\alpha)))^2$

)[2] In what follows, c will denote a generic strict positive constant with different values on different places.

for any m. It remains to prove that u solves $(\mathcal{P}(\alpha))$.

Let $\xi \in K(\alpha)$ be an arbitrary and let $\xi_j \in (H^1(\hat{\Omega}))^2$ be functions with properties given by Lemma 2.2 :

$$(2.13) \qquad \xi_j \to \hat{\xi} \quad \text{in} \quad (H^1(\hat{\Omega}))^2$$

and $\xi_j|_{\Omega_{n_j}} \in K(\alpha_{n_j})$. Denoting by u_{n_j} solutions of $(\mathcal{P}(\alpha_{n_j}))$ we have:

$$(2.14) \qquad (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{\Omega_{n_j}} + j_{n_j}(\xi_j) - j_{n_j}(u_{n_j})$$

$$\geq (F, \xi_j - u_{n_j})_{\Omega_{n_j}} + (P, \xi_j - u_{n_j})_{\Gamma_P} .$$

One can write:

$$(\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{\Omega_{n_j}}$$

$$= (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{G_m} + (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{\Omega_{n_j} \setminus \Omega(\alpha)}$$

$$+ (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{(\Omega(\alpha) \setminus G_m) \cap \Omega_{n_j}}$$

$$\leq (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{G_m} + (\tau(u_{n_j}), \varepsilon(\xi_j))_{\Omega_{n_j} \setminus \Omega(\alpha)}$$

$$+ (\tau(u_{n_j}), \varepsilon(\xi_j))_{\Omega(\alpha) \setminus G_m} ,$$

where $G_m \equiv G_m(\alpha)$. From (2.10), (2.12) and (2.13) it follows that

$$(2.15) \qquad \limsup_{j \to \infty} (\tau(u_{n_j}), \varepsilon(\xi_j - u_{n_j}))_{\Omega_{n_j}}$$

$$\leq (\tau(u), \varepsilon(\xi - u))_{G_m} + c \, \|\xi\|_{1, \Omega(\alpha) \setminus G_m} .$$

Analogously,

$$(F, \xi_j - u_{n_j})_{\Omega_{n_j}} = (F, \xi_j - u_{n_j})_{G_m} + (F, \xi_j - u_{n_j})_{\Omega_{n_j} \setminus \Omega(\alpha)}$$

$$+ (F, \xi_j - u_{n_j})_{(\Omega(\alpha) \setminus G_m) \cap \Omega_{n_j}} .$$

Arguing in the same way as before we have:

$$\text{(2.16)} \qquad \liminf_{j \to \infty} (F, \xi_j - u_{n_j})_{\Omega_{n_j}} \geq (F, \xi - u)_{G_m}$$

$$- c(\|F\|_{\Omega(\alpha) \setminus G_m} + \|\xi\|_{\Omega(\alpha) \setminus G_m}) .$$

Further,

$$(P, \xi_j - u_{n_j})_{\Gamma_P} = (P, \xi_j - u_{n_j})_{\Gamma_P \setminus M_m} + (P, \xi_j - u_{n_j})_{M_m} ,$$

where

$$M_m = \{(x_1, x_2) \in \mathbb{R}^2 \mid \ x_2 \in (\alpha(a), \alpha(a) + \tfrac{1}{m}) \quad \text{or}$$

$$x_2 \in (\alpha(b), \alpha(b) + \tfrac{1}{m})\} .$$

(This consideration can be omitted if $\operatorname{dist}(\Gamma_C(\alpha), \Gamma_P) > 0$.)
Then

$$\text{(2.17)} \qquad \liminf_{j \to \infty} (P, \xi_j - u_{n_j})_{\Gamma_P} \geq (P, \xi - u)_{\Gamma_P \setminus M_m}$$

$$- c(\|P\|_{M_m} + \|\xi\|_{M_m}) .$$

Finally we prove that

$$\text{(2.18)} \qquad j_{n_j}(u_{n_j}) \to j_\alpha(u) , \quad j \to \infty .$$

Indeed:

$$j_{n_j}(u_{n_j}) - j_\alpha(u) = g \int_{\Gamma_C(\alpha_{n_j})} |u_{1n_j}|\, ds - g \int_{\Gamma_C(\alpha)} |u_1|\, ds$$

$$= g \int_a^b |u_{1n_j} \circ \alpha_{n_j}| \sqrt{1+(\alpha'_{n_j})^2}\, dx_1 - g \int_a^b |u_1 \circ \alpha| \sqrt{1+(\alpha')^2}\, dx_1$$

$$\leq g \int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} - \hat{u}_1 \circ \alpha| \sqrt{1+(\alpha'_{n_j})^2}\, dx_1$$

$$+ g \int_a^b |\hat{u}_1 \circ \alpha| \, \left| \sqrt{1+(\alpha'_{n_j})^2} - \sqrt{1+(\alpha')^2} \right| dx_1 \equiv I_1^j + I_2^j .$$

From

$$\alpha_j \to \alpha \quad \text{in} \quad C^1([a,b]) ,$$

we immediately get

(2.19) $\quad I_2^j \to 0 , \quad j \to \infty .$

Further,

(2.20)
$$|I_1^j| \leq c \int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} - \hat{u}_1 \circ \alpha|\, dx_1$$

$$= c \int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} \pm \hat{u}_{1n_j} \circ \alpha - \hat{u}_1 \circ \alpha|\, dx_1$$

$$\leq c \int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} - \hat{u}_{1n_j} \circ \alpha|\, dx_1$$

$$+ c \int_a^b |\hat{u}_{1n_j} \circ \alpha - \hat{u}_1 \circ \alpha| \, dx_1$$

$$\leq c \int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} - \hat{u}_{1n_j} \circ \alpha| \, dx_1 + c \, \|\hat{u}_{1n_j} - \hat{u}_1\|_{\Gamma_C(\alpha)} .$$

As the imbedding of $H^1(\Omega(\alpha))$ into $L^2(\Gamma_C(\alpha))$ is completely continuous, one has

$$\|\hat{u}_{1n_j} - \hat{u}_1\|_{\Gamma_C(\alpha)} \to 0 \, , \quad j \to \infty \, . \tag{2.21}$$

Moreover:

$$\int_a^b |\hat{u}_{1n_j} \circ \alpha_{n_j} - \hat{u}_{1n_j} \circ \alpha| \, dx_1 = \int_a^b \Big| \int_\alpha^{\alpha_{n_j}} \frac{\partial}{\partial x_2} \hat{u}_{1n_j} \, dx_2 \Big| \, dx_1$$

$$\leq c \max_{[a,b]} |\alpha_{n_j} - \alpha| \, \|\hat{u}_{1n_j}\|_{1,\hat{\Omega}} \to 0 \, , \quad j \to \infty \, .$$

This, together with (2.19), (2.20), (2.21) yield (2.18). In the same way one can show that

$$j_{n_j}(\xi_j) \to j_\alpha(\xi) \, , \quad j \to \infty \, . \tag{2.22}$$

Taking into account (2.15), (2.16), (2.17), (2.18) and (2.22) we have that

$$(\tau(u) \, , \varepsilon(\xi - u))_{G_m} + c \, \|\xi\|_{1,\Omega(\alpha) \setminus G_m} + j_\alpha(\xi) - j_\alpha(u)$$

$$\geq (F \, , \xi - u)_{G_m} - c(\|F\|_{\Omega(\alpha) \setminus G_m} + \|\xi\|_{\Omega(\alpha) \setminus G_m})$$

$$+ (P \, , \xi - u)_{\Gamma_P \setminus M_m} - c(\|P\|_{M_m} + \|\xi\|_{M_m})$$

holds for any m and any $\xi \in K(\alpha)$. Passing to the limit with m, we are finally led to

$$(\tau(u)\,,\varepsilon(\xi-u))_{\Omega(\alpha)}+j_\alpha(\xi)-j_\alpha(u)\geq\langle L\,,\xi-u\rangle_\alpha$$

$$\forall\xi\in K(\alpha)\,,$$

i.e. $u\in K(\alpha)$ is a solution of $(\mathcal{P}(\alpha))$. □

Proof of Theorem 2.1. Denote by

$$p=\inf_{\alpha\in\mathcal{U}_{ad}}E(u(\alpha)\,,\alpha)$$

and by $\alpha_n\in\mathcal{U}_{ad}$ a minimization sequence, i.e.:

$$p=\lim_{n\to\infty}E(\alpha_n)\,,$$

where $E(\alpha_n)=E(u_n\,,\alpha_n)$, with $u_n\equiv u(\alpha_n)\in K_n\equiv K(\alpha_n)$ being the solutions of $(\mathcal{P}(\alpha_n))$. In view of the compactness of $\mathcal{U}_{ad}$ with respect to $C^1([a\,,b])$-topology, one can find a subsequence of $\{\alpha_n\}$ (denoted by $\{\alpha_n\}$) such that

$$\alpha_n\to\alpha^*\qquad\text{in}\quad C^1([a\,,b])\text{-norm}$$

and $\alpha^*\in\mathcal{U}_{ad}$.
According to Lemma 2.3, there exists a subsequence of $\{u_n\}$ (denoted by $\{u_n\}$ again) and an element $u\equiv u(\alpha^*)\in K(\alpha^*)$ such that

$$u_n\rightharpoonup u\quad\text{(weakly)}\qquad\text{in}\quad(H^1(G_m(\alpha^*)))^2$$

for any m, where $u(\alpha^*)$ solves $(\mathcal{P}(\alpha^*))$. Let m be fixed. Then for n sufficiently large

$$E(\alpha_n)=E_{G_m(\alpha^*)}(\alpha_n)+E_{\Omega_n\setminus G_m(\alpha^*)}(\alpha_n)\,,$$

where

$$E_{G_m(\alpha^*)}(\alpha_n)=\frac{1}{2}\,(\tau(u_n)\,,\varepsilon(u_n))_{G_m(\alpha^*)}+j_{\alpha_n}(u_n)$$

$$- (F , u_n)_{G_m(\alpha^*)} - (P , u_n)_{\Gamma_P \setminus M_m}$$

and

$$E_{\Omega_n \setminus G_m(\alpha^*)}(\alpha_n) = \frac{1}{2} (\tau(u_n) , \varepsilon(u_n))_{\Omega_n \setminus G_m(\alpha^*)}$$

$$- (F , u_n)_{\Omega_n \setminus G_m(\alpha^*)} - (P , u_n)_{M_n}$$

$$\geq - (F , u_n)_{\Omega_n \setminus G_m(\alpha^*)} - (P , u_n)_{M_m}$$

(with M_m defined in the same way as in the proof of the Lemma 2.2). As

$$\liminf_{n\to\infty} E(\alpha_n) \geq \liminf_{n\to\infty} E_{G_m(\alpha^*)}(\alpha_n) + \liminf_{n\to\infty} E_{\Omega_n \setminus G_m(\alpha^*)}(\alpha_n)$$

$$\geq E_{G_m(\alpha^*)}(\alpha^*) + \liminf_{n\to\infty} (-(F , u_n)_{\Omega_n \setminus G_m(\alpha^*)} - (P , u_n)_{M_m})$$

$$\geq E_{G_m(\alpha^*)}(\alpha^*) - c(\|F\|_{\Omega(\alpha^*) \setminus G_m(\alpha^*)} + \|P\|_{M_m})$$

holds for any m, we obtain for $m \to \infty$:

$$p = \liminf_{n\to\infty} E(\alpha_n) \geq E(\alpha^*) ,$$

i.e. α^* is a solution of $(\mathbb{P})$. $\square$

3. Sensitivity analysis

Let $\mathcal{T}_h(\alpha)$ be a triangulation of $\overline{\Omega(\alpha)}$, nodes of which, lying on $\Gamma_C(\alpha)$ will be denoted by $A_i = (a_i , \alpha(a_i))$, $i = 0 , N(h)$, $\alpha \in \mathcal{U}_{ad}$, $a_0 \equiv a$, $a_N \equiv b$, where $a_i \in [a , b]$ define its partition. With $\mathcal{T}_h(\alpha)$ we associate the finite-dimensional space $V_h(\alpha)$, of functions, defined on $\overline{\Omega}_{\alpha,h} = \cup \{T_i \in \mathcal{T}_h(\alpha) \mid T_i \cap \Omega(\alpha) \neq \emptyset\}$ (see Fig. 2) and given by

$$V_h(\alpha) = \{v_h = (v_{h1}, v_{h2}) \in (C(\overline{\Omega}_{\alpha,h}))^2 \mid v_h|_{T_i} \in (P_1(T_i))^2 ,$$
$$\forall T_i \in \mathcal{T}_h(\alpha) , \quad v_h = 0 \quad \text{on} \quad \Gamma_D \} \subset V(\alpha) ,$$

i.e. $V_h(\alpha)$ contains all piecewise linear functions over $\mathcal{T}_h(\alpha)$.
Let $K_h(\alpha) \subset V_h(\alpha)$ be the closed convex subset:

$$K_h(\alpha) = \{v_h \in V_h(\alpha) \mid v_{h2}(a_i , \alpha(a_i)) \geq -\alpha(a_i) , \quad i = 0 , \dots , N(h)\} .$$

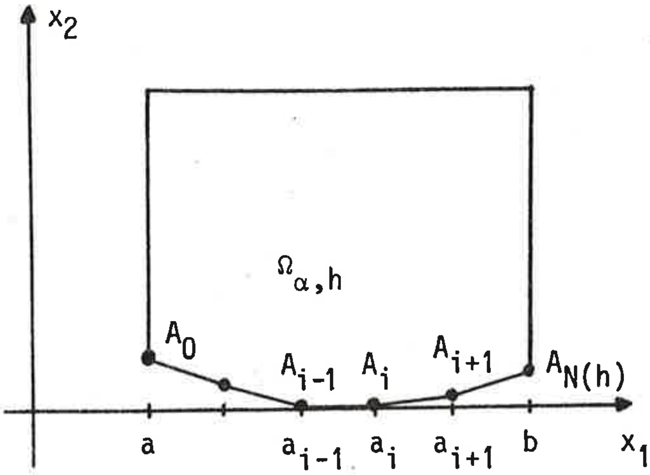

Fig. 2. $\Omega_{\alpha,h}$

By *the approximation* of $(\mathcal{P}(\alpha))$ we call the problem:

$$(\mathcal{P}(\alpha))_h \quad \begin{cases} \text{find} \quad u_h(\alpha) \in K_h(\alpha) \quad \text{such that} \\ (\tau(u_h) , \varepsilon(v_h - u_h))_{\Omega_{\alpha,h}} + j_{\alpha,h}(v_h) - j_{\alpha,h}(u_h) \\ \geq \langle L , v_h - u_h \rangle_{\alpha,h} \qquad \forall\, v_k \in K_h(\alpha) , \end{cases}$$

where

$$j_{\alpha,h}(v_h) = g \int_{\Gamma_h(\alpha)} |v_{h1}| \, ds , \quad \overline{\Gamma_h(\alpha)} = \bigcup_{i=0}^{N(h)-1} \overline{A_i A_{i+1}} ,$$

$$\langle L , v_h \rangle_{\alpha,h} = (F , v_h)_{\Omega_{\alpha,h}} + (P , v_h)_{\Gamma_P} .$$

An equivalent expression for $(\mathcal{P}(\alpha))_h$ is

$$(\mathcal{P}(\alpha))_h' \quad \begin{cases} \text{find} \quad u_h \in K_h(\alpha) \quad \text{such that} \\ J(u_h) \leq J(v_h) \qquad \forall\, v_h \in K_h(\alpha) \end{cases}$$

where

$$J(v_h) = \frac{1}{2}\,(\tau(v_h)\,,\varepsilon(v_h))_{\Omega_{\alpha,h}} + j_{\alpha,h}(v_h) - \langle L\,,v_h\rangle_{\alpha,h}\,.$$

Writing $(P(\alpha))_h'$ in a matrix form, we are led to the following problem:

$$(3.1)\qquad \begin{cases} \text{find} \quad x = x(\alpha) \in K(\alpha) \quad \text{such that} \\ J(x(\alpha)) \leq J(z) \qquad \forall z \in K(\alpha)\,, \end{cases}$$

where

$$J(z) = \frac{1}{2}\,(z\,,A(\alpha)\,z) - (y(\alpha)\,,z) + g \sum_{j_i \in I_1} \omega_i(\alpha)|z_{j_i}|\,,$$

$$K(\alpha) = \{z \in \mathbb{R}^n \mid z_{j_i} \geq -\alpha_i \equiv -\alpha(a_i) \qquad \forall_{j_i} \in I_2\}\,.$$

Here $\alpha = (\alpha_1\,,\ldots,\alpha_q) \in U \subset \mathbb{R}^q$, U closed and bounded, denotes the *design variables* - x_2 coordinates of A_i, $i = 0\,,\ldots,N(h)$ in our case, $A(\alpha)$ is a stifness matrix, $y(\alpha)$ a discretization of applied forces, both defending on $\alpha\,,\omega_i(\alpha) > 0$ are weights of a quadrature formula, used for the approximation of $j_{\alpha h}(v_h)$. $I_1\,,I_2$ are sets, containing indices of $x_1\,,x_2$ resp. components of the nodal displacement field at A_i, $i = 0\,,\ldots,N(h)$. By the approximation of the optimal shape design problem we mean the problem:

$$(\mathbb{P})_h \qquad \begin{cases} \text{find } \alpha^* \in U \quad \text{such that} \\ E(\alpha^*) \leq E(\alpha) \qquad \forall \alpha \in U \end{cases}$$

where

$$E(\alpha) \equiv J(x(\alpha)) = \frac{1}{2}\,(x(\alpha)\,,A(\alpha)\,x(\alpha)) - (y(\alpha)\,,x(\alpha))$$

$$+ g \sum_{j_i \in I_1} \omega_i(\alpha) \, |x_{j_i}(\alpha)|$$

and $x(\alpha) \in K(\alpha)$ solves (3.1).

Next we shall suppose that $A(\alpha)$ are *uniformly positive definite* with respect to $\alpha \in U$ and mappings

$$\alpha \to A(\alpha)$$
$$\alpha \to y(\alpha)$$
$$\alpha \to \omega_i(\alpha)$$

are one time continuously differentiable in $\tilde{U} \supset U$, $\tilde{U}$ open.

In what follows we prove that the mapping $\alpha \to x(\alpha)$ is *directionally differentiable,* i.e. there exists $x'(\alpha, \tilde{\alpha})$ for any $\alpha \in U$, $\tilde{\alpha} \in \mathbb{R}^q$, where

$$x'(\alpha, \tilde{\alpha}) = \lim_{t \to 0+} \frac{x(\alpha + t\tilde{\alpha}) - x(\alpha)}{t}$$

and this limit is finite. For sensitivity analysis of problems with unilateral constraints we refer to [1, 6, 12]. Here the situation is more involved than in those papers.

It is easy to see that the mapping $\alpha \to x(\alpha)$ is Lipschitz continuous. This is a direct consequence of

Lemma 3.1. For any $\alpha, \tilde{\alpha} \in U$ it holds

$$\|x(\alpha) - x(\tilde{\alpha})\| \le c \{ \|A(\alpha) - A(\tilde{\alpha})\| + \max |\omega_i(\alpha) - \omega_i(\tilde{\alpha})| + \|y(\alpha) - y(\tilde{\alpha})\| + \|\tilde{\alpha} - \alpha\| \} \tag{3.2}$$

where $c > 0$ doesn't depend on $\alpha, \tilde{\alpha} \in U$.

Proof. Following the definition of $x(\alpha), x(\tilde{\alpha})$ one has

$$(A(\alpha)\, x(\alpha), z - x(\alpha)) + g \sum_{j_i \in I_1} \omega_i(\alpha)(|z_{j_i}| - |x_{j_i}(\alpha)|) \tag{3.3}$$

$$\geq (y(\alpha), z - x(\alpha)), \qquad \forall z \in K(\alpha)$$

$$(3.4) \qquad (A(\tilde{\alpha})\, x(\tilde{\alpha}), \tilde{z} - x(\tilde{\alpha})) + g \sum_{j_i \in I_1} \omega_i(\tilde{\alpha})(|\tilde{z}_{j_i}| - |x_{j_i}(\tilde{\alpha})|)$$

$$\geq (y(\tilde{\alpha}), \tilde{z} - x(\tilde{\alpha})), \qquad \forall \tilde{z} \in K(\tilde{\alpha}).$$

It is readily seen that any element belonging to $K(\alpha), K(\tilde{\alpha})$ can be written in the form $\beta + K(0), \tilde{\beta} + K(0)$, respectively, where

$$K(0) = \{z \in \mathbb{R}^n \mid z_i \geq 0 \quad \forall i \in I_2\},$$

and $\beta \in \mathbb{R}^n$ is such that $\beta_{j_i} = 0$ if $j_i \notin I_2$, $\beta_{j_i} = -\alpha_i$ if $j_i \in I_2$ (analogously for $\tilde{\beta}$). Splitting $x(\alpha), x(\tilde{\alpha})$ as follows

$$x(\alpha) = \beta + \overline{x}(\alpha) \quad , \quad \overline{x}(\alpha) \in K(0)$$

$$x(\tilde{\alpha}) = \tilde{\beta} + \overline{x}(\tilde{\alpha}) \quad , \quad \overline{x}(\tilde{\alpha}) \in K(0),$$

one can write (3.3), (3.4) in the following form:

$$(3.3)' \qquad (A(\alpha)\, x(\alpha), z - \overline{x}(\alpha)) + g \sum_{j_i \in I_1} \omega_i(\alpha)(|z_{j_i}| - |\overline{x}_{j_i}(\alpha)|)$$

$$\geq (y(\alpha), z - \overline{x}(\alpha)) \qquad \forall z \in K(0),$$

$$(3.4)' \qquad (A(\tilde{\alpha})\, x(\tilde{\alpha}), z - \overline{x}(\tilde{\alpha})) + g \sum_{j_i \in I_1} \omega_i(\tilde{\alpha})(|z_{j_i}| - |\overline{x}_{j_i}(\tilde{\alpha})|)$$

$$\geq (y(\tilde{\alpha}), z - \overline{x}(\tilde{\alpha})) \qquad \forall z \in K(0).$$

Substituting $z = \overline{x}(\tilde{\alpha}), \overline{x}(\alpha)$ into (3.3)', (3.4)', respectively and then summing them up, we arrive at (3.2). □

Now we give the interpretation of $x(\alpha)$ (cf (2.1) - (2.6)).

<u>Lemma 3.2.</u> A vector $x(\alpha) \in K(\alpha)$ is a solution of (3.1) if and only if

$$(3.5) \qquad a_{ij}(\alpha)\, x_j(\alpha) = y_i(\alpha) \qquad \forall\, i \notin I_1 \cup I_2 \;;$$

$$(3.6) \qquad x_{j_i}(\alpha) \geq -\alpha_i \,,\quad N_{j_i} \geq 0 \,,\quad (x_{j_i}(\alpha) + \alpha_i)\, N_{j_i}(\alpha) = 0 \,, \qquad j_i \in I_2$$

where $N_i(\alpha) \equiv a_{ij}(\alpha) \cdot x_j(\alpha) - y_i(\alpha)$

$$(3.7) \qquad |T_{j_i}(\alpha)| \leq g\,\omega_i(\alpha) \,,\; j_i \in I_1 \;;\; T_i(\alpha) \equiv a_{ij}(\alpha)\, x_j(\alpha) - y_i(\alpha) \,,$$

$$(3.8) \qquad |T_{j_i}(\alpha)| < g\,\omega_i(\alpha) \;\Rightarrow\; x_{j_i}(\alpha) = 0$$

$$(3.9) \qquad |T_{j_i}(\alpha)| = g\,\omega_i(\alpha) \;\Rightarrow\; \exists\, \lambda_i \geq 0 : x_{j_i}(\alpha) = -\lambda_i\, T_{j_i}(\alpha)$$

Proof. Let $x(\alpha) \in K(\alpha)$ be a solution of (3.1), i.e.

$$(3.10) \qquad (A(\alpha)\, x(\alpha)\,, z - x(\alpha)) + g \sum_{j_i \in I_1} \omega_i(\alpha)(|z_{j_i}| - |x_{j_i}(\alpha)|) \geq (y(\alpha)\,, z - x(\alpha)) \qquad \forall\, z \in K(\alpha)$$

and set $z = x(\alpha) + t$, $t \in \mathbb{R}^n$. We shall show that (3.5) - (3.9) will be derived by a suitable choice of t.

If

$$t = (t_1, t_2, \ldots, t_n) \text{ with}$$

$$t_i = \begin{cases} a \in \mathbb{R}^1 & , \; i \notin I_1 \cup I_2 \\ 0 & , \; i \in I_1 \cup I_2 \end{cases}$$

we obtain (3.5).

Let

$$t_i = \begin{cases} a > 0 & \text{if } \; i \in I_2 \\ 0 & \phantom{\text{if }} \; i \notin I_2 \end{cases} .$$

Then for any $i \in I_2$ one has

$$a_{ij}(\alpha)\, x_j(\alpha) - y_i(\alpha) \equiv N_i(\alpha) \geq 0$$

as follows from (3.10). If $x_{j_i}(\alpha) > -\alpha_i$ for some $j_i \in I_2$, one can set $t_{j_i} = \delta \in \mathbb{R}^1$, δ sufficiently small and $t_k = 0$ for $k \neq j_i$ so that $N_{j_i}(\alpha) = 0$. As $x_{j_i}(\alpha) \geq -\alpha_i$ is automatically satisfied by the definition of $K(\alpha)$ we see that (3.6) holds.

Now set $t_i = 0$ for $i \notin I_1$. Then

$$\sum_{j_i \in I_1} T_{j_i}(\alpha)(z_{j_i} - x_{j_i}(\alpha)) + g \sum_{j_i \in I_1} \omega_i(\alpha)(|z_{j_i}| - |x_{j_i}(\alpha)|) \geq 0 \tag{3.11}$$

As (3.11) holds for any $z_i \in \mathbb{R}^1, i \in I_1$, (3.11) is equivalent with

$$T_{j_i}(\alpha)(z_{j_i} - x_{j_i}(\alpha)) + g\,\omega_i(\alpha)(|z_{j_i}| - |x_{j_i}(\alpha)|) \geq 0 \qquad \forall\, z_{j_i} \in \mathbb{R}^1, \tag{3.12}$$

$j_i \in I_1$. Setting $z_{j_i} = 0$, $2x_{j_i}$ in (3.12) one has

$$T_{j_i}(\alpha)\, x_{j_i}(\alpha) + g\,\omega_1(\alpha)|x_{j_i}(\alpha)| = 0\,. \tag{3.13}$$

This together with (3.12) yield

$$T_{j_i}(\alpha)\, z_{j_i} + g\,\omega_i(\alpha)|z_{j_i}| \geq 0 \qquad \forall\, z_{j_i} \in \mathbb{R}^1\,. \tag{3.14}$$

Relation (3.7) is now a direct consequence of (3.14). Finally, (3.7) and (3.13) lead to (3.8), (3.9).

The second part of the proof, namely if $x \in \mathbb{R}^n$ satisfies (3.5) - (3.9), then x is a solution of (3.1) is left as an easy exercise. □

Remark 3.1. System (3.5) - (3.9) is a discrete version of (2.1) - (2.6).

Before we prove that $x'(\alpha, \tilde{\alpha})$ exists, we give an equivalent formulation of (3.1), based on the saddle-point approach of (3.1). To this end we introduce two sets of Lagrange multipliers:

$$\Lambda_1 = \{\mu \in \mathbb{R}^p \mid |\mu_i| \le 1 \quad \forall i\}$$

$$\Lambda_2 = \{\mu \in \mathbb{R}^p \mid \mu_i \ge 0 \quad \forall i\} ,$$

$p = \mathrm{card}(I_1) = \mathrm{card}(I_2)$ and $L: \mathbb{R}^n \times \Lambda_1 \times \Lambda_2$ *the Lagrangian* defined by

$$L(x, \mu_1, \mu_2) = \frac{1}{2}(x, A(\alpha)x) - (y(\alpha), x)$$

$$+ g \sum_{j_i \in I_1} \omega_i(\alpha)\, \mu_i^1 x_{j_i} - \sum_{j_i \in I_2} \mu_i^2 (x_{j_i} + \alpha_i) ,$$

$\mu_1 = (\mu_1^1, \dots, \mu_p^1) \in \Lambda_1$, $\mu_2 = (\mu_1^2, \dots, \mu_p^2) \in \Lambda_2$.

Let $(\bar{x}, \lambda_1, \lambda_2)$ be a saddle-point of L on $\mathbb{R}^n \times \Lambda_1 \times \Lambda_2$; i.e.:

$$(3.15) \quad \begin{cases} (\bar{x}, \lambda_1, \lambda_2) \in \mathbb{R}^n \times \Lambda_1 \times \Lambda_2 \text{ is such that} \\ L(\bar{x}, \mu_1, \mu_2) \le L(\bar{x}, \lambda_1, \lambda_2) \le L(z, \lambda_1, \lambda_2) \quad \forall z \in \mathbb{R}^n \\ \qquad \forall \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2 \end{cases}$$

Lemma 3.3. $(\bar{x}, \lambda_1, \lambda_2)$ is a saddle-point of L on $\mathbb{R}^n \times \Lambda_1 \times \Lambda_2$ if and only if

$$\bar{x} = x(\alpha), \; -g\, \omega_i(\alpha)\, \lambda_i^1 = T_{j_i}(\alpha), \quad \lambda_i^2 = N_{j_i}(\alpha),$$

where $x(\alpha) \in K(\alpha)$ solves (3.1).

Proof. After differentiation of L with respect to x, μ_1, μ_2

we obtain the following relations for $\bar{x}, \lambda_1, \lambda_2$ (equivalent with (3.15)):

$$(3.16) \quad \begin{cases} a_{ij}(\alpha)\, x_j(\alpha) = y_i(\alpha) & i \notin I_1 \cup I_2 \\ a_{j_i k}(\alpha)\, x_k(\alpha) = y_{j_i}(\alpha) - g\,\omega_i(\alpha)\, \lambda_i^1 & j_i \in I_1 \\ a_{j_i k}(\alpha)\, x_k(\alpha) = y_{j_i}(\alpha) + \lambda_i^2 & j_i \in I_2 \end{cases}$$

$$(3.17) \quad g \sum_{j_i \in I_1} \omega_i(\alpha)(\mu_i^1 - \lambda_i^1)\, x_{j_i}(\alpha) \le 0 \qquad \forall\, \mu_1 \in \Lambda_1$$

$$(3.18) \quad \sum_{j_i \in I_2} (\mu_i^2 - \lambda_i^2)(x_{j_i}(\alpha) + \alpha_i) \ge 0 \qquad \forall\, \mu_2 \in \Lambda_2 .$$

From (3.17), (3.18) we easily deduce that $\bar{x} \in K(\alpha)$ is a solution of (3.1). Hence $\bar{x} = x(\alpha)$ because of the uniqueness. Taking into account the definition of $N_i(\alpha), T_i(\alpha)$ and (3.16) we finish the proof. □

<u>Consequence 3.1.</u> From Lemma 3.1 and (3.16) we see that $\lambda_1(\alpha), \lambda_2(\alpha)$ are continuous functions of α.

<u>Remark 3.2.</u> Replacing T_{j_i} by $-g\,\omega_i(\alpha)\,\lambda_i^1$ in (3.13) we obtain

$$(3.19) \quad |x_{j_i}(\alpha)| = \lambda_i^1\, x_{j_i}(\alpha) .$$

Let $x(\alpha) \in K(\alpha)$ be the solution of (3.1). With any $x(\alpha)$ we decompose sets I_1, I_2 as follows:

$$I_1 = I_1^+(\alpha) \cup I_1^0(\alpha) \cup I_1^-(\alpha)$$

$$I_2 = I_2^+(\alpha) \cup I_2^0(\alpha) \cup I_2^-(\alpha)$$

where

$$j_i \in I_1^+(\alpha) \iff x_{j_i}(\alpha) \neq 0$$

$$j_i \in I_1^0(\alpha) \iff x_{j_i}(\alpha) = 0 \ \& \ |\lambda_i^1(\alpha)| = 1$$

$$j_i \in I_1^-(\alpha) \iff x_{j_i}(\alpha) = 0 \ \& \ |\lambda_i^1(\alpha)| < 1 .$$

Analogously

$$j_i \in I_2^+(\alpha) \iff x_{j_i}(\alpha) > -\alpha_i$$

$$j_i \in I_2^0(\alpha) \iff x_{j_i}(\alpha) = -\alpha_i \ \& \ \lambda_i^2 = 0$$

$$j_i \in I_2^-(\alpha) \iff x_{j_i}(\alpha) = -\alpha_i \ \& \ \lambda_i^2(\alpha) > 0$$

Now we are able to prove

Theorem 3.1. The directional derivate $x'(\alpha, \tilde{\alpha})$ exists for any $\alpha \in \mathcal{U}$, $\tilde{\alpha} \in \mathbb{R}^q$. Moreover $x'(\alpha, \tilde{\alpha})$ is an element of $K(\alpha, \tilde{\alpha})$

$$\begin{aligned} K(\alpha, \tilde{\alpha}) = \{z \in \mathbb{R}^n \mid z_i &= 0 && \forall i \in I_1^-(\alpha) \\ z_{j_i} &\leq 0 && \forall j_i \in I_1^0(\alpha), \ \lambda_i^1(\alpha) = 1, \\ z_{j_i} &\geq 0 && \forall j_i \in I_1^0(\alpha), \ \lambda_i^1(\alpha) = -1, \\ z_{j_i} &= -\tilde{\alpha}_i && \forall j_i \in I_2^-(\alpha), \\ z_{j_i} &\geq -\tilde{\alpha}_i && \forall j_i \in I_2^0(\alpha)\} \end{aligned}$$

characterized through the relation)[3]

$$(3.20) \quad \begin{cases} L(x'(\alpha, \tilde{\alpha})) \leq L(z) \quad \forall z \in K(\alpha, \tilde{\alpha}), \ \text{where} \\ L(z) = \frac{1}{2}(z, A(\alpha)z) - (y'(\alpha) - A'(\alpha)x(\alpha), z) \\ \qquad + g \sum_{j_i \in I_1} \omega_i'(\alpha) T_{j_i}(\alpha) \end{cases}$$

)[3] If for a given α the set $K(\alpha, \tilde{\alpha})$ is empty, then $\alpha \mapsto x(\alpha)$ is continuously differentiable at α.

Proof. We shall show that $\{ \frac{x(\alpha + t\tilde{\alpha}) - x(\alpha)}{t} \}$, $t \to 0+$ has a unique cluster point, belonging to $K(\alpha, \tilde{\alpha})$ and satisfying (3.20). Let $t_n \to 0+$ and let us denote

$$\dot{x}_j(\alpha) = \lim_{t_n \to 0+} \frac{x_j(\alpha + t_n\tilde{\alpha}) - x_j(\alpha)}{t_n}$$

and similarly $\dot{\lambda}_j^1, \dot{\lambda}_j^2$. As $A(\alpha), y(\alpha), \omega_i(\alpha)$ are one time continuously differentiable,

$$a'_{ij}(\alpha) \overset{\text{def.}}{\equiv} \lim_{t \to 0+} \frac{a_{ij}(\alpha + t\tilde{\alpha}) - a_{ij}(\alpha)}{t}$$

$$= \lim_{t_n \to 0+} \frac{a_{ij}(\alpha + t_n\tilde{\alpha}) - a_{ij}(\alpha)}{t_n} = \nabla_\alpha a_{ij}(\alpha) \cdot \tilde{\alpha}$$

(analogously $y'(\alpha), \omega'_i(\alpha)$).

Writing the systen (3.16) for the design parameters α and $\alpha + t_n\tilde{\alpha}$, subtracting them and tending by $t_n \to 0+$ we obtain

$$(3.21) \quad \begin{cases} a_{ij}(\alpha)\, \dot{x}_j(\alpha) = y'_i(\alpha) - a'_{ij}(\alpha)\, x_j(\alpha), \quad i \notin I_1 \cup I_2 \\[2ex] a_{j_i k}(\alpha)\, \dot{x}_k(\alpha) = y'_{j_i}(\alpha) - a'_{j_i k}(\alpha)\, x_k(\alpha) - g\,\omega'_i(\alpha)\,\lambda_i^1 \\ \qquad\qquad - g\,\omega_i(\alpha)\,\dot{\lambda}_i^1, \quad j_i \in I_1 \\[2ex] a_{j_i k}(\alpha)\, \dot{x}_k(\alpha) = y'_{j_i}(\alpha) - a'_{j_i k}(\alpha)\, x_k(\alpha) + \dot{\lambda}_i^2(\alpha) \\ \qquad\qquad \forall j_i \in I_2 \end{cases}$$

Now we shall discuss the behaviour of each component $x_i(\alpha)$ according to which of the sets $I_1^+(\alpha), \ldots, I_2^-(\alpha)$ belongs.

Let $j_i \in I_1^+(\alpha)$. Then $j_i \in I_1^+(\alpha + t\tilde{\alpha})$ and

$$\lambda_i^1(\alpha) = \lambda_i^1(\alpha + t\tilde{\alpha}) \neq 0$$

for t sufficiently small so that

(3.22) $\dot{\lambda}_i^1(\alpha) = 0$.

Let $j_i \in I_1^-(\alpha)$. Then $j_i \in I_1^-(\alpha + t\tilde{\alpha})$ as follows from the continuity of the Lagrange multiplier λ_i^1 : $|\lambda_i^1(\alpha)| < 1$ implies $|\lambda_i^1(\alpha + t\tilde{\alpha})| < 1$ for $t > 0$ sufficiently small. From (3.19) one has

$$x_{j_i}(\alpha + t\tilde{\alpha}) = 0$$

and therefore

(3.23) $\dot{x}_{j_i}(\alpha) = 0$

Finally, let $j_i \in I_1^0(\alpha)$ and let $\lambda_i^1(\alpha) = 1$. As for $t > 0$ sufficiently small $0 < \lambda_i^1(\alpha + t\tilde{\alpha}) \leq \lambda_i^1(\alpha)$, necessarily $\dot{\lambda}_i^1(\alpha) \leq 0$ and at the same time

$$x_{j_i}(\alpha + t\tilde{\alpha}) \leq x_{j_i}(\alpha) = 0$$

from which follows that

(3.24) $\dot{x}_{j_i}(\alpha) \leq 0$.

Let us show that

(3.25) $\dot{\lambda}_i^1(\alpha)\, \dot{x}_{j_i}(\alpha) = 0$.

If $\dot{\lambda}_i^1(\alpha) = 0$, (3.25) holds. On the other hand, if $\dot{\lambda}_i^1(\alpha) < 0$ then $\lambda_i^1(\alpha + t\tilde{\alpha}) < 1$ for $t > 0$ sufficiently small. Hence $x_{j_i}(\alpha + t\tilde{\alpha}) = 0$ implies $\dot{x}_{j_i}(\alpha) = 0$ and (3.25) is true. One can proceed in the same way, if $j_i \in I_1^0(\alpha)$ and $\lambda_i^1(\alpha) = -1$. In such a case $\dot{\lambda}_i^1(\alpha) \geq 0$, $\dot{x}_{j_i}(\alpha) \geq 0$ and $\dot{\lambda}_1(\alpha)\, \dot{x}_{j_i}(\alpha) = 0$.

Now, let $j_i \in I_2^+(\alpha)$. Then $j_i \in I_2^+(\alpha + t\tilde{\alpha})$ for $t > 0$ sufficiently small and as

$$\lambda_i^2(\alpha) = \lambda_i^2(\alpha + t\tilde{\alpha}) = 0$$

we obtain $\dot{\lambda}_i^2(\alpha) = 0$.

Let $j_i \in I_2^-(\alpha)$. Then due to the continuity of λ_i^2 also $\lambda_i^2(\alpha + t\tilde{\alpha}) > 0$ for $t > 0$ sufficiently small. Hence $x_{j_i}(\alpha + t\tilde{\alpha}) = -\alpha_i - t\tilde{\alpha}_i$, which implies $\dot{x}_{j_i}(\alpha) = -\tilde{\alpha}_i$. Finally, let $j_i \in I_2^0(\alpha)$. Then for $t > 0$ sufficiently small:

$$\lambda_i^2(\alpha + t\tilde{\alpha}) \geq \lambda_i^2(\alpha) = 0 \Rightarrow \dot{\lambda}_i^2(\alpha) \geq 0$$

and

$$x_{j_i}(\alpha + t\tilde{\alpha}) \geq -\alpha_i - t\tilde{\alpha}_i = x_{j_i}(\alpha) - t\tilde{\alpha}_i .$$

This implies that

$$\dot{x}_{j_i}(\alpha) \geq -\tilde{\alpha}_i .$$

One can easy prove that

$$\dot{\lambda}_i^2(\alpha)(\dot{x}_{j_i}(\alpha) + \tilde{\alpha}_i) = 0 . \tag{3.26}$$

From the proof it immediately follows that $\dot{x}(\alpha) = (\dot{x}_1(\alpha), \ldots, \dot{x}_n(\alpha)) \in K(\alpha, \tilde{\alpha})$ and from (3.25), (3.26) we see that $\dot{x}(\alpha)$ minimizes the functional L (given by (3.20)) over the convex set $K(\alpha, \tilde{\alpha})$. Moreover, any cluster point of the sequence $\{\frac{x(\alpha + t\tilde{\alpha}) - x(\alpha)}{t}\}$, $t > 0$ has this property. As the minimizer of L over $K(\alpha, \tilde{\alpha})$ is determined in a unique way, $x'(\alpha, \tilde{\alpha})$ exists and is defened by (3.20). □

Let us denote by $J'(x(\alpha))$ the directional derivative of J, i.e.

$$J'(x(\alpha)) = \lim_{t \to 0+} \frac{J(x(\alpha + t\tilde{\alpha})) - J(x(\alpha))}{t} .$$

The main result of this section is

<u>Theorem 3.2.</u> J is one time continuously differentiable and

$$(3.27)\qquad J'(x(\alpha)) = \frac{1}{2}(x(\alpha), A'(\alpha)\,x(\alpha)) - (y'(\alpha), x(\alpha)) \;)^4$$

$$- \sum_{\substack{i=1\\ j_i\in I_2}}^{p} \tilde{\alpha}_i\, N_{j_i}(\alpha) + g \sum_{\substack{i=1\\ j_i\in I_1}}^{p} \omega_i'(\alpha)\, |x_{j_i}(\alpha)|\,.$$

<u>Proof.</u> As $x_{j_i}'(\alpha)$, $j_i \in I_1$ exists and $x_{j_i}(\alpha + t\tilde{\alpha})$ is monotone with aspect to $t \in (0,\delta)$, δ sufficiently small (see the proof of Theorem 3.1), there exists $|x_{j_i}(\alpha)|'$ as well. One can write

$$(3.28)\qquad J'(x(\alpha)) = (x'(\alpha), A(\alpha)\,x(\alpha) - y(\alpha)) + \frac{1}{2}(x(\alpha), A'(\alpha)\,x(\alpha))$$

$$- (y'(\alpha), x(\alpha)) + g \sum_{j_i\in I_1} (\omega_i(\alpha)\, |x_{j_i}(\alpha)|)'$$

$$= (x'(\alpha), A(\alpha)\,x(\alpha) - y(\alpha)) + \frac{1}{2}(x(\alpha), A'(\alpha)\,x(\alpha))$$

$$- (y'(\alpha), x(\alpha)) - \sum_{j_i\in I_1} (T_{j_i}(\alpha)\, x_{j_i}(\alpha))'\,.$$

Let us arrange the term $(x'(\alpha), A(\alpha)\,x(\alpha) - y(\alpha))$. From (3.16) it follows that

$$(3.29)\qquad (x'(\alpha), A(\alpha)\,x(\alpha) - y(\alpha)) = \sum_{j_i\in I_2} x_{j_i}'(\alpha)\, N_{j_i}(\alpha)$$

$$+ \sum_{j_i\in I_1} x_{j_i}'(\alpha)\, T_{j_i}(\alpha)\,.$$

But

$$(3.29)'\qquad \sum_{j_i\in I_2} x_{j_i}'(\alpha)\, N_{j_i}(\alpha) = - \sum_{\substack{i=1\\ j_i\in I_2}}^{p} \tilde{\alpha}_i\, N_{j_i}(\alpha)$$

)[4] By $A'(\alpha)$ we mean a matrix, components of which are given by $a_{ij}'(\alpha)$ (analogously $y'(\alpha)$).

as either $N_{j_i}(\alpha) = 0$ or $x'_{j_i}(\alpha) = -\tilde{\alpha}_i$.

(3.28), (3.29) and (3.29)' lead to

$$(3.30) \qquad J'(x(\alpha)) = \frac{1}{2}\,(x(\alpha)\,,A'(\alpha)\,x(\alpha)) - (y'(\alpha)\,,x(\alpha)) - \sum_{\substack{i=1\\ j_i\in I_2}}^{p} \tilde{\alpha}_i\,N_{j_i}(\alpha) - \sum_{j_i\in I_1} T'_{j_i}(\alpha)\,x_{j_i}(\alpha)\,.$$

Finally, let us arrange the last term of (3.30)

$$- \sum_{j_i\in I_1} T'_{j_i}(\alpha)\,x_{j_i}(\alpha) = g \sum_{\substack{i=1\\ j_i\in I_1}}^{p} \omega'_i(\alpha)\,\lambda^1_i(\alpha)\,x_{j_i}(\alpha) + g \sum_{\substack{i=1\\ j_i\in I_1}}^{p} \omega_i(\alpha)\,\dot{\lambda}^1_i(\alpha)\,x_{j_i}(\alpha) = g \sum_{\substack{i=1\\ j_i\in I_1}}^{p} \omega'_i(\alpha)\,|x_{j_i}(\alpha)|$$

because of (3.19) and the fact that or $x_{j_i}(\alpha) = 0$ or $\dot{\lambda}^1_i(\alpha) = 0$. □

Remark 3.3. Formula (3.27) can be received also by another way. Using Lagrange multipliers $\mu_1\,,\mu_2\,,$ problem (3.1) can be stated as follows

$$E(\alpha) = \inf_{x\in K(\alpha)} J\,(x) = \inf_{\mathbb{R}^n}\ \sup_{\Lambda_1\times\Lambda_2}\ L(\alpha\,,x\,,\mu_1\,,\mu_2)$$

$$\equiv \inf_{\mathbb{R}^n}\ \sup_{\Lambda_1\times\Lambda_2} \{(x\,,A(\alpha)\,x) - (y(\alpha)\,,x) + g \sum_{j_i\in I_1} \omega_i(\alpha)\,\mu^1_i\,x_{j_i} - \sum_{j_i\in I_2} \mu^2_i(x_{j_i} + \alpha_i)\}\,.$$

Then it is known by [3], that

$$(3.31) \qquad E'(\alpha) \overset{\text{def.}}{\equiv} \lim_{t\to 0+} \frac{E(\alpha + t\tilde{\alpha}) - E(\alpha)}{t}$$

$$= \inf_{x\in U_0} \sup_{(\mu_1,\mu_2)\in V_0} L'_\alpha(\alpha , x , \mu_1 , \mu_2)$$

$$= \sup_{(\mu_1,\mu_2)\in V_0} \inf_{x\in U_0} L'_\alpha(\alpha , x , \mu_1 , \mu_2) ,$$

where L'_α denotes the directional derivative of L with respect to α and $U_0 = \{x(\alpha)\}$, $V_0 = \{\lambda_i^1 , \lambda_i^2\}$; $x(\alpha) \in K(\alpha)$ is a solution of (3.1) and $\{\lambda_i^1 , \lambda_i^2\}$ are defined by Lemma 3.3. A direct calculation shows that (3.27) and (3.31) are equivalent.

The results of numerical tests where the gradient information (3.27) is applied be reported in a forthgoing paper.

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