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OPTIMIZATION OF THE DOMAIN IN ELLIPTIC VARIATIONAL INEQUALITIES[1]

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1. Introduction

An optimization problem for the system described by an elliptic equation defined in a domain $\Omega \subset \mathbb{R}^2$ with unilateral boundary conditions on a part $\Gamma(v)$ of the boundary is considered in this paper. We arrive at a nonsmooth shape optimization problem, i.e., in our optimization problem cost functional $J(v)$ fails to be continuously differentiable at any admissible designs $v \in \mathcal{U}_{ad}$, $\mathcal{U}_{ad}$ is a set of admissible designs. However, $J(v)$ is Lipschitz continuous on the set $\mathcal{U}_{ad}$.

We shall show the existence of an optimal solution for the optimization problem. Furthermore, we derive the form of directional derivative of the cost functional $J(v)$ for any $v \in \mathcal{U}_{ad}$. Finally, a numerical method of optimization of gradient type combined with the finite element method is used in order to solve numerically the optimization problem in question.

The method of sensitivity analysis of variational inequalities proposed by Mignot [12] combined with the results of Sokolowski [15, 16, 18] is used in this paper for the sensitivity analysis of unilateral problem. The related results on the shape sensitivity analysis of unilateral problems are presented in Sokolowski, Zolesio [19, 20, 21] and Zolesio [23]. The material derivative method applied in the shape sensitivity analysis of the classical boundary value problems is described in Zolesio [22]. We refer to Bendsøe, Olhoff, Sokolowski [1] for the sensitivity analysis of continuous as well as discrete problems of elasticity

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with unilateral constraints. See [7,10] for the case with friction.

A similar state problem but with other criterion functionals have been considered by several authors with various (variational inequality, method of penalization, dual method) methods: see Hlăváček, Nečas [11], Haslinger-Lovišek [6], Haslinger-Neittaanmäki [8,9]. For the shape optimization problems for contact problems in plane elasticity we refer to Benedic, Sokolowski, Zolesio [2], Haslinger, Horák, Neittaanmäki [7], Haslinger, Neittaanmäki, Kaarna, Tiihonen [10].

2. Notation

We introduce some notations needed in sequel.

The set of admissible functions (controls) is

$$(2.1)\qquad \mathcal{U}_{ad} = \{v \in H^1(]0,1[) \mid 0 < \alpha \leq v(s) \leq \beta\,,\ \left|\frac{dv}{dx_2}(s)\right| \leq C_1 \text{ for a.e. } s \text{ in }]0,1[\,,\ \int_0^1 v(s)\,ds = C_2\}\,,$$

where α, β, C_1 and C_2 are given positive constants. Let us consider domains $\Omega = \Omega(v) \subset \mathbb{R}^2$ with the following geometrical structure:

$$(2.2)\qquad \Omega(v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 < x_1 < v(x_2)\,,\ 0 < x_2 < 1\}\,,$$

$v \in \mathcal{U}_{ad}$; $\partial\Omega(v) = \Gamma_1(v) \cup \Gamma(v)$ the boundary of $\Omega(v)$ with

$$(2.3)\qquad \Gamma_1(v) = \partial\Omega(v) \setminus \Gamma(v)\,,$$

$$(2.4)\qquad \Gamma(v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = v(x_2)\,,\ 0 < x_2 < 1\}\,.$$

By $H^k(\Omega)$ ($k \geq 0$, integer) we denote the classical Sobolev space of functions the generalized derivatives of which up to order k are square integrable in Ω ($L^2(\Omega) := H^0(\Omega)$); the norm and scalar product is denoted by $\|\cdot\|_{k,\Omega}$ and by $(\cdot,\cdot)_{k,\Omega}$ respectively. $H_0^k(\Omega)$ is closure of space $C_0^\infty(\Omega)$ in the norm of space $H^k(\Omega)$.

Moreover, let

$$(2.5)\qquad K(v) = \{\phi \in H^1(\Omega(v)) \mid \phi = 0 \text{ on } \Gamma_1(v)\,,\ \phi \geq 0 \text{ on } \Gamma(v)\}$$

and

$$M = \{\phi \in H_0^1(]0,2[\times]0,1[) \mid \phi \geq 0 \text{ on }]0,2[\times]0,1[\,, \quad \|\phi\|_{H_0^2(]0,2[\times]0,1[)} \leq 1\}. \tag{2.4}$$

2.1. Variational inequality

For given element $v \in \mathcal{U}_{ad}$, we denote by $y(v) \in H^1(\Omega(v))$ the unique solution to the variational inequality:

$$\begin{cases} \text{find an element } y(v) \in K(v) \text{ such that} \\ \int_{\Omega(v)} \nabla y(v) \cdot \nabla(\varphi - y(v))\,dx \\ \geq \int_{\Omega(v)} f(\varphi - y(v))\,dx \qquad \forall \varphi \in K(v), \end{cases} \tag{2.5}$$

where $f \in L^2(\mathbb{R}^2)$ is a given element such that

$$\frac{\partial}{\partial x_1} f \in L^2(\mathbb{R}^2). \tag{2.6}$$

It can be verified that for any $f \in L^2(R^2)$ there exists a unique solution of (3.1). For the study of variational inequalities we refer to textbooks [5, 12].

2.2. Cost functional

Let $\phi \in M$ be given element. Denote

$$I_\phi(v) = \int_{\Gamma(v)} \frac{\partial}{\partial n} y(v;x)\,\phi(x)\,dx \tag{2.7}$$

and

$$J(v) = \sup\{I_\phi(v) \mid \phi \in M\}. \tag{2.8}$$

Observe that $\frac{\partial}{\partial n} y(s) \geq 0$, $\phi \geq 0$ on $\Gamma(v)$. Hence $I_\phi(v) \geq 0$ for all $\phi \in M(v)$ and $v \in \mathcal{U}_{ad}$.

Denote by $M^*(v) \subset H_0^1(]0,2[\times]0,1[)$ a set of the form:

$$M^*(v) = \{\phi \in M \mid I_\phi(v) = J(v)\}. \tag{2.9}$$

3. Shape optimization problems

We consider the following shape optimization problems for variational inequality (3.1):

$(\mathcal{P}_\phi)$ Minimize the cost functional $I_\phi(v)$ over the set $\mathcal{U}_{ad}$

and

$(\mathcal{P})$ Minimize the cost functional $J(v)$ over the set $\mathcal{U}_{ad}$.

3.1. Existence of an optimal solution

We have the following two lemmas:

Lemma 3.1 For every element $\phi \in M$ there exists an optimal solution $v_\phi \in \mathcal{U}_{ad}$ to the Problem $(\mathcal{P}_\phi)$.

Proof. Observe that

$$I_\phi(v) = \int_{\Omega(v)} [\nabla y(v) \cdot \nabla\phi - f\phi]\, dx . \tag{3.1}$$

Hence the existence of an optimal solution $v_\phi \in \mathcal{U}_{ad}$ to Problem $(\mathcal{P}_\phi)$ follows by classical argument ([3, 11]). □

We show the existence of an optimal solution of shape optimization problem $(\mathcal{P})$.

Lemma 3.2 There exists an optimal solution $\hat{v} \in \mathcal{U}_{ad}$ to the Problem $(\mathcal{P})$.

Proof. Let $\{v_n\}$ be a minimizing sequence for the problem $(\mathcal{P})$ then there exists an element $v \in \mathcal{U}_{ad}$ such that for a subsequence, still denoted $\{v_n\}$ we have

$$v_n \rightharpoonup v \quad \text{weakly in } H^1(]0,1[) \tag{3.2}$$

and by the result of Hlávăček, Nečas [11] it follows that convergence (3.2) implies the convergence of the sequence of solutions $\{y(v_n)\}$ in Sobolev spaces $H^1(\Omega')$, $\Omega' \subset \Omega(v_n)$ of variational inequalities of the form (2.5) defined in domains $\Omega(v_n)$.

On the other hand there exists a sequence $\{\phi_n\} \subset H^1(P)$, $P =]0,2[\times]0,1[$, $\phi_n \in M^*(v_n)$, $n = 1, 2, \ldots$, and an element $\phi \in M^*(v)$ such that

$$(3.3) \qquad \phi_n \to \phi \quad \text{in } H_0^1(P).$$

Therefore

$$(3.4) \qquad \inf\{J(v) \mid v \in U_{ad}\} = \lim_{n\to\infty} J(v_n)$$

$$= \lim_{n\to\infty} \int_{\Omega(v_n)} [\nabla y(v_n) \cdot \nabla\phi_n - f\phi_n]\, dx$$

$$= \int_{\Omega(v)} [\nabla y(v) \cdot \nabla\phi - f\phi]\, dx = J(v)$$

what completes the proof. □

4. Necessary optimality condition

4.1. Directional differentiability of the cost functionals

We denote

$$(4.1) \qquad dI_\phi(v;k) = \lim_{\tau\to 0} \frac{I_\phi(v+\tau k) - I_\phi(v)}{k},$$

for all $v \in U_{ad}$ and $k \in C^{0,1}([0,1])$.

We derive the form of directional derivative $dI_\phi(v;k)$ using the results of [15, 16, 18].

Lemma 4.1 The directional derivative (4.1) of cost functional I_ϕ (defined by (2.7)) takes the form

$$(4.2) \qquad dI_\phi(v;k) = \int_{\Omega(v)} \{\langle A\nabla y(v), \nabla\phi\rangle_{\mathbb{R}^2} + \langle \nabla w, \nabla\phi\rangle_{\mathbb{R}^2} - \frac{k}{v}\left(f + x_1 \frac{\partial f}{\partial x_1}\right)\phi\}\, dx .$$

Here $w \in H^1(\Omega(v))$ is a unique solution of the variational inequality: find an element $w \in S(\Omega(v))$ such that

$$(4.3) \qquad \int_{\Omega(v)} \nabla w \cdot \nabla(\varphi - w)\, dx$$
$$\geq \int_{\Omega(v)} \frac{k}{v} \left(f + x_1 \frac{\partial}{\partial x_1} f\right)(\varphi - w)\, dx$$
$$- \int_{\Omega(v)} \langle A\nabla y(v), \nabla(\varphi - w)\rangle_{\mathbb{R}^2}\, dx \qquad \text{for all} \quad \varphi \in S(\Omega(v)),$$

where

$$(4.4) \qquad S(v) = \{\varphi \in H^1(\Omega(v)) \mid \varphi = 0 \quad \text{on} \quad \Gamma_1(v),$$
$$\int_{Z(y(v))} \frac{\partial}{\partial n} y(v)\, \varphi\, d\Gamma = 0, \quad \varphi \geq 0 \quad \text{a.e. on} \quad Z(y(v))\},$$

$$(4.5) \qquad Z(y(v)) = \{x \in \Gamma(v) \mid y(v;x) = 0\}$$

and $A = A(k)$ is a matrix $[a_{ij}]_{2\times 2}$ with components

$$(4.6) \qquad a_{11} = -k \frac{1}{v^2}$$

$$(4.7) \qquad a_{12} = a_{21} = k \frac{x_1 v'}{v^2} - k' \frac{x_1}{v}$$

$$(4.8) \qquad a_{22} = k \left[1 - \left(\frac{x_1 v'}{v}\right)^2\right] + k' \frac{2x_1^2 v'}{v},$$

where

$$(4.9) \qquad v'(x_2) = \frac{d}{dx_2} v(x_2), \quad x_2 \in \,]0,1[\,.$$

<u>Proof</u>. For $t > 0$, t small enough we define the mapping

$$(4.10) \qquad T_t : \Omega(v) \ni (x_1, x_2) \to (\eta_1, \eta_2) \in \Omega(v + tk)$$

of the form

$$(4.11) \qquad \eta_1 = x_1(1 + tk(x_2)/v(x_2))$$

$$(4.12) \qquad \eta_2 = x_2 .$$

We denote

$$(4.13) \qquad v_t = v + tk$$

(4.14) $$A_t = \det(DT_t) \, {}^*DT_t^{-1} \cdot DT_t^{-1} .$$

We transport the cost functional defined on the domain $\Omega(v_t)$

(4.15) $$I_\phi(v_t) = \int_{\Omega(v_t)} \{ \langle \nabla y(v_t; \eta_1, \eta_2), \nabla\varphi(\eta_1,\eta_2) \rangle_{R^2} - f(\eta_1,\eta_2)\, \varphi(\eta_1,\eta_2) \} \, d\eta_1 \, d\eta_2$$

to the fixed domain $\Omega(v)$ using the mapping (4.10) and we obtain

(4.16) $$I_\phi(v_t) = \int_{\Omega(v)} \langle A_t(x_1,x_2) \nabla y(v_t; \eta_1(x_1,x_2),x_2), \nabla\varphi(\eta_1(x_1,x_2),x_2) \rangle_{R^2} \, dx_1 \, dx_2 - \int_{\Omega(v)} f(\eta_1(x_1,x_2),x_2)\, \varphi(\eta_1(x_1,x_2),x_2)\, v(x_2) \, dx_1 \, dx_2 .$$

Let us note that we can also transport the variational inequality defined in the domain $\Omega(v_t)$

(4.17) $$\begin{cases} y_t \in K(v_t) \\ \int_{\Omega(v_t)} \nabla y_t \cdot \nabla(\varphi - y_t)\, d\eta \geq \int_{\Omega(v_t)} f(\varphi - y_t)\, dx , \quad \forall \varphi \in K(v_t) \end{cases}$$

to the fixed domain $\Omega(v)$, here we denote $y^t = y_t \circ T_t$ and we obtain

(4.18) $$\begin{cases} y^t \in K(v) \\ \int_{\Omega(v)} \langle A_t \nabla y^t, \nabla(\varphi - y^t) \rangle_{R^2} \, dx \geq \int_{\Omega(v)} v f (\varphi - y^t)\, dx , \quad \forall \varphi \in K(v) . \end{cases}$$

It can be shown using the results of the papers [15, 16] that for $t > 0$, t small enough

(4.19) $$y^t = y(v) + \varepsilon w(k) + 0(\varepsilon) \quad \text{in } H^1(\Omega(v)) ,$$

where $\|0(\varepsilon)\|_{H^1(\Omega(v))}/\varepsilon \to 0$ with $\varepsilon \downarrow 0$, the element $w(k) \in H^1(\Omega))$ is given, for any $k \in C^{0,1}([0,1])$, by a unique solution of the following variational inequality

$$(4.20)\qquad \begin{cases} w(k) \in S(v) \\ \int_{\Omega(v)} \nabla w(k) \cdot \nabla(\varphi - w(k))\,dx \geq \int_{\Omega(v)} \{((k/v)\,f + x_1 \frac{\partial f}{\partial x_1})(\varphi - w(k)) \\ \quad - \langle A(k)\,\nabla y(v)\,,\nabla(\varphi - w(k))\rangle_{R^2}\,dx \qquad \forall\varphi \in S(v)\ . \end{cases}$$

Here we denote by $A(k)$ the limit

$$(4.21)\qquad A(k) = \lim_{t\downarrow 0} (A_t - I)/t\ ,$$

where the matrix $A(k)$ is given by (4.6) to (4.8). From (4.16) in view of (4.19), (4.21) we obtain the limit (4.1) in the form (4.2). □

Lemma 4.2 The directional derivative $dJ(v;k)$, $v \in \mathcal{U}_{ad}$, $k \in C^{0,1}([0,1])$ of the cost functional J (defined by (4.2)) takes the form

$$(4.22)\qquad dJ(v;k) = \sup\ \{dI_\phi(v;k)\ |\ \phi \in M^*(v)\}\ .$$

Proof. We can apply Theorem presented in [14], since the assumptions of this theorem, in view of Lemma 4.1, are verified, and we obtain (4.22). □

Lemma 4.3 The cost functional I_ϕ is differentiable on a dense subset of the set $\mathcal{U}_{ad}$. The gradient $\nabla I_\phi(\tilde{v})$, whenever it exists, can be calculated in the following way:

$$(4.23)\qquad dI_\phi(\tilde{v}\,;k) = \langle\nabla I_\phi(\tilde{v});k\rangle$$
$$= \int_{\Omega(v)} \{\langle A\nabla y(v)\,,\nabla(\phi - p_\phi)\rangle_{R^2} - \frac{k}{v}(f + x_1 \frac{\partial f}{\partial x_1})(\phi - p_\phi)\}\,dx\ ,$$

where the element $p_\phi \in H^1(\Omega(v))$ satisfies the variational equation: $p_\phi \in S(\Omega(v))$

$$(4.24)\qquad \int_{\Omega(v)} \nabla p_\phi \cdot \nabla\varphi\,dx = \int_{\Omega(v)} \nabla\phi \cdot \nabla\varphi\,dx$$

for all $\varphi \in S(\Omega(v))$,

$$(4.25)\qquad S(\Omega(v)) = \{\varphi \in H^1(\Omega(v))\ |\ \varphi = 0 \text{ on } \Gamma_1(v) \cup Z(y(v))\}\ .$$

Remark 4.1 The cost functional I_ϕ is not in general differentiable. dI_ϕ fails to exist when

$$\mu_1(\{x \in \Gamma(v) \mid y(v)(x) = 0\} \cap \{x \in \Gamma(v) \mid \frac{\partial}{\partial n} y(v)(x) = 0\}) > 0 ,$$

where μ_1 denotes the Lebesque measure on $\Gamma(v)$. In order to assure differentiability we should have $\frac{\partial}{\partial n} y > 0$ on $Z(y(v))$.

5. Numerical method and results of computations

5.1. An algorithm

Problem $(\mathcal{P}_\phi)$ must in practice be solved iteratively. In order to outline the method we give a naive algorithm (a steepest descent method) for solving Problem $(\mathcal{P}_\phi)$:

Algorithm 5.1 (For solving $(\mathcal{P}_\phi)$.)

Step 1 Given an element $v \in \mathcal{U}_{ad}$, calculate the solution $y(v) \in K(\Gamma(v))$ of variational inequality (2.5) and define the set $Z(y(v))$.

Step 2 Calculate "gradient" $\nabla_v I_\phi$ of the cost functional I_ϕ (for a given $\phi \in M$) given by formula (4.23).

Step 3 Use a gradient (or a subgradient) method for calculation of next element $\bar{v} \in \mathcal{U}_{ad}$ such that

$$I_\phi(\bar{v}) < I_\phi(v) .$$

If not possible then STOP, ELSE GO TO Step 1.

5.2. Discretization

In numerical realization the state problem (2.5) and the adjoint state problem (4.23) are solved by the finite element method.

The unknown boundary $\Gamma(v)$ is replaced by a piecewise linear approximation. Let $0 \equiv a_0 < a_1 < \dots < a_{n(h)} \equiv 1$ be a partition of $[0,1]$, $h := a_i - a_{i-1}$ and let

$$\mathcal{U}_{ad}^h = \mathcal{U}_{ad} \cap \{v_h \in C([0,1]) \mid v_h|_{[a_{i-1},a_i]} \in P_1([a_{i-1},a_i])\},$$

where P_1 denotes a set of linear functions. For any $v_h \in \mathcal{U}_{ad}^h$ we define

$$\Omega(v_h) := \{x \in \mathbb{R}^2 \mid 0 < x_1 < v_h(x_2)\ ,\ x_2 \in]0,1[\},$$

$$\Gamma(v_h) := \{x \in \mathbb{R}^2 \mid x_1 = v_h(x_2)\ ,\ x_2 \in]0,1[\},$$

$$\Gamma_1(v_h) := \partial\Omega(v_h) \setminus \overline{\Gamma(v_h)}.$$

As usual we suppose that the triangulation $T_h(v_h)$ of $\Omega(v_h)$ is uniformly regular with respect to h, $v_h \in \mathcal{U}_{ad}^h$. We assume that the nodes of $\Gamma(v_h)$ are allowed to move only in x_1 -direction. Taking into account this parametrization, we find that the shape of $\Gamma(v_h)$ (and hence also of $\Omega(v_h)$) is uniquely determined by the x_1 - coordinates of the nodes $A_i = (v_h(a_i), a_i)$ defined on $\Gamma(v_h)$. Consequently, the design variables (control) variables are $\alpha_i := v_h(a_i)$, $i = 0,\dots,n(h)$. We define the "design vector" $\alpha = (\alpha_0,\dots,\alpha_{n(h)})$.

In the following numerical examples, the state problem and the state problem have been discretized by FEM with linear elements. Our control problem (P_ϕ) leads to a nonlinear programming problem for finding $\alpha \in \mathbb{R}^{n(h)+1}$. The function evaluation means the solving of non-linear algebraic equation and the computation of the gradient the solving of linear system of equations . For technical details we refer to [9,10].

5.3. Numerical examples

In this chapter we study the performance of a gradient algorithm when it is used for solving optimal design problem $(\mathcal{P}_\phi)$. In minimization we will utilize, instead of a naive Algorithm 5.1, the NPSOL routine of SOL (System Optimization Laboratory). It is based on sequential quadrative programming together with linearization of contraints. The nonlinear state problem is solved by S.O.R. with projection.

The numerical tests presented in what follows have been carried using single precision arithmetic . The authors are indebted to A. Kaarna and T. Tiihonen for their help in carrying out the tests.

In Examples 5.1 - 5.4 the initial quess has been chosen to be $\alpha_i = 1$ for $i = 0,\dots,n(h)$, i.e. $\Omega(v_h^0) = (0,1)x((0,1)$. The discretization parameter h is 1/8 or 1/16 . Corresponding triangulation containts 128 or 512 elements. The dimension of the optimization problem is 9 or 17.

The state problem and adjoint state problem contain 56 or 224 unknowns . We assume in Examples 5.1-5.4 that the constraint parameters in the set of admissible controls $\mathcal{U}_{ad}^h$ are $\alpha = 0.5$, $\beta = 1.5$, $C_1 = C_2 = 1$. The function $\phi \in H_0^1$ $(]0,2[\times]0,1[)$ used in all examples can be seen in Fig 5.1 ($\phi \equiv 1$ for $x_1 \in [0.5,1.5]$ and $x_2 \in [0.25,0.75]$).

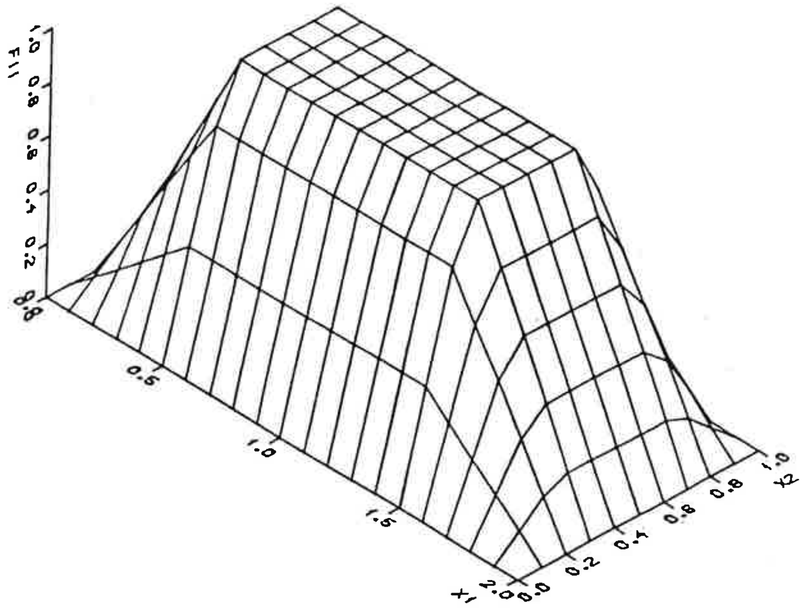


Fig 5.1 ϕ utilized in numerical tests.

Examples 5.1-5.4 differ only due to the right hand side in state, i.e. we consider the same problem for different f. In [9,10] similar examples have been analyzed but the costfunctional differs here. In [10] the costfunctionals considered have been

$$I^1(v) = \tfrac{1}{2} \int_{\Omega(v)} (y(v) - u_d)^2 \, dx \quad , \; u_d \in L^2(\Omega) \text{ given},$$

$$I^2(v) = \tfrac{1}{2} \int_{\Gamma(v)} (y(v) - z_d)^2 \, dx \quad , \; z_d \in L^2(\Gamma(v)) \text{ given},$$

$$I^3(v) = \tfrac{1}{2} \int_{\Omega(v)} (\nabla y(v))^2 \, dx \; ,$$

and

$$I^4(v) = \tfrac{1}{2} \left\| \frac{\partial}{\partial n} y(v) \right\|^2_{-1/2,\Gamma(v)}$$

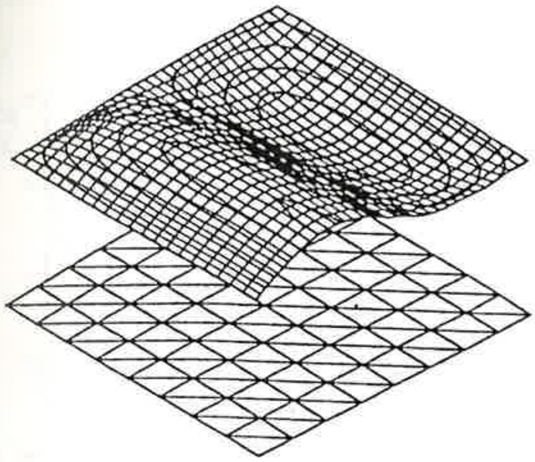
Our results can be compared with the last costfunctional. Here the results are somewhat better and the theoretical background differs from that of [6,10].

Table 5.2 contains the data for Examples 5.1-5.4 as well as value of I_ϕ for initial and final design. The number of iterations needed can be seen in Figures 5.4 - 5.7.

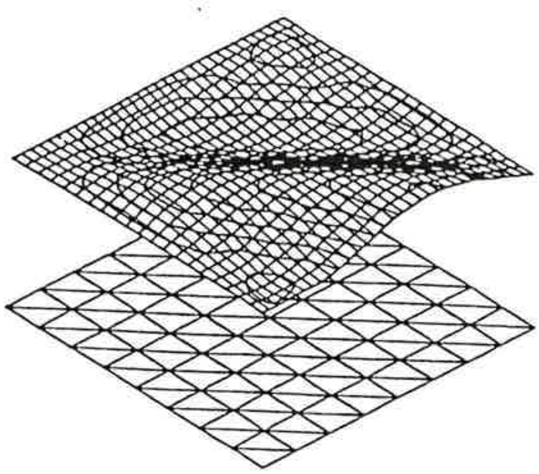
Example	force	Value of I_ϕ for initial design	Value of I_ϕ for final design
5.1	$4 \sin 2\pi x_2$	$0.618 \cdot 10^{-1}$	$0.216 \cdot 10^{-1}$
5.2	$4 \sin (2\pi(x_1 - x_2 + 1))$	$0.884 \cdot 10^{-2}$	$0.366 \cdot 10^{-8}$
5.3	$-8 \sin 2\pi x_1 \sin 2\pi x_2$	$0.629 \cdot 10^{-1}$	$0.148 \cdot 10^{-7}$
5.4	$8 \sin 4\pi x_1 \sin 4\pi x_2$	$0.179 \cdot 10^{-1}$	$0.158 \cdot 10^{-7}$

Table 5.2 Data and results for Examples 5.1-5.4.

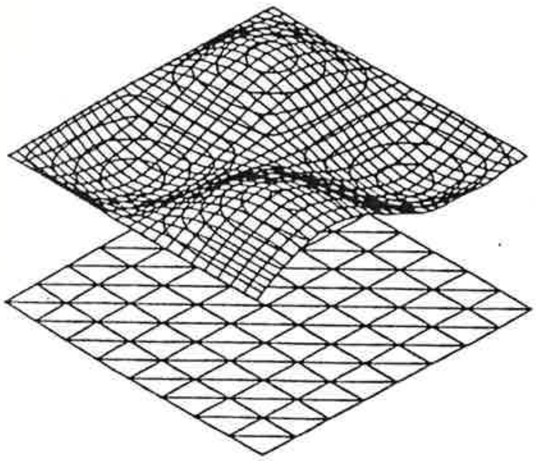
In Figures 5.3 a-d we see the initial triangulation of $\Omega(v^0)$ as well as spline-smoothed FE - solution of state problem of Examples 5.1-5.4.



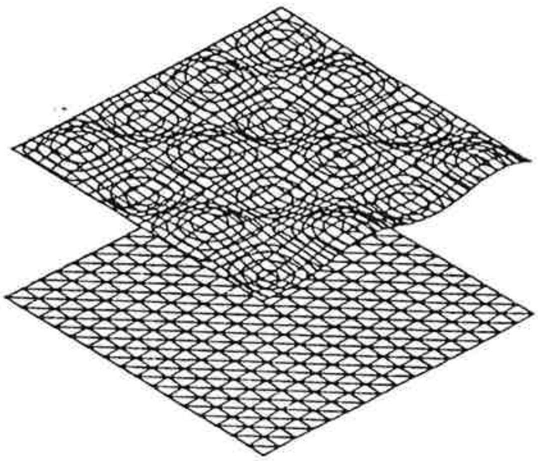
a) $f = 4\sin 2\pi x_2$



b) $f = \sin(2\pi(x_1 - x_2 + 1))$



c) $f = -8\sin 2\pi x_1 \sin 2\pi x_2$



d) $f = 8\sin 4\pi x_1 \sin 4\pi x_2$

Figure 5.3 Solutions of state problems for initial design $\Omega(v_h^0)$

In Figures 5.4-5.7 we see the numerical results for Examples 5.1-5.4: the diminution of I_ϕ versus iteration as well as spline-smoothed FE-solution of the state problem and the triangulation of "optimal" $\Omega(v_h)$ for last iteration. In Figures the value of I_ϕ is scaled by the factor 100.

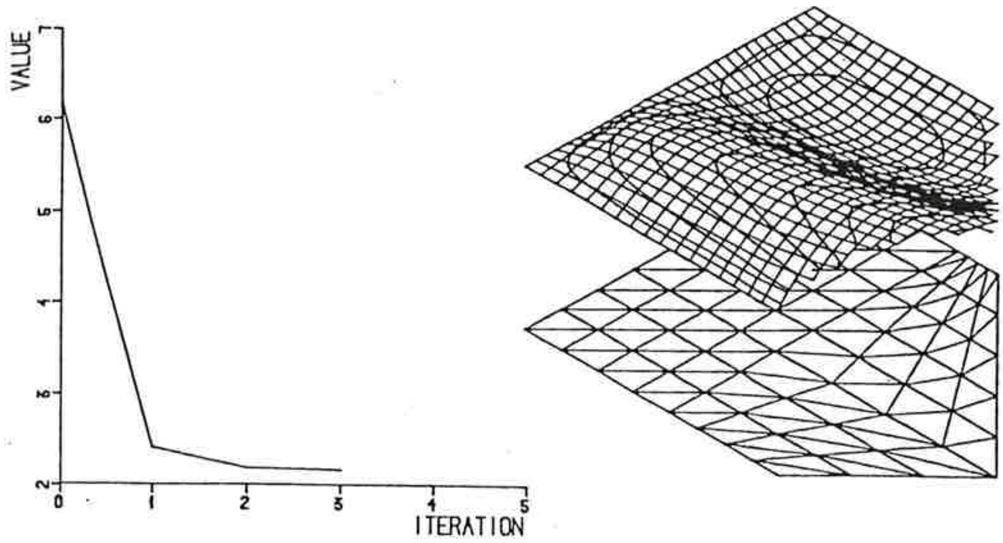


Fig 5.4 Numerical results for Ex. 5.1; $f = 4 \sin 2\pi x_2$

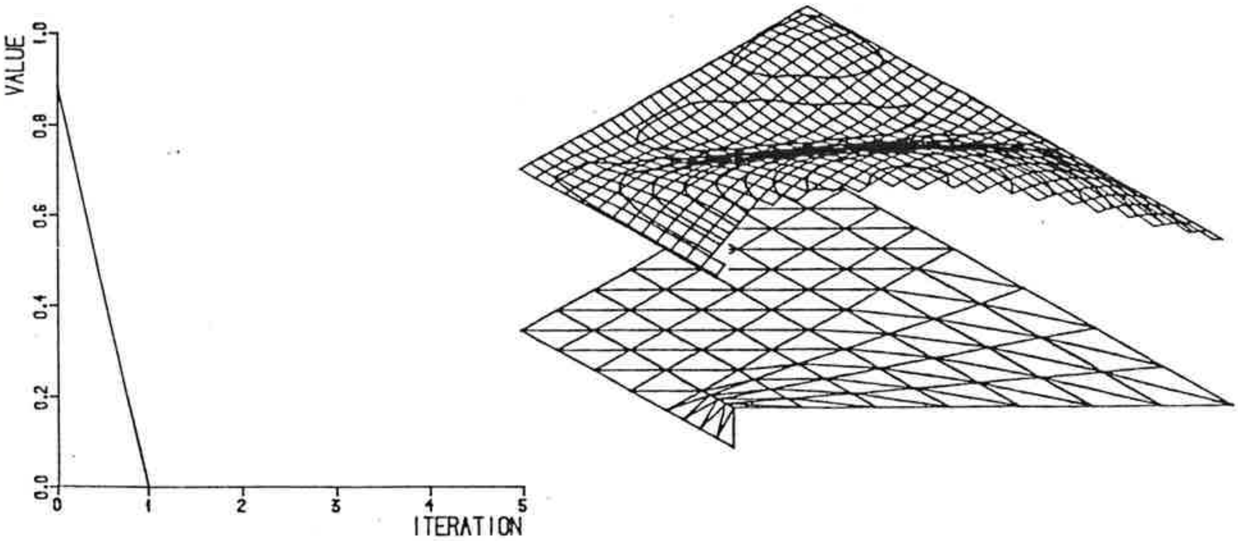


Fig 5.5 Numerical results for Ex. 5.2; $f = \sin (2\pi(x_1 - x_2 + 1))$.

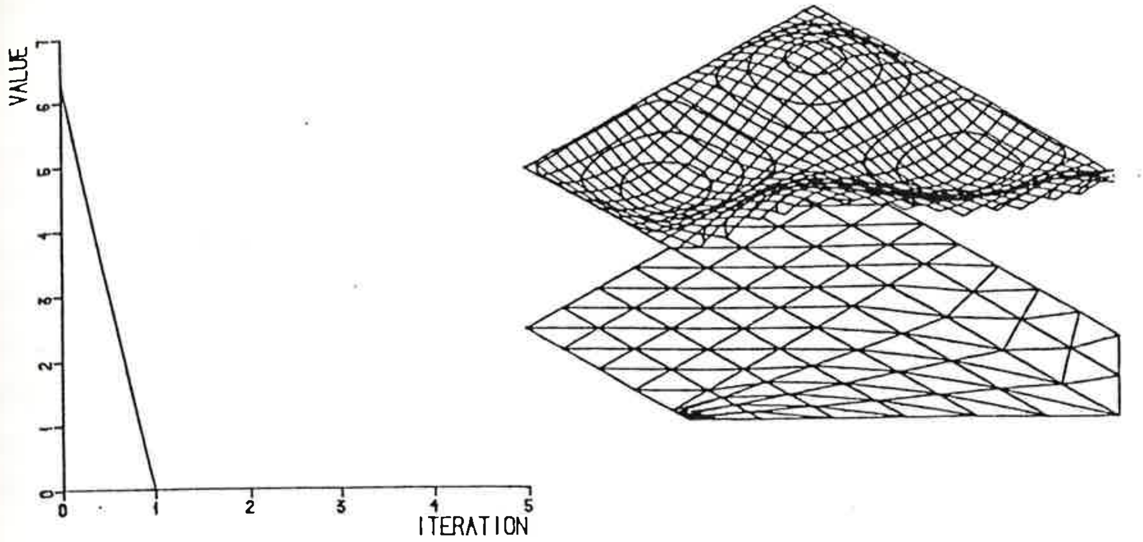


Fig 5.6 Numerical results for Ex. 5.3; $f = -8 \sin 2\pi x_1 \sin 2\pi x_2$.

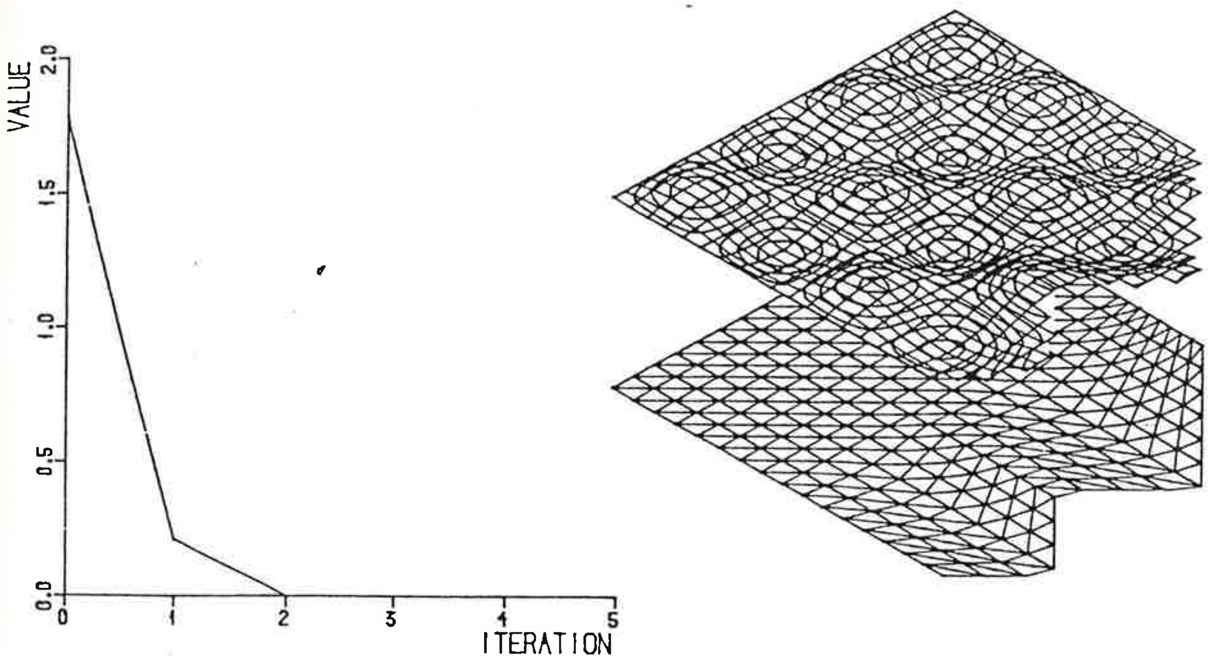


Fig 5.7 Numerical results for Ex. 5.4; $f = 8 \sin 4\pi x_1 \sin 4\pi x_2$.

From above given numerical results we find that the gradient algorithm could improve the design essentially in all cases. Only 1-3 iterations are needed to find a stationary point of I_ϕ.

In Examples 5.2-5.4 we could find for $\Gamma(v)$ such a design that $I_\phi \approx 0$. As we can see from Figures 5.5-5.7 $y(v) > 0$ (no contact) on $\Gamma(v)$ and the result $I_\phi \approx 0$ is then natural (in frame of numerical accuracy). Because of the nature of f in Example 5.1 it is not possible to find such $\Gamma(v)$ that $I_\phi = 0$. We could only enlarge the part of $\Gamma(v)$

where $y(v) > 0$. We note that in many cases the Lipschitz-constraint $|v'| < 1.0$ has become active.

As a conclusion of above considerations we find that the nonsmooth optimal shape control problems can be solved efficiently by using standard nonlinear programming packages. Before applying a gradient algorithm the nature and the source of the nonsmoothness must be analysed. As we have seen we were able to determine the gradient formula in spite of the fact that the mapping from the control to the state was only Lipschitz continuous. As the function $v \to I_\phi(v)$ is not convex the algorithm used guaranteed an achievement of a local minimum. However, in many practical problems it suffices to improve the performance of the system.

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