ON $O(h^4)$-SUPERCONVERGENCE OF PIECEWISE BILINEAR FE-APPROXIMATIONS

M. Křížek and P. Neittaanmäki

Preprint 75 November 1987

AMS Subject Classification (1985 Revision): 65 N 30

University of Jyväskylä, Jyväskylä 2017

978-951-39-7028-4 PDF

ISSN 0780-8569

ON $O(h^4)$-SUPERCONVERGENCE OF PIECEWISE BILINEAR FE-APPROXIMATIONS

M. KŘÍŽEK [1] AND P. NEITTAANMÄKI [2]

Abstract. This paper is supplementary to authors' survey paper [**44**] and to the paper [**43**]. We shall update the recent literature concerning the superconvergence of the finite element method. The second goal of this paper is to study the $O(h^4)$-superconvergence phenomenon obtained by combining the bilinear rectangular elements with the linear triangular elements. The connection between superconvergence of the FEM and higher order difference schemes is outlined. Results of numerical tests are presented.

1. INTRODUCTION

This paper is an extended version of our paper with the same title presented in the Second International Symposium on Numerical Analysis, Prague, August 1987.

The superconvergence of the finite element method is a quickly and dynamically developing field of research. Recently we have published a survey paper [44] on superconvergence techniques with 200 references. The recent literature to this subject is updated in this paper.

A big progress has been made especially in averaging techniques when recovering the gradient of a FE-solution of the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \quad (\Omega \subset \mathbf{R}^2 \text{ bounded}), \\ u &= 0 \quad \text{on } \partial\Omega \, , \end{aligned} \tag{1.1}$$

see e.g. [**2, 5, 19, 23, 25, 29, 33, 34, 43, 45, 46, 52, 66, 68**]. Some generalizations to the three-dimensional case can be found in [**14, 15, 40**]. For other superconvergence techniques for elliptic problems (including degenerated problems, nonlinear problems, etc.), we refer to [**6, 16, 17, 18, 21, 22, 32, 35, 36, 38, 47, 48, 49, 56, 60, 69, 70**]. For elliptic systems see [**39, 62, 63, 67**]. Superconvergence phenomena of FEM for two-point boundary value problems are treated in [**3, 7, 20, 28, 42, 55, 57, 58, 59**]. Further, for the Stokes problem see [**26**], for parabolic or hyperbolic equations see [**1, 4, 24, 30, 53, 54, 61**], and finally for integral equations we quote [**10, 11, 12, 13, 37, 65**].

The second goal of this paper is to continue the analysis of a superconvergence phenomenon which has been announced for the first time in [**41**], and which consists in combining the bilinear rectangular elements with the linear triangular elements. In Section 2, we introduce a finite element scheme which exhibits the global

---

Keywords: Superconvergence, post-processing of FE-scheme, combination of bilinear and linear elements.

1980 Mathematical Subject Classification. Primary 65 N 30.

[1] Czechoslovak Academy of Sciences, Mathematical Institute, Žitná 25, CS-11567 Prague 1, Czechoslovakia.

[2] University of Jyväskylä, Department of Mathematics, Seminaarinkatu 15, SF-40100 Jyväskylä 10, Finland.

$O(h^4)$-superconvergence at nodes employing special continuous and piecewise bilinear trial functions. We also present a post-processing which yields the local $O(h^4)$-superconvergence at nodes. In the both cases, a reduced integration is used. We shall see a close connection between superconvergence of the FEM and convergence of higher order difference schemes (see also [**41**, Rem. 1]). Section 3 is devoted to numerical tests.

Throughout the paper, $\|\cdot\|$ will be the Euclidean norm. The notations $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ are used for the standard norm and seminorm in the Sobolev space $W_p^k(\Omega)$, respectively. The spaces $L^2(\Omega)$ and $(L^2(\Omega))^2$ are equipped with the scalar product $(\cdot,\cdot)_{0,\Omega}$. The symbol $P_k(\Omega)$ stands for the space of polynomials of the degree k.

2. Nodal superconvergence arising from a combination of the linear and bilinear elements

Let M be a square mesh in the (x_1, x_2)-plane and let Ω be a bounded domain with a Lipschitz boundary $\partial\Omega \subset M$. Consider square refinements of M with the mesh size h and denote by $Z_h = \{z^i\}_{i=1}^n$ $(n = n(h))$ the set of their nodes lying in Ω. Let $\{b^i\}_{i=1}^n$ be the standard basis of piecewise bilinear functions over such a mesh (see Fig. 1), and denote by $\{p^i\}_{i=1}^n$ and $\{q^i\}_{i=1}^n$ the piecewise linear Courant basis functions over the mesh of Fig. 2 and 3, respectively; i.e. we assume that

$$b^i(z^j) = p^i(z^j) = q^i(z^j) = \delta_{ij}\ , \quad i,j = 1,\ldots,n\ . \tag{2.1}$$

Further, let us define

$$v^i = \frac{1}{2}b^i + \frac{1}{4}p^i + \frac{1}{4}q^i\ , \quad i = 1,\ldots,n, \tag{2.2}$$

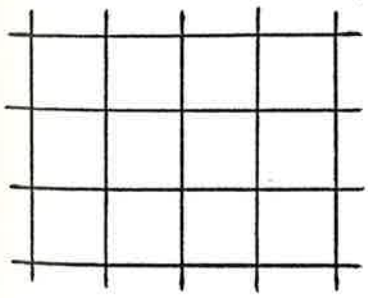

Figure 1

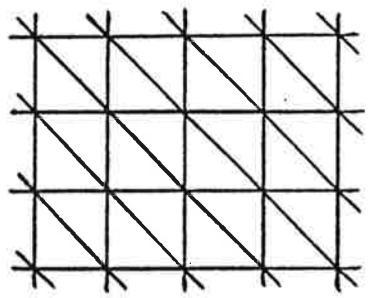

Figure 2

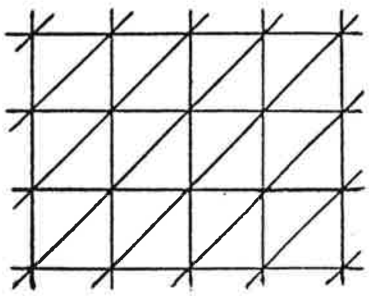

Figure 3

and let V_h be the linear hull of $\{v^i\}_{i=1}^n$. We shall look for $u_h \in V_h$, the values $u^j = u_h(z^j)$ of which satisfy

$$\sum_{j=1}^{n}\left(\frac{1}{2}\left(\nabla b^i, \nabla b^j\right)_{0,\Omega} + \frac{1}{4}\left(\nabla p^i, \nabla p^j\right)_{0,\Omega} + \frac{1}{4}\left(\nabla q^i, \nabla q^j\right)_{0,\Omega}\right) u^j = \left(f, v^i\right)_{0,\Omega} \tag{2.3}$$

for $i = 1,\ldots,n$. The corresponding matrix is thus a *weighted average* of stiffness matrices for the bases $\{b^i\}$, $\{p^i\}$, $\{q^i\}$. The width of the band of such a matrix will be clearly the same as for the bilinear elements.

As $(f, v^i)_{0,\Omega}$ in (2.3) cannot be computed exactly, in general, we employ the following integration formula which is exact (see [**51, 71**]) for all $g \in P_5(K)$,

$$(2.4) \quad \int_K g\,dx \approx \operatorname{meas} K(c_1(g(A)+g(B)+g(C))+c_2(g(D)+g(E)+g(F))+c_3 g(G)) .$$

Here K is an arbitrary triangle, $c_1 = (155 - \sqrt{15})/1200$, $c_2 = (155 + \sqrt{15})/1200$, $c_3 = 9/40$, the triangle coordinates of A, B, C, D, E, F, G are (a_1, b_1, b_1), (b_1, a_1, b_1), (b_1, b_1, a_1), (a_2, b_2, b_2), (b_2, a_2, b_2), (b_2, b_2, a_2), $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, respectively, and $a_1 = (9 + 2\sqrt{15})/21$, $a_2 = (9 - 2\sqrt{15})/21$, $b_1 = (6 - \sqrt{15})/21$, $b_2 = (6 + \sqrt{15})/21$ (see Fig. 4).

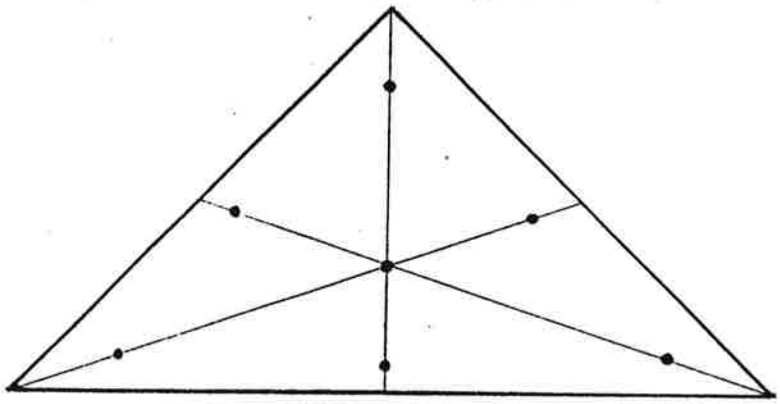

Figure 4

Applying (2.4) to each triangle of $\operatorname{supp} v^i$ where v^i is bilinear, we define the approximation $(f, v^i)^*_{0,\Omega}$ of $(f, v^i)_{0,\Omega}$. Further, let $\overset{*}{u}_h \in V_h$ ($\overset{*}{u}^j = \overset{*}{u}_h(z^j)$) be defined by (2.3), where the right-hand side is replaced by $(f, v^i)^*_{0,\Omega}$.

Before we prove that $|u - \overset{*}{u}_h| = O(h^4)$ at nodal points, we introduce two auxiliary lemmas.

LEMMA 2.1. *For any $f \in W^4_\infty(\Omega)$ and sufficiently small h it is*

$$\left| (f, v^i)_{0,\Omega} - h^2 f(z^i) - \frac{h^4}{12}(\Delta f)(z^i) \right| \le C h^6 \, |f|_{4,\infty,S^i} \ ,$$

where $S^i = \operatorname{supp} v^i$, $i \in \{1, \dots, n\}$.

PROOF: Let $\hat{S} = [-1,1] \times [-1,1]$ be the reference square. For $\hat{f} \in W^4_\infty(\hat{S})$ we define the linear functional

$$(2.5) \qquad L(\hat{f}) = (\hat{f}, \hat{v})_{0,\hat{S}} - \hat{f}(0) - \frac{1}{12}(\Delta \hat{f})(0) \ ,$$

where $\hat{v}$ is a continuous piecewise bilinear function ($\hat{v}(0) = 1$, $\hat{v}|_{\partial \hat{S}} = 0$) defined as in (2.2), i.e.

$$(2.6) \qquad \hat{v} = \frac{1}{2}\hat{b} + \frac{1}{4}\hat{p} + \frac{1}{4}\hat{q} \ .$$

We derive now that

$$(2.7) \qquad L(\hat{f}) = 0 \qquad \forall \hat{f} \in P_3(\hat{S}) \,,$$

showing that (2.7) holds for any basis function $\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2, \dots, \hat{x}_2^3\} \subset P_3(\hat{S})$. From (2.6),

$$L(1) = (1, \hat{v})_{0,\hat{S}} - 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} - 1 = 0 \,,$$

since $(1, \hat{b})_{0,\hat{S}} = (1, \hat{p})_{0,\hat{S}} = (1, \hat{q})_{0,\hat{S}} = 1$. Analogously, as $\hat{v}(\hat{x}_1, \hat{x}_2) = \hat{v}(\hat{x}_2, \hat{x}_1)$ we obtain by (2.6) that

$$L(\hat{x}_1^2) = L(\hat{x}_2^2) = \frac{1}{2}(\hat{x}_2^2, \hat{b})_{0,\hat{S}} + \frac{1}{4}\left(\hat{x}_2^2, \hat{p}\right)_{0,\hat{S}} + \frac{1}{4}\left(\hat{x}_2^2, \hat{q}\right)_{0,\hat{S}} - \frac{2}{12}$$
$$= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} - \frac{1}{6} = 0 \,.$$

Finally, for any $\hat{f} \in \{\hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2, \hat{x}_1^3, \hat{x}_1^2\hat{x}_2, \hat{x}_1\hat{x}_2^2, \hat{x}_2^3\}$ it is obviously

$$\hat{f}(0) = (\Delta \hat{f})(0) = 0 \,,$$

and $(\hat{f}, \hat{v})_{0,\hat{S}} = 0$, since $\hat{v}$ is an even function and $\hat{f}$ is an odd one with respect to the axis $\hat{x}_1 = 0$ or $\hat{x}_2 = 0$. Hence, (2.7) is valid.

Moreover,

$$(2.8) \qquad |L(f)| \le \|\hat{f}\|_{0,2,\hat{S}} \|\hat{v}\|_{0,2,\hat{S}} + C\|\hat{f}\|_{0,\infty,\hat{S}} + \frac{C}{12}\|\Delta \hat{f}\|_{0,\infty,\hat{S}} \le C'\|f\|_{4,\infty,\hat{S}} \,,$$

i.e. the functional L is continuous. Therefore, from (2.7), (2.8) and the Bramble-Hilbert lemma (see [**27**, Th. 4.1.3]) we come to

$$(2.9) \qquad |L(\hat{f})| \le C'' \, |f|_{4,\infty,\hat{S}} \,.$$

Further, let $z^i \in Z_h$ be a given node and let us consider the one-to-one mapping

$$(2.10) \qquad F^i(\hat{x}) = h\hat{x} + z^i$$

between $\hat{S}$ and S^i. Then by [**27**, Th. 3.1.2] we get

$$(2.11) \qquad |\hat{f}|_{4,\infty,\hat{S}} \le C h^4 |f|_{4,\infty,S^i} \,.$$

Now the combination of (2.5), (2.10), (2.9) and (2.11) yields

$$\left| \frac{1}{h^2}\left(f, v^i\right)_{0,S^i} - f(z^i) - \frac{h^2}{12}(\Delta f)(z^i) \right| \le C h^4 \, |f|_{4,\infty,S^i} \,.$$

□

LEMMA 2.2. *For any $f \in W^4_\infty(\Omega)$ and for h sufficiently small it is*

$$|(f,v^i)_{0,\Omega} - (f,v^i)^*_{0,\Omega}| \le Ch^6 |f|_{4,\infty,S^i} \ .$$

PROOF: The support $\hat{S}$ can be decomposed into 16 triangles $\hat{K} = \hat{K}^j$, $j = 1,\dots,16$, where the reference basis function (2.6) is bilinear. For a fixed j and $\hat{f} \in W^4_\infty(\hat{K})$ we define a linear functional

$$\ell(\hat{f}) = (\hat{f},\hat{v})_{0,\hat{K}} - (\hat{f},\hat{v})^*_{0,\hat{K}} \ .$$

We see that ℓ is continuous and that $\ell(\hat{f}) = 0$ for all $\hat{f} \in P_3(\hat{K})$, since $\hat{f}\hat{v}$ is a quintic polynomial. Therefore, by the Bramble-Hilbert lemma we again get (cf. (2.9))

$$|\ell(\hat{f})| \le C''|\hat{f}|_{4,\infty,\hat{K}} \ .$$

Summing this over all $\hat{K} \subset \hat{S}$ and using (2.10) and (2.11), we find that

$$\left|\frac{1}{h^2}(f,v^i)_{0,\Omega} - \frac{1}{h^2}(f,v^i)^*_{0,\Omega}\right| \le Ch^4|f|_{4,\infty,S^i} \ .$$

□

Let us recall the well-known 9-point finite difference scheme [**31, 50**] for the problem (1.1)

$$\frac{20U^5 - 4(U^2+U^4+U^6+U^8) - U^1 - U^3 - U^7 - U^9}{6} = h^2 f(z^i) + \frac{h^4}{12}\Delta f(z^i) \ , \qquad (2.12)$$

where $U^1, U^2, \dots, U^5, \dots, U^9$ correspond to the points $(z^i_1 - h, z^i_2 + h)$, $(z^i_1, z^i_2 + h)$, $\dots$, (z^i_1, z^i_2), $\dots$, $(z^i_1 + h, z^i_2 - h)$; if the k-th point lies on $\partial\Omega$ we set $U^k = 0$. Denoting by $U_h(z^i)$ the unknown at the nodal point $z^i \in Z_h$ and using the Gerschgorin trick for the above scheme (2.12), Bramble and Hubbard [**8, 9**] have proved that

$$|u - U_h|_h \equiv \max_{z\in Z_h} |u(z) - U_h(z)| \le Ch^4 \|u\|_{6,\infty,\Omega} \qquad \text{as } h \to 0 \ . \qquad (2.13)$$

This result and the foregoing lemmas are used to prove the $O(h^4)$-superconvergence at the nodes of the finite element scheme (2.3) using the numerical integration (2.4). Note that the optimal rate of convergence of the norm $\|u - \overset{*}{u}_h\|_{0,\infty,\Omega}$ cannot be better than $O(h^2)$, since $\overset{*}{u}_h$ is piecewise bilinear.

THEOREM 2.3. *Let the solution u of (1.1) belong to $W^6_\infty(\Omega)$. Then for sufficiently small h*

$$|u - \overset{*}{u}_h|_h \le Ch^4 \|u\|_{6,\infty,\Omega} \ . \qquad (2.14)$$

PROOF: Making use of (2.1), we easily find that

$$\begin{aligned}
&(\nabla b^i, \nabla b^i)_{0,\Omega} = \frac{8}{3} && \forall i\ ,\\
&(\nabla b^i, \nabla b^j)_{0,\Omega} = -\frac{1}{3} && \text{if } h \le \|z^i - z^j\| < 2h\ ,\\
&(\nabla p^i, \nabla p^i)_{0,\Omega} = (\nabla q^i, \nabla q^i)_{0,\Omega} = 4 && \forall i\ ,\\
&(\nabla p^i, \nabla p^j)_{0,\Omega} = (\nabla q^i, \nabla q^j)_{0,\Omega} = -1 && \text{if } \|z^i - z^j\| = h\ ,
\end{aligned}$$

and the other scalar products are zero. Hence, the matrices corresponding to the schemes (2.3) and (2.12) are the same, and we denote them by A_h. The difference between the right-hand sides of (2.3) and (2.12) has been established by Lemma 2.1. Moreover, by [**8, 9**]

$$\|A_h^{-1}\|_h \equiv \max_i \sum_j |A_{ij}^{-1}| \le Ch^{-2}\ , \tag{2.15}$$

where the matrix $\|\cdot\|_h$-norm is assosiated with the ℓ^∞-vector norm in (2.13). Thus according to (2.13), Lemmas 2.1, 2.2, (2.15), and (1.1), it is

$$\begin{aligned}
|u - \overset{*}{u}_h|_h &\le |u - U_h|_h + |u_h - U_h|_h + |u_h - \overset{*}{u}_h|_h\\
&\le Ch^4\|u\|_{6,\infty,\Omega} + C'h^6\|A_h^{-1}\|_h\ \|f\|_{4,\infty,\Omega}\\
&\le C''h^4\|u\|_{6,\infty,\Omega}\ .
\end{aligned}$$

□

REMARK 2.4. Employing the post-processing

$$(\widetilde{\nabla \overset{*}{u}_h(z)})_1 = \frac{1}{12h}(-\overset{*}{u}_h(z_1+2h,z_2) + 8\overset{*}{u}_h(z_1+h,z_2) - 8\overset{*}{u}_h(z_1-h,z_2) + \overset{*}{u}_h(z_1-2h,z_2))\ , \tag{2.16}$$

(the second component $(\widetilde{\nabla \overset{*}{u}_h(z)})_2$ is defined similarly) we can obtain by Theorem 2.3 even the interior $O(h^4)$-superconvergence of the gradient at nodes, i.e.

$$e_h \equiv \max_{z \in Z_h \cap \Omega_0} \|\nabla u(z) - \widetilde{\nabla \overset{*}{u}_h(z)}\| \le Ch^4(\|u\|_{7,\infty,\Omega_1} + \|u\|_{6,\infty,\Omega})\ , \tag{2.17}$$

where $\overline{\Omega}_0 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$. The proof of (2.17) is the same as in [**43**, Th. 4.1]. Note that mostly only the $O(h^2)$-superconvergence of the gradient has been obtained when applying the linear or bilinear elements (see the survey paper [44]).

3. NUMERICAL TESTS WITH SUPERCONVERGENCE

In the following simple examples we present numerical results for the technique described in Section 2.

EXAMPLE 3.1. Let $\Omega = (0,1) \times (0,1)$ and let f in (1.1) be chosen so that

$$u(x_1, x_2) = (x_1^3 - x_1) \sin \pi x_2$$

is the exact solution. The next table shows the global maximum errors at nodes for the standard Ritz-Galerkin method using bases $\{p^i\}$ and $\{b^i\}$, and for the scheme (2.3) with the basis $\{v^i\}$ and the numerical integration (2.4). It also illustrates the behaviour of the local error e_h given by (2.17) for $\Omega_0 = \left(\frac{1}{4}, \frac{3}{4}\right) \times \left(\frac{1}{4}, \frac{3}{4}\right)$.

h^{-1}	$\{p^i\}$	$\{b^i\}$	$\{v^i\}$	e_h
4	1.810E−2	1.944E−2	2.525E−4	—
8	4.988E−3	5.089E−3	1.658E−5	6.953E−4
16	1.262E−3	1.263E−3	1.031E−6	4.437E−5
32	3.170E−4	3.153E−4	6.437E−8	2.784E−6

EXAMPLE 3.2. Let $\Omega = (0,1) \times (0,1)$ and let f in (1.1) be chosen so that

$$u(x_1, x_2) = (x_1^{2.5} - x_1^{3.5}) \sin \pi x_2$$

is the exact solution. Analogously to the previous example we get:

h^{-1}	$\{p^i\}$	$\{b^i\}$	$\{v^i\}$	e_h
4	7.456E−3	8.490E−3	1.255E−4	—
8	2.130E−3	2.016E−3	9.075E−6	8.184E−4
16	5.392E−4	4.980E−4	7.210E−7	4.134E−5
32	1.352E−4	1.241E−4	6.077E−8	2.544E−6

In the last but one column we roughly have only the $O(h^{3.5})$-superconvergence, since $u \in W^2_\infty(\Omega) \setminus W^3_\infty(\Omega)$. However, this is still much more better than the $O(h^2)$-convergence from the two previous columns.

EXAMPLE 3.3. In this example we compare the superconvergence result of the previous section with superconvergence phenomena for quadratic elements. Let again $\Omega = (0,1) \times (0,1)$,

$$u(x_1, x_2) = x_2(x_2 - 1) \sin \pi x_1 \ ,$$

and let us denote by $\overset{*}{u}{}^h$ the standard Ritz-Galerkin approximation of u based on the triangular quadratic elements over the mesh of Fig. 2 (including the numerical integration (2.4)). Note that

$$\|u - \overset{*}{u}{}^h\|_{0,\infty,\Omega} = O(h^3)$$

is the best global rate. However, it is known (see e.g. the survey [44]) that the use of the quadratic elements yields the $O(h^4)$-superconvergence at vertices on uniform triangulations (i.e. when any two adjacent triangles form a parallelogram). The same phenomenon can be observed at the midpoints of sides as follows from the next table:

h^{-1}	$\max_{z\in Z_h} \lvert u(z) - \overset{*}{u}_h(z)\rvert$	$\max_{z\in Z_h} \lvert u(z) - \overset{*}{u}^h(z)\rvert$	$\max_{z\in M_h} \lvert u(z) - \overset{*}{u}^h(z)\rvert$
4	1.683E−4	6.580E−4	5.263E−4
8	1.031E−5	4.206E−5	3.593E−5
16	6.410E−7	2.646E−6	2.325E−6
32	4.001E−8	1.658E−7	1.468E−7

Here Z_h and M_h are the sets of vertices and midpoints, respectively.

References

1. A.B. Andreev, *Superconvergence of the finite element approximations of some parabolic equations. Constructive Theory of Functions*, (Proc. Conf., Varna, 1984).
2. A.B. Andreev and R.D. Lazarov, *Superconvergence of the gradient for quadratic triangular finite elements*, (to appear).
3. O.Axelsson and V.A. Barker, "Finite Element Solution of Boundary Value Problems," Academic Press, New York, London, 1984.
4. J.W. Barrett and Ch.M. Elliott, *Total flux estimates for a finite-element approximation of parabolic equations*, IMA J. Numer. Anal. **6** (1986), 253–264.
5. J.W. Barrett and Ch.M. Elliott, *Total flux estimates for a finite-element approximation of elliptic equations*, IMA J. Numer. Anal. **7** (1987), 129–148.
6. P.G. Binev, "Convergence and superconvergence in Hermite spline interpolation. Numerical Methods and Applications," (Proc. Int. Conf., Sofia, 1984), Izd. Bulg. Akad. Nauk, Sofia, 1985, pp. 179–184.
7. C. de Boor, "A Practical Guide to Splines," Applied Math. Sciences 27, Springer-Verlag, New York, Heidelberg, Berlin, 1978.
8. J.H. Bramble and B.E. Hubbard, *On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation*, Numer. Math. **4** (1962), 313–327.
9. J.H. Bramble and B.E. Hubbard, *A theorem on error estimation for finite difference analogues of the Dirichlet problem for elliptic equations*, Contributions to Differential Equations **2** (1963), 319–340.
10. H. Brunner, *High-order methods for the numerical solution of Volterra integro-differential equations*, J. Comput. Appl. Math. **15** (1986), 301–310.
11. H. Brunner and P.J. van der Houwen, "The Numerical Solution of Volterra Equations," North-Holland, Amsterdam, 1985.
12. G.A. Chandler, *Mesh grading for boundary integral equations*, in "Computational Techniques and Applications," (CTAC–83, Sydney, 1983), North-Holland, Amsterdam, New York, 1984, pp. 289–296. MR 86j:65149.
13. G.A. Chandler, *Superconvergent approximations to the solution of a boundary integral equation on polygonal domains*, SIAM J. Numer. Anal. **23**, 1214–1229.
14. C.M. Chen, *Optimal points of stresses for tetrahedron linear element*, Natural Science J. of Xiangtan Univ. **3** (1980), 16–24.
15. C.M. Chen, "Finite Element Method and its Analysis in Improving Accuracy," Hunan Sci. and Tech. Press, Changsha, 1982.
16. C.M. Chen, *Convergence and superconvergence of finite element method for axisymmetric problems*, Hunan Annals of Math. **3** (1983), 81–88.
17. C.M. Chen, *An estimate for elliptic boundary value problems and its application to finite element methods*, Numer. Math. J. Chinesse Univ. **5** (1983), 215–233. MR 86b:65120.
18. C.M. Chen, *Superconvergence of finite element methods*, Advances in Math. **14** (1985), 39–51.

19. C.M. Chen, *$W^{1,\infty}$-interior estimates for finite element method on regular mesh*, J. Comp. Math. **3** (1985), 1–7.
20. C.M. Chen, *Superconvergence of Galerkin solution for nonlinear singular two-point boundary value problem*, Comp. Math. **2** (1985), 113–123.
21. C.M. Chen, *The optimal points of stresses of triangular linear elements for quasilinear problems*, (to appear).
22. C.M. Chen, *Superconvergence and extrapolation of the finite element approximations to quasilinear elliptic problems*, (to appear).
23. C.M. Chen and J.Q. Liu, *Superconvergence of the gradient of triangular linear element in general domain*, Preprint Xiangtan Univ. (1985), 1–19.
24. C.M. Chen and V. Thomée, *The lumped mass finite element method for a parabolic problem*, J. Austral. Math. Soc. Ser. B **26** (1985), 329–354. MR86m:65117.
25. C.M. Chen and Q.D. Zhu, *A new estimate for finite element method and optimal point theorem for stresses*, Natural Science J. of Xiangtan Univ. **1** (1978), 10–20.
26. S.J. Cheng, *Superconvergence of finite element approximation for Navier-Stokes equation*, Proc. Conf., Bonn, 1983, Math. Schrift. No. 158, Bonn (1984), 31–45.
27. P.G. Ciarlet, "The Finite Element Method for Elliptic Problems," North-Holland, Amsterdam, New York, Oxford, 1978.
28. J.F. Ciavaldini and M. Crouzeix, *A finite element method scheme for one dimensional elliptic equations with high super convergence at the nodes*, Numer. Math. **46** (1985), 417–427. MR86g:65140.
29. R.Z. Dautov, *Accuracy estimation of schemes of the finite element method in mesh norms*, in "Variational-Difference methods in mathematical physics," (Proc. Conf., Moscow, 1984), Viniti, Moscow, 1985, pp. 92–97.
30. J.J. Douglas, T. Dupont and M.F. Wheeler, *Some superconvergence results for an H^1-Galerkin procedure for the heat equation*, in "Comp. Methods in Appl. Sci. and Engrg.," Part 1 (Proc. Sympos., Versailles, 1973), Springer-Verlag, Berlin, New York, 1974, pp. 288–311.
31. G. Forsythe and W. Wasow, "Finite-Difference Methods for Partial Differential Equations," Wiley, New York, 1960.
32. I. Fried, *On the optimality of the pointwise accurary of the finite element solution*, Internat. J. Numer. Methods Engrg. **15** (1980), 451–456.
33. G. Goodsell, *Gradient superconvergence properties for finite element approximations to two-dimensional Poisson problems*, MSc Dissertation, Brunel Univ. (1985).
34. G. Goodsell and J.R. Whiteman, *A unified treatment of superconvergent recovered gradient functions for piecewise linear finite element approximations*, IMA J. Numer. Anal. **7** (1987).
35. M.El Hatri, *Estimation de l'erreur de la méthode des éléments finis avec integration numérique pour un probleme aux limites dégénéré*, Ph. D. thesis, Univ. of Sofia.
36. M. El Hatri, *Estimation d'erreur optimale et de type superconvergence de la méthode des éléments finis pour un problème aux limites dégénéré.*, RAIRO Modél. Math. Anal. Numér. **21** (1987), 27–61.
37. F.K. Hebeker, J. Mika and D.C. Pack, *Application of the superconvergence properties of the Galerkin approximation to the calculation of upper and lower bounds for linear functionals of solutions of integral equations*, IMA J. Appl. Math. **38** (1987), 61–70.
38. B. Heinrich, *Integrabilanzmethode für elliptische Probleme II. Fehlerabschätzung und Konvergenz.*, Beiträge Numer. Math. **12** (1984), 75–94. MR 85h:65215.
39. I. Hlaváček and M. Křížek, *On a superconvergent finite element scheme for elliptic systems*, in "Software and Algorithms of Numerical Analysis," (Proc. Conf., Doksy, 1985), VŠSE Plzeň, 1986, pp. 5–14.
40. V. Kantchev and R.D. Lazarov, *Superconvergence of the gradient of linear finite elements for 3D Poisson equation*, in "Optimal Algorithms," (Proc. Conf., Blagoevgrad, 1986), Izd. Bulg. Akad. Nauk, Sofia, 1986, pp. 172–182.
41. M. Křížek, *Superconvergence results for linear triangular elements*, in "Proc. Conf. EQUADIFF VI, Brno, 1985," Springer-Verlag, Berlin, Heidelberg, 1986, pp. 315–320.
42. M. Křížek and P. Neittaanmäki, *A phenomenon arising from the use of harmonic and biharmonic elements.*, Research Report 5/1984, Lappeenranta Univ. of Technology (1984), 1–11.

43. M. Křížek and P. Neittaanmäki, *Postprocessing of a finite element scheme with linear elements*, in "Numerical Techniques in Continuum Mechanics," (Proc. of the Second GAMM Seminar, Kiel, 1986) W. Hackbusch, K. Witsch (Eds.), Friedr. Vieweg & Sohn, Braunschweig, 1987, pp. 69–83.

44. M. Křížek and P. Neittaanmäki, *On superconvergence techniques*, Acta Appl. Math. 9 (1987), 175–198.

45. N. Levine, *Pointwise logarithm-free error estimates for finite elements on linear triangles*, Numer. Anal. Report No. 6/84, Univ. of Reading (1984).

46. N. Levine, *Superconvergent estimation of the gradient from linear finite element approximations on triangular elements*, Numer. Anal. Report 3/85, Univ. of Reading (1985), 1–202.

47. Q. Lin, *High accuracy from the linear elements*, in "Proc. Fifth Sympos. on Differential Geometry and Differential Equations, Beijing, 1984," pp. 1–5.

48. Q. Lin and R. Xie, *Some advances in the study of error expansion for finite elements*, J. Comput. Math. **4** (1986), 368–382.

49. Q. Lin and R. Xie, *Error expansions for finite element approximation and its applications*, Preprint of Inst. of Systems Science, Beijing (1987), 1–15.

50. W.E. Milne, "Numerical Solution of Differential Equations," John Wiley, New York, London, 1953.

51. T. Moan, *Experiences with orthogonal polynomials and "best" numerical integration formulas on a triangle; with particular reference to finite element approximations*, ZAMM **54** (1971), 501–508.

52. M. Nakao, *Superconvergence of the gradient of Galerkin approximations for elliptic problems*, in "Proc. Congress on Computational and Applied Mathematics, 1986," North-Holland, 1987 (to appear).

53. M. Nakao, *Error estimates of a Galerkin method for some nonlinear Sobolev equations in one space dimension*, Numer. Math. **47** (1985), 139–157. MR 86j:65134.

54. M. Nakao, *Some superconvergence of Galerkin approximations for parabolic and hyperbolic problems in one space dimension*, Bull. Kyushu Inst. Tech. (Math. Natur. Sci.) **32** (1985), 1–14.

55. M. Nakao, *Some superconvergence for a Galerkin method by averaging gradients in one dimensional problems*, J. Information Processing **9** (1986), 130–134.

56. M. Nakata, A. Weiser and M.F. Wheeler, *Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains*, in "Proc. Conf. MAFELAP 1984, Brunel Univ.," Academic Press, INC., London, New York, 1985, pp. 367–389.

57. E. O'Riordan and M. Stynes, *An analysis of a superconvergence result for a singularly perturbed boundary value problem*, Math. Comp. **46** (1986), 81–92.

58. T. Regińska, *Superconvergence of external approximation for two-point boundary problems*, Apl. Mat. **32** (1987), 25–36.

59. T. Regińska, *Error estimates for external approximation of ordinary differential equations and superconvergence property*, (to appear in Apl. Mat.).

60. A.H. Schatz, *An introduction to the analysis of the error in the finite element method for second-order elliptic boundary value problems*, in "Numerical Analysis, Lancaster, 1984," LN in Math. 1129, Springer-Verlag, Berlin, New York, 1985, pp. 94–139. MR86k:65114.

61. M.C. Squeff, *Superconvergence of mixed finite elements for parabolic equations*, Thesis, Univ. of Chicago (1985).

62. R. Stenberg, *A family of mixed finite elements for the elasticity problem*, Report MAT-A243, Helsinki Univ. of Technology (1986), 1–37.

63. R. Stenberg, *On the postprocessing of mixed equilibrium finite element methods*, in "Efficient Numerical Methods for Problems in Continuum Mechanics," Vieweg Verlag, 1986 (to appear).

64. V. Thomée, J.C. Xu and N.Y. Zhang, *Superconvergence of the gradient in piecewise linear finite element approximation to a parabolic problem*, Chalmers Univ., Dept. Math., Preprint 1987–07.

65. G. Vainikko and P. Uba, *A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel*, J. Austral. Math. Soc. Ser. B **22** (1981), 431–438. MR 82h:65100.

66. M.F. Wheeler and J.R. Whiteman, *Superconvergent recovery of gradients on subdomains from piecewise linear finite element approximations*, Numer. Methods Partial Differential Equations **3** (1987), 65–82.
67. J.R. Whiteman and G. Goodsell, *Superconvergent recovery for stresses from finite element approximations on subdomains for planar problems of linear elasticity*, in "Proc. Conf. MAFELAP VI, 1987," J.R. Whiteman (Ed.), Academic Press, London, 1988 (to appear).
68. Q.D.Zhu, *The derivative good points for the finite element method with 2–degree triangular element*, Natural Science J. of Xiangtan Univ. **4** (1981), 36–45.
69. Q.D. Zhu, *Uniform superconvergence estimates for the finite element method*, (to appear).
70. Q.D. Zhu and Q. Lin, "The theory of superconvergence for finite elements," Lectures in Xiangtan Univ., 1986.
71. O.C. Zienkiewicz, "The Finite Element Method in Engineering Science," McGraw-Hill, London, 1971.

51. P. NEITTAANMÄKI and D. TIBA
A variational inequality approach to constrained control problems
(Preprint 51, 1986, 20 pp.)

52. J. TERVO
On weak solutions of linear coercive integral equations in spaces $L^2(X; H_k)$
(Preprint 52, 1986, 22 p.)

53. O. MARTIO
Counterexamples for unique continuation
(Preprint 53, 1986, 32 pp.)

54. J. HASLINGER, P. NEITTAANMÄKI and D. TIBA
On state constrained optimal shape design problems
(Preprint 54, 1986, 12 pp.)

55. O. MARTIO and J. VÄISÄLÄ
Global L^p-integrability of the derivative of a quasiconformal mapping
(Preprint 55, 1986, 15 pp.)

56. J. HEINONEN and T. KILPELÄINEN
A-superharmonic functions and supersolutions of degenerate elliptic equations
(Preprint 56, 1986, 34 pp.)

57. P. NEITTAANMÄKI
On the control of the secondary cooling in the continuous casting process
(Preprint 57, 1986, 17 pp.)

58. J. HASLINGER and P. NEITTAANMÄKI
On the design of the optimal covering of an obstacle
(Preprint 58, 1986, 20 pp.)

59. J. HEINONEN
Boundary accessibility and elliptic harmonic measures
(Preprint 59, 1987, 12 pp.)

60. J. HASLINGER, P. NEITTAANMÄKI and K. SALMENJOKI
On optimal mesh design for FEM in unilateral boundary value problems
(Preprint 60, 1987, 11 pp.)

61. T. TIIHONEN
On Korn's inequality and shape optimization
(Preprint 61, 1987, 6 pp.)

62. T. TIIHONEN
Abstract approach to a shape design problem for variational inequalities
(Preprint 62, 1987, 12 pp.)

63. O. PEKONEN
A short proof of a theorem of Ahlfors
(Preprint 63, 1987, 8 pp.)

64. J. HEINONEN and T. KILPELÄINEN
Polar sets for supersolutions of degenerate elliptic equations
(Preprint 64, 1987, 24 pp.)

65. J. TERVO
On solvability of linear partial differential operators in local spaces $B_{p,k}^{loc}(G)$
(Preprint 65, 1987, 28 pp.)

66. K. ASTALA and J. HEINONEN
On quasiconformal rigidity in space and plane
(Preprint 66, 1987, 21 pp.)

67. O. PEKONEN
The Einstein field equation in a multi-dimensional universe
(Preprint 67, 1987, 5 pp.)

68. O. MARTIO
Sets of zero elliptic harmonic measures
(Preprint 68, 1987, 13 pp.)

69. O. PEKONEN
Secondary invariants of flat bundles
(Preprint 69, 1987, 9 pp.)

70. J. HASLINGER, P. NEITTAANMÄKI, T. TIIHONEN and A. KAARNA
Optimal shape design and unilateral boundary value problems I
(Preprint 70, 1987, 23 pp.)

71. J. HASLINGER, P. NEITTAANMÄKI, T. TIIHONEN and A. KAARNA
Optimal shape design and unilateral boundary value problems II
(Preprint 71, 1987, 27 pp.)

72. E. LAITINEN and P. NEITTAANMÄKI
On numerical simulation of continuous casting process
(Preprint 72, 1987, 19 pp.)

73. O. MARTIO and J. VÄISÄLÄ
Elliptic equations and maps of bounded length distortion
(Preprint 73, 1987, 42 pp.)

74. J. L. FERNÁNDEZ, J. HEINONEN and O. MARTIO
Quasilines and conformal mappings
(Preprint 74, 1987, 26 pp.)