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J. Haslinger and P. Neittaanmäki

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ON THE EXISTENCE OF OPTIMAL SHAPES IN CONTACT PROBLEMS - PERFECTLY PLASTIC BODIES

J. HASLINGER
Charles University, Faculty of Mathematics and Physics,
KAM MFF UK, Malostranské 2/25, CS-11800 Prague,
Czechoslovakia

and

P. NEITTAANMÄKI
Department of Mathematics, University of Jyväskylä,
Seminaarinkatu 15, SF-40100 Jyväskylä, Finland

Abstract The optimal shape design of a two-dimensional elastic perfectly plastic body (a punch) on a rigid frictionless foundation is analyzed. The problem is to find the boundary part of the body where the unilateral boundary condition are assumed in such a way that certain energy integral of the system in the equilibrium will be minimized. It is assumed that the material of the body is elastic perfectly plastic, obeying the Hencky's law. The variational formulation in terms of stresses is utilized. The existence of optimal shapes is proved.

1. Introduction

It is the aim of this paper to continue the analysis of [4-7], where the optimal shape design of two dimensional elastic bodies on a rigid foundation was analyzed. In [4] the case of frictionless foundation and in [5] the model with given friction is analyzed. The variational inequality approach in terms displacement is utilized. The works [6,7] contain numerical realization with numerical examples. of the problems presented in [4,5].

If the material of the bodies is elastic perfectly plastic, obeying the Hencky's law, the formulation in terms of stresses is more suitable than that in displacements. Thus, we first present the well known Haar-Kármán principle in the case of a unilateral contact on the boundary, [3], [8]. The present paper is concerned with the existence of a solution to the contour design problem for a planar punch, material of which is elastic-plastic. In our optimum design problem the contact boundary of the punch with unilateral boundary conditions must be redesigned in such a way that certain energy integral will be minimized (see chapter 2 problem ($\mathbb{P}$)).

Approximation of the problem in question can be done by finite element method. For example, in the simplest approach, piecewise constant external approximation of the set of statically admissible stress field can be applied. When the state problem together with the set of admissible controls are discretized we are led to a nonlinear programming problem where the evaluation of the objective function involves the solving of the nonlinear state problem. The details will be discussed in a fortcoming paper.

Throughout this paper we shall use the Cartesian coordinate system $x = (x_1, x_2)$. The summation is implied over the range 1, 2 if an index is repeated. $H^j(\Omega)$ will denote the usual Sobolev space $W^{j,2}(\Omega)$ of functions with square integrable derivatives up to the order j in the sense of distributions. Especially we write $L^2(\Omega) = W^{0,2}(\Omega)$, with the scalar product $(\cdot,\cdot)_\Omega$.

2. The problem

Let us assume a punch, material of which is elastic - perfectly plastic, obeying the Hencky's law ([1], [2]). The punch occupies a bounded plane domain $\Omega(\alpha) \subset \mathbb{R}^2$:

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in]a,b[\ , \ \alpha(x_1) < x_2 < \gamma\} \ ,$$

where $\alpha \geq 0$ is a function such that the boundary $\partial\Omega(\alpha)$ is Lipschitz continuous (other conditions on α will be specified later). $\Omega(\alpha)$ is subjected to a body force $F = (F_1, F_2)$ and surface tractions $P = (P_1, P_2)$ on a top of $\Omega(\alpha)$. Moreover, $\Omega(\alpha)$ is unilaterally supported by a rigid frictionless foundation - here by the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ (see Fig. 2.1). This means that the punch is pushed onto a rigid frictionless matrix.

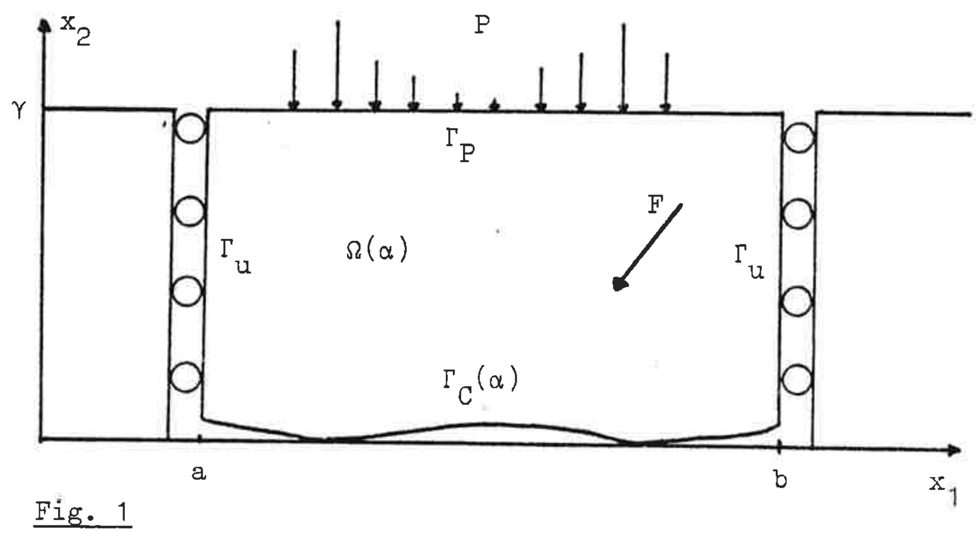

Fig. 1

We start with some notations:

$$S(\Omega(\alpha)) = \{\tau = (\tau_{ij})_{i,j=1}^{2} \in (L^2(\Omega(\alpha)))^4 \mid \tau_{ij} = \tau_{ij} \quad \text{a.e. in } \Omega(\alpha)\} \ ,$$

$$\langle\tau,\varepsilon\rangle_{\Omega(\alpha)} = \int_{\Omega(\alpha)} \tau_{ij}\,\varepsilon_{ij}\,dx \qquad \forall\tau,\varepsilon \in S(\Omega(\alpha)) ,$$

$$\|\tau\|_{\Omega(\alpha)} = \langle\tau,\tau\rangle^{1/2}_{\Omega(\alpha)} .$$

$S(\Omega(\alpha))$ can be equipped with an equivalent, energy norm $[\tau]_{\Omega(\alpha)} = (\tau,\tau)^{1/2}_{\Omega(\alpha)}$, where

$$(\tau,\varepsilon)_{\Omega(\alpha)} \equiv \langle\Lambda^{-1}\tau,\varepsilon\rangle_{\Omega(\alpha)} , \qquad \tau,\varepsilon \in S(\Omega(\alpha)) .$$

Λ^{-1} is the inverse of Λ: $S(\Omega(\alpha)) \to S(\Omega(\alpha))$, which is the isomorphism on $S(\Omega(\alpha))$, given by the generalized Hooke's law:

$$\sigma(x) = \Lambda(x)\,\varepsilon(x) \Longleftrightarrow \sigma_{ij}(x) = c_{ijkl}(x)\,\varepsilon_{kl}(x) \qquad \text{a.e. in } \Omega(\alpha) .$$

$c_{ijkl} \in L^{\infty}(\hat{\Omega})$ satisfy the usual symmetry conditions:

$$c_{ijkl} = c_{jikl} = c_{klij} \qquad \text{a.e. in } \hat{\Omega}$$

and

$$\exists\alpha = \text{const.} > 0 : \langle\Lambda\varepsilon,\varepsilon\rangle_{\hat{\Omega}} \geq \alpha\|\varepsilon\|^2_{\hat{\Omega}} \qquad \forall\varepsilon \in S(\hat{\Omega}) ,$$

where $\hat{\Omega} = (a,b) \times (0,\gamma)$.

Let $\mathbb{R}^{\sigma}$ be the space of all symmetric 2×2 matrices, $f: \mathbb{R}^{\sigma} \to \mathbb{R}^1$ a continuous and convex yield function. The set

$$B = \{\tau \in \mathbb{R}^{\sigma} \mid f(\tau) \leq 1\}$$

is called the set of plastically admissible stresses. The set of plastically admissible stress fields is now given by

$$P(\Omega(\alpha)) = \{\tau \in S(\Omega(\alpha)) \mid \tau(x) \in B \quad \text{a.e. in } \Omega(\alpha)\} . \tag{2.1}$$

Let $\pi_B(x): \mathbb{R}^\sigma \to B$ be the projection on the closed, convex set B with respect to the scalar product $(\Lambda^{-1}(x)\,\sigma)_{ij}\,\tau_{ij}$. π_B induces the projection $\pi_{\Omega(\alpha)}$ of $S(\Omega(\alpha))$ on $P(\Omega(\alpha))$ with respect to the scalar product $(\tau,\varepsilon)_{\Omega(\alpha)}$, namely

$$(\pi_{\Omega(\alpha)}\,\tau)(x) = \pi_B(x)\,\tau(x) \qquad \text{a.e. in } \Omega(\alpha)$$

(see [1], [2]).

Let $\varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^{2} \in S(\Omega(\alpha))$,

$$\varepsilon_{ij}(u) = 1/2\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

be the linearized strain field corresponding to the displacement field $u = (u_1,u_2) \in (H^1(\Omega(\alpha)))^2$. Then the Hencky's law of plasticity can be stated in the form:

$$\sigma(u) = \pi_{\Omega(\alpha)}(\Lambda\varepsilon(u)). \tag{2.2}$$

Set $\partial\Omega(\alpha) = \bar{\Gamma}_u \cup \bar{\Gamma}_P \cup \bar{\Gamma}_C(\alpha)$, where

$$\Gamma_u = \{(a,x_2) \mid \alpha(a) < x_2 < \gamma\} \cup \{(b,x_2) \mid \alpha(b) < x_2 < \gamma\},$$

$$\Gamma_P = \{(x_1,\gamma) \mid x_1 \in]a,b[\},$$

$$\Gamma_C(\alpha) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = \alpha(x_1),\ x_1 \in]a,b[\},$$

i.e. $\Gamma_C(\alpha)$ is given by a graph of α.

Let us assume there exists a displacement field $u = (u_1,u_2)$ sufficiently smooth and satisfying:

<u>the displacement condition on</u> Γ_u:

$$u_1 = 0, \quad T_2(\sigma) \equiv \sigma_{2j}(u)\,n_j = 0, \quad \text{on } \Gamma_u \tag{2.3}$$

<u>realizes tractions on</u> Γ_p :

$$T_i(\sigma) \equiv \sigma_{ij}(u)\, n_j = P_i \ , \quad i = 1,2 \tag{2.4}$$

$n = (n_1, n_2)$ is the unit outward normal with respect to $\partial\Omega(\alpha)$;

<u>the unilateral conditions on</u> $\Gamma_c(\alpha)$:

$$\left.\begin{array}{ll} u_2(x_1, \alpha(x_1)) \geq -\alpha(x_1) & \forall x_1 \in]a,b[\\ T_2(\sigma) \equiv \sigma_{2j}(u)\, n_j \geq 0 \ , & T_2(\sigma)(u_2 + \alpha) = 0 \\ T_1(\sigma) = 0 & \end{array}\right\} \tag{2.5}$$

The stress field $\sigma(u)$, related to u by means of (2.2) satisfies the <u>equilibrium equations</u>:

$$\frac{\partial \sigma_{ij}}{\partial x_j}(u) + F_i = 0 \qquad i = 1,2 \quad \text{in} \quad \Omega(\alpha) \tag{2.6}$$

The displacement field u , satisfying (2.2) - (2.6) (if any) is called <u>the classical solution</u> of the punch problem for an elasto-perfectly plastic material, occupying $\Omega(\alpha)$.

Next we derive the variational principle, satisfied by the stress field $\sigma(u)$. Let

$$V(\Omega(\alpha)) = \{v = (v_1, v_2) \in (H^1(\Omega(\alpha)))^2 \mid v_1 = 0 \quad \text{on} \quad \Gamma_u\} \ ,$$

$$K(\Omega(\alpha)) = \{v \in V(\Omega(\alpha)) \mid v_2(x_1, \alpha(x_1)) \geq -\alpha(x_1) \quad \text{in }]a,b[\} \ ,$$

$$K_0(\Omega(\alpha)) = \{v \in V(\Omega(\alpha)) \mid v_2(x_1, \alpha(x_1)) \geq 0 \quad \text{in} \quad]a,b[\} \ .$$

By $K^+_{F,P}(\Omega(\alpha))$ we denote the closed, convex subset of $S(\Omega(\alpha))$ given by:

$$K^{+}_{F,P}(\Omega(\alpha)) = \{\tau \in S(\Omega(\alpha)) \mid \langle\tau,\varepsilon(v)\rangle_{\Omega(\alpha)} \geq \langle L,v\rangle_{\Omega(\alpha)}$$

$$\forall v \in K_0(\Omega(\alpha))\} \ ,$$

$$\langle L,v\rangle_{\Omega(\alpha)} \equiv \int_{\Omega(\alpha)} F_i v_i \, dx + \int_{\Gamma_P} P_i v_i \, ds$$

with $F \in (L^2(\hat{\Omega}))^2$, $P \in (L^2(\Gamma_P))^2$.

Let $u_0(\alpha) \in (H^1(\Omega(\alpha))^2$ be such that

$$\left.\begin{aligned} &u_{01}(\alpha) \equiv 0 \quad \text{in} \quad \Omega(\alpha) \\ &u_{02}(x_1,\alpha(x_1)) = -\alpha(x_1) \quad \text{a.e. in} \quad]a,b[\end{aligned}\right\} \qquad (2.7)$$

<u>Theorem 2.1</u> Let u,σ be related by means of (2.2)-(2.6). Then $\sigma \in K^{+}_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha))$ and

$$(\sigma,\tau-\sigma)_{\Omega(\alpha)} \geq \langle\varepsilon(u_0(\alpha)),\ \tau-\sigma\rangle_{\Omega(\alpha)} \qquad (2.8)$$

holds for any $\tau \in K^{+}_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha))$.

<u>Proof</u> (see [3]). We first note that (2.2) implies $\sigma \in P(\Omega(\alpha))$. Multiplying (2.6) by $v \in K_0(\Omega(\alpha))$ and using the Green's formula we are led to

$$\langle\sigma_{ij}(u),\varepsilon_{ij}(v)\rangle_{\Omega(\alpha)} = \langle L,v\rangle_{\Omega(\alpha)} + \int_{\Gamma_C(\alpha)} T_2 v_2 \, ds \geq \langle L,v\rangle_{\Omega(\alpha)}$$

making the use of (2.3)-(2.5).

Let us write $u = u_0(\alpha) + w$. Then $w \in K_0(\Omega(\alpha))$ and (2.2) implies

$$(\sigma(u),\tau-\sigma(u))_{\Omega(\alpha)} \geq \langle\varepsilon(u),\tau-\sigma\rangle_{\Omega(\alpha)} = \langle\varepsilon(u_0(\alpha)),\tau-\sigma\rangle_{\Omega(\alpha)}$$
$$+ \langle\varepsilon(w),\tau-\sigma\rangle_{\Omega(\alpha)} \ . \qquad (2.9)$$

But

$$\langle \varepsilon(w),\sigma\rangle_{\Omega(\alpha)} = \langle L,w\rangle_{\Omega(\alpha)} + \int_{\Gamma_C(\alpha)} T_2(\sigma)\, w_2\, ds$$

$$= \langle L,w\rangle_{\Omega(\alpha)} + \int_{\Gamma_C(\alpha)} T_2(\sigma)(u_2+\alpha)\, ds$$

$$= \langle L,w\rangle_{\Omega(\alpha)} .$$

Hence

$$\langle \varepsilon(w),\tau-\sigma\rangle_{\Omega(\alpha)} = \langle \varepsilon(w),\tau\rangle_{\Omega(\alpha)} - \langle \varepsilon(w),\sigma\rangle_{\Omega(\alpha)}$$

$$\geq \langle L,w\rangle_{\Omega(\alpha)} - \langle L,w\rangle_{\Omega(\alpha)} \geq 0 \qquad \forall \tau \in K^+_{F,P}(\Omega(\alpha)) .$$

From this and (2.9), the relation (2.8) follows. □

Consequence 2.1 Set

$$S_\alpha(\tau) = 1/2[\tau]^2_{\Omega(\alpha)} - \langle \varepsilon(u_0(\alpha)),\tau\rangle_{\Omega(\alpha)} .$$

Then $\sigma \in K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha))$ satisfies (2.8) if and only if

$$(\mathcal{P}(\alpha)) \qquad S_\alpha(\sigma) \leq S_\alpha(\tau) \qquad \forall \tau \in K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha)) .$$

$\mathcal{P}(\alpha)$ can be taken as the definition of the variational formulation in terms of stresses of a punch problem for an elastic - perfectly plastic material. Using the well-known results, one can prove

Theorem 2.2 Let $K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha)) \neq \emptyset$. Then there exists a unique solution σ of $(\mathcal{P}(\alpha))$.

Remark 2.1 Let us mention that even $(\mathcal{P}(\alpha))$ has a unique solution σ, the existence of the displacement field $u \in (H^1(\Omega(\alpha)))^2$, related to σ by means of (2.2) is not guaranteed, in general.

Remark 2.2 Necessary condition for $K^+_{F,P}(\Omega(\alpha))$ to be non-empty is that

$$\int_{\Omega(\alpha)} F_2\,dx + \int_{\Gamma_P} P_2\,ds \leq 0 \;.$$

Up to now, a function α, describing $\Gamma_C(\alpha)$ has been given. Let us suppose now that $\Gamma_C(\alpha)$ may vary, i.e. α is a variable, belonging to an admissible set $\mathcal{U}_{ad}$, given by:

$$\mathcal{U}_{ad} = \{\alpha \in C^{1,1}([a,b]) \mid 0 \leq \alpha(x_1) \leq C_0 < \gamma\,,\ |\alpha'(x_1)| \leq C_1\,,$$
$$|\alpha''(x_1)| \leq C_2 \text{ a.e. in }]a,b[\,,\ \text{meas } \Omega(\alpha) = C_3\} \;.$$

C_0, C_1, C_2, C_3 are positive constants, chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. Let $\tilde{\mathcal{U}}_{ad} \subseteq \mathcal{U}_{ad}$ be such that

$$\alpha \in \tilde{\mathcal{U}}_{ad} \Leftrightarrow K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha)) \neq \emptyset \;.$$

Next we suppose that

(A.1) $\tilde{\mathcal{U}}_{ad} \neq \emptyset$;

(A.2) $\exists r > 0 \quad \forall \alpha \in \tilde{\mathcal{U}}_{ad} \quad \exists \tilde{\tau}(\alpha) \in K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha))$:

$$\|\tilde{\tau}(\alpha)\|_{\Omega(\alpha)} \leq r \;.$$

For any $\alpha \in \tilde{\mathcal{U}}_{ad}$, the state problem $(\mathcal{P}(\alpha))$ has a unique solution $\sigma = \sigma(\alpha)$ (to emphasize the dependence of σ on α, we shall write α as the argument).

Our aim will be to find $\alpha^* \in \tilde{\mathcal{U}}_{ad}$ in such a way that

($\mathbb{P}$) $\quad E(\alpha^*) \leq E(\alpha) \quad \forall \alpha \in \tilde{\mathcal{U}}_{ad}$

where

$$E(\alpha) = \mathcal{S}_\alpha(\sigma(\alpha)) = 1/2[\sigma(\alpha)]^2_{\Omega(\alpha)} - \langle \varepsilon(u_0(\alpha)), \sigma(\alpha) \rangle_{\Omega(\alpha)}$$

and $\sigma(\alpha)$ solves $(\mathcal{P}(\alpha))$ on $\Omega(\alpha)$.

3. Existence result

The main result of this section is

Theorem 3.1 Under (A.1), (A.2) there exists at least one solution $\alpha^* \in \tilde{\mathcal{U}}_{ad}$ of $(\mathbb{P})$.

Before we prove this theorem, we present some auxiliary results, which will be useful in what follows.

Lemma 3.1 Let $\alpha_n \to \alpha$ in $C^0([a,b])$, $\alpha_n, \alpha \in \mathcal{U}_{ad}$ and let $\varphi \in K_0(\Omega(\alpha))$. Then there exist $\varphi_j \in (H^1(\hat{\Omega}))^2$ and a subsequence $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ such that

- $\varphi_j|_{\Omega(\alpha_{n_j})} \in K_0(\Omega(\alpha_{n_j}))$; (3.1)

- $\varphi_j \to \hat{\varphi}$ in $(H^1(\hat{\Omega}))^2$, (3.2)

where $\hat{\varphi}$ denotes the Calderon extension of φ from $\Omega(\alpha)$ on $\hat{\Omega}$.

Proof See [4]. □

Let a function $u_0(\alpha) = (u_{01}(\alpha), u_{02}(\alpha))$, appearing in the linear term of $\mathcal{S}_\alpha(\tau)$ be chosen as follows:

$$u_{01}(\alpha) \equiv 0 \quad \text{in } \hat{\Omega}$$

$$u_{02}(\alpha)(x_1,x_2) = -\alpha(x_1) \qquad (x_1,x_2) \in \hat{\Omega} .$$

It is easy to see that $u_0(\alpha) \in (H^2(\hat{\Omega}))^2$, provided $\alpha \in \mathcal{U}_{ad}$ and

$$\|u_0(\alpha)\|_{1,\hat{\Omega}} \leq C\|\alpha\|_{C^1([a,b])} , \qquad (3.3)$$

where $C > 0$ doesn't depend on $\alpha \in \mathcal{U}_{ad}$.

Proof of Th. 3.1 Let

$$q = \inf_{\alpha \in \tilde{\mathcal{U}}_{ad}} E(\alpha) = \lim_{n\to\infty} E(\alpha_n) ,$$

i.e. $\{\alpha_n\}$, $\alpha_n \in \tilde{\mathcal{U}}_{ad}$ is a minimizing sequence of $(\mathbb{P})$. Following the definition of $\mathcal{U}_{ad}$, there exists a subsequence of $\{\alpha_n\}$, (denoted by $\{\alpha_n\}$ again) and an element $\alpha^* \in \mathcal{U}_{ad}$ such that

$$\alpha_n \to \alpha^* \quad , n \to \infty \quad \text{in} \quad C^1([a,b]) . \qquad (3.4)$$

Denote by $\Omega_n = \Omega(\alpha_n)$ and $\sigma_n = \sigma(\alpha_n)$ the solution of $(\mathcal{P}(\alpha_n))$ on Ω_n:

$$(\sigma_n, \tau - \sigma_n)_{\Omega_n} \geq \langle \varepsilon(u_0(\alpha_n)), \tau - \sigma_n \rangle_{\Omega_n} \qquad \forall \tau \in K^+_{F,P}(\Omega_n) \cap P(\Omega_n) . \qquad (3.5)$$

Using (A.2), (3.3), (3.4) and (3.5) we see that $\{\sigma_n\}$ is bounded in the following sense:

$$\exists C > 0 \qquad \|\sigma_n\|_{\Omega_n} \leq C \qquad (3.6)$$

where $C > 0$ doesn't depend on n. Let

$$G_m(\alpha^*) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a,b) , \ \alpha^*(x_1) + 1/m < x_2 < \gamma\} .$$

Let m be fixed. Then there exists $n_0(m)$ such that

$$\Omega_n \supset \overline{G_m(\alpha^*)} \qquad \forall n \geq n_0(m)$$

and

$$\|\sigma_n\|_{G_m(\alpha^*)} \leq C \tag{3.7}$$

due to (3.6). Thus there is a subsequence $\{\sigma_{n_1}\} \subset \{\sigma_n\}$ and an element $\sigma^{(m)} \in S(G_m(\alpha^*))$ such that

$$\sigma_{n_1} \rightharpoonup \sigma^{(m)} \quad \text{(weakly)} \quad \text{in} \quad S(G_m(\alpha^*)) .$$

Analogously, there exists $n_0(m+1)$ such that

$$\Omega_n \supset \overline{G_{m+1}(\alpha^*)} \supset \overline{G_m(\alpha^*)} \qquad \forall n \geq n_0(m+1)$$

and

$$\|\sigma_{n_1}\|_{G_{m+1}(\alpha^*)} \leq C .$$

One can extract a subsequence $\{\sigma_{n_2}\} \subset \{\sigma_{n_1}\}$ such that

$$\sigma_{n_2} \rightharpoonup \sigma^{(m+1)} \quad \text{in} \quad S(G_{m+1}(\alpha^*)) .$$

Clearly $\sigma^{(m)} \equiv \sigma^{(m+1)}$ a.e. in $G_m(\alpha^*)$. Repeating the same procedure for any m integer, one can take a diagonal sequence $\{\sigma^D_{n_k}\}$ determined by $\{\sigma_{n_k}\}$, which satisfies:

$$\sigma^D_{n_k} \rightharpoonup \sigma \quad \text{in} \quad S(G_m(\alpha^*)) \quad \text{for any} \quad m , \tag{3.8}$$

where $\sigma \equiv \sigma^{(m)}$ a.e. in $G_m(\alpha^*)$. Next, we shall simply write σ_{n_k} instead of $\sigma^D_{n_k}$. For the moment let us suppose that we have already proven that $\sigma \in K^+_{F,P}(\Omega(\alpha^*)) \cap P(\Omega(\alpha^*))$. Now we show that $\alpha^* \in \tilde{\mathcal{U}}_{ad}$ solves $(\mathbb{P})$ and $\sigma = \sigma(\alpha^*)$ is a solution of $(\mathcal{P}(\alpha^*))$. Indeed, let m be fixed. Then

$$\mathcal{S}_{\alpha_n}(\sigma_n) = 1/2[\sigma_n]^2_{\Omega_n} - \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{\Omega_n}$$

$$= 1/2[\sigma_n]^2_{G_m(\alpha^*)} + 1/2[\sigma_n]^2_{\Omega_n\setminus G_m(\alpha^*)} - \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{G_m(\alpha^*)}$$

$$- \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)}$$

$$\geq 1/2[\sigma_n]^2_{G_m(\alpha^*)} - \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{G_m(\alpha^*)}$$

$$- \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)} .$$

From (3.3), (3.4) and (3.7) one gets:

$$q = \liminf_{n\to\infty} \mathcal{S}_{\alpha_n}(\sigma_n) \geq 1/2[\sigma]^2_{G_m(\alpha^*)} - \langle\varepsilon(u_0(\alpha^*)),\sigma\rangle_{G_m(\alpha^*)}$$

$$- \limsup_{n\to\infty} \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)} .$$

On the other hand,

$$\limsup_{n\to\infty} \langle\varepsilon(u_0(\alpha_n)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)}$$

$$\leq \limsup_{n\to\infty} \langle\varepsilon(u_0(\alpha_n)) - \varepsilon(u_0(\alpha^*)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)}$$

$$+ \limsup_{n\to\infty} \langle\varepsilon(u_0(\alpha^*)),\sigma_n\rangle_{\Omega_n\setminus G_m(\alpha^*)} \leq C\|\varepsilon(u_0(\alpha^*))\|_{\Omega(\alpha^*)\setminus G_m(\alpha^*)} ,$$

where $C > 0$ doesn't depend on m. Thus

$$q \geq 1/2[\sigma]^2_{G_m(\alpha^*)} - \langle\varepsilon(u_0(\alpha^*)),\sigma\rangle_{G_m(\alpha^*)}$$

$$- C\|\varepsilon(u_0(\alpha^*))\|^2_{\Omega(\alpha^*)\setminus G_m(\alpha^*)} \tag{3.9}$$

holds for any m integer.

Letting $m\to\infty$ in (3.9) we finally get

$$q \geq 1/2[\sigma]^2_{\Omega(\alpha^*)} - \langle \varepsilon(u_0(\alpha^*)), \sigma \rangle_{\Omega(\alpha^*)} = S_{\alpha^*}(\sigma) .$$

Let $\sigma(\alpha^*) \in K^+_{F,P}(\Omega(\alpha^*)) \cap P(\Omega(\alpha^*))$ be a solution of $(\mathcal{P}(\alpha^*))$. Then

$$q \geq S_{\alpha^*}(\sigma) \geq S_{\alpha^*}(\sigma(\alpha^*)) \geq q ,$$

i.e. $\alpha^* \in \tilde{\mathcal{U}}_{ad}$ is a solution of $(\mathbb{P})$ and $\sigma = \sigma(\alpha^*)$ solves $(\mathcal{P}(\alpha))$. □

It remains to verify that $\sigma \in K^+_{F,P}(\Omega(\alpha^*)) \cap P(\Omega(\alpha^*))$. This follows from

<u>Lemma 3.2</u> Let $\alpha_n \to \alpha$ in $C^0([a,b])$, $\alpha_n, \alpha \in \mathcal{U}_{ad}$. Let $\sigma_n \in K^+_{F,P}(\Omega(\alpha_n)) \cap P(\Omega(\alpha_n))$ be such that

$$\sigma_n \rightharpoonup \sigma \quad \text{in } S(G_m(\alpha)) \text{ for any } m \text{ integer,} \tag{3.10}$$

where

$$G_m(\alpha) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in]a,b[,\ \alpha(x_1) + 1/m < x_2 < \gamma\} .$$

Then $\sigma \in K^+_{F,P}(\Omega(\alpha)) \cap P(\Omega(\alpha))$.

<u>Proof</u> It is readily seen that $\sigma \in P(\Omega(\alpha))$ if and only if $\sigma|_{G_m(\alpha)} \in P(G_m(\alpha))$ for any m integer. But this is true because of (3.10) and the fact that $P(G_m(\alpha))$ is the closed, convex subset of $S(G_m(\alpha))$. Let us prove that $\sigma \in K^+_{F,P}(\Omega(\alpha))$, i.e.

$$\langle \sigma, \varepsilon(v) \rangle_{\Omega(\alpha)} \geq \langle L, v \rangle_{\Omega(\alpha)} \qquad \forall v \in K_0(\Omega(\alpha)) .$$

Let $v \in K_0(\Omega(\alpha))$ be fixed. According to Lemma 3.1 there exists a sequence $v_j \in (H^1(\hat{\Omega}))^2$ and a subsequence $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ with properties given by (3.1), (3.2). Let $\{\sigma_{n_j}\} \subset \{\sigma_n\}$ be a subsequence such that $\sigma_{n_j} \in K^+_{F,P}(\Omega_{n_j})$. Then

$$\langle\sigma_{n_j},\varepsilon(v_j)\rangle_{\Omega_{n_j}} \geq \langle L,v_j\rangle_{\Omega_{n_j}} .$$

Let m be fixed. Then

$$\langle\sigma_{n_j},\varepsilon(v_j)\rangle_{\Omega_{n_j}} = \langle\sigma_{n_j},\varepsilon(v_j)\rangle_{G_m(\alpha)} + \langle\sigma_{n_j},\varepsilon(v_j)\rangle_{\Omega_{n_j}\setminus\Omega(\alpha)} + \langle\sigma_{n_j},\varepsilon(v_j)\rangle_{(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega_{n_j}} . \tag{3.11}$$

From (3.10) and (3.2) it follows that

$$\langle\sigma_{n_j},\varepsilon(v_j)\rangle_{G_m(\alpha)} \to \langle\sigma,\varepsilon(v)\rangle_{G_m(\alpha)} \quad , \ n_j\to\infty \tag{3.12}$$

$$\langle\sigma_{n_j},\varepsilon(v_j)\rangle_{\Omega_{n_j}\setminus\Omega(\alpha)} = \langle\sigma_{n_j},\varepsilon(v_j-v)\rangle_{\Omega_{n_j}\setminus\Omega(\alpha)} + \langle\sigma_{n_j},\varepsilon(v)\rangle_{\Omega_{n_j}\setminus\Omega(\alpha)} \to 0 \quad , \ n_j\to\infty . \tag{3.13}$$

Finally,

$$\limsup_{n_j\to\infty} \langle\sigma_{n_j},\varepsilon(v_j)\rangle_{(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega_{n_j}} \leq C\|v\|_{\Omega(\alpha)\setminus G_m(\alpha)} , \tag{3.14}$$

where $C>0$ doesn't depend on m.

Analogously,

$$\langle L,v_j\rangle_{\Omega_{n_j}} = \langle L,v_j\rangle_{G_m(\alpha)} + \langle L,v_j\rangle_{\Omega_{n_j}\setminus G_m(\alpha)} = \langle L,v_j\rangle_{G_m(\alpha)} + \langle L,v_j-v\rangle_{\Omega_{n_j}\setminus G_m(\alpha)} + \langle L,v\rangle_{\Omega_{n_j}\setminus G_m(\alpha)}$$

so that

$$\liminf_{n_j\to\infty} \langle L,v_j\rangle_{\Omega_{n_j}} \geq \langle L,v\rangle_{G_m(\alpha)} - C\|v\|_{\Omega(\alpha)\setminus G_m(\alpha)} . \tag{3.15}$$

From (3.11) - (3.15) we obtain

$$\langle\sigma,\varepsilon(v)\rangle_{G_m(\alpha)} + C\|v\|_{\Omega(\alpha)\setminus G_m(\alpha)} \geq \langle L,v\rangle_{G_m(\alpha)} - C\|v\|_{\Omega(\alpha)\setminus G_m(\alpha)} .$$

Letting $m \to \infty$ we arrive at the assertion of Lemma 3.2. □

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