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**Title:** On FE-grid relocation in solving unilateral boundary value problems by FEM

**Year:** 1992

**Version:**

**Please cite the original version:**

Haslinger, J., Neittaanmäki, P. & Salmenjoki, K. (1992). On FE-grid relocation in solving unilateral boundary value problems by FEM. Applications of Mathematics, 37 (2), 105-122. Retrieved from https://eudml.org/doc/15703

37 (1992) APPLICATIONS OF MATHEMATICS No. 2, 105–122

# ON FE-GRID RELOCATION IN SOLVING UNILATERAL BOUNDARY VALUE PROBLEMS BY FEM

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(Received July 9, 1990)

*Summary.* We consider FE-grid optimization in elliptic unilateral boundary value problems. The criterion used in grid optimization is the total potential energy of the system. It is shown that minimization of this cost functional means a decrease of the discretization error or a better approximation of the unilateral boundary conditions. Design sensitivity analysis is given with respect to the movement of nodal points. Numerical results for the Dirichlet-Signorini problem for the Laplace equation and the plane elasticity problem with unilateral boundary conditions are given. In plane elasticity we consider problems with and without friction.

*Keywords*: FE-grid relocation, unilateral boundary value problem.

*AMS classification*: 65N30

## 1. INTRODUCTION

In present literature on numerical analysis concerning the theory and application of FEM one can see a growing interest in improving the accuracy achieved by FEM (mesh design [1, 2, 11, 12], extrapolation methods [3], superconvergence [14]). In this paper we consider the improvement of accuracy by a-priori relocation of the FE-grid ($r$-method) with fixed number of nodes and fixed degree $p$ of elements (order of approximation).

The traditional way to achieve higher accuracy is to use the $h$-method, refining the mesh either uniformly or locally with a fixed degree $p$ of elements [4, 2, 6]. Another possibility is the $p$-method, which means increasing the degree $p$ of the elements either uniformly or selectively with a fixed mesh parameter $h$ [2, 6]. Remarkable results have been obtained with the $hp$-method, which combines the best effects of the $h$- and $p$-methods [6].

In this paper we state the relocation problem for the FE-grid and consider especially contact problems. In [5] the $r$-method for linear, elliptic, second order partial differential equations with classical boundary conditions was considered with total potential energy as the criterion of optimality. We shall show that this criterion is useful for our purposes as well: Moreover we show that this criterion enables us to minimize the discretization error $\|u - u_h\|^2$ or to improve the approximation of the contact set $\{u(x) = \Gamma(x) \text{ on } \Gamma_C\}$ or to improve the approximation of the sliding and non-sliding parts where $\Gamma(x)$ is the obstacle (or foundation) and $\Gamma_C$ is the part of the boundary of the domain $\Omega$ with contact conditions. We shall present sensitivity analysis needed in order to apply an efficient NLP(NonLinearProgramming)-solver to minimize the criterion function.

## 2. Formulation of the optimal mesh design problem

**2.1. The regular moving grid.** Let $\Omega \subset \mathbf{R}^2$ be a polygonal domain and let $\mathcal{T}$ be its triangulation. We denote by $N = \{N_i\}$, $N_i = (x_i, y_i)$, the set of all nodes of $\mathcal{T}$. In order to emphasize the dependence of $\mathcal{T}$ on $N$, we write $\mathcal{T}(N)$ in what follows. We also assume that all $\mathcal{T}(N)$ are topologically equivalent. To keep the triangulation admissible (i.e. regular) during the computation, some restrictions on the location of $N_i$ must be added.

Let $\bar{\mathcal{T}}$ be an initial triangulation, defined by nodes $\bar{N}_i = (\bar{x}_i, \bar{y}_i)$. Then we consider relocations of $\bar{N}_i$, denoted by $N_i$, satisfying

$$N_i \in U_\delta(\bar{N}_i) \text{ for interior points } \bar{N}_i; \tag{2.1}$$

$$N_i \in U_\delta(\bar{N}_i) \cap \partial\Omega \quad \begin{array}{l} \text{in the case of boundary nodes } \bar{N}_i, \\ \text{which are not vertices of } \overline{\Omega}, \end{array} \tag{2.2}$$

where $U_\delta(\bar{N}_i)$ denotes a $\delta$-neighborhood of $\bar{N}_i$ ($\delta > 0$ sufficiently small).

The set of all $N$ with the above mentioned properties will be denoted by $\mathcal{M}$.

**2.2. Optimal mesh design problem.** In the FE-grid optimization problem with a fixed number of nodes we wish to relocate the nodes $N \in \mathcal{M}$ into a new position $N' \in \mathcal{M}$ in order to obtain a more close FE-solution $u_h$ for approximating $u$.

By a $\delta$-neighborhood of a point $\bar{N}_i = (\bar{x}_i, \bar{y}_i)$ we mean the square $[\bar{x}_i - \delta, \bar{x}_i + \delta] \times [\bar{y}_i - \delta, \bar{y}_i + \delta]$, $\delta > 0$.

In the next section we shall show that one possible way to do this is to solve the following NLP-problem

$$\textbf{(P)} \qquad \begin{cases} \text{Find } N^* \in \mathcal{M} \text{ such that} \\ \mathcal{L}(N^*) \leqslant \mathcal{L}(N) \quad \forall\, N \in \mathcal{M}, \end{cases}$$

where $\mathcal{L}(N) = J(u(N))$, with $u(N)$ being the FEM-solution of the problem considered depending on the positions of the nodes $N$.

There are several possibilities for the functional $J$. The most appropriate choice turns out to be the total potential energy of the system. To see this for the linear elliptic problems we refer to [5]. In Section 4 we shall show that this is the case suitable also for the elliptic unilateral boundary value problems.

It is easy to see that Problem (P) has at least one solution as $\mathcal{M}$ is compact and $\mathcal{L}$ is continuous.

## 3. Formulation of the model problems

**3.1. Case 1: Unilateral boundary value problem—scalar case.** We shall consider the problem of optimal mesh design for FEM in the case of the following unilateral boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)\times(0,1) \\ u = 0 & \text{on } \Gamma_1 \\ u \geqslant 0, \quad \dfrac{\partial}{\partial n}u \geqslant 0, \quad u\dfrac{\partial}{\partial n}u = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.1}$$

where $\Gamma_2 = \{x \in \mathbf{R}^2 \mid x_1 = 1,\ 0 < x_2 < 1\}$, $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}_2$, $f \in L^2(\Omega)$.

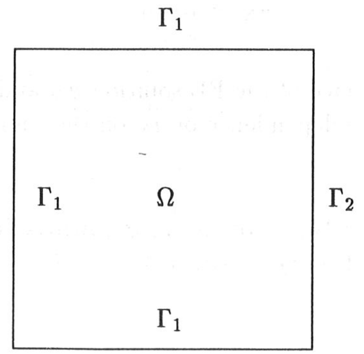


Fig. 3.1

Weak formulation of (3.1) is the following:
Find $u \in K = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_1,\ \varphi \geqslant 0 \text{ on } \Gamma_2\}$ such that

$$a(u, v-u) \geqslant \langle L, v-u\rangle \qquad \forall\, v \in K, \tag{3.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad \langle L, v\rangle = \int_\Omega f v \,\mathrm{d}x.$$

It is well-known that the solution of (3.2) can be characterized as a minimizer of the total potential energy functional $J$:

$$u = \underset{v \in K}{\arg\min}\{J(v)\}, \tag{3.3}$$

where $J(v) = \frac{1}{2}\, a(v,v) - \langle L, v\rangle$.

Next, with any $\mathcal{T}(N)$, $N \in \mathcal{M}$, we associate a closed, convex subset of a finite dimensional space, defined by:

$$\begin{aligned} K_N = \{v_N \in C(\overline{\Omega}) \mid\ & v_N|_T \in P_1(T)\ \forall\, T \in \mathcal{T}(N), \\ & v_N = 0 \text{ on } \overline{\Gamma_1},\ v_N \geqslant 0 \text{ on } \Gamma_2\} \end{aligned} \tag{3.4}$$

i.e. $K_N$ contains all piecewise linear continuous functions, vanishing on $\Gamma_1$ and satisfying unilateral boundary conditions on $\Gamma_2$. The finite element approximation of (3.2) is defined in the usual way:

$$\begin{aligned} &\text{Find } u_N \in K_N \text{ such that} \\ &a(u_N, v_N - u_N) \geqslant \langle L, v_N - u_N\rangle \quad \forall\, v_N \in K_N, \end{aligned} \tag{3.5}$$

or equivalently

$$u_N = \underset{v_N \in K_N}{\arg\min}\, J(v_N). \tag{3.6}$$

For the proof of convergence of the FE solution $u_N$ and the error estimates we refer to [10]. To emphasize the dependence of $u_N$ on the choice of $N \in \mathcal{M}$, we write $u(N)$ in what follows.

**3.2. Case 2: Elastic body on a rigid foundation.** Consider a body $\Omega$ supported by a rigid foundation $Q = \{(x_1, x_2) \mid x_2 \leqslant 0\}$.

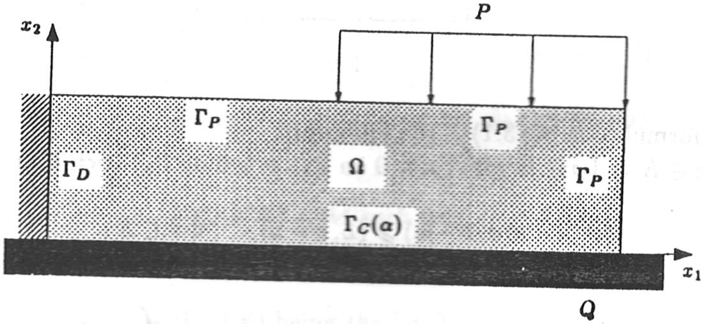


Fig. 3.2

Let $u = (u_1, u_2)$ denote the displacement field. We assume that the stress tensor $\tau(u) = \{\tau_{ij}(u)\}_{i,j=1}^2$ is related to the strain tensor $\varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^2$ by means of the linear Hooke's law

$$\tau_{ij}(u) = C_{ijkl}\varepsilon_{kl}(u),\ ^1 \tag{3.7}$$

where

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\right) \tag{3.8}$$

and the elasticity coefficients $C_{ijkl}$ are symmetric and uniformly elliptic in $\Omega$, $C_{ijkl} \in L^\infty(\Omega)$.

The equilibrium equation reads

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \qquad \text{in } \Omega,\ i = 1,2, \tag{3.9}$$

where $F = (F_1, F_2)$ denotes the vector of the body force.

We suppose $\alpha\colon [a,b] \to \mathbf{R}$ is a non-negative piecewise linear continuous function defining $\Gamma_C = \{x \in \mathbf{R}^2 \mid x_2 = \alpha(x_1),\quad x_1 \in (a,b)\}$. $\Gamma_C$ is the part of the boundary which may come in contact with the rigid foundation, see Fig. 3.2. Moreover, $\Omega = \{(x_1,x_2) \in \mathbf{R}^2 \mid a < x_1 < b,\quad \alpha(x_1) < x_2 < \gamma\}$, where $\gamma$ is a positive constant. Let the boundary $\partial\Omega$ be splitted up as follows: $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_P \cup \overline{\Gamma}_C$, where $\Gamma_D$ is a non-empty part of the boundary, where the body is fixed, $\Gamma_P$ is the part of the boundary, where the surface tractions $P = (P_1, P_2)$ are given, and on $\Gamma_C$ the unilateral boundary conditions with given friction are prescibed:

$$\begin{cases} u_i = 0 \quad \text{on } \Gamma_D, \qquad i = 1,2; \\ T_i(u) = \tau_{ij}(u)n_j = P_i \qquad \text{on } \Gamma_P,\ i = 1,2; \\ u_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1) \qquad \forall\, x_1 \in (a,b); \\ T_2(u) \geqslant 0 \text{ and } (u_2 + \alpha)T_2(u) = 0 \quad \text{on } \Gamma_C, \qquad \text{(contact)} \\ \begin{cases} |T_1(u)| \leqslant g \qquad \text{on } \Gamma_C; \\ |T_1(u)(x)| < g \Rightarrow u_1(x) = 0; \qquad \qquad \text{(friction )} \\ |T_1(u)(x)| = g \Rightarrow \exists\lambda(x) \geqslant 0\colon u_1(x) = -\lambda(x)T_1(u)(x), \end{cases} \end{cases} \tag{3.10}$$

where $g$ is the coefficient of friction and $n_j$ are the components of the outward unit normal. If $g \equiv 0$ we have the case without friction.

Let

$$K = \{v \in V \mid v_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1) \quad \forall\, x_1 \in (a,b)\}, \tag{3.11}$$

[1] Summation convention is used.

where

$$V = \{v = (v_1, v_2) \in (H^1(\Omega))^2 \mid v_i = 0 \text{ on } \Gamma_D, \quad i = 1, 2\}. \tag{3.12}$$

The weak formulation of the problem given by (3.9) and (3.10) is:

$$\begin{aligned} &\text{Find } u \in K \text{ such that} \\ &a(u, v - u) + j(v) - j(u) \geqslant \langle L, v - u\rangle \quad \forall\, v \in K, \end{aligned} \tag{3.13}$$

where

$$a(u, v - u) = (\tau(u), \varepsilon(v - u))_{0,\Omega} = \int_\Omega \tau_{ij}(u)\varepsilon_{ij}(v - u)\, \mathrm{d}x,$$

$$\langle L, v\rangle = \int_\Omega F_i v_i\, \mathrm{d}x + \int_{\Gamma_P} P_i v_i\, \mathrm{d}s$$

and

$$j(v) = g \int_{\Gamma_C} |v_1|\, \mathrm{d}s.$$

The problem (3.13) can be formulated equivalently as

$$u = \arg\min_{v \in K} J(v), \tag{3.14}$$

where

$$J(v) = \frac{1}{2} a(v, v) + j(v) - \langle L, v\rangle\,. \tag{3.15}$$

Next, with any $\mathcal{T}(N)$, $N \in \mathcal{M}$, we associate a closed, convex subset of a finit dimensional space, defined by:

(3.16)

$$\begin{aligned} K_N = \{v_N \in (C(\overline{\Omega}))^2 \mid v_N|_T \in (P_1(T))^2\ \forall\, T \in \mathcal{T}(N),\ v_{N,j} = 0 \text{ on } \overline{\Gamma}_D,\ j = 1, 2, \\ v_{N,2}(x_1, \alpha(x_1)) \geqslant -\alpha(x_1)\ \forall\, x_1 \in [a, b]\}, \end{aligned}$$

i.e. $K_N$ contains all piecewise linear continuous functions, vanishing on $\overline{\Gamma}_D$ an satisfying unilateral boundary conditions on $\Gamma_C$. The finite element approximatio of (3.13) is defined in the usual way:

$$\begin{aligned} &\text{Find } u_N \in K_N \text{ such that} \\ &a(u_N, v_N - u_N) + j(v_N) - j(u_N) \geqslant \langle L, v_N - u_N\rangle \qquad \forall\, v_N \in K_N \end{aligned} \tag{3.17}$$

or equivalently

$$u_N = \arg\min_{v_N \in K_N} J(v_N) \tag{3.18}$$

with $J$ given by (3.15). For the proof of convergence of the FE-solution and th error estimates we refer to [10]. To emphasize the dependence of $u_N$ on the choic of $N \in \mathcal{M}$, we write $u(N)$ in what follows.

## 4. THE COST FUNCTIONAL OF $r$-METHOD

As mentioned in the introduction it is not evident (as in the case of classical boundary value problems) how to choose the criterion function. In the case of Dirichlet boundary conditions (for example), the minimization of the total potential energy functional at the equilibrium state with respect to $N \in \mathcal{M}$ evidently minimizes the discretization error in energy norm.

The case with unilateral boundary conditions is more involved. The aim in solving Problem (**P**) is to control the following functionals:

$$I_1(N) = \frac{1}{2}\,a(u(N) - u, u(N) - u) \tag{4.1}$$

and

$$\begin{aligned} I_2(N) &= \left\langle \frac{\partial}{\partial n} u, u(N) \right\rangle \quad \text{in Case 1 (problem (3.1))}, \\ I_2(N) &= \langle T_2(u), u_2(N) + \alpha \rangle \quad \text{in Case 2 (problem (3.9))}, \end{aligned} \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ means the duality pairing between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$, $H^{-1/2}(\Gamma_C)$ and $H^{1/2}(\Gamma_C)$, respectively [10]. In Case 2 we also control

$$I_3(N) = \int_{\Gamma_C} (g|u_1(N)| + T_1(u)u_1(N))\,\mathrm{d}s.$$

For convenience we set $I_3(N) = 0$ in Case 1.

To see this we prove

**Theorem 4.1.** *Let $N \in \mathcal{M}$. Then*

$$I_1(N) + I_2(N) + I_3(N) = \mathcal{L}(N) - J(u),$$

*where $J$ is given as in* (3.3) *or in* (3.15) *and $\mathcal{L}(N) = J(u(N))$.*

Proof. In Case 1 we obtain by the Taylor-expansion

$$\mathcal{L}(N) = J(u) + J'(u, u(N) - u) + \frac{1}{2}\,a(u(N) - u, u(N) - u), \tag{4.3}$$

where

$$J'(u, u(N) - u) = a(u, u(N) - u) - \langle L, u(N) - u \rangle. \tag{4.4}$$

Choosing $v = 0$ and $v = 2u$ in (3.2), we see that

$$a(u, u) - (f, u)_{0,\Omega} = 0, \tag{4.5}$$

so

$$J'(u, u(N)-u) = a(u, u(N)) - (f, u(N))_{0,\Omega}. \tag{4.6}$$

Now using Green's formula in (4.6) for problem (3.1) we have

$$J'(u, u(N)-u) = \left\langle \frac{\partial}{\partial n} u, u(N) \right\rangle. \tag{4.7}$$

In Case 2 we have for $J_0(u(N)) = J(u(N)) - j(u(N))$ as in (4.3) that

$$\begin{aligned} J(u(N)) &= J_0(u(N)) + j(u(N)) \\ &= J_0(u) + J_0'(u, u(N)-u) + \frac{1}{2} a(u(N)-u, u(N)-u) + j(u(N)) \end{aligned} \tag{4.8}$$

with $J_0'(u, u(N)-u)$ analogous to (4.4). Using Green's formula for $J_0'(u, u(N)-u)$ and making use of the equations (3.9) and boundary conditions on $\Gamma_D$ and $\Gamma_P$ we have

$$\begin{aligned} J_0'(u, u(N)-u) &= \int_\Omega \left[ \left( -\frac{\partial}{\partial x_j} \tau_{ij}(u) \right) (u_i(N) - u_i) \right] \mathrm{d}x \\ &\quad + \int_{\partial\Omega} \tau_{ij}(u) n_j (u_i(N) - u_i) \, \mathrm{d}s - \langle L, u(N) - u \rangle \\ &= \langle T_2(u), u_2(N) - u_2 \rangle + \int_{\Gamma_C} T_1(u)(u_1(N) - u_1) \, \mathrm{d}s. \end{aligned} \tag{4.9}$$

So we have

$$\begin{aligned} J(u(N)) - J(u) &= \frac{1}{2} a(u(N)-u, u(N)-u) + \langle T_2(u), u_2(N) - u_2 \rangle \\ &\quad + \int_{\Gamma_C} T_1(u)(u_1(N) - u_1) \, \mathrm{d}s + j(u(N)) - j(u) \\ &= \frac{1}{2} a(u(N)-u, u(N)-u) + \langle T_2(u), u_2(N) - u_2 \rangle \\ &\quad + \int_{\Gamma_C} (g|u_1(N)| + T_1(u) u_1(N)) \, \mathrm{d}s - \int_{\Gamma_C} (g|u_1| + T_1(u) u_1) \, \mathrm{d}s \\ &= \frac{1}{2} a(u(N)-u, u(N)-u) + \langle T_2(u), u_2(N) + \alpha \rangle \\ &\quad + \int_{\Gamma_C} (g|u_1(N)| + T_1(u) u_1(N)) \, \mathrm{d}s \end{aligned} \tag{4.10}$$

again using the contact and boundary conditions (3.10).

Consequently,

$$I_1(N) + I_2(N) + I_3(N) = \mathcal{L}(N) - J(u)$$

in both cases. □

R e m a r k 4.1. For problem (3.1), if $\frac{\partial u}{\partial n} \in L^2(\Gamma_2)$ then

$$I_2(N) = \left\langle \frac{\partial u}{\partial n}, u(N) \right\rangle = \int_{\Gamma_2} \frac{\partial u}{\partial n} \cdot u(N) \, dx_2 = \int_{\tilde{\Gamma}_2} \frac{\partial u}{\partial n} \cdot u(N) \, dx_2,$$

where $\tilde{\Gamma}_2 = \left\{ x \in \Gamma_2 \mid \frac{\partial u}{\partial n}(x) > 0 \right\}$. For problem (3.9)

$$I_2(N) = \int_{\Gamma_C} T_2(u)(u_2(N) + \alpha) \, ds = \int_{\tilde{\Gamma}_C} T_2(u)(u_2(N) + \alpha) \, ds$$

where $\tilde{\Gamma}_C = \{x \in \Gamma_C \mid T_2(u(x)) > 0\}$ provided $T_2(u) \in L^2(\Gamma_C)$. For the exact solutions of problems (3.1) and (3.9) we have

$$\left\langle \frac{\partial u}{\partial n}, u \right\rangle = 0 \qquad \text{in Case 1 (problem (3.1))}$$

and

$$\int_{\Gamma_C} T_2(u)(u_2 + \alpha)) \, ds = 0 \qquad \text{in Case 2 (problem (3.9))},$$

and $I_2(N) \geqslant 0$ for all $u(N)$ in both cases as $\Omega$ is polygonal. So in both cases 1 and 2 reduction of $I_2$ in fact means a better approximation of the contact surface obtained by the FE-solution $u(N)$.

For problem (3.9) we control by $I_3$ better approximation of the sliding and non-sliding zones, as

$$I_3(N) = \int_{\Gamma_C} (g|u_1(N)| + T_1(u)u_1(N)) \, ds \geqslant 0$$

and for the exact solution of (3.9) we have

$$\int_{\Gamma_C} (g|u_1| + T_1(u)u_1) \, ds = 0.$$

**5.1. Case 1: Problem (3.1).** In order to efficiently apply NLP-algorithms for solving numerically Problem (**P**) we need information on the gradient of $\mathcal{L}$ with respect to the change of $N$. We shall use a similar technique as in [5, 8] and compute the gradient of $\mathcal{L}$ in terms of a nodal displacement vector $V \in \mathcal{K} = \{V \in \mathbb{R}^{D(N)} \mid V_i \geqslant 0,\ i \in I_{\Gamma_2}\}$, where $I_{\Gamma_2}$ refers to the indices of nodes lying on $\Gamma_2$, $D(N) = p + \operatorname{card} I_{\Gamma_2}$ and $p$ is equal to the number of internal nodes of $\mathcal{T}(N)$.

The algebraic analogue of (3.6) is

$$U = U(N) = \operatorname*{arg\,min}_{V \in \mathcal{K}} \left\{ J(V, N) = \tfrac{1}{2}(V, A(N)V) - (F(N), V) \right\}. \tag{5.1}$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{D(N)}$,

$$A(N) = \Big( \sum_{T \in \mathcal{T}(N)} \int_T \nabla\varphi_i(N) \cdot \nabla\varphi_j(N)\, \mathrm{d}x \Big)_{i,j=1}^{D(N)} \tag{5.2}$$

denotes the global stiffness matrix and

$$F(N) = \Big( \sum_{T \in \mathcal{T}(N)} \int_T f\varphi_j(N)\, \mathrm{d}x \Big)_{j=1}^{D(N)} \tag{5.3}$$

is the force vector; $\varphi_i(N)$ are the Courant basis functions corresponding to $\mathcal{T}(N)$. We emphasize the dependence of $A$ and $F$ on $N$ by writing $N$ as an argument. By using this notation, the criterion function of problem (**P**) takes the form

$$\mathcal{L}(N) = J(U(N), N). \tag{5.4}$$

Let $N = (N_1, N_2, \dots, N_d) = (x_{1,1}, x_{2,1}, \dots, x_{1,d}, x_{2,d}) \in \mathbb{R}^{2d}$, let $N_j = (x_{1,j}, x_{2,j})$ be the nodes of $\mathcal{T}(N)$ which can be relocated, i.e. all nodal coordinates forming $\mathcal{T}(N)$ with the exception of the corners of $\overline{\Omega}$. Let $Q \in \mathbb{R}^{2d}$. By $\mathcal{T}(N + tQ), t > 0$ we denote the new triangulation of $\overline{\Omega}$, nodes of which are given by

$$\begin{aligned} x'_{1,i} &= x_{1,i} + tv_i \\ x'_{2,i} &= x_{2,i} + tw_i, \end{aligned} \tag{5.5}$$

where $Q = (v_1, w_1, \dots, v_d, w_d)$. We denote by

$$\mathcal{L}'(N) = \mathcal{L}'(N)Q = \lim_{t \to 0+} \frac{\mathcal{L}(N + tQ) - \mathcal{L}(N)}{t} \tag{5.6}$$

the directional derivative of $\mathcal{L}$ at $N$ in the direction $Q$; analogously $U'$, $A'$ and $F'$ are directional derivatives of $U, A$ and $F$ at $N$ in the direction $Q$. Let us mention that the corresponding derivatives exist (see [8]).

From (5.4) we obtain

$$
\begin{aligned}
\mathcal{L}'(N) &= (U'(N), A(N)U(N) - F(N)) \\
&\quad + (U(N), \tfrac{1}{2}A'(N)U(N) - F'(N)) \\
&= (U(N), \tfrac{1}{2}A'(N)U(N) - F'(N))
\end{aligned} \tag{5.7}
$$

as $(U'(N), A(N)U(N) - F(N)) = 0$ (see [8]). The terms $A'(N)$ and $F'(N)$ can be calculated in several ways: either one could use the finite difference or the analytic derivation of the local stiffness matrix and the force vector in the usual manner ([8, 15]).

We close this chapter with a remark. As is well known in the case of variational inequalities, the mapping $N \longrightarrow U(N)$ is not differentiable in general. However, in the case of the cost functional of the total potential energy it was possible to compute the gradient of $\mathcal{L}$. This was possible due to the vanishing of the term where $U'(N)$ appears.

**5.2. Case 2: Problem (3.9).** Again, let $N = (N_1, N_2, \dots, N_d) = (x_{1,1}, x_{2,1}, \dots, x_{1,d}, x_{2,d}) \in \mathbb{R}^{2d}$ be the nodes of $\mathcal{T}(N)$ which can be relocated, i.e. all in $\mathcal{T}(N)$ with the exception of the vertices of $\overline{\Omega}$ and let $Q = (v_1, w_1, \dots, v_d, w_d) \in \mathbb{R}^{2d}$. The symbol $\mathcal{T}(N + tQ)$, $t > 0$ has exactly the same meaning as before.

Let $F$ and $P$ denote the discretization of the body force and surface traction, respectively, and let $A$ be the FEM-stiffness matrix corresponding to the elasticity problem. Problem (3.18) can be written in a matrix form

$$
\begin{cases}
\text{Find } U(N) \in K(N) \text{ such that} \\
J(U(N), N) \leqslant J(Z, N) \qquad \forall\, Z \in K(N),
\end{cases} \tag{5.8}
$$

where

$$
J(Z, N) = \frac{1}{2}(Z, A(N)Z) + g \sum_{j_i \in I_1} \omega_i(N)|Z_{j_i}| - (F(N) + P(N), Z), \tag{5.9}
$$

with $\omega_i(N) > 0$ being the weights of the quadrature formula used for numerical approximation of $j$. The set $K(N)$ is given by

$$
K(N) = \{Z \in R^{D(N)} \mid Z_{j_i} \geqslant -\alpha(a_i) \quad \forall\, j_i \in I_2\}, \tag{5.10}
$$

where $a_i$ denotes the $x_1$-coordinate of the node lying on $\Gamma_C$. Here $I_1, I_2$ are two disjoint sets, containing indices of $x_1$- and $x_2$-components of the displacement field $(u_1(N), u_2(N))$ at nodes on $\Gamma_C$.

Let $\mathcal{L}(N) = J(U(N), N)$, where $U(N)$ is the solution of (5.8). Following [8] we get the directional derivatives of $\mathcal{L}$ at $N$ in the direction $Q \in R^{2d}$:

$$\begin{aligned}\mathcal{L}'(N, \tilde{N}) = \frac{1}{2}(U(N), A'(N)U(N)) - (F'(N) + P'(N), U(N)) \\ - \sum_{j_i \in \tilde{I}_2} w_i r_{j_i}(N) + g \sum_{j_i \in \tilde{I}_1} \omega_i'(N)|U_{j_i}(N)|.\end{aligned} \tag{5.11}$$

Here $\tilde{I}_1 \subset I_1$, $\tilde{I}_2 \subset I_2$ result from $I_1$, $I_2$, respectively, by deleting the indices corresponding to the $x_1$- and $x_2$-components of the displacement field $(u_1(N), u_2(N))$ at fixed vertices lying on $\overline{\Gamma}_C$, and $r(N)$ is the residual vector $r(N) = A(N)U(N) - F(N) - P(N)$.

## 6. Numerical examples

**6.1. Case 1: Problem (3.1).** We have tested the optimal grid design method presented above.

**Example 6.1.** Suppose that $\Omega$ is the unit square and

$$f(x_1, x_2) = \begin{cases} \sin(2\pi x_1)\big[4\pi^2(\frac{1}{2} - x_2)^3 x_2(-2x_2 - 1) + 2 - (1 - 2x_2)(6x_2^2 - 4x_2 + \frac{1}{2}) \\ \quad - (\dfrac{1}{2} - x_2)^2(12x_2 - 4) - 4x_2^3 + 4x_2^2 - 4\big], & \text{if } x_2 \leqslant \dfrac{1}{2} \\ \sin(\frac{1}{2}\pi x_1)(\frac{1}{2} - x_2)^2\big[16 - 20x_2 \\ \quad - \frac{1}{4}\pi^2 \sin(\frac{1}{2}\pi x_1)(1 - x_2)(\frac{1}{2} - x_2)^2\big], & \text{if } x_2 \geqslant \frac{1}{2} \end{cases} \tag{6.1}$$

For this $f$ the exact solution $u$ of the unilateral boundary value problem (3.1) is

$$u(x_1, x_2) = \begin{cases} 100(\frac{1}{2} - x_2)^2 \sin(2\pi x_1)(2x_2^3 - 2x_2^2 + \frac{1}{2}x_2), & \text{if } x_2 \leqslant \frac{1}{2} \\ 100 \sin(\frac{1}{2}\pi x_1)(1 - x_2)(\frac{1}{2} - x_2)^4, & \text{if } x_2 \geqslant \frac{1}{2}. \end{cases} \tag{6.2}$$

In Figure 6.1 the solution $u$ is shown.

As the initial triangulation $\mathcal{T}_h$ a uniform grid with mesh size $h = \frac{1}{8}$ has been used, which in the grid optimization problem (**P**) gave 126 degrees of freedom.

For the tolerance parameter $\delta$ in conditions (2.1) and (2.2) we used the value $\delta = 0.0275$. For optimization we used the E04VDF-routine of NAG-library (SQP-method). The initial value of $J_0$ was $J_0 = -1.1246$ and the value $J_{\text{opt}}$ of the optimized grid was $J_{\text{opt}} = -1.4445$. For $N_{\text{opt}}$ the box constraints became active. The value of the energy for the exact solution can be computed symbolically to be $-(\frac{625}{2016} + \frac{125\pi^2}{202752} + \frac{125(220+7\pi^2)}{22176}) \approx -1.9456$.

In Figures 6.2 and 6.3 we see the initial grid and the optimized grid together with the contour plots of the FE-solutions.

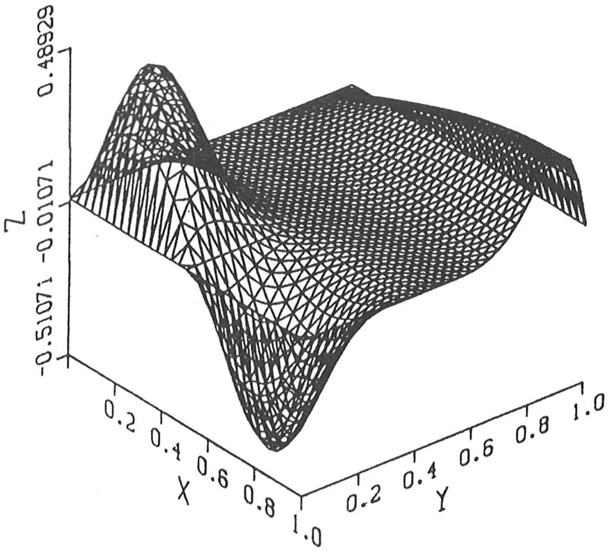


Fig. 6.1 The solution $u$ of (3.1) for $f$ given by (6.1).

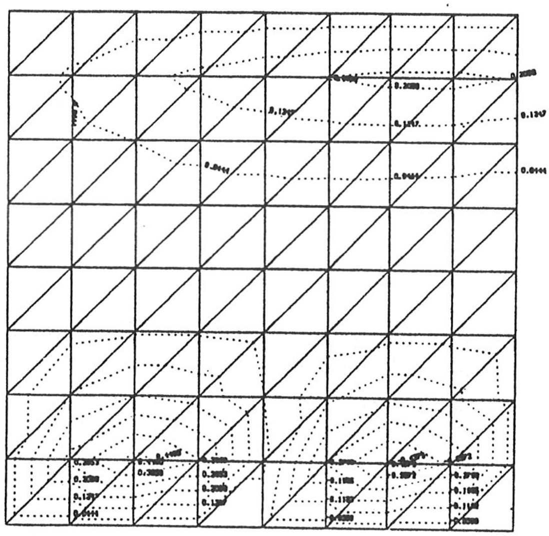

Fig. 6.2 $u(N_0)$

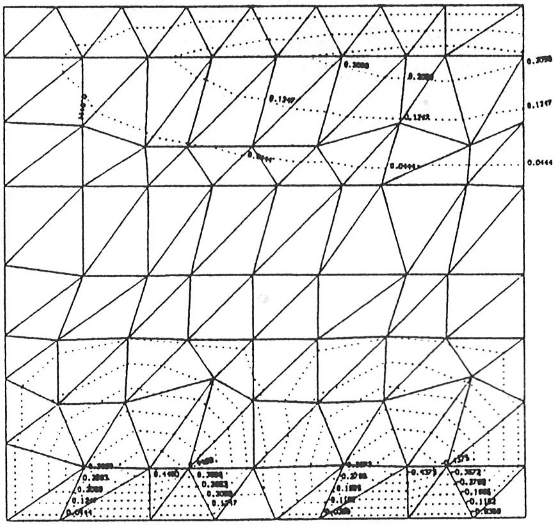

Fig. 6.3 $u(N_{\text{opt}})$

The minimum and maximum values of the solution of (3.1) for $f$ given by (6.1) are

$$u_{\text{min}} = -\frac{64}{125} = -0.512 \qquad u_{\text{max}} = \frac{64}{125}.$$

The corresponding values computed by using the initial grid and the optimized grid are

$$u_{\text{min}}(N_0) = -4.549 \cdot 10^{-1} \qquad u_{\text{max}}(N_0) = 4.753 \cdot 10^{-1}$$
$$u_{\text{min}}(N_{\text{opt}}) = -5.178 \cdot 10^{-1} \qquad u_{\text{max}}(N_{\text{opt}}) = 5.263 \cdot 10^{-1}.$$

We find that the FE-solution has been improved in this sense as well.

**6.2. Case 2: Problem (3.9).** Here we use simple rectangular domains as examples, but the analysis applies to any polygonal domains provided the corners defining the domain are kept fixed. Solving the grid relocation problem (**P**) one has a non-linear minimization problem with constraints. As above, the SQP (Sequential Quadratic Programming)-method was used (NAG routine E04VDE).

The material was assumed to be homogeneous and isotropic with Lamé's coefficients $\lambda = 1.15 \cdot 10^4 Nm^{-2}, \mu = 0.83 \cdot 10^4 Nm^{-2}$. The body force $F$ is taken to be $F = (0, 0)$ and surface tractions $P$ are applied on the top of $\Omega$, only. Boundary nodes (except corners and nodes lying on the top of $\partial\Omega$) are allowed to move in one direction. Here we assume that not only the corners, but also the nodes on the top of $\partial\Omega$ are kept fixed. In such a case $P$ is the same for all $N \in \mathcal{M}$.

**Example 6.2.** Let $\Omega_1 = (0,4) \times (0.05, 1)$ , $P = \begin{cases} (0,0) \text{ on } (0,2) \times \{1\} \\ (0,-57.5) \text{ on } (2,4) \times \{1\} \end{cases}$. We take the $\delta$-neighbourhood of $\overline{N}_i$ (in (2.1)) to be the square $[\overline{x}_i - \delta_x, \overline{x}_i + \delta_x] \times [\overline{y}_i - \delta_y, \overline{y}_i + \delta_y]$, where $\delta_x = 0.15667, \delta_y = 0.07333$. In Fig. 6.4 we see the domain $\Omega_1$ with the initial grid both before the deformation and after the deformation. In Fig. 6.5 we have the FE-grid after the application of grid optimization, plotted both before and after the deformation. The energy value of the initial grid was $J_0 = -4.8541$ and the value for the optimized grid was $J_{\text{opt}} = -4.9188$.

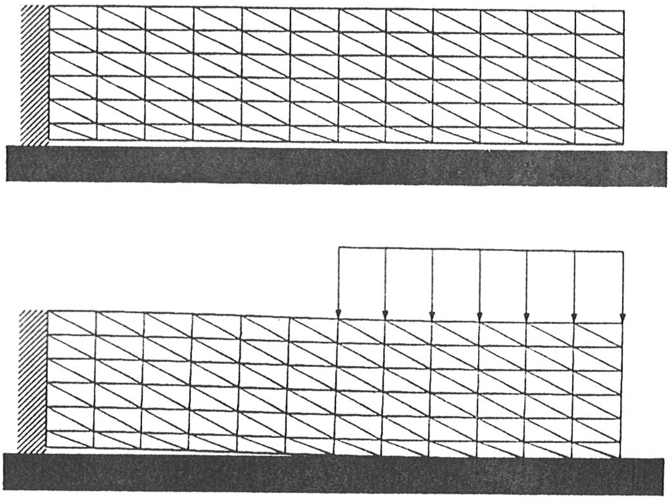

Fig. 6.4

In Fig. 6.6 the contact stresses for the initial grid and the optimized grid are shown.

**Example 6.3.** In the last example we consider a case with friction. We take $\Omega_2 = (0,4) \times (0,1)$, $P = \begin{cases} (0,0) \text{ on } (0,2) \times \{1\} \\ (-57.5,-57.5) \text{ on } (2,4) \times \{1\} \end{cases}$ and $g = 10$. Set $\delta_x = 0.08333$, $\delta_y = 0.04166$. Figs. 6.7 and 6.8 show the initial grid and the optimized grid for $\Omega_2$ both in the undeformed and deformed state. Fig. 6.9 shows the corresponding contact stresses and Fig. 6.10 shows the corresponding tangential stresses. For $\Omega_2$ the energy value of the initial grid was $J_0 = -0.6099$ and the value for the optimized grid was $J_{\text{opt}} = -0.6211$.

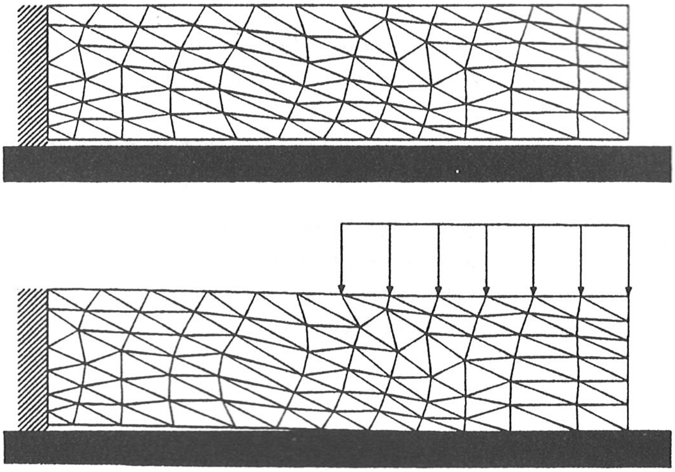

Fig. 6.5

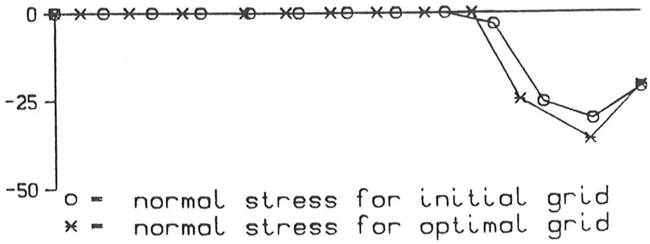


Fig. 6.6

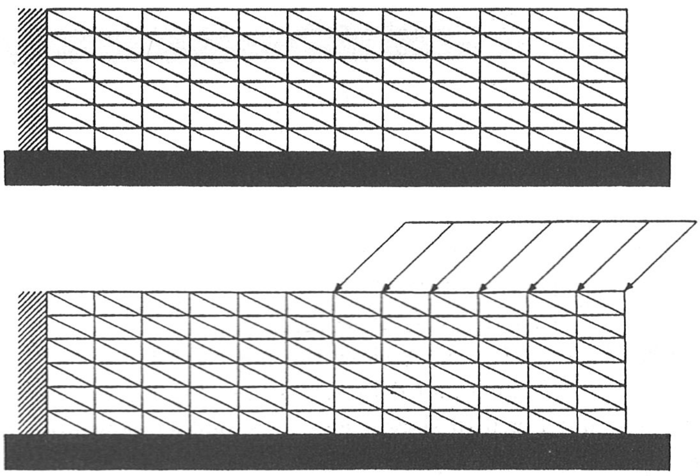

Fig. 6.7

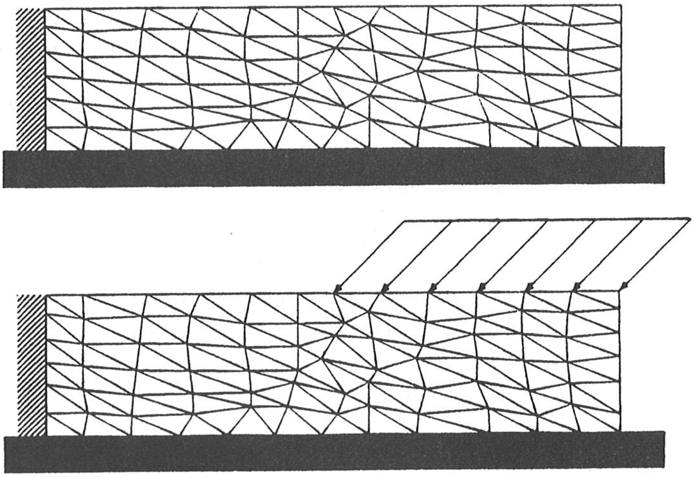

Fig. 6.8

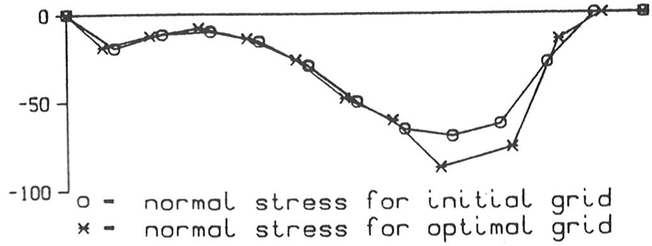


Fig. 6.9

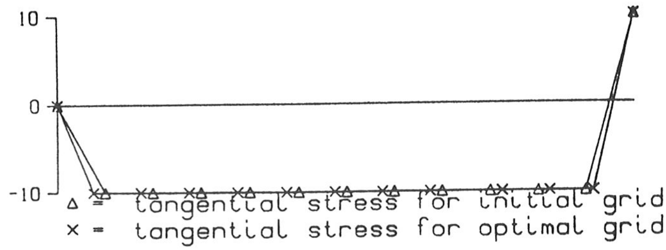


Fig. 6.10

*References*


[1] *Babuška, I., Rank, E.*: An expert-system-like feedback approach in the *hp*-version of the finite element method, Finite Elements in Analysis and Design *3* (1987), 127–149.

[2] *Babuška, I., Zienkiewicz, O.C., Gago, J., de A. Oliveira, E.R. (eds.)*: Accuracy Estimates and Adaptive Refinements in Finite Element Computations, J. Wiley & Sons, Chichester, 1986.

[3] *Blum, H., Lin, Q., Rannacher, R.*: Asymptotic error expansion and Richardson extrapolation for linear finite elements, Numerische Mathematik *49* (1986), 11–37.

[4] *Ciarlet, P.G.*: The Finite Element Method for elliptic problems, North-Holland, Amsterdam, 1978.

[5] *Delfour, M., Payre, G., Zolésio, J.P.*: An optimal triangulation for second-order elliptic problems, Computational Methods in Applied Mechanics and Engineering *50* (1985), 231-261.

[6] *Guo, B., Babuška, I.*: The *h-p* version of the finite element method, Computational Mechanics, Part 2 *1* (1986), 203-220.

[7] *Eriksson, K., Johnson, C.*: An adaptive finite element method for linear elliptic problems, Mathematics of Computation *50* (1988), 361–383.

[8] *Haslinger, J., Neittaanmäki, P.*: Finite Element Approximation of Optimal Shape Design. Theory and Approximation, John Wiley & Sons, London, 1988.

[9] *Haslinger, J., Neittaanmäki, P., Salmenjoki, K.*: On optimal mesh design for FEM in unilateral boundary value problems, MAFELAP VI, (J.R. Whiteman ed.) (1988), 103–114, Academic Press, London.

[10] *Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.*: Numerical Solution of Variational Inequalities, Springer Series in Applied Mathematical Sciences 66, Springer-Verlag, Berlin, 1988.

[11] *Jarusch, H.*: On an adaptive grid refining technique for finite element approximations, SIAM Journal of Scientific and Statical Computations *4* (1986), 1105-1120.

[12] *Kikuchi, N.*: Finite Element Methods in Mechanics, Cambridge University Press, London, 1986.

[13] *Kikuchi, N., Oden, J.T.*: Contact Problems in Elasticity, SIAM Studies in Applied Mathematics, Philadelphia, 1988.

[14] *Křížek, M., Neittaanmäki, P.*: On superconvergence techniques, Acta Applic. Math. *9* (1987), 175-198.

[15] *Pironneau, O.*: Optimal shape design for elliptic systems, Springer-Verlag, Berlin, 1984.

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