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SOLVABILITY OF A FIRST ORDER SYSTEM
IN THREE-DIMENSIONAL NON-SMOOTH DOMAINS

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1. INTRODUCTION

In this article we first deal with the validity of the inequality

$$(1.1) \quad \|v\|_0 \leq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0),$$

where v is a vector function defined on a bounded and generally non-smooth domain $\Omega \subset \mathbb{R}^3$, and the vanishing normal component $n \cdot v$ on the boundary $\partial\Omega$ is assumed. Following some preliminary lemmas in the next section, we show that (1.1) holds if and only if Ω is simply connected (Section 3). The inequality (1.1) was established earlier for a smooth domain which is homeomorphic to a ball even for the $\|\cdot\|_1$ -norm on the left-hand side (see [3]). Other proofs are given in [8, 18–21]; they are mainly based on contradiction arguments. Estimates analogous to (1.1) for plane non-smooth domains are treated in [10] and in [11], where also mixed boundary conditions are prescribed. We also recall [15] that in the case of vanishing tangential components of v on $\partial\Omega$, the inequality (1.1) is valid iff $\partial\Omega$ is connected (in \mathbb{R}^2 and \mathbb{R}^3).

In Section 4 we apply (1.1) to the problem of solvability of the first order system of four partial differential equations

$$(1.2) \quad \begin{aligned} \operatorname{div} u &= f & \text{in } \Omega, \\ \operatorname{rot} u &= g & \text{in } \Omega, \\ n \cdot u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

which play an important role in fluid flow and magnetostatic problems [4, 5, 16–22].

2. SOME FUNCTION SPACES

Throughout the paper, $\Omega \subset \mathbb{R}^3$ will always be a bounded domain with a Lipschitz boundary $\partial\Omega$ (see [14], p. 17) and with the outward unit normal n . Notations $H^k(\Omega)$, $k = 0, 1, \dots$, are used for the (real valued) Sobolev spaces. The usual norm in $H^k(\Omega)$

and also in $(H^k(\Omega))^3$ will be denoted by $\|\cdot\|_k$. The scalar product on $(L^2(\Omega))^m, m = 1, 3$, will be written as $(\cdot, \cdot)_0$ and we set

$$L^2_0(\Omega) = \{\chi \in L^2(\Omega) \mid (\chi, 1)_0 = 0\}.$$

Further, $H^{1/2}(\partial\Omega)$ is the space of traces of functions from $H^1(\Omega)$, and $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with a compact support in Ω .

We note (see [9], p. 16) that the functional $v \mapsto n \cdot v|_{\partial\Omega}$ defined on $(C^\infty(\bar{\Omega}))^3$ can be extended by continuity to a linear continuous mapping from the space

$$H(\operatorname{div}; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists F \in L^2(\Omega) : (v, \operatorname{grad} z)_0 + (F, z)_0 = 0 \ \forall z \in \mathcal{D}(\Omega)\}$$

into $H^{-1/2}(\partial\Omega)$, the latter being the dual space to $H^{1/2}(\partial\Omega)$. The function F is called the divergence of v (in the sense of distributions) and the Green formula can be rewritten as

$$(2.1) \quad (\operatorname{div} v, z)_0 + (v, \operatorname{grad} z)_0 = \langle n \cdot v, z \rangle_{\partial\Omega} \quad \forall v \in H(\operatorname{div}; \Omega), \quad \forall z \in H^1(\Omega).$$

Here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Let $\partial\Omega_1, \dots, \partial\Omega_r$ be the components of $\partial\Omega$. For $v \in H(\operatorname{div}; \Omega)$ we define the functional $n \cdot v \in H^{-1/2}(\partial\Omega_i)$, $i \in \{1, \dots, r\}$, by

$$(2.2) \quad \langle n \cdot v, z \rangle_{\partial\Omega_i} = (\operatorname{div} v, z)_0 + (v, \operatorname{grad} z)_0, \quad z \in Z_i,$$

where

$$Z_i = \{z \in H^1(\Omega) \mid z = 0 \text{ on } \partial\Omega_j \ \forall j \in \{1, \dots, r\} - \{i\}\}$$

and $\langle \cdot, \cdot \rangle_{\partial\Omega_i}$ is the duality pairing between $H^{-1/2}(\partial\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$.

Let us further introduce the space

$$H(\operatorname{rot}; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists G \in (L^2(\Omega))^3 : (v, \operatorname{rot} z)_0 = (G, z)_0 \ \forall z \in (\mathcal{D}(\Omega))^3\}$$

endowed with the norm

$$\|\cdot\|_{H(\operatorname{rot}; \Omega)} = (\|\cdot\|_0^2 + \|\operatorname{rot} \cdot\|_0^2)^{1/2}.$$

The function G introduced above is called the rotation of v (in the sense of distributions) and the following Green formula holds:

$$(2.3) \quad (\operatorname{rot} v, z)_0 - (v, \operatorname{rot} z)_0 = \langle n \times v, z \rangle_{\partial\Omega} \quad \forall v \in H(\operatorname{rot}; \Omega) \quad \forall z \in (H^1(\Omega))^3.$$

Here the vector product $n \times v$ is from $(H^{-1/2}(\partial\Omega))^3$ (see [9], p. 21) and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $(H^{-1/2}(\partial\Omega))^3$ and $(H^{1/2}(\partial\Omega))^3$.

Now, we define several subspaces of $H(\operatorname{div}; \Omega)$ and $H(\operatorname{rot}; \Omega)$:

$$\begin{aligned} H_0(\operatorname{div}; \Omega) &= \{v \in H(\operatorname{div}; \Omega) \mid n \cdot v = 0 \text{ on } \partial\Omega\}, \\ H(\operatorname{div}^0; \Omega) &= \{v \in H(\operatorname{div}; \Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega\}, \\ H_0(\operatorname{div}^0; \Omega) &= H_0(\operatorname{div}; \Omega) \cap H(\operatorname{div}^0; \Omega), \\ H_0(\operatorname{rot}; \Omega) &= \{v \in H(\operatorname{rot}; \Omega) \mid n \times v = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

$$\begin{aligned}
H(\text{rot}^0; \Omega) &= \{v \in H(\text{rot}; \Omega) \mid \text{rot } v = 0 \text{ in } \Omega\}, \\
H_0(\text{rot}^0; \Omega) &= H_0(\text{rot}; \Omega) \cap H(\text{rot}^0; \Omega), \\
\mathcal{H}_{\mathcal{D}} &= H_0(\text{div}^0; \Omega) \cap H(\text{rot}^0; \Omega), \\
\mathcal{H}_{\mathcal{A}} &= H(\text{div}^0; \Omega) \cap H_0(\text{rot}^0; \Omega), \\
V &= H_0(\text{div}; \Omega) \cap H(\text{rot}; \Omega), \\
D &= \{v \in H(\text{div}^0; \Omega) \mid \langle n \cdot v, 1 \rangle_{\partial\Omega_i} = 0, i = 1, \dots, r\}.
\end{aligned}$$

From (2.1) we can easily derive

$$(2.4) \quad \text{grad } z \in H(\text{rot}^0; \Omega) \quad \text{for } z \in H^1(\Omega).$$

Henceforth, we shall present some other properties of the above spaces.

Lemma 2.1. *The following inclusions hold:*

$$(2.5) \quad \text{rot } v \in D \quad \text{for } v \in H(\text{rot}; \Omega),$$

and

$$(2.6) \quad \text{rot } v \in H_0(\text{div}^0; \Omega) \quad \text{for } v \in H_0(\text{rot}; \Omega).$$

Proof. Let $v \in H(\text{rot}; \Omega)$ and $z \in \mathcal{D}(\Omega)$ be given. Then by (2.3) we obtain

$$(2.7) \quad (\text{rot } v, \text{grad } z)_0 = (v, \text{rot grad } z)_0 + \langle n \times v, \text{grad } z \rangle_{\partial\Omega} = 0.$$

Hence, (2.1) yields

$$(2.8) \quad \text{rot } v \in H(\text{div}^0; \Omega).$$

Let us choose $i \in \{1, \dots, r\}$ arbitrarily and let $\eta \in C^\infty(\bar{\Omega})$ be such that $\eta = 1$ in a neighbourhood of $\partial\Omega_i$ and $\eta = 0$ in some neighbourhoods of the other components $\partial\Omega_j$, $j \neq i$, that is $\eta \in Z_i$. Thus (2.2), (2.8) and (2.3) imply

$$\langle n \cdot \text{rot } v, 1 \rangle_{\partial\Omega_i} = (\text{rot } v, \text{grad } \eta)_0 = \langle n \times \text{grad } \eta, v \rangle_{\partial\Omega} = 0.$$

Consequently, (2.5) is valid. The relation (2.7) holds for any $v \in H_0(\text{rot}; \Omega)$ and $z \in C^\infty(\bar{\Omega})$ as well. Therefore, $\text{rot } v \in H_0(\text{div}^0; \Omega)$. \square

Lemma 2.2. *The identity*

$$(\text{rot } \varphi, \text{rot } \varphi)_0 = (\varphi, \text{rot rot } \varphi)_0$$

holds for all $\varphi \in H_0(\text{rot}; \Omega)$ such that $\text{rot } \varphi \in H(\text{rot}; \Omega)$.

Proof. Let $\varphi \in H_0(\text{rot}; \Omega)$ with $\text{rot } \varphi \in H(\text{rot}; \Omega)$ be given. As $(C^\infty(\bar{\Omega}))^3$ is dense in $H(\text{rot}; \Omega)$ (see [6, 9]), there exists a sequence $\psi_j \in (C^\infty(\bar{\Omega}))^3$ such that

$$(2.9) \quad \|\text{rot } \varphi - \psi_j\|_{H(\text{rot}; \Omega)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Applying the Green formula (2.3), we get

$$(\text{rot } \varphi, \psi_j)_0 - (\varphi, \text{rot } \psi_j)_0 = \langle n \times \varphi, \psi_j \rangle_{\partial\Omega} = 0,$$

since $\varphi \in H_0(\text{rot}; \Omega)$. From (2.9) we conclude that

$$(\text{rot } \varphi, \psi_j)_0 \rightarrow (\text{rot } \varphi, \text{rot } \varphi)_0$$

and

$$(\varphi, \operatorname{rot} \psi_j)_0 \rightarrow (\varphi, \operatorname{rot} \operatorname{rot} \varphi)_0$$

for $j \rightarrow \infty$, which yields the result as required. \square

3. STUDY OF THE INEQUALITY (1.1)

First, let us recall the definition of a simply connected domain (see e.g. [2, 7, 12, 14]).

Definition 3.1. A domain Ω in \mathbb{R}^d is said to be simply connected if it has the following property: Given any simple closed curve $\gamma: x = h(t)$, $t \in [a, b]$, with range in Ω , there is a continuous function $x = F(s, t)$ defined for $s \in [0, 1]$, $t \in [a, b]$ such that:

- (i) $F(0, t) = h(t)$, $t \in [a, b]$;
- (ii) $F(1, t) = P$, $t \in [a, b]$, where P is some point in Ω ;
- (iii) $F(s, t)$ lies in Ω for all $s \in [0, 1]$, $t \in [a, b]$.
- (iv) $F(s, a) = F(s, b)$ for all $s \in [0, 1]$.

Defining (closed) curves γ_s by $x = F(s, t)$, $t \in [a, b]$, we say that the family $\{\gamma_s\}$ represents a continuous deformation of γ into a point P .

Domains which are not simply connected are called multiply connected.

The main task of this section will be to prove the following theorem.

Theorem 3.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then

$$(3.1) \quad \|v\|_0 \leq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0) \quad \forall v \in V = H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$$

if and only if Ω is simply connected.

The proof is based on an auxiliary lemma:

Lemma 3.3. Let Ω be a simply connected domain with a Lipschitz boundary and let $\psi \in H_0(\operatorname{div}^0; \Omega)$. Then there exists exactly one stream function $\varphi \in D \cap H_0(\operatorname{rot}; \Omega)$ such that

$$\psi = \operatorname{rot} \varphi.$$

Moreover,

$$(3.2) \quad \|\varphi\|_0 \leq C\|\operatorname{rot} \varphi\|_0,$$

where $C > 0$ does not depend on φ (and ψ).

Proof. For the existence of precisely one divergence-free stream function $\varphi \in D \cap H_0(\operatorname{rot}; \Omega)$ corresponding to $\psi \in H_0(\operatorname{div}^0; \Omega)$ see e.g. [1, 24]. We only prove the inequality (3.2).

From the unicity of φ and (2.6), the linear operator

$$(3.3) \quad \operatorname{rot}: D \cap H_0(\operatorname{rot}; \Omega) \rightarrow H_0(\operatorname{div}^0; \Omega)$$

is bijective. The space $H_0(\operatorname{div}^0; \Omega)$ equipped with the $\|\cdot\|_0$ -norm is a Banach space. One can easily find that the space $D \cap H_0(\operatorname{rot}; \Omega)$ with the norm $\|\cdot\|_{H(\operatorname{rot}; \Omega)}$ is a Banach space as well. As the operator (3.3) is continuous, i.e.

$$\|\operatorname{rot} \varphi\|_0 \leq C' \|\varphi\|_{H(\operatorname{rot}; \Omega)},$$

by the closed graph theorem the inverse (closed) operator is continuous as well. Thus (3.2) holds. \square

Proof of Theorem 3.2. \Rightarrow : It is known (see e.g. [1], p. 153) that $\Omega \subset \mathbb{R}^3$ is simply connected if and only if the components of $\mathbb{R}^3 - \bar{\Omega}$ are simply connected. Suppose that Ω is multiply connected. Then there exists a component ω of $\mathbb{R}^3 - \bar{\Omega}$ which is also multiply connected, and we show that (3.1) does not hold.

In accordance with Definition 3.1 there exists a simple closed curve $\gamma \subset \omega$ which cannot be continuously deformed into a point without leaving the domain ω . Clearly, γ can be chosen in such a way that it is smooth enough. Let $\tilde{\Gamma}$ be a sufficiently smooth orientable surface bounded by γ (see Fig. 1) and let

$$\Gamma = \tilde{\Gamma} \cap \Omega.$$

By a regularization technique (see e.g. [13], p. 58), it is easy to construct a function $q \in C^\infty(\Omega - \Gamma)$ with bounded derivatives such that $q = 1$ in an exterior neighbourhood of Γ (with respect to a given orientation of $\tilde{\Gamma}$), and $q = 0$ in an interior neighbourhood of Γ . Setting

$$w = \begin{cases} \operatorname{grad} q & \text{in } \Omega - \Gamma, \\ 0 & \text{on } \Gamma, \end{cases}$$

we see that $w \in (C^\infty(\bar{\Omega}))^3$ and that w is not a potential field globally on Ω .

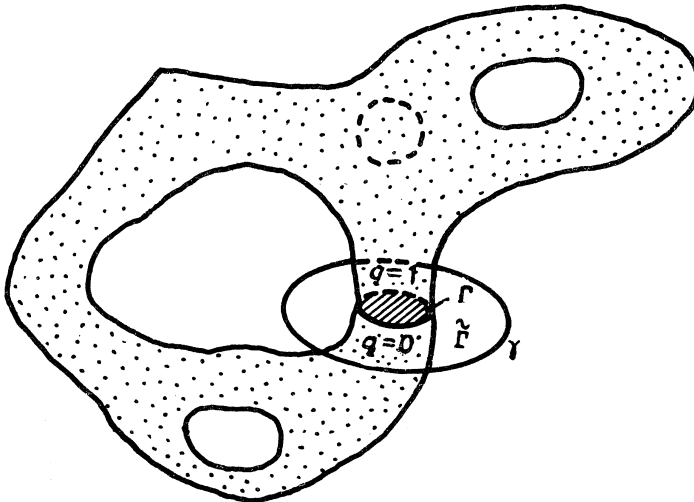


Fig. 1.

Consider the Neumann problem: Find $p \in H^1(\Omega)$ such that

$$(3.4) \quad \begin{aligned} \Delta p &= \operatorname{div} w \quad \text{in } \Omega, \\ \partial_n p &= n \cdot w \quad \text{on } \partial\Omega, \end{aligned}$$

(∂_n being the normal derivative), which is solvable because by (2.1)

$$(\operatorname{div} w, 1)_0 = \langle n \cdot w, 1 \rangle_{\partial\Omega}.$$

Now, let us define

$$(3.5) \quad v = \operatorname{grad} p - w.$$

Making use of (2.4) and (3.4), we arrive at

$$(v, \operatorname{grad} z)_0 = (\operatorname{grad} p - w, \operatorname{grad} z)_0 = \langle \partial_n p - n \cdot w, z \rangle_{\partial\Omega} = 0 \quad \forall z \in H^1(\Omega),$$

that is $v \in H_0(\operatorname{div}^0; \Omega)$.

Furthermore, $v \in H(\operatorname{rot}^0; \Omega)$ which follows from (3.5), (2.4) and the fact that $w \in (C^\infty(\bar{\Omega}))^3$ vanishes in some neighbourhood of Γ . Consequently, v satisfies (1.2) with zero right-hand sides. On the other hand $v \neq 0$, since it is not a potential field by (3.5). So the inequality (3.1) is not valid for multiply connected domains.

\Leftarrow : Let Ω be simply connected and let $v \in V$ be given. Consider the problem

$$(3.6) \quad \begin{aligned} \Delta z &= \operatorname{div} v \quad \text{in } \Omega, \\ \partial_n z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which has exactly one weak solution z in $L_0^2(\Omega) \cap H^1(\Omega)$, because $\operatorname{div} v \in L_0^2(\Omega)$ by (2.1), and it holds that

$$(3.7) \quad \|z\|_1 \leq C_1 \|\operatorname{div} v\|_0.$$

The relations (2.1), (2.4) and (3.6) give $\operatorname{grad} z \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}^0; \Omega)$, i.e. again by (3.6)

$$(3.8) \quad \psi = v - \operatorname{grad} z \in H_0(\operatorname{div}^0; \Omega) \cap H(\operatorname{rot}; \Omega).$$

In accordance with Lemma 3.3 there exists exactly one stream function $\varphi \in D \cap H_0(\operatorname{rot}; \Omega)$ such that

$$(3.9) \quad \psi = \operatorname{rot} \varphi.$$

Applying now Lemma 2.2 and (3.2), we come to

$$(3.10) \quad \begin{aligned} \|\operatorname{rot} \varphi\|_0^2 &= (\operatorname{rot} \varphi, \operatorname{rot} \varphi)_0 = (\varphi, \operatorname{rot} \operatorname{rot} \varphi)_0 \leq \\ &\leq \|\varphi\|_0 \|\operatorname{rot} \operatorname{rot} \varphi\|_0 \leq C_2 \|\operatorname{rot} \varphi\|_0 \|\operatorname{rot} \operatorname{rot} \varphi\|_0. \end{aligned}$$

So by (3.8), (3.9), (3.10), (3.7) and (2.4) we obtain

$$\begin{aligned} \|v\|_0 &\leq \|\operatorname{grad} z\|_0 + \|\operatorname{rot} \varphi\|_0 \leq \|z\|_1 + C_2 \|\operatorname{rot} \operatorname{rot} \varphi\|_0 \leq \\ &\leq C_1 \|\operatorname{div} v\|_0 + C_2 \|\operatorname{rot} \psi\|_0 \leq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0). \quad \square \end{aligned}$$

Remark 3.4. The spaces \mathcal{H}_Ω and $\mathcal{H}_\mathbb{R}$ are finite-dimensional (cf. [18, 19, 22, 23]). From Theorem 3.2 we see that \mathcal{H}_Ω is trivial iff Ω is simply connected; (note that $\mathcal{H}_\mathbb{R}$ is trivial iff $\partial\Omega$ is connected [15]). The proof of the inequality (3.1) can be modified for $v \in V \cap (\mathcal{H}_\Omega)^\perp$ without any assumptions on the connectivity of Ω (the symbol \perp denotes the orthocomplement in $(L^2(\Omega))^3$). This was proved e.g. in [21] for smooth domains.

4. APPLICATION TO A VARIATIONAL PROBLEM

In this section we shall deal with a variational formulation of the problem (1.2).

For $f \in L^2(\Omega)$ and $g \in (L^2(\Omega))^3$ we define the linear form

$$(4.1) \quad b(v) = (f, \operatorname{div} v)_0 + (g, \operatorname{rot} v)_0, \quad v \in V,$$

and the bilinear form

$$(4.2) \quad a(w, v) = (\operatorname{div} w, \operatorname{div} v)_0 + (\operatorname{rot} w, \operatorname{rot} v)_0, \quad w, v \in V.$$

Assume that a sufficiently smooth u satisfies (1.2) in the classical sense. Then we immediately see that $u \in V$ and

$$\begin{aligned} (\operatorname{div} u, \operatorname{div} v)_0 &= (f, \operatorname{div} v)_0, \\ (\operatorname{rot} u, \operatorname{rot} v)_0 &= (g, \operatorname{rot} v)_0 \end{aligned}$$

for all $v \in V$. Consequently,

$$(4.3) \quad a(u, v) = b(v) \quad \forall v \in V,$$

and moreover, by (2.1) and (2.5) we have

$$(4.4) \quad f \in L^2_0(\Omega), \quad g \in D.$$

Conversely, let (4.4) hold and let (4.3) be satisfied for a sufficiently smooth $u \in V$. Assuming that Ω is simply connected, we show that u fulfils (1.2).

So let $\chi \in L^2_0(\Omega)$ be arbitrary and let $z \in L^2_0(\Omega) \cap H^1(\Omega)$ be the weak solution of the problem

$$(4.5) \quad \begin{aligned} \Delta z &= \chi \quad \text{in} \quad \Omega, \\ \partial_n z &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Then $v = \operatorname{grad} z \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}^0; \Omega) \subset V$ and from (4.5), (4.2), (4.3) and (4.1) we get

$$(\operatorname{div} u, \chi)_0 = (\operatorname{div} u, \operatorname{div} v)_0 = a(u, v) = b(v) = (f, \operatorname{div} v)_0 = (f, \chi)_0.$$

Hence, $\operatorname{div} u = f$ in $L^2_0(\Omega)$.

Furthermore, let $\psi \in D$ be arbitrary. Then by [9], p. 28, there exists a divergence-free stream function $v' \in H(\operatorname{div}^0; \Omega) \cap (H^1(\Omega))^3$ (not uniquely determined) such

that $\psi = \text{rot } v'$. As $\langle n \cdot v', 1 \rangle_{\partial\Omega} = 0$ due to (2.1), the following problem is solvable:

$$\begin{aligned} \Delta\eta &= 0 & \text{in } \Omega, \\ \partial_n\eta &= n \cdot v' & \text{on } \partial\Omega. \end{aligned}$$

Then clearly the function $v = v'$ -grad η is from $H_0(\text{div}^0; \Omega) \cap H(\text{rot}; \Omega) \subset V$ and v is also a divergence-free stream function to ψ , that is

$$\psi = \text{rot } v$$

(cf. [1, 15, 24]). Using (4.2), (4.3) and (4.1), we arrive at

$$(\text{rot } u, \psi)_0 = (\text{rot } u, \text{rot } v)_0 = a(u, v) = b(v) = (g, \text{rot } v)_0 = (g, \psi)_0,$$

i.e. $\text{rot } u = g$ in D .

Thus we have justified the following definition.

Definition 4.1. Let Ω be simply connected. The problem of finding $u \in V$ which satisfies (4.3) is called the variational formulation of the problem (1.2).

Theorem 4.2. Let Ω be simply connected. Then the variational formulation of the problem (1.2) has precisely one solution.

Proof. By Theorem 3.2 the bilinear form (4.2) is a scalar product on V . It is easy to show that V is a Hilbert space and that the linear form (4.1) is continuous on V . Now the assertion follows from the Riesz theorem. \square

Remark 4.3. When Ω is multiply connected the bilinear form (4.2) is a scalar product on $V \cap (\mathcal{H}_\varnothing)^\perp$ (cf. Remark 3.4), i.e. the solution of (1.2) exists and is unique apart from a function of \mathcal{H}_\varnothing .

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Souhrn

ŘEŠITELNOST JISTÉHO SYSTÉMU PRVNÍHO ŘÁDU NA TROJROZMĚRNÝCH NEHLADKÝCH OBLASTECH

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Je studován systém parciálních diferenciálních rovnic prvního řádu, který je definován pomocí operátorů divergence a rotace na ohraničené oblasti $\Omega \subset \mathbb{R}^3$ s nehladkou hranicí. Na hranici $\partial\Omega$ je předepsána nulová normálová složka řešení. Je podána variační formulace a vyšetřována její řešitelnost.

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