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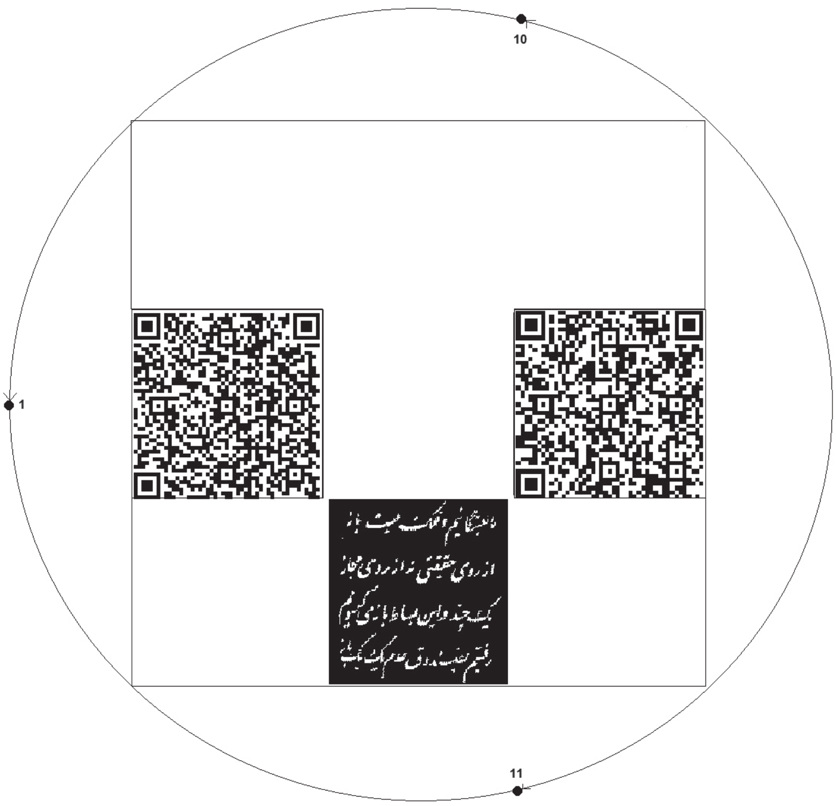
Jorma Kyppö

The N-dimensional N-person Chesslike Game Strategy Analysis Model



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UNIVERSITY OF JYVÄSKYLÄ

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But helpless Pieces of the Game
He plays Upon this Chequer-board of Nights and Days;
Hither and thither moves, and checks, and slays,
And one by one back in the Closet lays...

Omar Khayyam
(Rubaiyat, translation by Edward Fitzgerald)

ABSTRACT

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In this research *a mathematical, symmetric n-player game model, based on chess* is designed. Symmetry in this context refers to players' positions with respect to each other. While the order of move naturally violates the symmetry, this problem may also be solved. The motivation for building this kind of game model stems from the difficulty of finding mathematical solutions for multi-player games in general. The number of varying factors is so huge, that finding optimal strategies is mathematically almost impossible. The best way to attempt this is to use simulation. Once the model has been built, it can be applied in many ways by using computational algorithms based on the created model. Chess in this design is the basic structure around which the model is built. The players' weighting values can later be changed, as well as the weighting values of the pieces, in order to better reflect the variety of real-life situations. While chess is a board game, it can mirror various types of interactions between a number of different participants. Thus the game, in a larger extent, may play a role in understanding such things as politics, ecology and weather forecasting. During this research a great number of spin-off results and observations were discovered.

The main objective and result of this research was, however, to create a symmetric n-person strategy game, because currently there is no simple mathematical model for symmetric n-player, strategy games.

Keywords: N-player strategy game, combinatorics, tiling, topology, chess, multinomial formula, tetrahedron, game theory, graph theory

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Besides my advisor, I would like to thank my colleague, friend and supervisor in the initial stage of my PhD. study, Prof. Frank Harary from the University of New Mexico. After passing away in 2005, my research work was put aside for a while and my focus was turned over to teaching. Coincidentally, Frank Harary was born in the same year as my father Ensio Kyppö, whose motivating and inspiring approach has created the overall basis for my research work.

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Jyväskylä 23.11.2016
Jorma Kyppö

FIGURE

TABLE

CONTENTS

1 INTRODUCTION

Chess has been known as a game for some two thousand years. The basic form of this game was called chaturanga, and the movements of the pieces in this game differed significantly from their movements in the modern game of chess. There are various theories related to the primary structure of chess and particularly to its predecessor. Chess arrived in Europe around 1000 AD, and after several changes reached its present form. Ever since then, one of the most interesting pieces has been the knight. The strangeness of the knight's moves in relation to the moves of the other chess pieces often led to people to connect chess with the rituals and religious magic of ancient India and with number theory and prime numbers.

The knight's moves on a chessboard can be described by using so-called knight's trails and paths. The construction of knight's trails and paths are classic problems and special cases of the Hamiltonian path problem in graph theory.

This problem can be generalized, for example, by changing the number of cells or squares where the chess pieces are placed and, by changing the shape of the cells. In addition to the usual rectangular chessboard, two-dimensional triangular or hexagonal chessboards may also be used, for there are exactly these three different ways to tile an infinite, two-dimensional, plane surface by regular polygons.

Why then does the knight move so strangely on the chessboard? Let's imagine that we are developing a chess game on a square board. Based on the details in Chapter 6, we will arrive at a game, which can be played on an 8x8 rectangular board between four players. Historical references indicate that, in the beginning, chess was thought to have four players, but the movements of the chess pieces in early chess resembled, to some extent, the movements of the pieces in our theoretical chess models.

One can also create three- or multidimensional variations of chess by means of the cube and hypercube. In all the variations, the knight's moves may easily be defined. Not only the size and the form of the chessboard, but the knight itself and its moves may be generalized. The term *hyperofficer* is used for the *hyperknights* and *hyperbishops* situated on n-dimensional chessboards with various tilings. The movements of hyperofficers may be depicted by so-called knights graphs. Knights graphs have the cells of the boards as vertices and the knight's movements define the edges.

1.1 Basis of the research

The original idea for this research grew from research concerning the famous Four Color Conjecture (4CC), which appears now as the Four Color Theorem (4CT). This difficult problem, which emerged in 1878, has defeated hundreds of researchers over the past century. At the same time, however, it has given rise to a great deal of progress in the field of graph theory. Although the author of this research has also spent a few years trying to solve this problem, he did obtain enough results that led to his master's thesis and orientation to graph theory, particularly to topological graph theory.

The graph theoretical starting point for this work was suggested by studies concerning the knights trails mentioned above. Research on the generalizations of these problems suggested the rules of so-called abstract chess, and this built the basis for new kinds of chess games. With the same boundary conditions as those for the normal 64-square chessboard, we developed another game: hexagonal chess on an 87-hexagon board, made for three players. It was the number of players in the game that provided exciting new strategic dimensions, which have no direct connection with the movement of the pieces. The movements of the chessmen follow, in line with the abstract chess idea, the normal rules of chess as far as possible. This led to other games built by the same principles: special kinds of chess games that are presented in this thesis' 6th and 7th chapters.

1.1.1 Background

Scientific research is largely motivated by natural human curiosity, an attraction of some kind of will to find something new, despite the fact that reaching the limits of knowledge is a hopeless task. Infinity and eternity do not feel pity towards our often-failed attempts to find enlightenment. However, to expand mankind's knowledge is a fascinating task: to take one more step, to walk one more street, just to get a chance to look around the corner and see if there is something new to find.

To make a new discovery in this world is really difficult. Every thought, every idea, every theory has probably already been conceived by someone else. When science is taking steps from one level to the next, it always creates conditions for new discoveries and inventions. The question is not whether we will find them or not, but who will, and when? Always when a new discovery is made, our culture goes up a little higher on the stairs of knowledge and opens up opportunities for completely new kinds of achievements. It is rarely that we get very large results: the greatest part of the scientific work consists of small results on one level. And it is quite usual that more than one researcher will offer their own tiny variation of any particular result.

When starting his studies in the university, the foremost target of the author of this thesis was to reach a point on the frontier, where knowledge meets the unknown. This could provide an opportunity to step into new territory and add something new, no matter how small, to science. However, over time, one becomes aware that even the border area itself is so blurred that finding it is more than difficult. Every now and then you may think that you've got a marvelous idea that no one else has come across before. Later on, however, an examination of the literature will quite likely dispel that false notion. The result might have been published already a century ago, or just a few years ago. One of the most disheartening situations is when you are in the process of putting your newly obtained results in writing and you find that a journal has just published identical results. You can always take solace by seeing that at least the direction seems to be correct.

In Jyväskylä University there was in the 90's a seminar, "Graphs and Knots," which was attended by visiting Professor Frank Harary, an internationally-known graph theorist. During his visit, we considered a game theoretic problem, a possible solution to which we discussed by email correspondence. However, the solution was a proof similar to another problem, belonging to the field of graph theory. This problem concerned a generalization of the classical problem of finding a Hamiltonian path on a chessboard by a knight. The first proof of this problem was presented by Leonhard Euler in the 18th century. Constructing a new proof was done quickly enough and soon there were only a few special cases to solve. However, when this proof was almost ready, another research group published their proof of the same problem, using a different method, and our work remained unfinished.

However, this research did have a spin-off, because during the process we wondered if the problem could be generalized. This led to a further thought: why does a knight's move differ so much from the moves of all of the other chess pieces? A few years earlier, the author of this thesis had been thinking about the same peculiarity and came to the conclusion that chess might originally have been a four-person game. This conclusion was based solely on the knight's move and the shape of the game board, but very soon it became evident that this could really be true, as shown by some historians (Benton & Benton 1977, Bidev 1986, Bird 2004, Bornet 2012).

This led to further research, and the realization that chess was divided into "prime elements" and its rules were made for the previously-mentioned abstract game of chess. The next stage of chess was rebuilt from these prime elements. All of the subsequent changes in the rules took place over the millennial journey of chess from India, to Persia, to Arabia, and then to Europe. As a result, two new games were created. The first one consisted of triangles, but was not playable by abstract chess rules, although it was a good theoretical model. The second game was composed of 87 hexagons and was quite an interesting and fascinating chess game for three players.

Because the second game seemed to be so interesting, it was reasonable to specify the rules in detail. However, the idea was to stay absolutely faithful to the rules of traditional chess. During its millennial history, chess had been going through several evolutionary stages, which made it so perfect, that even the smallest changes would cause unpleasant surprises. Some of the rules were easy to apply, but every now and then there came a situation, where we had to choose the most appropriate rule among several choices. In such a situation, the third game model, with triangles, was useful, as comparing it with traditional chess brought up a rule option, where the isomorphism requirement was best achieved. The game was born in two weeks, and was called Trichess by Professor Harary.

The next step was to find out what kinds of games had previously been developed. It turned out that some quite similar games existed. In 1912 an Austrian engineer, *Siegmund Wellisch*, invented a primitive game of chess on a board of 91 hexagons. It was primitive because in this three-player game, there were no bishops. The knights moved in same way as bishops in Trichess. In the earlier versions of Trichess, we also reached the same conclusion regarding bishops, but later found out, that the conclusion wasn't correct. The most famous similar game was probably Polish chess, which was called Glinsky Chess, by its inventor, Wladyslaw Glinski, who developed this game in 1936. This game is also played on a board of 91 hexagons, but it was designed for only two players. The movement of pieces corresponded to those of our Trichess pieces. For this game, which is also called Hexagonal Chess, world championship tournaments have been held since the 1980's. (Gik & Määttänen 1988)

However, it should be pointed out that these so-called fancy chess games have been extensively developed. Trichess differs from most of these games, because it was never developed for the purpose of generating a new game, it was just a spin-off product of the research concerning the mathematical structure of chess.

We did, however, make several tests of Trichess. The first test games, which were the basis for making one rule change, were attended by brothers Pasi and Harri Halttunen, and a little later by a national level chess player, *Risto Nevanlinna* from Jyväskylä and a computer science teacher, *Heikki Saastamoinen*. After this, the game was ready.

After that, it was left in a drawer for three years into a drawer, until there was an innovation competition, to which we entered it after a suggestion from a colleague. To our pleasant surprise, it became rewarded in this innovation competition. This led to the commercialization of the game, but also explicitly to its further development, which resulted in to a broader generalization of chess, to the universal chess model.

The universal chess model, and consequently this study, might never have been on this stage if the patenting process would have not been such a problem. Already in the beginning of the process we found some slightly similar, but basically very different variations like the above-mentioned Wellisch chess, which we present also as an example in this thesis. However, the patent process revealed, that a person called Dana Rewega had received a patent (Rewega 1992) in the United States just a few months before the idea of Trichess was born. Rewega's game was almost identical to Trichess. This did stop the direct patenting of the game, but not its development and marketing.

A most interesting aspect of Rewega's game is the realization that some of the same ideas can be born in different places around the world at same time. Rewega's patent was approved at the end of 1992 and the idea of Trichess was born in May 1993. This innovation process is documented in e-mail correspondence between the author and Professor Frank Harary (Appendix 1.) The birth of the process is explained in more detail in Section 6.1.1. How Rewega discovered his model remains a mystery because no other information about him is known after 1992.

Later on we investigated whether the game could get some kind of patent. The first patent was about a boardless chess, a chess game without a chessboard (Kyppö 1997). This game was based on same ideas, which gave rise to the Trichess model. After this a patent application was made about the board numbering system (Kyppö 1999). Other similar games were found, but the game's extensive patenting outside Finland would have required the support of a clear marketing plan. However, even if an international patent application was made, it was not economically feasible to pay for it, because there were no clear and visible markets. The experience was interesting nevertheless, because it showed how difficult it is to publish ideas so that you can keep the copyright.

However, the development of boardless chess was a part of the idea of universal chess, which got its birth from the patent problems of *Trichess*. Trichess was a byproduct, or spin-off, of trying to solve a graph theoretical chess problem. In a way the patenting problems led to the continuing development of Trichess, and to deeper research into the elements of chess and finally to the model of universal chess, and to this research project.

1.1.2 The object of the research

The object of this research is to create a symmetric, n-person game model, which can be used to simulate, and solve, n-person game problems. Although the results of this study are initially used for constructions of n-person game strategies, the results can be later applied to understand and simulate other systems, which have several participants.

One partial motivation for this research is the fact that in game theory there are no simple mathematical solution models for n-player strategy games. Therefore, one goal is a mathematical model, which can be used as a basis for a simulation model, because in game theory n-player problems are usually solved by simulation algorithms.

Chess is suitable as a basis for this model, because it has clear rules, it is not based on randomness, and it has a finite number of solutions. The problem is the number of the solutions, which is so exponentially large that it has not been possible to benefit from this. However, this thesis also presents some smaller chess game variations, where a perfect definition tree is possible to present. In these multiplayer chess variations, the rules define them cooperative games, which means that it is possible to find allies in these games. This also makes the solution models more complex.

Chess and checkers involve two players each of whom has a finite number of available strategies. Each player knows the other player's moves, and there is no chance involved. Considering games with more than two players, one value will no longer suffice in representing the outcome. A cooperative game is one in which communication and coalition formation is allowed between players. A coalition is a subset of the n players among whom a binding agreement exists. For cooperative games with a given coalition structure, max"[1] will find an equilibrium point as a possible solution of the game and determine a strategy for a coalition. (Luckhart & Irani 1986)

The theory of the general n-person game, in contrast to that of the zero-sum two-person game, remains in an unsettled state. The chief problem seems to be that of determining the proper definition of a solution for such games. The efforts in this direction divide themselves into two groups, the cooperative theory in which the players are expected to form coalitions, and the non-cooperative theory in which such coalitions are forbidden. (Gale 1953)

It is currently not known if Nash equilibria can be computed efficiently. For two-person games the known algorithms either have exponential worst-case running time or it is unknown whether they run in polynomial time. For three player games, the problem seems to be even more difficult. While two player games can be formalized as a Linear Complementarity Problem (LCP) the problem for three player games is a Non-linear Complementarity Problem. Algorithms for approximating equilibria in multiple player games are also believed to be exponential. The problem of computing Nash equilibria is of considerable interest in the computer science community and has been called one of the central open problems in computational complexity. (Lipton, Markakis & Mehta 2003)

There are several fields in science and society, like mathematics, economics, meteorology and politics, where you can find use for the solution models of n-person games.

One example is the Kyoto Summit described by Dementieva: Climate change is the first among the global environmental threats to civilization. We discuss a real-life cooperative game. Flexible mechanisms of the Kyoto protocol are the basis of the cooperative model. In our example there are three players: the European Union, the Russian Federation and the new members of the European Union. (Dementieva 2004)

This study examines the question: can the game of chess, or some other similar complex strategy game, be generalized using the same clear rules in different dimensions and in different tilings (tessellations) of the game board? The reason for this is that in a two-dimensional world, only three players, and in a three-dimensional world, only four players, can be placed in symmetrical positions to each other so that each player is in the same position relative to the oth-

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1 The algorithm of the authors

er players. In the higher dimensions, however, the number of different tilings is limited. After this, in case a suitable structure is found, the aim is to build a model, by which it is possible to achieve a simulating game software. This study is limited to chess because it is a widely known game, which has clear rules and already has been used to create strategies in politics, war, business, and other fields.

1.2 Research problems

Our aim here is to build a strategy game where n players can play against each other, so that they are all on an equal footing with each other as far as their positions on the board are concerned. We use the word *symmetrical* to describe this situation. The research question:

> *Is it possible to create a multi-person-strategy board game, where n players are symmetrically positioned with respect to each other?*

Regardless of the form a two-dimensional game board has, it is not possible to place more than two or three players symmetrically to each other on it. By symmetry we mean here the strategic positions of the points in relation to each other for all the players. Two and three points can be set on a plane so that they all would play the same strategic role in relation to each other. For four or more points this is not possible. The opposite points' positions differ from those of the adjacent points, regardless of what kind of game board we choose to consider. Adding a fourth player requires a three-dimensional game board, and generally an n-player game board must be n + 1-dimensional. Four points are symmetrically in three-dimensional space, three on a plane and one above them. The shape of the game board is n-simplex, also known as n-dimensional hypertetrahedron. A problem arises when we tile the game board into similar cells. In higher dimensions, this kind of tiling can be done only by hypercubes (honeycombs), which have a shape that differs from hypertetrahedra. This means, that we can get a suitable shape for a multidimensional game board by using the hypertetrahedra, but a workable inside structure is achieved by using hypecubes. Neverthless, it is quite complicated to use these two structures together.

We decided to use the traditional game of chess as a base for the rules for this game, because these rules have a millennial history and the game has been widely used in game strategy research. Consequently *another aim of this research is to build for this game such movements of pieces, which are isomorphic with the chess rules*.

Deep Blue was the first chess computer to defeat a reigning human world chess champion in a regulation match. A number of factors contributed to the system's success, including its ability to extract useful knowledge from a database of 700,000 Grandmaster chess games. Deep Blue takes a different approach for using the opening information in its database. Instead of subjecting a few selected positions to detailed analysis, it creates an additional database we call the "extended book." The extended book allows the system to quickly summarize previous Grandmaster experience in any of the several million opening positions in its game database. Deep Blue uses knowledge extracted from the Grandmaster game database to improve its performance in actual play. The extended book technique, involving summarized human decisions to bias a search, also appears to be of general value in non-chess domains with access to large databases based on expert decisions. These databases include other games, medical diagnoses and stock trading. They are handy, especially when there is a useful similarity measure between differing situations for a given domain. (Campbell, Hoane & Hsu 2002)

In our earlier research, we embedded the rules of chess in a 3-person game, which gives a good starting point to our n-person model. The extended book technique used by Deep Blue cannot be directly applied to n-person models because there doesn't exist any database of such games. However, their similar rules provide some opportunities to also apply this technique in subprograms. And if the symmetrical model is workable, it can later be applied to different kinds of real-world game positions, for example by changing the powers of the players.

1.3 Main Concepts

Game theory, graph theory, chess, strategy games, combinatorics, arithmetical triangle, tiling (tessellation), hypercube, simplex – here we need all of them. An important main concept is naturally the graph. Figure 1 shows the three main parts of a graph G = (V, E), having vertices (a), edges (b) and faces (c). In the literature a vertex is also known as a node or a point, an edge as a line or arc and a face as a region.

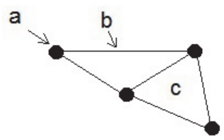

Figure 1 The basic elements of a graph

Other basic concepts in this thesis are *the degree of a vertex, the degree of a face, planar graphs, dual graphs and directed graphs (digraphs)*. The degree of a vertex v is denoted *deg(v)* and equals the number of edges connected with or incident to the vertex, and a face degree, deg(f), is the number of edges surrounding the face. For example, Figure 1 shows that deg (a) = 2 and deg (c) = 3.

A digraph or directed graph is a graph with each of its edges directed. (Gross et al. 2004). The directions are usually drawn as arrows.

A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect. A graph is *planar* if it can be embedded in the plane. The planar graph has also a *dual graph*. Given a plane graph G, its geometric dual G* is constructed as follows: place a vertex in each region of G, including the exterior region, and if two regions have an edge x in common, join the corresponding vertices by an edge x* crossing only x. The result is always a plane *pseudograph*. A 2-connected plane graph always has a graph or a *multigraph* as its dual, while the dual of a 3-connected graph is always a graph. A multigraph is a graph, where no loops are allowed, but more than one edge can join two vertices. If both loops and multiple edges are permitted, we have a pseudograph. A loop is an edge joining a vertex to itself. The connectivity κ(G) of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a graph having a single vertex. A graph G is *n-connected* if $\kappa(G) \geq n$. (Harary 1969)

1.4 The Results

The main result is *the symmetrical n-person strategy game model*, presented in Chapter 8. However, there are several spin-off results in different chapters. In Chapter 4 there is *a hypothesis about the origins of famous Phaistos Disk* in Crete. In same Chapter 4 there is also *Fjögratafl, a hypothetical four–person tafl*. The Chapter 6 consist some previous ideas and innovations of the author including *Trichess, Bridge chess, and Chess without a board*. Also all the game models, including *Large Chess* and *Four Dimensional Chess* in Chapter 7 are made by the author. Some of the constructs of this chapter have been previously published or patented. In Chapter 8 there is the main result, but also a small result concerning *a generalization of Pascal's rule*. Chapter 9 includes only spin-off results for the continuing research of the author, including *the generalization of the Euler-Poincare characteristic*, some *generalizations of Fibonacci sequences*, *strategy networks of small chesslike games*, *induced cycles in Pascal's polytopes*, and some notes on *the odd and even Euclidean dimensions* including Euler's characteristics, Pi, and *a generalization of the Golden Ratio*.

1.5 The structure of the thesis

The thesis is divided into four main sections, from the general background theory and previous results to the actual research problem and its solution. At the end of the thesis and before the summary, there is a chapter for the concluding remarks. It includes other results, which were obtained during the research process. They are also possible topics for further research. In Chapters 2 and 3, we give the tools to build the model. Chapters 4 and 5 discuss the history of board games and chess variations, which provides the background for this research. Chapters 6 and 7 describe our earlier research in this field, and in Chapter 8 we present the model and its solution. In Chapter 9, we present some spin-off results of this work for future research. In Chapter 2, *Game Theory and N-person Games*, we discuss the problems and motivations for this research. Chapter 3, *Tiling in Different Dimensions*, deals with the background theory, which is needed in Chapter 7. In Chapter 4, *History and Prehistory of Chess*, we explain the background of board games in relation to chess, and in Chapter 5, *Later Chess Variations*, we study chess variants occurring through the centuries. From Chapters 4 and 5, there is a direct continuation to the game designs in Chapters 6, 7 and 8. Chapter 6, *The Basic Model of Universal Chess*, deals with our earlier published results, and Chapter 7, *The Extensions of Universal Chess*, concerns our earlier unpublished results. In Chapter 8, *Symmetric N-person Chess*, we present the model we were searching for and the solution of this of this research problem. Chapter 9, *Concluding Remarks*, presents other results found during this research. In a few words, we could say that Chapters 2 and 3 are the soil from which this research grows, Chapters 4 and 5 create its roots and Chapters 6 and 7 the body or the trunk. Chapter 8 is the treetop and Chapter 9 the branches.

The structure of the work and the connection between its chapters are illustrated in Figure 2, where some of the results are also shown inside the boxes. The box of Chapter 9 is connected by lines to other boxes to show from which part of the research the spin-off results were obtained.

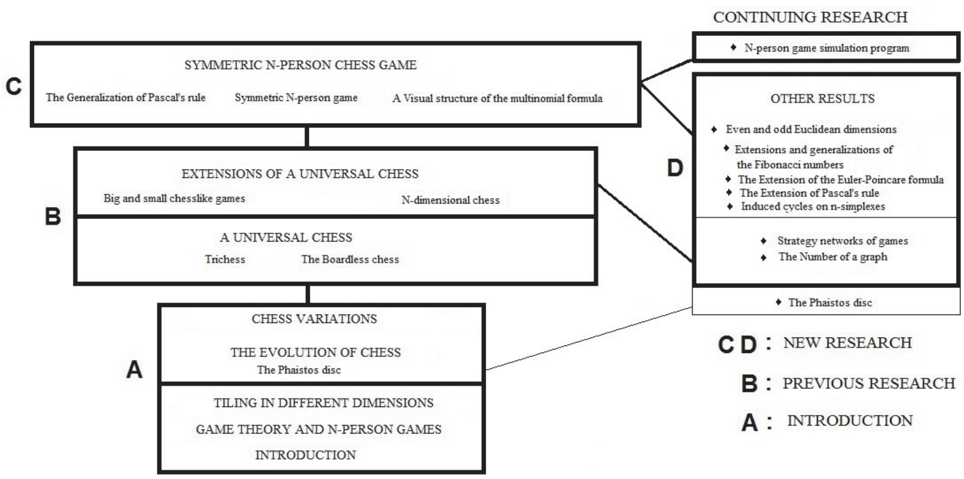

Figure 2 The structure of the thesis

1.6 Relation with other research

This thesis is a monograph; however, parts of the contents are based on earlier works either published or patented by the author. The thesis is connected, to a certain extent, with two patents *Game especially chess* and *The numbering system of game board (Kyppö 1997, Kyppö 1999)*. This work also has some fairly thin interfaces with the author's licentiate thesis (Kyppö 1994). Some parts of the licentiate work can be found in Chapter 3, which deals with the theory of tiling. The workings of the patents are described in Chapter 6.

2 GAME THEORY AND N-PERSON GAMES

Game theory is a mathematical theory which deals with the interactions of individuals in some situation, usually that of conflict. There are choices available for each individual, and the eventual outcome of the game depends upon the choices of all the players involved. Each player has some influence on the eventual result of the game and will get the appropriate reward or suffer a loss. The key feature of the most interesting games is that the best choice depends upon the choices of the other players, and this is what differentiates the game theory from optimization theory. (Broom & Rychtar 2013)

Each of us has to constantly make decisions, and these decisions must be based on something. Decision-making under uncertainty is, from a mathematical point of view, a complex process, where you must consider a huge number of different methods and theories. All these are connected by the fact that one-person decision-making happens in a complex environment, in which the other participants are only variables that do not make decisions. In this environment, the decision-maker is only trying to find an optimal solution.

The situation changes significantly if the decision-maker must also consider the acyions of other decision-makers. In this case, we move into the field of game theory.

Game theory is a subfield of mathematics, but it also has connections with the social and behavioral sciences. It studies interactions and decision-making between two or more participants in situations where their interests and goals are usually opposed but often also parallel, either partially or fully. In n-person games, the goals can be contradictory or shared by some of the participants.

The game theory that is used to support decision-making is a wide subfield of operations research. The difference between the game theory and the optimization problems in operations research is the number of decision-makers with opposing interests. This number in game theory is usually two or more.

2.1 Background of the game theory

Although people have been playing various games for thousands of years, when did we begin to analyze games?

The Babylonian Talmud is a compilation of ancient law and tradition set down during the first five centuries AD and serves as the basis of Jewish religious, criminal and civil law. One problem discussed in the Talmud is the so-called marriage contract problem: a man has three wives whose marriage contracts specify that in the case of his death they receive 100, 200 and 300 units of the property, respectively. The Talmud gives apparently contradictory recommendations. When a man dies leaving an estate worth of only 100 units, the Talmud recommends equal division. However, if the estate is worth 300 units proportional division (50,100,150) is recommended, while for an estate worth of 200 units, its recommendation of (50,75,75) is a complete mystery. This particular Mishna has baffled Talmudic scholars for two millennia. In 1985, it was recognized that the Talmud anticipates the modern theory of cooperative games. Each solution corresponds to the nucleolus of an appropriately defined game. (Walker 1995) *The Nucleolus method* is one of the classic methods in game theory (Balog et al. 2012).

In such games as chess where players alternate moves, but where there is effectively only a finite number of game sequences, there will be a determined an outcome with best play, usually assumed to be a draw. Thus if a player is sufficiently intelligent, he or she could work out the best play. This is formalized in the original theorem of game theory, Ernst Zermelo's theorem. Zermelo (1913) stated that in chess there is either a forced win for white, or a forced win for black, or the result is a draw, both sides can force a draw. (Broom & Rychtar 2013)

Poker involves not only strategy but also the luck of the draw. If you get a poor hand, you're likely to lose no matter how clever your strategy is. In chess, on the other hand, all the moves are chosen by the players. Zermelo's paper on chess apparently confused some of its readers, many secondary reports of his results being vague and contradictory. Zermelo tried to show that if the White player managed to create an advantageous arrangement of pieces it would then be possible to end the game within fewer moves than the number of possible chessboard arrangements. By a "winning configuration" or "advantageous arrangement" Zermelo meant to achieve a situation from which White would be sure to win assuming that White didn't make any dumb moves. Zermelo proved his proposition by using principles of set theory. (Siegfried 2006)

Modern game theory was born probably in 1944, when John von Neumann, together with Oskar Morgenstern, published *Theory of Games and Economic Behavior*. The book was the foundation of much of modern game theory, including the concept of cooperative games. But before this book, Emile Bore gave in the 1920's an explicit demonstration of mixed strategies and minimax solution for two-player games. The classical game of Prisoner's Dilemma was first formulated in 1950, by Melvin Dresher and Merrill Flood. The name of the game and its associated story was invented by Albert W. Tucker as late as 1980. (Broom & Rychtar 2013)

2.2 Two person games

Two-player games are played by just two players. Before going to n-person games we study some classic two-player games.

The popularity of pairwise games is not simply due to their relative simplicity, but also to the wide applicability of this idea. (Broom & Rychtar 2013)

2.2.1 Two-person zero-sum games

The two-person zero-sum game is a simple game model. John von Neumann and Oscar Morgenstern developed a theory of how two-person zero-sum games should be played. There are two players: the row player and the column player. Both can choose among a number of strategies: the row player has 1, .., m and the column player 1, ..., n strategies. The use of each strategy in relation to the other player's strategy brings a reward. The zero-sum game got its name from the fact that the amount of these rewards is zero for all the strategies. (Winston 1994) In a matrix, zero sum means that the rewards are the row player's profits while the column player loses an equivalent amount. Figure 3 shows an example of a reward matrix of the zero-sum game. Here the row player has five and the column player six strategies.

	Player Y: 1	2	3	4	5	Row min
Player X: 1	5	2	8	5	3	2
2	6	2	7	2	2	2
3	2	7	5	0	4	0
4	5	3	7	1	3	1
5	4	1	3	3	2	1
6	8	9	6	(6)	9	6 Max
Column max	8	9	8	6 Min	9	

Figure 3 The payoff matrix of zero-sum game

The solution for a zero-sum game is obtained by finding the minimum reward in every row of the row player. It is added to the last column on the right as a row minimum. After that - although here the order does not matter – we seek from the column player's reward columns the maximum value and place it on the last row as the column maximum. Once this has been done, the row player chooses from the column of the row minimum the maximum value and selects the indicated strategy. In the example of Figure 3, the row player chooses the 6th strategy. In other words the row player chooses from each strategy the reward with which he could at least win and places it at the strategy row on a column on the right side of the matrix. From this column, the row player then chooses the strategy which has the greatest reward. Next, the column player searches the minimum value of the column maximum, and that will be his/her strategy. The reason for this is that the column player's rewards are in fact losses which should be minimized. Therefore the column player selects from the options the worst one in each strategy, finally choosing the strategy where the worst option is the least bad. In this case, the choice would be the 4th strategy. Because all the rewards are positive in this example, it seems that the column player always loses, but the figures are relative. The matrix would look more reasonable if, for example, we reduced five credits from each reward. If we did so some of the rewards would get a negative value. Consequently, the negative numbers would be losses for the row player but profits of the column player.

If the maximum reward of the minimum row has the same value as the minimum reward of the column maximum, then min (column maximum) = max (row minimum), the game has a *saddle point*. The saddle point is also the value of the game for the row player. If the game has no saddle point, then the value of the game cannot be calculated. In the example of Figure 3, there is a saddle point, because min {8,9,8,6,9} = 6 = max {2,2,0,1,1,6}. The saddle point is also the *equilibrium point* of the game. In that case, neither player should change the strategy, as a one-sided change would risk getting worse result for the player doing it. (Winston 1994)

If the reward of X is reduced in point (X,Y) = (strategy 6, strategy 3) to five, then the situation will change (Figure 4). The players would continue by choosing the same strategies, but the saddle point would be lost and min = {8,9,8,6,9} = 6 ≠ 5 = max {2,2,0,1,1,5}.

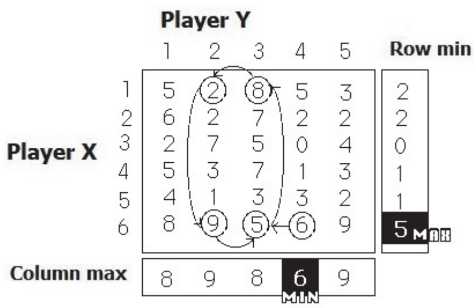

Figure 4 A Zero-sum game without a saddlepoint

We might speculate how the players would behave if they could guess the next strategy of the opponent. Such a "chain of speculations" (if "X knows, that Y knows"), however, creates a loop after a few steps, and the strategies that neither of the players would choose would remain outside of this loop. A situation where there is no saddle point in a zero-sum game can be solved by eliminating first so-called dominating strategies and finally by solving the problem, either graphically or as an LP-problem.

There are also games which don’t have a unique minimum of the column maximum or a unique maximum of the row minimum. Instead there may be several of them. A typical example of such game is the children's favorite game, stone-paper-scissors, where the players keep their hands first hidden and then show them, both at the same time. Each player can choose from among three different hand strategies (paper: palm is open, scissors: two fingers point out, stone: the hand is closed as a fist). The reward matrix can be formed by using the following rules: stone can break scissors, scissors can cut paper, and paper can cover stone. As a result, the reward matrix will look like in Figure 5.

	ST	PA	SC	min
STONE	0	−1	+1	-1
PAPER	+1	0	−1	-1
SCISSORS	−1	+1	0	-1
max	+1	+1	+1	

Figure 5 The Stone-, paper, scissors-game

The task can be solved as a linear optimization problem (LP-problem) where every strategy gets the value of 1/3. The value of the game will be zero. However, solving a game situation as an LP-problem does not belong to this thesis as it has no relevance with the present research problem.

2.2.2 Two –person constant-sum games

The basic idea in a two-person constant-sum game is the same as in the zero-sum game. The sum in the two-person constant-sum game is some constant c, whereas in the zero-sum game it is always zero. In other words, the zero-sum game in fact is a constant-sum game where c = 0. Rewards and strategies are presented in the same way in both, that is, by using a reward matrix. Also in this matrix the rewards reflect the row player's profits and the column player’s loss. If all the rewards are positive, then the column player always loses some amount.

The solution of this game is similar to that of the zero-sum game. We search from each row the row minimum, and from each column the column maximum. After that one has to find the maximum value among the row minimum values and the minimum value among the column maximum values. These results give the optimal strategy for both players. So the strategy is to choose the option of “the best bad result”. The basic assumption is that the game is always played in the best possible way and that your opponent is familiar with this strategy. Figure 6 is an example of a constant-sum game.

Column Player

Row Player	A	B	C	D	Min
A	720	250	480	440	250
B	670	550	600	630	550
C	370	270	330	450	270
D	560	510	590	610	510
Max	720	550	600	630	

Figure 6 A Constant-sum-game

In this example, we have a game where two imaginary politicians, here the column player and the row player, are struggling over the votes of a million voters. The numbers in the game matrix have to be multiplied by 1000. Thus the value of 720 means 720 000 voters. The numbers are based on Gallup surveys made before the elections. In the election campaign, there are four possible political themes and the candidates choose their electoral strategies from among them. Each candidate must think which of the campaign strategies A, B, C or D he/she should emphasize.

The numbers in the matrix refer to the row player's voters. If each player chooses the A strategy, then as a result the row player will get 720 000 votes and the column player 1 000 000 - 720 000 = 280 000 votes. Each player tries to find a game strategy where the smallest possible reward is the greatest. Thus, the row player lists for each strategy the worst possible outcome and places it on the right side of the matrix, in the Min-column. From among these rewards, the row-player selects the greatest one, in this case, the B strategy, which adds up to 550 000 votes. The column-player acts in the opposite way by listings for each strategy the best possible outcome, which is then placed below the matrix in the Max-row. From among these rewards, the column-player selects the smallest one, in this case, the B strategy, which has 550 000 votes. For the column-player this means, that he/she will get 1 000 000 – 550 000 = 450 000 votes. Because both players have chosen strategy B, the row-player will win the election by 550 000 votes against 450 000. Naturally, the column-player could have chosen a different strategy, but in all of them the loss would have been greater against the B strategy the row player chose. If the row-player had chosen some other strategy in the case where the column-player selects the B strategy, then the row-player would have lost in two of them, in A and C and won in D. But also with the D strategy he/she would have won less than in the chosen strategy. In addition, in this game Max (row minimum) = 550 = Min (column maximum), which means that the game also has a saddle point and hence 550 is also the value of the game for the row-player. (Winston 1994)

2.2.3 Two-person non-constant-sum games

Most of the business game models are not constant sum games. They are usually non-constant sum games because, in the world of business, the competing sides are rarely in full conflict amongst themselves (Winston 1994).

Two-person non-constant-sum games can be divided into so-called co-operative-sum and non-co-operative-sum games. We will observe as an example a small non-co-operative-sum game where the players are not allowed to jointly plan their strategies, and we will see what could happen when the game becomes co-operative. The reward matrix differs in some detail from the zero-sum game reward matrix. The rewards are expressed with coordinates of two numbers (x, y). The number on the left (x) gives the reward of the row player, and the number on the right (y) is the reward of the column player. Because the given numbers now stand for profits for each player, also the **column** player will get the maximum reward from among the minimum rewards (not the other way around).

Let's take again the example in politics. Two ministers from different parties, A and B, are fighting for power inside the government and both have two main strategies: to increment one's own power by a dirty game behind the back and at the expense of the other player (X) or to create a consensus with the other party (Y). The rewards of these strategies have been estimated in such a way that taking all the powers for yourself will give five points because of the advantages brought by your increased power, but at the same time the attempt to try to achieve this goal brings two minus points because of the lost political reputation and possible disintegration of the government, and other consequences as well. Correspondingly, loss of your own power brings three minus points. A consensus with the other party in maintaining the government provides neither credit nor loss points to your tally.

With this information, a reward matrix can be created (Figure 7, on the left). Here A is the row player and B the column player. It should be noted that, if both choose the attacking strategy X, then the government will break down and both players will lose. If both sides select a consensus, then the government will stay put and the situation will remain unchanged, resulting in status quo. The profit towards securing a victory can be obtained by the player only if the player selects an attacking strategy but the opponent does not do same, and will lose the game. However, here the matrix tells us that in this case the equilibrium point is not the point where the government can be maintained but where both players choose the offensive strategy while losing at the same time political points. This example, from real life warns us about circumstances where the balance of power could eventually lead to a negative outcome for the state.

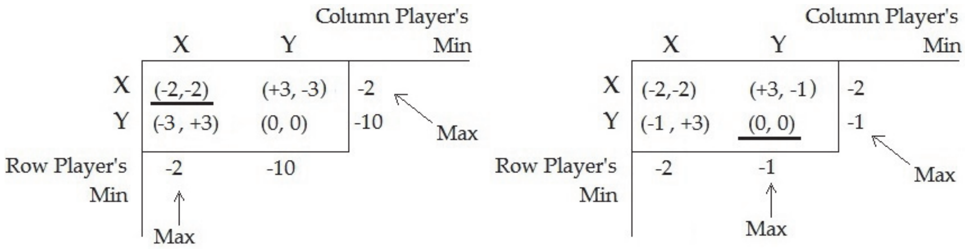

Figure 7 A Non-constant-sum game

If, instead, this game was played co-operatively, then the players would probably choose a strategy that would give the most favorable outcome to all the participants. In that case, the government would stay and neither of the participants would experience any great gain or loss.

If we evaluate the rewards in different strategies, we notice that the equilibrium point remains the same also in a situation where victory would bring better rewards. On the other hand, if the loss, which means a loss in the power position, is reduced from three points to one point, then the equilibrium point will change its position to where it supports the government (Figure 7, on the right). In Figure 8, we have the same game analyzed graphically. Also in this illustration, we can see the change of the equilibrium point. The equilibrium point is on the centroid of the polygons. The centroid a polygon is the arithmetic mean position of all the points.

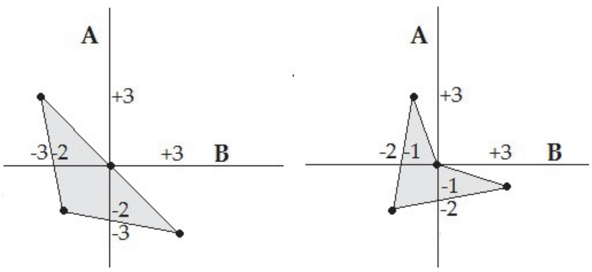

Figure 8 A Non-constant-sum-game seen in the coordinate system

A classic example about two person non-constant-sum game without co-operation is the famous *prisoner's dilemma*. Two escaped prisoners who had participated in a robbery have been recaptured and a new trial is waiting for them. Before their capture, they had decided not to confess. They both are guilty, but the police does not have enough evidence. To get them to testify against each other, the police tells each prisoner: "If only one of you confesses and testifies against the partner in crime, the person who confesses will go free while the person, who doesn't confess will be convicted and given a 20-year jail sentence. If neither of you confesses, you will each get a 1-year prison sentence. If both confess, the sentence will be 5 years for each. After this declaration, the prisoners are not able to discuss the matter and hence are unable co-operate among themselves (Figure 9).

prisoner 2	prisoner 1: confess	prisoner 1: don't confess
confess	(-5,-5)	(0,-20)
don't confess	(-20,0)	(-1, -1)

Figure 9 Prisoner's dilemma

For each prisoner, the "confess" strategy dominates the "don't confess" strategy. If each prisoner follows his "confess" strategy, both get five years in jail, but if they keep on their original plan, and do not confess, both get only one year in prison. In this game, (-5,-5) is the equilibrium point because if either prisoner changes his strategy his reward will decrease from -5 to -20. However, each prisoner would be better off at point (-1, -1). (Winston 1994)

The paradoxical quality of this result helps explain part of the fascination of the dilemma and the game. But the major reason for the interest is purely practical. Comparable outcomes in social life are often less advantageous than we might hope, and the prisoners' dilemma provides one possible key to their understanding.

It is tempting to think that the problem here arises because the prisoners cannot communicate with one another. If they could get together, they would quickly see that the best result for both comes from 'not confessing'. But communication is not all that is needed. Each still faces the choice of whether to hold to an agreement that they have struck over 'not confessing'. Is it in the interest of either party to keep to such an agreement? A quick inspection reveals that the best action in terms of pay-off is still to 'confess'. (Heargrave Heaps & Varoufakis 1995)

2.3 N-person games

In most competitive situations, there are more than two competitors. Any game with n players is an n-person game. We begin with several citations starting from the year 1950.

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over pure strategies, the pay-off functions are the expectations of the players. This gives rise to polylinear forms of the probabilities with which the various players play their various pure strategies. (Nash 1950)

The complexity of the mathematical work needed for a complete investigation increases rather rapidly with increasing complexity of the game. So it might only be feasible to use approximate computational methods. A less obvious type of application is the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies and payoffs as usual; but with the assumption that the players can and will collaborate. This means the players may communicate and form coalitions which will be enforced by an umpire. (Nash 1951)

The theory of the general n-person game, in contrast to that of the zero-sum two-person game, remains in an unsettled state. The chief problem seems to be that of determining the proper definition of a solution for such games. The efforts in this direction divide themselves into two groups, the cooperative theory in which the players are expected to form coalitions, and the non-cooperative one in which such coalitions are forbidden. The most general theorem in the literature concerning n-person games with perfect information states that all such games possess an equilibrium point among their pure strategies. This equilibrium point will not in general be unique, nor will such games be solvable in the sense of Nash. (Gale 1953)

If the object of game theory were to uncover effective strategies in situations involving conflicts of interest, then the investigations would center on the construction of the game tree and on examining the outcomes resulting from the combinations of strategies by several players. Enormous difficulties would be encountered here. To see this, let us take one of the simplest games of strategy, Tic-Tac-Toe, and see what is involved in constructing its game tree. Issuing from the root, there are 9 branches, which represent the 9 first moves open to Player 1. Each of the new branch points will have 8 branches. We must continue this branching process for at least 5 moves, since no game can end before the fifth move. By the time we get to the fifth move, we have 9x8x7x6x5 = 15120 branches. To be sure, we can drastically reduce this number by taking into account the symmetries of the Tic-Tac-Toe grid. For example, on his first move, Player 1 has essentially only 3 alternatives: center, corner, and side. Were he choose center, Player 2 would have essentially two alternatives: corner or side; if Player 1 chose side, Player 2 would essentially have five alternatives (since one degree of symmetry remains); and so on. Nevertheless, even taking symmetries into account, we would have a rather large tree; and although some effort would be saved in reducing the number of branches, more effort would be needed to examine the sets of situations which are equivalent by symmetry. (Rapoport 1970)

Figure 10 (left and center) shows the two situations, where Player 1 (x) chooses the center and a side. Player 2 (o) chooses a corner and a side in this figure. The rightmost picture shows a situation where Player 1 wins the game. This simple game and the problematics we face when building its game tree has connections with the issue we discuss in sub-section 9.2, which deals with small chesslike games and their game trees.

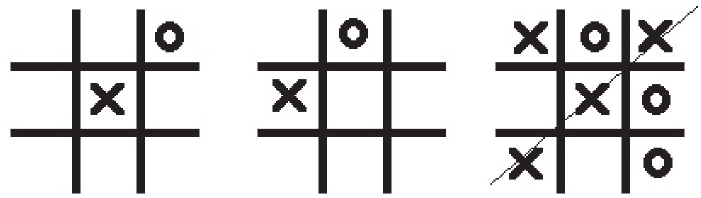

Figure 10 Tic-Tac-Toe

When we are dealing with real life situations, a preliminary problem must be solved before the game tree is constructed. One must ascertain the "rules of the game". The essential function of rules is to delimit and specify the available alternatives and several situations which can result from the player's choices. If the number of "moves" and the number of alternatives available at each move are quite small, it may be possible to list each player's available strategies. The representation of the game by its strategies alone is called representation in *normal form*. Once the game is represented in the normal form, the rules of the game become irrelevant. The rules are important only to the extent that they determine the structure of the game tree and through it the available strategies and outcomes associated with the combined strategy choices. A two-person game represented in the normal form is a game matrix with rows and columns. A three-person game would have to be represented by a three-dimensional grid and an n-person game by an n-dimensional grid with an n-tuple of payoffs in each "box". Once the game is represented in the normal form, the game matrix (or grid) rather than game tree becomes the mathematical object of interest. (Rapoport 1970)

Trees are often used as models of decision making in artificial intelligence (AI) and game theory. From the rules or definition of a game, the game tree representation can be specified for an n-person game. Because most games of interest have combinatorially explosive game trees, AI programs tend to analyze partial game trees in order to determine best moves.

Game theory solutions to non-cooperative games are usually a set of strategies for each player that are in some sense optimal: the player can expect the best outcome given the constraints of the game and assuming the other players are attempting to maximize their own payoffs. A solution for an n-person, perfect information game is a vector which consists of a strategy for each player. For the player a strategy defines the move to make for any possible game state. (Luckhart & Irani 1986)

There are examples of small three-player games, with rational payoff matrices, in which all Nash equilibria are irrational. Algorithms for approximating equilibria in multiple player games are believed to be exponential (Lipton, Markakis & Mehta 2003).

There is a huge amount of literature about two-person games among N participants. We, however, believe that the participants of a genuine N-person game should simultaneously play with all the other players in the game. The appearance of powerful personal computers has made the modeling and simulation of N-person games possible. Several papers have appeared describing simulations of some practical situations. (Szilagyi 2012)

The lack of attention from biological disciplines on multi-player games has two reasons. Firstly, real conflicts often comprise pairwise games, and a lot can be learnt from considering them. Secondly, the mathematics involved in the analysis of multi-player games is more complex, and it is harder to come up with generalizable results. (Broom & Rychtar 2013)

Citing Rapoport above: "Two-person game is represented as a game matrix and N-person game as an n-dimensional grid" (Figure 11). Broom and Rychtar concentrate to analyze multi-player games through biology, but these rules are general. The complete payoffs to the three-player, two-strategy game can be written as:

$$\begin{pmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{pmatrix} \begin{pmatrix} a_{211} & a_{212} \\ a_{221} & a_{222} \end{pmatrix}$$

Similarly for the four-player, two strategy game we have:

$$\begin{pmatrix} a_{1111} & a_{1112} \\ a_{1121} & a_{1122} \end{pmatrix} \begin{pmatrix} a_{1211} & a_{1212} \\ a_{1221} & a_{1222} \end{pmatrix}$$

$$\begin{pmatrix} a_{2111} & a_{2112} \\ a_{2121} & a_{2122} \end{pmatrix} \begin{pmatrix} a_{2211} & a_{2212} \\ a_{2221} & a_{2222} \end{pmatrix}$$

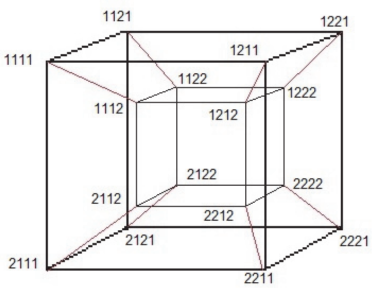

Figure 11 Four-player game represented in a 4-grid

Some games are directly formulated as multiplayer games, and some games can be easily modified for many players. Killer whales, *Orcinus Orca* have been observed to use a collective hunting technique, the so-called carousel feeding. A small group of whales releases bursts of bubbles to round the prey into a tight defensive ball close to the surface and the whales then slap the ball with their tails, stunning or killing up to 10-15 fish with successful slaps. The technique requires good cooperation by a number of whales. This can be modelled as a multi-player stag hunt game since whales may, instead of cooperating, feel tempted to start feeding on their own. (Broom & Rychtar 2013)

2.4 Summary

Game theory is much too large a field to be explained in a single chapter, and hence here is only a part of it. This thesis presents an n-person game model. The role of game theory becomes significant just at the stage when the model is to be tested by simulations to obtain information about the processes and events in the games. It is however appropriate already at this stage to deal with game theory, particularly because of its bearing on further research.

Different games, their strategies and game theory form a complex context, but the big picture it is even more complex. Richard Guy (1996, page 45) has these thoughts about games:

It is hard to draw the line between mathematics and psychology. There are even cases where one should prefer a bad move to a good one! We often refer to a move as being "good" if it wins, and "bad" if it doesn't. In theory it usually suffices to find any good move, or to show that no good move exists. But in real-life games there are many other criteria for choosing between various options. If you are losing, then all your options are bad in the above sense, but in practice they are not all equal, and you might prefer one that makes the situation too complicated for your opponent to analyze (the Enough Rope Principle). (Guy 1996)

This kind of thinking might work for example in computer games. The author did once succeed in making a simple chess program to go out of its mind for more than ten moves. This was due simply to starting the game in a "stupid" way by moving only pawns one step and one by one starting from the left side. The program didn't "understand" what was happening.

Here we have been briefly discussing some basic matters of game theory, relevant to this thesis. However, the object of this thesis is to present a mathematical model where it is possible to embed symmetric games with more than two players. These symmetric games should, if needed, later transform into asymmetric ones just by changing their parameters. This model can later be used to simulate n-person games. By simulation, it is possible to find optimal strategies and their outcomes. The representations of these games by their strategies give their *normal forms* (see Rapoport 1970, Figure 11), which can be investigated by game-theoretical tools.

3 TILING IN DIFFERENT DIMENSIONS

It is essential to discuss in this thesis about tiling, because we need this concept when we create different game boards. *The text and figures in the first three sections 3.1, 3.2 and 3.3. are based on our earlier publications (Kyppö 1993, Kyppö 1990, Kyppö 1993).*

3.1 Concepts

In *tiling* or *tessellation*, or sometimes *paving*, a plane is divided into different geometric shapes and patterns. In this thesis, we use the term tessellation also when dividing a three- or multidimensional space. The words tiling and tessellating are often used interchangeably, and there isn't a great difference between these two words. Tiling might be a bit more restricted term used to describe how polygons tile the plane. Therefore, in this research, tiling is used for two dimensions and tessellation for higher dimensions.

Graph G = (V,E) is said to be *regular* if all its vertices have the same degree and is *complete regular* if its dual graph G' is also regular (Ore & Wilson 1963). This means that each face in G is bounded by the same number of edges, which implies that the degree in each vertex of G' is the same. Briefly, for all connected planar graphs we have the formula:

$$\sum_{k=2}^{n}(4-k)v_k + \sum_{r=2}^{m}(4-r)f_r = 4E(S).$$

In this formula, f_r is the number of faces with r neighbors and v_k is the number of vertices of degree k. E(S) is the Euler characteristic for the surface S. On a two-dimensional plane, E(S) = 2. (White 1973)

We can try this formula for the simple graph in Figure 1, where there are four vertices and three faces. Two of the vertices have degree 2, the two others have degree 3. There are only three faces: two of them have degree 3 and one (outside the graph) has degree 4. In the given formula we get: (4-2)*2 + (4-3)*2 + (4-2)*0 + (4-3)*2 + (4-4)*1 = 4 + 2 + 0 + 2 + 0 = 8 = 4*2 = 4*E (S) when S is a plane.

We get the general formula for regular plane graphs when we give only one value for r, and we get: $\sum_{k=2}^{n}(4-k)v_k + (4 - r)f_r = 4E(S)$. This formula can also be derived to the form $\sum_{k=2}^{n}(2r-rk+2k)v_k = 2rE(S)$. We won't go deeper in these formulas here, but in Figure 12, on the left, is an example of a regular planar graph and its dual graph, which is not regular graph. All the faces of this dual graph have three edges, but the degree of vertices varies. This is also an example of a triangular planar graph, and hence the formula simplifies to $\sum_{k=2}^{n}(6-k)v_k = 12$. This formula is also known as Kempe's formula, and it has played a key role in solution trials and solutions of the famous Four Color Conjecture (4CC) (Appel & Haken 1977). Kempe's formula reveals that if we begin to draw a regular triangular graph where each vertex has a degree greater than 6, the graph will soon become denser on the border and hence impossible to draw. If the degree is less than 6 the graph will close and become a projection of a regular polyhedron on the plane. There are two projections of this kind: a projection of the icosahedron, when the degree is 5, and a projection of the tetrahedron, when the degree is 3. If the degree is equal to 6, then the surface is divided into an infinite number of similar triangles, which is an example of regular tiling.

For complete regular graphs, where k and r are fixed, the formula gets a simpler form: $(4 - k)v_k + (4 - r)f_r = 4E(S)$ or $(2r-rk+2k)v = 2rE(S)$. The formulas can easily be verified from the planar graph on the right in Figure 12.

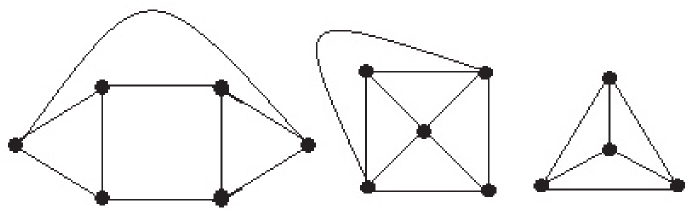

Figure 12 A regular graph, its dual graph and a complete regular graph

3.2 Regular -tiling

In tiling, we divide the plane to polygons in such a way that there will be no free space between the polygons (Grünbaum & Shephard 1987). In regular tiling, we do the same by using regular polygons. So the edges of a tiled plane form part of a complete regular planar graph. There are only three polygons that make regular tiling possible. These polygons (Figure 13) are the triangle, quadrilateral and hexagon (Maor 1987). Why they are precisely these becomes clear when you look at the formula $(2r-rk+2k)v = 4r$ for infinite complete regular planar graphs. In this formula, r is the number of edges in the polygon and k is the degree of vertices of the graph. This formula gives the number of vertices v = 4r/(2r-rk+2k). We can see that the number of vertices in the graphs of Figure 12 is infinite, for in each graph 2r-rk+2k=0, and hence v = 4r/(2r-rk+2k) = ∞.

Let's consider 2r+rk-2k=0 ⇒ r=2k/(k-2). Because k is an integer and k ≥ 3, formula 2k/(k-2) will get integer values only when k = 3, 4, and 6, wherein the corresponding values of r are 6, 4 and 3. Therefore, for only these polygons, the set of pairs (r, k) is the set {(6,3), (4,4), (3,6)} and the number of vertices can increase to infinity in complete regular graphs. The pairs (r,k) are mirror images of each other, and hence we can conclude that triangular and hexagonal graphs are dual graphs to each other and the graph of quadrilateral (here square) faces is self-dual.

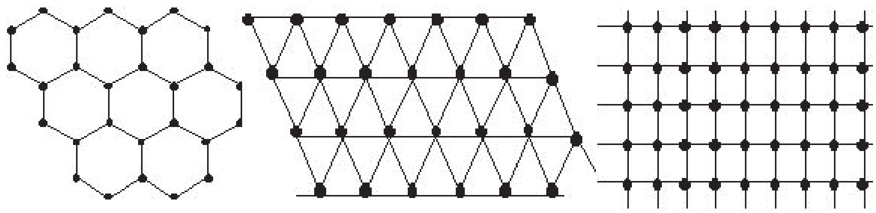

Figure 13 The three ways of planar regular tiling

Another way to find these graphs is to look at the size of the angles of a regular r-gon. The sum of all angles of the r-gon is, in radians, (r-2) π/2, wherein one corner angle is (r-2)π/2r if all the angles are of the same size. For example, the sum of the angle degrees in a triangular graph is (3-2) π/2 = (3-2)*180°/2 = 90°. If this kind of r-gon graph divides a plane, the angle of the corner must have the same degree at each point. Because the length of the circle is always 2π (=360°), the sum of all angles around one point is 2π and thus in each point the degree of one corner angle is 2π/((r-2) π/r)=2r/(r-2). This expression gets the values 6, 4, 3.33…, 3, 2.8, …, 2, where r = 3, 4, 5, 6, 7, 8, ..., ∞. The division gets integer values only if r = 3, 4 or 6. For example the graph, that consists of regular pentagons cannot tile the plane regularly because one corner angle has the size (5-2)π/10 = 3*360°/10=108°, which means that in one point or vertex only three pentagons can touch and one 36° angle would be left to be filled.

This example implies that the plane can be tiled regularly also by using different kinds of combinations of various polygons. It has been shown that the following polygon sets can also form a regular tiling on plane: {4,8}, {4,6,12}, {3,4,6}, {3,12}, {3 4} and {3,6}. (Maor 1987)

As we can see, there are six polygon sets of this kind, but it has been shown that the last two sets, {3,4} and {3,5}, both can form two different sets. This will provide eight regular tiling types. One of these, {4,8}, using squares and octagons, can be seen in Figure 14.

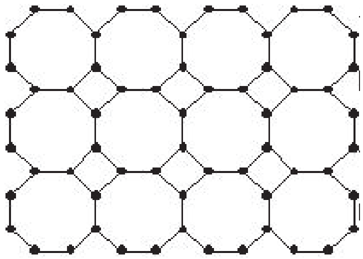

Figure 14 Tiling of the plane by regular squares and octagons

Finally we discuss the differences between, *a face* and *a 2-cell*. Among the basic concepts we have defined athe vertex, an edge and a face. When we examine more complex topological surfaces and graphs embedded in them, we need to extend the definition of a face. Next, a more precise description of a face is defined as well as a more general concept of a 2-cell.

A graph is *embedded* in a surface S if it is possible to draw it on S so that the edges intersect each other only in vertices. A graph is planar if it can be embedded on a plane. *A pseudograph* is a graph that allows multiple edges and loops. Let a pseudograph G be embedded in surface S. The components of S - G are called faces of the embedding. A face of an embedding of graph G is said to be a 2-cell if it is homeomorphic to the open unit disk. If the cell for an embedding is a 2-cell, then the embedding is said to be a 2-cell embedding. (White 1973)

A circular disk (unit disk) is the surface of a circle. Each topological image of a circular disk is called a 2-cell. Similarly an edge is called a 1-cell and a vertex a 0-cell. An embedding of a graph into a closed surface is called a cellular embedding (2-cell embedding) if G divides S into 2-cells. (Ringel et al. 1974)

3.3 Tessellation of the three-dimensional space

When we leave the plane and go to three-dimensional space, we find out that there are fewer possibilities of tiling this space, than tiling the plane. This is because the set where the regular polygons for tiling could be selected is infinite. There are an infinite number of regular polygons of which triangle, quadrilateral and hexagon are suitable for regular tiling. But in third dimension there exist only a finite number of regular polyhedra, the well-known Plato's five solids. The polyhedra in Figure 16 from left to right are the tetrahedron, cube, octahedron, icosahedron and dodecahedron. Tetrahedron is self-dual, which means that if a corner point (vertex) is placed inside its every face and the points are connected to each other then another tetrahedron is generated inside. Similarly, the cube and octahedron are each other's duals, as are the icosahedron and dodecahedron. The faces of the polyhedrons (polyhedra) are triangles, quadrilaterals or pentagons only.

There are very few complete regular graphs in the sphere. The equation (2r-rk+2k)v=4r gives those graphs. Since v, k and r are positive integers, 2r-rk+2k must also be positive, and the expression 4r/(2r-rk+2k) must be a positive integer. Thus the following condition for r is obtained:

$2r-rk+2k>0 \Rightarrow r<2k/(k-2)$

When k=2, then r<∞ and the graph is a usual polygon.
When k>2, it results in r<6 for k=3 and r < p = 2k/(k-2)=2/(1- $\frac{2}{k}$), when k→ ∞.
This implies, that r ∈ (2,6) ⇒ r ∈ [3,5], when k>2.

These values of r give the following results:

$$v = \begin{cases} 2, & k = 2, \\ \frac{12}{6-k}, & k > 2 \text{ and } r = 3\ (a), \\ \frac{8}{4-k}, & k > 2 \text{ and } r = 4\ (b), \\ \frac{20}{10-3k}, & k > 2 \text{ and } r = 5\ (c). \end{cases}$$

Possible positive values of v for these equations are obtained when k=3, 4 and 5 (a), and k=3 for (b) and (c). These five graphs are the only complete regular planar graphs (Figure 15), when k≥3, and the embeddings of these graphs on the sphere are *regular polyhedrons (polyhedra), the five Platonic solids* (Figure 16). (Maor 1987, Grünbaum 1967) In case of other values of r, when k>2, there exist only negative or infinite values of v.

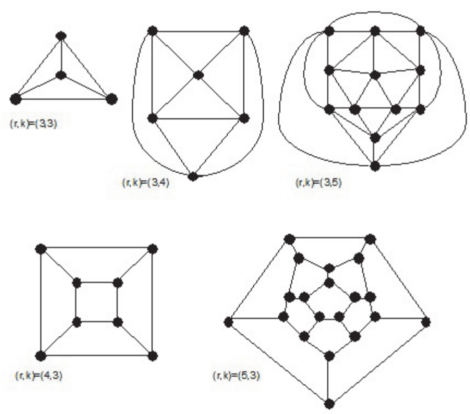

Figure 15 The five complete regular planar graphs

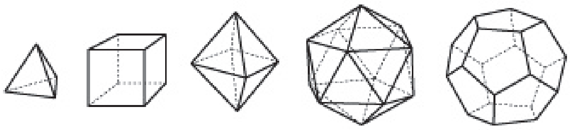

Figure 16 The Regular Platonic solids

Only one of these five solids, the cube, tiles the space regularly (Figure 17). This result can be calculated in basically the same way as we calculated the regular tessellation, but the calculations become more complicated because the formula has to be built in a different way. For example, one of the pillars of the formula, the condition 2e = 3f, is no longer valid in three-dimensional space, because several polyhedrons are connected to each other.

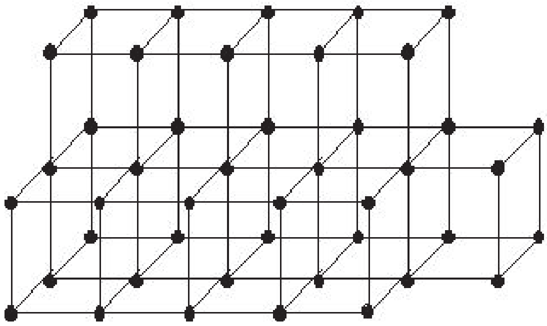

Figure 17 The regular tessellation of three-dimensional space by cubes

As on the plane, also in three-dimensional space it is possible to find different combinations of polyhedrons which divide the space without leaving any empty spaces in between. However, this is not a regular tessellation. In three-dimensional space, a regular tessellation would mean that every polyhedron has to be identical with each other, each face must touch the same number of other faces (usually one) and each vertex must have the same degree.

If the first restriction (identical polyhedrons) is removed and more types of polyhedrons are allowed, there will be more possibilities to tile the space with similar solids. The chances of this combinatorial tessellation are limited, taking into account the number of different regular polyhedrons. One example is the 14-polyhedron, which consists of 6 quadrilaterals and 8 hexagons, 36 edges and 24 vertices, so that all the neighbors of quadrilaterals are hexagons, and every second of the hexagons' neighbors is a quadrilateral (Maor 1987). This kind of polyhedron is called *truncated octahedron* (Figure 18, A13) and it is one of the 13 *Archimedean solids* (Torquato & Jiao 2009). Structures of this type are belong to crystallography.

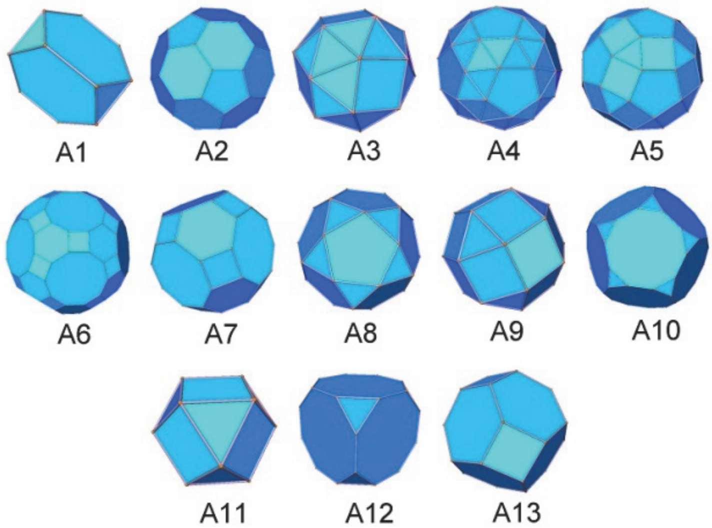

Figure 18 The Archimedean solids (Torquato & Jiao 2009)

3.4 Tessellation of N-dimensional space

As in three-dimensional space, tessellation in higher-dimensional spaces has a limited number of options. Tessellation in n-dimensional space, where n ≥ 3, is only possible with hypercubes. If we use combinatorial tessellation, slightly more options can be found. The options can be best understood with the aid of the Schläfli symbol. However, first we must clarify what are the polytopes.

3.4.1 Polytopes

The definition of regularity needs three statements: one on regular faces, another on equal faces, and a third on equal solid angles. There is another definition involving only two statements but with the same effect. We replace the consideration of solid angles by that of *vertex figures*. The vertex figure of vertex O of a polygon is the segment joining the mid-points of two sides through O (see Figure 19 on left). The vertex figure at the vertex O of a polyhedron is a polygon whose sides are vertex figures of all the faces that surround O, thus its vertices are the mid-points of all the edges through O (see Figure 19 on right). So, a polyhedron is regular if its faces and vertex figures are all regular.

If the mid-points of all the edges that emanate from a given vertex O of a polytope lie on one hyperplane, then these mid-points are the vertices of an (n-1)–dimensional polytope called the vertex figure of the polytope at O. Beginning with the regular polygons (triangle, quadrilateral, pentagon, …) we can define a regular polytope inductively as follows. A polytope (dimension n > 2) is said to be regular if its cells are regular and there is a regular vertex figure at every vertex. (Coxeter 1973). See different kind of vertex figures in Figure 19.

A polygon is a finite planar figure, which is surrounded by straight lines. All the sides of a regular polygon are equal in length, and the angles between them are of same. A polygon is called an n-gon if the number of its corner angles is n. The first part of the word polygon, "poly-", means many and the last part "-gon" an angle (from Greek *polugōnon* via Latin).

A polyhedron is a solid surrounded by faces, which are polygons. The faces are surrounded by edges and vertices (points). All the faces of regular polyhedrons are identical regular polygons. There are exactly five regular polyhedrons; the tetrahedron, cube, octahedron, dodecahedron and icosahedron. These are also known as Plato's solids. The ending –hedron (from Greek polyedron via Latin; 'hedra' = 'seat', 'base') in the word polyhedron, refers to a tip or a face of a geometrical solid.

A polytope is a finite area surrounded by hyper-planes in n-dimensional space. So, polytope is a general term for n-dimensional geometric solids, which are separated from outer space by 0, 1, 2, ..., n-1 multi-dimensional polytopes. An N-dimensional polytope is regular if the surrounding (n-1)-dimensional polytopes are regular. A polyhedron is a three-dimensional polytope, and a polygon is a two-dimensional polytope.

The concept polytope was first presented in 1882 by German Reinhard Hoppe as *polytop* in its German form (Effenberger 2010). Alicia Boole Stott, an Irish-English mathematician, made it known more widely in its English form (Ball & Coxeter 1987). By it, Stott originally meant four-dimensional polytopes. There exist exactly six regular four-dimensional polytopes, which have surfaces that consist of identical regular polyhedrons. Their interfaces consist of 5, 16 or 600 tetrahedrons, or 24 or 12 dodecahedrons, or 8 cubes. The ending of the word polytope, *tope* means a surface.

As told, polyhedrons and polytopes can be defined more accurately by the *vertex figure.* A polygon's vertex figure is a line segment that connects the centers of the two edges of a vertex O. The vertex figure of a polyhedron is a polygon, which is generated when the centers of the adjacent outgoing edges of the vertex O are joined together. For example, the vertex figure of a cube is a triangle. A polyhedron is regular if all its faces and vertex figures are regular. Similarly, the vertex figure of an n-dimensional polytope is n (n-1)-dimensional polytope, which is generated between the center points of the edges of the vertex O. The n-dimensional polytope is regular if it is surrounded by (n-1)-dimensional polytopes and its vertex figures are regular (Coxeter 1973).

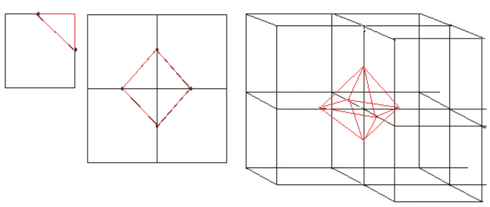

Figure 19 Vertex figures

3.4.2 Schläfli symbol

In geometry, the Schläfli symbol is a notation that defines regular polytopes and tessellations. By using the Schläfli symbol, it is possible to define several different kinds of polytopes and also tessellations in various dimensions. The Schläfli symbol is used often in scientific articles in the field of chemistry. This notation was developed in the 19th century by a Swiss mathematician Ludwig Schläfli, and it is of the form {p, q, r, ...}, where the number of attributes gives the dimension of a polytope (Coxeter 1973). Next we explain this notation in detail.

If the Schläfli symbol consists of only one variable, {p}, it is a polygon, that is, a p-gon. For example, the quadrangle is {6}. When there are two parameters {p, q}, the polytope is a polyhedron with p-gons as its faces. Around one vertex (point) there are q faces. So, for example, {4,3} is a cube, the faces of a cube being squares and each vertex is joint to three squares. In the four-dimensional space, the Schläfli symbol has the form {p, q, r}, where p and q are as above and r is the number of the polyhedrons around one edge. For example, for the tesseract, which is a four-dimensional polytope, the Schläfli symbol is {4,3,3}, where the first figures, 4 and 3 mean the cube. The last number, 3, tells that every edge is surrounded by three cubes. In Figure 20 there is the Schlegel's diagram of a four-dimensional hypercube. Schlegel's diagram illustrates the polytopes as projections to space that is lower by one dimension. For example in Figure 17, there is a projection of a four-dimensional hypercube in a three-dimensional Euclidean space. In case of hyper cubes the Schläfli symbol notation is the following. The symbol of a five-dimensional hypercube is {4,3,3,3}, because in this dimension each two-dimensional face of the five-dimensional hypercube is surrounded by three four-dimensional hypercubes. In general, $\{4, 3^{n-2}\}$ is the Schläfli symbol[2] of an n-dimensional hypercube. The first number is the number of the edges in the two-dimensional quadrilateral. The second number indicates how many 2-dimensional quadrilaterals are limited to the vertices, of the hypercube. The third number indicates how many three-dimensional cubes are limited to the two-dimensional quadrilaterals of a four-dimensional hypercube, etc. (Coxeter 1973)

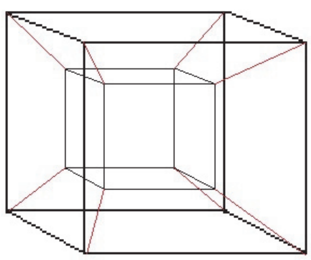

Figure 20 The Schlegel diagram of the four-dimensional hypercube

2 n-2 is not an exponent in this presentation, but the number of digits 3.

Another example could be a triangle, which has a simple Schläfli symbol {3}. The tetrahedron has four faces, which are all triangles, and each vertex is in contact with three triangles, so the tetrahedron has the Schläfli symbol {3,3}. In the Figure 16 we presented, five regular polyhedrons, the Plato's solids. Also the cube and the tetrahedron belong to this group of solids. The other three, as it is possible to count from that figure, have the following Schläfli symbols: octahedron {3,4}, icosahedron {3,5}, and dodecahedron {5,3}. (Coxeter 1973, Ball & Coxeter 1987)

In the case of tessellations and honeycombs the Schläfli symbol differs slightly. We will explain this more closely in the next section.

3.4.3 Honeycombs

Honeycomb is a name used in geometry for space filling. It means also close packing of higher-dimensional cells. Close packing means, that there are no gaps between the cells. A honeycomb is a kind of tessellation in varying numbers of dimensions.

The honeycomb as a structure means a space-saving construction, and its name comes from the hexagonal honeycomb structure made by bees. In geometry the honeycomb structure has, however, a more general meaning. It means filling or packaging the space in such a way that there remain no empty spaces. In this case of polyhedrons and polytopes of higher dimension, a honeycomb is one form of tessellation.

A three-dimensional honeycomb consists of an infinite number of polyhedrons which are connected to each other and fill the space so that the face of each polyhedron is also a part of another polyhedron. A honeycomb is called regular if all its polyhedrons are regular and identical.

The structure of the honeycomb can also be represented by a Schläfli symbol. Let's take again as examples, a quadrilateral, cube, and hypercube. The Schläfli symbol of a quadrilateral is {4}, and for cube it is {4,3} because in the cube each corner point is limited to three edges. The quadrilaterals can also tile a two-dimensional plane, whereby each quadrilateral is surrounded by four quadrilaterals against the edges, and another four quadrilaterals touch each corner point. Each corner point of the quadrilateral is thus in contact with four edges instead of three. For this reason the Schläfli symbol is {4,4}. When we extend this to three-dimensional space and tessellate it with cubes, we get the value of {4,3,4}, where the first 4 and 3 are the Schläfli symbols of a cube. The third number, 4, tells us to how many cubes one edge belongs. The number of these cubes is four. In four-dimensional space, which is formed by hypercubes, the Schläfli symbol of a honeycomb is {4,3,3,4}, where the first three numbers represent the tesseract, the four-dimensional hypercube {4,3,3}, and the last number, 4, tells us that in the four-dimensional honeycomb one quadrilateral is a member of four four-dimensional hypercubes. *So the fourth number 4 tells us to how many hypercubes one quadrilateral face is limited.* In general, $\{4, 3^{n-2}, 4\}$ is the

Schläfli symbol[3] of a honeycomb made by n-dimensional hypercubes. (Coxeter 1973)

The structure of honeycombs, polygons and polyhedrons may be determined by a vertex figure as we have already shown. The vertex figure is created, as explained earlier, when a vertex is placed in the center of each edge that is incident to a given vertex (point) of each polygon, polyhedron, the n-dimensional solid or honeycomb and all these vertices are connected to each other. (Coxeter 1973)

For example, the vertex figure of each polygon is always a triangle, and the vertex figure of a cube is a tetrahedron. The vertex figure of a two-dimensional quadrilateral tessellation, a two-dimensional honeycomb, is a quadrilateral and the vertex figure of a three-dimensional honeycomb, which consists of cubes, is an octahedron. The Schläfli symbols of these three vertex figures are: {4}, {3,4} and {3,3,4}. They are presented in Figure 19 from left to right. The vertex figure of a four-dimensional honeycomb that consist of four-dimensional hypercubes is a 16-cell, also known as a hexadecachoron.

In other words, the vertex figure of a two-dimensional honeycomb, which is also a two-dimensional tessellation and tiling, is constructed by connecting the vertex figures of the elements of the honeycomb to each other. For example, in a honeycomb that consist of quadrilaterals we connect four triangles to each other (Figure 19, center), and in a honeycomb, that consist of cubes, we connect eight tetrahedrons to each other (Figure 19, right).

A three-dimensional honeycomb is an infinite set of polyhedral fitting together to fill all space just once, so that every face of each polyhedron belongs to one other polyhedron. A honeycomb is regular if its cells are regular and equal. (Coxeter 1973)

In this study the most relevant aspect is the honeycomb model that is based on hypercubes, because it is the only one in the n-dimensional Euclidean space that is regular. The regularity means that it consists of identical elements, which in the lower dimensions are quadrilaterals and cubes, and in the higher dimensions, when $n > 3$, are hypercubes.

3.4.4 Kissing number

The kissing number problem is a geometric problem that got its name from billiards: two balls "kiss" if they touch (Pfender & Ziegler 2004). The Kissing number is also sometimes called the Newton number, the contact number, the coordination number, or the ligancy. (Conway & Sloane 1988)

In mathematics the kissing number means the number of n-dimensional hyperspheres (n-balls) that can touch at the same time, one n-dimensional ball of the same size. For example, in two-dimensional space, the kissing number is 6, which means that a circle can touch six circles of the same size. In the three-dimensional Euclidean space, the number is 12, and in the four-dimensional Euclidean space it is 24. In 5, 6 and 7 dimensions, the number is known to be to

[3] n-2 is not an exponent in this presentation, but the number of digits 3

40, 72 and 126, respectively. In higher dimensions, the exact number is known only for dimensions 8 and 24. In these dimensions, the numbers are 240 and 196,560. In geometry, the kissing number is generally defined as the number of such hyper-manifolds that may contact other hyper-manifolds without intersecting them. (Pfender & Ziegler 2004)

In the kissing number problem, the idea is to seek the maximum kissing number for n-dimensional Euclidean spaces. The two-dimensional kissing number 6 (Figure 21 left) is trivial, and is easily determined. Also, in three-dimensional space, it is easy to arrange 12 balls of the same size around a ball of a similar size and thus conclude that the kissing number is at least 12. To prove that the 13th ball cannot be added is difficult, though there is a lot of space left. Isaac Newton proved that 12 balls can be added around one ball. His contemporary, David Gregory, believed a 13th ball could be introduced among these 12. This did cause a disagreement between these two mathematicians in 1694 (Pfender & Ziegler 2004). The final proof for the number 12 was obtained as late as in 1953. In Figure 21, there are 12 balls arranged around a single ball, which creates a kind of icosahedron configuration (graphics: Detlev Stalling, ZIB Berlin) (Pfender & Ziegler 2004). It was known already that the number would be either 24 or 25 in four-dimensional space. Of these options, 24 was clearer since it is possible to place that number of balls around the four-dimensional sphere. The problem turned to be similar as in the three-dimensional case: would it be possible to place a 25th ball on that sphere also? The answer is no: in 2003, Oleg Musin proved, by using a subtle trick, that the correct answer is 24. (Musin 2003).

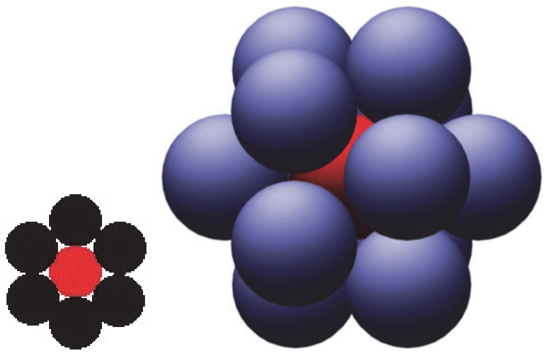

Figure 21 Visualization of 2D and 3D kissing numbers

The exact value of the kissing number in n dimensions is not known for n > 4. The exceptions are n = 8 (240), and n = 24 (196,560), as shown in the table below (Mittelmann & Vallentin 2010). The results in these dimensions are connected with the existence of the E8 lattice and the Leech lattice. These two are highly symmetrical lattices.

Table 1 First 24 Kissing numbers (Conway & Sloane 1988)

Dimension	Kissing number
1	**2**
2	**6**
3	**12**
4	**24**
5	at least 40; no more than 44
6	at least 72; no more than 78
7	at least 126; no more than 134
8	**240**
9	at least 306; no more than 364
10	at least 500; no more than 554
11	at least 582; no more than 870
12	at least 840; no more than 1,357
13	at least 1130; no more than 2,069
14	at least 1582; no more than 3,183
15	at least 2564; no more than 4,866
16	at least 4320; no more than 7,355
17	at least 5346; no more than 11,072
18	at least 7398; no more than 16,572
19	at least 10688; no more than 24,812
20	at least 17400; no more than 36,764
21	at least 27720; no more than 54,584
22	at least 49896; no more than 82,340
23	at least 93150; no more than 124,416
24	**196,560**

Figure 22 shows the growth of the kissing number in different dimensions. The red line shows the known maximum value and the blue line the known minimum value. Black circles are the dimensions the exact value of which is known.

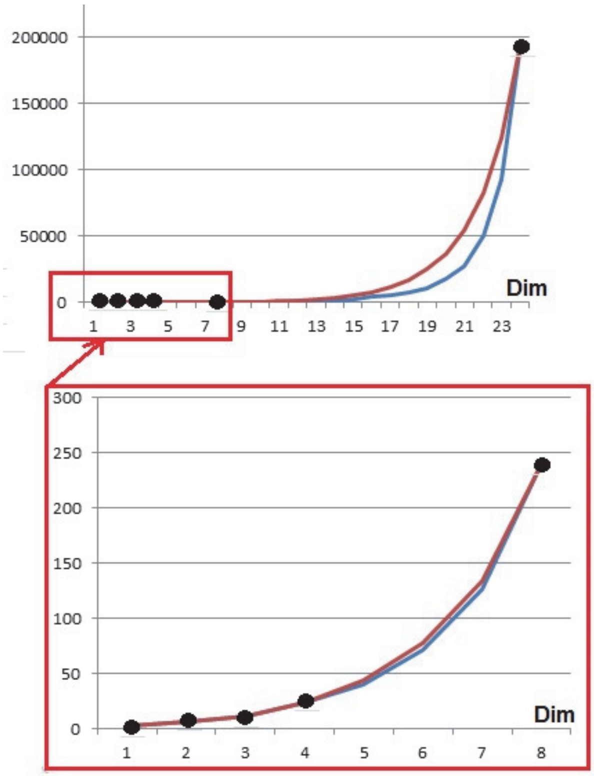

Figure 22 The kissing number in dimensions 1 – 24

3.4.5 Packing problems

The topic of cutting and packing is characterized by the fact that problems of essentially the same logical structure appear under different names in the literature: for example, cutting stock and trim loss problems, bin packing, dual bin packing, strip packing, vector packing, and knapsack problems, vehicle loading, pallet loading, container loading, and car loading problems, assortment, depletion, design, dividing, layout, nesting, and partitioning problems, capital budgeting, change making, line balancing, memory allocation, and multiprocessor scheduling problems. (Dyckhoff 1990)

Over time, the variety of the names using in cutting and packing problems (C&P problems) has been quite large in the literature, as can be seen from Table 2 (Dyckhoff 1990)

Table 2 The variety in C&P names (Dyckhoff 1990)

Surveys on special aspects of C&P

Author(s)	Year	Notion(s)	Discipline
Brown	1971	Packing, depletion	Computer Science
Salkin/de Kluyver	1975	Knapsack	Logistics
Golden	1976	Cutting stock	Industrial Engineering
Hinxman	1980	Trim loss, assortment	Operational Research
Garey/Johnson	1981	Bin packing	Combinatorial Optimization
Israni/Sanders	1982	Cutting stock, layout	Manufacturing
Rayward-Smith/Shing	1983	Bin packing	Mathematics
Coffman et al.	1984	Bin packing	Computer Science
Dowsland	1985	Packing	Operational Research
Dyckhoff et al.	1985	Trim loss	Management
Israni/Sanders	1985	Parts nesting	Production
Berkey/Wang	1987	Bin packing	Operational Research
Dudzinski/Walukiewicz	1987	Knapsack	Operational Research
Martello/Toth	1987	Knapsack	Mathematics
Rode/Rosenberg	1987	Trim loss	Engineering/Production
Dyckhoff et al.	1988	Cutting stock	Production

However, in this research project we are primarily interested in geometric packing and especially in sphere packing. In mathematics, ball packing, or more precisely sphere packing, is used to arrange n-dimensional balls of n-dimensional Euclidean space in such a way that the balls do not intersect each other. The spheres are usually supposed to have the same size. Sphere packing problems can be generalized to packing of different sizes of spheres, and packing in n-dimensional Euclidean space and in non-Euclidean spaces, such as hyperbolic space. In a typical sphere packing problem, we try to find an arrangement where the spheres would fill as large an amount of space as possible. The ratio of the space that spheres fill to the space they don't fill is called the density of the sphere arrangement.

This kind of packing is also related to the Kepler's conjecture (Figure 24, on the right (Mainz & Girolami 2012)) in the case of three-dimensional Euclidean space which must be filled by balls as efficiently as possible to minimize the empty space. Kepler conjecture can be formalized as the following theorem: *No packing of congruent balls in Euclidean three space has density greater than that of the face-centered cubic packing and the best packing density* of $\frac{\pi}{3\sqrt{2}} = 74\%$. Kepler conjecture is also the 18th of the famous David Hilbert's 23 mathematical problems. The optimal type of packing is the hexagonal close packing and the face-centered cubic packing (Figure 24, on the left (Hales 2005)), where the balls are in a form of a tetrahedron, just as Kepler assumed. The problem was proved by Thomas Hales. He used the so-called "proof by exhaustion" by separately proving each sub-problem. The proof remained incomplete, and, in 2014, Hales made a new proof by using large computer programs. (Hales 2005, Hales et al. 2015)

Here are some notes of Hales about the proof process. It is not essential to explain in this context more closely the term *tame graph* and its quite long definition, which includes eight conditions.

The combinatorial structure of each possible counterexample to the Kepler conjecture is encoded as a plane graph satisfying a number of restrictive conditions. Any graph satisfying these conditions is said to be tame. A list of all tame plane graphs up to isomorphism has been generated by an exhaustive computer search. The tame plane graphs encode the possible counterexamples to the Kepler conjecture as plane graphs. The archive is a computer-generated list of all tame graphs.

The original proof classifies tame plane graphs by a computer program written in Java. Therefore, as a first step in the formalization, we recast the original Java program for the enumeration of tame plane graphs in Isabelle/HOL. The archive that came with the original proof contained over 5,000 graphs. The first formalization resulted in a reduced archive of 2,771 graphs.

However, a bug caused two graphs to be missed in an early draft of the blueprint proof. (Hales et al. 2015)

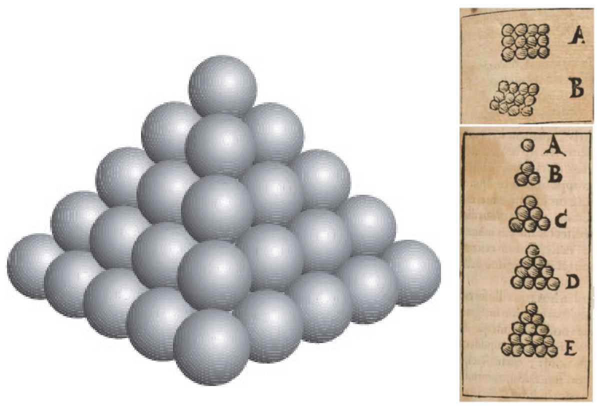

Figure 23 Space packing and Kepler conjecture

A direct proof using just a paper-and-pen has not been done, but the result has been checked by a logical HOL –proof program, called Isabelle. Kepler's problem, as well as the way it was solved, is comparable to the classic four-color conjecture (4CC). After more than 100 years, a long and complicated computer solution was found. Hales did again use the proof assistant Isabelle, while Georges Gonthier and Benjamin Werner used the proof assistant Coq in 2004 for the 4CC. (Gonthier 2005)

3.5 Summary

This chapter primarily considers tiling, tessellation and packing, the aim being to find a model that can be used to construct a game board of Euclidean dimensions that can be varied while the mathematical structure of the board stays similar. Similarity here means that the model is scalable to different dimensions in a regular manner, whereby it is possible to generalize. Sections 3.1 - 3.3 dealt with the basic concepts as well as tiling and tessellation in the two-dimensional plane and three-dimensional space. In Section 3.4 we started to examine a general solution for the problem in n-dimensional Euclidean spaces. In Subsections 3.4.1 and 3.4.2 we reviewed some basic concepts and tools needed for the research. In these subsections, we defined the manifolds and studied the Schläfli symbol. The Schläfli symbol was important in helping us to understand how tessellations and manifolds differ in differing n-dimensional spaces. In Sections 3.4.3, 3.4.4 and 3.4.5, we briefly explained honeycombs, the kissing number and packing problems.

These concepts will be necessary in Chapter 8, where we try to find an optimal working model for a symmetrical n-player game. The possible models are based on these structures.

4 THE HISTORY AND PREHISTORY OF CHESS[4]

This thesis deals with games based on chess. The aim is to use them to study n-person game models. Accordingly, it is reasonable to look back in history to find out how the game of chess was born. This chapter is divided into three parts. The first part studies board games that are older than chess. The motive for this is to get an idea of the kinds of elements used and the way chess may have evolved. The second part examines in particular, the early stages of the chess and the various theories related to it. The third part provides a further review of the more recent, as well as historical, chess variations. The motive for this third part is to get a historical overview of chess variants, and hence to get a perspective on the variants presented later in this thesis.

The oldest known board games are at least 5000 years old. One of the oldest and most famous is chess, dating back nearly 2000 years. We start this study with board games that are older than chess.

4.1 Old chesslike strategy games

It is possible to categorize the world's oldest board games in various ways. One way is to classify them into war games, race games (e.g, backgammon), position games (e.g, Go), mancala games (e.g, Mancala), dice games and domino games (Bell 1979). Of today's games, backgammon is probably the most famous among the race games, chess and draughts among the war games, and Go among the position games. Below, in the first subsection, we will consider the four oldest race games, and then, in the following subsection, three old war games, and finally, in the third subsection, the position game Go and two other very old strategy games. We do this because these games might have some connections with the prehistory of chess. Also the mancala games belong to the oldest

4 *This chapter includes a couple observations placed in the summary. The first observation deals with the famous Phaistos Disk and the second one with the evolution of board games.*

games, but we will talk about them only briefly in the summary. In the summary you can find some speculations about the Phaistos Disk and its possible connections with the race games.

4.1.1 Four race games

We consider first the race games, among which are also the oldest known board games: Egyptian Mehen and Senet as well as the Royal Game of Ur, from Sumer. As the fourth race game, we study in this chapter also the Ashtapada, because it might provide a link between race games and chess.

4.1.1.1 Mehen, an ancient n-person game

The earliest mention of Mehen, an ancient Egyptian game, was found in Hesy-Ra's tomb, from the period of 2800 - 2600 BC. Shore (1963) explains this game in detail below.

Given the playing-board and pieces of some ancient game, scenes showing the game in progress, and specialist knowledge of games, it should be a matter of deduction to reconstruct the rules of the game. Attempts to do so for the games of ancient Egypt have not proved satisfactory; and in the case of the so-called "serpent" game, it has not even been possible to set up the pieces.

The requirements for the game of Mehen are represented in a painted scene from the tomb of Hesy-re, close to the step pyramid of Djoser, and dated to the period of the Third-Dynasty (2800 B.C.). The board is circular, and its upper surface is covered with the figure of a coiled snake, its head at the centre and its tail at the perimeter. The figure of snake is divided into nearly 400 squares along its length. The playing-pieces kept in a box at the side comprise a set of six animals (three lions and three lionesses) and thirty-six colored marbles. (Shore 1963)

Mehen, which means "coiled", was originally a mythical Egyptian snake god. On the game board, the snake is thought to present the "world serpent" related to the mythical story of Osiris and his journey to the underworld. The world serpent, a mythological symbol of *Ouroboros,* is well-known throughout the world. It is a snake, which is eating its own tail, in the form of a ring. It is also known in Scandinavia. In the mythological tales of Vikings, it has the Old Norse name *jörmungandr* or *jormagund.* In Egypt, the game had religious and mythological meanings related to the passage to the land of the dead, and back. Mehen's mythical aspects can also be found in another game, Senet, which is presented in Section 4.1.1.2. Benedikt Rothöhler says: "Ancient Egyptian boardgames normally have a religious symbolism, although it is often very difficult to find the exact meaning. One cannot examine the subject of Egyptian boardgames without taking into account this background. The symbolism of Egyptian boardgames, especially Mehen and Senet, is directly connected with the functions of this deity." (Schädler 1999, page 10, Rothöhler 1999)

The game itself consists of a round disk, inside which there is a snake (Figure 24, below left). The serpent's head is in the middle of the disc, Shore

(1965), and on its back there are some 400 compartments, supposedly meant for game pieces. Other versions of the game, with different number of compartments, are also found. In addition, the game includes throwing sticks as well as balls and lions as game tools. No rules have been found for Mehen, but, in Sudan, and in some Arabic countries, a game called Hyena, with quite similar rules, is played. The rules of Mehen are built on the basis of this game. In Mehen there are six lions of different colors and for each lion there are six similarly colored balls. This adds up to 36 balls. Based on these facts, it has been assumed that Mehen has been *a game for six players* and **thus the oldest known multi-player game** (Rogersdotter 2011).

In Figure 24, there are three pictures depicting Mehen. At the top, four men are playing the game (Rothöhler 1999). Below on the left, is a board where the snake can be clearly seen (Figure by Rob Koopman, Mehen, old Egyptian serpent game, WikiCommons). Below on the right, is a board kept in the British Museum and with the following description: “A figure of a crouching lion in ivory: intended for use as a gaming-piece, originally carved with great detail but now in rather poor condition. The front paws are missing, as is the side of the head, and the surface of the ivory has decayed. The features still visible include the eyes and details of the mane, as well as the slightly open mouth, in which the teeth are delineated. The detail is only preserved in a small area beneath the chin and also beside the surviving ear. The underside of the lion is worn smooth from use.” (British Museum number EA66216)

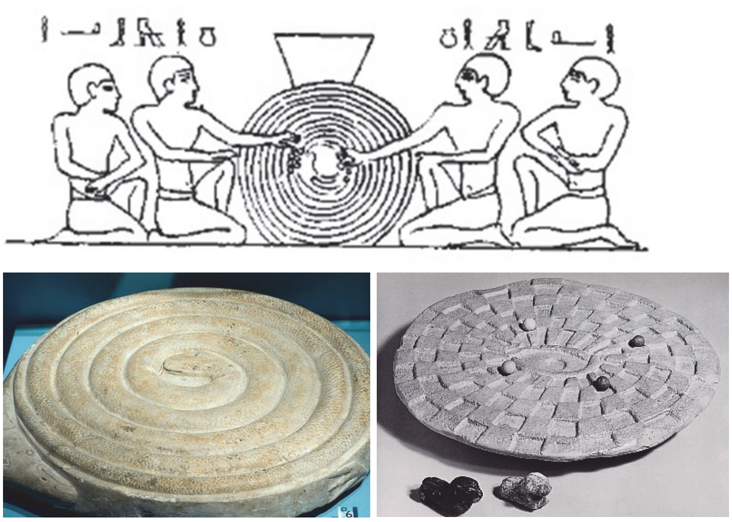

Figure 24 Mehen game from ancient Egypt

The process of the game and the rules can be summarized as follows: Each player throws his sticks (which play the role of dice) and when he gets an acceptable number, he is allowed to move the first ball to the first square, which is located at the tail of the serpent. After the balls travel to the center, they turn back. When one of the balls has reached the starting point, the player is allowed to send a lion on its way on the game board. When the lion has arrived in the center, where the serpent's head is, it turns back. On the way back, the lion can eat the other players' balls if it lands on the same square with them. The winner is the one, who has eaten all the opponents' balls or most of them.

4.1.1.2 Senet

Senet is an ancient Egyptian game, the earliest mention of it is found around 3500 BC (Cazaux 2003, Piccione 1980). Senet is a race game like Mehen. The game is not only one of the oldest known in the world, it also deserves to be mentioned because a painting in which the pharaoh is playing Senet with his queen was found among the drawings in the pyramids. In Figure 25, there are three pictures: on the top left, there is a senet board from Tutankhamun's tomb; on the top right, Queen Nefertari plays senet in a scene from her tomb; and below there is a senet board with numbered squares to indicate the sequence of play, the squares 15 and 26-30 having special significance (Hageman 2005). Earlier, on the basis of these figures, it was assumed that the game could have been chess and the origins of chess were attributed to Egypt (Ferlito & Sanvito 1990). But just by looking at the game pieces in the painting, it is easy to conclude that the game is, in fact, Senet. There are only two kinds of game pieces visible.

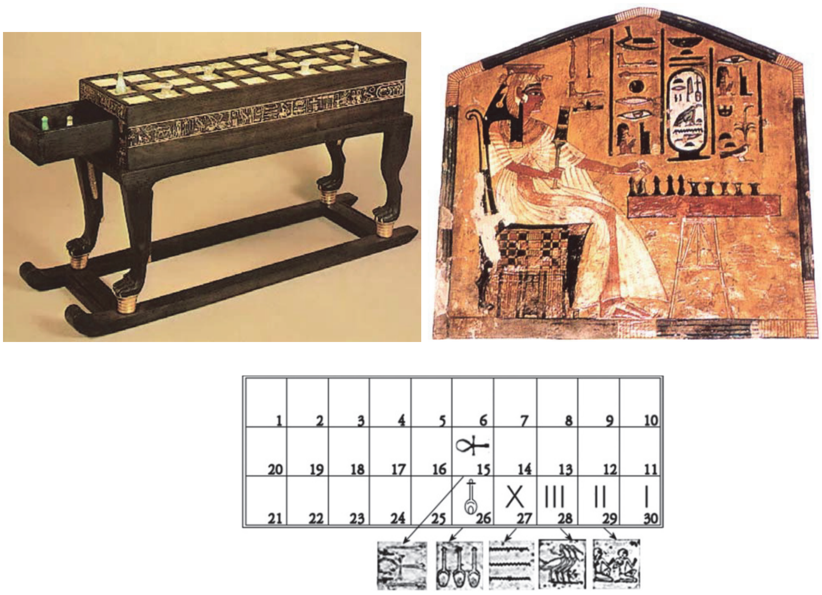

Figure 25 Senet

The play of the game is briefly as follows. On the game board, there are 30 squares, and each player has five game pieces. The pieces are moved on the board by the numbers given by the four sticks that are used as dice to decide the movements of the pieces in same way as in Mehen. The results provided by these four sticks also determine who can make the next move, but there is no need to explain that in detail in this context. If a piece gets in same square with the opponent's piece, then it has to be moved to the initial screen. However, if two pieces are in successive squares, then they are safe. Three pieces in successive squares form a barrier that the opponent cannot breach. The pieces move along the board in an "s" shaped pattern: first on the top 10 squares, then back 10 squares along the center line, and finally on the bottom 10 squares. The last five squares on the board are marked with symbols as special squares. One of these would send the piece to the 15th square in the center of the board, the other four are safety squares. Also the 15th square is safety square with a symbol figure. The aim of the game is to be the first to get one's own game pieces off the board. A piece is allowed to be removed from the board if it gets the exact number value of the last, 30th square. (Hageman 2005, Botermans et al. 1990).

4.1.1.3 The Royal Game of Ur

The Royal Game of Ur has many similarities with Senet. It is a race game, where the players follow a certain coiled route. However, there are no preserved rules of the game and the rules were arrived at on the basis of the game board and game pieces. The earliest discoveries of this game dated to 2600 BC from Mesopotamia, and the game can be found later, in 1800 BC, also in Egypt. In the Egyptian game, the shape of the board is a bit different (Cazaux 2003), as can be seen in Figure 26 [5]. We shouldlook at a couple of the rule options proposed by researchers.

Each player has seven pieces and the moves are determined by throwing small, tetrahedron-shape dice. Players start from the larger section of the board made of flower patterns, each player from the corner flower of his own (Figure 26, on the left)[6]. On the boards in Figure 26, the route of the lower-positioned player runs along the long edge, through a narrow pass and then rotates counter-clockwise at the smaller end of the board and returns back along the middle squares. The route of the other player is a mirror image of this. The game in Figure 26, suggested by RC Bell[7] (Bell 1979), is slightly different. The figure on the left depicts an original game stored in the British Museum. Here it is shown as a mirror image in order to make it easier to compare with the other two boards. In Figure 26, the board on the bottom is the modified version of the later game in Egypt.

If a piece gets on the square where there is an opponent's piece, then the opponent's piece has to start the game from the beginning. The flower squares are safety places, and they may contain several game pieces. The winner is the first to get his pieces off the board (Bell 1979, Botermans et al. 1990), just as in Senet.

5 http://www.luckydog.pwp.blueyonder.co.uk/games/ur/
6 By BabelStone (Own work), CC0, https://commons.wikimedia.org/w/index.php?curid=10861909
7 Figure 27, on the right

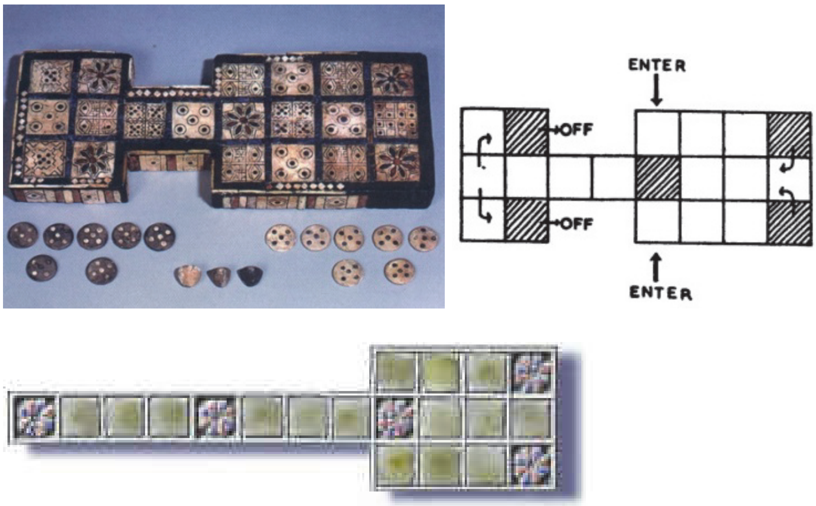

Figure 26 The Royal Game of Ur

4.1.1.4 Ashtapada

Ashtapada is a game which is thought to be the predecessor of chaturanga[8], and hence also the predecessor of chess, especially because of the shape of the game board. The earliest mention of the word ashtapada, which means "eight feet" or "eight fields" in Sanskrit, can be dated to ca. 400 BC (Cazaux 2003). Ashtapada could be played by 2, 3 or 4 players, each of whom had two game pieces. Each player placed his two game pieces on the two center squares of his side (Figure 27). Using dice, the pieces were moved along the marked route, pictured as a red line in Figure 27. When the piece reached one of the four center squares, it was allowed to be removed from the board. If a piece lands on the same square was one of the opponent's piece, then the opponent's piece would be returned to the beginning. The marked squares are safety places, where the pieces are safe from the opponents. The winner is the player who is the first to get all of his pieces off of the board. (Botermans et al. 1990)

8 Section 4.2.1 Chaturanga

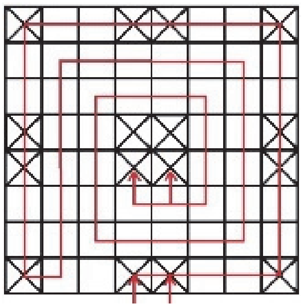

Figure 27 The Ashtapada-game

4.1.1.5 Summary

It is easy to see similarities among these four race games, although their geographical distance from each other is long, from Egypt to India. The ashtapada that was presented is a kind of unifying link between mehen and chess. The game play in ashtapada is comparable to the race on a snake's back in mehen, while the game board of ashtapada is similar to the present chess board.

4.1.2 Tafl games, Latrunculi and Petteia

Tafl games were brought by the Vikings to Scandinavia, British Isles and Russia. They are strategy games thought to have links with the birth of chess, the first tafl games having been found in the same era. The origins of this game are not known with certainty: it may have been developed by the Vikings, but it has also been claimed that the Vikings could have picked up the game from the British Isles, possibly from the Picts (Lawrence 2013). However, it is generally assumed that this game is based on a game called Latrunculi, which was played by Roman soldiers. Latrunculi is a game that is probably based on a Greek game called Petteia. On the other hand, it has been thought that Petteia had some influence on the emergence of chess. This is the reason why these three games are discussed together in this chapter.

4.1.2.1 Hneftafl and other tafl games

Hneftafl is the oldest known of these tafl games; the initial findings of it are known from 600 - 700 AD. In Denmark there has been found even an older game board, known as the Vimose discovery, from the 5th century. However, the Vimose board also could have been a Roman latrunculi game. Hneftafl, that is, *the king's board,* spread with the Vikings in the 9th – 11th centuries, after which chess replaced it. The most recent discoveries of the game, dated to 1587, are from Wales and from Lapland, dated to 1723. On the basis of these games, also the rules of hneftafl have been determined (Page 1969).

The oldest confirmed hneftafl discovery is quite recent. It was made in 2008 and 2011 in Viking ships, in Estonia, Salmi (Konsa et al. 2009, Peets, Allmäe & Maldre 2011). The older of these discoveries is timed in the period of 650 - 720 AD, but the ship was probably built already in the beginning of the 7th century. So we can assume that the game board was probably made in the 7th century. What makes this interesting is that the assumed predecessor of chess, i.e. chaturanga, was mentioned for the first time also in that same century. In the ships, no game board was found, but in the older ship there were 72 game pieces and one King piece. The number of pieces indicates the existence of two game sets, as in one hneftafl there should be 36 pawn pieces. On the King piece, a human figure was drawn. In Figure 28 (Peets, Allmäe & Maldre 2011), there is a dice on the left and a King piece on the right found on the older ship. From another ship, was found a King piece wearing a metallic helmet.

Figure 28 Hneftafl game pieces from Salmi, Estonia

The rules are best known about the Sami tablut-game, which was still played, according to Carl Linnaeus's documentation, in 18th century Lapland (Lawrence 2013, Helmfrid 2005, Bayless 2005). In tablut, in the center of the game board, instead of the Vikings there are Swedish army soldiers wearing 18th century uniforms. The troops that siege them are Russian soldiers. The number of the besieging soldiers is 16, and there are 8 soldiers who defend the king in the centre (Figure 29 on the left, Wilhelm meis, en.wikipedia Public Domain). It might be added that Irish had a game called fidchell the Breton Celts gwezboell and Welsh had gwyddbwyll (Niehues 2014), in which the size of the board and the positions of the pieces were exactly the same as in tablut (Figure 29, center).

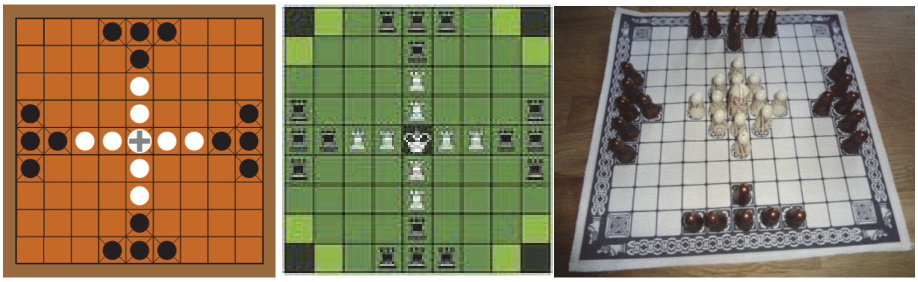

Figure 29 Tablut, gwyddbwyll[9] and hneftafl

Next we will explain the rules of the tafl-game that are mostly applicable to hneftafl. In other tafl games, the rules are quite similar, the main differences being the starting positions and the board size. To explain the rules, we use the board in Figure 29, on the right (own photo). This one is an exact copy of the board found from the Norwegian Gokstad Viking ship (Helmfrid 2005), and the pieces of this game are copies of the pieces found in the Outer Hebrides, on the coast of Scotland. This game board was bought in 2014 by the author from The National Museum of Iceland with the attached information: "The ornamenst on the textile board originate from the Gokstad vikingship where a Viking game board with pieces was found. The pieces are based on the design of the famous Lewis chess pieces found on the Outer Hebrides west of Scotland, an area under Viking rule at the time the pieces were made (about 1100 AD)".

The rules are simple. The game board has 11x11 squares. The sieging player (black) has 24 warriors, six on each side, as shown in the figure. The defending player (white) has the king in the center of the board, defended by 12 warriors.

The goal of the game for White is different from that for Black. White is winner if the king manages to escape to one of the castles in the corners. Black wins if it manages to capture the white king.

All the pieces move in the same way as the rooks in the chess, which is towards the sides of squares. A game piece can move as far as there are free squares, but it cannot jump over other pieces. The center square of the board is a place of safety, where the king is not threatened and where warriors are not allowed to stop. The king is captured if it is surrounded on all four sides by the opponent's warriors or the edge of the game board. The warrior is captured and removed from the board if it is surrounded on two opposite sides by the opponent's warriors or if it is between an opponent's warrior and one of the castles in the corners.

Figure 30 depicts four other versions of tafl games. Tawlbwrdd is Welsh, Alea Evangelista was a game of Saxons, Ard Ri (Figure: François Haffner on fr.wikipedia Public Domain) as a Scottish game and Brandubh (Figure: Luis

[9] By Llydawr, https://br.wikipedia.org/wiki/Gwezboell#/media/File:Gwyddbwyll.jpg

Dantas, en.wikipedia Public Domain) an Irish version of Hneftafl (Helmfrid 2005).

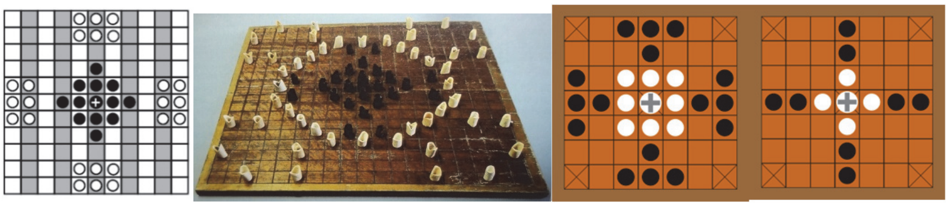

Figure 30 Tawlbwrdd[10], Alea Evangelii[11], Ard Ri and Brandubh

When chess came to Europe, it was often mistaken as one of the tafl games. Because of this, Vikings began to refer to chess as Skáktafl to distinguish it from the other tafl games. Also the names of other tafl games had meanings: hneftafl meant “the board game of the fist, tawlbwrdd “tall-board” and alea evangelii “board game of the Gospels" (Bayless 2005).

Interestingly, there is also a theory, that hneftafl could have been of Pictish origin (Lawrence 2013), which would make this game much older.

4.1.2.2 Latrunculi

The Roman board game Latrunculi *(soldiers game or little soldiers)* has been considered as a game, which formed the basis for the Vikings' Tafl games. The oldest mention of this game is from Varro 116 - 27 BC (Botan & Nuțu 2009). There were different sizes of game boards, from 8x8 to 8x12 squares. Also in the proposed rules, there are small nuances. In general, the rules are as follows: If the game board is as shown in Figure 31[12], then each player has 12 soldiers on the back row. In the centre, in front of them, the *dux*, director, is placed. All the pieces move in the direction of the sides as far as they can go, in same way as the rook in chess or the pieces of hneftafl. The pieces are also “eaten” in the same way as in hneftafl, which means that a game piece is eaten if on its both sides there is an opponent's game piece. The exception is the dux, which must be sieged from all sides. Normally, the dux moves just like the soldiers do, but it also can jump over another piece, which is not allowed for the soldiers. The game ends when the dux is under siege from all sides, which is 2, 3 or 4 sides depending on its location, or when the opponent’s army is totally destroyed. (Perkis 2010)

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10 Helmfrid 2005

11 Botermans et al. 1990

12 Author's Note & Disclaimer: Permission is granted to any and all who wish to link or borrow images and text, they are used under the Internet Fair Use policy for educational resources. Dr. Wladyslaw Jan Kowalski, www.aerobiologicalengineering.com/wxk116/Roman/BoardGames/latruncu.html

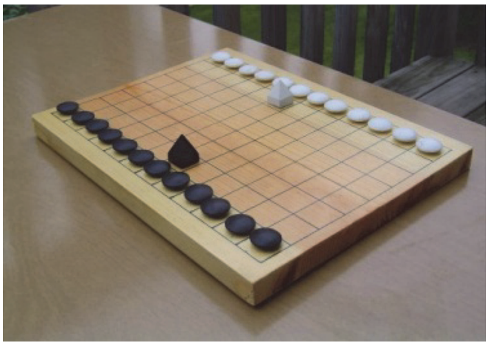

Figure 31 Latrunculi

4.1.2.3 Petteia

Petteia is an interesting mystery in the history of board games: the game can be found in pictures of Achilleus and Ajax playing it in ancient Troy. The paintings (Figure 32)[13] on pots are from the period of 550 - 500 BC, so the game must be at least that old. Also the oldest written sources of the game are from that period, for example Plato and Aristotle mentioned this game. Petteia may be even older because the Trojan war, which destroyed the town, most likely took place about 1190 BC. The pictures do not tell anything about the shape of the game board because they are shown from the side (Figure 32)[14]. In this game dice were not used, for it was a pure strategy game. It is thought that the game was taken to India by Greek soldiers, who accompanied Alexander the Great during his wars. There, this game could have affected chaturanga, a game played with dice in India at that time. So petteia might have affected chaturanga to make it more like a strategy game, hence giving an impetus to the development of chess. However, this is only a conjecture, as no rules of petteia have survived to our day. (Austin 1940, Mark 2007)

The rules of this game have, however, been derived reasonably well, based on some assumptions. On the basis of these assumptions, petteia is thought to be very much like Roman latrunculia and perhaps developed from it.

13 http://www.gamesmuseum.uwaterloo.ca/Archives/Austin/

14 Public domain, Workshop of Diosphos Painter - Marie-Lan Nguyen (2011), https://commons.wikimedia.org/wiki/File%3AAchilles_Ajax_dice_Louvre_MNB911.jpg

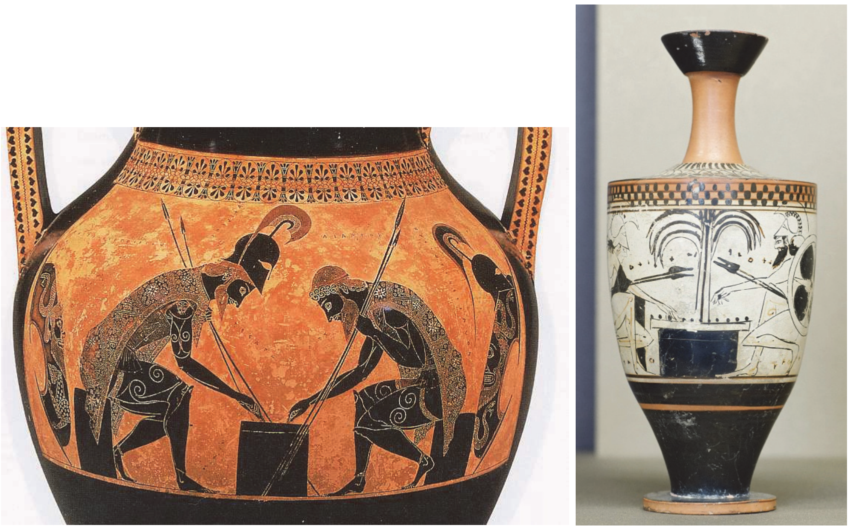

Figure 32 Petteia

The Greek philosopher Plato, who lived between the 4^{th} and 5^{th} centuries BC, wrote that the game originally came from Egypt. More information can be found about the game of *poleis* (cities), and sometimes the names petteia and poleis (also known as polis) have been used in parallel. Also the rules of poleis were used in deducing the rules of petteia. The board is usually 8x8 or 8x12. All the game pieces are placed on the opposite longer sides of the board in one row. The game pieces are moved in the same way as in latrunculi, or as the rook is moved in chess. Also, the capture of the opponent's game piece happens in the same way as in latrunculi. However, in this game there is no *director* piece, so the game was probably won by the one who first captured the opponent's pieces. The game may also have ended in a kind of checkmate, like chess. In one of his essays written in about 380 BC, Plato describes: "Bad petteia players, who are finally cornered and made unable to move, by clever ones." So it seems that in this game all the losing opponent's pieces end up checkmated. Plato wrote this sentence as a metaphor about good and bad philosophy students, at the same time opening the rules of petteia for us. (Samsin 2002)

4.1.3 Seega, Checkers, and Alquerque

In Sections 4.1.1 and 4.1.2, we introduced race games and war games. In this section, there are three games that are also strategy games but are structurally slightly different. However, these three games have some common characteristics, which we will discuss in the summary section.

4.1.3.1 Seega

The rules of seega are simple, but the game has been said to be as difficult as the game of Go if not more complex. Games such as chess, backgammon, checkers, Othello and Go have been of interest to the AI research community. The ancient Egyptian board game of Seega is a challenging game that, in some ways, is more difficult than chess and may even be comparable to Go in difficulty (Abdelbar, Ragab & Mitri 2004). The game board is relatively small, only 5x5 squares (Figure 33, left)[15]. However, other board sizes up to 9x9 squares are known. Thus, this kind of board was easy to set up anywhere, even by drawing it in the sand. On the smallest, 25-square board, each player has 12 game pieces, leaving only one free square in the center. At the beginning of the game, players decide who is going to start. Then they put their game pieces alternately, two at the same time, in any square except the center one. When all the pieces are on the board, then the player who has set the last two pieces can start the game. The game pieces move horizontally and vertically as rooks do in chess. The opponent's piece can be captured just as in tafl and latrunculi, that is, when surrounded by two of the opponent's game pieces. The game piece can be moved as long as you can capture an opponent's piece with it. It is obligatory to capture the opponent's piece if that is possible. The center square is a safety square. If you cannot move any of your game pieces, then the turn passes to the opponent. The player achieves a "Great Victory" if he can capture all the opponent's pieces. This doesn't happen very often. Mostly the result is a draw, which is referred to as a "Small Victory". (Botermans et al. 1990)

Figure 33 Seega and its variation from Somalia

Plato assumed that the Greek petteia was of Egyptian origin, and Egyptian seega has been compared with petteia (Austin 1940). Because of this, there are theories according to which Petteia had its roots in seega. Later it was found that seega is from a later period, from the Roman era in Egypt. However, it is possible, that seega and petteia are based on some unknown older Egyptian game, perhaps the one which Plato referred to. Seega is also known by other names: seeja and siga. This is a bit confusing, for Murray used the term siga also for the Indian saturanga game (Samsin 2002). Another interesting point is the possible connection between seega and checkers. Checkers differs from the other previously discussed games in one detail: capturing opponent's pieces does not happen by surrounding them but by jumping over them to a square which

15 www.etsy.com/listing/94083139/egyptian-seega-game-board-with-midnight
Etsy has been a Certified B Corporation since 2012.

is of the same color as the starting square. But also checkers have different variations in different cultures.

In Somalia, seega is played on the same game board, but has with some modifications to the rules. The game is called *high jump* (Figure 33, right)[16], and it differs from seega in that the opponent's piece is captured by jumping over it. This is permitted only vertically and horizontally, not diagonally as in checkers. In addition, the centre square is the safety square. One major difference is that the game pieces are already on the board when game begins. (Bell 1979, Botermans et al. 1990)

4.1.3.2 Checkers

Checkers (Draughts) is one of the best-known board games, and it is necessary to write a few words about it in this research. It is assumed that checkers was born in southern France, sometimes between the 11th and 13th centuries (Botermans et al. 1990). But it is possible, that the historical roots of checkers are earlier, than the 11th century. It might also be a variant of ancient Greek or Egyptian games. The original French name came from the expression "jeu de dames", which means "women's game". In most languages, this game has a name that refers to this French expression. The English terms for it, draughts and checkers, which are derived from chess, differ from the terms in other languages. There are several variations of this game known around the world. The differences are related to the board size, how the game pieces are set on the board, and the right to move pieces also backwards. The most common checker versions are English checkers and international checkers. Below, are brief descriptions of both versions.

English checkers is played on an 8x8 chessboard, where each player has 12 pieces. The popularity of this game may be due to the fact that it can be played on a chess board, which usually is easily available. The pieces are set on black squares, leaving two free rows in the middle. The game board is placed so that in the right-hand corner of the player there is a white square. The player with black pieces starts the game, and the pieces move only diagonally on black squares. If there is an empty square behind the opponent's piece, then the piece can be captured. If after this capture there are still free squares behind the opponent's pieces, the player can capture as many opponent's pieces during one of his turns as possible, using the same game piece. During this kind of chain-capturing, each captured piece must be removed immediately because it might influence the game if left in place. The pieces must not capture or move backwards. If a piece reaches the opponent's back row, then it becomes a king that can move backwards. The winner of the game is the player who has captured all the opponent's pieces or besieged them so that they cannot move. In this game, it is obligatory to capture if possible. If the player doesn't do that, there is a variety of sanctions to be applied.

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16 www.etsy.com/listing/108300196/seega-game-board-with-greek-geometric
Etsy has been a Certified B Corporation since 2012.

International checkers, or Polish draughts, is played on a 10x10 board (Figure 34, on the left, Michel32Nl, en.wikipedia Public Domain), where each player has 20 pieces. Another major difference with the English checkers is that, in this game, pieces can move from the start and capture also backwards. (Botermans et al. 1990)

The game that is popular in North America and the British Commonwealth, English draughts, has pieces moving forward one square diagonally, kings moving forward or backward one square diagonally and a forced-capture rule on 8 x 8 board. In 2007, all the moves of English checkers were completely analyzed by computers for the first time. This game has roughly 500 billion possible positions (5×10^{20}). The task of solving the game, that is, determining the final result in a game with no mistakes made by either player, is daunting. Almost continuously since 1989, dozens of computers have been working on solving checkers, applying state-of-the-art artificial intelligence techniques to the proving process. An important finding of this research was that perfect play by both sides leads to a draw. (Schaeffer et al. 2007)

4.1.3.3 Alquerque

It is thought that the alquerque game (Figure 34, right)[17] was a role model for checkers. Alquerque, where the capturing was carried out almost in the same way as in checkers, was brought to Spain in the 11th century by the Moors. In alquerque, a piece could be captured if there was a free square behind it. Thus, it was possible to move diagonally, vertically and horizontally.

Figure 34 Checkers and Alquerque

4.1.3.4 Summary and discussion

In Figure 33 on the left, there is a seega board. It is very similar to the alquerque board shown in Figure 34 on the right. In seega, at the beginning of the game, the pieces are placed by the players on squares chosen by them, but in the Somalian variant the pieces are placed in the starting position as shown in Figure

[17] http://mittelalter-lagerbedarf.de/Spiele/
http://mittelalter-lagerbedarf.de/images/product_images/popup_images/254_1.jpg

33, on the right. As can be seen, the setup is the same as in alquerque. These two properties suggest that Somalian seega could be an intermediate form between seega and alquerque and leading further to checkers.

So it seems that Egyptian seega might have developed from Roman latrunculi. Later came the variant of seega which led to alquerque and checkers. On the other hand, it is thought by some that latrunculi developed from petteia, which was inspired by some Egyptian board game. Even though it is possible that this intermediate board game was seega, no older evidence than that from the Roman era exists.

4.1.4 Go and Liubo

Weiqi is better known in the West under its Japanese name Go. Liubo was contemporaneous with weiqi, though some claim that it dates back to the Shang dynasty in the second millennium BC (Papineau 2001). Because of this relation, we discuss these two games in this same section.

4.1.4.1 Go

Go and seega games have some surprising similarities, and therefore also the Japanese go is included in our historical review of war and strategy games. Go is different from the strategy games presented in previous sections: the players do not try to destroy the opponent's pieces but to conquer new areas. Go is a positional game. Go is the name of the game in Japan; however, the game has its origins in the Chinese game called Weiqi. The game is very old, and the first records of it are in Zuozhuan written texts from as early as 500 BC (Papineau 2001). The rules are simple, but the game itself extremely complex. The game description is summarized below. Go, also known as Baduk, is frequently considered to have more strategic depth than any other board game commonly played, more even than Chess. In Go, two players, Black and White, alternatively place stones on a 19 x 19 board. Unlike in chess or in checkers, pieces are played on the corners of the squares. A group of stones consists of a set of stones of one color connected by means of direct orthogonal connections. (Johnson 2014)

On the game board (Figure 35) [18] is a 19x19 grid, where are 361 stones as game pieces. The number of stones is exactly the same as the number of line intersections on the board lattice. Of these, 180 are white and 181 black. All the stones are identical except for the color. At the beginning of the game the board is empty. The black player starts by putting a stone on one of the line crossings, which can include a crossing on the board edge. After this, the players alternately put their stones on some free crossing point. A player can also pass his/her turn, and the stones, which already are on the board, cannot be moved. The aim of the game is to conquer regions on the board, which is done by plac-

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18 By Goban1 - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=15223468

ing stones on the board in such a way that they make chains within which there are as many *free points* as possible. The term chain in this context refers to the stones of the same color which are connected to each other by a line between the points. All the points without a stone are considered free. A stone or a stone chain can be threatened by conquering free points around it. Once the last free point has been occupied, the captured stones are removed from the board. After the removal, each player is allowed to place his/her stones on the released points. It is strategically advantageous to build a chain in such a way that inside there are only single free points. The opponent is not allowed to place stones on such points. If there are more points available, then the opponent may try to take over the inside area. It is also not always reasonable or economical to remove all the free points from the threatened stones. If the last free point is not occupied, then the stone or stone chain is imprisoned. The stones remain on the board, but they will be removed at the end of the game. (Botermans et al. 1990)

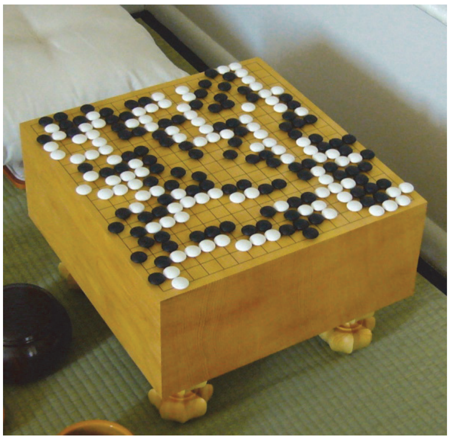

Figure 35 Go

At the end of the game, the free points inside the chains and the captured stones will be added up. The sum of these two is the final score (Botermans et al. 1990).

4.1.4.2 Liubo

Liubo is a Chinese game from the era of the Han Dynasty around 200 BC - 200 AD. The name Liubo consists of "liu", which means six, and "bo", which means a stick or a game played with dice. Liubo has been thought by some to be a predecessor of Chinese chess, the xianqi, and this is the reason for this short presentation. There are no clear rules left of this game either, although the game is still played. Our knowledge about this game is restricted to the game board

(Figure 36)[19] and the game equipment, the six sticks, which were used as dice. Both players had six game pieces. It is thought that the game pieces went around the game board, and hence liubo was a race game. On the board, there were twelve paths and, in the middle, the water, just like in xianqi. (Li 1998)

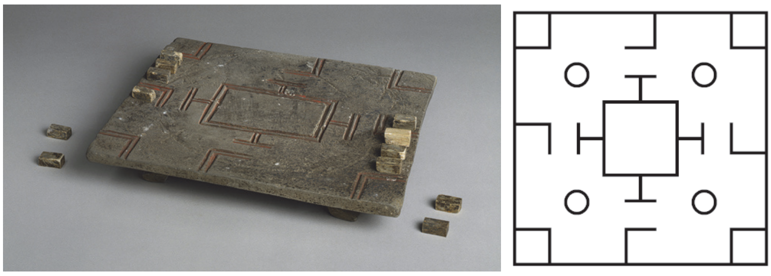

Figure 36 Liubo-game

It has been claimed that liubo was mentioned already during the Shang Dynasty (1600 BC - 1028 BC). The group of scholars who claim so think that the origins of chess are in China, where liubo disappeared in the 6th century and xianqi took its place (Li 1998).

4.1.5 Konane and Agon

In the end of this section, we will introduce two very special kinds of board games, Konane and Agon. Konane (Figure 37) was developed in Hawaii, which is an isolated island far from the rest of the world, and hence might shed light on the evolution or birth processes of board games. Agon (Figure 38 (Walker 2014)) has, as its peculiarity, a hexagonal board which might be the oldest one known.

4.1.5.1 Konane

In Hawaii, an ancient board game called konane can be found (Ernst 1995, Hearn 2005). This game was added to this research because of its background. Hawaii had its first contact with the rest of the world in 1778. Captain Cook visited the islands and gave a report about a game, most probably konane that he saw there: "One of their games resembles our game of draughts (checkers), but from the number of squares, it seems to be much more intricate. The board is of the length of about two feet, and it is divided into two hundred and thirty-eight squares, fourteen in a row (hence a 14-by-17 board). In this game they use black and white pebbles, which they move from one square to another." (Ernst 1995)

[19] http://america.pink/liubo_2725490.html
AMERICA.PINK is free online article encyclopedia . All images and text on this site are used legally and does not violate copyright law. Our site is an educational resource and not a commercial.

This game has some connections with checkers, but also has several differences. The isolation of Hawaii leads us to two interesting questions: A. Did the Hawaii people invent the game by themselves? B. Did they bring it to the islands with them? Answering "yes" to these questions opens up fascinating speculations in both cases.

A. If they did invent it by themselves, it means that the human brain has some processes that enable it to find the same kinds of games independently. Cognitive science might be able to help us in this research. Another, but quite different example is the Maori game Mu Torere in New Zealand (Botermans et al. 1990).

B. If they brought the game from somewhere else, it tells us that the evolution of the board games is indeed very old. The Hawaiians came to the islands from the Southern Pacific in two waves. The last one was in about 1000 AD and the first one ca. 300 AD. To follow their migratory path, I had to (paradoxically) add to my research also a paper from a totally different field of science. Just "lately" (2003 - 2009), DNA research has found out that the Polynesians, including Hawaiians and Maoris (in New Zeeland) left South East Asia (Philippines, Taiwan) already 5,000 years ago. (Soares et al. 2011, Su et al. 2000) This means that, if Konane belongs to a large board game family with a common evolution, the basic forms of these games must be that old.

”The ancient Javanese/Malayan game of main chuki or tjuki is similar to Konane in that it is a kind of checkers played with 60 white beans and 60 black beans on the 120 points formed by intersections of lines [Wilken 1893, 162; Wilkinson 1925, 60].” (Ernst 1995)

ScienceDaily (January 27, 2009): Pacific people spread (Figure 37, below) from Taiwan, language evolution, study shows. New research into language evolution suggests most Pacific populations originated in Taiwan around 5,200 years ago. Scientists at the University of Auckland have used sophisticated computer analyses of vocabulary from 400 Austronesian languages to uncover how the Pacific was settled. [20] The colonization of Polynesia has remained a controversial topic. Two hypotheses, one postulating Taiwan as the putative homeland and the other asserting a Melanesian origin of the Polynesian people, have received considerable attention. Surprisingly, Taiwanese Y haplotypes are rarely found in Micronesia and Polynesia. Likewise, a Melanesian-specific haplotype is not prevalent among the Polynesians. However, all the Polynesian, Micronesian, and Taiwanese haplotypes are present in the extant Southeast Asian populations. Evidently, the Y-chromosome data does not lend support to either of the prevailing hypotheses. Rather, we postulate that Southeast Asia provided a genetic source for two independent migrations: one towards Taiwan and the other one towards Polynesia through the islands of Southeast Asia. (Su et al. 2000)

[20] https://www.sciencedaily.com/releases/2009/01/090122141146.htm

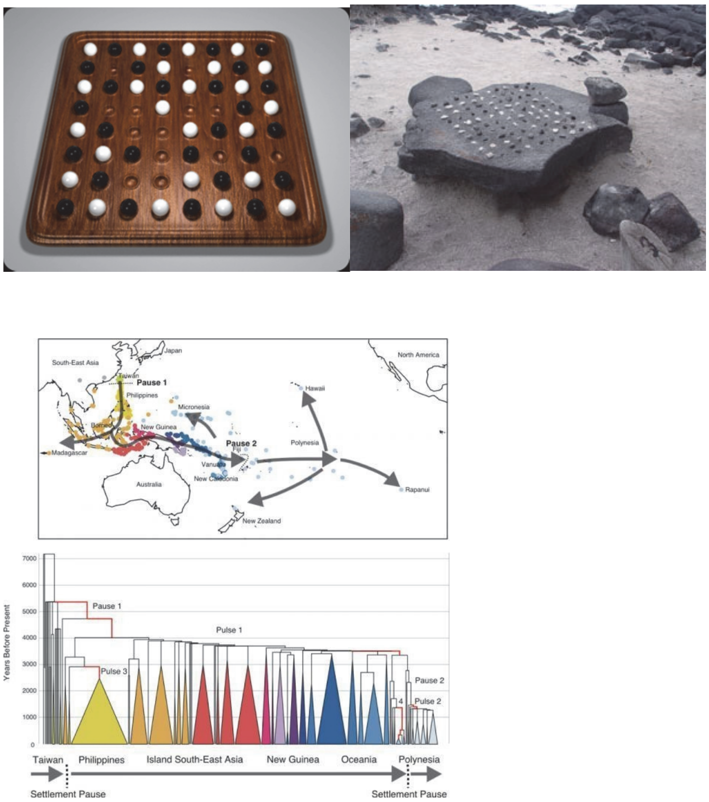

Figure 37 Konane game from Hawaii [21] and Polynesian migrations

There are other ways to investigate the colonization of Polynesia, using rats. People have been in the Pacific for over 40 000 years. The first people to arrive in the region were part of one of the first 'Out of Africa' migrations. Sometime around 60,000 years ago, during a period when sea levels were much lower than today, they most likely followed the coastline through southern Asia, along the landmass of Sunda. The Lapita people are believed by most prehistorians to be the ancestors of the Polynesians. The Lapita colonists were the first humans to arrive on the islands of Remote Oceania. Archaeological evidence suggests that these colonists moved from the Bismarck Archipelago out into the previously uninhabited islands of Remote Oceania sometime around 3000 years ago. Lapita sites appear at about that time from the Reef Santa Cruz Islands through Vanuatu, New Caledonia, Fiji, Tonga, and Samoa. Like the Polynesians, the Lapita peoples carried with them their familiar and important food items and introduced these to the pristine island environments they settled. The ani-

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21 Left, http://tabtop.blogspot.fi/2009/04/konane.html
© Ian Henry 2009 Contact via email ianhenry@tbltop.com
Right, http://www.pricepages.org/Hawaii/week9/
Photo by Seth Price, seth@pricepages.org

mals that were introduced by Lapita peoples to Remote Oceania included the dog, the pig, the chicken, and the Pacific rat. The Pacific rat, also known as kiore, does not belong to the same species as the rats introduced by Europeans – and does not interbreed with them. Therefore the kiore found on Pacific islands today should be the direct descendants of those introduced by Lapita peoples. A mitochondrial DNA analysis of Polynesian populations of Rattus exulans, kiori, tells also how Polynesia was populated. (Matisoo-Smith 2009)

4.1.5.2 Agon

The agon board game is included in this presentation primarily because of the shape of its board. Agon is probably the oldest known strategy game on a hexagonally tiled board.

Figure 38 Agon

The origins of this game are probably in the 18th century. Agon is a game for two players and was first seen in France in the 1780s. It is notable for being one of the earliest games played on a hexagonal grid and one of the breed of race games which relies purely on skill rather than on an element of luck such as dice. Each player has a queen and her six guards. The players strive to be the first to get their queen and her guards to the centre of the board. The name comes from an ancient Greek word for contest, or challenge. (Walker 2014)

4.1.6 Summary

There are innumerable old board games, and it is not reasonable to discuss all of them in this thesis. In this section (4.1), we have, however, gathered a group of games, which may in one way or another have connections to the birth of chess. In the following section (4.2), we will examine the games that are related to the history of chess, while in this section (4.1), we examined games that might have been related to the "prehistory" of chess. Some conclusions are drawn in Section 4.2.5.

4.2 Origins of chess

We have been discussing board games, some older than chess, and some of their predecessors. Now we are going to focus on the birth of chess.

Chess has been known for two thousand years. The ancient sources of chess can be found in *Karnamak-i-Artak-Hatr-i-Papakan (the deeds of Ardershiri's son Papakan).* This manuscript was written during the Sasanian Dynasty between 224 AD and 651. AD (Mark 2007)

At that time, the game was called chaturanga (Figure 41), where the movements of chessmen were different. The board was called ashtapada (see Section 4.1.1.4). The word chaturanga means, in Sanskrit, quadripartite and was used to describe the Indian army, which had four divisions made up of elephants, cavalry, chariots, and infantry (Hooper & Whyld 1987). Since that time, the most fascinating chesspiece has without any doubt been the Knight. The peculiarity of the Knight's movements and its numerical symbolism have been the main factors of the hypothesis connecting the origins and structure of chess with the secret magical religious rituals of ancient India (Bidev 1986). On the account of Forbes' false trail, the ancestor of chaturanga was once considered to be a form of four handed-chess (Hooper & Whyld 1987). We continue with this theory in Sub-section 4.2.2.

Chess arrived in Europe in 1000 AD. After several modifications, medieval chess achieved its present form. Several theories about the origin and structure of chess appeared in the following years. Probably the oldest known chessmen were those found by Burjakov in 1977 in Afrasiab, Uzbekistan, close to Samarkand. These chessmen have been dated on the 8^{th} century. In Figure 39, there are some Afrasiabian chessmen.

Figure 39 Possible chessmen found in Afrasiab, Iran

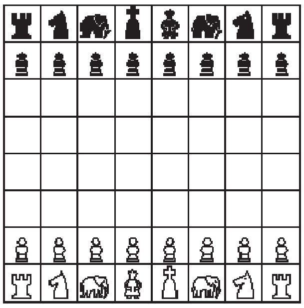
On top there are horses; the riding man below right has been assumed to be the king. (Cazaux 2003, Cazaux 2003, Mark 2007)

The early Indian form of chess arrived in China in 400-300 BC. The game was transformed into a variation which is still known as Chinese Chess (Xiang-chi). Chinese chess is in many ways the most important variety of chess. For example, the pieces do not move on the squares, but on the line-intersections of the squares. Neither of the fighting forces are on the same battle field, as they are placed on the opposite banks of a river. It has also been claimed that the origins of chess are in China instead of India. From China, chess moved to Japan, where it is called Shogi. The size of the one-colored game board varies from a 7x7 board to a 25x25 board, with 177 pieces on each side. The best-known of these is the Middle Shogi with a 12x12 board having 46 pieces on each side.

4.2.1 Chaturanga

During its journey of over thousand years to Europe, chaturanga went through several transformations before it got its final form as chess. For example, the Queen, called *firzan* or *fers*, and at the very beginning *mantri*, moved only one step diagonally. The Bishop, *fil*, originally *gaja*, meaning an elephant, moved exactly two steps diagonally (Hooper & Whyld 1987). Also the dice were often used in the game (Gollon 2013).

We will now observe how all of the game pieces move. The King, *raja*, moved as the king in chess does: one step to any of the eight neighboring squares. The Queen, or firzan, moved only one step in the four corners on the board. The rook or *ratha* moved, just like the modern rook, to vertical or horizontal directions as far as it could. The Bishop, or fil, moves to the direction of the corners, jumping over one square. However, also some other types of movements have been documented. The knight, or *asva*, moved as the knight in current chess does. The pawn, or *sippoy,* moved as today, that is, one step forward. In Figure 40, on the left, is the starting position for the pieces. On the right, we can see Krishna and Radha playing chaturanga.[22]



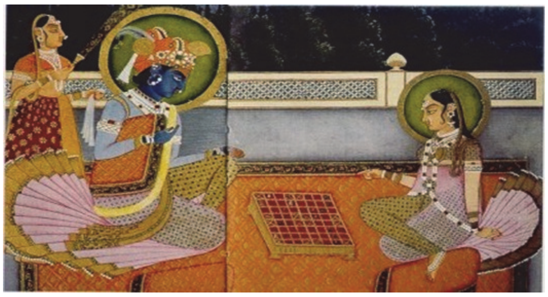


Figure 40 Chaturanga

Chaturanga later on spread to Persia and Arabic countries, where it was called *shatranj*, and later chess (Cazaux 2003). We'll discuss this evolution more closely in Chapter 6.

4.2.2 Chaturaji, the four-person chaturanga

There also exists a four-player version of chaturanga (Figure 42), which has been suggested to be chaturanga's basic form. (Benton & Benton 1977, Bird 2004) The name of the game is chaturaji, which means the four kings (p.299) (Forbes 1860). This theory, however, has not received wide support, but was one of the most important reasons why our research on universal chess began.

The idea that the four-player chaturanga could have been the basic form of chess is based on the Cox-Forbes theory. Hiram Cox claimed, in 1801, that chess is based on four-handed chaturanga of Hindu people, which had been developed as early as 3600 BC. Forbes wrote that the developer of the game was the wife of the legendary King Ravan of Sri Lanka. She might have developed the game to entertain her spouse, when he was besieging the town of Lanka. (Forbes 1860, Cox & Harington 1807)

The game was later changed to a two-player game, because there were not always enough players. Forbes developed the theory further after Cox. However, the Cox-Forbes theory has now been rejected because there is no evidence of four-player chaturanga before the 11th century (Hooper & Whyld 1987).

The game was played as follows. Two cube-shaped dice were used, but without the numbers 5 and 6; the corresponding faces were blank. The starting position is shown in Figure 41. The figure on the left is taken from the web pages of Cazaux[23] (whose book is one of our references) and the figure on the right is Cox's sketch of the game.

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22 By Darkness1089 at English Wikipedia - Transferred from en.wikipedia to Commons by Laurens using CommonsHelper., Public Domain, https://commons.wikimedia.org/w/index.php?curid=7330681

23 http://history.chess.free.fr/chaturanga.htm

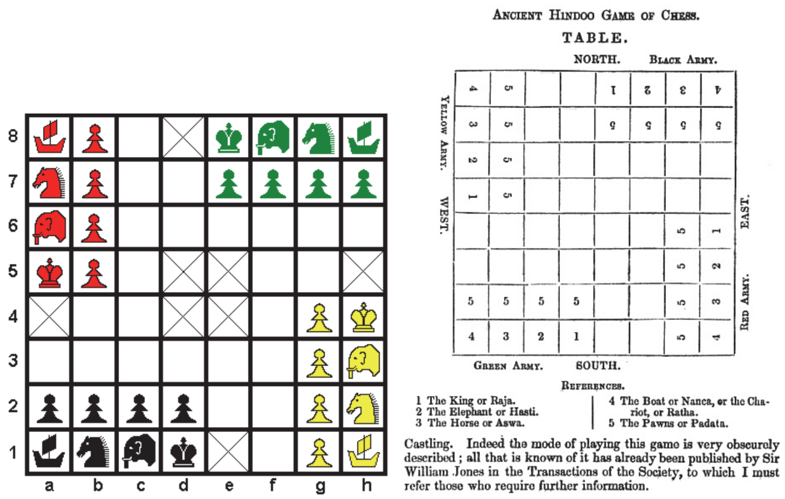

Figure 41 Chaturaji, the four-person chaturanga

The game pieces moved in the same way as in chaturanga, and hence the pawns and knights moved just like in modern chess. Firzan was the queen but moved like the king in modern chess. The elephant, which in chaturanga moved like the bishop in modern chess, was moved in this game like the rook in the modern game. The rook moved in a very special way, diagonally, but only one step at a time and leaping over one square. So the rook was moving exactly in the same way as the queen in chaturanga. On the playing board, the ship among game pieces was the rook. Each player had four pawns. The dice decided which game pieces the players could move. If the number was one, the player was allowed to move a firzan or a pawn, number two was for rooks, number three for knights and number four for elephants. If the dice show one of the blank sides, where ordinary dice have five or six, then the former is one and the latter is four. Firzan (king/queen) can be captured, but it does not need to be removed from the board. The winner will be the player who obtains most points from the captured the pieces. The points given were: for firzan 5, elephants 4, knights 3, rooks 2 and pawns 1. If a player had captured the other three firzans, he got 3x5 = 15 points in case he had lost his own firzan. If that firzan was preserved, he got 54 points. (Sachau 1910)

4.2.3 Byzantine circular chess

Byzantine chess (Figure 42, Cazaux[24]), or circular chess, round chess, was an odd chess variant played on a circular board with 64 squares and 4 citadels (Hooper & Whyld 1987, Josten 2001). The moves were similar to those of shatranj, except that there was no pawn promotion. The Arabs called this Roman chess (Josten 2014, page 88).

24 http://history.chess.free.fr/byzantine.htm

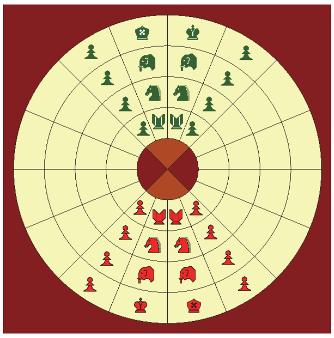

Figure 42 Byzantine chess

Byzantine chess, invented about 1000 years ago, is one of the most interesting variations of the original chess game Shatranj. It was very popular in Byzantium since the 10th century (and possibly created there). Princess Anna Comnena tells that the emperor Alexius Comnenus played 'Zatrikion' - as Byzantine scholars called this game. Zatrikion or Byzantine chess is the first known attempts to play on the circular board instead of a rectangular one. The board is made up of four concentric rings with 16 squares (spaces) per ring giving a total of 64 squares - the same as in a standard 8x8 chessboard. It also contains the same pieces as its parent game – and most of the pieces with almost the same moves. In other words, we can divide the normal chessboard into two halves and make a closed round strip out of it. (Hooper & Whyld 1987, Khalfine & Troyan 2007)

4.2.4 Xiangqi, Chinese chess

A majority of researchers consider that the origins of chess are in India, but a large number of thes believe that chess was developed from xiangqi, a Chinese chess (Bidev 1986, Li 1998).

In Chinese chess, the game board has 8x8 squares just like a traditional chess board, but it is divided into two 4x8 regions between which there is a river. Instead of placing the game pieces on the squares, they are placed on the intersection points of lines, which means that in practice there are 9x10 places for the game pieces (Figure 43, Inductiveload, en.wikipedia Public Domain). In addition, there are two castles of four squares, which are marked with diagonal lines on the game board. The kings are in the castles. (Li 1998)

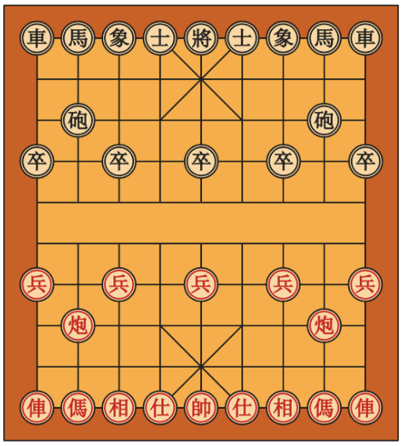

Figure 43 Chinese chess

Next, we will look at the movements of pieces in relation to traditional chess. The king moves like in chess, but it cannot leave his castle. On the board (Figure 43), the king is down and up in the centre. The two *counselors*, who are beside the king, are also inside the castle, from which they cannot leave. Counselors can move only diagonally. There are two bishops on the side of the counselors outside of the castle. The bishop is allowed to move diagonally exactly two steps but may not jump over another game piece and cross the river. There are two knights on the side of the bishops. The knight moves just like a knight in chess but with the difference that it does not jump over another piece. Two rooks are placed at the corners, and they move exactly like rooks in chess. Two cannons are placed on the third line. The cannon moves just like a rook in chess, but it differs from a rook because it captures the opponent's pieces in a peculiar way. The cannon can move only by jumping over another piece, but just over a single one. It can jump also over the player's own game piece. Pawns move on their own territory one step forward and capture an opponent's pieces in the same way. After passing the river, they can also move horizontally. Pawns are not coronated on the opposite row but can continue moving horizontally. They can also capture horizontally. (Li 1998) The other chessmen can pass the river without any exceptional rules.

A game is won by checkmating the opponent's king or by bringing it to a position where it is threatened and cannot stop the threat. In contrast to a traditional chess, the game can also be won by making a draw, which means that the king is not under a direct threat, but – as well as any other piece – cannot be moved. Chinese Chess ends in a real draw if neither of the players can make a checkmate or winner-draw. (Li 1998)

4.2.5 Shogi, the Japanese chess

Shogi ("shogi" = "general game") is very similar to chess, but differs from it in some details. One great difference is the possibility to take your opponent's pieces for your own. Also, the board size is different: shogi is played on a 9x9 board. In the beginning, there were also larger versions of shogi, of which the largest was a 25x25 board game, *Grand shogi* (*shogi and one)*, which had 354 game pieces, with several different types of officers. In Figure 44[25], the shogi board on the left is marked with its original Japanese names for the game pieces and, on the right, with chess symbols. (Botermans et al. 1990, Li 1998)

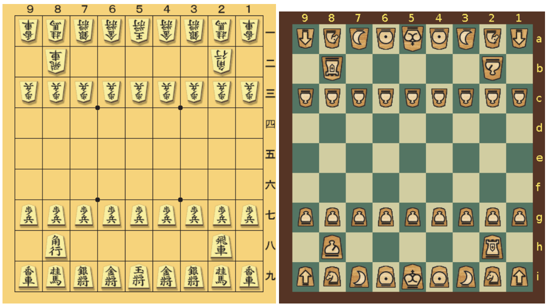

Figure 44 Shogi

Shogi rules can be summarized as follows: Each player has 20 game pieces on the 81-square board. The first three lines are its own territory. Both players have game pieces that are of the same color. The narrow end of the piece points towards the opponent. The reason for this rule is that during the game the player can also take the opponent's game pieces for his own pieces. Kyosha or spearman, moves like a rook, but only straight ahead. Keima or knight moves like a knight in chess, but only forward. *Ginsho or silver general* can move one step forward or diagonally, but not to left, right or backwards. *Kinsho or gold general* can move one square to any direction except diagonally backwards, so it has six directions to move. *Osho or jewel general or king* moves exactly like a king in chess. *Kakugyo or* bishop moves just like a bishop in traditional chess. *Hisha or rook* moves just like a rook in traditional chess. *Fuhyo or soldier* moves like a pawn in traditional chess. The difference from chess is that in shogi the soldier cannot make a double step in the beginning, and it captures directly in the next square forward, not diagonally. (Botermans et al. 1990)

So in shogi there is no queen, and in addition, the knight's and pawn's moving directions are limited as compared with those in chess. On the other hand, there are three other officers in shogi: spearman, silver general and gold general. Another difference is in coronation: in shogi all the other pieces except the king and the gold general can be coronated. The spearman, the knight, silver general and the pawn can be coronated to gold general. The bishop can be coronated to a *dragon horse* (*dragon knight*), which has the same movements as a king and a bishop. The rook can be coronated to the *dragon king,* which can move as a combined rook and king. A game piece will be coronated, when it

25 http://www.chessvariants.com/shogi.html

arrives in the opponent’s territory, which means rows 7-9, moves there and then leaves from there. It is not necessary to coronate a game piece immediately; it can be made at an appropriate time. However, if a game piece reaches the last row of the opponent side, it cannot be moved before coronation. (Botermans et al. 1990)

One of the major differences between shogi and chess is that the player can take the captured opponent’s game piece at any time during the game back on the board as his own piece. The piece returns to where it was on the board when the game started. So any possible coronation is not taken into account. The place where a piece is placed on the board is determined separately, but here we have no need to explain this part of the rules in detail. The game ends in checkmate like traditional chess. The starting position on the board is shown on the right in Figure 44. The king is back in the middle, and next to it are the gold generals, silver generals, knights and spearmen. The bishop is in front of the knight at the left, and the rook in front of the knight at the right. (Botermans et al. 1990, Hooper & Whyld 1987)

4.2.6 The Magic squares

Magic squares are not games, but because of one peculiar piece in chess, the knight, we should also consider the magic squares in this chapter about the origins of chess.

Although chess is supposed to have been modeled after ancient Indian armies and their battles, the game might have had some religious meaning connected with a fertility cult, mainly by numerology and the use of dice (Bidev 1986).

The idea itself does not seem far-fetched with the knight’s move on a chessboard, and the so-called magic squares may have something in common. We are going discuss this briefly in Section 6.1.1, which is about knight’s paths.

Magic squares first appeared in Arabic sources in AD 900. They were figures in a square grid that would add to the same number in four directions. The number was the total of the numerological values of the consonants in a particular Hebrew name, each Hebrew consonant having been assigned a numerical value in cabala. (Roos 2008)

This method of construction was subject to a number of variations. For example, the knight's move in its path may be upwards and to the left instead of to the right, or it may be made downward and either to the right or left, and also to other directions. There are in fact eight different ways in which the knight's move may be started from the center cell in the upper line. Some might prefer another method for locating numbers in their proper cells, which involves the conception of a double cylinder. This method consists of having auxiliary squares around two or more sides of the main square and temporarily writing the numbers in the cells of these auxiliary squares when their regular placing carries them outside the limits of the main square. The temporary location of these numbers in the cells of the auxiliary squares will then indicate into which cells of the main square they must be permanently transferred. (Bouisson 1985)

So the magic square is, in a way, twisted to a cylinder horizontally or/and vertically. After this twisting, the knight's move can continue over the edges of the main magic square. This helps the knight to find its final goal in the original magic square. In Figure 45, there are four magic squares. Each of them has the same sum on every row, column and diagonal. In the magic square on left this sum is 15, and in the magic square in the middle the sum is 111. On the right, there is a magic square of Mercury on ”a chess board” with the middle sum 260. The squares have numbers 1 … n, where n is the number of squares. (Bouisson 1985)

4	9	2
3	5	7
8	1	6

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

10	18	1	14	22
11	24	7	20	3
17	5	13	21	9
23	6	19	2	15
4	12	25	8	16

Figure 45 Four magic squares

The magic squares are generally composed so that their body of figures is numerically balanced. For example, each line will add up to the same total as any other line. If done well, even the diagonal lines will add up correctly. Through connecting their integers in a certain way, graphic representations of that square's master figure are born. In the course of their mastering the creation of theses squares, the ancient cosmologists discovered not merely harmonious numerical arrangements, but interesting patterns and unique figures that sometimes repeated themselves in other magic squares, mystically connecting the collection. In this, each magic square was considered a numerical house or domain from where important references to the mysteries of numerical nature might be divined. Obviously, the magic square science may have been an important stage in the development of understanding fractions, decimals, and square roots. But perhaps more important was their probable influence in the derivation of units of measure. If you were working with the circle or hexagon, the measure of square of six was practical. (Hamilton 2001) The square of six is also called the square of Sun, and one of this kind can be found in Figure 46, where it is the second from left.

These magic squares did have religious meaning in ancient times, at least among Chinese, Arabs and Hebrews. Magic squares reveal, by the sum of their letters, the names of planetary genii. Thus, the simplest, the magic square of 15, or the square of Saturn, gives in each direction a total of 45, the secret number of Saturn. In Hebrew, letters and figures correspond to the numerical value of the letters. Jupiter's square is set out in four columns, each one of which gives a total of 34, and the total sum of 136. Mars has five columns with the sum of 65, and the total of 325. Venus has seven columns, with the sum 175, and the total of 1225. Moon has nine columns, with a base of 369, and the total of 3321, Mercury has eight columns, with a base of 260, and the total of 2080. In a similar way, the magic square of 111, that of the sun, gives us the number of SVRTh, Sorath – the demon of the sun, 666, as that of the Beast in the Apocalypse. The number 111 is also the number of Nakiel, a mystical number whose meaning has been noted especially in Arabic magic. (Bouisson 1985)

The beast of the Sun, Sorath (spelled *Samech*, *Vau*, *Resh*, *Tau* in Hebrew) has the numerical value 60 + 6 + 200 + 400 = 666. Nachiel, with the numerical value 111 was the angel of the Sun. (Roos 2008, Powell 2010)

The magic square of six was called "Squarer of the Sun". From within its body of cells, certain other figures could be extracted, all derived from the father figure of 666. Why then is 666 considered so dark and nefarious? The answer is simply that it is associated with pagan belief systems and cosmology that was adopted and improved by the Pythagorean School - controversial even to this day. (Hamilton 2001)

We may also note that the magic square of Mercury (Figure 45, on the right), was designed on 64 squares' "chessboard", but there is no evidence that the size of the chessboard has anything to do with Mercury.

4.2.7 Speculations about the origins of chess

The dominating view is that chess has its roots in India and the game is based on chaturanga, which was described in Section 4.2.1. Hyde placed the game to India already in 1694, and the focus later moved to northwestern parts of India. However, there are very different opinions among researchers about the origins of the game. This is one reason why we observe here several games related to chess. Among the countries which have been suggested to be the birth place of chess are China, Persia, Afghanistan and Babylon, but there are also numerous other candidates (Mark 2007).

Of these theories about the birthplace of chess, particularly that on China has gained popularity. The argument is based on the theory that a Chinese general Han Xin developed the Chinese chess, Xiangqin, in 204 BC, when two opposite armies were in their winter positions. The same theory claims that another old Chinese game, Liubo (Li 1998), which we already discussed in Section 4.1.4.2, would have formed the basis for chess. Lin's view is based on Irwin's *The Origin of Chess* paper dating back as far as to 1793. Irwin's view, on the other hand, is based on the material of his Chinese friend of Pan Zhen-guan. Nevertheless, no better evidence for Lin's theory exists. This Chinese origin has also been supported by Needham and Bidev. The theories of these three were created during the years 1962 to 2007. (Bidev 1986, Mark 2007)

Arguments in favour of a Chinese influence have been put forward in Needham 1962, and Bidev 1986 and 1987, and a Chinese origin has been argued in Li 1998; a Persian origin has been propounded in Bland 1851 and Yekta in

1970, and an origin somewhere on the Silk Road or in the Kushan Empire has been suggested in Josten 2001. (Mark 2007)

In the life of Timur, by Ibn Arabshah, a more complicated game is described on a larger board, several additional pieces, and of a hundred and ten squares, with fifty-six men, while Chess, in its usual form, had but thirty-two pieces on sixty-four squares. Chess is supposed to have been invented in India, and brought to Persia in the sixth century of our era. To this opinion, the author of our Persian manuscript places himself in direct opposition, maintaining chess, in its perfect and original form, was invented in Persia and taken to India, from whence it returned in its abridged and modern state. Whether the game existed first in a larger or smaller form, of course, affects. (Bland 1851)

Josten supports Linder's theory of 2001, according to which the game would have been born somewhere on the side of the Silk Road or in the ancient Kushan Empire, located in the area of current Pakistan and North-East India. If this was the case, the game would have emerged sometime between the years 50 BC and 200 AD. In Figure 46, there are objects thought to have been chess pieces, found in Kushan. On the left side, there is an elephant. The ancient Indian army consisted of infantry, cavalry, chariots and elephants. Today these have been transformed to pawns, knights, bishops and rooks in current chess. (Mark 2007, Josten 2001)

In Figure 39, we presented game pieces found in Afrasiab, which today is in an Uzbek territory. Also, Afrasiab was earlier a part of the kingdom of Kush.

Figure 46 Possible chess pieces from Kushan (Josten 2001)

Also the four-handed chaturanga, chaturaji (Section 4.2.2), has been suggested to be an early form of chess. This is not very probable as the earliest references to this game are from as late as the 11th century. Also Kraaijeveld, in 2000 in his interesting research, in which he compared chaturanga, chaturaji and Chinese chess from an evolutionary view point, reached the conclusion that chaturaji is not the basic form of chess. He made a list of all various chess-like games played in the world and family trees for them, the kinds of family trees usually made for animals and plants. (Kraaijeveld 2000) The method is referred to as *phylogenetic*. When applying this method for biological species, family trees are made by comparing fossils and the characteristics of currently living organisms. Several variants have been suggested as the ancestor of all the other chess variants. The ancient Indian Chaturanga is usually seen as at least close to the ancestor of chess, but the 4-sided Indian dice-variant, Chaturaji, has also been regarded as ancestral at various points in time. More recently, Li (1998) presented a hypothetical reconstruction of the original form of Xiangqi. Other ideas on the identity of the ancestor of chess have been suggested, but as no full details on the pieces, moves and other rules of these hypothetical ancestors have been given, it is not possible to include them here. Therefore, phylogenetic analyses are performed using Chaturanga, Chaturaji and Li's proto-Xiangqi as the hypothetical ancestors. To illustrate that using the wrong ancestor gives demonstrably wrong results. An analysis was first carried out with a candidate ancestor of which we can be absolutely sure that it is not the correct one: Federation chess[26], a 3-dimensional fictional variant from Star Trek. From the two trees with an Indian ancestor, the one with Chaturanga gives a better match with historical knowledge than the one with Chaturaji. This suggests that the 4-sided dice-game Chaturaji is less likely than Chaturanga to be the ancestor of chess. This is in accordance with historical sources, which suggest that the 4-sided dice-game was an experiment of a much later date than the 2-sided game. The lower probability of Chaturaji to be the ancestor of chess cannot solely be caused by it being a dice-controlled game. Whether play is controlled by dice or not is only one of 109 features used in the analyses. The analyses take the full set of characteristics into account, without focusing on a single one.

Comparison of the Chaturanga-based tree (being the better of the two "out of India" trees) with the proto-Xiangqi-based tree shows that the former is in a better agreement with historical events than the latter. In other words, the phylogenetic analyses performed here suggest that the ancestor of the range of chess variants used here resembles more Chaturanga than Xiangqi. This does not necessarily mean that chess originated in India. (Kraaijeveld 2000)

The relationship between chess and shogi thus points to a common origin, that is, to chaturanga, which was born in India around the 7th century AD. From there, chaturanga moved to the west and the north, going through several transformations on its way. The western branch was called Shatranj in Arabia and eventually referred to as chess in Europe. From the northern branch became Xiangqi in China and Janggi in Korea. Sometimes during the 11th to 13th centuries, chess arrived in Japan, where became Shogi. (Benton & Benton 1977, Cazaux 2003)

4.3 Three viewpoints for the boardgames

In the following three Sections we present three viewpoints and ideas in connection with the evolution process and structure of board games, especially chess. We also present two experimental game models.

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[26] Known better as Tri-Chess

4.3.1 Evolution of chess

The oldest board games, as presented earlier, before the chess, and also before chaturanga, were the Egyptian Senet (3500 BC) and Mehen (2800 BC), as well as the Royal Ur 20-square game in Mesopotamia, Sumer (2600 BC). This Sumerian game can also be found later in Egypt (1800 BC). These three games are clearly the world's oldest known board games. The next age group (about 500 BC -.... 500 AD) includes the ones we have already described in this chapter: Petteia, Latrunculi, Ashtapada, Seega, Tafl, Go and Liubo.

The games above can be divided into two main groups in at least three ways. In some of them, a rectangle-shaped game board is used, while some use a circle-shaped game board. Some games have squares for the game pieces, some have holes. And, finally, some of them are race games and some war one games. When it comes to the places for the game pieces, a hole is a natural choice: you can always make a hole in the ground and put your game piece, e.g. a stone, there. It is the same with a square board, which is easy to draw in the sand. Interestingly, a hexagonal game board seems to have appeared only two or three hundred years ago. The oldest hexagonal game we could find is Agon, presented in Sub-section 4.1. The next games of this kind seem to have been the fairy chess games in the 20th century, which are presented in Chapter 5.

Also, hexagons in tiling seem to appear in mosaic art rather late, at the earliest in the 15th and 16th centuries (Grünbaum & Shephard 1987). This seems quite strange, for there are plenty of hexagons in nature. The best-known example is obviously the honeycomb.

The oldest board games are race games, which differ completely from chess and its predecessors not only by the game idea but the shape of the board. The only common factor is the capture of the opponent's pieces. An interesting connection between chess and race games is Indian Ashtapada, which also resembles a race game, but is played on an 8x8-square chess board. The game board of the predecessor of chess, Chaturanga, had same game board, which was also called ashtapada, based on Sanskrit word for eight. Ashtapada can be connected with the race games by its safe squares and the race trail, which is like the snake in Mehen.

The war games, Petteia, Latrunculi and Tafl in Europe, which are some two thousand years younger than these race games, but older or of the same age as chaturanga, are similar to chess except for the board shape. The idea of the game and how the game pieces move are also similar. In these games, the pieces move like rooks in chess and in Chaturanga. Also in these games the aim is to capture the king by besieging it with other pieces, which is similar to checkmating in chess.

Chaturanga is generally considered to be the common "ancestor" of chess, and therefore the previous games should be compared to it. One connection between Chaturanga and Mehen can be found in the game board. In Mehen, the game board was a disc depicting Mehen, the snake god, in the form of a snake. The turning point was the serpent's head in the middle of the disc. The game board of Chaturanga was Ashtapada, which was used also in the Ashtapada game. In Ashtapada, the race started from the edge of the board and ended at a snake-like circle in the center of the board (see Figure 27). In Indian Ashtapada, the location of the game pieces was given by the squares of the 8x8-square board, whereas in Mehen the game pieces were in holes on the back of the snake. Another similarity between Mehen, Ashtapada and Chaturanga was that of capturing opponent's pieces, which happened if a piece went into the same square or hole as the opponent's piece.

A game of chaturanga end in checkmate just like chess, which meant that the opponent's king has been immobilized. The king was only imprisoned, not captured. A similar system operated in the Vikings' Hneftafl, where the game ended when the king was surrounded on four sides and thus imprisoned. In addition, in Hneftafl warriors moved just like the rook (elephant) in Chaturanga. On the other hand, Hneftafl seems to have been based on the Roman Latrunculi, which probably was based on the Greek game Petteia. And Plato assumed that the origin of Petteia was in Egypt. This opens up an interesting hypothesis about the development chains of these games.

On the basis of these considerations, Chaturanga, and hence also Chess, might be a combination, or a hybrid, of some race and war games. This idea is backed not only by the author of this thesis but the same thoughts have also occurred to other researchers.

According to this idea, Chess would be a hybrid game combining western characters inherited from Greco-Roman or Indian games with some eastern elements which have also led to Xiangqi. Others before me have suggested links with board games such as the Liubo, Polis or Ashtapada. (Cazaux 2003).

4.3.2 Was the Phaistos Disk a game?

In Section 4.1.1 of this thesis, we discussed the ancient Egyptian board game Mehen. We also found an interesting theory concerning an old mystery, the famous Phaistos Disk found in Crete. Figure 47[27] shows both sides of this disc: on the left, the A-side, and on right, the B-side.

A question arises concerning the purpose of the Phaistos Disk: Could it have been a board game? Some researchers have reached this conclusion. At one time, it was thought that the Phaistos Disk contains old unknown writing. The writing code fascinated researchers and they tried to crack it in the way the code of the famous Rosetta Stone had been cracked.

The Rosetta Stone is a fragment of a stela inscribed with a priestly decree in honor of Ptolemy V. The main significance of the text lies not in its content, however, but in the fact that it is written in three scripts: hieroglyphic, demotic, and ancient Greek. Early Orientalists recognized immediately the potential of the Stone for the decipherment of Egyptian hieroglyphs. Thomas Young made great advances, especially with the demotic text, but it was Jean-François

27 By PRA - Own work, CC BY 1.0, https://commons.wikimedia.org/w/index.php?curid=6878117

Champollion who made the final breakthrough in 1822. In so doing, he cracked much more than two Egyptian scripts: he opened up Egyptian culture as a whole to historians. (Parkinson 1999)

The Phaistos Disk has been dated roughly to the period of the Minoan culture. The Minoan civilization, which is the oldest known European high culture, began about 4000 years ago on the island of Crete. This culture came into being and existed before the Indo-European people arrived in Europe and before the cultures of ancient Greece, including Athens and Sparta, had appeared. The Minoan people are today best remembered by the labyrinth in the palace of Knossos and the Minotaur. During that time, three different writing systems were known in Crete: the ancient Cretan hieroglyph system, the Minoan Linear A, and, as the youngest, the Mycenaean Linear B, which was created after the Minoan period. However, the Phaistos Disk does not contain any of these. The linear writing systems were developed from the hieroglyphs and are more advanced. The hieroglyphic figures were replaced by stripe-shaped marks. Cretan linear-to-linear B has been interpreted, but not its linear-to-A(Castleden 2012, Whittaker 2013).

Figure 47 The Phaistos Disk

The Phaistos Disk has always been an attractive object for researchers, and it has been interpreted in countless, from an astronomical map, to merchant, to music notes. That it could have been a game is only one interpretation among these.

The only known object which has similarities to the Phaistos Disk is an Etruscan Magliano Disc (Figure 48). This disc dates back to about 500 BC, and it includes writing, an Etruscan text, which has only been partly interpreted. A small lead plate found at Magliano, probably dating from the 5th century BC, has a strange spiral inscription on each side, running from the exterior margin inwards toward the center. There are about seventy words. The word for "gods", *aiser*, which occurs here and elsewhere, seems to refer to a group or "college" of gods. (De Grummond & Simon 2009)

The Magliano Disc is about 1000 years younger than the Phaistos Disk, but nevertheless it has led researchers to interesting interpretations about the Phaistos Disk and its possible connections to ancient Troy. There are also suspicions that the Phaistos Disk could be a fake.

Jerome Eisenberg claims the following: The Phaistos Disk is a small clay disk stamped with a series of unique 'hieroglyphs' purportedly excavated in July 1908 by Luigi Pernier in the palace of Phaistos on the Island of Crete. It may not ever rank in the public's mind with the Piltdown Man as an object of great renown in the field of man's attempt to fool both the public and countless numbers of scholars. However, its exposure as the most famous fabrication of an ancient script should certainly end the long-standing controversy over its origins and the translation of its intriguing hieroglyphs. On this 100th anniversary of its 'discovery', the writer hopes to bring to light its dubious origin. The interpretations of the script range from scholarly discussions of its relationship to ancient Greek scripts such as Proto-Ionian and, obviously, Minoan, to Anatolian (Hittite and Luwian), as well as often far-fetched links to Basque, Indo-European, Proto-Slavonic, Rhodian, Coptic, Semitic, Proto-Byblic, Tatarish-Turkish, scripts from the Black Sea area (South Caucasian/Georgian, Kartvelian, Colchian, Mingrelian-Laz), and even West Finnish or Old Estonian, Indian, Chinese, and Polynesian. (Eisenberg 2008)

Jerome Eisenberg continues: Luigi Pernier - in addition to his possible wish to compete with the spectacular discoveries of Federico Halbherr at Gortyna and Arthur Evans at Knossos - may have created and planted the disk to excite the sponsors of the excavation to encourage them to supply further backing. Dr Jean Faucounau kindly sent the writer a copy of his book Le Déchiffrement du Disque de Phaistos (1999) which the writer had previously quoted from other authors only. He was gratified to find several excellent sources illustrating at least four of the signs. The shields closely resemble the shields carried by the Sea People on the Kadesh battle reliefs on the walls of the Ramasseum at Thebes. (Eisenberg 2008)

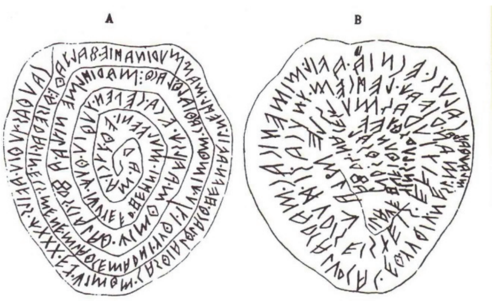

Figure 48 The Magliano disk[28]

[28] Eisenberg 2008

However, these points are not the main focus of this thesis, so we return to the idea of the game. Peter Aleff (2002) has listed several points that interconnect the Phaistos Disk and the race games of Mehen and the Royal Game of Ur. The first and most obvious is, of course, the shape of the disc, the Phaistos Disk being of the same shape as the Mehen game and the circulating loop inside it resembling a snake. There is another similar connection with the Royal Game of Ur and with some other ancient games. This connection is the rosette, a flower with eight leaves, which occurs on the Phaistos Disk. On the Phaistos Disk, there are four rosettes; three on the A-side: at the beginning at the side of the circle, in the middle at the end and just before the center; and one on the B-side, just at the beginning, at the edge. The rosette is as a very old symbol, and it can be found for example on numerous Babylonian wall reliefs. It is considered to symbolize the beginning of life and its end. In Mesopotamia, the goddess Ishtar is the symbol of Ishtar, the planet Venus. In Mesopotamian mythologies, the rosette was considered to be a flower in the Tree of Life in the Biblical garden of Eden. A rosette can also be found in games, for example, in the Royal Game of Ur, which has 20 squares and five rosettes, as well as in its Egyptian variant (Figure 26). A game board inlaid with lapis lazuli, shell and bone, one of several similar ones, was found in the Royal Graves of Ur in Sumer, dating to about 2500 BC. The "Game of 20 Squares" and boards of this type have the same track as in Ur, but one of its ends is unfolded into a straight line. They appeared in Egypt only under the foreign Hyksos rulers, but remained popular there even after these unpopular invaders had been driven out. (Aleff 2002)

It is interesting that the oldest discovered rosette objects dating back almost 30,000 years, were found in a mammoth hunters' grave east of Moscow. The earliest example here, and one of the most beautiful, is an 8-leaf rosette carved on an ivory disk from about 28,000 years ago. It was found in a child's burial place at the Aurignacian site of Sungir in Russia, and its funerary context suggests that it may have been associated with a cluster of rebirth and regeneration ideas. (Aleff 2002)

A rosette can be found also at the centre of the Senet game, where it is a symbol of the 15th square, which is a safety square. Sometimes this rosette is replaced by the ankh symbol, which has the same symbolic meaning – life, planet Venus or sun – as the rosette (Figure 25). The ancient Egyptian ankh-symbol appeared later as the Coptian ankh, crux ansata (Evans 2004). The rosette can also be found in the ancient Egyptian Seega game (Figure 33), where it was also the safety square at the centre of the board.

Aleff's observations might have a point, keeping in mind that there are also other connecting factors which he did not mention. The Minoan civilization in Crete had a strong connection with the Egyptian culture of that time, because the geographical distance was short between them. In addition, the last references to the Egyptian Mehen game during the Middle Kingdom of Egypt (2000 BC - 1400 BC) were found in Cyprus, close to Crete. During the same time (2200 BC – 1300 BC), the Minoan culture flourished in Crete.

Furthermore, the Royal game of Ur, as well as the Egyptian version, had in addition to the rosette also another corresponding figure. The Phaistos Disk has 17 rings, which are called “shields”. Each shield has seven holes: one in the centre and six on the edge. Fifteen of them are on the A-side and two on the B-side. In the Royal Game of Ur, there are seven ring-shaped game pieces. All of them have a point in the center, and four on the edge. Also the game board has five squares with similar points. A similar kind of point pattern can also be found in the other squares, larger in two of them, and smaller in four of them. Altogether, there are 15 patterns of this kind.

So, the Phaistos Disk can be connected with the Royal Game of Ur by two figures (the rosette and the “shield”) on the board, and with Mehen by its serpentine structure. In Mehen, the game starts from the tail of the snake, and the turning point is in the centre, by the head of the snake; in the Royal Game of Ur, both the starting and the ending squares have rosettes; and in the Phaistos Disk there are rosettes in the center and in the edge, where the path starts. But could there be something else, something more obvious that connects it to Mehen and the Royal game of Ur?

Indeed, there may be. A 5000 year-old 20-square game was found in Shahr-i Sokhta in the eastern part of Iran, near the border of Afghanistan. The interesting point is that instead of squares there were twenty loops. These loops were formed by a twisted long snake (Figure 49) (Piperno & Salvatori 1983), similar to the one in the Egyptian Mehen game (Figure 25). This game was actually 200 - 300 years older than the oldest so-far found the Royal 20-square Game of Ur. It has been referred to as "the world's oldest backgammon game" (Jarrige, Didier & Quivron 2011, Schädler & Ulrich 2014). In this game, the snake’s tail is in its mouth; it is just like the legendary world serpent, the *ouroboros* (Sheppard 2013).

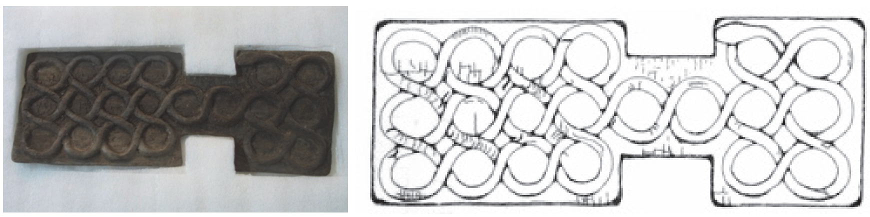

Figure 49 The Royal Ur’s game found in Iran

However, the most interesting thing in this game is that it builds a connection between Mehen and the Royal Game of Ur. In addition, in Mehen the snake’s head is in the middle of the disk-shaped game board, the place where the players turn back. In the game of Shahr-i Sokhta, the serpent's head is also located at the point where, according to Bell was the point of departure (Bell 1979) (Figure 26). If we look at the Sumerian Royal game of Ur, (Figure 26), we will notice that there is a rosette in the same square.

The Phaistos Disk also has a snakelike path to the center, the location of the head of the snake in Mehen. Instead of a snake's head on the A-side of the Phaistos Disk, there is a rosette just like in the Royal Game of Ur in the end square or in the Shahr-i Sokhta version of this game.

One might therefore assume that the Phaistos Disk indeed may have been an ancient race game where the figures are instructions on what the player has to do in different squares. This theory could be strengthened if game pieces of the same shape as the shield figures on the Phaistos Disk could be found in Phaistos. In the Royal Game of Ur, there was a similar shield figure except that number of the points in the shield was five, not seven. In Figure 50, there is a sketch of what the Phaistos Disk might look like as a game. As game pieces, there are circular discs with seven points. On the left (own photo), there is the start position, and in the middle we see an imaginary game situation. The picture on the right is the Mehen game (Robinson 2015).

Figure 50 The Phaistos disk as a game, and Mehen

4.3.3 Fjögratafl, a hypothetical four–person tafl

In this summary, we have been comparing ancient board games in order to understand the evolution of board games and in particular, the evolution of chess. One point of interest has been the four-player chaturaj game and the European hneftafl of a similar age. A similarity between these two games is the game board, divided by a square, the center region and four teams, one on each edge. A four-player Hneftafl game is not known, but we present it here in the last part of this chapter as a hypothetical game. This is because the idea of tri-chess, is based on this logic. We call this new game in Icelandic Fjögratafl (Figure 52).

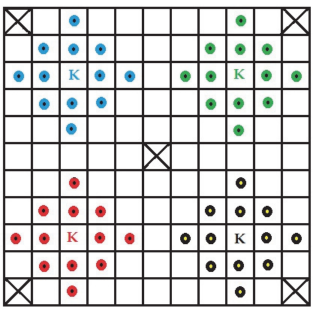

Figure 51 Fjögratafl

The rules are as follows. The game pieces, warriors and kings, move just as in Hneftafl – like rooks in traditional chess. In this game, each team has a king, which can be captured in the same way as in Hneftafl. Also the warriors can be captured in the same way as in Hneftafl. If the king dies, then his warriors become disabled and cannot move. They can be removed from the board by capturing them. Also, if all the warriors of one king die, then the king will be disabled. The winner is either the team which has the last remaining king or a king who succeeds to escape to the safety castle in centre of the board. The king can also move to the safety castle in the corner, but when there he cannot move his warriors.

4.4 Summary

As we saw in the previous chapters, there are several different opinions about the origins of chess. The dominant theory suggests that India is the home of chess, followed by the claim that China is the birth place of chess. These theories are based on various ancient texts, archaeological discoveries and logical reasoning concerning different rules, game board sizes and the development of the game in different periods. In this fourth chapter, we also present some other old board games in order to estimate the evolution of chess when compared to other board games.

This thesis is not attempting to provide the right answer to the origins of chess, although we did present several viewpoints. The motive for these observations is an attempt to understand this development and evolution, as the expansions of chess, presented in this thesis, are based on the basic structures of chess.

5 LATER CHESS VARIATIONS

An enormous number of chess variants have been developed through the ages. Today, knowledge of these is rapidly increasing because of the internet. In this chapter we examine some old variants which were invented before the 20th century. We also take a look at some of the best-known modern variants, focusing on variations with more than two players, because they are relevant to the contents of this thesis. Here we list only a few games, although during this research we found almost 150 chess variants in the literature.

It has been interesting to find out that in all older variants, the game board consisted of squares. It seems that hexagons and triangles came to be used more widely in the 1970s and 1980s at the earliest. The earliest hexagonal chess game we found was published in 1912 (Jelliss 1992). An older, and perhaps the oldest, hexagonal board game could be *Hexagonia*, which we will briefly explain in this chapter. *In Section 5.3, we will consider the similarity and origins of some modern three-handed chess variations. This discussion also forms a small sub-result and a part of the research we carried out when investigating the background for this thesis.*

5.1 Ancient N-handed chess games

When we (Kyppö, based on research with Frank Harary) developed a three-handed chess game in 1993, it seemed unique. During the development, we found in the literature a similar type of game, which had been developed in 1912. It was a smaller three-handed chess game (Gik & Määttänen 1988). After releasing our game in 1995, we found a couple of other games of that kind. The World Wide Web began to grow strongly at the same time, and because of it, an increasing number of chess variants became more widely known. For this thesis, it was reasonable to study the history of chess variants more broadly. It is amazing how far back in time, multi-player variations of chess were created. The oldest record we found for this thesis is dated back to 1762**.** However, the Three Kingdoms game that we present in the following Section (5.1.1), is clearly even older, although the first mention of it dates back just to 1876 (Deutsche Gesellschaft für Natur- & Völkerkunde Ostasiens 1876). The first reference to the game was made at least by 12th century (Li 1998).

5.1.1 Three Kingdoms and Three Friends -games

Chinese Chess, San Guo-qi, means Three Kingdoms (Figure 52, right)[29]. This game was created in China, probably in the beginning of the second millennium.

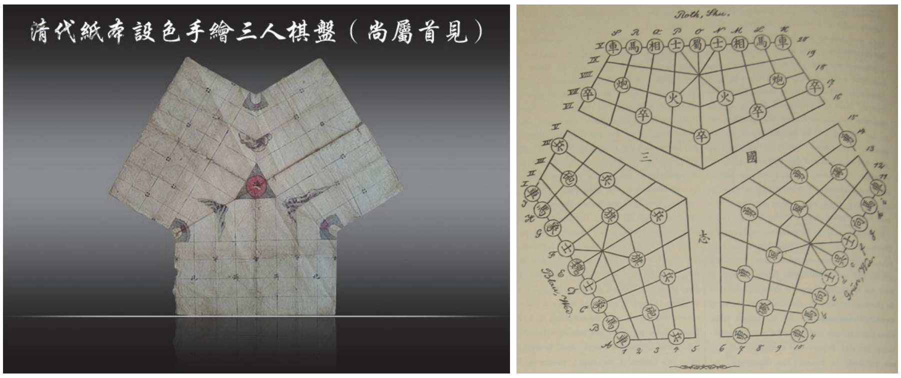

Figure 52 Three Friends and Three Kingdoms games

Information about this game can be found in Dr. O. von Möllendorf's article "Schachspiel der Chinesen", dating back to 1876 (Deutsche Gesellschaft für Natur- & Völkerkunde Ostasiens 1876). According to this source, the idea of the game is based on 3rd century China, when a war was waged between three kingdoms. After the fall of the Han dynasty, three kingdoms, Wei, Shu and Wu, were fighting against each other for about 40 years, over the control of China, and these wars gave the name of this game. Möllendorf's article appeared in the 19th century, but the game was already mentioned in Yao Kung-wu's work *Notes on Books Read in the Jun Study* in 1151. (Li 1998) The Three Kingdoms game is a clear variant of Chinese chess, Xiangqi. The game pieces are similar, and also in this game the board is divided by a river, though in this case it branches into three. The game also has another variation, the *Three Friends game, San-you-qi*, which was invented by Zheng Jinde between the 16th and 17th centuries. (Figure 52, left)[30].

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29 Deutsche Gesellschaft für Natur- & Völkerkunde Ostasiens 1876

30 Cazaux, http://history.chess.free.fr/sanguoqi.htm

5.1.2 Seven Armies game

This game is also a variation of Chinese chess: the game pieces and rules are partly the same, but there are seven players, as the name implies, and the game is played on a 19x19 Weiqi (Go) board (Figure 53). The developer of this game was Sima Guang, at the beginning of the second millennium. The Seven Armies game was inspired by fights between warlords around 400 - 200 BC. An interesting detail is that the queen and the bishop already at that time moved like they do in modern chess. (Lo & Wang 2004) The game was probably developed between 1071 and 1085 (Li 1998).

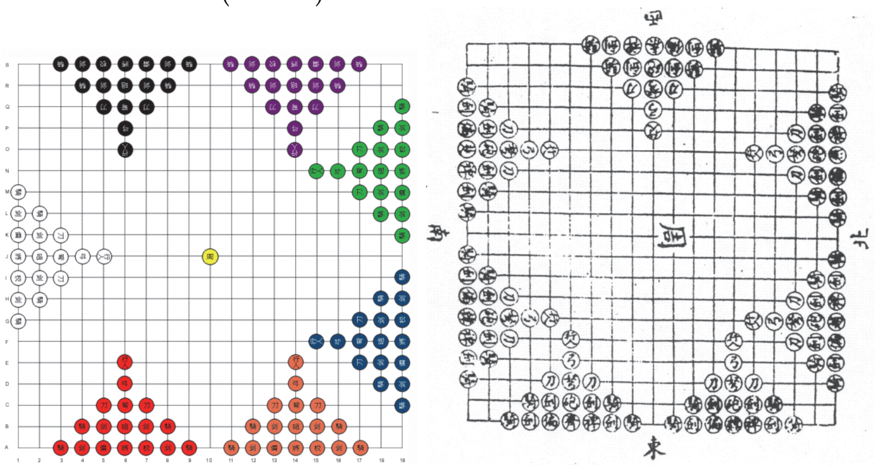

Figure 53 Seven Armies[31]

5.1.3 Four seasons chess

During Alfonso X Wise's regime in 13th century Spain, a four-handed chess game known by the name of Four Seasons was played. This game had some similarities with Chaturaji, the four-handed Chaturanga, but did differ from it by the positions of game pieces and by some of the rules. In Chaturaji (Figure 42), the officers were placed on queues along the sides, starting from the corners, and the soldiers next to them. In Four Season Chess (Figure 54)[32] the officers were placed at the corners in a quadrilateral formation and the soldiers around them. This formation resembles the so-called Spanish Square, better known as Tercio, used by Spanish Army in the 15th and 16th centuries. Tercio was created in 1493, when the company was divided into squadrons led by sergeants. One of the chief promoters of this reform was "the Great Captain", Gonzalo Fernández de Córdoba. (López & López 2012)

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31 Cazaux, http://history.chess.free.fr/qiguoxiangxi.htm
(Li 1998)

32 http://www.chessvariants.com/historic.dir/4seiz.html

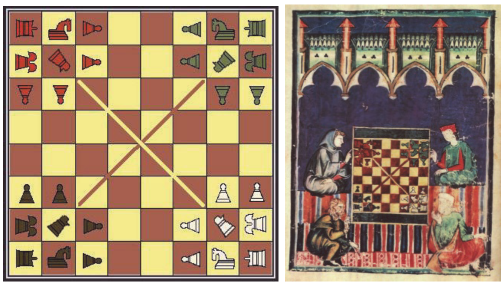

Figure 54 The Four Seasons Chess

In this game, Kings, Rooks, and Knights move as in traditional chess, and Pawns capture diagonally as in traditional chess. Bishops move as in Chaturanga, diagonally, jumping over one square. When a Pawn reaches the opposite edge, it will be coronated as a Queen (the General), which moves like the Queen in Chaturanga, diagonally one square. It is not difficult to understand why the Bishop and the Queen move in such a peculiar way, as in Chaturanga and Shatranj, because the game is quite an old variation of chess and, hence, might have been developed from these two old chess games. The King can become checkmated just as in usual chess. In that case the King was removed from the board and the player who made the checkmate earned the King's army. It was also possible to use one or two dice in the game and play it as a money game, where the profits were calculated by the captured game pieces. The pieces were rated according to their value, so that the King had the value of 6 points, the Queen 5, the Rook, 4, the Knight 3, the Bishop 2 and, the Pawns 4, one point each. The game pieces were always green, red, black and white, reflecting the four seasons, spring, summer, autumn and winter, but also the four elements, air, fire, earth and water. The player usually attacked the opponent on the right hand side, but was also allowed to attack the left side. This imitated the movement of the sun and was, together with the colors of the game pieces, part of the mystical symbolism of games in that time. (Burckhardt 1969, Golladay 2007, Verney 1885)

5.2 Variations from the 18th and 19th century

This Section provides an overview of chess variations between 100 and 300 years ago. Numerous chess variations were developed during this period, but we focus only on variations that are either multi-player games or games played on a hexagonal board, or both. The games presented in this Section are: Marinelli's three-handed chess, four-handed chess games from the 18th and 19th centuries, the Gala-game from 19th century Germany, Wellisch's three-handed chess, Glinsky's hexagonal chess and a game called Hexagonia.

5.2.1 Marinelli's three-handed chess

The earliest known European three-handed chess is Philip Marinelli's game (Figure 55)[33] from 1722 (Verney 1885).

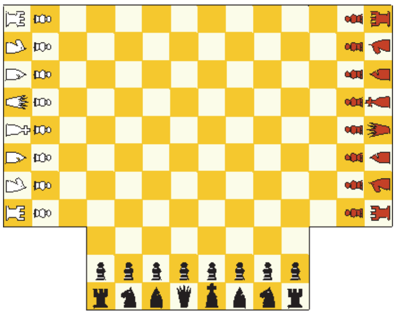

Figure 55 Marinellis's three-handed chess

Marinelli's game is extended in a rather simple way by adding 3x8 "wings" on three sides of a standard chess board. The same pieces as a player has in normal chess are placed on three sides. Each player has a queen located on the left side of the king. The game is not symmetrical, because the position of Black is different from that of White or Red. In addition, the black pawns have a shorter distance to coronation, which takes place, as in traditional chess, on the opposite side. The winner is the player who has checkmated both opponent's Kings. Once the first player has been checkmated, the game pieces remain immobilized on the board. However, all of them except the king can be captured. Marinelli's three-handed chess was dedicated to the Savoy's Prince Eugene, who was inspired by the game and encouraged Marinelli to make it known around the world (Marinelli 1826). It is interesting that even though Marinelli's game was created about 300 years ago new kinds of three-handed chess games are still being created. (Rewega 1992, Verney 1885)

5.2.2 Four-person cross-shaped chess games from Europe

Several cross-shaped, four-player chess games appeared in Europe during the 18th and 19th centuries. The oldest of these, which dates back to 1784, was created by "K.E.G." in Dassau, and another can be dated back to 1792 Altenburg in Germany. We can call these games as *Dassau's game* and *Altenburg's game*. The oldest game for which the rules and the game itself are documented was Captain George Hope's *Verney's game* dating back to 1881. (Verney 1885)

These games were visually very similar in appearance, as shown in Figure 56.

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33 http://www.chessvariants.com/historic.dir/marinelli.html

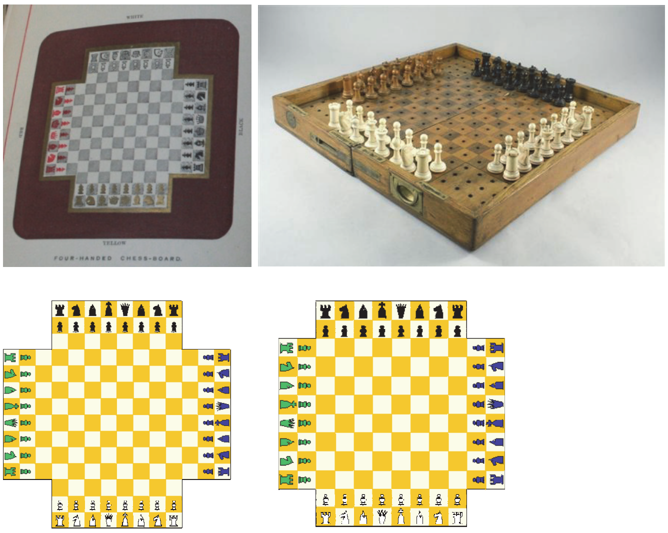

Figure 56 Four-person cross-shaped games

Figure 56[34] (Verney 1885) shows the Verney's chess game at the top; at the bottom on the left, there is the Altenburg's game; and on the right we can see the Dassau's chess game. As we can see, the starting position and the board of the Altenburg's game are identical to that of Verney's. There is one row less at the edges in Dassau's game.

In Verney's game (Verney 1885), not all play against each other: the players on the opposite sides of the board are allies. Their aim is to checkmate the other two players. The checkmate has to be done against both of the opponents. Yellow and White therefore play against Black and Red. Each player has the same number and same kind of game pieces as in traditional chess. The moves are made clockwise. There are some peculiar rules in this game. The two allied players are not allowed to communicate with each other, and their kings can be placed on adjacent squares because they do not threaten each other. Because the persons involved in the game have uneven skill levels, there is a special rule by which the players whose turn comes second can change their allies in the beginning of the game. A player is not allowed to make a move that would put his partner's king in checkmate. A special situation arises when pawns of the allies are in front of each other. In such a case, pawns are allowed to jump over their ally's pawns. Castling is not allowed. (Verney 1885)

Altenburg's game is quite similar: the opposing players are allies in it as well. Once one of the kings has been checkmated, no game piece of this player is allowed to be moved any longer. In addition, this game has a large number of special rules. However, in Altenburg's chess, castling is allowed. In this game

34 http://www.chessvariants.com/multiplayer.dir/4players.html
http://www.thisnext.com/item/9A7C52FA/B9566C41/Victorian-4-Player-Chess-Set

he strictest silence as to the game must be preserved during the play. (Verney 1885)

5.2.3 Gala-game

One of the historical chess games is the 19th century Gala game from Germany. Gala was once quite popular in Germany, especially in Schleswig-Holstein. The game board had 10x10 squares, and the game pieces were placed in the corners of the board as shown in Figure 57[35]. Although the game seems to have been made for four players, it was in fact a two-player game. Both of the players had two groups of game pieces, including two kings. Therefore the pieces had only two colors. In this game, there were no queens and knights. The yellow squares were on the higher level on the game board. The reason for this was that the pieces that moved to yellow squares changed their direction. The aim of the game was to capture the opponent's kings. (Botermans et al. 1990)

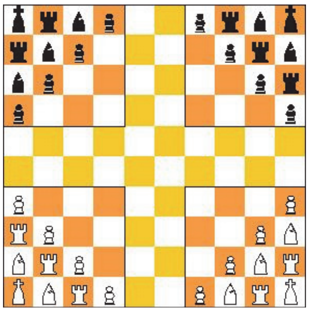

Figure 57 Gala-game

The movement of the pieces differed quite a lot from those in the traditional chess. The pawns moved diagonally on their own territories, the 4x4 orange regions in the picture. If a pawn went out from its territory on the region of yellow squares, it had to stop first on the first square. After that it was able to move to each neighboring square, exactly like the king in chess. If a pawn returned back to its original territory, it could move only diagonally as in the beginning. The rook moved on the orange-colored regions just like the rook does in traditional chess and on the yellow-colored region just like the bishop in traditional chess. The movements of bishops are mirror images of rooks' moves. On the orange region, bishops move just like they do in chess, and on the yellow region, bishops move like rooks in traditional chess. The king had also one peculiar feature compared with traditional chess. It could move all over the board like the king does in traditional chess, but if it got to one of the four centre squares, it could be placed in any free square, except the starting square! When a player threatens the king, he says, "gala". If the king cannot move to a safe square, it is captured. The player who has captured all the opponent's

[35] http://www.chessvariants.com/historic.dir/gala.html

kings wins the game. The game ends in draw if each of the players has only one king left. (Botermans et al. 1990)

5.2.4 N-person chess game variations in 19th century Europe

In 19th century Europe, many really peculiar chess variants can be found. One of these is G.R. Neumann's four-hand chess dating back to 1867. Also in this game, White and Yellow play together against Black and Red, but the shape of the game board is different: 8x16 squares. Each team is placed side by side on their own side of the board (Figure 58).

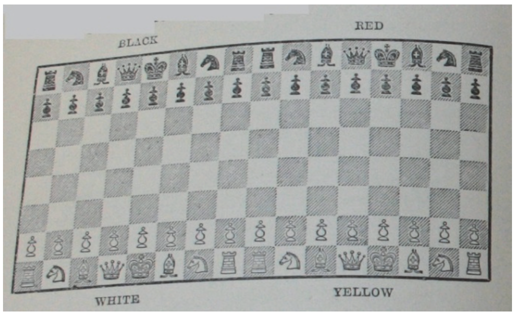

Figure 58 A Four-person chess

The team that is the first to checkmate the opponent team's both kings is the winner. When one of the kings has been checkmated, the other pieces become "dead", but will remain on the board. However, the other player of the team can "bring them back to life" by releasing the king from checkmate. Also in this game, the kings who are on the same side may be on adjacent squares because they do not threaten each other.

There is another classic four-handed 19th century chess game we should mention: Russian chess. Russian chess has an interesting "fortress" solution: in every corner, there is a fortress, a bit outside of the main board (Figure 59). We find this solution fascinating, because the concept of universal chess, to be discussed in the next chapter, originally had its origins in the fortress idea.

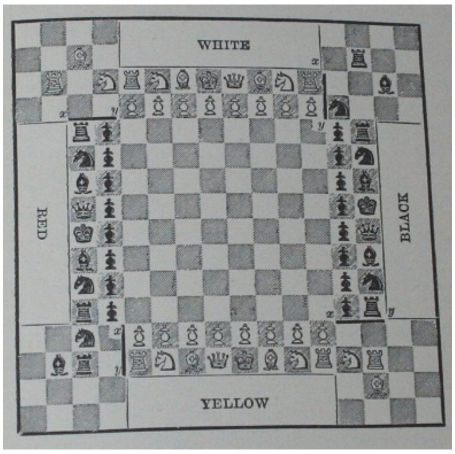

Figure 59 A Russian four-person castle-chess

Russian chess might be even older, but the earliest document about it can be found in a chess magazine that appeared in Berlin in the year 1850. Even this game is not truly for four persons, because the opposing players play together as allies. White and Yellow are partners and play against Black and Red. The game's specialty is the fortress system. Each player has, in his fortress reserve, an extra Bishop, Knight and Rook, which pieces, at the commencement of the game, the player can place in any position in the fortress. The rules are quite similar to those presented earlier, but we should pay attention to an example of an initial situation shown in Figure 59. Because the officers in the fortresses are not arranged in the same way, this affects the game. For example, Black can move the Bishop, which is in reserve, only after moving one of the pawns. It takes longer for the other players to move their Rook and Bishop. (Verney 1885)

Next we present two bigger chess games, which do not differ that much from the four-player games. Figure 60 depicts Max Lang's six-player chess dating back to 1881. It has exactly the same kinds of rules as those in Neumann's six-player chess. However, there are only four differently colored pieces in this game, for the players on the edge have pieces with the same color. This, however, should not cause confusion, according to the documentation (Verney 1885).

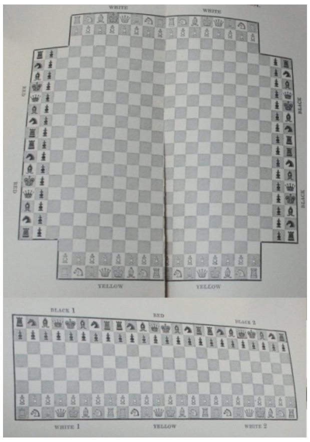

Figure 60 Eight- and six-handed chess games in 19th century

Of the large older games we examine now Verney's eight-handed chess (Figure 60), which is actually only an extension of his four-handed chess and has exactly the same rules. A similarity with the six-handed chess is with the colors of the pieces: players on the same side have pieces of the same color. Thus, there are only four different colors in this game. (Verney 1885)

5.2.5 A Hexagonal three person chess by Siegmund Wellisch

In 1912, Austrian engineer Siegmund Wellisch created a three-handed chess with a board of 91 hexagons (Figure 61). This was the earliest three-handed chess played on hexagons in the Variant Chess 1992 issue (Jelliss 1992). In this game there are no Bishops, and Knights move diagonally stepwise, in same way as Bishops in Glinski's chess (next chapter) and in tri-chess (Chapter 6). Due to this, the game has three knights. (Hooper & Whyld 1987)

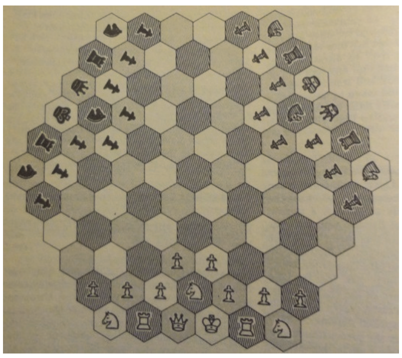

Figure 61 Three-person hexagon chess by Wellisch

5.2.6 Glinski's hexagonal chess

The most famous of these kinds of chess games is *Polish chess* or *Glinski's hexagonal chess,* developed by Wladyslaw Glinski in the 1950's (Figure 62). This game is played on a 91-hexagon board. However, the game is not three-handed; it is intended only for two players. The movements of the pieces correspond to the moves in tri-chess, except for the pawn, as discussed in Chapter 6. The game is also known as *hexagonal chess*, and since the 1980's it has had its World Championships. (Gik & Määttänen 1988, Hooper & Whyld 1987)

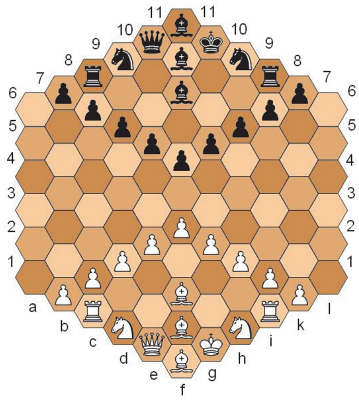

Figure 62 Glinski's hexagonal chess (Chess, Ramesh & Anand)

5.2.7 Hexagonia-game

Perhaps the earliest game of chess with a hexagonal board, Hexagonia was created by 1862 (Figure 64)[36]. However, the rules of the game related to the ending of the game differed from chess. In this game, there was no checkmate: the winner was the player who first got his King to the golden safety hexagon in the centre of the board. This kind of ending in Hexagonia has an interesting similarity with the Tafl games

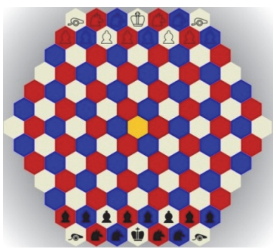

Figure 63 Hexagonia

36 https://boardgamegeek.com/image/1931516/hexagonia

The game board of Hexagonia consisted of 127 hexagons, which were blue, red and white, except for the middle safety box, which had the color of gold. Each player had eight pawns, four knights, two cannons and a king. No definite information has been preserved about the moves of the game pieces. The game was developed by British John Jacques along with several other games. (Routledge 1866)

5.3 Modern variations

In recent years, three-handed chess has spread more widely throughout Europe. The background of this game is quite interesting, since the same game appears to bear the hallmark of several different developers. It was difficult to find the original inventors of these games. However, we got to the conclusion, that the first of these variations were invented by Robert Zubrin, who was working in NASA. Probably he didn't know about some quite similar games (Figure 65) [37] (Verney 1885) created already in the 19th century. Figure 64[38](Zubrin 1972) presents four variants, the oldest of which goes back to 1972. The inventor of this game was Robert Zubrin (Zubrin 1972). On the left in the figure is Zubrin's game and to the right are games designed by Patton, Rasmussin and Langronier, respectively. These three are from the end of the 1990's and after 2000. In recent years, the Patton's game has spread widely, although the developer's name is not always clear in its marketing. All four games have clearly the same structure. All three players have the same number of pawns and officers as in ordinary chess. The boards are assembled so that a chess board has been cut in the middle and then three of these kinds of "half chess boards" are placed together and connected to the center by bending and stretching the squares. Each board has thus 3x32 = 96 squares and in the centre a hexagram-shaped "star".

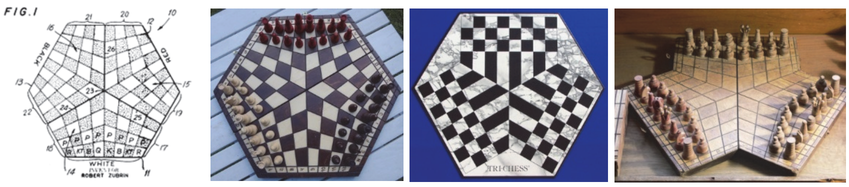

Figure 64 Three-person chess games, with 96 squares

These games are modern variations and do not differ greatly from some three-player chess games of the 19th century (Verney 1885). Two examples are

[37] http://www.chessvariants.com/historic.dir/self.html
[38] https://boardgamegeek.com/image/395516/tri-chess
By Slip - 3 players chessboard, CC BY 2.0,
https://commons.wikimedia.org/w/index.php?curid=4433248

Waider's and Self's games (Figure 65). Waider's game dates back to 1837 and Self's chess was created in 1894. These two games have a fairly similar board, differing only in the centre. However, in these games three ordinary chessboard halves have been combined, just like in the previous four games. At the time, Waider also presented a four-handed chess, which was quite similar to previous ones.

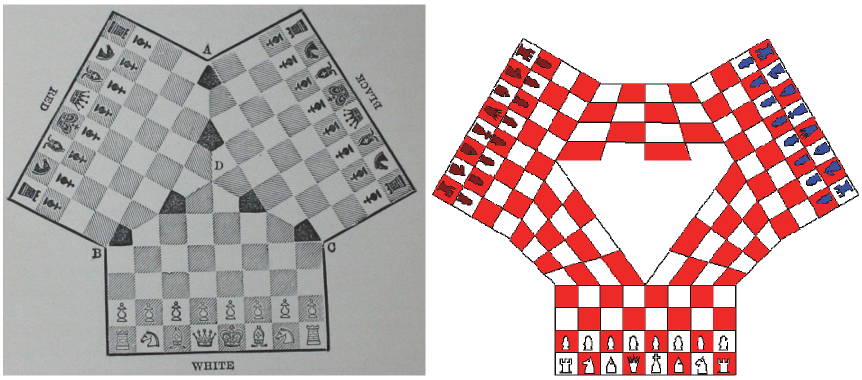

Figure 65 Waider's and Self's three-person chess games[39]

5.4 Summary

This chapter considers some of the numerous chess variations and chess-like games from different eras up to the present day. The modifications shown here have been based mainly on the form of the game board and the number of players. This gives some perspective on how many generally unknown innovations have been made in this field and also how difficult it is to develop completely new game ideas or new ideas at all. Most of the things have been invented and forgotten far in the past.

The variants in this chapter give a background for the idea of universal chess, which we will present in the following three chapters.

39 http://www.chessvariants.com/historic.dir/self.html
(Verney 1885)

6 BASIC MODEL OF UNIVERSAL CHESS

The basic principles described earlier can be generalized to ″universal chess″ rules. These rules remain mostly the same despite the number of officers and pawns in the game and the size of the game board. For larger board types, we create new officer types to cover wider defense zones. A game board can also be transformed into multiple dimensions. Also, if we want to increase the number of players while preserving equality, then higher dimensions are necessary. Finding a suitable multidimensional geometric structure is a problem, because in higher dimensions regular tiling can be done only by hypercubes. Due to the limited ability of humans to ″give form″ to higher-dimensional objects, the three- or four-dimensional versions of chess are mainly used to simulate game-theoretical models. The Knight is a chess piece the moves of which seem to differ the most from the moves of the other chess pieces. Attempts to understand this led to the invention of three-person hexagonal chess and later to the rules of the universal chess. The Knight has always been considered to be the most peculiar chess piece, when it comes to the movements.

As discussed in the introductory section, the starting point of this thesis is three-handed chess, which is greatly influenced by the Knight's odd movements on the chess board. From these studies was born our own *theory of prime chess* (), which may never be proven correct or incorrect. However, this theory allows one to decompose chess into its “prime elements”, and to rebuild it again from these elements. From these elements, we built two distinct games: a standard, 64-square, two-player board game (chess) and an 87-hexagon three players board game. How this happened is explained in this chapter. To differentiate between the two-handed and the three-handed chess, we developed, on the basis of the universal chess idea, our game, *Trichess*, and, as a general term for it, we use *three-handed chess*.

6.1 A theory about the origins of chess

The strangeness of the Knight's move and its possible numerical symbolism have been two of the main reasons for the hypothesis connecting the origins and structure of chess with secret magical and religious rituals of ancient India. Several theories have been proposed during the years about the birth of chess and its structure.

Suppose that we want to create a game where kingdoms or kings are struggling against each other. Each player has a fortress of his own, and outside of this fortress there is an area where players can attack. Let's think of a situation where we have an open field for the game pieces to move. If we want to determine the places of the pieces on this field, then we must divide the field into regular parts. The two-*dimensional* plane can be divided, as told in Chapter 3, to identical and similarly-sized regions exactly in three forms: triangles, squares and hexagons.

In this chapter, we examine first the classic Knight's Path problem and Knight's Tour problem, because this problem was one of the factors which led to the idea of universal chess. In the background, was an attempt to solve, together with professor Frank Harary, a generalization of the Knight's Tour problem. After this, we discuss the birth process of the trichess board, as well as the defense zones that led to the determination of the officers in the game.

6.1.1 The Knight's tour, and background of the research

The Knight differs from the other chess pieces in modern chess because of its "right-angled" move. Often the explanation for this has been connected with the way a horse or horseman moves around on a real battlefield.

The Knight is the piece the movement of which probably surprises everyone who becomes familiar with this game. In the hot climate of India, the rider's weapon was a spear or a mace by Kainulainen (1984). Hence Knight's moves on the board resembled jumping. But on the other hand it has been thought that the Knight's move might have been taken from some other games, perhaps the old magical temple games. In the so-called magic number squares, there can be found routes which could be interpreted as the Knight's moves. (Kainulainen 1984)

The problem of the Knight's Tour, on a traditional 8x8 chess board was solved (Figure 66, on bottom)[40] already by Euler in 1759 (Biggs, Lloyd & Wilson, 1986). The problem can be divided into two problems: the Knight's Tour and the Knight's Path, also called as the Knight's open Tour. In the first one, the Tour, you always finish where you started. In the second one, the Path, it is enough just to go through all the squares. After Euler's proof, numerous publications, which calculated a variety of ways to cover a chess board by knight's

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40 http://www.renneslechateau.it/index.php?sezione=studi&id=greatparchment, © 2016 Mariano Tomatis Antoniono

tours, were written. The Knight's Tour and the Knight's Path are special cases of Hamiltonian cycles and Hamiltonian paths in the graph theory, which in turn is a special case of the Traveling Salesman Problem, which has many practical applications. In the Hamiltonian Path Problem, we investigate whether all the vertices in a graph could be visited. To be more exact, a Hamiltonian path, or spanning path, in a digraph D, is a path that includes all the vertices of digraph D. A Hamiltonian cycle, or spanning cycle, in a digraph D is a cycle that includes all the vertices of D. (Gross et al. 2004)

In the Travelling Salesman Problem, we are looking for the minimum length of a spanning walk in an edge-weighted graph. Also in the Travelling Salesman Problem all the vertices must be visited, but it is not forbidden, unlike in the Knight's Tour, to visit a single node more than once The Knight's Tour (as well as the Knight's Path) problem can be generalized by changing the shape of the chessboard. Schwenk proved in 1991 that on an m x n chess board, that consist of squares where $n \geq m \geq 5$, a Knight's Tour can be found if m or n is an even number. If m <5, a Knight Tour is found only if m = 3 and $n \geq 10$ and it is even (Schwenk 1991).

This result of Schwenk was used as the basis when we began to look for a generalization. Frank Harary wrote: "I shall e-write to Allen Schwenk and ask him to airmail to you his pretty paper on Knights move Hamiltonian cycles for all even b>=6. Then we will have a better idea where we can send our article." (Frank Harary, Appendix 1)

The aim was to prove the theorem, according to which a Knight's Path always exists on m x n square tiled boards iff $m \geq n \geq 5$. The work that began in 1993 remained unfinished and unpublished because another proof was made about the same theorem in 1994. However, this process finally led to the invention of Trichess, which we are going to discuss in Section 6.2. Therefore, we take also a brief look at the method of proof we used.

A Knight's Tour, abbreviated as KnT, as well as a Knight's Path abbreviated as KnP, can be represented as a Knight's Graph. In Figure 66 there is a Knight's Graph in which we can search for KnT_4 or KnP_4, when m = n = 4. On the top left side of the graph in Figure 66, there is a 4x4 game board, where a Knight's Path covers only 15 of the 16 squares. There is a reason for this. On the 4x4 gameboard, which consists of squares, it is not possible to find a Knight's Tour, which is easy to verify if we look at the graph. The graph can be divided into four sub-graphs of four vertices each as shown in the middle figure. There are two types of these sub-graphs. In one of them, the degrees of the vertices are 2 and 4, and, in the other, all the vertices have degree 3. This means that, in the sub-graphs where there are two vertices of degree 2, both of these vertices must be visited before leaving the sub-graph. This is not possible because at least to one of these two sub-graphs we must go via a vertex of degree 4. And from a vertex of degree 4 it is possible to go only to a vertex of degree 2. After that it is no longer possible to move to another vertex of degree 2 in a same sub-graph because the route goes via a vertex of degree 4.

Figure 66 on the top right shows a Knight's Path KnP_5 on a 5x5 chessboard by numbers 1 to 25. Starting from a corner of the game board, the Knight moves along a two squares wide strip and then spirally towards the center. It is easy to find out the route by following the numbers on the board. So, on this board there is a Knight's Path.

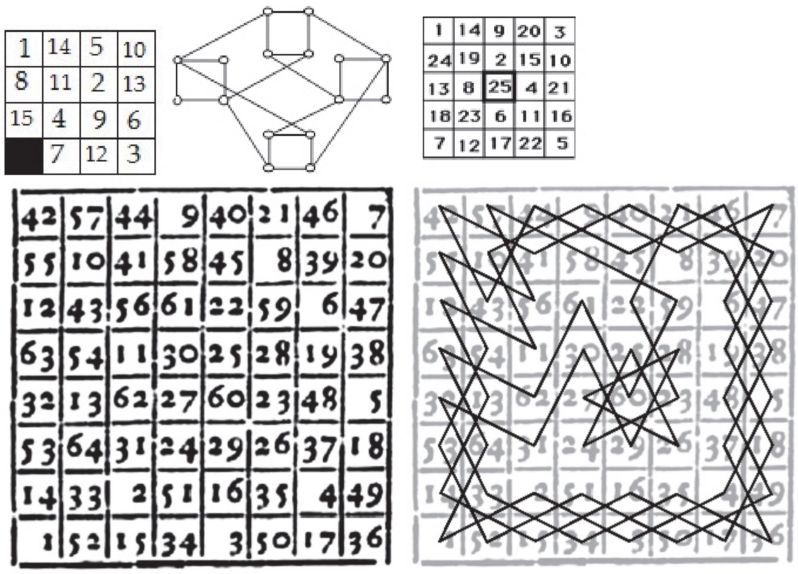

Figure 66 Knight's Graph on 4x4 board, Knight's Path on 5x5 board and The first Knight's Tour suggested by Euler

The complexity of finding a Knight's Tour and Path increases exponentially as we increase the board size, as can be seen from figure 67, which shows the graph KnP_6 of a 6 x 6 game board (Watkins 2004). At the same time, we may note, however, that the basic structure of the graph remains as the same. In each corner of the board, an interesting sub-graph is created. It is useful in trying to find a Hamiltonian path. These sub-graphs and their structure are the key to a more general proof if we continue with the kind of reasoning we used previously with KnT_4. It is this structure of a graph that led us to generalize the same problem on tilings other than rectangles and squares.

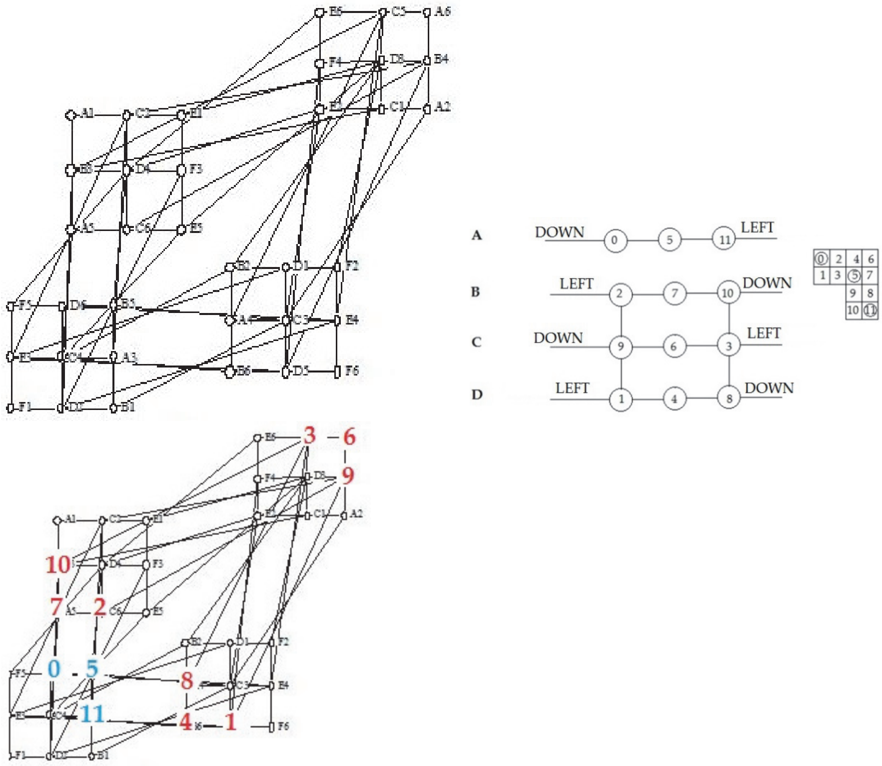

Figure 67 KnP_6, Knight’s Path on 6x6 board

Figure 67 on the left depicts the Knight’s movements as a graph on a 6x6 square chessboard. On the right of the figure, there is a detailed picture of one of the corners of the board and the relevant sub-graph. On the graph at the bottom, it is possible to find by numbers how the vertices of this corner are situated in the main graph.

In order to determine the structure of the graph when the board size increases, a new idea was born. We began to generalize the Knight’s Tour and Path also on game boards with other kinds of tilings. At first, we applied this to a board which consisted of hexagons, where we determined the knight’s movements. Even though this generalization did not make any progress with the proof, in about two weeks it led to the creation of a game we refer to as Trichess. Figure 68 shows an example of a Knight’s Path on a small game board of 27 hexagons. How we got to these kinds of Knight’s moves will be explained in Sections 6.1.2 and 6.1.3 that follow. Next we explain the details of Figure 68.

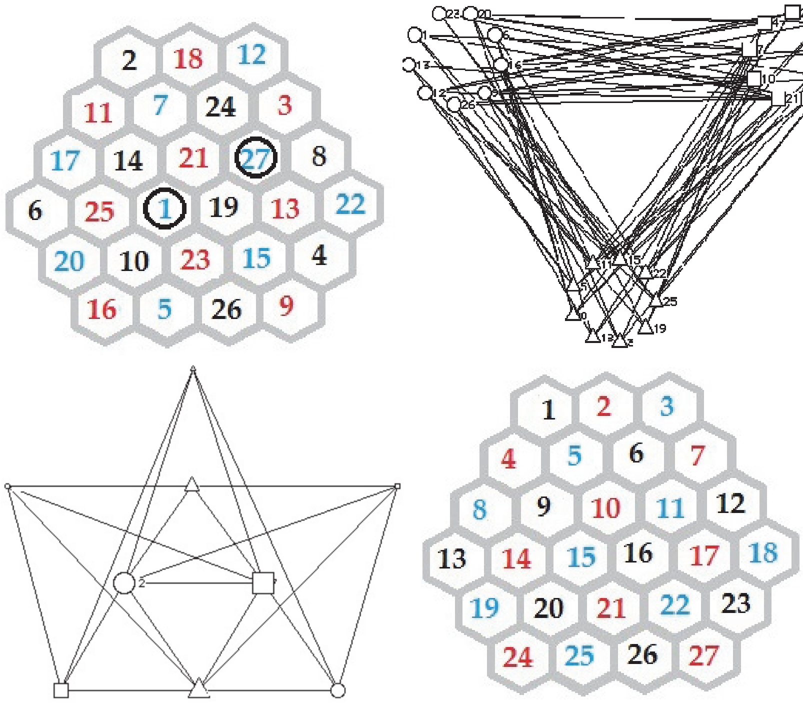

Figure 68 Knight's Path on hexagonal board

On top left of Figure 68, there is a solution, where we can see one Knight's Path. The colors of the numbers describe the colors of the hexagons. On the top right, there is a graph of the Knight's movements on this same game board. This graph was made in same way as the graph for a square board in Figure 67. In this graph, circles correspond to black hexagons, squares to red hexagons and triangles to blue hexagons. The vertices that correspond to the numbers are shown in the hexagonal board that is under the graph in this figure. There are three different kinds of vertices (of degrees 4, 5, and 6). Those with degree 4 are situated on the short three-hexagon sides of this game board; the vertices of degree 6 (deg(v) = 6) are at the center of the board (vertices 9, 10, 11, 15, 16 and 21); the remaining vertices (nodes) have the degree 5 (deg(v) = 5). As we can see, the graph is tri-partite. This means that no two vertices of the same color are adjacent and that the number of the colors is three. The graph on the bottom left describes the colors of this game board, and there are connections (seen as edges) between. In the graph, the shape of the vertex describes the color of a hexagon cell on the game board. The size of the vertex is indicated by its degree: for example, on the top, a small triangle refers to a set of three blue hexagons with degree 4, and a larger triangle refers to a set of four blue hexagons, which have degree 5. On the bottom, the large triangle represents two blue hexagons of degree 6.

The next phase of this study would, of course, be a trial to find Knight's Tours and Paths on different-size hexagonal boards and then generalize the results and look for similarities and general rules of Knight Paths and Tours as the size of the board varies.

However, this thesis deals with a symmetric n-player game model, and the reason for explaining the process of the Knight's Path or Trail was to find out how the process of Trichess got started. Next, we continue with TriChess.

6.1.2 The Game board

There are exactly three ways of tiling the plane by means of similarly shaped regular polygons. Rectangular chessboards with square cells are not the only ones: it is also possible to form triangular chessboards or hexagonal chessboards with well-defined Knight's moves. So not only the chessboard but also the Knight's move can be generalized. This may give rise to several new problems related to artificial chess boards in the field of graph and game theory.

For the designer of the game board in ancient times, the natural way was to divide a plane into sections of the same size and shape, into areas that indicate locations. Thus we once again are dealing with the concept of tiling. As mentioned before, there are only three polygons which can tile a plane evenly: a triangle, a square and a hexagon (Grünbaum & Shephard, 1987).

This can be seen in Section 3.2, where we deal with the infinite, complete, regular, planar graphs in which the degrees of vertices and faces are constants. There the formula $v = 4r / (2r + rk-2 k)$, where k is the area of the node and r the degree, gave the number of vertices. By degree we mean the number of the edges which are connected with a vertex or surround a face. Next we define the concepts of zone and territory.

As the first step, we select a 3x3 square, which is divided into 9 smaller squares, for the game board. The small square in the center of the board is the King's fortress (K). The first *zone* consists of 8 squares which surround the center square (Figure 69 on the left). We call this zone *the first territory* of the King. Next we take two new game pieces, A and B, which can move in various ways. Piece A moves horizontally or vertically and piece B diagonally. These two new pieces cover the whole first zone when the starting square is the center square. After this we extend the territory with *the second zone* of small squares which are in contact with the first zone. This zone has 16 squares (Figure 70 on the right, the gray squares). Eight of these squares, that is, a half of them, can be covered with pieces A and B if they continue directly to the direction they can move. This means that there will still be 8 uncovered squares. For these squares, we need a new game piece, marked with C in the figure. When we now observe these three pieces, we notice that they correspond to the rook (A), the bishop (B) and the knight (C) in the game of chess. These three pieces cover the whole of the *King's second territory* (Figure 69, on the right), which consists of the first and second zones.

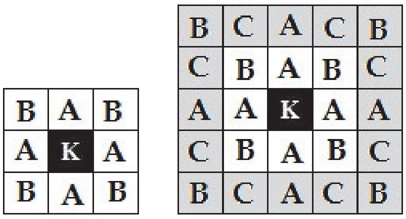

Figure 69 King’s territories

When a cell to be defended is chosen, its surroundings can be divided into different zones by distance (Figure 73 a). The cell chosen can have a shape of any of the previous three polygons. A two-dimensional plane can be filled regularly only with three regular polygons: triangle, square or hexagon, as explained earlier. When the cells are squares, then the game pieces of type A and B, which move like the rook and the bishop in chess, cover only the lowest one of these zones, as shown in Figure 69. When moving into the second zone, some cells stay out of the bishop’s and rook’s reach, and especially these cells become the possible destination cells for a third piece C, the knight (Figure 69). This means that a piece like the knight is essential for a good and complete defense of the central cell. Let’s now assume that the central cell of the defense zone is called a fortress.

Next we place, on the side of this fortress and its territories, another similar fortress with two zones. We do this in such way that both central fortresses will be unreachable by the territories of the neighboring fortress. So the distance between the two fortresses (K) will be two squares, and the total size of both fortresses and their territories is 5x8 = 40 squares (Figure 70). Then we place these fortresses so close to each other that both will have two defensive zones but will remain outside the second territory of the neighboring fortress. This will result in an 8x8-square board, which in fact is the Ashtapada board (Figure 27) or standard chess board.

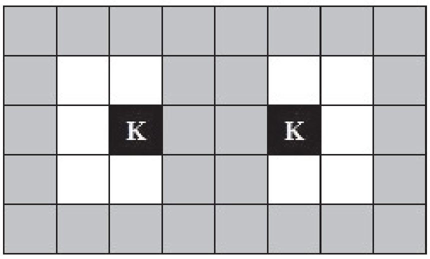

Figure 70 Two fortresses

Next, we make the board symmetrical by placing another fortress above each existing fortress outside their second territory. In this way the game board grows to 8x8 = 64 squares, which is the same size as that of the ancient Indian game board *Ashtapada* ("eight squares"). So now we have a 64-square game board of *four* fortresses, where the basic chess pieces – the rook, bishop and knight – are the game pieces (Figure 71). This may lead us to think that chess might have been a four players' game originally.

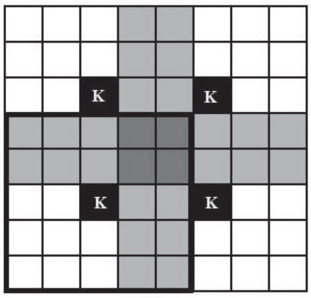

Figure 71 The Symmetry of Kingdoms

Let us return to Section 4.2 and the discussion about the origins of chess. When we study the history of chess we can see that chess came to Europe after 1000 AD from the Arabian countries, where it was known as *Shatranj* (Mark 2007, Eales 1985). There are other possible arrival routes: for example, the eastern route through Russia. The Arabs got the game from Persia *(shatrang),* the latter country having received it from the 8th century India. In India, the game might have evolved from a four-person board game, Chaturanga (see also Section 4.2), "the four branches of army service", which was played on ashtapada boards. These four branches of military service were the cavalry (Knight), elephants (Elephant, later Bishop), the chariots (Rook) and the infantry or ordinary soldiers.

The Chaturanga signifies the game of four angas, or four species of forces, which, according to the Amira Kosha of Amara Sinha and other authorities means elephants, horses, chariots and foot soldiers, which, in the native tongue is Hasty, aswa, ratha and padatum. It was first brought to notice by the learned Dr. Thomas Hyde of Oxford, in his work De Ludus Orientalibus, 1694. (Bird 2004)

During the development of chess into its current form, the movements of the game pieces have changed considerably, and the information on their movements in the earlier forms of the game supports the idea about the zones. For example, *the Bishop (fil) was able in the past only to jump diagonally over one square,* that is, from the central square straight to the second zone. The first zone's corner squares were covered by the *Queen's predecessor, a significantly weaker "senior officer" (firzan), which was able to move just one step and only diagonally.* These early moves are still a part of today's Chinese chess. Chinese chess also has a special rule according to which the King and two senior officers cannot leave the fortress. There are nine squares in this fortress.

This kind of zone-oriented thinking led us to our conclusion that a game piece like the Knight was essential to the defense of the assumed central fortress. Figure 72 shows how the officers of Chaturanga covered the first two stones, the second territory, around the center square (black). In the figure, we use the same abbreviations as in the modern chess game. This kind of thinking might have motivated the creation of a knight-like piece; there might have been a need to cover the squares which the fil, firzan and rook (originally *ratha)* did not cover.

B	Kn	R	Kn	B
Kn	Q	R	Q	Kn
R	R		R	R
Kn	Q	R	Q	Kn
B	Kn	R	Kn	B

Figure 72 The Officers of Chaturanga and the zones

Next we consider the trajectories of the officers from a slightly different perspective, based on the direction of movement, and at the same time we define two new concepts, the file and the sector, for our future needs. The file is a common concept in chess, but here we are going to use it with a slightly more general meaning.

The rook, bishop and knight can be defined in a more general way by using files. The rook moves from the fortress to the directions of the sides of the square in a straight line to four directions: 0°, 90°, 180° and 270°. We call these directions the *files* of the rook and the squares on these files the *sectors* of the rook. When we draw between these files new files to directions 45°, 135°, 225° and 315°, they determine the files of the bishops. The squares under these files are the sectors of the bishops. The sectors of the rook and the bishop cover all the squares on the first zone and territory, and half of the squares on the second zone. When we make new files between those of the rooks and bishops in such way that they cut the rest of the squares on the second zone, we get the directions: 26.6° + k*90° and 64.4° + k*90°, where k = 0, 1, 2 or 3. These files go through the center points of eight squares on the second zone. Those are the squares where the knight moves from the center fortress. Now, the second territory is also completely covered. If we enlarge the group of territories by adding new zones, we need new kinds of game pieces or new officers. This principle is relevant for the next step when we begin to generalize the game board.

6.1.3 The Zones and sectors on different boards

The theory of prime chess, and how this chess-like game was invented, stimulated from the study of knight's movements and why the game board for chess has 64 squares (Figure 73). This gave rise to a strange thought: *perhaps the predecessor of chess was a game for four persons?*

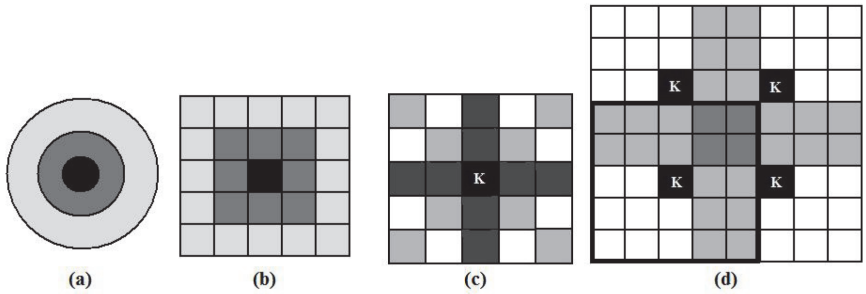

Figure 73 The two-step-zone on the square chess board

During the development of the game, these four kingdoms would have been united to two kingdoms and two kings. Two of the four kings could have been changed to queens (firzan). However, this was only an hypothesis. The important thing is that this reasoning led to ideas about new kinds of chess variations.

Interesting from an historical point of view, however, is the fact that the same hypothesis was also entertained in the past (Bidev 1986). The name of the predecessor of chess in Sanskrit was *shaturanga,* which translates to four sets; however, this is interpreted generally as referring to the four branches of service, which are represented by three officers and soldiers. In Germany there is an old chess variant, Gala, which we described in Section 4. In Gala the teams were divided into the four corners of the board. A two-dimensional, planar, four-person game usually degenerates to a game of two players, because four points cannot be set on a plane so that they all would play the same strategic role in relation to each other. The positions of neighbors will always be different from those of their opponents. For three players, this would be possible. In Figure 74, on the left, there are three points, symmetrically placed on the plane; in the middle, there are four points asymmetrically placed; and, on the right, all four points are symmetrically placed in three-dimensional space. By *symmetry* we mean here the strategic positions of the points in relation to each other for all players. In the middle of Figure 74, the opposite points' positions differ from those of the adjacent points, regardless of what kind of game we choose to consider.

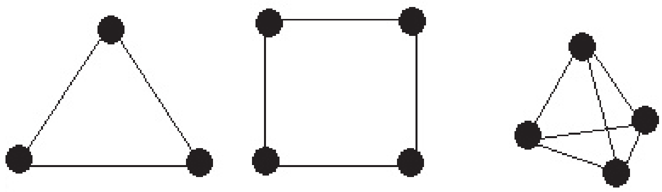

Figure 74 The Symmetry between three and four points

6.2 Trichess[41]

As discussed earlier, different chess variants have been developed through the ages, as explained in Section 5. Among these variants there are many which have more than two players. These include three-player games. The three-player chess we present in this section results from a trial to solve a certain mathematical problem, commented upon in the introduction of this thesis. This ancient mathematical problem, the Knight's Path, was explained in Section 6.1.1. One of the ideas regarding it was to generalize the knight's movement, and the original chess game had to be structured to the basics. We traced the evolutionary steps of chess back to Chaturanga and then transformed the game from the viewpoint of general chess theory. As a result, three different chess games were found: traditional chess, a theoretical game based on triangles, and a three-players chess which is played on a hexagonal board. The Rules of three players chess were specified and completed from the basic rules of chess. The name Trichess was given to it by Professor Frank Harary in May 1993.

The aim wasn't to create a game for three players; we only to study the movements of the game pieces on other than square-shaped cells of a game board. When we finally decided to take a hexagonal model instead of a triangural or rectangular one, then also the number of players became naturally and logically three, due to the shape of the board. Next we will explain the principles of Trichess.

6.2.1 The Hexagonal board

A kind of fortress/zone-based logic might help us to picture in our minds how the first game boards were created 2000 years ago. We could then build a chess board also on triangular and hexagonal boards, where the location of a game piece could be determined in a manner resembling that on the traditional square board. If the number of zones outside the fortress is two, then we will get territories such as those in Figure 75. When we create the game board using the same principles as the square board in Figure 73, we get new game boards (Figures 76 and 78).

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41 *Trichess was awarded an innovation prize in 1995 by the Jyväskylä Technology Center and the University of Jyväskylä. In 1999. Mika Vesterholm wrote a Java-program for this game, and Figures 80 – 81 and 84 - 88 are from that program.*

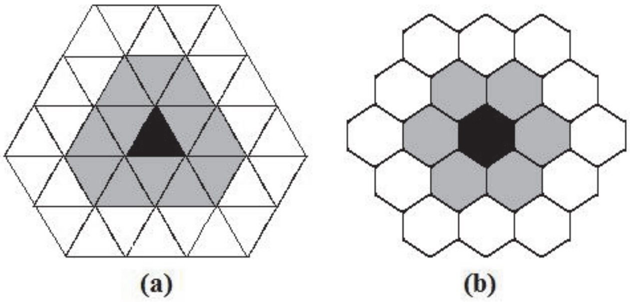


Figure 75 The first zone on triagonal and hexagonal boards

It is impossible to set up 4 players in equal positions in relation to each other on the plane. In the case of three players this is possible. When the movements of chess pieces are transferred to a board consisting of hexagons, the two innermost defense zones may be covered withthe rook's and bishop's moves. The knight becomes essential in the third zone. The game with 3 fortresses consists of 87 cells/hexagons (Figure 78). These cells on a trichess board are colored with three different colors (usually black, white and brown), and that is also the reason why three bishops are required for each player. The number of other officers equals the number of the officers in regular chess (Figure 81). By using these same principles, it should also be possible to construct a game of chess on a triangular board, but the movements of these chess pieces cannot be transformed identically to the way they move on hexagonal and traditional chess. For example, in the starting position, none of the officers (the knight included) could move before one of the pawns had been moved. We will briefly return to this detail in Section 7.3.2. Figure 76 shows the fortresses (K) on a triangular board with two defense zones. The number of fortresses is six.

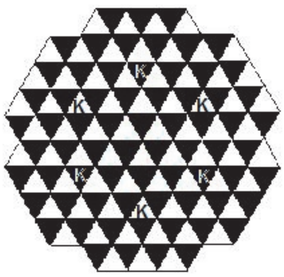


Figure 76 The symmetry between kingdoms on a triangular board

As a next step we build a chess board about hexagons and then show how the rules of the traditional chess can be applied to this kind of board. The first defense zone around this kind of fortress consists of six hexagons. This defense zone is colored gray in Figure 75 b. The second, the outer defense zone, is made up of twelve hexagons (Figure 75 b).

The rook can move towards any of the six sides, which means that there will be six sectors which are separated by angles of 60°. In Figure 78, the rook, marked with the letter A, can move to directions 30° + k*60 when k ∈ [0,5] k ∈ N. All of the hexagons of the first and every second on the second defense zone can be covered with a rook. On the second zone, six hexagons still remain uncovered. These uncovered hexagons are intermediate sectors, and they are for bishops. Bishops, marked with letter B in Figure 77, can move to directions 0° + k*60° when k ∈ [0,5] and k ∈ N. If we want to create a game with the same number of different officers as in ordinary chess, we must create a third defense zone to get the knight in the game. On this zone, there are 18 hexagons. The rook covers six of them. The bishop doesn't cover any of the hexagons on this zone: but it will cover hexagons next on the fourth zone. The uncovered 12 hexagons are for the knight. In Figure 77, the knight is marked by letter C and moves to the directions of approximately 10° + k*60° and directions of approximately 50° + k* 60 when k ∈ [0,5] and k ∈ N. Now the number of different officer types is the same as in the traditional chess.

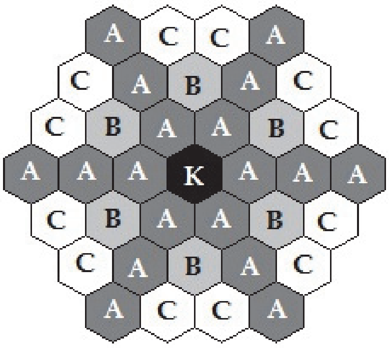


Figure 77 The officers and their sectors in Trichess

Next we add the fortress in the same way as in a square board, so that the territories don’t reach the neighboring fortress, which is the hexagon with K. In this way, the distance between the fortresses will be three hexagons and the total number of hexagons will be 65. If we want to add more fortresses so that they are in same position with these two, it is possible only by adding one more fortress above them. In this way, there will be three fortresses and the number of hexagons will increase to 87 (Figure 78). (Honkela 1999)

So, now we have a game board and the game pieces as defined. The next step is to decide the number and positions of the pieces on the game board.

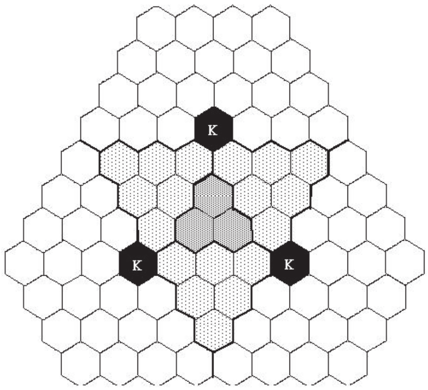


Figure 78 Symmetry of kingdoms on Trichess board

This board has six sides, three long ones of 8 hexagons and three short ones of 4 hexagons. We choose the short side with four hexagons for the player's corner positions. The reason for this is that the zones of the other players do not intersect these shorter sides and the number of chessmen is as close as possible to what it is in traditional chess. In traditional chess, the total number of chessmen is half of the number of squares. On the long sides, we can introduce only one full row, that is, 8 hexagons. In case we also used the next row, then it would mean that no free hexagons would remain between the neighboring teams. A reasonable number of chess pieces can be obtained from traditional chess: half of the squares (32/64) are free. The exact number of one player's pieces in three players' game would be then 87/2/3 = 14.5 => 14 or 15. When 8 officers (there are 3 bishops and the king among them) are deducted from this number, then the remaining number of pawns is 6 or 7. The three rows (Figure 79) on the short side have 4 + 5 + 6 = 15 hexagons. On the two bottom rows, there are 9 hexagons and thus the places for the king and the officers. On the third row, there are 6 hexagons for the pawns. The officers will be ordered as follows: on the back row, there are 4 hexagons, in the center of the row are the king and the queen. On the two corners are the rooks. In this way, castling is made possible. On the next row, we have the knights at the sides and the bishops in the center, each on a different color. This kind of order between knights and bishops is necessary because only these positions protect all the chessmen on their starting places. A different kind of order would lead to a situation where bishops could capture opponents' pieces already during their first moves. The bishop can move at the beginning of the game, just like the knight, even though none of the pawns have been moved. This is the biggest difference from the traditional chess. In Figure 79, the arrow shows how one of the bishops can move from its starting position.

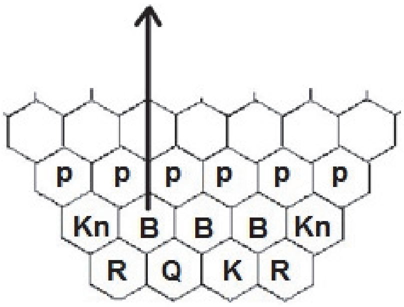


Figure 79 The officers and pawns in the opening position of Trichess

6.2.2 The Chessmen of hexagonal Trichess

When we start to describe the movements of chess pieces on a hexagonal board, we begin by taking a look at the traditional square board where the rook moves to the directions of its sides, and hence covers the whole of the innermost defense zone. The sectors of the bishops are between the sectors of the rooks, and the second defense zone will be covered by the rooks and bishops. From this, it follows that the resulting board of 46 hexagons (Figure 80) can be formed without the knights. They will become necessary only on the third defense zone.

Figure 80 Hexagonal Trichess with 46 hexagons

When we also add the knight, the board will grow to the size of 87 hexagons, due to the third defense zone and the three fortresses (Figure 81).

The development of these games began when we started looking for solutions for generalizations of the Knight's Tour problem. It is possible to search for Knight's Tours and Knight's Paths on hexagonal boards as well as on square boards, but the structure of the graphs will change significantly. However, this plays no essential role in this section, where we concentrate on explaining the birth process of Trichess. In the development process, we divided the chess game into kinds of "initial elements", which means that we explored ancient Chaturanga and then followed the assumed evolution process of chess. We assembled the game from its basic elements, but we also took into account the rule changes which have occurred in chess during the two millennia, and we formulate "abstract chess". From this abstract chess model, we built Trichess and later on, traditional chess. The latter was done to ensure the correctness of the model. The third transformation, made on the triagonal board in this "evolution", was so unique that it could be considered a game different from chess. However, the triangular model provided useful support, when we tried to find a better isomorphism in the details between Trichess and traditional chess. After all, during its millennial history, chess became so perfect that even the smallest changes over time would bring unpleasant surprises. Most of the rules we found easy to project to the new game, but every now and then we came across a situation in which we had to choose the correct rule among several options. In such cases, the triangular model was used for verification (Figure 76).

6.2.3 The Numbering system on the hexagonal board

Figure 81 shows the Trichess game board and the placement of chessmen before the game begins. The white player is on the bottom and the black player on the right.

Because this chessboard consisted of hexagons, the determination of the numbering was clearly a problem. We decided to define the numbering in the same way as in traditional chess but applied to a board of 87 hexagons. The hexagon rows are marked by letters from a to k, starting from the bottom, which is the white player's corner. The individual hexagons on each row are numbered from left to right. This means that the greatest number of the shortest row is 4 (a4) and the greatest number for the longest row is 11 (h11). The corner hexagons on the opposite side of white player are k1 on the left and k8 on the right. On k1 there is brown rook, and on k8 there is black rook.

Figure 81 The Opening positions on the Trichess board of 87 hexagons

The practical implementation of the numbering

However, the problem is the marking in the numbering of the board so that it would be
a) unbiased for any of the three players,
b) always visible regardless of the locations of chessmen,
c) observable to make the location of chessmen immediately apparent, and
d) appropriate for the board layout and not disturbing.

The following solution we have presented (Kyppö 1999) fulfills these criteria. In this solution, the coordinate for each hexagon is marked on three of its six corners so that each player can see one of these coordinates written in the correct way round for the player. The coordinates are compact in size, which means that they cannot be seen from faraway and thus remain non-disturbing. However, the coordinates are large enough, so that the players themselves can read them easily. The coordinates are written on the corners, so that they are not hidden by the chessmen. Coordinates are positioned in such a way that every second intersection of the three hexagons has three coordinates and the rest of the intersections are free (Figure 82). The figure shows an enlarged view of the intersection of three hexagons: F8, F9 and G9.

In the utility model, this is specified as follows: There is a game board for three players, formed by hexagonal boxes, which are arranged in a truncated equilateral triangular shape. In the game, each player is positioned behind the triangular tip of the extension, the player's viewing direction being essentially the game board. Each hexagon has three coordinates, and for each player one coordinate is placed in the player's viewing direction from the hexagonal to the nearest corner of the hexagon.

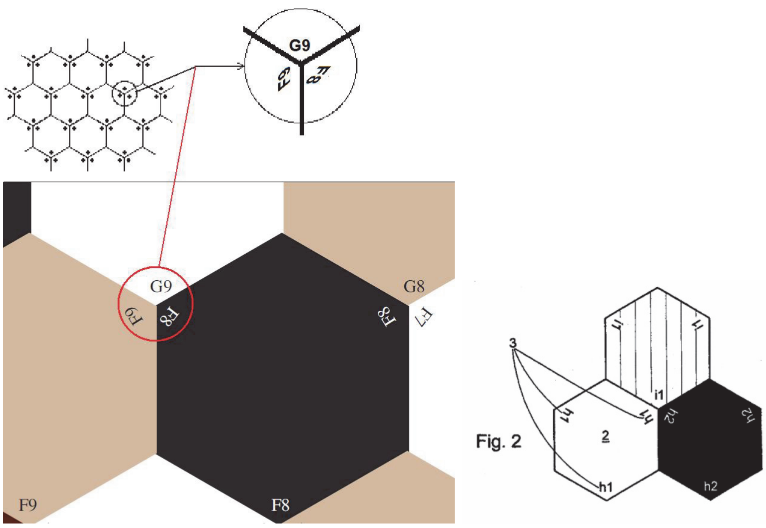


Figure 82 Numbering of a hexagonal board

6.2.4 Trichess as a game

Even though Trichess is based on standard chess, there are differences in the characters of the games. This is because the density of sectors is greater in Trichess, and hence also the power of officers. Another great difference is the dynamics of game: this is a multiperson game. However, the greatest difference is at the strategic level. Above the normal game there is also a second-level of game: being not a two-player game, the players can form allies with other players during the game. This creates interesting game-theoretical situations, similar to those in politics. As an example, we can take the political game in 1999 between the Finnish Social Democrats, the Centre Party and the National Coalition Party. For the parliamentary elections in the spring of 1999, all these three parties had boldly transferred their best "chess officer", the queen, to the central area, which means to real life, in the largest electoral district, Helsinki, in South Finland. Other well-known examples can be found from the time of the Cold War. The Soviet Union v USA v China, the political situation which formed the background for George Orwell's science fiction novel *1984* (Orwell & Mattila 1999). In this book, the super-states were Oceania, Eurasia and the Far East. As another example, we can raise the crisis in Yugoslavia, which developed in the 1990s. From the viewpoint of Trichess, the center of this crisis was Bosnia-Herzegovina. For a long time, that state had been the meeting point of the East, West and South, or Orthodoxy, Roman Catholicism and Islam, or the Russian sphere of interest, Western countries and the poor South. In 2014, the center crisis point was around the Black Sea.

In Trichess, the moves are based on traditional chess. The only difference is that the squares are replaced by hexagons. This means, that for example the rook can be moved to in any of six directions instead of four, because a hexagon has six sides. It is the same with the bishop, which moves to the direction of the corners as in traditional chess. The queen's movement combines the movements of the rook and the bishop. The king moves like the queen, but only one step at one move. The knight moves to the nearest hexagon, which the rook or the bishop cannot reach. The knight's movement can also be defined similarly to those in traditional chess: one step forward like the rook and one step diagonally like the bishop. Based on conventional chess, the movements of all chessmen are easy to learn, but at first it might be difficult to get used to a game that is not played on a square board. In particular, bishops might cause problems. Figure 83 below shows the former world chess champion, Anatoly Karpov, solving a game situation where his king is threatened by a bishop.

Figure 83 Anatoly Karpov solves a checkmate in Trichess[42]

The top of Figure 83 shows the game process (left to right). At first, Black moves one of his bishops in the center, where it threatens White's king. This was a threat for Karpov, but it was easy to cancel as White's rook can capture this Black's bishop. The threat was created to demonstrate how the bishop can surprise opponents' chessmen in the game. Photos represent the situation on the board on the top right. Next we explain the rules in more detail.

The Rules

The Board. The Trichess board differs from the traditional chessboard in that the 64 squares with two colors are replaced by three different colors (usually black, white and brown). We naturally need three bishops for each player, but the number of other officers equals their number in regular chess.

The Trichess chessboard consist of 87 hexagons, called *cells*, which are colored alternately: white, brown and black. The board has three long sides (8 hexagons) and 3 short sides (4 hexagons) arranged so that the opposite sides are always of different size. The three players place themselves on the short sides of the board. The cells (hexagons) form *rows*, which have three different directions. From the viewpoint of a player, the vertical rows are called *files* and the two (left and right) horizontal rows are called *ranks*. This means that all three rows are both files and ranks depending on the player's viewpoint. That is why a row

42 *The photos of Karpov were taken in August 1996 by Matti Turpeinen, the photographer of the Keskisuomalainen newspaper. The movements above are made by a trichess program created by the Vision project in 1999 – 2000 (the Department of Computer Science and Information Systems, University of Jyväskylä).*

may also be called black-white rank, white-brown rank or brown-black rank. The rows, or lines, of hexagons of the same color that cross the chessboard are called *diagonals*.

The players are called White, Brown (or Red) and Black, because these three colours are most commonly used with traditional chess boards. The players are seated in such way that on the left side of White is Brown and on the right side of White is Black. Of the three short sides, the White player selects the side with black hexagons on its corners. In the illustrations in this thesis, White can always be found at the bottom of the board.

The Game process. The game is played between three opponents, who take turns making their moves in the mentioned order. The player with the white pieces begins the game. At the beginning of the game, each player has a set of 15 chessmen, one set of light color for the White side, the other black for the Black side and the third of brown color for the Brown side. No piece can be moved to a cell occupied by a piece of the same color. If a piece moves to a cell occupied by an opponent's piece, the latter is captured and removed from the chessboard as a part of the same move. One exception is en-passant, about which we will discuss later. A piece is said to attack a cell if the piece could make a capture on that cell.

The Chess pieces. Each player has a set of 15 pieces: one set light in color for the White side, the other brown for the Brown side and the third black for the Black side. The first two rows are occupied by five kinds of "pieces", distinguished by their shapes: King, queen, rook, bishop and knight. These are abbreviated in conventional notation: K, Q, R, B, N. The chessmen in front of these pieces, positioned on the third row, are called pawns, abbreviated by P. Behind the pawns are officers. On the back row, rooks are on corners, the king and the queen in the center. Each queen is placed on a cell of its own color. On the second row, there are three bishops in the center and knights on the left and right side.

The Moves. The King moves both in any rank direction and in any diagonal direction, one cell at a time (Figure 85, on the left). The Queen, rook, and bishop are long-range pieces that can cover any distance across the board in any direction, as long they are not obstructed. The Rook moves any distance on files or rank (Figure 84, left), and the bishop on the diagonals (Figure 84, on the right). This means that both the rooks and bishops have six possible directions, supposing the way is free, to leave their cell. The Queen combines the powers of a rook and a bishop. The Knight moves over two cells on a same file or rank, at a time. One of these cells is on the file, the other on the diagonal (Figure 85, on the right): thus "jumping" carries the Knight over any Black, Brown or White chessman that may occupy any of these intermediate cells. In other words, the Knight passes the first diagonal cell and then continues to one of the two cells which touch the first and second diagonal cell on same direction. Equivalently a knight takes three steps on two different files, at first jumping two steps in the same direction and then one step to the cell which is 60 degrees to the left or right from the original direction. Any piece other than a Pawn captures as it moves; that is, an enemy piece can be captured by the capturing piece on a cell to which it may legally move. The capturing piece replaces the captured piece on the same cell, and the captured piece is removed from the board.

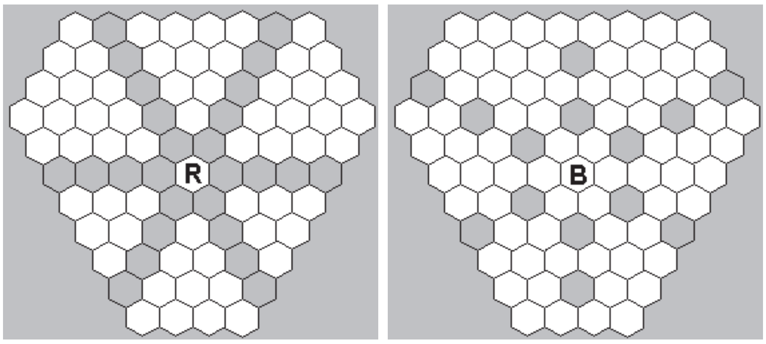


Figure 84 The Moving directions of Rook and Bishop

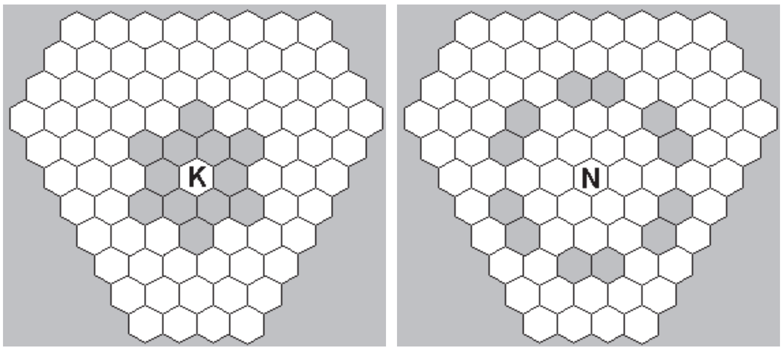


Figure 85 The Moving directions of King and Knight

Moving directions in detail:

The Rook moves to any cell along this file or rank on which it stands. It cannot leap over any piece. The Rook captures on the same direction it moves. When it captures an enemy piece, it occupies the cell of the captured piece.

The Bishop moves along the diagonals it stands on, always on the cells of the same color. It cannot leap over any piece. The Bishop captures on the same direction it moves. When it captures an enemy piece, it occupies the cell of the captured piece.

The King moves from its cell to one of the contacting cells or to the nearest cell of the same color if the cell is not threatened by any of the opponent's pieces. The kings of two opponents can never be on cells touching each other or on adjacent cells of the same color. The King is the most important piece in the game. If the king is checkmated, the game is over. Winner is the player, who made the checkmate, but there might be cases, where the solution is more complicated (see Chapter 8.4. and case, when there are three players). The King can move to any rank or file direction and to any diagonal direction as well, one cell at a time. The King captures the same way as it moves. When it captures an enemy piece, it displaces that piece. The King may not move to a cell that is threatened by one or more of an opponent's pieces.

The Queen moves along the ranks, files, and diagonals it stands on. The Queen is the most powerful piece of all the chessmen. Its moves are the combination of the moves of a rook and a bishop. The Queen cannot leap over any piece. The Queen captures the same way as it moves. When it captures an enemy piece, it occupies the cell of the captured piece.

The Knight moves at first one step like a rook and then another step diagonally forward like a bishop, or the other way round. In the cells that are between the departure and arrival cells, there may also be other pieces. The color of the departure cell is always different from the color of the arrival cell. So the Knight is the only piece that can leap over other chessmen (either his own or the opponent's). The Knight is also, with the King, the only officer whose move is of a fixed length.

The Pawn is different from the pawn in traditional chess, because one player has two opponents. Thus, the pawn can move to two of the surrounding six cells, toward its opponents. The pawn can change its moving direction between the two opponents any time during the game. It moves forward only one cell at a time. The exception is the first move - even if this occurs late in the game: the pawn may, but it does not have to, advance two cells along the same file provided both cell squares are unoccupied.

A pawn can diagonally capture cells of the same color on the right and left side. As the pawn has two directions to move, due to its two opponents, the number of cells where it can capture grows to three (Figure 86). It should be noted that of these three cells the pawns can capture the left-hand opponent's piece only in two left-side cells and the right-hand opponent’s piece only in two right-side cells. This rule is necessary: otherwise we might face a situation, where there are pawns of two different players in an attacking position, but only one of them could capture the other.

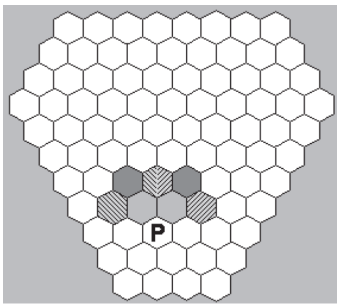


Figure 86 The Moving directions of Pawn

The pawn will be coronated when it reaches one of the eight cells on the opposite side. When the pawn reaches the eleventh rank, farthest from the starting cell, it is immediately replaced by a queen, rook, bishop, or knight of the same color, at the option of the owner. This transformation is called promotion or queening, because it is usual to promote the piece to a queen. If the chosen substitute is not a queen, the change is called under-promotion. A player may have two or more queens or other pieces, except a king. Promotion to a king is not permitted.

If a pawn makes its first move known as a double advance, an adversary's pawn that could have captured it had the first pawn moved only one cell may capture it in passing (*en passant*) if the pawns end up occupying the neighboring cells (Figure 87). This in passing capture may be made only on the immediate turn, not later.

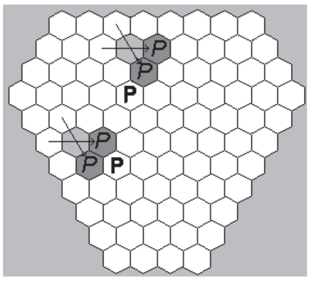


Figure 87 En passant in Trichess

Castling is a compound move of the king and one of the rooks. It can be done only once in a game. The move is executed by moving the king one cell towards the left rook and then placing that rook over the king on the cell (Figure 88, below). The move is legal only if neither the king nor the rook has yet moved from its original cell, and if the king is not in check. Castling can also be carried out by simply placing the right rook on the cell over the king (Figure 88, above). Castling is prevented for the time being if the cell on which the king stands, or the cell which it must cross, or the cell which it is to occupy is attacked by one or more of the opponent's pieces, or if there is any piece between the king and the rook with which castling is effected.

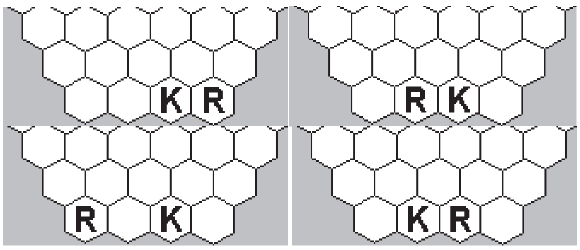


Figure 88 Castling

The objective and the game termination. The aim is to make the king of one of the opponent's unable to move or, in another variation of the game, to make the kings of all the opponents unable to move. All the other pieces can be captured and removed from the board. The king is said to be 'in check' if it is under attack by one or more of on opponent's pieces, even if such pieces cannot themselves move. This situation is possible, when there are three players. The objective of each player is to place an opponent's king 'under attack' in such a way that the opponent has no legal move which would avoid the 'capture' of the king on the following move. The only ways of countering a check are to move the king, capture the attacker, or interpose a man on the line of check. Capture of a king is never consummated and remains symbolic – when the king is attacked, in check, and cannot escape by any of the aforementioned means, then he is checkmated. The game is won by the player who has checkmated an opponent's king with a legal move. This immediately ends the game. Games can also end by resignation when it is obvious that ultimate defeat cannot be avoided. The game is drawn when the player to move has no legal move left and his king is not in check. Such a game is said to end in a stalemate. The game can end in a draw by agreement between all three players during the game. The game may be drawn if an identical position is about to appear or has appeared on the chessboard three times. The game may be drawn if the last 50 consecutive moves have been made by each player without the movement of any pawn and without the capture of any piece. The game can be, if it is agreed in advance, also continued between the last two players and to the second checkmate. More about this next.

Two different ways to finish the game

A. The player who makes the first checkmate of an opponent's king, will be the winner ("first checkmate to win").

B. The player whose king has been checkmated, will be out of the game, and the last player left on the board is the winner ("the last checkmate wins"). There is one addition of rule to this variant. That additional rule can be found below in the section *Some related rules for the game,* ***the defeat.*** The player whose king is threatened and has no way to escape can also resign and finish the game.

If method B is used in tournaments, then the game can be activated by giving two points to the winner who is the last one to effect a checkmate and one point to the player who made the first checkmate. If both checkmates are made by the same player, then he/she gets three points. If one of the players resigns, and the two others continue, then the winner gets all three points. Hence, the game can end with the points as (3, 0, 0), (2, 1, 0) or (1, 1, 1) among the players. The last one is the case, when all three players end up with a draw.

Some related rules for the game

The opening procedure. The White player makes the first move, after which the moves are made clockwise on the board. The Brown player has the second turn, and the Black one the last turn.

The game. In the starting position, each player can move only bishops, knights and pawns.

The move is finished a) when the piece has been moved to an empty cell and the player has released it from his hand, b) during the capturing process, when the captured piece has been removed from the board and replaced by the capturing piece and the player has released it from his hand, c) in castling, when the player has released the king, or d) in coronation, when the pawn has been removed from the board and the player has released the coronated piece.

Check and checkmate. The king is in check, when it is threatened by one or both opponents. Check must be removed immediately during the next move of the player whose king is threatened: a) by moving the king to a cell, where it is not threatened, b) by capturing the piece that threatens the king or c) by moving another piece on the threat line for blocking. If the check cannot be prevented, then the king is in checkmate.

The defeat. This rule is relevant only if the game has been agreed to be played with method B, as discussed earlier in this section. Any player whose king has been checkmated, has lost the game. Similarly, in case of resignation, the resigned player has lost the game. Matti ja antautumisilmoitus ovat tehtävissä missä pelin vaiheessa tahansa. The player who has lost first is eliminated from the game, but the *pieces remain, "powerless, on the board.* "Powerless" means that they do not threat any other piece but that they, including the king, can be captured from the board. So the game can end in two different ways, but the players must agree about this before they start the game. "Voimaton", joka muuten oli myös sanan "matti" merkitys persian kielessä, tarkoittaa sitä että nappuloita ei voi liikuttaa, mutta ne eivät myöskään uhkaa muita nappuloita.This rule is necessary to avoid a situation where the strategic positions would change on the board if some of the chess pieces are removed from the board during the game process.

Draw between two players. If only two players are left, then the game between these players ends to a draw in situations where it would end in draw in traditional chess.

Draw between three players. The game can end in a draw also between all three players if the king of the player who has to do the next move is not in check, but this player cannot make any legal move.

Irregularities. If during a game, it is discovered that there has been a violation of the rules, then all chess pieces must be returned to their positions before this violation occured. If this is not possible, then the game shall be void, and a new game must be played. All the rules above are consistent with the traditional chess rules.

The algebraic notation of moves. The chess pieces use standard symbols, capital letters, that are used in traditional chess: K for a king, Q for a queen, R for a rook, B for a bishop and N for a knight (In Section 6.1.1, though, we used symbol Kn for a knight because Kn is widely used in graph theory). Usually in chess a pawn doesn't have a symbolic letter, but in the figures of this thesis we use the letter P.

The columns (called files) are labeled by letters a to k from left to right from the white player's point of view, and the rows (called ranks) by numbers 1 to 11, with 1 being closest to the White player. This closely resembles the standard algebraic chess notation, only the numbers and letters differ. This means that the largest number of the shortest row is 4 in the cell a4, and the greatest number for the longest row is 11 in the cell h11. On the opposite of the White player, on the k^{th} row, the left cell is k1 and the right cell k8.

In the starting position, White's pieces are on rows a, b and c, Brown's and Black's pieces are on rows f … k, so that Brown has cells f1, g1-2, h1-3, and k1-3. Black's pieces are in cells f9, g9-10, h9-11, i8-10, j7-9 and k6-8.

6.3 An asymmetric four-way chess

Here we present an asymmetric chess game for four players. During the history of chess, several four-handed chess games have been developed, as already discussed in Sections 4 and 5. These games were all asymmetrical because they were two-dimensional.

The *asymmetrical four-handed chess* we introduce here is a combination of traditional chess and trichess. In this game, we have four players, and the game is played on the traditional 64-square chessboard. The game is based on our universal model of chess, and was invented about the same time as trichess, which was introduced in Section 6.2. The idea for this game was to apply the principles of trichess to a traditional chessboard, but for four players. The motive for this comes from the same knowledge we had of trichess, the ancient four-handed chaturanga, which was introduced in Section 4.2.2.

When we invented trichess in the early 90's, we got the idea from four-handed chaturanga, but we didn't know how that game played like. When we were writing this thesis, we found a medieval chess-like game called Four Seasons, which is presented in Section 5.13 and in Figure 54. It resembles this game to a surprising degree, particularly because of the initial position of the chess pieces.

The chessboard is similar to that in traditional chess, with 64 white and black squares. The White player takes the corner where the square is named a1. On White's left side is the Red player, on the opposite side the Black player and on the right side the Yellow player (Figure 89 - The colors are not seen in this figure.). The horizontal lines are called *ranks* and the vertical lines are called *files.* The diagonal lines are called diagonals.

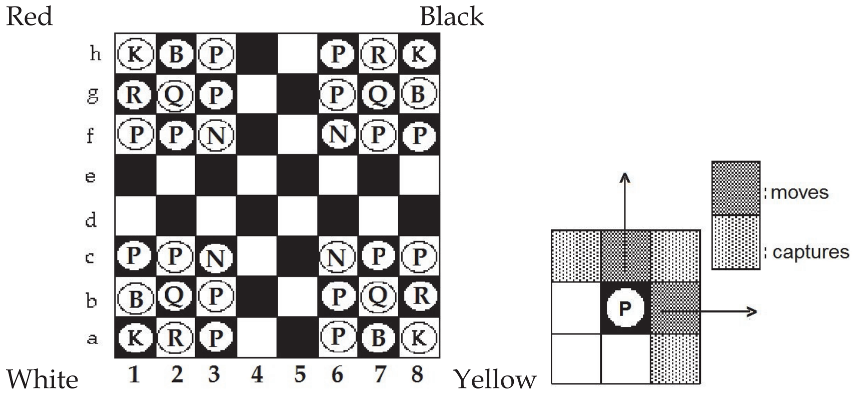


Figure 89 An asymmetric four-handed chess game (bridge chess)

Bridge chess. The game can be played as Trichess, where all play against each other. As such, the game works well as a social game. Another way is to play in two-person teams, the adjacent or opposite players on the same side. In this case, communication is prohibited during the game. This kind of game can be called bridge chess after a well-known card game, bridge.

Rules of the game. The rules of the game are a direct application of the rules of trichess presented in Appendix 2.

6.4 Chess without a board

Chess without a board is based on the model of Universal chess. The game of chess usually includes a game board marked with squares or identical cells. The squares or other markings show the permitted locations of the pieces. In the game of chess, only one piece can fit on a square, and each piece must remain on the game area marked with squares.

Another idea (Kyppö 1997) is to create a boardless game, where no squares or other markings are required to show the permitted places of the pieces on the board or a base. Actually this idea was invented already in 1993 (see Appendix 1, post number 16). In that kind of game no squares are required because the pieces always move a measurable distance in a set direction. Contact between the pieces is defined by means of the intersection of the individual areas as replacements to squares.

6.4.1 The Game field and the chess pieces

The game area is a limited two-dimensional surface, which is formed by game board 1, in which the boundaries 2 are defined by a feature (Figure 90). In this example, the edge of the circular game board 1 forms the aforesaid boundary. Each player has a group 3 and 3' (Figure 90) of sixteen pieces, as in normal chess, each comprising a king, a queen, two rooks, two bishops, two knights, and eight pawns. Each piece has a circular area around it, which is defined by the area of the base of the piece.

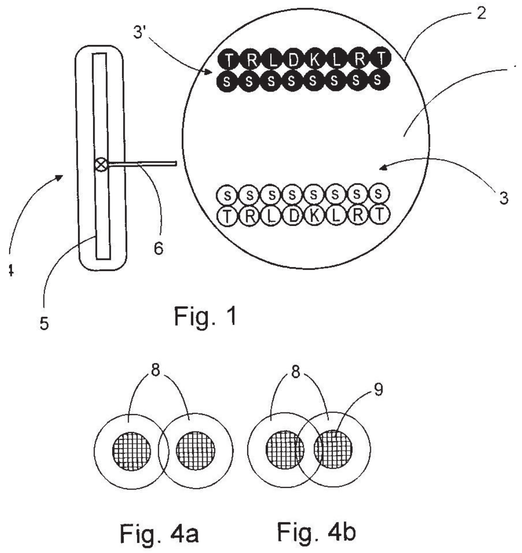


Figure 90 Opening position, the guider and areas of chess pieces[43]

In the initial arrangement, the pieces are placed so that on the White side the king and queen are in the middle, with the queen on the left. To the left and right of them there are, in order, a bishop, a knight, and a rook. Each is in its own area, so that the areas border one another. In front of these so-called noblemen are eight pawns, so that the area of each pawn touches the area of the nobleman behind it. At a distance of five area measures at right-angles to the pawns (there are four completely empty areas between), there is a straight line of Black pawns, behind which are correspondingly Black officers. The Black queen directly faces the White queen.

In the initial setup, there are six area measures between the Black and White kings. The extent of the area of the game is defined by drawing a circle, the centrepoint of which is the centrepoint of the line segment between the Black queen and the White king. Though the game can be played on any level base whatever, the most advantageous game generally has a base, a particular game board, in which this delimitation has been ready made, as in this case.

In Figure 91, the construction of a knight is used to exemplify the construction of the pieces. The definition of the location of a piece takes place according to a direction scale 11 printed on the upper surface of the base 8 of the piece (numbers in Figure 91). The *direction device* 10, a measuring rule or tape, at the base of the piece, which is attached to a rotating ring, is turned in the chosen

43 Fig.1 , Fig. 4a ja Fig 4b, and the small numbers are taken from a patent (Kyppö 1997).

direction and the movement of the piece is read from the set scale of the direction device. Its length is a measure of 10 areas, here called also belts, for a bishop, castle, and queen, a measure of 3 areas for a knight, and a measure of 2 areas for other pieces. The direction device is harmonized with the guider on left side of the game area (see Figure 90).

Before the direction is set, the piece must be placed at right-angles to the setup of the game. This takes place with the aid of direction indicator 4, see Figure 90. Direction indicator 4 is on the left when viewed from the White player's position and it has a button moving in a groove 5 in the center, from which there is a measuring rule 6 that can be moved to the right. At the beginning of the game, direction indicator 4 is set parallel to a straight line drawn between the White and Black kings.

Chess pieces. At the beginning of the game, both players have sixteen 'Black' or 'White' pieces. Each has one king, one queen, two rooks, two knights, two bishops, and eight pawns. The number and setup of the pieces are thus the same as in 'conventional' chess.

The Aim. The game's aim, opening procedure, and rules for moves are the same as in normal chess, except for the alterations described later.

The Moves. Each piece can move from the point of departure within the framework of the predetermined rules. A piece cannot move to such a place that belongs to the area of another piece on its own side, nor to such a place that belongs to the area of one of the opponent's pieces. If the area of a moved piece intersects the area of one of the opponent's pieces, this latter is 'taken' and removed from the area of the game. Only the knight and bishop may move over the areas of the other pieces. The area of a piece may not intersect the boundary of the area of the game.

The Area. The definition of an area takes place in practice by the piece having a flat circular disc as a base, i.e. the aforesaid base 8, the surface area of which is the same as the area of the piece. In the piece, there is also a concentric inner ring 9, the so-called inner area, the diameter of which is half the diameter of the entire area (Figure 90). The aggressiveness of the pieces can be selected according to figures 4a and 4b. At the separately agreed start of a game, the aggressiveness of the pieces can be reduced by moderating the intersection rule concerning the intersection of the area (base 8) and the inner area (ring 9) (Figure 91). The result of this moderation is that the pieces can also be located next to one another so that the ring-shaped zones remaining between the edge of the inner area and the edge of the entire area of the piece intersect one another.

In Figures 92 and 93, there is an example oft another officer, the bishop.

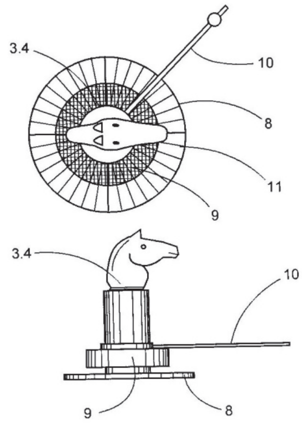


Fig. 2

Figure 91 The Knight of the Boardless chess[44]

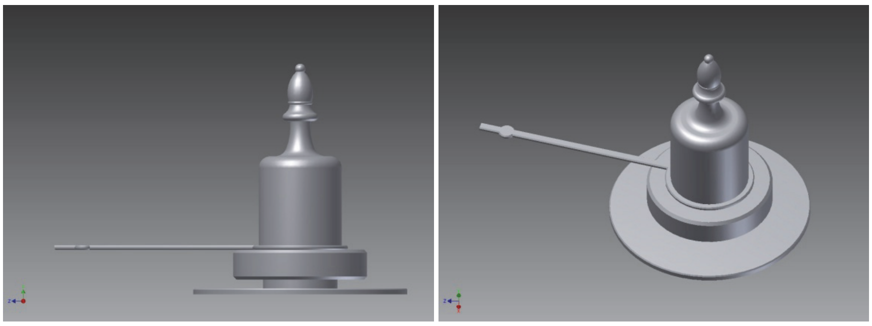


Figure 92 The Bishop of the Boardless chess

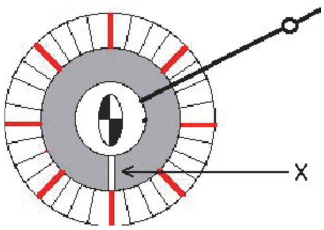


Figure 93 The Sectors of the Bishop

44 Fig.2 , and the small numbers are taken from a patent (Kyppö 1997).

6.4.2 Moves of individual pieces

The rook moves from the north in directions 0°, 60°, 120°, 180°, 240° and 300°. The number of steps in its move is not limited, provided the route is otherwise unobstructed (Figure 94, blue lines).

The bishop moves two areas at a time in the directions 30°, 90°, 150°, 210°, 270° and 330° (Figure 94, green lines). If there are other pieces in the area between, the bishop jumps over them. As in the case of the rook, the number of steps in bishops move is not limited.

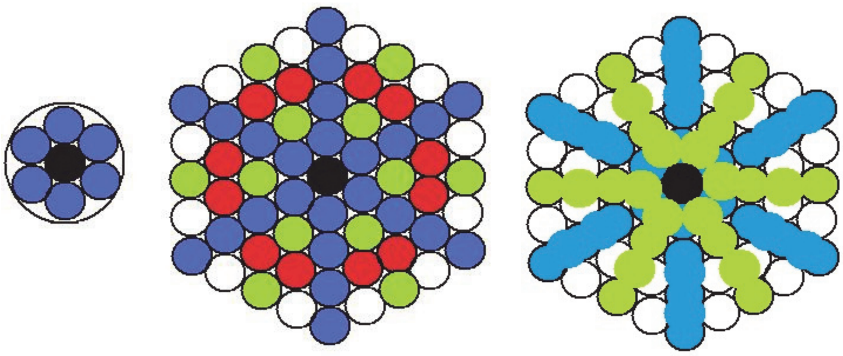


Figure 94 The sectors of the officers

The knight moves over three areas at time in the directions of 20°, 40°, 80°, 100°, 140°, 160°, 200°, 220°, 260°, 280°, 320° and 340°, one step at a time (Figure 94, red circles). If there are other pieces in the areas in between, the knight jumps over them.

The queen moves in the same way as the rook and bishop combined.

The king moves like the queen, but only one area measure at a time.

Castling is a king's move complemented by the rook's (castle's) move. It is counted as a single move and takes place as follows: the king remains in place or else moves next to a rook that is on the same horizontal level, after which the rook moves over the king if the area on the other side of the king is free. Castling is impossible if one or other of the pieces, the king or the castle has already been moved. Castling is temporarily prevented if the king is threatened by one of the opponent's pieces.

The pawn moves only by the amount of its own area. Only the initial move may span two areas if the area is completely free. The pawn moves from its area to a distance of one step in a direction of ±30° forwards. A pawn, which threatens an area that a pawn from the opponent's original area has passed in a move of two steps, may, immediately on next move (and only then), take the opponent's pawn, just as if the latter had moved only one step. This special move is termed taking *en passant*.

Every pawn that has been able to move more than six areas from its original position must during the same move and irrespective of the remaining pieces, be changed into a queen, rook, bishop, or knight of the same color, as the player wishes. The transformation of the pawn is termed coronation and comes into force immediately and the pawn is replaced by another piece, the coronated piece.

6.4.3 The Rules

The opening procedure. The White player makes the first move.
The move is finished a) when the piece has been moved to an empty cell and the player has released it, b) during the capturing process, when the captured piece has been removed from the board and replaced by the capturing piece and the player has released the hold from it, c) in castling, when the player has released the hold of the king, or d) in coronation, when the pawn has been removed from the board and the player has released the hold of the coronated piece.
Check, checkmate and stalemate. The king is in check, when it is threatened by the opponent. The check must be made ineffective immediately during the next move of the player, whose king is threatened: a) by moving the king to a location, where it is not threatened, b) by capturing the piece that threatens the king or c) by moving another piece on the threat line. If the check cannot be prevented, then the king is in the checkmate. If the king is not threatened, but it is not possible to move to safe place and there is no other chess piece available to move, then the game becomes a draw and is referred to as stalemate.
Irregularities. All the rules are consistent with traditional chess rules and with the rules of asymmetric, four-handed chess, previously introduced.
The algebraic notation of moves. As in traditional chess, the chess pieces use capital letters as standard symbols: K for the king, Q for the queen, R for a rook, B for a bishop and N for a knight. For a pawn we use in this thesis P. The movements of the pieces are given as directions and lengths. The length unit is the diagonal of a zone of one chess piece. In case of a knight and a pawn, the length of a step is not given because it remains standard. The only exception is the first move of a pawn in case it is two steps. R90,5 means, that a rook moves five steps to the direction of 90°. N80 means, that a knight moves to the direction 80°.

Wall version without the guider

In the wall version, the pieces are attached to the base by magnets or by some other method. Gravity replaces a separate direction indicator, the guider. As a gameboard we use a flat metal plate where a circle shows the game area. The gameboard is set to hang in a vertical position, for example against a wall. On the bottom of chess pieces there are magnets and the pieces are placed on the area as shown in Figure 90. White's pieces are at the bottom and Black's on the top. The position of the pieces is given by the directions which are drawn on the surface of the bottom of the pieces. At the beginning of the game, the orientation is specified by turning the piece in a position where the basic line x (Figure

93) is pointing straight down. This is the case if the free-hanging direction device is on the basic line.

After this the piece is made to move by turning the direction device to the selected direction. The length of the move is read from the scale inserted on the direction device. The length of the direction device move is 10 area measures for bishops, rooks and queens, 3 for knights, and 2 for other pieces. (Kyppö 1997)

6.5 Summary

This chapter has dealt with the issue of how our idea of universal chess was established, and especially the specific instance of it, a kind of three-handed chess we call trichess. This chapter explains the reasoning which led to game, and the game itself. In the beginning of this section we introduced the classic Knight Path’s problem, because it played a central role in the birth of trichess. In addition, this chapter introduces two other games based on the same idea, boardless chess (a chess without a gameboard) and an asymmetrical four-handed chess. Of these two, the boardless chess was also patented. All of these games were developed before the writing commenced for this research and thesis.

As we already discussed in the earlier chapters, it is really hard to invent something totally new. This has also been the case with trichess. One very good example is the 1000-year old Chinese three-player chessgame introduced in Chapter 5.

We add to this summary section a picture, which shows that even Walt Disney's legendary cartoonist, Carl Barks, the creator of the Donald Duck character, thought about this kind of game already in the 1940s (Figure 95). It shows how most good ideas have already been considered by someone else in the past. In science, it is very difficult to find your way to the so-far never visited “last shore”, which is thought even not to exist.

Figure 95 Three-handed chess suggested Carl Barks in 1947

7 EXTENSIONS OF UNIVERSAL CHESS

In this chapter we extend at first (Section 7.1) the square-tiled chess game by changing the size of its board, then (Section 7.2) we extend the size in different dimensions by using cubes and hypercubes, and finally (Section 7.3) we explore models that are not based on square, hypercube or hexagonal tilings.

7.1 Extension of chess on the plane

The first chess extension on the plane generalizes the number of the officer types and, in connection with that, the board size. As the number of the defense zones around the center fortress grows, this will create new types of officers. For example, if we add a third defense zone to the square board model, then the number of officers will rise by two and the gameboard will grow to 144 squares. This is due to the fact that the third defense zone brings in two new officer types. The squares where these officers can move with one step from the central square (fortress) are marked in Figure 96 with blue and red. The number of all officer types is considered to be even, as in the traditional chess, and therefore the length of the back rank with these officers will increase from 8 to 12, and, as a result, the entire size of the board will consist of 12x12 = 144 squares. We name these officers in Section 7.1.1.

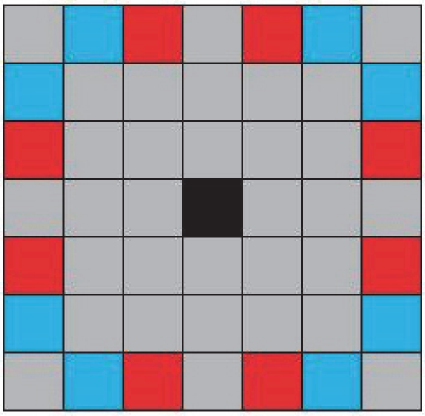


Figure 96 Kn_{32} (blue) and B_{31} (red) on the 3rd defense zone

We give the coordinates for the officers. The central square (fortress) has the coordinates (0,0), and in the coordinate point (x, y), the first parameter x is the distance from the center square vertically upwards in the direction files and y is the distance from the center square center horizontally to the right in the direction of the ranks. The new officers move the same way as the knight, but further away from coordinate (0,0) to squares (3,1) and (3,2). The knigth of traditional chess ends up in square (2,1).

Other changes, besides the growth of the gameboard and officer types, take place in further extensions. However, they follow the same logic, which can be defined on the basis of universal, abstract chess. The same logic also works in traditional chess. On the basis of these rules, it is possible to build a general algorithm and a computer program, which works with all the variants of universal chess. In the following sections, the first models are presented through illustrations of the first universal chess game program[45] (Figures 98, 101-104, 107 and 110 on the right).

7.1.1 Extensions of chess on large square boards

Next we define some new concepts. When the boards are extended, new officers for covering the wider defense zones are created. A game board can also be transformed to multiple dimensions. The chess pieces of officers that move on the squares of the same color may be renamed *hyper bishops*. Officers that move from white to black squares or vice versa are called *hyper knights*. Therefore, on a usual, two-dimensional ,8x8 board, the rook is a hyper knight.

We can determine the moves of hyper knights and hyper bishops on two-dimensional rectangular boards by adding the new officers to every new defense zone level. Hyper knights and hyper bishops are generally called *hyperofficers*. Presumably all officers of the lower defense zones are capable of moving as far as they can in their directions (Figure 97).

45 Made by Mika Vesterholm in 1998

Thus, for example, the knight may move on a 12x12-board and larger boards along its direction as far as it can go, that is, like the bishop in the traditional chess, but to the direction (2n, 1n). The knight's direction on larger boards can be found as line number 3 in Figure 97.

Hyper bishops' and hyper knights' moves are defined on two-dimensional square boards by adding a new officer for each new defense zone. Adding a new defense zone increases the size of the square board. On the traditional 64-square board, the number of defensive zones is two: the first for the rook and the bishop, the second for the knight. The officers, which are on the outermost defense zone, move only one step in every move. On the lower levels of the defense zone, officers can thus move to the specified direction on the game board as long as there are no other pieces on the sector. The directions are given as (x,y)-coordinates, between the lines x = 0 and y = x. Hyperofficers have abbreviations Kn_{xy} and B_{xy}, wherein Kn is the hyper knight, B the hyper bishop and xy the coordinates of one move. The sum of x+y is always even for hyper bishops and odd for hyper knights. The traditional knight in chess is $Kn_{xy,}$ where x = ±1 or ±2, y = ±1 or ±2 and $|x+y|$ = 3. The knight has 8 directions: Kn_{21}, Kn_{12}, $Kn_{2,-1}$, $Kn_{1,-2}$, $Kn_{-1,-2}$, $Kn_{-2,-1}$, $Kn_{-2,1}$ ja $Kn_{-1,2}$. In the following, we list as an example the hyperofficers of the first 6 defense zones. On the list, the traditional rook, bishop and knight are marked as follows.

Table 3 Defense zones I -VI

Rook = $Knight_{10}$, Bishop = $Bishop_{11}$, and Knight = $Knight_{21}$. Moving from the starting square (0,0):

Defense zone I	**$Knight_{10}$: (1,0) ... (n,0),** **$Bishop_{11}$: (1,1) ... (n,n)**
Defense zone II	$Knight_{21}$: (2,1) ... (2n,n)
Defense zone III	$Bishop_{31}$: (3,1) ... (3n,n), $Knight_{32}$: (3,2) ... (3n/2n)
Defense zone IV	$Knight_{41}$: (4,1) ... (4n,n), $Knight_{43}$: (4,3) ... (4n/3,n)
Defense zone V	$Bishop_{51}$: (5,1) ... (5n,n), $Knight_{52}$: (5,2) ... (5n/2,n) $Bishop_{53}$: (5,3) ... (5n/3,n), $Knight_{54}$: (5,4) ... (5n/4,n)
Defense zone VI	$Knight_{61}$: (6,1) ... (6n,n) $Knight_{65}$: (6,5) ... (6n/5,n)

In Figure 97, hyperofficers are marked as follows: 1: $Knight_{10}$, 2: $Bishop_{11}$, 3: $Knight_{21}$, 4: $Bishop_{31}$, 5: $Knight_{32}$, 6: $Knight_{41}$, 7: $Knight_{43}$, 8: $Bishop_{51}$, 9: $Knight_{52}$, 10: $Bishop_{53}$, 11: $Knight_{54}$, 12: $Knight_{61}$, and 13: $Knight_{65}$.

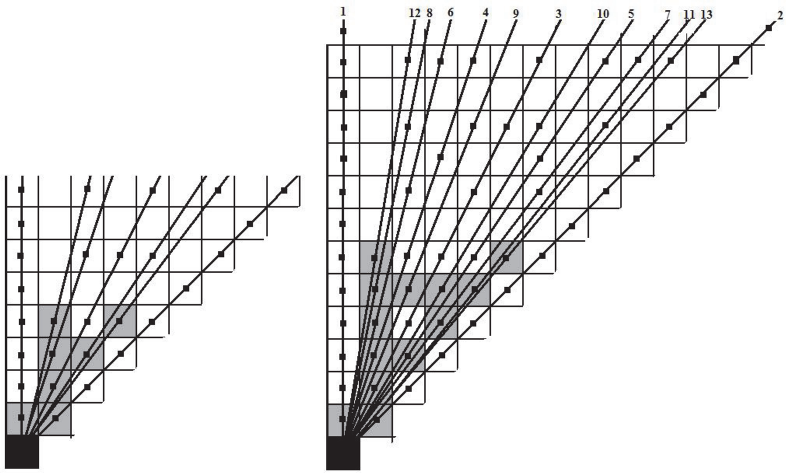


Figure 97 Hyperofficers' moving trajectories, when there are 4 and 6 zones

When we start to build larger game boards by increasing the number of officers and defense zones, we use the following rules. The size of the board increases with the number of new officers, so that $N^2 = (2k + 2)^2$, where k is the number of officers. Officers are always placed on the back row, so that the king and the queen are in the middle as in traditional chess, where the white queen is on a white square and the black queen towards it on the opposite side. On the left and right side of the king and queen are the bishops. The rooks are placed on the corner ends of the last line. The order of all the other officers, from the direction of bishops to that of rooks, is: first the knight, then the new officers from the fourth defense zone, and so on. The last added officers move on the board always only one step at a time; the other officers can move in the same direction as far as there are available squares and no occupied squares. The number of pawn rows grows each time when the size of the board is increased, so that the number of squares that remain unoccupied is always 50%. As a consequence, on an NxN-sized board each player always has (N/2-2)/2 = N/4 - 1 rows with pawns, that is, Nx(N/4 -1) pawns all in all. When we apply this formula to traditional chess, we get, 8x(8/4 - 1) = 8 pawns, as it should be. *We increase the size of the board* by adding the length of the pawn's first move. The rule says that the first step can reach halfway across the board. This rule works also with traditional chess. Thus, in traditional chess, the initial step can have a length of up to 2 squares, on a 12x12 board the first step can be up to 3 squares, on a 16x16 board the first step can be up to 4 squares, and so on, when we increase the size of the board. All other rules are the same as in traditional chess.

In Figures 98 and 99 there are examples of the 12x12 and 16x16 game boards. The left image shows on the 12x12 board, the starting positions of the game pieces, and the hyper knight's ($Knight_{32}$) move is shown by a green square. The right-hand image shows how a hyper bishop ($Bishop_{31}$) has been moved to the opponent's side (purple square on 4th back row), and the colored squares represent the squares where it could move next. The green squares are free, but on red squares there is the opponent's piece, and hence there is an option to capture a piece.

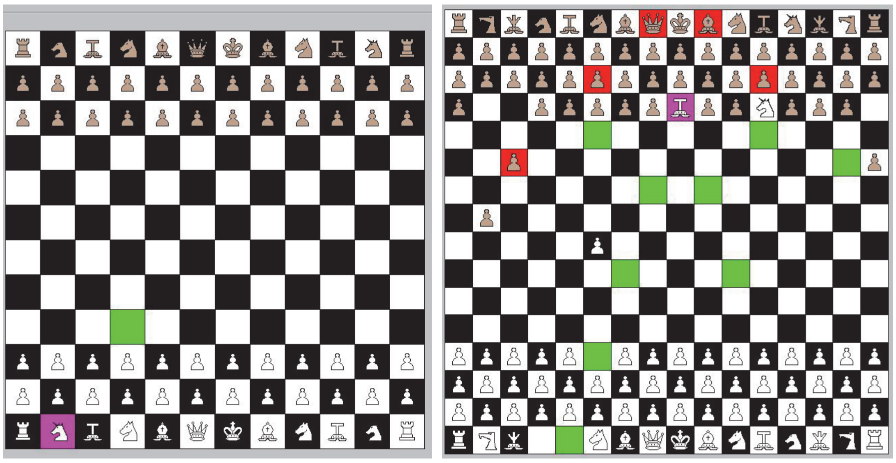


Figure 98 Kn_{32} on 12x12 board (left) and B_{31} on 16x16 board (right)

On the first six defense zone levels, the number of officers will increase: 2, 1, 2, 2, 4 and 2, which is 13 new officers. There is no clear pattern for the number of new officers, but naturally if the number of defense level k is prime, there will be k-1 new officers. This is because the lower-level officers move to same direction, and every defense level number they reach must be a divisible number. If the number of level k is a prime number, then all the officers on that level are of a new type.

Figure 99 A chess game on a 16x16 board

Table 4 lists the number of new officers, hyperknights, and hyperbishops on 31 first defense zones. Except for the 2nd zone, the number of officers grows by even numbers. Up to the 9th zone, the sum of officers is always a prime number. Of the 10 first primes, 9 is included. The prime 17 is not included.The primes are marked as bold just for the interest. The width of the board is always 2k + 2, where k is the sum of the officers.

Table 4 The officer types on the first 31 levels

Zone	New	Knights	Bishops	Sum	Board width
1	2	1	1	**2**	6
2	1	1	0	**3**	8
3	2	1	1	**5**	12
4	2	2	0	**7**	16
5	4	2	2	**11**	24
6	2	2	0	**13**	28
7	6	3	3	**19**	40
8	4	4	0	**23**	48
9	6	3	3	**29**	60
10	4	4	0	33	68
11	10	5	5	**43**	88
12	4	4	0	**47**	96
13	12	6	6	**59**	120
14	6	6	0	65	132
15	8	4	4	**73**	148
16	8	8	0	81	164
17	16	8	8	**97**	196
18	6	6	0	**103**	208
19	18	9	9	121	244
20	8	8	0	129	260
21	12	6	6	141	284
22	10	10	0	**151**	304
23	22	11	11	**173**	348
24	8	8	0	**181**	364
25	20	10	10	201	404
26	12	12	0	213	428
27	18	9	9	231	464
28	12	12	0	243	488
29	28	12	0	**271**	544
30	8	8	0	279	560
31	30	15	15	309	620

In Figure 100, the growth in the number of new officers is shown graphically up to 31 levels, as in Table 4. The blue squares are hyper bishops, and the red squares are hyper knights. The squares on which no new officers will be placed are white. These squares are covered by lower level officers able to move long distances, not just one step. In other words, in the image on the right, each colored square represents a new officer. On the left-side image, the white squares are colored with a black dot added in the centre. The red or blue color tells if the square is on the sector for a hyper knight or a hyper bishop when the starting point is the black square in the corner.

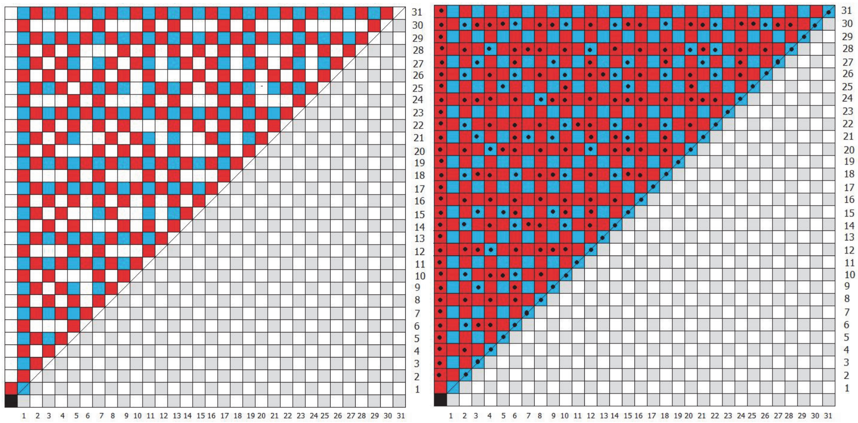


Figure 100 The growth of the number of new officers

7.1.2 Extensions of chess on small square boards

The chess game can be generalized also to a more simple game using the same principles we used in enlarging the game. Earlier, we designed a traditional 8x8 chessboard by using the principles of the universal chess so that two 5x5 defense zones intersected each other (Figure 70, in Section 6.1.2). The central squares were left outside of this intersection. Of these two zones, the inner one defined the rook and the bishop, the second one the knight. When we enlarged the scheme to larger gameboards, then the increase of each new zone brought in new officers. The number of these officers defined the size of the gameboard.

Small chess

When the second defense zone is omitted, we get a 3x3-square grid. When the grids are overlapped with each other, as was done previously, we get a 5x5 chess board. Following the idea of universal chess, there must be two officers of every type (except the queen and the king), and this affects the size of a gameboard. On the first defense zone, there are the two officer types – the rook and the bishop – and hence four officers. Therefore, the size of the board is $(2k + 2)^2$ = 6x6 = 36 squares. On the board, there are now pieces used in traditional chess but with three differences: there are no knights, the number of pawns is six, and the pawns don't have the possibility to take a double step from the initial position (Figure 101). On such a small board, we do not apply the formula N/4 – 1, which was applied in the previous section with large boards, to the number of pawns, because there must be at least one full rank of pawns. For the number ranks for pawns, this formula would give only ½, which is three squares. Because of this, the number of free squares on this board is only 33%. However, the rule which says that the initial step is always to the middle of the board works also on this board.

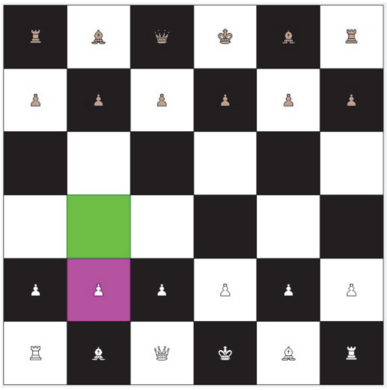


Figure 101 The opening position in the smallchess

This type of game is well suited for educational use when we want teach chess. The rook and the bishop move just like in traditional chess, along long lines, though on the basis of the officers' moving rule, that we mentioned in Section 7.1.1 previously they should move only one step. Long lines are a better choice for making the game more educational and more playable. The duration of a game of this kind is by experience about 15 to 20 minutes. One special note: at the beginning of the game, the White player will have an opportunity for a second move checkmate. This is the case if White opens by moving the pawn that stands in front of the king and Black opens by moving the pawn which stands in front of the bishop of black squares. When the White's queen moves now to the right side of the board, it will be a checkmate.

Primitive chessgames

Chess games can be simplified to make them even smaller than small chess, but then we have to be more flexible with respect to the universal chess model. These smallest games here are referred to as *primitive chess games.* We had three main reasons to begin the design of these kinds of games. These games can be used for educational purposes and for studying learning processes. Because of their small size, it is also possible to identify all game strategies. This is a topic which we are going to discuss more in Chapter 9, the last chapter of this thesis. The survey of the strategies of these simple games might later help us in our research on the strategies of more complex games, as there are some similarities between the structures of the strategy networks.

Next we list the primitive chess games from the largest to the smallest. The simplest primitive chess games are suitable for children up to 3-5 years of age, even for 3-year-old children.

The next game after small chess is the *primitive 5x6 chess game*. Primitive 5x6 chessgame is played on a board the size of which is 5x6 squares. Each player has a king, two rooks, two bishops and five pawns (Figure 102). For these chessmen the rules are the same as in normal chess, except that the pawns do not have the possibility of taking two steps in the beginning, as in small chess. In this game the pawn can be coronated to a rook or a bishop.

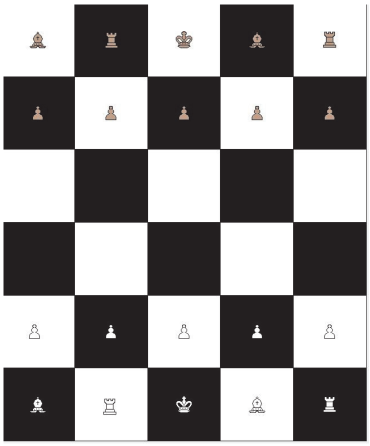


Figure 102 Primitive 5x6 chessgame, the opening position

The primitive 3x6 chessgame is illustrated in Figure 103. Each player has a king, two rooks and three pawns. The rules are same as in chess, except that the pawns do not have the possibility to take two steps in beginning. The pawn can be coronated only to rook.

The following games are even smaller and therefore they must have one very fundamental change in the rules, to make them playable. In these games, the pawns can be coronated to kings, and so one player can have more than one king.

In *the primitive 3x4 chessgame,* each player has a king, and next to it two soldiers, one on both side (Figure 103). The pawns can be coronated to kings if they reach the opponent's side of the board.

In *the Primitive 3x3 chessgame*, both players have three pawns, which can be coronated to kings. Figure 103 shows the initial positions of 3x6, 3x4, and 3x3 primitive chessgames. The green square shows, where the leftmost white pawn will next move.

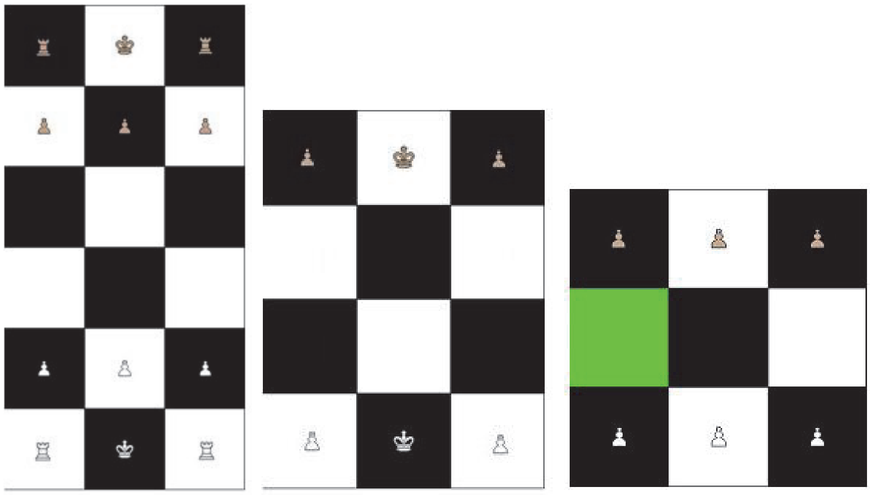


Figure 103 Primitive 3x6, 3x4 and 3x3 chess games, the opening positions

There are some differences with the traditional chess in these very small games. One great difference is that there might be more than one king. Another difference is in how the game ends. The loser is the player who cannot legally move any chesspiece. As in traditional chess, also in these games the king is not allowed to be moved into a square which is threatened. The game ends to a draw when both players can still move their pieces but only by repeating the same patterns endlessly.

Finally, we introduce the most simpliest of these games, *the primitive 2x3 chess game* in which both players have only two pawns, and they can be coronated to kings.

Figure 104 shows, on the left, the initial position (White pawn moves to the green square), and, on the right, a situation where another of the Black player's pawns has been coronated to a king. In that same situation, the White player's piece threatens (red square) one of the Black player's pawns.

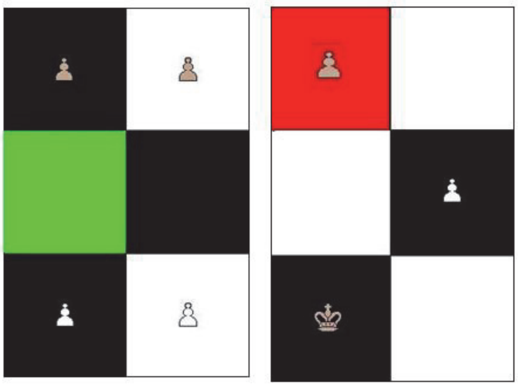


Figure 104 Primitive 2x3 chessgame, two positions

Summary of the rules

Two white and two black chess pieces are placed on a 2x3 chessboard with 6 squares (Figure 104). The pieces move in the same way as the pawn in an ordinary chess game. When reaching the opposite square, any of the chess pieces may get upgraded and become a king. In the following games, the kings follow the same rules as in ordinary chess. Nevertheless, the course of the game differs from ordinary chess in two ways: 1) A player may have more than one king; 2) the player who either loses all the chess pieces or cannot move any pieces any more (by his own move), loses the game. The game ends in a draw when both players may still move their chesspieces, even though the moves are endlessly repeated. The game on a 3x3 board (Figure 103 on the right) has the same rules. The game on a 6x6 chessboard (Figure 101) is like a game on an ordinary chessboard, but with two exceptions: there are no Knights, and the Pawn is not allowed to move two squares on its first move. The primitive 5x6 chessgame and the primitive 3x6 chess game are similar to small chess game. The only difference is that the small chess is smaller than traditional chess and has no Knights. The Primitive 5x6 chessgame is similar, but has no Bishops.

7.2 The Extension to dimension N

When we leave planar games and move to the third dimension, we will find that the tessellations are less regular than in the second dimension. On a two-dimensional plane, there are infinitely many regular polygons, and three of them, a triangle, a square and a hexagon are suitable for regular tiling. As we already discussed in Chapter 3, tessellation and tiling mean the same thing, but in this thesis we use tiling only in dimension two, and tessellation in dimensions greater than two. In the third dimension, there are only a finite number of regular polyhedral, consisting of the five Platonic solids: tetrahedron, cube, octahedron, icosahedron and dodecahedron. Of these, only the cube can form a regular tessellation in three-dimensional space. But just like on a two-dimensional plane, also in three-dimensional space it is possible to find different combinations of regular polyhedra which can tessellate the space. Strictly speaking, there are only five different polyhedra: cube, a combination of two octahedra and two tetrahedra, a combination of a tetrahedron and three truncated tetrahedra, a combination of three truncated octahedra, and a combination of an octahedron and cuboctahedron (Ball & Coxeter, 1987). Cuboctahedron is a polyhedron with eight triangular faces and six square faces. It has 12 corner points, and in each of them two triangles and two squares touch each other. In Figure 105 (Hisarligil 2012), there are two very old drawings of this polyhedron. The one on the left was made by Pappus and the one on the right by Leonardo da Vinci (Hisarligil 2012). In this case, it is not the kind of tessellation we are looking for a three-dimensional game board, where all the polyhedra should be similar and every face should touch the same number (most probably just one) of other faces. Therefore, the only option is to fill the space with cubes, because the cube is the only regular polyhedron that can tessellate three-dimensional space.

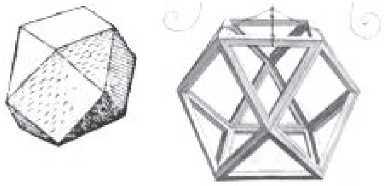


Figure 105 The Cuboctahedron

The defense zones around a cube are made in a similar way as around a square on a plane. When we build the first defense zone to cover one cube, we need a zone of 26 cubes. This is easy to account for. If a cube has on every side, (against every face and every corner) another cube, then there must be altogether 3x3x3 = 27 cubes, and hence around the central cube 26 cubes. We also notice that, in this zone of 26 cubes, there are three officer types, whose movement directions are (1,0,0), (1,1,0) and (1,1,1). The first one of these is the rook, which moves to the direction of faces to six different sectors. The other two are bishop_1 and bishop_2. The latter one we can also refer to as a kind of "knight". Bishop_1 moves toward the directions of 12 edges of the cube, and bishop_2 moves towards the 8 corner points of the cube.

The next dimension, the 4D-model, will include a rook that moves to 8 different directions, and also three other officer types. This 4D-model can be projected on two- and three-dimensional spaces. We are going to discuss this model in Section 7.2.2.

7.2.1 Three-dimensional two-person chess

In this chapter, the concept of *cubic game board* means a rectangular cuboid which consists of cubes. Each cube is a cell of this three-dimensional game board. The moves of hyperbishops and hyperknights are defined on three-dimensional cubic game boards by adding a new officer on each new level of defense zone. These new defense zones naturally increase the size of the cubic game board. When the first defense zone is added around the central cube, the central fortress, we get, as explained in the previous section, a game board of 27 cubes, where the first defense zone includes 26 cubes. There are three different kinds of movement from the center cube: namely, 6 to the direction of faces, 12 to the direction of edges and 8 towards the corners (Figure 106). This means, that already on the first defense zone we need three different officers. We call them at this stage rook (moves towards the faces), bishop (moves towards the edges) and knight (moves towards the corners). Later on, the names of the officers will be clarified and set in a final form. If we build a game board of this kind with one defense zone, using in this three- dimensional model, and the same principle as previously in the two-dimensional chess game, we get a game board of 5x5x5 = 125 cubes. When we follow the idea that the officers, whose number must be even, are on the edge of the game board, as in traditional two-dimensional chess, and the pawns are on the next level, on the first defense zone, protecting the officers, then the size of this three-dimensional game board is at least 6x6x6 = 216 cubes. This is the minimum size, because between the players there must be at least two empty zones. We can make this game board even a bit smaller, but yet playable, if we use a rectangular cuboid instead of a cube. This kind of cuboid has the size of 4x4x6 = 96 cubes. In the game of this 3D smallchess (Figure 107), each player has 4x4x2 = 32 chess pieces, which are placed on the both sides of the cuboid, so that on both back faces of the cuboid, there are 16 officers and in front of them on the first defense zone 16 pawns. Between these two players, there are two empty zones.

One of the basic rules of universal chess is that all officers on the outermost defense zone move only one step in the determined direction. The officers of inner defense zones are allowed to move in their sectors as far as they can if there are no other chess pieces blocking their way. The directions are given as (x,y,z) coordinates. The hyperofficers can be marked as $Knight_{xyz}$ and as $Bishop_{xyz}$, but from now on we use abbreviations where Kn represents a knight and B denotes the bishop, as we did in the two-dimensional case. Index xyz means the first step in the directions x, y and z in the xyz coordinate system. The sum of x+y+z is always even for hyperbishops and odd for hyperknights. For example, the traditional knight's move would be Kn_{210}: two steps forward, one on side and 0 tells that these steps are made on same planar level.

To make three-dimensional smallchess more playable, our rules also allow the possibility of choosing a variant in which the rook (Kn_{100}) and bishop (B_{110}) move along long lines and the third officer called knight (Kn_{111}), on the first defense zone, moves only one step at a time. Figure 107 shows the two different moves of Kn_{111}. The starting cell is of pink color and the destination cells are green.

The following list concerns the hyperofficers of the first two defense zones. The moves of hyperknights and hyperbishops can be determined by vectors on the three-dimensional honeycomb boards.

Defense zone I	Rook = Knight1 = Kn_{100}: from (0,0,0) to (1,0,0) … (n,0,0),
	Bishop = Bishop1 = B_{110}: from (0,0,0) to (1,1,0) … (n,n,0)
	Knight = Knight2 = Kn_{111}: from (0,0,0) to (1,1,1)… (n,n,n)
Defense zone II	Knight3 = Kn_{210}: from (0,0,0) to (2,1,0) … (2n,n.0)
	Bishop2 = B_{211}: from (0,0,0) to (2,1,1) … (2n,n,n)
	Knight4: (2,2,1) = Kn_{221}: from (0,0,0) to (2,2,1)… (2n,2n,n)

The officers of the defense zone II are not included in Figure 108.

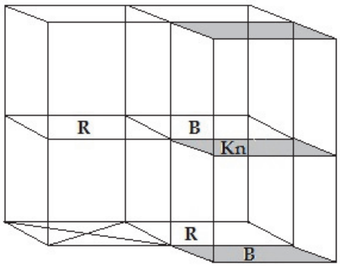


Figure 106 The moving directions of officers in 3D Chess

Figure 107 illustrates the movement directions of the first defense zone. The bottom of the starting cube is marked with a cross. In the other cubes, the letters indicate where the officers (Kn_{100} = rook, B_{110} = bishop and Kn_{111} = knight) can move. Thus the rook moves towards the faces, the bishop toward the edges, and the knight toward the corners. Figure 108 shows the positions of the officers on the side of the board. These are all the officers of the first defense zone. The second-zone officers are not used in this game example.

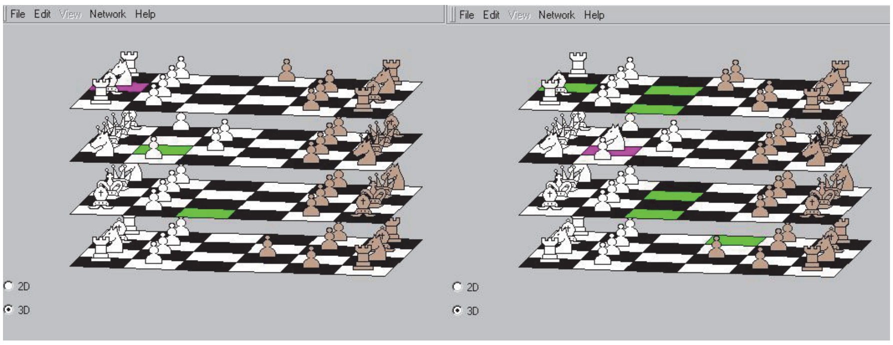


Figure 107 Three-dimensional chess

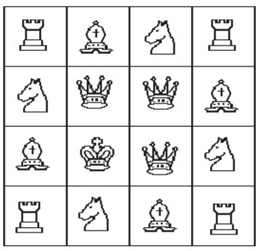


Figure 108 The officers of the 3D board

7.2.2 Four-dimensional two-person chess

The rules of universal chess can be applied to higher dimensions by using a hypercube model. In the previous section, we used cubes with three-dimensional chess.

We define the new types of hyperbishops and hyperknights and their moves on four-dimensional hypercube boards when we add new defense zones. Naturally, the new defense zones mean larger game boards. When we add the first defense zone around the central fortress (central cube), we get a group of 81 hypercubes forming a 3x3x3x3 hypercube board where the first defense zone includes 80 hypercubes. For officers, there are 4 types of moving directions (in 4 dimensions) from the central cube: 8 are towards the cubes that surround the four-dimensional hypercube, 24 towards the two-dimensional faces, 32 towards the edges, and finally 16 towards the corners. This means, that the first defense zone of the four-dimensional hypercube requires four different officers.

This can be explained better by a picture. In Figure 109 on the left, there is a rook inside a three-dimensional cube or cuboid. The figure shows the directions where the rook can move. These possible moving directions consist of the six faces of the cube: left, right, forward, backward, up, and down. The drawing in the center shows the same cube with a rook in a "flattened" form on a plane, i.e. projected to a two-dimensional plane, which lacks the space for a three-dimensional rook. For this reason, the rook is drawn as a gray "shadow image". However, it is possible to find those six directions of movement also from this image. These directions are the four squares or quadrangles around the rook in the center, the square with the grey figure of the rook, and the region outside of the figure. The rightmost drawing in Figure 109 is easier to understand after examining the two other drawings to the left of it. The drawing on the right introduces a four-dimensional hypercube: each cell in the four-dimensional hyper-cuboid board is also this kind of hypercube. Inside the hypercube in this drawing, a "shadow" of a rook shows its position in the cell. The moving directions of the rook are towards the eight three-dimensional cubes that surround the rook. These eight cubes are the one inside the hypercube, the six cubes touching the inner cube and the large cube in the space outside.

The rightmost drawing in Figure 109 also shows the number of faces (24), edges (32) and corners (16) that surround this hypercube. They determine the directions of other officers in the first defense zone.

The same drawing helps us to see the three-dimensional cubes (8) forming this hypercube: the small cube inside, the great cube outside, and the rest six cuboids between these two.

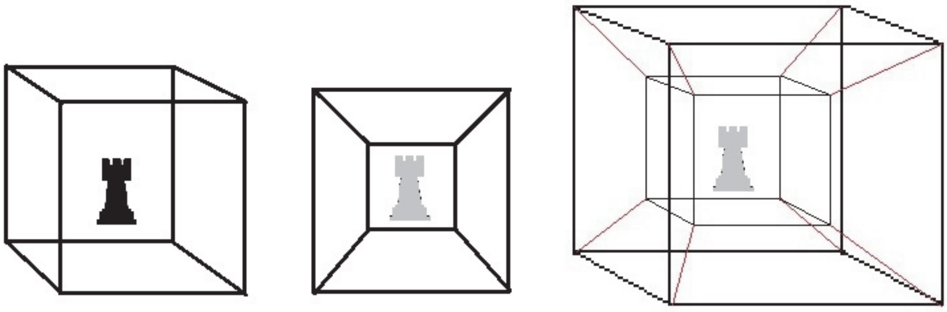


Figure 109 Rook in a cube and in a hypercube

If we build a four-dimensional game board and use in it only the first defense zone while applying the same principles as with the three-dimensional game, then we would get a hypercube board with 5^4 = 625 hypercube cells. If we follow the principle of the three-dimensional game, namely that all the officers are at the back, the pawns are in front, and there are two free zones between the players, then the size of the game board must be at least 6^4 = 1296 cells. However, we can build a smaller, yet playable version of this game if we do as we did in the case of the three-dimensional small chess and leave one side longer than the others. In this way, we get a four-dimensional small chess on a 4x4x4x6 = 384 -cell game board. In the game, each player has 4x4x4x2 = 128 chessmen, which are placed in both of the opposite ends of the hyper-cuboid board in such a way that on the back in the 4x4x4 = 64 cells there are 64 officers and on the next zone 64 pawns. Hence, between the players there are two empty zones (Figure 110, on the left).

As in the previous games, the officers on the outermost defense zone move only one step at a time in a specified direction. The officers on the inner defense zones are able to move as far as they can if there are no other chessmen in the way. The directions are given as (x,y,z,q)-coordinates. The hyperofficers have the abbreviations Kn_{xyzq} and B_{xyzq}, where Kn is a hyperknight, B a hyperbishop and the index xyzq is first step in the xyzq -coordinate system.

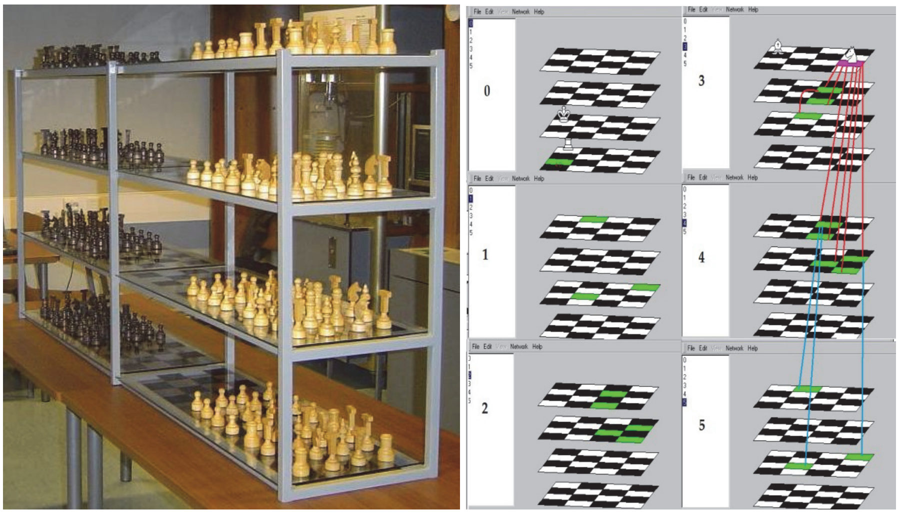


Figure 110 Four-dimensional smallchess

Here the traditional rook is Knight1 and the traditional bishop is Bishop1. In other words, the traditional rook belongs to the group of hyperknights.

Defense zone I

Knight1 = Kn_{1000}: from (0,0,0) to (1,0,0,0) … (n,0,0,0),
Bishop1 = B_{1100}: from (0,0,0) to (1,1,0,0) … (n,n,0,0),
Knight2 = Kn_{1110}: from (0,0,0) to (1,1,1,0) … (n,n,n,0),
Bishop2 = B_{1111}: from (0,0,0) to (1,1,1,1) … (n,n,n,n).

Defense zone II

Knight3 = Kn_{2100}: from (0,0,0) to (2,1,0,0) ... (2n,n,0,0),
Bishop3 = B_{2110}: from (0,0,0) to (2,1,1,0) ... (2n,n,n,0),
Knight4 = Kn_{2111}: from (0,0,0) to (2,1,1,1) ... (2n,n,n,n),
Knight5 = Kn_{2210}: from (0,0,0) to (2,2,1,0) ... (2n,2n,n,0),
Bishop5 = B_{2211}: from (0,0,0) to (2,2,1,1)) ... (2n,2n,n,n),
Knight6 = Kn_{2221}: from (0,0,0) to (2,2,2,1) ... (2n,2n,2n,n).

If we take a look at the left image of Figure 110, there is Kn_{1000} - the traditional rook – that can move to eight different directions on the game board, which consists of six three-dimensional "shelves". Each of these shelves has four 4x4 planar game boards, one upon the other. The directions on a single 4x4 board are forward/backward, left/right, up/down, and also to the corresponding cells in the neighboring “shelves”. A “shelf” actually is a zone, but it is easier to understand the figure when we use this expression.

Figure 111 introduces the officers’ positions in the outermost zones, on the “shelves” of the game. The highest level is on the left. In the figure, there is also an explanation of the game icons, including the previous 3D game.

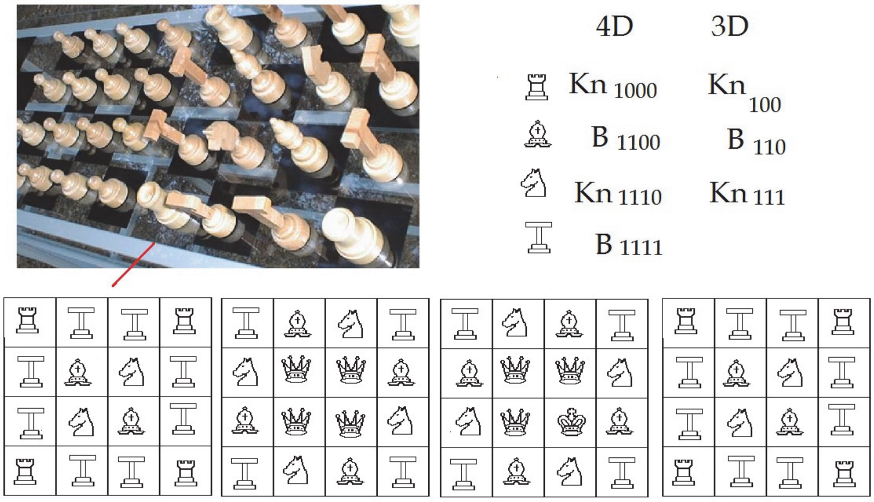


Figure 111 The officers of a 4D board

The image on the right of Figure 112 shows how Kn_{1110} moves. Kn_{1110} is placed on the highest level, the 3rd zone or "shelf" (numbered 3), and the green squares indicate the locations where it can move. The red lines show the squares which can be reached by only one step. These squares are on the same 3rd zone and also on the 4th zone. The blue lines indicate the 5th zone squares, which are in the same direction but need two steps. The two-step moving paths to the zones 0, 1 and 2 are not explicitly shown with blue lines, but these directions to "upwards" are identical to the previous ones. This can be easily seen by comparing the 2nd zone with the 4th zone and the 1st zone with the 5th zone. The green squares are in the same places. In addition, this knight reaches also one square on the 0 zone.

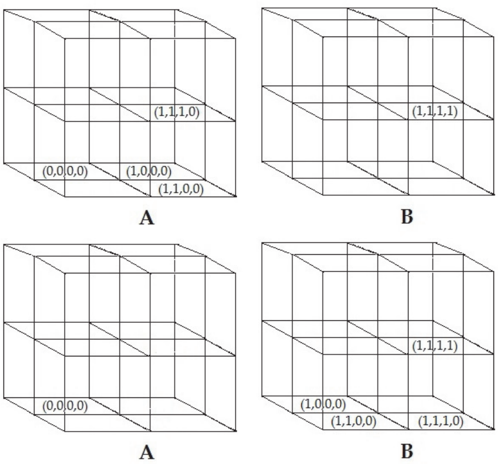


Figure 112 Movement directions of B_{1111} in 4D Chess

To gain a better understanding, we can consider the movements of B_{1111}: first, it can move on the same 4x4-square planar level one step forward or backward, then, on the same planar level, one step left or right, and then one step up or down, to the next planar level on the same zone ("shelf"). After these three moves, there is still one move to another zone (shelf) to the square which has the same coordinate position on that zone. This is just the length of one step of B_{1111}. If B_{1111} moves further than one step, the movement continues in the same vector direction. The one-step movement is shown in Figure 112. It starts from cube (0,0,0,0), and the final cube is (1,1,1,1). The route between these two cubes can go through all the cubes which are marked by a coordinate number in Figure 112. Here, A and B are parts of different zones ("shelves"), both being three-dimensional projections from the four-dimensional game board.

7.2.3 N-dimensional two-person chess

This chapter extends the two-player chess game to the n-dimensional Euclidean space by using the same universal chess principles as in the previous chapters, where we observed this game in the third and fourth dimensions. The moves of hyperbishops and hyperknights are defined on n-dimensional hypercube boards by adding a new officer type on every new defense zone, just as we did in the lower dimensions.

When we add the first defense zone around the central hypercube (fortress), we get a board which consists of 3^n hypercubes. On the first defense zone, we have 3^n-1 hypercubes. There are n different directions from the central hypercube. These directions are (n-1)-dimensional hypercubes that surround the n-dimensional hypercube. The number of them is $2^{n-k}\binom{n}{k}$. So the number of the largest surrounding n-1 hypercubes is $2^{n-(n-1)}\binom{n}{n-1}$ = 2n, and the number of smallest "hypercubes", which in fact are points, is $2^{n-0}\binom{n}{0} = 2^n$. (Coxeter 1973)

Coxeter wrote his formula mainly for an n-dimensional *parallelotope*. An n-dimensional hypercube is a special case of an n-dimensional parallelotope. When a point is moved along a line from an initial to a final position, it traces out a segment. When the segment is translated, but not along its own line, from an initial to a final position, it traces out a parallelogram. Similarly a parallelogram traces out a parallelepiped. The n-dimensional generalization is known as a parallelotope. (Coxeter 1973)

Going back to the formula, for example in the case of the four-dimensional hypercube the numerical figures are $2^{4-1}\binom{4}{1} = 4*2^3 = 32$ (towards edges), $2^{4-2}\binom{4}{2} = 6*2^2 = 24$ (towards faces), $2^{4-3}\binom{4}{3} = 4*2 = 8$ towards cubes) and $2^{4-0}\binom{4}{0} = 2^4 = 16$ (towards corner points). Because there are n different moving directions, the number of different officer types on the first defense zone is also n. On the higher defense zones, it becomes more complicated to estimate the number of officer types, just like in the two-dimensional case (see Table 4).

In the following, we use the symbol #X for the size or *cardinality* (the number of elements) of set X. Because these sets are finite, *size* as a term might be better, but we use cardinality to avoid confusion with other meanings. For example, #{a, b, c} = 3. In the following presentation, #{x} means the number of elements x in the index. For example, if #{1} = 1, 3, it means that in the index there can be either one number 1 or three consecutive numbers 1. So in this case we are given the number of 1s in the index.

We clarify this notation a bit more. Generally # means the number of units. In this case for example #{3} means the number of the units in index 3. If #{3} is for example 2, 5 or 8, it means that there are 2, 5 or 8 consecutive pieces of 3 there. The index tells the position of a chess piece in the xyz…n coordinate system.

To continue with the game, when we apply this notation to the n-dimensional game board, we have the officers on different defense zones as follows:

In defense zone I (the first officers, numbering n), the initial cell is (0,0,0, ..., 0), where the number of zeros is n. The hyperknights are $Kn_{\{\#1\}\ \#\{0\}}$, where #{1} = 1, 3, 5, ... n, #{1} is an odd number and #{0} = n - {#1}. The hyperbishops are $B_{\#\{1\}\#\{0\}}$, where #{1} = 2, 4, 6, … n, #{1} is an even number and #{0} = n - #{1}. So, in this presentation, #{1} gives the number of consecutive 1s and #{0} gives the number of consecutive 0s in the index.

For example, in a four-dimensional game of the previous section, we would represent the hyperknights as $Kn_{\#\{1\}\#\{0\},}$ where #{1} = 1, 3, and #{0} = 4 - 1 = 3 and 4 - 3 = 1. The hyperknights are Kn_{1000} and Kn_{1110}. The first hyperknight in this defense zone has in its index one number ‘one’ and another hyperknight of this zone three ‘ones’. The number of zeros of the first hyperknight of this zone is three because 3 = 4-1, and the second hyperknight has one zero because 1 = 4-3. Similarly, for hyperbishops $B_{\#\{1\}\#\{0\}}$, #{1} = 2, 4 and #{0} = 2 and 0. The hyperbishops are B_{1100} and B_{1111}.

The index of the hyperofficer shows to which cells it is able to move. For example, if #{1} = 5 and #{0} = 2, then hyperknight $Kn_{\#\{1\}\#\{0\}}$ can move from the initial cell (0,0,0,0,0,0,0) to cells (1,1,1,1,1,0,0) ... (n,n,n,n,n,0,0) in a seven-dimensional game of chess. The number of the digits in the coordinate tells the dimension. As an example we can take the previous four-dimensional case where #{1} = 1 and #{0} = 3. Here we got hyperknight Kn_{1000}, which can move from (0,0,0,0) to cells (1,0,0,0) … (n,0,0,0).

When we add the second defense zone, we get a hypercube board, which consists of 5^n hypercubes, where in the second defense zone there are $5^n - 3^n$ hypercubes. So, if we have, for example, a three-dimensional game and the hypercube is an ordinary cube, then on the second defense zone there are $5^3 – 3^3$ = 125 – 27 = 98 cubes. This is even easier to see on a two-dimensional board, where the “hypercube” is a square. The number of squares on the second defense zone is $5^2 – 3^2$ = 25 – 9 = 16 squares. This is easy to see for example in Figure 74.

Thus, on the first defense zone, the number of new, and also the first, officers is n. In traditional two-dimensional chess, where n = 2, the officers on the first defense zone are the rook and the bishop. On the second defense zone, there is one new officer, the knight. In Sections 7.2.1 and 7.2.2 we showed that the number of new officers on the second defense zone is 3 when n = 3, and it is 6 when n = 4. The first defense zone officers, $B_{\#\{1\}\#\{0\}}$ and $Kn_{\{\#1\}\#\{0\},}$ correspond on the second zone to $B_{\#\{2\}\#\{0\}}$ and $Kn_{\#\{2\}\#\{0\}}$. All the new officers are again in the form of $B_{\#\{2\}\#\{1\}\#\{0\}}$ and $Kn_{\#\{2\}\#\{1\}\#\{0\}}$ or $B_{\#\{2\}\#\{1\}}$ and $Kn_{\#\{2\}\#\{1\}.}$ For example, when n = 3, then the "old" officers are Kn_{200} (the rook), B = B_{220} (the bishop) and Kn_{111} (the knight) and the new officers are Kn_{210}, Kn_{221}, and B_{211}. The length of the index is thus n. On the second defense zone, there are no new officers which have in their index only the numbers 2 and 0, or only 2. They are the moving directions for the officers of the first defense zone. The number of index units for 1 and 2 are distributed in such a way that when there is the maximum number n-1 for 2, then there is only 1 index unit for 1. When the number of the index unit 2 is n-2, then the number of the index unit 1 is one or two, ... and when there is only one index unit 2, then the number of the index unit 1 is between 1 and n-1. The number of such combinations is $\binom{n}{2}$. Therefore, the number of new officers at the first level $\binom{n}{1}$ = n and at the second level $\binom{n}{2}$. When we get to the third level, the determination of the number of new officers will become more complicated, as we saw in Section 7.1.1, where n = 2. This would be an interesting issue for further research, but it is not essential for this thesis.

7.3 Some special cases of Universal chess

Because the goal of this research is to find a symmetric model for the n-player game, a couple of counterexamples of universal chess merit consideration. Some of these models might be usable for finding a symmetric, multi-person game, and though some of them are useless for that purpose, they also provide valuable information. Next, we briefly introduce three games: a boardless three-handed chess, a triangular chess and a fullerene chess, based on a dodecahedral model.

7.3.1 Boardless trichess

The rules of this game (Figure 113) are similar to the rules of trichess and the movements of the chesspieces are the same as in boardless chess. Because this game is played on a two-dimensional planar board, symmetry is not possible between more than three players. This lack of symmetry makes it useless to extend this game model to any n-player game.

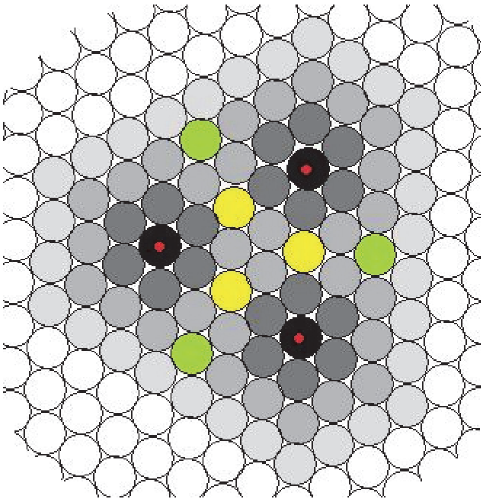


Figure 113 Boardless trichess

7.3.2 Triangular chess

Triangular chess was the third game invented by us when we started our research on the universal chess. In Section 6.2.2, we wrote: "we built Trichess and traditional chess … the third transformation made on the triangular board was quite unique". Despite of the uniqueness of that transformation, the game board can also be converted to three-handed chess.

The game board was created in the same way as in the hexagonal Trichess, as was explained in Section 6.2.1. The difference was in the number of central fortresses, which was six (Figure 76). Of these six fortresses, it is possible to build not only a six-handed, but also a three-handed game. A three-handed game is symmetric unlike a six-handed one, because the game board is planar. In Figure 114, on the left, the officers of the first defense zone are marked by green and blue circles.

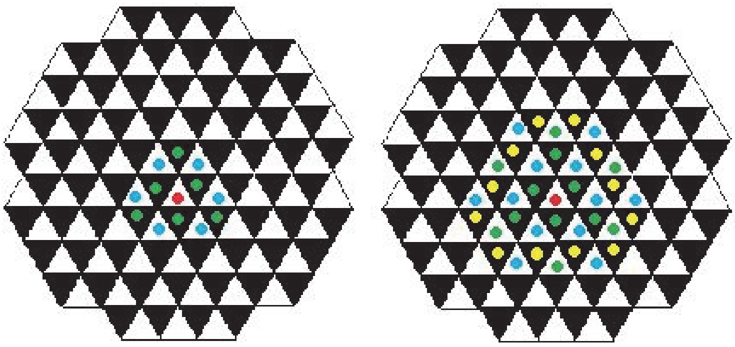


Figure 114 Triangular chess

The drawing on the right of the figure shows also the officers of the second defense zone; the new officers are marked with yellow circles. In this way, we can build a game which has similarities with the Trichess. The green circles are rooks, the blues circles are bishops and the yellow circles are knights. One difference with the Trichess is that the rooks move both towards the corners and the sides of the triangles.

7.3.3 Fullerene chess

The planar games presented above can be played on triangular, quadrilateral and hexagonal cells. In addition to these three, there are also pentagonal cells. We will now examine whether it is possible to embed a chesslike game on pentagons. For that we need a dodecahedron, which consists of pentagons. The surface of a two-dimensional dodecahedron is planar, but on a three-dimensional dodecahedron it is curving. Therefore, it might be possible to position not only three, but four players, in a symmetric configuration with respect to each other. A four-color dodecahedron (on the left side of Figure 115, own drawing) is very small for a game board, and only some simple game can be played on its surface. However, it is possible to enlarge the surface by adding five hexagons around each pentagon. This model has several names, one of them the Goldberg polyhedron G (2,0) (Goldberg 1937), but it is best known as a soccer ball. The dodecahedron itself is often referred to as Goldberg polyhedron G (1,0). In the middle and on the right of Figure 115, we can see soccer balls made by two artists, Milan Mikulastik and Gabriele De Santis. These two soccer balls are art artifacts and not created to be playable games.

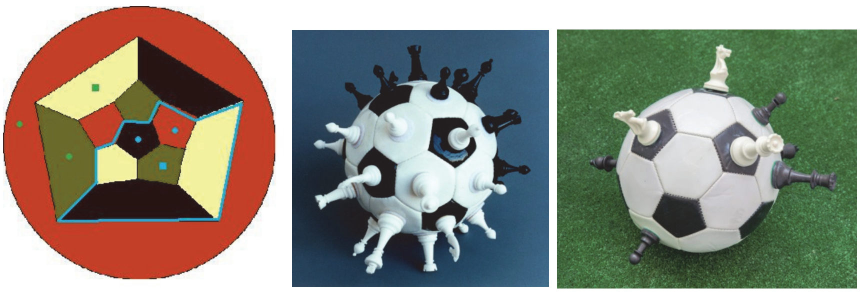


Figure 115 A pentagonal game and “chessballs”

The number of hexagons can be indefinitely increased for they can tile the plane and the 12 pentagons of the dodecahedron close the sphere of hexagons. This was explained in Sections 3.1 and 3.2. Figure 116 shows an example of a polyhedron Goldberg G(7,0) (Goldberg 1937). In chemistry, these structures are known as fullerenes. A Goldberg polyhedron G(7,0) is a fullerene C180, while the smallest fullerene, C20, is the dodecahedron (Wirz et al. 2016). On this kind of fullerene ball it would be easy to embed a chess game with the same kind of rules and movements as in trichess. The pentagons – the red cells in Figure 116 - would be central fortresses, and the chessmen would be placed around them in circles. To get a four-person game, we just choose four of these twelve pentagons.

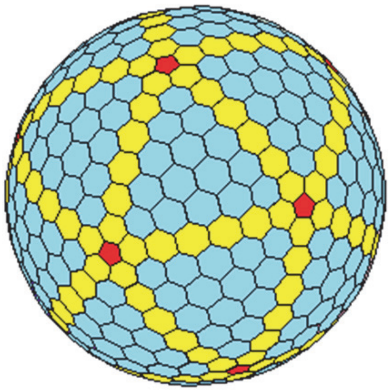


Figure 116 A Goldberg polyhedron G(7,0)[46]

But even if we could select four points symmetrical to each other on the sphere, four-point symmetry doesn't work in the Goldberg polyhedra. The explanation for this is the dodecahedron's independence number, which is three, even though its chromatic number is four. The independence number means that, on a dodecahedron, there cannot be more than three pentagons which do not have a common edge. This can be easily understood by observing the two-dimensional projection of a dodecahedron on the left of Figure 116. In case of the Goldberg polyhedron, this means that it is not possible to select the fourth pentagon in such a way that its distance from the other three would be the same as the mutual distance between those three.

A set $X \subseteq V(G)$ is independent if there are no edges between vertices in X. The cardinality of a largest independent set in G is called the *independence number* of G. A vertex-coloring of a graph G is a function from its vertex-set Vg vertices to a set C whose elements are called colors. A graph is c-colorable if it has a proper vertex-coloring with c or fewer colors. The *chromatic number* of a graph G is the smallest number c of colors such that G is c-colorable. (Gross et al. 2004)

The dual graph of dodecahedron is an icosahedral graph, which has a chromatic number four and an independence number three.

7.4 Summary

In this chapter, as well as in the chapter 6, we examined some extensions of universal chess that we had constructed already before this thesis. Their differ-

[46] https://commons.wikimedia.org/wiki/File:Goldberg_polyhedron_7_0.png#/media/File:Goldberg_polyhedron_7_0.png

ence to the games in this chapter is that the games in this chapter belong to the second, later phase of development. In this chapter, we continued extending the game boards with more officers and with more dimensions. However, these games used only square, cube, and hypercube models as boards. The exception was Boardless chess, where the movements of the chessmen were encoded in the pieces themselves. This game was also patented, but not published in other ways.

In Section 7.1, we first enlarged the board size and increased the number of officers. After that, we developed chesslike games, smaller than chess. All these games followed the basic design of universal chess. In Section 7.2 we continued with higher dimensions. We extended the games, which were built of cubes and hypercubes, to n dimension. The number of players was two. Also two concrete models of these games were made, both of them four-dimensional hypercube chessgames. One of them was placed in the Science Park of the City of Oulu (Figure 111, left).

In Section 7.3 we briefly introduced two unfinished game models and a dodecahedral model. The motive for the dodecahedral model was to allow us to increase the number of players from three to four. However, this failed, and hence we continue in Chapter 8, with a slightly different concept.

8 SYMMETRIC N-PERSON CHESS

In the beginning of this study, our goal was to find out whether it is possible to create a strategy game in which n players are positioned in a symmetric position. On a two-dimensional plane, only two or three points can be completely symmetric with respect to each other. If there are more points, then we have to increase the number of dimensions. When we increase the number of dimensions, we also have to choose the shape of the board, which is probably a polytope of some kind as discussed in Section 3.4.1. In this thesis, we refer to this shape as *the outer structure*. Next, we must find a tessellation of the board. This we call *the inner structure*. The problem is to find a workable solution between these two structures. In the game, the number of players is equal to the number of symmetric points, and the inner structure must be regular to allow it to be generalized and expanded to higher Euclidean dimensions.

In Section 8.1, we present three alternate solutions. One of them, which provides an answer to the symmetry problem and a solution to it, is presented in sub-section 8.2. In Section 8.3, we place a chesslike game on a board which is based on this solution, and, in Section 8.4, we discuss its impact on the rules of chess.

8.1 Different models for symmetric n-person chess

When we construct a board game with pieces, moved on the basis of certain rules, a clear way to define the movements of the pieces is to place the game on a surface where the positions of the pieces and their directions of movement can be clearly defined. This is how the traditional chess board was constructed: it is made up of squares, which act as coordinates. In the game, there are 8x8 = 64 squares on which 16 black and 16 white pieces are placed. There are altogether 32 pieces, covering exactly 50% of the board. There are six kinds of pieces, each with a defined system for moving on this grid-shaped board.

The reason why the chess board has 64 squares is hidden in the darkness of history. Alternate explanations were presented in Section 6.1.2. In the same section, we also presented our own theory, *the territorial model*, about the dawn of chess.

The territorial model can be used quite consistently to explain the moving directions of the chess officers. If we place each officer in turn in the center of the black square in Figure 117, then the rook covers 8 (letter A in the figure) of the 24 squares and the bishop covers 8 as well (the squares with letters B and F in the figure). After these operations, there will still be 8 uncovered squares on the board. As shown in Section 6.1.2, these squares can be covered by the knight. With this hypothesis, the knight's peculiar movements have a logical explanation. This theory gets support from the ancient games of chaturanga (Section 4.2.1), whose bishop, *fil*, moved only one step at a time (Figure 117, piece B) (Eales 1985). A bishop moving like the ancient fil of chaturanga would have covered only 4 squares, namely the ones in the corners, on the territory of the first two defense zones. The 4 corner squares of the inner defense zone were covered by the queen's predecessor, firzan (Figure 117, piece F), which moved only one step diagonally.

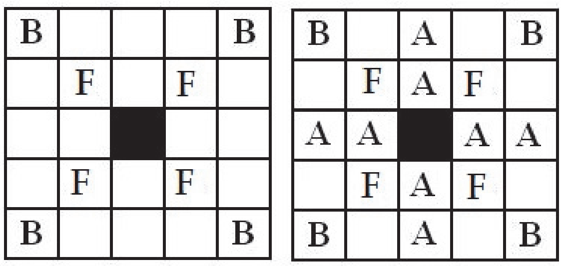


Figure 117 Fil and firzan of the chaturanga

By using this game architecture, it is possible to transfer chess with its rules to a different environment and to different game boards, as was done in Chapters 6 and 7. A plane can be evenly divided by regular polygons exactly in three different ways, as pointed out in Chapter 3: by squares, triangles and hexagons. When the game board is built from hexagons, with the same fortress principle, we get three fortresses as mentioned in Section 6.2.1 (see Figure 81). In case the number of zones in the territory is two, as above, the number of hexagons in the chess board will be 46. However, two officers, the rook and the bishop, are enough to cover a territory of two zones on a board built of hexagons. To introduce a knight to this game model, we must add a third defense zone, which will enlarge the board to 87 hexagons. When the pieces are placed at the corners of the board, we get a symmetric, genuinely three-handed chess game with the same rules as those in traditional chess. Symmetry refers to the fact that all three players are positioned symmetrically in their positions relative to each other. The only asymmetry is in the order the players move their pieces.

If three players, a, b and c, make their moves in order a - b - c - a ..., player a will be in a different position in relation to the other two players when we compare this order with the order of b - a - c - b ... It is also possible to remove this asymmetry if we use randomness or some other rule to change the order during the game.

On the planar plane, it is not possible to construct a symmetrical game board for more than three players. Because of this, we will examine next, what kinds of game boards it is possible to build if the number of dimensions is n> 2. And an even more complicated investigation is required to figure out whether there can be boards of more than three players that can be placed symmetrically in relation to each other.

8.1.1 The Hypercube model

When we begin to build a game board in higher Euclidean dimensions, the problem will be how to tessellate (tile) it. The plane can be tiled in three different ways, but a three-dimensional space only in one way, by using cubes. It will be the same with higher dimensions, where tessellation can be done only by hypercubes. A two-dimensional square has 4 corner points and 4 one-dimensional edges which border a two-dimensional plane. A three-dimensional cube is bordered by 8 points, 12 edges and 6 rectangles. In general, an n-dimensional hypercube is enclosed by k-dimensional hypercubes, where k = 0, ..., n-1 and the number of these hypercubes is $2^{n-k}\binom{n}{k}$, where $\binom{n}{k}$ = n!/k!(n-k)! (Coxeter 1973). For example, a four-dimensional hypercube is bordered by $2^{4-0}\binom{4}{0} = 2^4 = 16$ corner points, $2^{4-1}\binom{4}{1} = 32$ edges, $2^{4-2}\binom{4}{2} = 24$ squares and $2^{4-3}\binom{4}{3} = 8$ cubes.

Table 5 shows the number of k-dimensional hypercubes (k = 0, …, n-1) that border an n-dimensional hypercube.

Table 5 The number k-dimensional cells of n-dimensional hypercubes

k =	**0**	**1**	**2**	**3**	**4**	**5**
n						
1	**2**	**1**				
2	**4**	**4**	**1**			
3	**8**	**12**	**6**	**1**		
4	**16**	**32**	**24**	**8**	**1**	
5	**32**	**80**	**80**	**40**	**10**	
6	**64**	**192**	**240**	**160**	**60**	**12**
…						

It is extremely complicated to place more than two players symmetrically in this hypecube model. To do that in different dimensions requires different approaches, which means that a general rule cannot be found. As an easy example, let's consider a cube. On a square board, there are two players facing each other on opposite sides. In the three-dimensional space, it is possible to place four points in symmetrical positions with respect to each other.

When the number of k-dimensional cells that border an n-dimensional hypercube is divided by the number of the players, the result is an integer number only when the dimension is n-1. However, in this dimension there are only two border cells that are disjoint from each other.

8.1.2 The Tetrahedral Model

Another way to naturally tile a three-dimensional space to build a game board is to use tetrahedra. This method could be compared with using triangles on the plane. A tetrahedron has a structure with which the space can be tiled and hence filled by similar regular objects, therefore tessellating it.

Also Aristotle (384 -322 BC.) came to this conclusion in his book "De Caelo", which when translated in English means "In the heavens". He concluded that a plane can be tiled only by three polygons (a triangle, a square and a hexagon) but the space only by two (a cube and a pyramid). By a pyramid he meant a tetrahedral pyramid, the tetrahedron, in which all faces are equilateral triangles. However his deduction failed. Aristotle's error was definitively disproved 1800 years later by Paulus van Middelburg (1445-1534), a professor of astrology (!) in 1478–1481, in Padua, Italy. (Lagarias & Zong 2012)

In the drawing on the left of Figure 118, there are five tetrahedra connected to each other by one edge.

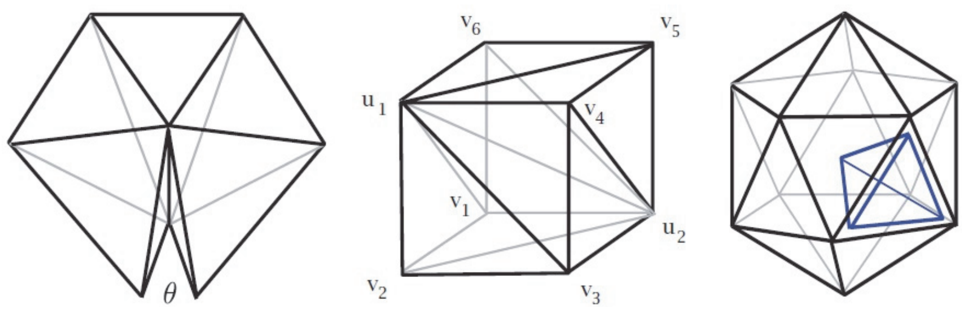


Figure 118 Three different tetrahedral packings [47]

The outermost two tetrahedra do not touch each other and hence leave a gap in between them. The size of this gap is 7° 21', because the angle between the two faces of these is 70° 32′ and 360° - 5*7° 32′ = 7° 21′. So if five tetrahedra are fitted around an edge, then there remains a small gap whose angular meas-

[47] Lagarias & Zong 2012

ure satisfies 360° - 5*7° 32′ = 7° 21′. We conclude that regular tetrahedra cannot fill the space when arranged face-to-face. (Lagarias & Zong 2012).

In the drawing in the middle, 6 tetrahedra are packed inside of the cube. Because the cube tiles the three-dimensional space, it might be possible to use this kind of cube/tetrahedron combination to build a three-dimensional game board. However, its internal structure would be quite complicated and tricky. We seriously considered this design in the early stages of this study.

In the rightmost drawing, 20 tetrahedra have been packed inside the icosahedron. The structure is interesting because more tetrahedra have been packaged in a single space than in the other two models. The problem of this model is that tetrahedra, unlike cubes, do not tile the space. (Lagarias & Zong 2012).

There is also a possibility to form combinations with tetrahedra and other polyhedra. For example, regular tetrahedra do not tessellate space, but with regular octahedra they form a cubic honeycomb, which is a tessellation in three-dimensional space. Also in this model the inside structure is problematic when we want embed a chesslike game into it.

It seems that the tetrahedral model could be used for a three-dimensional game board. However, this model would have a construction which would make it hard to find a clear continuum with the previous two-dimensional games in order to define the rules. In higher dimensions, it will be even more complicated. As a conclusion, it seems that the tetrahedral model is not a solution for our problem.

8.1.3 The Kissing number model

A planar game board consisting of hexagons could be replaced by a game board made of circles, each circle touching six other circles just like a hexagon touches six hexagons (Figure 94). In a three-dimensional Euclidean space, the circles could be replaced by spheres and placed in a tetrahedron, around which four players could be arranged in symmetrical positions with respect to one another. Around one sphere, 12 other spheres can be placed. They are the neighboring cells in a game. However, the generalization to higher dimensions creates a problem: we must deal with the so-called *Kissing number problem*, which we introduced in Chapter 4. On a two-dimensional plane, a circle can be touched by exactly six circles of the same size. This means that its kissing number is 6. Thus, on the plane it is possible to form, by using circles, the same kind of game board as was created by using hexagons. When we expand this to a three-dimensional space, then the number of balls will be exactly 12, as we mentioned earlier. Similarly, a four-dimensional sphere can be covered by using exactly 24 four-dimensional spheres. However, a mathematical proof for this has been very difficult to formulate. The kissing number for three-dimensional Euclidean space was proved as late as 1953, and for four-dimensional space in 2003 (Musin 2003). Apart from these, the exact value of the kissing number was proven only in 8- and 24-dimensional spaces, as mentioned in Chapter 4. These proofs were based on the four-dimensional space case. In all the other dimensions, we know only the largest and the smallest possible value of the kissing number. Because of this, the use of this kind structure in n-simplexes is problematic.

8.1.4 A model based on the multinomial formula

As we have already stated, it is possible to position four players symmetrically with respect to each other in a three-dimensional space. So the game board should be transformed to multiple dimensions if we want to raise the number of players and at same time preserve the symmetry. The four corner vertices of a three-dimensional tetrahedron are in symmetrical positions to each other. If there are five players around a symmetrical game board, we need a four-dimensional tetrahedron, the pentachoron, where all five corner points are located symmetrically in relation to each other.

This model can also be represented by using *the simplex*. A simplex is a generalization of the notion of a triangle or tetrahedron for different dimensions in geometry. A point is 0-simplex, a line is a 1-simplex, a triangle a 2-simplex, a tetrahedron a 3-simplex, a pentachoron a 4-simplex, and a k-dimensional polytope is a k-simplex. When the number of dimensions increases by one, so does also the number of simplex vertices. This means that, if we place n players symmetrically in relation to each other, it can be done with the (n-1)-simplex structure, for which we need an (n-1)-dimensional polytope.

This same thing was considered with the tetrahedral model in Section 8.1.2. In that section, the problem was the inner structure of the model. The next step was to find a better inner structure by using other tools. Finally, we did find one. This happened at a kind of ”eureka” moment one Monday morning during the research project. The solution is a generalization of the binomial formula, known as the multinomial formula, and its geometrical visualization.

8.1.5 Summary

In this Section 8.1, we introduced four different candidate models for a game board in which we could embed a symmetric multiplayer game, a symmetric multiplayer chess-like game in particular. The problem was to find a good inner and outer structure. The outer structure here refers to a model in which more than two players are in the same symmetrical position. Because this is possible only in higher dimensions, we must know the structure in those dimensions. If the number of players is n, we need a polytope the structure of which is known to us at least in Euclidean dimension n-1. The inner structure in turn requires that we can find a regular tessellation in all different dimensions.

When we observe these models, we find that the one which consists of hypercubes has a clear outer structure and a good inner structure as well; thus it is possible to tessellate it also by hypercubes. For this model, it is easy to generalize the movements of chess pieces, and it works well as a two-player game board. However, a multi-player format is very difficult to embed in this model in a reasonable way.

As a second model, we introduced the tetrahedral structure, which is, in a way, an opposite of the hypercube model. The outer structure of a tetrahedron is suitable for a symmetrical multi-player implementation, but the problem is its inner structure.

The kissing number model was investigated as the third model. It has both good outer and inner structure, but its generalization to higher dimensions is not working because exact data about the kissing number doesn't exist in it.

As the fourth and the last model, we presented one based on the multinomial formula. This formula can solve the inner structure of the tetrahedral model, and therefore we continue with it in Section 8.2, which follows.

8.2 Embedding chess in the simplex platform

The arithmetical triangle has a long history from ancient China to the time of Blaise Pascal and his binomial formula. (Edwards 1987)

The *Pascal's arithmetical triangle* can be generalized as three-dimensional "Pascal's" *arithmetical tetrahedron* and as four-dimensional *arithmetical pentachoron. In general, we refer to these extensions as the arithmetic of three- and four-polytopes.*

Multinomial coefficients are coefficients in the multinomial formula, which is an extension of the binomial formula. The coefficients of the binomial formula can be represented by the Pascal's triangle. In same way we can also represent multinomial coefficients by arithmetical n-polytopes.

8.2.1 Pascal's rule

The binomial coefficients of Pascal's triangle are arranged as follows:

1
1, 1
1, 2, 1
1, 3, 3, 1
1, 4, 6, 4, 1
1, 5, 10, 10, 5, 1
...

Above are the first lines of Pascal's triangle, and it is easy to see the well-known Pascal's rule (Edwards 1987). According to which, the sum of two neighboring numbers on the same row can be found below them on the next row. Thus, for example, 1 + 2 = 3 and 4 + 6 = 10 in the pictured triangle.

The binomial coefficients are k-combinations C(p,k) = p!/k!(p-k)!, where p is one of the natural numbers {0, 1, 2, 3, ...} and k ∈ {0, 1, 2, ..., p}. The Pascal's triangle presented by these coefficients is as follows:

C(0,0)
C(1,0) C(1,1)
C(2,0) C(2,1) C(2,2)
C(3,0) C(3,1) C(3,2) C(3,3)
C(4,0) C(4,1) C(4,2), C4,3), C(4,4)
C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)
...

In general, we notice that C(p+1, k+1) = C(p, k) + C(p, k+1), where p (∈ N) is the number of the row and k (k∈ N, k = 0, … p+1) is the k_{th} coefficient of that row.

8.2.2 Three person game embedded on Pascal's triangle

We can actually use Pascal's arithmetical triangle when solving the inner structure problem. Pascal's triangle can also be embedded in a hexagonal model, where within each hexagon there is one binomial coefficient. When the shape of three-handed chess, which we introduced in Chapter 6, is slightly changed, it can be embedded in Pascal's triangle so that the game itself remains unchanged. Figure 119 shows how the traditional chess board is transformed to a hexagonal board and how the hexagonal board is transformed to Pascal's triangle.

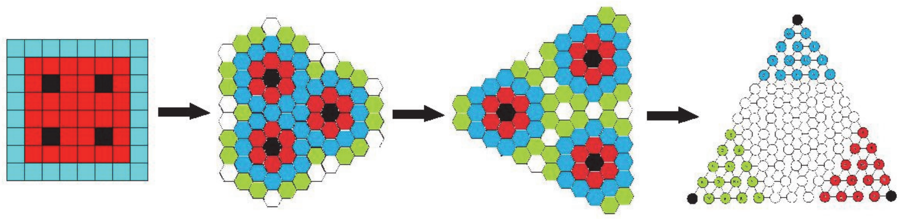


Figure 119 Transforming chess board into Pascal's arithmetical triangle

Since the location of each binomial coefficient of Pascal's triangle can be determined on the basis of the binomial formula, then, respectively, also the locations of the multinomial coefficients of the "Pascal's" n-dimensional tetrahedrons can be exactly defined on the basis of the multinomial formula. Thus, the multinomial coefficients form the coordinates of the game board, and therefore the movements of game pieces can be exactly represented on gaming boards of different dimensions. Next, we examine in more detail the structure of the game boards, starting from the simplest one, which can be accommodated on a two-dimensional plane.

As explained in Section 8.2.1, the coefficients on the row are sums of the two coefficients above them. Because of this, every coefficient number in Pascal's triangle can be determined by its neighbouring numbers: by its neighboring numbers on same row or by the sum of two numbers on the row above. For example, on the fifth row coefficient 6 has two neighbors on the row above. Both numbers have the value of 3 on that row, and 3 + 3 = 6. In addition, the neighbors include both numbers 4 on the same row. On the row below these, the neighbors both have the value of 10, because 4 + 6 = 10 and 6 + 4 = 10. There are six neighbors all in all close together. This is easy to see in Figure 120, where Pascal's triangle is drawn as a triangular graph. These six numbers also form the first defense zone for number 6, if we think of this model as a game board.

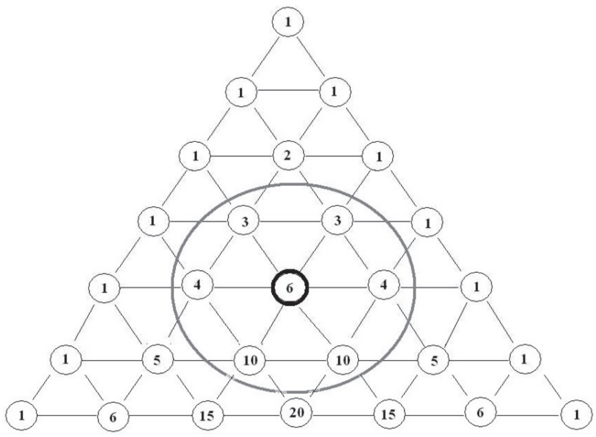


Figure 120 The first defense zone in Pascal's triangle

Represented as a combination, binomial coefficient 6 is marked as C(4,2). Its neighbors are C(3,1) and C(3,2) on the upper row, C(5,2) and C(5,3) on the lower row and both C(4,1) and C(4,3) on the same row. In general, we notice that coefficient C(p,k) has C(p-1,k-1) and C(p-1,k) as neighboring coefficients on the upper row, C(p+1,k) and C(p+1, k+1) on the lower row and both C(p,k-1) and C(p,k+1) on the same row.

Here we have used a shorter notation for this combination. When using a longer notation, C(4,2) becomes C(4,2,2) and C(4,1) becomes C(4,1,3), where the first variable is the sum of the other numbers following it. The added extra number has been obtained by subtracting the second number from the first one. Later on we are going to use only this longer notation. The first rows of Pascal's arithmetical triangle using the longer notation are shown below.

C(0,0,0)
C(1,0,1), C(1,1,0)
C(2,0,2), C(2,1,1), C(2,2,0)
C(3,0,3), C(3,1,2), C(3,2,1), C(3,3,0)
C(4,0,4), C(4,1,3), C(4,2,2), C(4,3,1), C 4,4,0)
C(5,0,5), C(5,1,4), C(5,2,3), C(5,3,2), C(5,4,1), C(5,5,0)
...

8.2.3 Multinomial formula and n-simplex

Just as the binomial coefficients can be represented by Pascal's triangle, we can represent the trinomial coefficients by “Pascal's pyramid", which is also known as “Pascal's tetrahedron”. As a geometrical object, the triangle is a 2-simplex and tetrahedron a 3-simplex. In the same way, quadratic coefficients can be represented as a four-dimensional simplex, the 4-simplex, which is also known as pentachoron or pentatope. In general, the multinomial coefficients can be represented in an n-dimensional simplex model.

We can determine also the other coefficients in the n-simplex game board model by using the sums of higher level coefficients in the way we did in Pascal’s triangle. Every *level* of a simplex is one dimension smaller than the simplex itself, and hence the levels of an n-simplex are (n-1)-dimensional. In our two-dimensional Pascal’s triangle, a level is a one-dimensional row. In Pascal’s triangle, the value of any coefficient or example is the sum of the two coefficients on the row above it; in same way, the value of any coefficient of Pascal's tetrahedron is the sum of the three coefficients on the level above it. Similarly, in the pentachoron every coefficient is the sum of the four coefficients on the level immediately above it.

Next we will examine in more detail how this method works in dimensions three, four and five.

8.2.4 The Symmetric 4-players model on 3-simplex board

The coefficients of the trinomial formula are $C(p,k_1,k_2,k_3)$, where $k_1 + k_2 + k_3 = p$. For example the trinomial coefficients with exponents 0-3 are as follows:

Table 6 The first four trinoms

Trinom	Coefficients
$(x+y+z)^0 = 1$	1
$(x+y+z)^1 = x+y+z$	1, 1, 1
$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$	1, 1, 1, 2, 2, 2
$(x+y+z)^3 = x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3yx^2 + 3yz^2 + 3zx^2 + 3zy^2 + 6xyz$	1, 1, 1, 3, 3, 3, 3, 3, 3,6

Combination $C(p,k_1,k_2,k_3)$	Coefficients
$x+y+z)^0 = 1$	C(0,0,0,0)
$(x+y+z)^1 = x+y+z$	C(1,1,0,0), C(1,0,1,0), C(1,0,0,1)
$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$	C(2,2,0,0), C(2,0,2,0), C(2,0,0,2), C(2,1,1,0), C(2,1,0,1), C(2,0,1,1)
$(x+y+z)^3 = x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3yx^2 + 3yz^2 + 3zx^2 + 3zy^2 + 6xyz$	C(3,3,0,0), C(3,0,3,0), C(3,0,0,3), C(3,2,1,0), C(3,2,0,1), C(3,1,2,0), C(3,0,2,1), C(3,1,0,2), C(3,0,1,2), C(3,1,1,1)

The trinomial coefficients can be placed in a 3-simplex (tetrahedron) in exact positions as shown in Figure 121. The top vertex is a trinom $(x+y+z)^0$, which has the value of 1. Under the top vertex on the next level, the exponent of the trinom is 1, and we get three coefficients, which all have the value of 1 (see the second equation on Table 6). This value labels the three nodes on this level (see the graph of three nodes in Figure 121). On the third level, which is numbered level 2 (the numbering of the levels starting from 0), the exponent of the trinom is 2 and there are six coefficients, three of which have the value of 1 and another three which have the value of 2. These coefficients are placed in such a way that those with the value of 1 are in corner nodes and those with the value of 2 in the center (see the graph of six nodes in Figure 121). We also make an observation that on the sides of this triangle, which is a slice of a tetrahedron, there is the third row, 1, 2, 1, of the Pascal's arithmetical triangle. When we examine Figure 121, we see that a two-dimensional Pascal's arithmetical triangle can be found on all the faces of this tetrahedron.

Generally speaking, we place the coefficients on the triangle of the level k in the tetrahedron in the following way. On the corners we place the coefficients which have the value of 1 and a variable which has the k exponent. On the sides of the triangle, we place the coefficients which have the variables with exponent k-1. We will continue by placing coefficients on the sides of the triangle in accordance with the highest (decreasing) exponent of the variables. If two coefficients have the greatest exponent of equal size, then the order is determined by the next highest exponent. The trinom has three variables, and all of them have exponents. This is easier to figure out with the help of the last equation in Table 6 and by comparing it with the triangular graph of ten nodes in Figure 121.

When all the sides of one triangular graph have the coefficient values, then we move to the next inner triangle on the same level and repeat the same process. We continue until we are at the center of the triangle. There the exponents of the variables are at their smallest, whereas the coefficients are at their greatest. In Figure 121, the positions of the trinomial coefficients are shown for levels 0–6 in the tetrahedron. These positions are more exactly determined by combinations in Section 8.2.6.

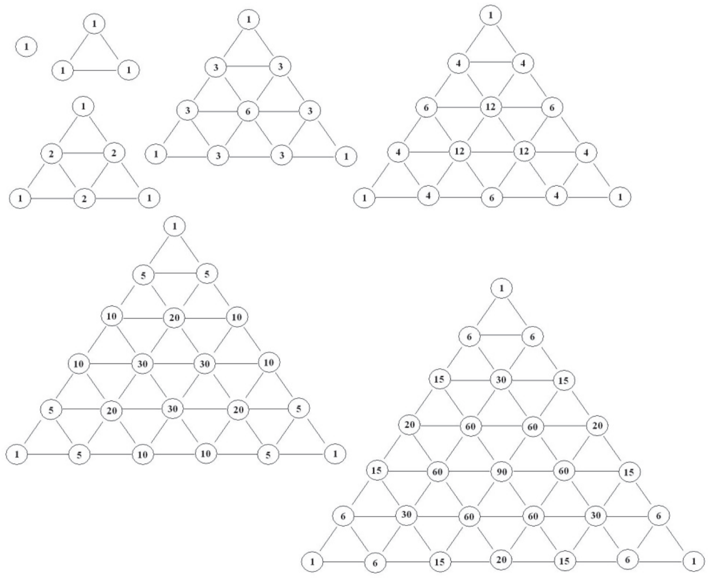


Figure 121 The coefficients of the trinomial formula in Pascal's Tetrahedron

If we consider the relationship between the coefficients in this three-dimensional model in the same way as in the two-dimensional case, we will find that the coefficients on the lower level can also be obtained from the sum of the coefficients on the upper level. In the two-dimensional model, the sum was that of two coefficients, but, in the three-dimensional model, this sum is the sum of three coefficients. This is shown in Figure 122. The upper level is marked by red nodes, and at the center of three red nodes (the triangles with two nodes above and one below) there is the sum of the numbers of these nodes in a white node. In the tetrahedron, the white nodes are on a level that is lower than that of the red nodes. The 0-nodes on the left outside the triangles were added to the figure to show that the same method works also with edge nodes. All these sums consist of triangles which have two red nodes above and one below. The highest triangle at the top is the one with nodes 0, 0, 1.

The triangles, which have two red nodes below and one on top, do not matter in this research. However, we may note that the sums of these triangles, starting from the top and from right to left are 1, 1, 3, 7, 3, 3, 12, 12, 3, 1, 7, 12, 7 and 1. Also these sums might have significance in some other contexts.

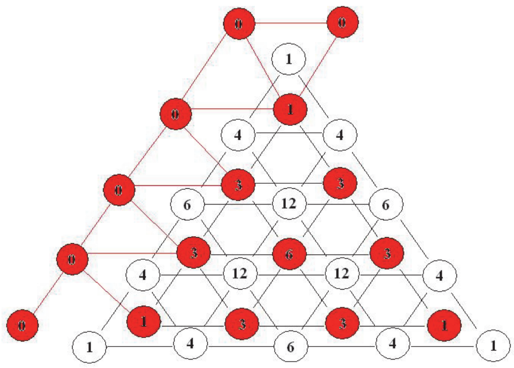


Figure 122 The summation of coefficients in Pascal’s Tetrahedron

Also the neighbors in the three-dimensional model can be defined by this same method. We refer to Figure 123, which shows the inner sums on some of the first seven levels. As in Pascal's triangle, and also in (Pascal's) arithmetical tetrahedron neighboring coefficients can be defined for a coefficient, depending on whether the coefficient is a part of the sum of other coefficients or if its neighboring nodes on the same level. For example, the node on level six with coefficient 30 has upper level neighbors 6, 12 and 12, because 6 + 12 + 12 = 30. On the lower level, the neighbors are 60, 60 and 90, because 30 + 10 + 20 = 60 and 30 + 30 + 30 = 90. Here 30 is one of the factors in these sums. The adjacent neighbors on the same level are 10, 20, 30, 30, 20 and 10. So the total number of the neighbors is 12.

As combinations, the neighbors of the trinom coefficient 30 = 5!/2!2!1! = C(5,2,2,1) can be defined more exactly: on the upper level, 6 = C(4,2,2,0), 12 = C(4,2,1,1), and 12 = C(4,1,2,1); on the lower level, 90 = C(6,2,2,2), 60 = C(6,2,3,1), and 60 = C(6,3,2,1); on the same level, 10 = C(5,3,2,0), 10 = C(5,0,2,3), 20 = C(5,3,1,1), 20 = C(5,1,3,1), 30 = C(5,2,1,2) and 30 = C(5,1,2,2). See Figure 123.

In general, the neighbours of the coefficient $C(p,k_1,k_2,k_3)$ are on the upper level $C(p-1,k_1,k_2,k_3-1)$, $C(p-1,k_1,k_2-1,k_3)$ and $C(p-1,k_1-1,k_2,k_3)$, on the lower level $C(p+1,k_1,k_2,k_3+1)$, $C(p+1,k_1,k_2+1,k_3)$ and $C(p+1,k_1+1,k_2,k_3)$, and on the same level $C(p,k_1+1,k_2,k_3-1)$, $C(p,k_1-2,k_2,k_3+2)$, $C(p,k_1+1,k_2-1,k_3)$, $C(p,k_1-1,k_2+1,k_3)$, $C(p,k_1,k_2-1,k_3+1)$ and $C(p,k_1-1,k_2,k_3+1)$.

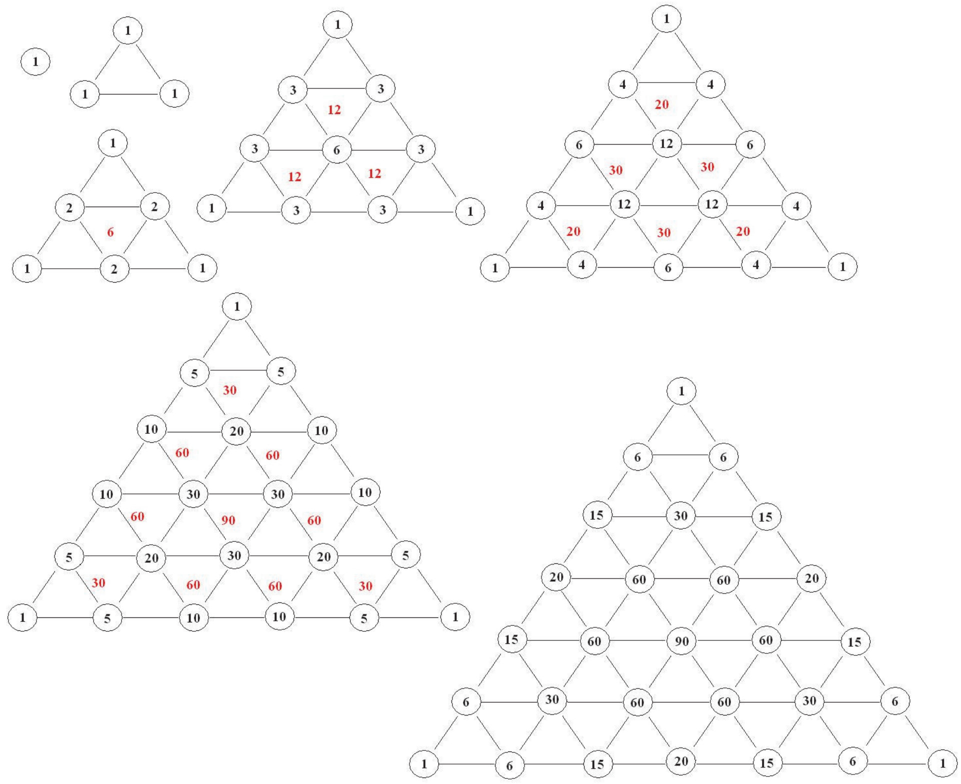


Figure 123 Summation on levels 1-6 in Pascal's Tetrahedron

8.2.5 The Game platform and variety of hyperofficers

A symmetrical game platform for four players can also be implemented concretely in the real world by a game board shown in Figure 124. The levels from left to right run from 1 to 9.

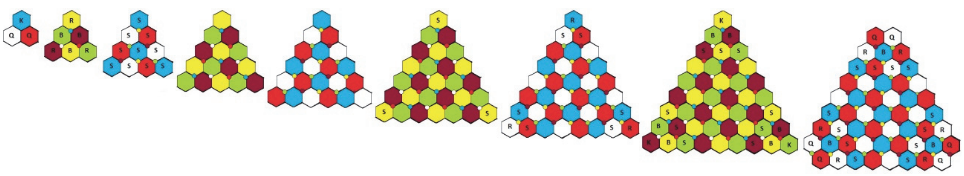


Figure 124 A Game board model for four-person symmetric chess game

As it can be seen, we use six different colors to get the neighboring cells colored by different colors. A tetrahedral game board is six-colorable, which can be easily verified by Figure 124. As an example we can take the node with coefficient number 6 in the center on level 3. If that node is marked by color A, then its neighboring nodes on the same level, level 3, can be colored by two colors, B and C, which makes a chain B - C - B - C - B – C around the central node. Two colors is enough because the number of nodes is even. If it were odd, then we would need three colors. The neighboring nodes of node 6 on the lower level, level 4, are the three nodes with coefficient number 12. Each of these three nodes is also a neighbor to the two (with coefficient 3) adjoining nodes of 6. Therefore, none of them can have any of the colors A, B or C. Since the nodes with the value of 12 are also neighbors to each other and have different colors, there is a need for additional colors D, E and F. Since for one level, three colors are sufficient, no more colors are required on that level. Because level 2 above level 3 is not in contact with level 4, these two levels can use the same colors.

In Chapter 7, we generalized the officers into two types, hyperknights and hyperbishops, depending on whether they stay or not on the same colored cells during their moves. Hyperknights and hyperbishops are generally called hyperofficers. On the extended, two-dimensional, hexagonal board and on multi-dimensional simplex boards, the number of colors grows, which means that hyperknights can move to cells of several different colors. So there will be different kinds of combinations. This is not essential to this thesis, nevertheless it provides an interesting field for future research.

8.2.6 Defining the coordinates on a simplex board

In the following, we explore how the position of an individual vertex is determined on a simplex board in different dimensions.

On a two-dimensional (2-simplex) board embedded on the Pascal's arithmetical triangle, the combinations were placed as follows:

C(0,0,0)
C(1,0,1), C(1,1,0)
C(2,0,2), C(2,1,1), C(2,2,0)
C(3,0,3), C(3,1,2), C(3,2,1), C(3,3,0)
...

In combination $C(p,k_1,k_2)$, variable p indicates the row. Parameter k_1 tells the location of the combination, starting from the left, and k_2, starting from the right. The locations are k_1+1 and k_2+1. So, for example, C(3,1,2) = 3 is located in the Pascal's triangle on the 3rd row as the second number starting from left and as the third number starting from right. Naturally, this numbering system differs from the system we used in the earlier chapters with trichess, but it can be easily converted to it. This numbering is derived from the arithmetic triangle and Pascal's binomial formula, while in trichess we wanted to find a numbering system similar to the one used in the traditional chess.

Next we refer Figure 125. In the three-dimensional model, which can be referred to as 3-simplex or a tetrahedron, the combinations have the form $C(p,k_1,k_2,k_3)$. The positions of combinations $C(p,k_1,k_2,k_3)$ are determined such that in one corner of the tetrahedron all the variables p, k_1, k_2 and k_3 get the value of 0 and so C(0,0,0,0) = 1.

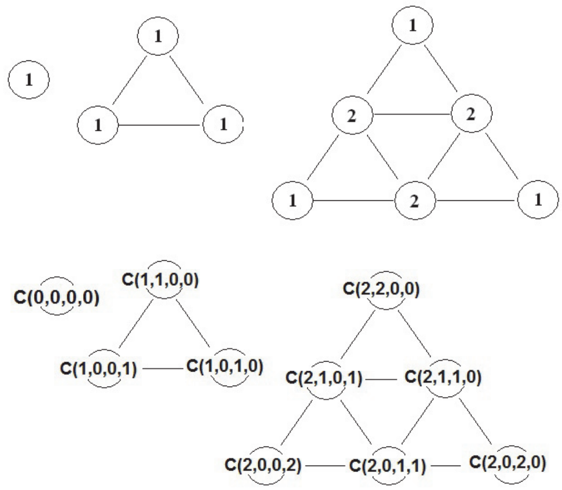


Figure 125 Position of coefficients in the trinomial formula

To make this numbering system easier to understand, let us assume that C(0,0,0,0) is the top vertex of a tetrahedron, a three-sided pyramid. After this, we continue onto the next levels, and next triangles, under the top vertex. The value of p increases from 0 to 2 during the process. On the lowest triangle level, the combinations are in the form of C(2,k_1,k_2,k_3). In the combinations on each side of this triangle, one of the k_1, k_2 or k_3 variables gets a value of 0, and the same variable gets a value 2 in the corner vertex opposite to the side. In the vertices that are between the side and the corner vertex, the variable value is increased from 0 to 2. This is illustrated in Figure 125. The coefficients of the trinomial formula are placed in "Pascal's" arithmetical tetrahedron. In the figure, k_3 gets the value of 0 on the right, k_2 is 0 on the left and k_1 is 0 on the bottom. On this lowest level triangle (on the right in Figure 125), the values of the variables in combinations increase from 1 to 2 when moving from a side towards the corner vertex.

In the four-dimensional model (Figure 126), which can be referred to as a four-simplex or a pentachoron, the combinations have the form C(p,k_1,k_2,k_3,k_4). In one of the five corners of the pentachoron, we place a vertex with combination C(0,0,0,0,0) = 1. This top vertex determines also the 0 level, which is represented by the first parameter in the combination. The next levels of this pentachoron consist of tetrahedra, of which the smallest is on level 1. This tetrahedron has only four vertices, which are its corner vertices. Their coordinates are C(1,1,0,0,0) = 1, C(1,0,1,0,0) = 1, C(1,0,0,1,0) = 1 and C(1,0,0,0,1) = 1.

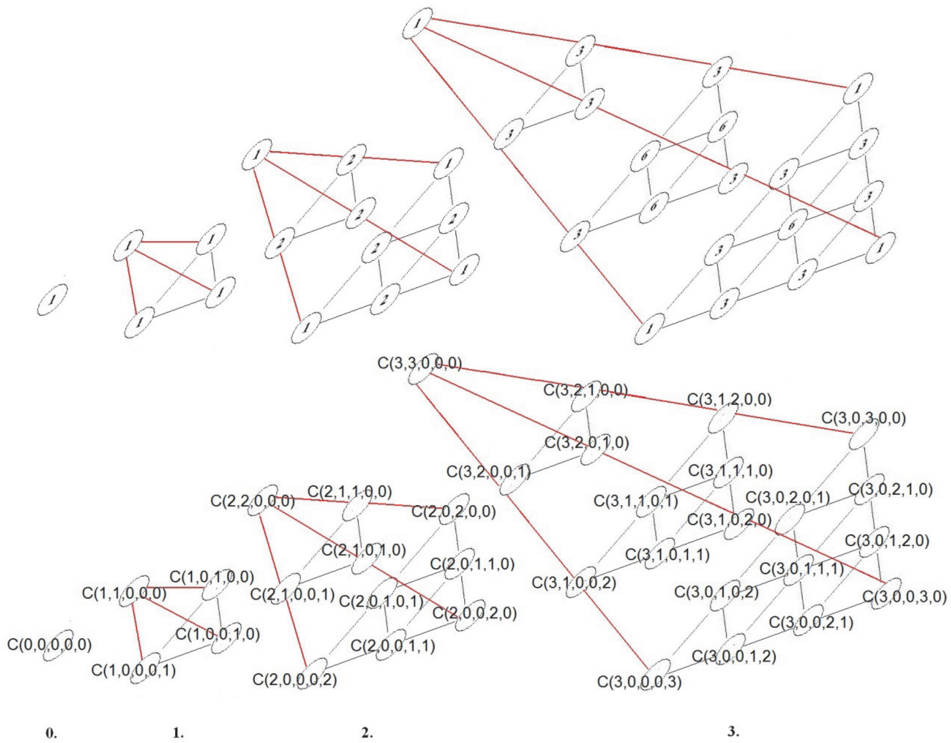


Figure 126 The position of coefficients in quadrinomial formula

Generally, in an n-dimensional simplex, where combinations are of the form of C(p,k_1,k_2,…,k_n) and $p = \sum_{i=1}^{n} k_i$, the location for each combination is determined in the following way.

In one of the corners, there is always a combination of C(0,0,0,0 , ... 0) when the number of variables (here zeros) is n + 1. On the opposite side of this combination vertex, there is one of the border levels of the n-simplex. This border level is an (n-1)-dimensional simplex, and in its combination vertices p = s - 1. Here s is the "size" of the simplex, which is the number of vertices on its edges.

The first variable p of C(p,k_1,k_2,…,k_n), $p \in [0, s-1]$, indicates on which (n-1)-level in the n-simplex the combination is located when the starting vertex is the combination of C(0,0,0,0, ... 0). Parameter k_1 indicates the distance from the (n-2)-simplex, starting from one border level, where $k_1 = 0$. Parameter k_2 indicates the distance from another border level, where $k_2 = 0$. And so on, until parameter k_n indicates the distance from some border level where $k_n = 0$. For the parameters, $k_i \in [0, s]$ and $i \in [0, n]$.

Parameter p differs from the other parameters: its starting level (zero level) is a corner vertex, whereas the other parameters use some of the (n-1)-dimensional border simplexes of the n-simplex as their starting level. The reason for this is illustrative, but, in algorithms, p can always be replaced by parameter $k_0 = s - p$. The starting level of k_0 also starts from an (n-1)-simplex on the border.

Figure 126 clarifies the method above and shows how the combinations are located in the pentachoron. In the four-dimensional 4-simplex (pentachoron), where the combinations have the form of $C(p,k_1,k_2,k_3,k_4)$, the location of each combination is determined in the following way.

One of the vertices of the simplex always has the form of C (0,0,0,0,0), where the number of variables, in this case zeros, is 4 + 1 = 5. On the opposite side of this corner vertex, there is a three-dimensional border level, a tetrahedron, where $p = s - 1$. As mentioned, here the size of s is that of the size of the simplex, which is the number of vertices on its edges. In Figure 126, parameter p is 4.

The first variable, $p \in [0, 3]$, of combination $C(p,k_1,k_2,k_3,k_4)$, determines on which three-dimensional tetrahedral level in the pentachoron (4-simplex) the combination is located, when the staring vertex is combination C(0,0,0,0,0). Parameter k_1 tells the distance to one 2-simplex on one of the border levels, in this case, to one triangular face inside the tetrahedron, where $k_1 = 0$. Parameter k_2 indicates the distance to another 2-simplex, which is another triangular border face. In the same way, k_3 and k_4 indicate the distances to the triangles, where their value is 0.

8.2.7 Symmetric 5-players model on 4-simplex board

We continue the review started in Section 8.2.4 by taking yet one more dimension under consideration. The object is now the four-dimensional pentachoron. In a pentachoron (a 4-simplex), the quadrinomial coefficients are placed so that each level is a tetrahedron, and the size of the tetrahedra increases on higher levels. Figure 127 presents the first four levels. The level-2 tetrahedron is shown separately as a three-dimensional model as well. It is the rightmost graph with four vertices. Level 4 is on the right in the figure.

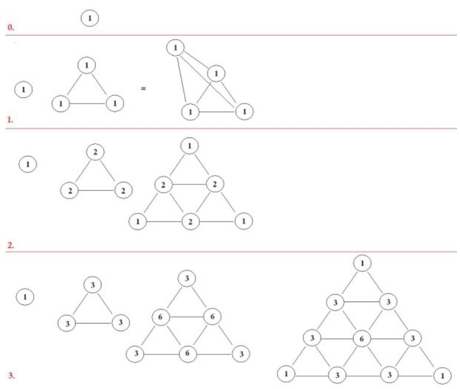


Figure 127 Levels 1-4 on Pascal's Pentachoron

Also in this dimension, the coefficients are derived from the sums of the previous levels. On the 2-dimensional level, the sum consists of two coefficients on the upper row; and in the 3-dimensional space it consists of the coefficients of the corner vertices of a triangle with three vertices on the upper level. In a 4-dimensional pentachoron, the sum is derived from the coefficients in the corner vertices of an upper level tetrahedron with four vertices. This is illustrated in Figure 128, where a part of the fourth level tetrahedron of the pentachoron is on the left and a part of the fifth level tetrahedron is on the right. Colors show which fourth level coefficients are factors of the fifth level coefficients with a value of 60.

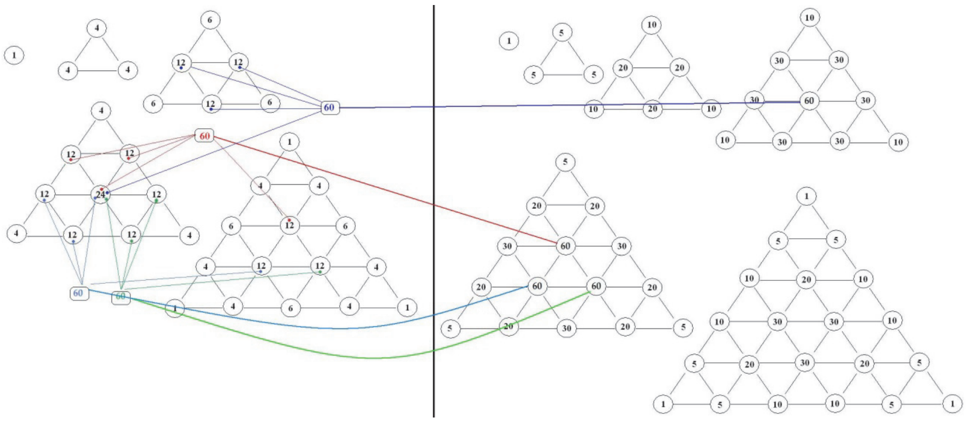


Figure 128 Summation of coefficients on level 5 in “Pascal’s” pentachoron

In the pentachoron, the number of neighboring coefficients on the same level is 12, which is exactly same as the number of all the neighboring coefficients in a tetrahedron. This means that in the 4-dimensional pentachoron, the total number of neighboring coefficients is 4 + 12 + 4 = 20. Figure 129 shows the positions of the players in the 4-simplex model. Four of the players have their starting positions in the corners of the three big tetrahedra on the right: Player 1 is on the top and players 2, 3 and 5 on the bottom corners. The fifth player (number 4) has the three small black tetrahedra on the left.

In this model, the exact places of the chessmen have not yet been determined, but their locations are on three levels in the corners. The corner vertex doesn’t form a part of the game board. Thus there is one free level between all the players.

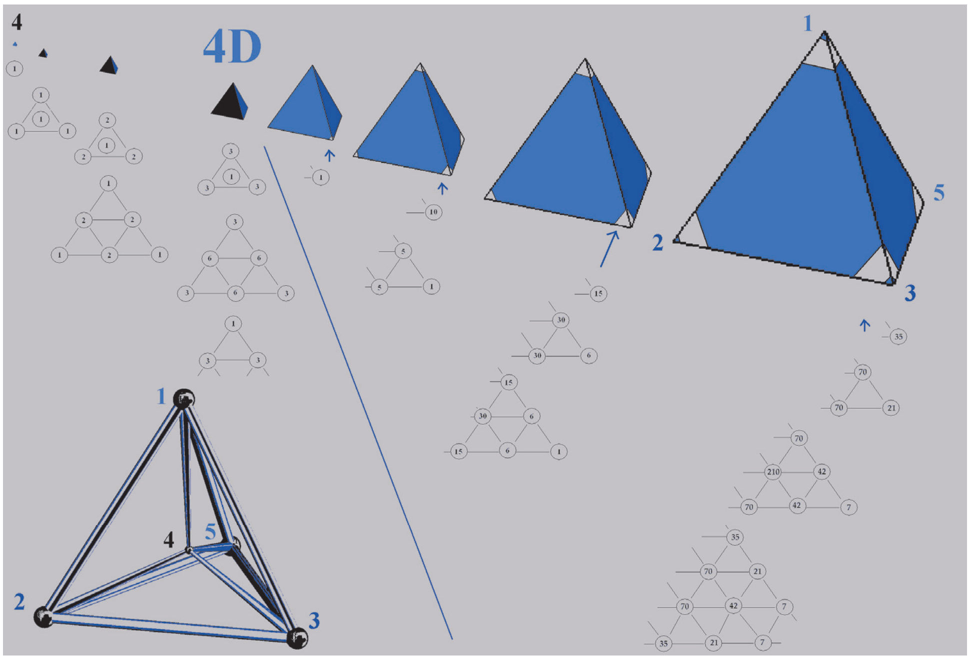


Figure 129 Five players in the corners of a pentachoron

The previous four-dimensional projection figure might be easier to understand with a description similar to the two-dimensional Pascal's triangle model (a projection to one-dimensional space from two-dimensional space in Figure 130). Compare this with Figure 119, where the same model shown as two-dimensional is on the right.

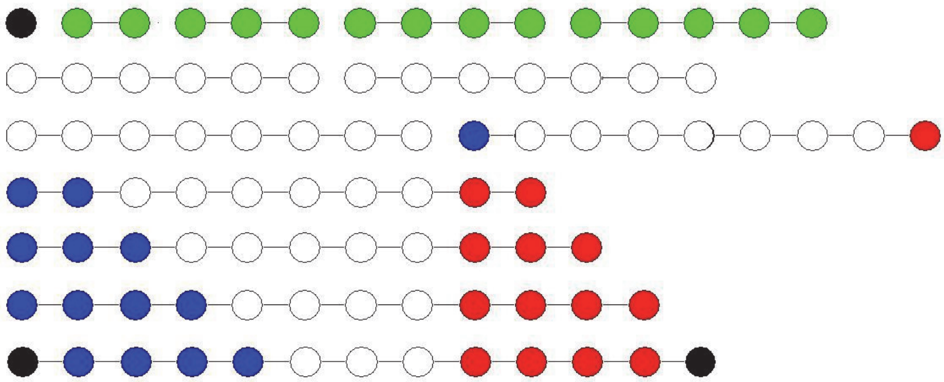


Figure 130 Pascal’s Arithmetical triangle as a projection on dimension 1

8.2.8 The Symmetric 6-player model on 5-simplex board

Our last example is a model embedded in a five-dimensional hexateron (Figure 131). In the Pascal's arithmetical triangle, the coefficients of the binomial formula were placed on rows, which got longer when going from a higher level to a lower level. In the same way, in the hexateron, or 5-simplex, the coefficients of the pentanomial formula are placed on pentachoron levels, the size of which gets greater when going from higher levels to lower levels. In this example, we have a game board model, which has 8 levels.

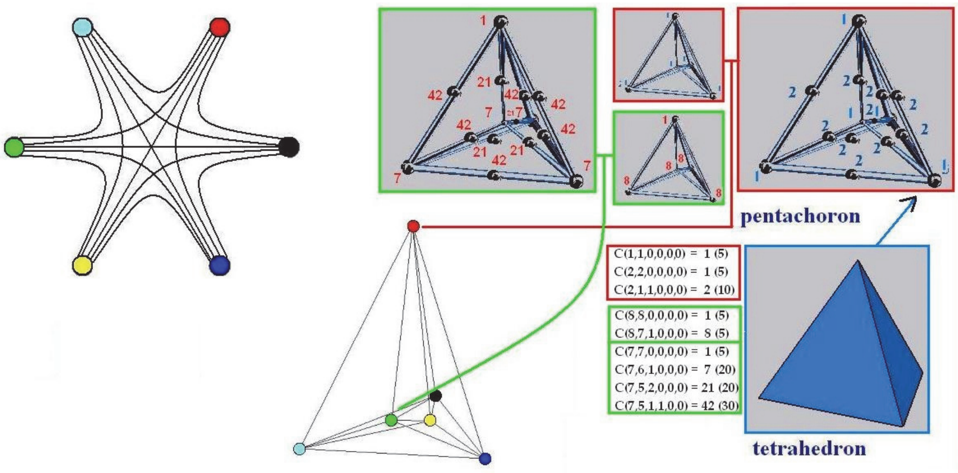


Figure 131 Six-player symmetric game on hexateron

A hexateron is a regular, 5-dimensional manifold which is defined by 6 pentachorons, 15 tetrahedra, 20 triangles, 15 edges and 6 vertices.

Just like in previous cases, we get the coefficients of the sums of the lower-level coefficients. The levels are 4-dimensional pentachorons, and the sum consists of five coefficients, which are in the corner vertices of a pentachoron. In this game board model, we have 8 levels.

Figure 131 illustrates the idea of how the coefficients are placed in a hexateron. The two drawings framed by red lines are the two highest pentachoron levels, and below them, in red rectangles, are the coefficients (the number of each in brackets). On the first level, there are five coefficient vertices, with numerical values C(1,1,0,0,0) = 1. On the second level, the five coefficient vertices in the corners of the pentachoron also have the value of 1, for C(2,2,0,0,0) = 1. In pentachoron, there are 10 edges and for the central vertices of these edges we have C(2,1,1,0,0,0) = 2. And indeed, the number of combinations C(2,1,1,0,0,0) is also 5!/2!3! = 10.

So this hexateron has 8 levels, which are all pentachorons, and six corners, which all have pentachorons as game levels. From each corner, there are 8 levels to the opposite side border level of the hexateron.

The previous explanation is easier to understand if we apply it to the Pascal's triangle in Figure 119 on the right. The Pascal's arithmetical triangle here has 13 levels, which are all rows, and three corners, which all have rows as game levels. From each corner, there are 13 levels to the opposite side border level of the triangle.

In Figure 131, the red vertex shows the top corner and, "under" it, 5 bottom corners. The bottom corners are shown in Figure 131 as drawings in the green squares. One of the smaller pentachoron's five corner vertices gets C (8,8,0,0,0) = 1 as its value, and the four others get the value C(8,7,1,0,0) = 8. In the figure, the numbers in brackets mean the number of combinations in all five corners. In the greater pentachoron, the combinations are C(7,7,0,0,0,0) = 1 in one corner, and in four other corners, C(7,6,1,0,0,0) = 7. A pentachoron has 10 edges, and the vertices in the middle of them have C(7,5,2,0,0,0) = 21 combinations on four edges and C(7,5,1,1,0,0) = 42 on six edges.

8.2.9 Symmetric n-players model on (n-1)-simplex board

In this section, we define the symmetrical multi-player model in a general level. A general formula to count the upper level coefficient from the sum of lower level coefficients is $C(p,k_1,k_2,\ldots,k_n) = C(p-1,k_1-1,k_2, \ldots,k_n) + C(p-1,k_1,k_2-1, \ldots, k_n) + \ldots C(p-1,k_1,k_2, \ldots,k_n-1)$, where $k_1 + k_2 + \ldots k_n = p$, and $k_i \geq 0 \ \forall\ i \in N$.

Proof:

$C(p-1,k_1-1,k_2, \ldots,k_n) + C(p-1,k_1,k_2-1, \ldots, k_n) + C(p-1,k_1,k_2, \ldots,k_n-1)$
$= (p-1)!/(k_1-1)!k_2! \ldots k_n! + (p-1)!/k_1!(k_2-1)!\ldots k_n! \ldots + (p-1)!/k_1!k_2! \ldots(k_n-1)!$
$= (p-1)!\, k_1/k_1!k_2! \ldots k_n! + (p-1)!\, k_2/k_1!k_2!\ldots k_n! \ldots + (p-1)!k_n/k_1!k_2! \ldots k_n!$
$= (p-1)!(k_1 + k_2 + \ldots k_n)/k_1!k_2! \ldots k_n! = (p-1)!p/k_1!k_2! \ldots k_n! = p!/k_1!k_2! \ldots k_n!$
$= C(p,k_1,k_2,\ldots,k_n)$ q.e.d.

In the text corresponding to Figure 123, we listed the neighbors of a single coefficient vertex on the sixth level in a three-dimensional model. Their number (#n) is 12. In the four-dimensional model, #n = 20. Moving from a two-dimensional model to higher-dimensional models, from an n-simplex to an (n+1)-simplex, the number of neighbors increases, and on the upper and lower levels the number of neighboring vertices will always be one more than on the previous level. On the same level, the number of neighboring vertices is the same as the total number of neighboring vertices in a lower dimension. We can explain this in a simple way. On a one-dimensional row, a vertex has two neighboring vertices. On a two-dimensional plane, one vertex has two neighboring vertices on the same level as well as on the upper and the lower levels, and thus the total number 2 + 2 + 2 = 6. In a three-dimensional space, there are six neighboring vertices on the same level, the same number as the total number of neighboring vertices on the two-dimensional plane; in addition, there are three neighboring vertices both on the upper and lower levels. So the total number 6 + 3 + 3 = 12. In a four-dimensional space, there are 12 neighboring vertices on the same level, the same number as the total number of neighboring vertices in the three-dimensional space; in addition, there are four neighboring vertices both on the upper and lower levels. So the total number here 12 + 4 + 4 = 20.

The number of neighboring vertices is always equals to 2 × (the third term on Pascal's arithmetical triangle on row n+1), where n is the dimension. So the number of the neighboring vertices #n = 2×C(n+1, n-1). We note that #n = n×(number of players). The number, #n, grows from a two-dimensional plane to higher dimensions: 6, 12, 20, 30, 42, 56, 72, etc. We also should note that the first two numbers are the same as the kissing numbers (see Table 1) in the same dimensions. However, after that the kissing numbers grow faster.

8.3 Extension of Pascal’s rule

Pascal's rule can also be seen as a part of a more general rule, where each coefficient can be counted not only by the coefficients of the previous level, but by the coefficients of all the previous upper levels.

8.3.1 Extension in traditional Pascal’s triangle

Figure 132 shows an example of how the fourth coefficient (10) on the fifth row and the fourth coefficient (20) on the sixth row can be counted by using a reverse Pascal's triangle (red numbers) and its product sums on the preceding rows.

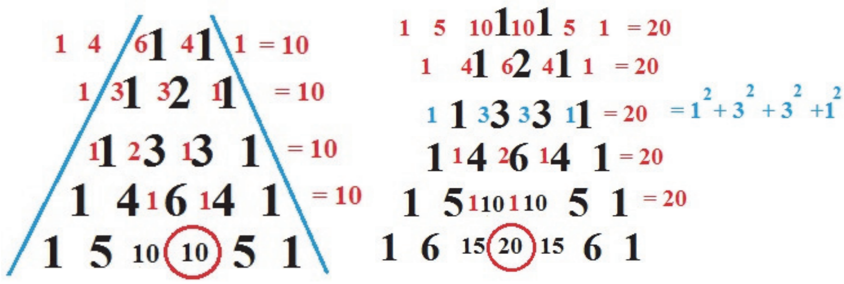


Figure 132 Extension of Pascal’s rule

In Figure 132, the triangle on the left results in: C(5,3) = C(4,2) + C(4,3) = 1*C(3,0) + 2*C(3,1) + 1*C(3,2) = 1*0 + 3*C(2,0) + 3*C(2,1) + 1*C(2,2) = 1*0 + 3*0 + 6*C(1,0) + 4*C(1,1) + 1*0 = 10.

When we write also the coefficients of the reverse triangle as combinations, we will get:

C(5,3) = C(1,0)C(4,2) + C(1,1)C(4,3)
= C(2,0)C(3,0) + C(2,1)C(3,1) + C(2,2)C(3,2)
= C(3,0)*0 + C(3,1)C(2,0) + C(3,2)C(2,1) + C(3,3)C(2,2)
= C(4,0)*0 + C(4,1)*0 + C(4,2)C(1,0) + C(4,3)C(1,1) + C(4,4)*0 = 10.

Thus, the Pascal's rule C(p+1, k+1) = C(p,k) + C(p, k+1), can be extended in the following way:

C(p+1, k+1) = C(1,0)C(p,k) + C(1,1)C(p, k+1)
= C(2,0)C(p-1,k-1) + C(2,1)C(p-1,k) + C(2,2)C(p-1,k+1)
= C(3,0)C(p-2,k-2) + C(3,1)C(p-2,k-1) + C(3,2)C(p-2,k) + C(3,3)C(p-2,k+1)
…
= C(p,0)C(1,k-p+1) + C(p,1)C(1,k-p+2) + … + C(p,p)C(1,k+1),
where C(x,y) = 0 if y < 0 or y > x.

As another example, we may consider the right-hand triangle of Figure 133. There we use a longer notation for combinations: We get:
C(6,3,3) = C(5,2,3) + C(5,3,2) = C(4,1,3) + 2*C(4,2,2) + C(4,3,1) = C(3,0,3) + 3*C(3,1,2) + 3*C(3,2,1) + C(3,3,0) = C(2,-1,3) + 4*C(2,0,2) + 6*C(2,1,1) + 4*C(2,2,0) + C(2,3,-1) = C(1,-2,3) + 5*C(1,-1,2) + 10*C(1,0,1) + 10*C(1,1,0) + 5*C(1,2,-1) + C(1,3,-2) = 20.

When this extended rule is generalized from the binomial formula to a multinomial formula, we get a generalized extended Pascal's rule that can be used to calculate a single coefficient by using the sums more distant than on the previous level. This will be useful later when we define the movements of the officers.

8.3.2 Extension of Pascal's rule for 3D simplex model

In Section 8.2.6, for the determination of coordinates, we defined the neighboring vertices of a single coefficient vertex. Using the extension of Pascal's rule and its generalization for a trinom, we can directly define also the neighboring vertices on the levels at the distance of two or more steps.

Let's start again with an example. In Section 8.2.4, coefficient vertex 30 = C(5,2,2,1) (Figure 123). Its neighbors on the upper level were vertices 6, 12 and 12, as combinations C(4,2,2,0), C(4,2,1,1) and C(4,1,2,1). The extension of Pascal's rule in a three-dimensional model gives 30 = 6 + 12 + 12. The coefficient, 30, is on level 5, and thus the rule was applied on level 4. Next we apply this extension on level 3. In Figure 133, on level 3, we select the vertices whose sums on level 4 give the above-mentioned combination vertices, 6, 12 and 12, that is, their neighboring vertices. Of these, we get 6 by using 0, 3 and 3. Here 0 is the vertex outside the triangle (Figure 133). The second vertex, 12, is obtained from 3, 3 and 6 and another, 12, from 3, 6, and 3. As combinations, these vertices translate to 6 = C (3,2,1,0) + C(3,1,2,0) + 0, 12 = C(3,2,1,0) + C(3,2,0,1) + C(3,1,1,1) and 12 = C(3,1,1,1) + C(3, 1,2,0) + C(3,0,2,1). Consequently, vertex C(5,2,2,1) on level 5 can be calculated already on level 3 with the help of five vertices: C(3,2,1,0), C(3,2,0,1), C(3,1,2,0), C(3,0,2,1), and C(3,1,1,1). So we can calculate vertex C(5,2,2,1) over two levels : C(5,2,2,1) = C(3,2,1,0) + C(3,1,2,0) + 0 + C(3,2,1,0) + C(3,2,0,1) + C(3,1,1,1) + C(3,1,1,1) + C(3,1,2,0) + C(3,0,2,1).

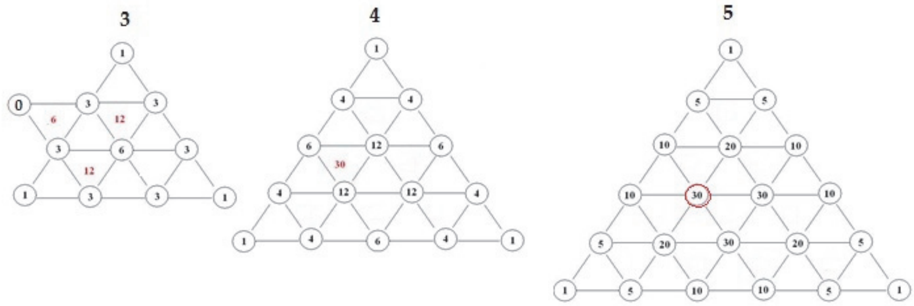


Figure 133 The extension of Pascal’s rule and trinom

8.4 The effect on rules, when there are n players

The n-player symmetric game model deals with the structure formed on the game board, the coordinates of positions, the types of game pieces and the directions of their movements. When the number of players increases, then definitions and rules about game strategy are needed outside of the game model. One of the major issues is how the game is finished. In an n-player game of chess, this requires definitions of check and checkmate. In a chess game involving more than two players, some rules must be clarified, including those needed for check and checkmate situations that cannot occur in two-player chess. Two of these situations have been dealt with in the original rules, and a couple of others have come up in concrete game-playing situations. The same rules must work logically in all the multiplayer chess games introduced in this research. In the following, we consider ten specific cases. Six of these concern the 3-player game.

Three players

The game rotation sequence is A, B, C, A, B, C, … and so on.

1. Player A checkmates C. The result: A is the winner of the game since B is not allowed to terminate the checkmate. In general (also applicable for two-person chess), a checkmate is done immediately when the move is made, not after C's theoretical next move. So B is not allowed to undo the checkmate.
2. Player A puts C in check, and Player B complets it with a checkmate. The first one making a check, in this case A, is the winner.
3. Player A makes an open checkmate to C, that is, A reveals one of B’s chess pieces in such a way that B will threaten C’s king, and a checkmate is made. Here, the winner is B.
4. Player A makes an open check to C: that is, A reveals one of B’s chess pieces in such a way that B threatens C’s king, and a check is made. Also this is a checkmate, even though C could undo the check during its own turn. The rea-

son for this is that B could capture C’s king, which is not allowed in chess. Just like in case 3, the winner is B.
5. Player A makes an open checkmate both to B and C. Interpretation: A is the winner even though also B has a checkmate over C, and the other way round. In order to avoid a conflict with the case 3, we establish a definition: a player is the winner only if its own king is not threatened during the checkmate.
6. Player A makes an open check both to B and C: that is, A reveals one of player’s B chess pieces in such a way that it will threaten C’s king and create a check, revealing at the same time some player’s C chess piece in such a way that it will threaten B’s king, which also creates a check. Interpretation: the game continues as B cannot capture C’s king because it has to undo its own check. It doesn’t seem to be possible to do this and at same time capture C’s king.

Four players

The game rotation sequence is A, B, C, D, A, B, C, D, … and so on.

7. Player A makes an open checkmate to all the other players. A is the winner.
8. Player A makes an open checkmate to B and C. D doesn’t threaten anyone. A is the winner.
9. Player A makes an open checkmate to B and C so that both A and D are checking after A’s move. A is the winner.
10. Player A makes an open checkmate to B and C so that only D is checking. On the basis of case 3, D is the winner.

N players

Here we discuss a general case where the number of players is n > 2. Player X who has made the move that leads to a checkmate is the winner if some of X’s pieces is checking the king in the checkmate and X’s own king is not in check. If Player X does not threaten the king, which is in the checkmate, then the winner is the next player whose piece threatens that king and whose own king is not threatened. If all the players who are threatening the king in checkmate have their own kings also threatened, Player X who caused the checkmate can be the winner even when none of X’s pieces is threatening the king who has been checkmated.

Some notes about transfer order

Although the players' positions in relation to each other are symmetrical, some asymmetry is left in the order of moves. Also this could be avoided by defining a transfer order that takes place randomly. The checkmate rules would work also in that case, but naturally the nature of the game would be totally different, as chance would play part in the game.

8.5 Multi-simplex chess

Next, we have a look at the three first defense zones and their officers on an n-simplex game board. This gives the elements and rules which will later make it easy to build on the algorithm for the simulation model.

8.5.1 First defense zone in varying dimensions

The combinations of neighboring vertices of $C(p, k_1, k_2,\ldots, k_n)$ in an n-dimensional simplex are the following.

Upper level: $C(p-1, k_1-1, k_2, \ldots, k_n)$, $C(p-1, k_1, k_2-1, \ldots, k_n)$, …,$C(p-1, k_1, k_2, \ldots, k_n-1)$.
Same level: $C(p, k_1+1, k_2, \ldots, k_n-1)$, $C(p, k_1-1, k_2, \ldots, k_n+1)$, $C(p, k_1+1, k_2, \ldots k_{n-1}-1, k_n)$, $C(p, k_1-1, k_2, \ldots k_{n-1}+1, k_n)$, …, $C(p, k_1+1, k_2-1, \ldots, k_n)$, $C(p, k_1-1, k_2+1, \ldots, k_n)$, …,$C(p, k_1, k_2+1, \ldots, k_n-1)$, $C(p, k_1, k_2-1, \ldots, k_n+1)$, …, $C(p, k_1, k_2, \ldots k_{n-1}+1, k_n-1)$, $C(p,k_1, k_2, \ldots k_{n-1}-1, k_n+1)$.
Lower level: $C(p+1, k_1+1, k_2, \ldots, k_n)$, $C(p+1, k_1, k_2+1, \ldots, k_n)$, …,$C(p+1, k_1, k_2, \ldots, k_n+1)$.

We can explain this by a simple example, where $n = 2$. In this case, the neighboring vertices of $C(p, k_1, k_2)$ are $C(p-1, k_1-1, k_2)$ and $C(p-1, k_1, k_2-1)$ on the upper level, $C(p, k_1+1, k_2-1)$ and $C(p, k_1-1, k_2+1)$ on the same level and $C(p+1, k_1+1, k_2)$ and $C(p+1, k_1, k_2+1)$ on the lower level. In Figure 121 of Section 8.2.2, there is an example in which $p = 4$ and $k_1 = 2$. In that case, for vertex C(4,2,2) or C(4,2) when using the short notation, the neighbors are C(3,1,2) and C(3,2,1) (upper level), C(5,2,3) and C(5,3,2) (lower level), and C(4,1,3) and C(4,3,1) (same level).

8.5.2 How do rooks move on an n-simplex board?

If there are no other pieces on the way, the rook can move to every position in the first dense zone. It can then continue moving a maximum of q steps ($q \in (0,r)$ and r is the number of levels). The number of directions available, #n, is explained in Section 8.2.9. When the rook starts from cell $C(p, k_1, k_2,\ldots, k_n)$, we get the following directions:

Upper level : $C(p-q, k_1-q, k_2, \ldots,k_n)$, $C(p-q, k_1, k_2-q, \ldots, k_n)$, …,$C(p-q, k_1, k_2, \ldots, k_n-q)$.
Same level: $C(p, k_1+q, k_2, \ldots, k_n-q)$, $C(p, k_1-q, k_2, \ldots, k_n+q)$, $C(p, k_1+q, k_2, \ldots k_{n-1}-q, k_n)$, $C(p, k_1-q, k_2, \ldots k_{n-1}+q, k_n)$, …, $C(p, k_1+q, k_2-q, \ldots, k_n)$, $C(p, k_1-q, k_2+q, \ldots, k_n)$, …, $C(p, k_1, k_2+q, \ldots, k_n-q)$, $C(p, k_1, k_2-q, \ldots, k_n+q)$, …, $C(p, k_1, k_2, \ldots k_{n-1}+q, k_n-q)$, $C(p, k_1, k_2, \ldots k_{n-1}-q, k_n+q)$.
Lower level: $C(p+q, k_1+q, k_2, \ldots, k_n)$, $C(p+q, k_1, k_2+q, \ldots, k_n)$, …,$C(p+q, k_1, k_2, \ldots, k_n+q)$.

As an example, we choose again the simplest two-dimensional case on a plane, where n = 2, and #n = 2 x C(n+2, n+2-2) $\Rightarrow$ #2 = 2 x C(4,2) = 6. Thus the rook moves to the upper level to the vertices in directions C(p-q, k_1-q, k_2) and C(p-q, k_1, k_2-q), to the lower level to the vertices in directions C(p+q, k_1+q, k_2) and C(p+q, k_1, k_2+q) and on the same level to vertices in directions C(p, k_1+q, k_2-q) and C(p, k_1-q, k_2+q). If we again take the vertex C(4,2) = 6 in Figure 121 as the starting point, then p = 4, and $k_1 = k_2 = 2$. So the rook moves on the upper level to vertices C(4-q, 2-q, 2) and C(4-q, 2, 2-q), on the lower level to vertices C(4+q 2+q, 2) and C(4+q, 2, 2+q) and on the same level to vertices C(4, 2+q, 2-q) and C(4, 2-q, 2+q). When moving to the upper level, q can have the values of 1 or 2; so in the direction of the upper left, we have vertices C(3,1,2) and C(2,0,2), and in the upper right, the vertices C(3,2,0) and C(2,2,0). Moving to the lower level, q can only have the value of 1 and the vertices are C(5,2,3) and C(5,3,2). Also the sums for q on the same level can have the values of 1 and 2. When moving to the left, the vertices are C(4,1,3) and C(4,0,4), and, when moving to the right, they are C(4,3,0) and C(4,4,0).

In general, when the rook moves upwards or on the same level, in the last vertex, k_1=p or k_2 =p. When it moves downwards, then in the last vertex, p = r. Figure 134 describes how the rook moves in a four-dimensional simplex, a pentachoron. The starting point is in vertex 24 (in the orange square).

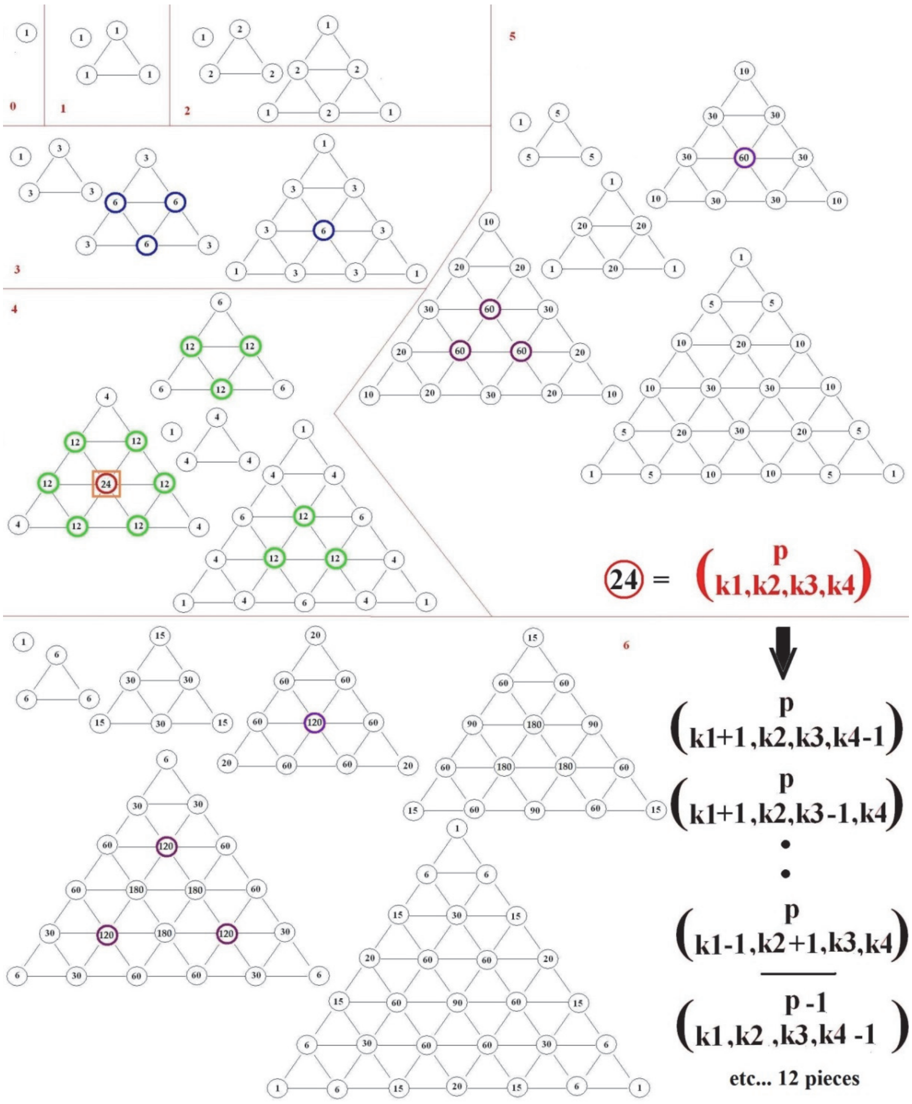


Figure 134 Directions of the rook's move in a 4-simplex

8.5.3 Second zone officers on n-simplex board

In all dimensions, there is exactly one officer, the rook, which holds all the cells of the first defense zone. On the second defense zone, we add new officers on those cells, which the rook cannot reach. In Section, 8.5.1, we presented the game cells as vertices and mapped the neighboring vertices for a single vertex on a first defense zone in an n-dimensional space. After that, in Section 8.5.2, we examined the directions of the rook's movements in an n-dimensional space. On the second defense zone, there will be new officers filling the directions that are between the rook's directions. We use the following notation. In the

$C(p,k_1,k_2, \ldots, k_n)$ combination, variable p indicates the level of the combination on a single n-1 dimensional level. In the combination, $p \in N$, where N is a natural number or 0. Variables $m \in N$ and $k_m \in N$, $k_m = 0, \ldots p+1$ tell the location of the combination in an n-simplex. The number of steps that officers can move to every direction is not more than q, where $q \in (0,r)$ and r is the number of the levels, if there are no other pieces blocking the way.

We start our observation from lower dimensions. When dimension n = 2, then the cells that remain between the rook's directions on the second defense zone are: $C(p-2, k_1-1, k_2-1)$, $C(p-1, k_1+1, k_2-2)$, $C(p+1, k_1+2, k_2-1)$, $C(p+2, k_1+1, k_2+1)$, $C(p+1, k_1-1, k_2+2)$ and $C(p-1, k_1-2, k_2+1)$. This can be seen in Figure 123 by setting values p = 4 and $k_1 = 2 \Rightarrow k_2 = 2$. From this it follows that the second defense zone of vertex C(4,2,2) = 6 in the center is clockwise and starting from the top: C(2,1,1) = 2, C(3,3,0) = 1, C(5,4,1) = 5, C(6,3,3)= 20, C(5,1,4) = 5 and C(3,0,3) = 1. The moving directions are: $C(p-2q, k_1+q, k_2)$, $C(p-q, k_1+q, k_2-2q)$, $C(p+q, k_1+2q, k_2-q)$, $C(p+2q, k_1+q, k_2+q)$, $C(p+q, k_1-q, k_2+2q)$ and $C(p-q, k_1-2q, k_2+q)$.

8.5.4 The next steps

When the size of the game board is extended, new officers must be added. This process doesn't depend on the number of dimensions. The new officers will be added on the sectors, which are between the sectors where the officers on the lower dense zones are able to move. The system is similar that which we were discussing in the Section 7.1.1. and is easy to count and generalize.

So what is next? We will discuss the simulation program and its applications in the real world.

8.6 Summary

Figure 135 explains, better than words the basic idea and evolution of this simulation model.

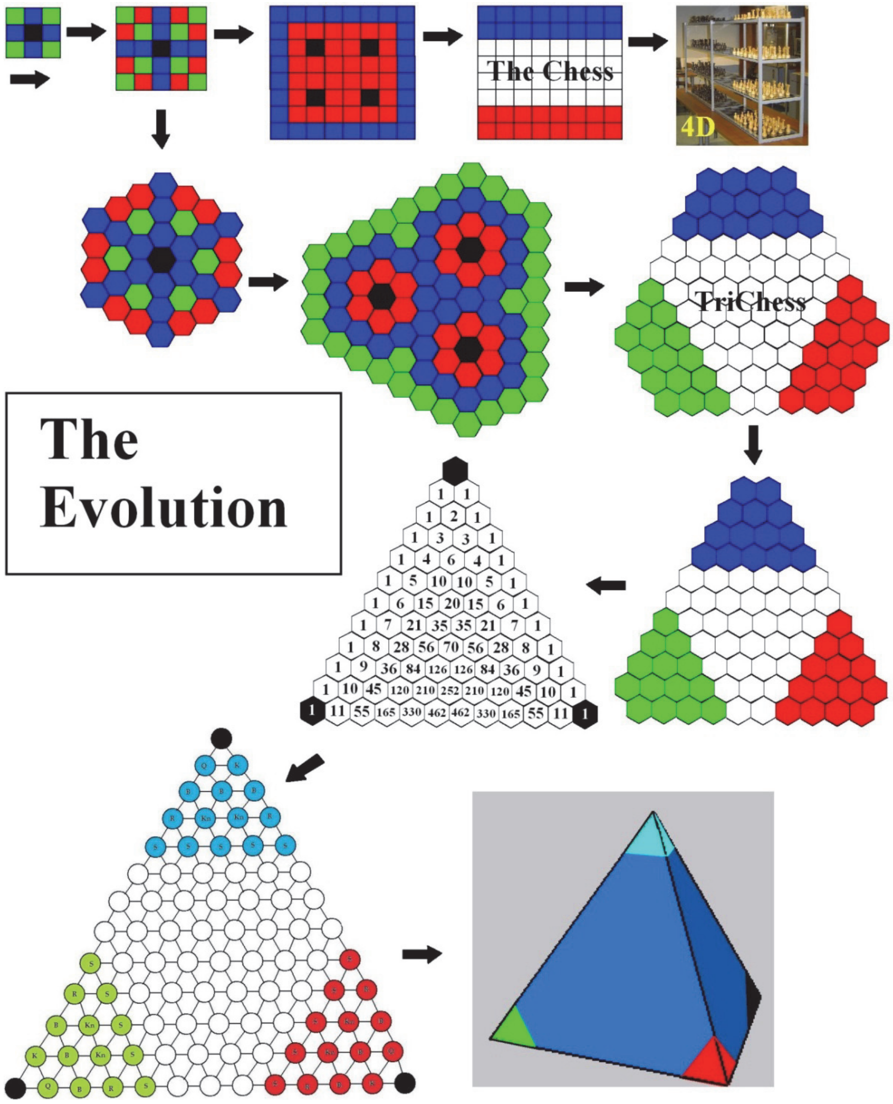


Figure 135 The Game model idea in one picture

The ideas about 4-player chess led us to 3-player chess. It was possible to embed Trichess on Pascal's triangle, which was our 3rd step. The 4th step was to enlarge the game to higher dimensions by using the multinomial formula, which is an extension of the binomial formula.

9 CONCLUDING REMARKS

In this chapter we present additional results obtained during this research.

9.1 The Number of a Graph

The results of this chapter are based on a joint work by the author and Professor Frank Harary. References have been added where results of others have been used.

In the following, we introduce a method for representing any graph by a unique number and any number as a unique graph.

A graph G of order n consists of a finite nonempty set, V=V(G), of n vertices together with specified set X of unordered pairs of distinct vertices. There are several ways to present the structure of G, which is an abstract graph. One of the ways is by drawing G and labeling the vertices of G with the integers 1 to n. The adjacency matrix of G is another form of representing a graph, as is the set of unordered pairs of adjacent vertices. We propose to assign a unique positive integer to G. The examples are displayed for labeling of the graphs of order 4 and size 5. After obtaining the numbers of all the connected graphs of order 4, we continue with a list of open questions and possible applications.

9.1.1 Labelling a graph

A simple graph, which is a graph without loops and multiple edges and is of order p, is a finite non-empty set of V = V (G) with p vertices and set X of unordered pairs. A pair x = {u,v} ∈ X, is called an edge of graph G, which connects vertices u and v. We can take the following graph G with set V of vertices as an example, where V = {v_1, v_2, v_3, v_4}, and X = {{v_1, v_2}, {v_2, v_3}, {v_3, v_4}, {v_1, v_4}, {v_1, v_3}}. In this labelled graph (Figure 136, b), the order is p = 4 and the integers from 1 to 4 are associated with the vertices. This graph can be labelled in six different ways, which are shown in Figure 137. In general, a graph with p vertices and k edges has n possible labeled graphs where for p = 4 and k = 5,

$$n = \binom{\binom{p}{2}}{k} \Rightarrow \binom{\binom{4}{2}}{5} = 6$$

9.1.2 Connected undirected graphs

There are several ways of presenting the structure of abstract graph G (Figure 136). The first (1) is drawing G, and labeling G adds the numbers 1 to n at its vertices. The list presentation (2) of a labeled graph has the vertex number 1 plus the ordered labels of its adjacent vertices, and vertices 2 and 3 have only the larger labels of its adjacent vertices. The adjacency matrix (3) of G is another kind of presentation formed from the set of unordered pairs of adjacent vertices. All these are displayed for a labeling of a graph of order 4 and size 5.

Once the graph has been labeled, its adjacency matrix A(G) can be created and the sequence of 0s and 1s formed by the rows of this matrix may be written. For an undirected graph, a shorter code can be constructed by taking only the elements above the diagonal. Such a code can be regarded as a binary integer. One way of coding an unlabeled graph would be by considering all possible labeling and the corresponding codes (Read & Corneil 1977, Nagle 1966).

(1) Figure 136 (a) shows a drawing of G
(2) The adjancency list presentation of the labeled graph of Figure 137 (b) is
1: 2, 3, 4
2: 4
3: 4
(3) Its adjacency matrix is A, on the right.

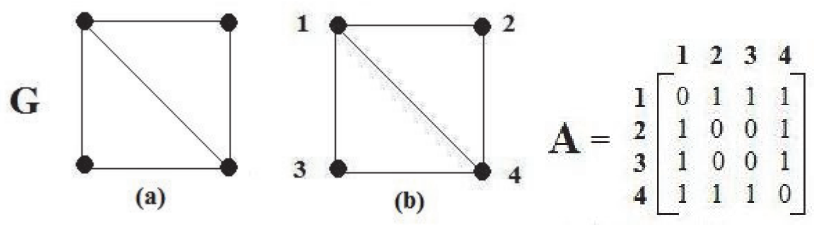


Figure 136 A graph of order 4 and size 5, with the adjancency matrix A

The upper diagonal matrix of A is	1	1	1
		0	1
			1

When written as a binary number, 111011 is obtained, which is 59 = 32 + 16 + 8 + 2 + 1 in binary notation. Harary and Read, as well as Harary, Palmer and

Read, proved that a graph with s automorphisms has exactly n!/s different labelings (Harary & Palmer 1973). One of them must have the largest number. This is called the number of the graph, written #H (H stands for Harary). We note that 59 is not the canonical number since its labeling in Figure 137 does not produce the maximum number.

The corresponding numbers are:

Binary	Decimal
111 011 =	59 (G_1)
101 111 =	47 (G_2)
111 110 =	**62 (G_3)**
011 111 =	31 (G_4)
111 101 =	61 (G_5)
110 111 =	55 (G6)

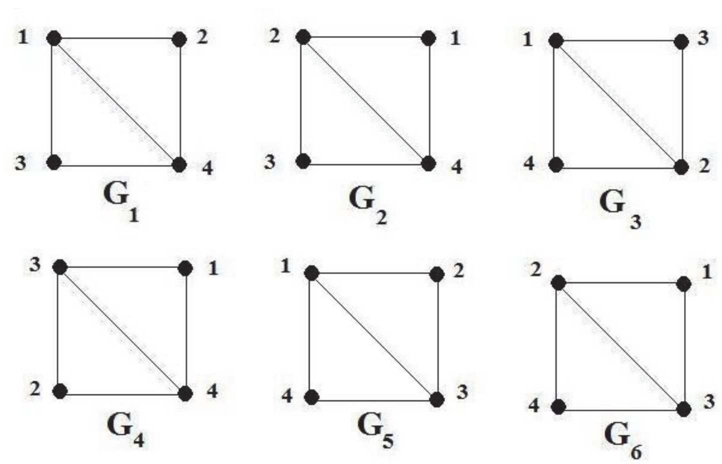


Figure 137 All the labelings of graph G

It should be mentioned that each decimal number determines the same abstract graph, but we prefer the biggest number for uniqueness. Hence #H(G) = 62.

Generally, every binary number (n,2) can be represented as a unique graph of order n. We refer to these numbers as *graphical* numbers.

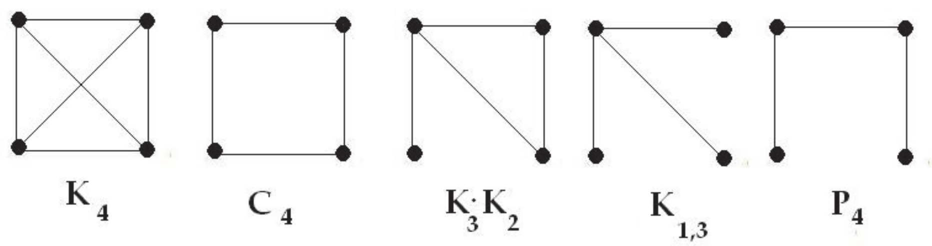


Figure 138 Five other graphs with order 4

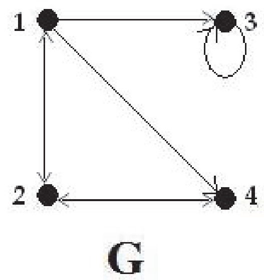
The binary numbers of the upper diagonal matrices are: K_4 = 111111, C_4 = 110011, K_3K_2 = 111100, $K_{1,3}$ = 111000 and P_4 = 110011. The decimal numbers are: K_4 = 63, C_4 = 51, K_3K_2 = 60, $K_{1,3}$ = 56 and P_4 = 50. See Figure 138.

9.1.3 A Directed graph

A directed graph, also called a digraph, G = (V, E), consists of a set of vertices V and a set of directed edges E, also known as arrows. The vertices represent the basic elements of a graph, and the relations between the vertices are given by the directed edges. A directed edge is located between the two vertices u and v (an ordered pair of (u,v)). For the presentation of directed graphs, we can use the same data structures we used with undirected graphs. In this context, we use an adjacency matrix.

The canonical number of a digraph

The adjacency matrix of a directed graph is not always symmetrical on both sides of the diagonal axis as is the case with an undirected graph. Also loops and multiple edges are allowed for directed graphs we discuss here. Because multiple edges and loops are allowed in pseudographs (Harary 1969), we will also consider pseudodigraphs. The diagonal elements in the matrix are not necessarily zeros. Figure 139, is an example of digraph G and its adjacency matrix.



	1	2	3	4
1	0	1	1	1
2	1	0	0	1
3	0	0	1	0
4	0	1	0	0

Figure 139 A digraph with a loop, and its adjacency matrix

For the numbers of digraphs we use notation #K. The graph in Figure 139 provides a binary number 0111100100100100 => #K(G) = 31012 for this labeling.

The graph of a number

Each number can be presented uniquely as a directed graph with loops allowed, so that, if the digraph has 1 ... n vertices, then all the numbers from 1 to ... $2^{n^2} - 1$ can be represented.

The digraphs in Figure 140 provide binary numbers 011000000, 011101110 and 111111111. The corresponding decimal numbers are 192, 238 and 511.
In Figure 141, digraph H can be represented by the matrix on the right. The matrix gives a binary number 1100100010010000000100010, and the corresponding decimal number is 26 288 162.

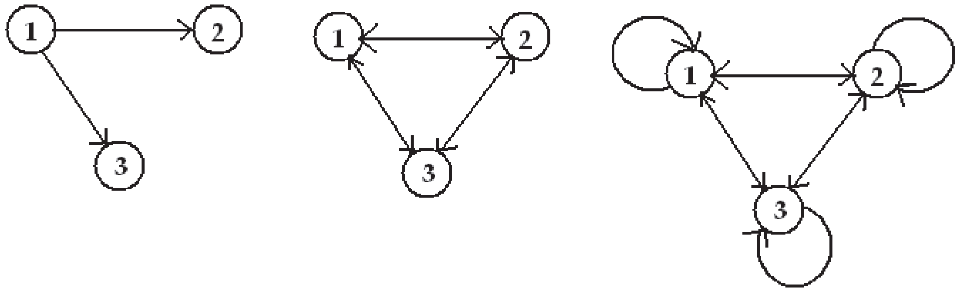
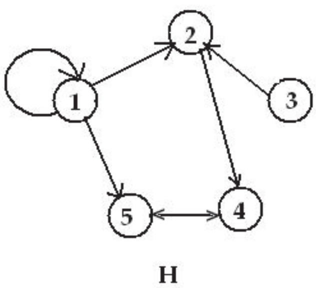


Figure 140 The digraphs of numbers 192, 238 and 511

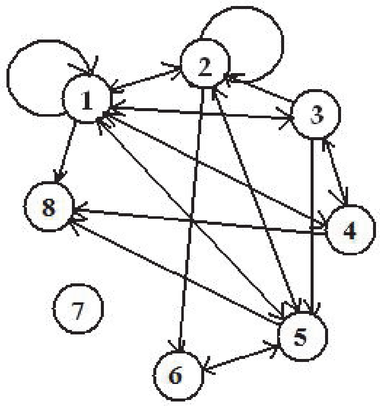


	1	2	3	4	5
1	1	1	0	0	1
2	0	0	0	1	0
3	0	1	0	0	0
4	0	0	0	0	1
5	0	0	0	1	0

Figure 141 Number 26288162 as a digraph and an adjacency matrix

Very large numbers

Very large numbers can be presented uniquely as directed and labelled pseudodigraphs. Figures 142 and 143 present numbers up to the size of 18 trillion by digraphs of only eight vertices. A well-known Edward Kasner's playfully invented fictional large number "googol" = 10^{100}, can be represented by a directed pseudograph of 330 vertices. This means that the classification of directed graphs is practical only if the graphs are not large.



	1	2	3	4	5	6	7	8
1	1	1	1	1	1	0	0	1
2	1	1	0	0	1	1	0	0
3	1	1	0	1	1	0	0	0
4	1	0	1	0	0	0	0	1
5	1	1	0	0	0	1	0	1
6	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0

Figure 142 18 A trillion as a digraph and an adjacency matrix

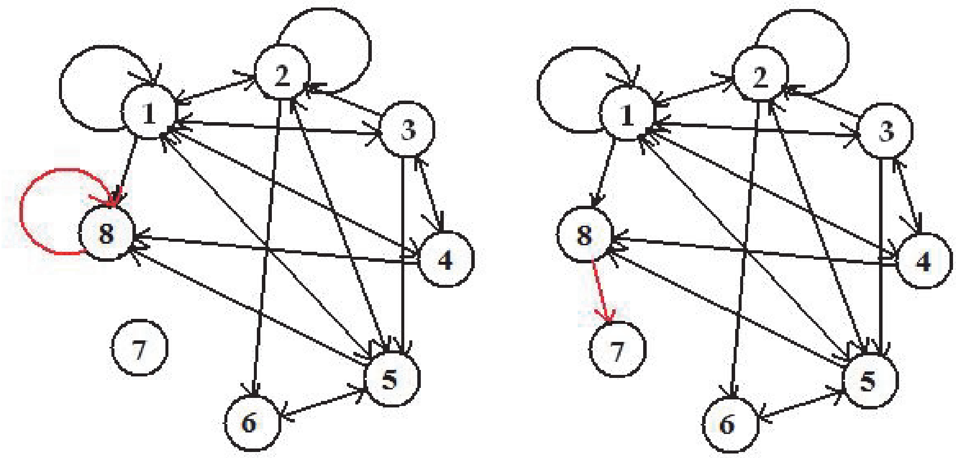
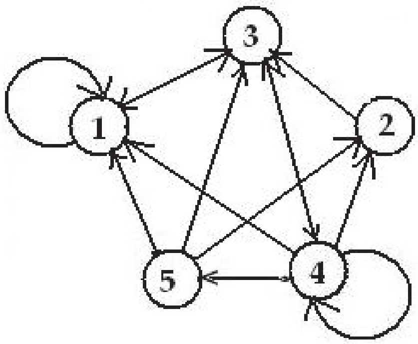


Figure 143 Numbers $18x10^{18}+1$ and $18x10^{18}+2$ as digraphs

Coding a number graph in other ways

Naturally, a binary number placed in an adjacency matrix can be represented in many ways other than as a graph. One way is to use hand signals. As an example, we take number 21 122 012, which as a binary number is 1010000100100101111011100. Since this binary number has 25 digits, it can be represented by a 5x5 adjacency matrix and, further, as a digraph of five vertices. Hence, this number can be represented also by using the fingers of one hand. In this system, the first vertex, the vertex number one, which is the first row in the matrix, is displayed by the first finger (the thumb). After that we display those other vertices which are the end vertices of the directed edges from the first vertex. This process is continued until all the five vertices are shown. In Figure 144, the given number is presented as a matrix, as a digraph and by hand signals.

	1	2	3	4	5
1	1	0	1	0	0
2	0	0	1	0	0
3	1	0	0	1	0
4	1	1	1	1	0
5	1	1	1	0	0



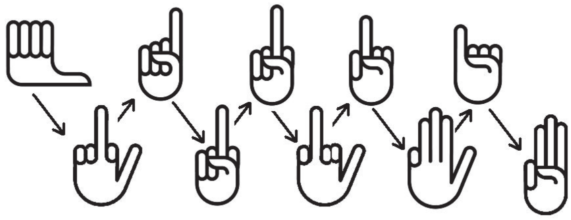


Figure 144 Number 21 122 012 as digraph and finger code

When we use hand signals, the place of the finger shows the row of a matrix (the hands above) and the place of the digit one on every row (the hands below). In practice, the hands that are uppermost are not even needed, because the order of signals tells us the row. We added this hand signal example in the end of this chapter just to illustrate how simple codes can show quite large numbers and hence other kinds of complex information.

Short Summary

This is a continuing work. We don't know yet, if it leads to any useful results. In 1977 Read was quite skeptical. Another way of representing graphs is by means of an algebraic concept of some kind, such as an adjacency matrix; but a graph can have as many adjacency matrices as there are ways of labelling its vertices. (Read & Corneil 1977)

9.2 Strategy networks of small chesslike games[48]

The main focus of this chapter is to examine some of the winning strategies of small board games. Game flow, the strategy network, is modeled using directed graphs, and the situation in every move is presented as a code in vertices. Here we present only a shortened version of the idea, this being just a spin-off of this thesis. The basic idea was to develop as simple chesslike game as possible, expand it to larger ones by a logic rule. The smallest one is a "chessboard", the size of which is 2x3 = 6 squares. The motive for these games was to teach very young children to play and to allow the monitoring of their learning. We call these games Primitive Chessgames (or "Babychess"). In the two smallest versions, there is one significant difference from the rules of traditional chess, namely, there are several kings. This study is aimed at exploring winning strategies of small board games which have been developed from chess. The progress of the games is modeled by means of digraphs, and the information about the game situations is depicted by the codes in vertices. The possibilities of generalizing the results and applying them for more complicated games were investigated. The coding system, introduced here, was designed especially for small games. By means of this system, the position of chess pieces on the chess board as well as the players' moves may be monitored directly from the vertex codes.

Figure 145 is a complete representation of a strategy network for 2x3 size primitive chess. In the starting position of this game, both players have two pawns (see Figure 104, in Section 7.1). All pawns can be coronated to kings if they reach the opposite side of the board.

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48 *This topic emerged when developing the games in Chapter 7, where not only larger square board chess games but also smaller ones were created.*

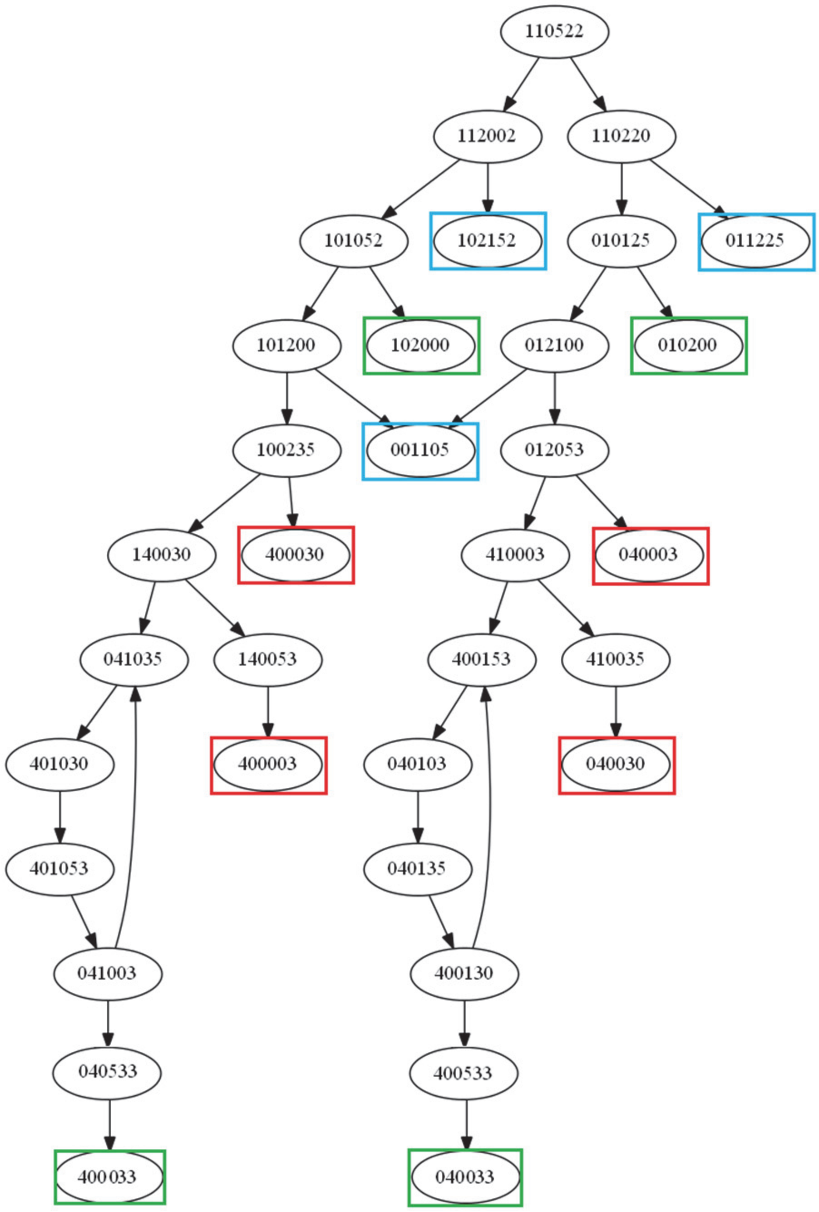

Figure 145 A digraph depicting different moves in 2x3 chess

Every vertex on the graph is coded in such a way that the code perfectly displays the overall situation as well as next moves on the chess board. Both the winning and drawn games are depicted by the coloredl boxes on the graph. The code in a vertex determines the place of the chess pieces on the board. Figure 146 shows the idea of the coding system.

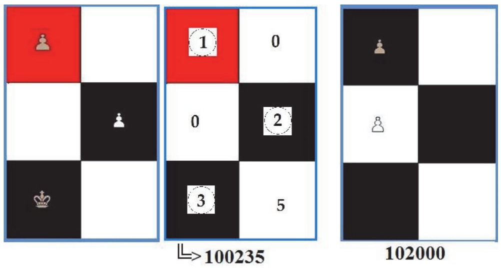

Figure 146 The Coding system of vertices

Two vertices of the digraph in Figure 145 are presented. Every code has as many digits as there are squares on the board (six in this case), and the numbers tell the contents of a square in the moment of observation. In the vertex, which has the code of 100235, the digits are read line by line from top to left down so that 1 = black pawn, 2 = white pawn, 3 = black king and 0 = empty square. If there were also a white king, then its number would be 4. If white is next to move, then, in the last empty square, 0 is replaced by 5, as in this case. To avoid misunderstandings, we note that, in the digraph, the root vertex tells the starting position, the last empty square and hence the number 5 is the fourth digit. The case on the right with the code of 102000 is a checkmate for white, because black cannot make any moves. This is shown in Figure 145 by a green box around the vertex.

When the number of squares increases by three, then each player has two pawns and a king. Also the digraph grows enormously greater, how much, that we do not know yet exactly. Figure 147 shows the beginning of the graph and the coding of vertices. The letter E means a checkmate.

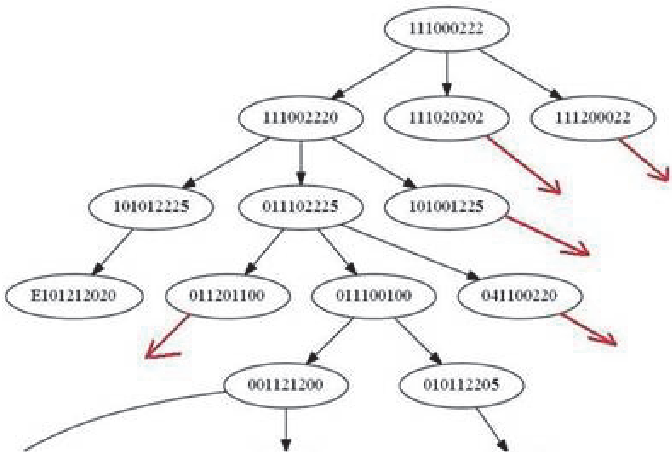

Figure 147 Part of the strategy network of a 3x3 primitive chesslike game

The digraph in Figure 148 is larger subgraph of the graph which comprises all the strategies of this game. It must be noted that the game is very simple and, though all the moves are taken into account, only few of them are reasonable for an intelligent player. The optimal winning strategy for this game on the 3x3 board is represented by the red vertices.

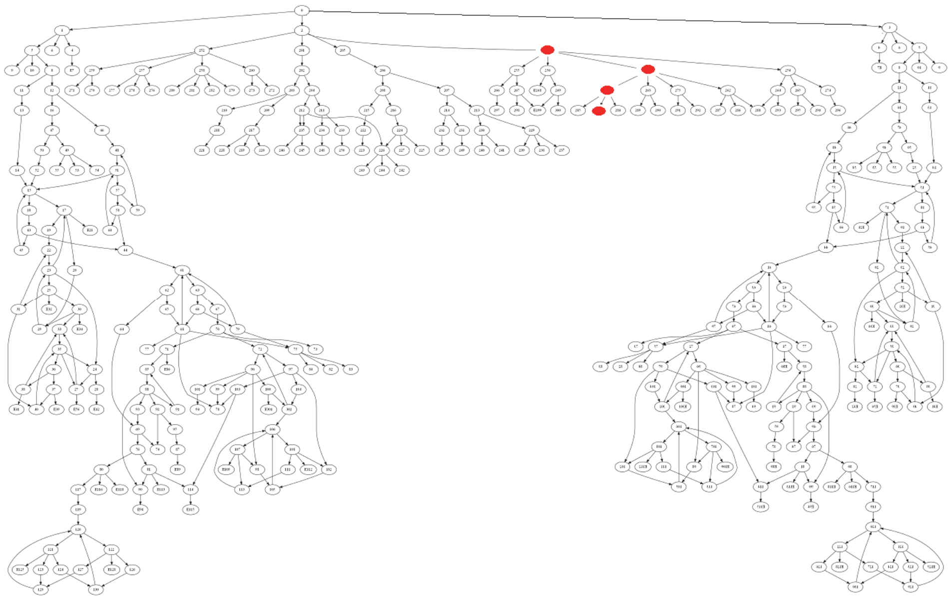

Figure 148 A larger part of the strategy network of Figure 146

The main purpose in developing these games was to research cognitive learning in small children. The games are played by computers and information of the moves is saved. Once data has been collected from a large number of players, its classification will be easy on the basis of the numerical coding, and it will be possible to find different player types and different developments in learning.

9.3 Generalization of Euler-Poincare characteristic[49]

9.3.1 Background

The Study of polyhedra can be traced back four thousand years to Ancient Egypt and later to Ancient Greece. The Greeks were interested in the mathematical properties of regular polyhedra, and they discovered five solids, which became later known as the Platonic solids: the cube, tetrahedron, octahedron, ico-

[49] *This subject came out in Chapter 8, where we determined the moving directions of officers on an N-simplex game board. The research has connections with the author's earlier works in the field topological graph theory (Kyppö 1994, Kyppö 1993).*

sahedron and dodecahedron. In 1750 - 1752, Leonhard Euler discovered and published a formula which could be used to prove that the number of these solids was five. In 1811, the first generalizations of Euler's formula were published by Simon Lhuilier and Augustin-Louis Cauchy. (Biggs, Lloyd & Wilson 1986).

Finally, in 1899, Henri Poincaré generalized Euler's formula to the higher dimensions. (Alama 2009a, Alama 2009b, Dufourd 2008, Sachs 1970)

There is less information about the genus of higher-dimensional solids. Gagliardi made in 1981 a proposition and two corollaries about a four-dimensional genus. He also claimed that the same result could be generalized to higher dimensions (Gagliardi 1981).

Euler's polyhedral formula is:

$$V - E + F = 2 \tag{1}$$

where V is the number of vertices, E is the number of edges, and F is the number of faces. Cauchy connected the study of polyhedra to planar graphs (Alama 2009a).

Euler's polyhedral formula generalizations, by Euler-Lhuilier-Cauchy are:

$$V - E + F = 2 - 2g \text{ and } V - E + F - C = 1, \tag{2}$$

where g is a given genus and C is the number of components.

So the Euler characteristic of a closed orientable surface can be calculated from its genus g or, intuitively, from the number of its "handles" or "holes". Euler's polyhedral formula was generalized by Schläfli to higher dimensions (Coxeter 1973). It was also known as the Euler- Poincaré formula, because of Poincaré's role in it (Alama 2009a).

Euler-Poincaré formula:

$$N_0 - N_1 + N_2 - N_3 + \ldots \pm N_k = \begin{cases} 2, k := even \\ 0, k := odd \end{cases} \tag{3}$$

Euler-Poincaré characteristic:

For every simply connected polyhedron p of dimension k+1, we have

$$\chi(E) = \sum_{i=0}^{k} (-1)^i N_i = \begin{cases} 2, k := even \\ 0, k := odd \end{cases} \tag{4}$$

where N_i is the number of polytopes p of dimension i. (Alama 2009b)

A polytope is a finite region of n-dimensional space bounded by hyperplanes, the geometrical entity represented by the general term of the infinite sequence: point (node), line (edge), polygon (face), polyhedron (cell), polychoron (4-facet), polyteron (5-facet), etc.

Euler's polyhedral formula (Eq. 1) was originally defined for polyhedra by Leonhard Euler, and it was used to prove the classification of the Platonic solids, the five regular convex polyhedra. This formula has also been widely used in graph theory: for example, the proof of the famous Four Color Theorem's solution was based on the applications of Euler's formula. Euler's polyhedral formula is defined on the sphere, and the value of Euler characteristic is $\chi = 2$. On the closed orientable surface, the Euler characteristic generally corresponds to $\chi = 2 - 2g$, where g is the genus, the number of holes or handles of the surface. (Biggs, Lloyd & Wilson 1986)

For example, in case of a torus, g = 1 and the Euler characteristic $\chi = 0$. If there are C components, it can be shown that V - E + F - C = 1. These are the first generalizations (Eq. 2) of Euler's characteristic (Sachs 1970).

The third generalization of Euler's characteristic is the Euler-Poincare formula (Eq. 3) and the Euler-Poincare characteristic (Eq. 4), which is also defined on dimensions higher than three. The Euler-Poincare characteristic $\chi(E)$ is an alternating sum, and it is always equal to zero on any closed even-dimensional manifold; it is two on any closed odd-dimensional manifold if genus g = 0 (Alama 2009a, Alama 2009b, Sachs 1970).

The value of the genus cannot be seen in case of an even-dimensional manifold. This may be proved in two-dimensional Euclidean space, where the manifolds are perceived as polygons. The formula in that case is V - E = 0. A two-dimensional hole into the polygon cannot be made without dividing the polygon into two components, unlike the case of a three-dimensional polyhedron, where it is possible to make a three-dimensional hole. In the four-dimensional Euclidean space, we have V - E + F - C = 0. And, as can be seen, there is no genus in this formula either. Also Carlo H. Séquin did write about this topic in his article *Generalized Euler-Poincaré Theorem:* "I normally deal with more complex objects that also may have holes or tunnels, and sometimes the data file contains a description of several objects. Thus I need a more general formula that can accommodate all these cases" (Séquin 2008).

We introduce the Euler-Poincare generalization (Eq. 5), where the value of the characteristic $\chi(E)$ may be other than zero on the closed odd-dimensional, n-dimensional manifolds, if genus g_i, where $i \leq n$ and $g_i \notin g_j$, when $i < j$. When $g_i \in g_j$ and $i < j$, the result will be different! *The notation $g_i \in g_j$ means, that in a j-dimensional hole there is an i-dimensional sub-hole*[50]. To serve as an example of the five-dimensional manifold (Eq. 6), we may have the five-dimensional holes (handles) g_4 and three-dimensional pseudoholes g_2, which may change the value of $\chi(E)$, unlike the four-dimensional pseudohole g_3, which does not display any changes. *As a conclusion, it seems that the holes of the manifolds differ from each other, depending on whether the dimension N is odd or even*.

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50 For example if we remove one of the three faces from the hole, which has in Figure 150 a shape of a triangular prism.

9.3.2 Generalizing genus in the Euler-Poincare characteristic

We can generalize the Euler-Poincaré characteristic by adding, in the formula of an n-polytope, holes which have a dimension smaller than n. We call these kinds of holes *pseudoholes*.

For every simply connected polyhedron p of dimension k, we have

$$\chi(E) = \sum_{i=0}^{k}(-1)^i N_i = 1 + (-1)^k - \sum_{i=1}^{k}[1 + (-1)^i]g_i \qquad (5)$$

where N_i is the number of polytopes of p of dimension i and where g_i is the genus – intuitively, the number of "handles" or "holes" of dimension i. Genus g_i tells us the number of holes (handles) of the manifold. If $i < k$, we say that the hole is a *pseudohole*. The same word is used also in operator theory with a slightly different meaning ()(Bosch et al. 1982, Jung, Ko & Pearcy 2001).

The boundary of a two-dimensional hole is a compound of vertices and edges. The boundary of a three-dimensional hole is a compound of vertices, edges and faces. The boundary of a four-dimensional hole is a compound of vertices, edges, faces, and three-dimensional cells. In general: The boundary of an N-dimension hole is a compound of 0,..., N-1-cells.

Considering the problem in 5D:

$$N_0 - N_1 + N_2 - N_3 + N_4 = 2 - 2g_2 - 2g_4 \qquad (6)$$

Some examples may be demonstrated with a cube (Example 1), where $\chi(E) = N_0 - N_1 + N_2 = 2 - 2g_2$, and with a four-dimensional pentachoron (Examples 2 and 3), where $\chi(E) = N_0 - N_1 + N_2 - N_3 = 2 - 2g_2 + 0g_3$. As can be seen from the formula, a four-dimensional genus g_3 does not display any changes, but a three-dimensional genus g_2 does.

Example 1. A cube with genus $g_2 = 1$.

A cube consists of 8 points, 12 edges and 6 faces. Hence $V = N_0 = 8$, $E = N_1 = 12$, and $F = N_2 = 6 \Rightarrow \chi(E) = 8 - 12 + 6 = 2$. If a two-dimensional triangle is added to the two opposite two-dimensional faces and both triangles are connected to one corner by edges (Figure 149), the following results are obtained: 2*3 = 6 more points, 2*4 = 8 more edges and 2*1 = 2 more faces. For Euler's characteristic, the value is $\chi(E) = 14 - 20 + 8 = 2$.

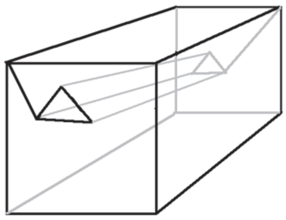

Figure 149 A Three-dimensional hole in cube

If the triangle points are connected through the cube, it may be seen that the cube is penetrated by a triangular prism. If the prism is removed, a hole will remain in the cube and a polyhedron with genus $g = g_2 = 1 \Rightarrow \chi(E) = 0$ will be obtained. To verify this result, the number of elements is calculated. The number of faces has increased from 8 to 9. That is, while the three faces of the triangular prism inside the cube have been added, the two triangle ends have been removed from the number of faces of the cube => $F = 8 – 2 + 3 = 9$. The number of edges has grown by three, due to the three edges of the prism inside the cube. The number of points remains the same. Now we get the value for Euler's characteristic $\chi(E) = 14 – 23 + 9 = 0$.

Example 2. A Pentachoron with genus $g_3 = 1$.

A pentachoron is covered by 5 points, 10 edges, 10 faces and 5 tetrahedrons. Hence $N_0 = 5$, $N_1 = 10$, $N_2 = 10$ and $N_3 = 5 \Rightarrow \chi(E) = 5 – 10 + 10 - 5 = 0$. If a three-dimensional tetrahedron is added to two opposite three-dimensional tetrahedrons of the pentachoron (*compare with the triangles in the previous example*), which are connected by an edge to one corner, the following values are obtained: 2*4 = 8 more points, 2*(6+1) = 14 more edges, 2*4 = 8 more faces and 2*1 = 2 more tetrahedrons. We get the value of Euler's characteristic $\chi(E) = 13 – 24 + 18 - 7 = 0$.

If the points of the two added tetrahedrons are connected through the pentachoron, it may be seen that the pentachoron is penetrated by a tetrahedral prism (Figure 150, edited by the author using some public figures)[51]. The tetrahedral prism consists of 8 points, 16 edges, 14 faces, and 6 three-dimensional cells consisting of 2 tetrahedrons and 4 triangular prisms. If the tetrahedral prism is removed, a hole in the pentachoron is made and thus a 4-polytope with the genus $g = g_3 = 1$ is obtained. Because genus g_3 does not exist in $\chi(E)$, when $N = 4$, the formula is to be checked by counting the number of the elements. The number of three-dimensional cells has increased from 7 to 9. Inside the pentachoron, four triangular prisms have been added. However, two tetrahedron ends were removed from the tetrahedral prism => $N_3 = 7 – 2 + 4 = 9$. Inside the pentachoron, six more faces of the four triangular prisms have been added. All of them form a part of the two triangular prisms. Due to the four edges of the

51 Attribution must be given to Robert Webb's Stella software as the creator of this image along with a link to the website: http://www.software3d.com/Stella.php.

prism inside the pentachoron, the number of edges has increased by four. The number of points remains the same. Now we get the value for Euler's characteristic $\chi(E) = 13 - 28 + 24 - 9 = 0$. This result is expected, as the four-dimensional genus g_3 does not display any changes.

Example 3. A Pentachoron with genus $g_2 = 1$.

The same pentachoron $\chi(E) = 5 - 10 + 10 - 5 = 0$, as in Example 2, is used. If two triangular faces which are connected by an edge to one corner of a tetrahedron are added on two opposite tetrahedrons of the pentachoron, we shall get the following values: 2*3 = 6 more points, 2*(3+1) = 8 more edges and 2*1 = 2 more faces. The value of Euler's characteristics is $\chi(E) = 11 - 18 + 12 - 5 = 0$.

Figure 150 A Pentachoron with genus $g_2 = 1$

If we connect the points of the two triangles through the pentachoron, we can see that there is a triangular prism (Figure 151) inside the pentachoron. If we remove this prism, the pentachoron will be penetrated by the triangular prism and we shall get a 4-polytope with a three-dimensional pseudohole and genus $g_2 = 1$. To verify this result, we may count the number of elements. The number of three-dimensional cells remains the same. Inside the pentachoron three faces of the triangular prism have been added while two triangle ends of the prism have been removed => $N_2 = 12 - 2 + 3 = 13$. Three more edges of the triangular prism have also been added. The number of points remains the same. Now we shall get the value for Euler's characteristic $\chi(E) = 11 - (18 + 3) + (12 - 2 + 3) - 5 = 11 - 21 + 13 - 5 = -2 = -2g_2$.

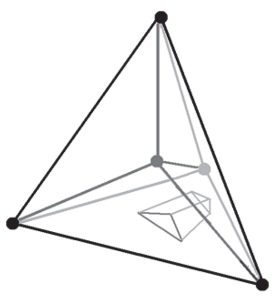

Figure 151 A three-dimensional pseudohole in tetrahedron

Thus, if a 4-dimensional polytope has a 3-dimensional genus of value 1, then => $\chi(E) = -2$. Finally, to understand Example 3, we may consider a simple case demonstrated by a cube and a 2-dimensional genus.

Example 4. A Cube with genus $g_1 = 1$.

A cube is a three-dimensional object, and hence we can apply the original Euler's polyhedral formula $\chi(E) = N_0 – N_1 + N_2 = 2$. We add in the formula one parameter, genus g_1, which is the number of two-dimensional pseudoholes. Now we get the following formula: $\chi(E) = N_0 – N_1 + N_2 = 2 – 0g_1$.

A cube consists of 8 points, 12 edges and 6 faces. Hence $V = N_0 = 8$, $E = N_1 = 12$ and $F = N_2 = 6 \Rightarrow \chi(E) = 8 – 12 + 6 = 2$. If one-dimensional edges (a,b) and (c,d) are added on two opposite two-dimensional faces which are connected to two corners (10) by an edge, we get the following values: 2*2 = 4 more points, 2*3 = 6 more edges and 2*1 = 2 more faces. For Euler's characteristic we get the result $\chi(E) = 12 – 18 + 8 = 2$.

If the end points of the edges (a,b) and (c,d) are connected through the cube with the opposite edge, it may be seen that the cube is penetrated by a two-dimensional plane (Figure 152).

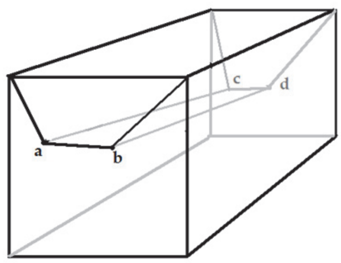

Figure 152 A two-dimensional pseudohole in cube

If the plane is reduced, a two-dimensional "hole", a pseudohole, in the cube will be made and a polyhedron will have the genus $g_1 = 1$. This is a hole which cannot be penetrated by any three-dimensional object as it has the width but its thickness equals to zero. The number of faces on the cube remain the same. Two edges (a,c) and (b,d) inside the cube have been added, and edges (a,b) and (c,d) have been removed from the faces of the cube. The number of points remains the same. The value for Euler's characteristic is then $\chi(E) = 12 – (18 - 2 +2) + 8 = 12 – 18 + 8 = 2$.

We can see that, even though the thickness of the two-dimensional hole (a,c,d,b) is zero, it separates the faces, which are inside the cube, on its both sides.

9.3.3 Determining holes in a k-simplex

As an example, we take a pentachoron, where Euler's characteristic has the following form: $\chi(E) = N_0 - N_1 + N_2 - N_3 = 5 - 10 + 10 - 5 = 0$.

The pentachoron is surrounded by five tetrahedra. Inside two of them, we add a smaller tetrahedron (N_3), and then we add an edge from one corner of these small tetrahedra to one corner of the greater tetrahedron. In this way, the edges of the pentachoron will stay connected. Thus the number of tetrahedra of the pentachoron is increased by two, the number of faces by eight, the number of edges by 14 and the number of vertices by eight. Figure 154 in the next Section 9.4 illustrates how the hole is built. Figure 153 illustrates a bit simplier 3D hole in a pentachoron.

The formula will now have the following form:

$\chi(E) = (5 + 8) - (10 + 14) + (10 + 8) - (5 + 2) = 13 - 24 + 18 - 7 = 0$

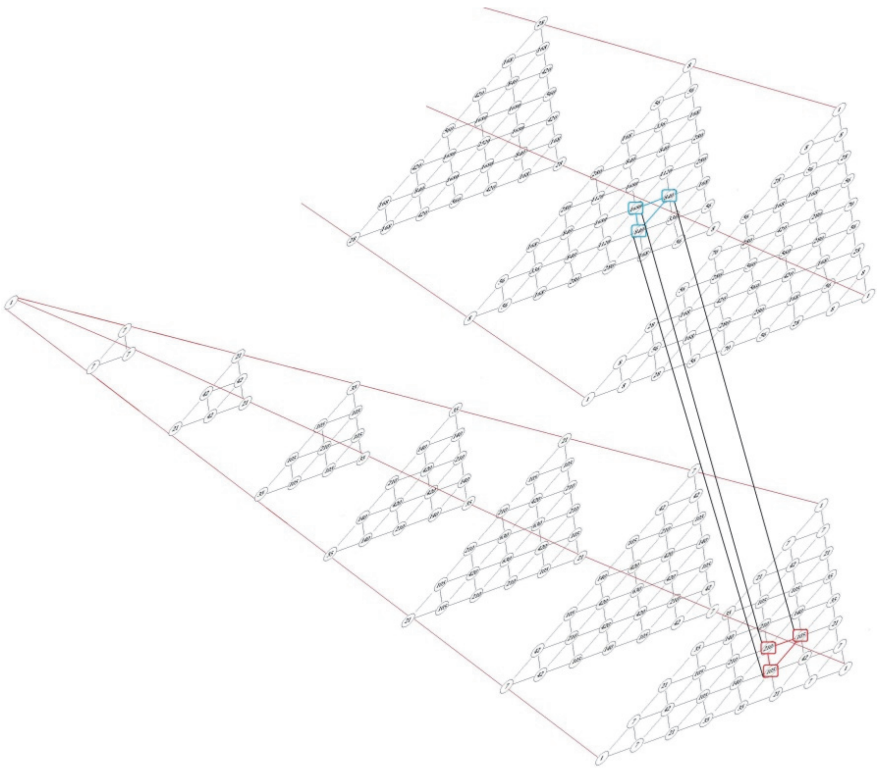

Figure 153 A 3D-hole in a pentachoron

More about holes and their building can be found in Section 9.4.

9.3.4 Conclusions

In general, when we make an i-dimensional hole, M_i, on a k-dimensional object, N_k ($i \leq k$), we get the following: $N_0 – N_1 + N_2 – N_3 + ... \pm N_k – M_1 + M_2 – M_3 + ... \pm (M_{i-1} – 2)$. Here in short are the results of examples 1 – 4.

Result of example 1: **V - E + F** - e + (f-2) = **14 - 20 + 8** - 3 + (3-2) = 14 - 23 + 9 = 0.
Result of example 2: **$N_0 – N_1 + N_2 – N_3$** – $M_1 + M_2 – (M_3 – 2)$ = **13 - 24 + 18 - 7** - 4 + 6 - (4-2) = 13 - 28 + 24 - 9 = 0.
Result of example 3: **$N_0 – N_1 + N_2 – N_3$** – $M_1 + (M_2 – 2)$ = **11 – 18 + 12 - 5** - 3 + (3-2) = 11 - 21 + 13 - 5 = -2
Result of example 4: **V - E + F** - (e-2) = **12 - 18 + 8** - (2 - 2) = 12 – 18 + 8 = 2.

For genus g_i: $i < n$ and $g_i \notin g_j$, when $i < j$.

9.4 Odd and even Euclidean dimensions[52]

In Section 9.3, it was possible to see the different values of the Euler characteristic, depending on whether the dimension of the Euclidean space was odd or even. The Euler characteristic is null when the dimension is even but two if the dimension is odd. For every simply connected polyhedron p of dimension k+1, we have

$$\chi(E) = \sum_{i=0}^{k}(-1)^i N_i = \begin{cases} 2, k := even \\ 0, k := odd \end{cases}$$

The same variability can be seen also when we observe the generalizations of the Euler characteristic and pseudo-holes:

$\chi(E) = N_0 – N_1 + N_2 – N_3 + N_4 - \ldots + N_k = 2 - 2g_2 - 2g_4 \ldots - 2g_k$, if k is even,
$\chi(E) = N_0 – N_1 + N_2 – N_3 + N_4 - \ldots - N_k = - 2g_2 - 2g_4 \ldots - 2g_{k-1}$, if k is odd.

Why is this? Let's see what happens in the Euler characteristic if we add one hole into the N-simplex, when N = 2, 3 and 4. Thus we explain what happens in the formula in cases of a triangle, tetrahedron and pentachoron.

Triangle: $\chi(E) = N_0 – N_1 = 3 – 3 = 0$
Tetrahedron: $\chi(E) = N_0 – N_1 + N_2 = 4 – 6 + 4 = 2$
Pentachoron: $\chi(E) = N_0 – N_1 + N_2 – N_3 = 5 – 10 + 10 – 5 = 0$

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52 *This topic emerged when writing Section 9.3 describing the generalization of the Euler characteristic.*

Next, on the outer edge of each of these we add the borders of the hole. On the two edges of the triangle, we add two points (N_0), which means that the number of edges (N_1) increases by two. On the two faces of the tetrahedron, we add triangles (N_2). This operation increases both the number of points and edges by six. In addition, both triangles are connected from one corner point to one corner of the corresponding face. In this way, all the edges on the tetrahedron are linked to each other, and the corresponding graph stays connected. The pentachoron is surrounded by five tetrahedra. Inside two of them, we add a smaller tetrahedron (N_3), and then we add an edge from one corner of these small tetrahedra to one corner of the greater tetrahedron. In this way, the edges of the pentachoron will stay connected. Thus, the number of tetrahedra of the pentachoron is increased by two, the number of faces by eight, the number of edges by 14 and the number of vertices by eight. The formulas will now have the following form:

Triangle: $\chi(E) = (3 + 4) - (3 + 4) = 7 - 7 = 0$
Tetrahedron: $\chi(E) = (4 + 6) - (6 + 8) + (4 + 2) = 10 - 14 + 6 = 2$
Pentachoron: $\chi(E) = (5 + 8) - (10 + 14) + (10 + 8) - (5 + 2) = 13 - 24 + 18 - 7 = 0$

After this, we make a hole or opening in each of the above items. In the case of the triangle, we connect the points on the opposite sides by edges (Figure 154, center). In the tetrahedron, we connect the triangles on the opposing faces by the edges from the corner points of the triangles (Figure 154, right). In the pentachoron, we connect the four corner vertices of the two smaller tetrahedra placed on opposite tetrahedral sides (Figure 154, left).

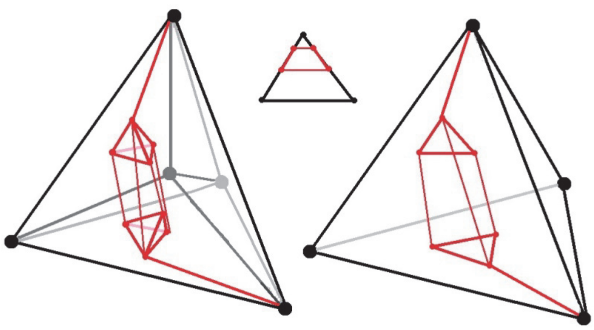

Figure 154 Holes in dimensions 4, 2, and 3.

Next we add the new elements to the above formulas, which present new characteristics for these manifolds, and then we "open" the holes. This is done with the triangle by removing the two edges from the both sides. In Figure 154 (center), these are the red edges on the left and right. The points of these edges are not removed. In case of the tetrahedron, we remove the two triangles in the center of faces. The corner points and edges are not removed. Inside the tetrahedron there will be three rectangular faces, which surround the hole that has the shape of a triangular prism. From the pentachoron, we remove "the inner space", which means that the corner points, edges, and faces of the tetrahedron are not removed. As with the tetrahedron and the triangle, the manifolds, borders of a hole. In this way, we have made through the pentachoron a hole, which has a shape of tetrahedral prism. The structure of the tetrahedral prism is explained in Section 9.3 referring to Figure 150. In this way, the formulas will have the following forms:

Triangle: $\chi(E) = 7 – (7- 2 + 2) = 7 - 7 = 0$
Tetrahedron: $\chi(E) = 10 – (14 + 3) + (6 – 2 + 3) = 10 - 17 + 7 = 0$
Pentachoron: $\chi(E) = 13 – (24 + 4) + (18 – 8 + 14) – (7 – 2 + 4) = 13 – 28 + 24 - 9 = 0$

We notice, as previously, that in case of the three-dimensional tetrahedron, the value of the Euler-Poincare characteristic is changed from 2 to 0, but in two- and four-dimensional Euclidean spaces, which are even, the value remains the same. In the even dimensions, the Euler-Poincare formula doesn't give a specified genus, and the value of the characteristic remains the same. In the two-dimensional Euclidean space "a hole" in practice splits the plane into two parts (Figure 154, in center). But does this also happen in other even dimensions? Next, we consider the pentachoron to see how the formula works in four-dimensional space. Figure 155 shows the four-dimensional hole referred to in the Euler-Poincare formula above. In this figure, the hole is described with the help of the coordinates of the multinomial formula. On the left, the red and blue lines show the tetrahedra, which form the openings into a hole. On the right, black lines connect the corner points of the red and blue tetrahedra. As a result, we get a tetrahedral prism, which pierces the pentachoron and hence creates a hole through it.

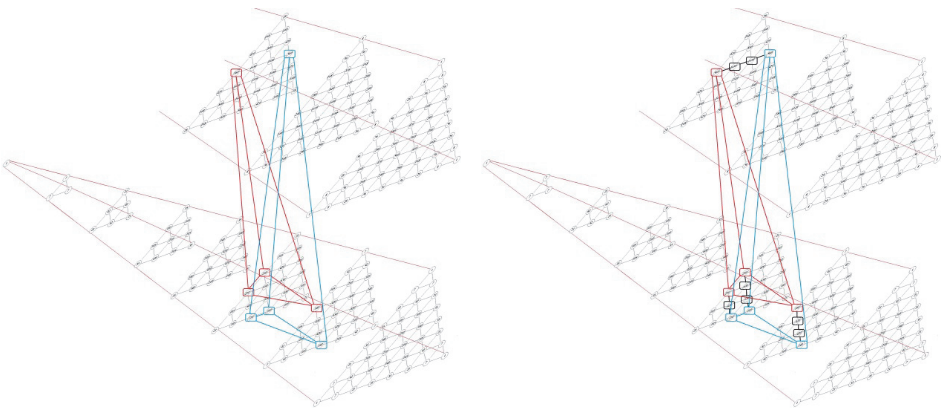

Figure 155 The structure of 4D-hole in pentachoron

We can examine the structure of the hole with the help of its coordinates. There are two openings to the hole, two three-dimensional tetrahedra, as shown in Figure 155 in blue and red colors. Below are the coordinates of the corner points and the coordinates of the edges that connect these corner points through the pentachoron.

Blue tetrahedron : C(8,2,3,3,0) = 560, C(7,2,0,2,3) = 210, C(7,2,0,3,2) = 210 and C(7,1,0,3,3) = 140.
Red tetrahedron: C(8,2,3,0,3) = 560, C(7,2,2,0,3) = 210, C(7,2,3,0,2) = 210 and C(7,1,3,0,3) = 140.

The edges that connect tetrahedra are:
C(8,2,3,0,3) - C(8,2,3,1,2) - C(8,2,3,2,1) - C(8,2,3,3,0) in Figure 155 from red to blue on the 8th level 560 – 1680 – 1680 – 560; C(7,2,0,2,3) – C(7,2,1,1,3) – C(7,2,2,0,3) from blue to red on the 7th level 210 – 420 – 210; C(7,2,0,3,2) – C(7,2,1,2,2) – C(7,2,2,1,2) – C(7,2,3,0,2) from blue to red on the 7th level 210 – 630 – 630 – 210; and C(7,1,0,3,3) – C(7,1,1,2,3) – C(7,1,2,1,3) – C(7,1,3,0,3) from blue to red on the 7th level 140 – 420 – 420 – 140.

If, in the case of a pentachoron, we also removed the faces of the tetrahedra, then the Euler-Poincare formula would not work, because the result would be

$$\chi(E) = 13 - (24 + 4) + (18 - 8 + 6) - (7 - 2 + 4) = 13 - 28 + 24 - 9 = -8$$

This means, that the opening of the four-dimensional hole is a three-dimensional tetrahedron which is closed by four faces just the same as the three-dimensional opening in Figure 153 is a triangle, which is closed by three edges. Further, we note that the hole in the pentachoron does not split it the way the hole in the two-dimensional triangle does. It remains as a single piece, for all the vertices are connected to each other, which can be seen in Figure 155. The hole is a tetrahedral prism, all the coordinates of which were given above.

9.4.1 Generalized Pi

Surprisingly, we can also see the same alteration in case of Pi's ratios. In two-dimensional Euclidean space, the ratio of a circumference to its diameter in a circle is π. In other words, when the radius is r, the diameter $H_1 = 2r$ and the circumference $K_2 = 2\pi r$, where index 2 is a dimension. In the same way, we can see that the ratio of a three-dimensional sphere, K_3, to its two-dimensional circle-diameter is 4 ($K_3/H_2 = 4\pi r^2/\pi r^2 = 4$). In general we get the following formulas for V_n and S_n (Weisstein 2002, Huang & He 2008, Lasserre 2001):

$$V_n = C_n r^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} r^n, \text{ where } \Gamma \text{ is the gamma function and } \Gamma(n) = (n-1)! \qquad (1)$$

We can compute the n-dimensional volume V_n of the unit n-ball in n-dimensional Euclidean space also without the gamma function (Parks 2013).

When n is even, volume $V_n = C_n r^n = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} r^n$ (2)

and the area of sphere $S_n = nD_n r^{n-1} = n \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} r^{n-1} \Rightarrow S_{n+1} = (n+1) \frac{\pi^{\frac{n+1}{2}}}{\left(\frac{n+1}{2}\right)!} r^n$ (3)

In this case, the ratio $S_{n+1}/V_n = \frac{2^{n+1}[\left(\frac{n}{2}\right)!]^2}{n!}$ (McDonald 2003).

When n is odd, we can use the property $\Gamma(1/2) = \sqrt{\pi}$ of the gamma function and the formula $\Gamma(x+1) = (x+1)\,\Gamma(x)$. Now we have $\Gamma(1+\frac{n}{2}) = \left(\frac{n}{2}\right)! = \sqrt{\pi}\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}\ldots\frac{n}{2} = \ldots = \sqrt{\pi}\frac{(n+1)!}{\left(\frac{n+1}{2}\right)!2^{n+1}}$. When we replace the gamma function with this formula, we finally get the volume $V_n = \frac{2^n\left(\frac{n-1}{2}\right)!\pi^{\frac{n-1}{2}}}{n!} r^n$ of the n-ball, (4)

and the area of sphere $S_n = \frac{2n\left(\frac{n-1}{2}\right)!(4\pi)^{\frac{n-1}{2}}}{n!} r^{n-1} \Rightarrow S_{n+1} = \frac{2(n+1)\left(\frac{n}{2}\right)!(4\pi)^{\frac{n}{2}}}{(n+1)!} r^n$ (McDonald 2003). (5)

In this case the ratio $S_{n+1}/V_n = \frac{(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\pi$

It is possible to evaluate the factorial function n! by using the Stirling approximation (Parks 2013, Marsaglia & Marsaglia 1990), which gives $n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$.

Next we insert the Stirling approximation to the ratio when n is even:

$$S_{n+1}/V_n = \frac{2^{n+1}[\left(\frac{n}{2}\right)!]^2}{n!} \approx \frac{2^{n+1}[\sqrt{n\pi}\left(\frac{n}{2e}\right)^{\frac{n}{2}}]^2}{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n} = \frac{2^{n+1}n\pi\left(\frac{n}{2e}\right)^n}{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n} = \frac{2^{n+1}n\pi\left(\frac{n}{e}\right)^n 2^{-n}}{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n} = \frac{2n\pi}{\sqrt{2n\pi}} = \sqrt{2\pi n}.$$

Stirling approximation, when n is odd, is as follows:

$$S_{n+1}/V_n = \frac{(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\pi \approx \sqrt{\frac{2\,\pi\,(n+1)^{n+1}}{(n-1)^n}\frac{1}{e}}$$

Generalized Pi converges towards $\sqrt{2\pi n}$, when even n → ∞ and towards $\sqrt{\frac{2\,\pi\,(n+1)^{n+1}}{(n-1)^n}\frac{1}{e}}$, when odd n → ∞. In Table 7, the values are given with two decimals. The Stirling approximations are compared with S_{n+1}/V_n. The schema of even numbers is used also in case of odd numbers, and the result is written in cursive within brackets. The decimal number, S_{n+1}/V_n, is a rational number in all even dimensions n, and it is an irrational number in all odd dimensions n. The irrational number in odd dimensions is kπ, where k is a rational number. In the fourth column, there is the coefficient C_n of volume V_n and, in the fifth column, there is the coefficient D_n of the area S_n. The values of C_n and D_n are irrational when n > 1, while C_n grows until n= 5 and then gets smaller. The value of D_n grows until n = 7, and after that it gets smaller. It is remarkable, and interesting as well, that the volume of the unit n-ball approaches 0 as $n \to \infty$ (Parks 2013). This is obvious when we observe the formula of V_n (1), where the numerator is the power function of π and the denominator is a factorial function that grows faster than the nominator. In the table, also $S_{n+1}/V_n = D_{n+1}/C_n$.

Table 7 The growth of generalized Pi

n	$\sqrt{2\pi n}$	S_{n+1}/V_n	$\frac{1}{e}\sqrt{\frac{2\,\pi(n+1)^{n+1}}{(n-1)^n}}$	C_n	D_n
0	0	2		1	
1	*2.50*	3.14 (π)	∞	2	2
2	3.5	4		3.14 (π)	6.28
3	(*4.34)*	4.71	5.21	4.19	12.57
4	5.01	5.33		4.93	19.74
5	(*5.60*)	5.89	6.22	**5.26**	26.32
6	6.14	6.40		5.16	31.01
7	(*6.63)*	6.87	7.14	4.72	**33.07**
8	7.09	7.31		4.06	32.47
9	(*7.52)*	7.73	7.96	3.39	29.69
10	7.93	8.13		2.55	25.50
11	(*8.31)*	8.50	8.71	1.88	20.73
12	8.68	8.87		1.34	16.02
13	(*9.04)*	9.21	9.39	0.91	11.84
14	9.38	9.55		0.59	8.39
15	(*9.71)*	9.87	10.04	0.38	5.72
16	10.03	10.18		0.24	3.77
…					
30	13.73	13.84		0.00…	0.003
…					
75	(*21.71)*	21.78	21.85	0.00…	0.00…

The ratio S_{n+1}/V_n, which includes π, is a rational number when n is even, and is irrational when n is odd. This is a natural result from the schemas (2) – (5) of the volume and surface area, where the denominator of π is 2. Because of this, π has the same exponent always in two dimensions. Because the dimension of the surface area is always one dimension higher, π disappears in every second dimension. Figure 156 shows the variation given in Table 7.

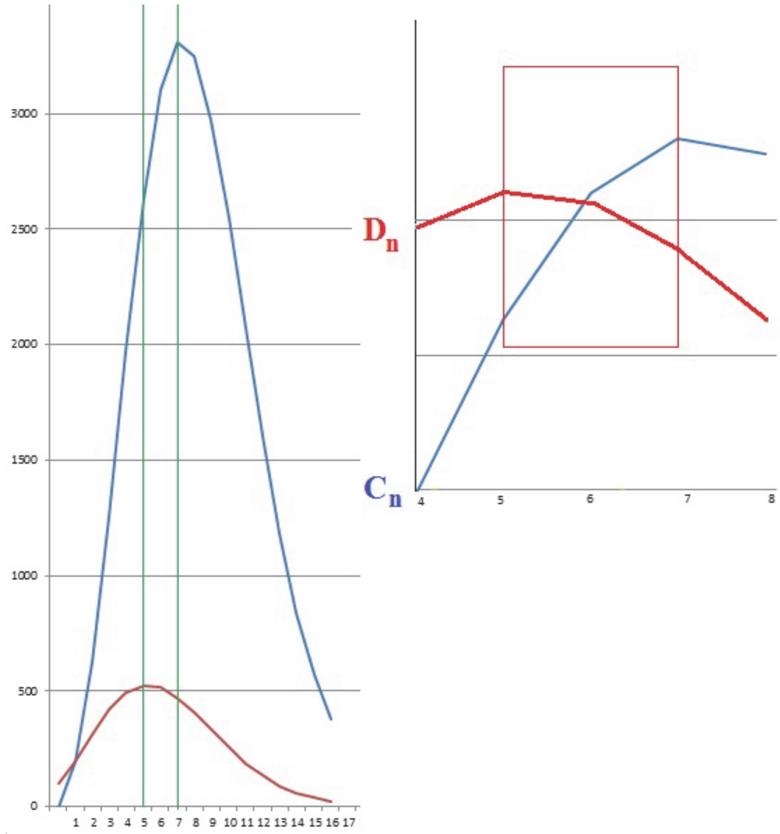

Figure 156 N-ball's volume and surface area

The N-ball includes several special features, one of them being the N-ball's ratio with a hypercube with same diameter and in the same dimension. If a circle with radius r is embedded in a square so that it touches all the four sides of a square, it follows that this circle, which has the area of size πr^2, takes 79% of the area of the square. In the same way, a ball in a cube fills 52% of the cube. In general, when the growth of n from 1 to ∞, the ratio of N-hypercube/N-ball grows as follows:

n = 1	2	3	4	5	6	7 …
100%	79%	52%	31%	16%	8%	4%

We now take a brief look at the growth of the N-ball's volume and surface area when N is growing. C_n of the volume grows to dimension five, and D_n of the surface area grows to dimension seven. From this it follows that, between dimensions 5 and 7, Cn gets smaller and D_n becomes larger. After observing the volumes of the n-sphere and the hypercube in different dimensions, we take a look at the volume of an n-simplex, because this polytope has been playing a central role in this research. The general schema for n-simplexes is V_n =

$\frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}}$ (Buchholz & Smith 1996, Buchholz 1992), where a is the length of an edge in the simplex. For example, if n = 2, then $V_2 = \frac{\sqrt{3}}{4}a^2$ and V_2 is the area of a triangle. If n = 3, then $V_3 = \frac{1}{6\sqrt{2}}a^3$ and V_3 is the volume of a tetrahedron. If a = 1, we get following values for V_n:

n =	1	2	3	4	5	6 ...
Vn =	0.500	0.433	0.118	0.023	0.004	0.0004

9.4.2 Pi and Napier's number

Pi and Napier's number

$$S_{n+1}/V_n = \frac{(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\pi \approx \sqrt{\frac{2\,\pi(n+1)^{n+1}}{(n-1)^n}}\frac{1}{e}$$

Next we approximate Pi in relation with the Napier's number by using the Stirling formula: $n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$.

This gives $n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \Rightarrow \pi \approx \frac{[(n-1)!]^2 e^{2n}}{2n^{2n-1}}$, where with small values of n we get for π the next approximations (π) and the distance (Δ) Pi:

n =	1	2	3	4	5	6 …
$\pi\approx$	3.694	3.412	3.320	3.275	3.248	3.230
$\Delta\approx$	0.552	0.270	0.178	0.133	0.106	0.088

If we approximate the Napier's number by π, it is easier by using another formula, namely the one we used earlier with odd numbers:

$$S_{n+1}/V_n = \frac{(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\pi \approx \sqrt{\frac{2\,\pi(n+1)^{n+1}}{(n-1)^n}}\frac{1}{e} \Rightarrow e = \frac{2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!\sqrt{2(n+1)^{n+1}}}{(n+1)!\sqrt{\pi(n-1)^n}}.$$

e ≈ 2.718281…, and this formula when n = 3, 5, and 7 gives the approximations e ≈ 3.009..., 2.871… and 2.827…

9.4.3 Summary

The ratio of a circumference to its diameter in a circle is Pi. The ratio of the area of sphere to its diameter circle (the great circle) is four. Generally the ratio of an N-sphere to its (N-1)-diameter grows with N, but very slowly. This ratio as N increases partly resembles the logarithm function (Figure 157). Interestingly, Pi occurs in the ratio only if N is even. If N is odd, the ratio is a rational number. This may be connected with the fact that in the higher-dimensional, generalized Euler-Poincare formula the characteristic is 0 if N is even.

In other words, Pi exists in the N-sphere when N = 2, 4, 6, 8, …, and the genus in the Euler-Poincare formula does not exist if N = 2, 4, 6, 8, … In the N-sphere, both the volume and the surface area schemes are especially interesting in dimensions from 5 to 7. Between these dimensions, the volume gets smaller, even though the surface area grows when radius r = 1. Naturally we are dealing here only with Euclidean dimensions, but as a curiosity we may recall that in quantum physics the fifth dimension has a connection, via Kaluza-Klein's theory, to gravitation theory (Pope 2002).

The intermediary theory was developed in 1919 by Theodor Kaluza, and later, in the 1920's, it was made more complete by Oskar Klein. The same theory was presented even earlier, in 1914, by Finnish Gunnar Nordström, who discussed his results with Albert Einstein, as did also Kaluza. Einstein seized Nordström's theory, but was not able to apply it and hence tried to find some other kind of solution, but without success. However, Nordström's ideas are still relevant, because the Great Unified Theory (GUT) and the Theory of Everything (ToE) are not complete, and the higher dimensions play a central role in their possible solution. (Norton 1993, Ravndal 2004)

9.5 Generalizations of the Fibonacci sequence

This issue was raised in Section 8.2 when defining the n-simplex game board coordinates.

In defining how the pieces of the simplex model move, we were able to show that the Fibonacci numbers can also be found in Pascal's triangle. These can also be traced by the movement directions of the bishop in the hexagonal trichess game.

9.5.1 Planar generalizations

A question arises concerning the knight's movement directions, which results in another number sequence. This particular sequence, which is called Fibonacci 2-numbers, was published by Stakhov and Roz in 2005 and forms a special case among Fibonacci p-numbers. For example, Fibonacci 4-numbers can be produced by a "hyper knight" on a hexagonal trichess board (Section 6.2.2), the hyper knight being the next new officer on the 4^{th} defense zone. It moves on a hexagonal board one step, like a bishop, and then two steps, like a rook. Fibonacci-2 numbers are created by the sums to the right top direction. (Stakhov & Rozin 2006)

Generalized Fibonacci numbers are the sums of the elements found on successive diagonals of Pascal's triangle, written in a left-justified form, beginning in the left-most column and moving up (c-1) and right throughout the array (Bicknell-Johnson & Spears 1996).

Modern natural science requires the development of a new mathematical apparatus. The generalized Fibonacci numbers, or Fibonacci p-numbers, (p = 0,1,2,3, . .), which appear in the ''diagonal sums'' of Pascal's triangle and are assigned in the recurrent form, are a new mathematical discovery. The aim is to derive analytical formulas for the Fibonacci p-numbers. We can use the derivation of the Binet formula in order to calculate the Fibonacci 2-numbers. If p=2, the recurrence relation takes the following form: $F_2(n) = F_2(n-1) + F_2(n-3)$; $F_p(0) = 0$, $F_p(1) = F_p(2) = 1$. For p = 4, the formula takes the following form: $F_4(n) = F_4(n-1) + F_4(n-4)$; $F_4(0) = 0$, $F_4(1) = F_4(2) = F_4(3) = F_4(4) = 1$. (Stakhov & Rozin 2006)

The famous Binet formula is used to count the n^{th} Fibonacci number, and the previous sequences, where p= 2 and p= 4, give the numbers 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, … and 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, … Several sequences of these kinds, made of generalized Fibonacci numbers from the Pascal's triangle, can be found. For example, the knight's move on a hexagonal board placed on Pascal's triangle gives also the sequence Kn(n) = Kn(n-1) + Kn(n-2) + Kn(n-4) = 1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, … This sequence, among many others, can be found in several publications, formulated in slightly different ways (Sharp 2000, Krcadinac 2006). For these sequences, see Figure 157.

This division has been named High-Phi by Martin Gardner to match the often-used symbol for the Golden Section whose logical child it is. The associated recurrence sequence, like the Fibonacci sequence for the Golden Section, is $u_n = 2u_{n-1} - u_{n-2} + u_{n-3}$ = 0, 1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, … (Sharp 2000)

By the binomial theorem we get, for the k^{th} upper Fibonacci sequence, $F_n^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i+1} F_{n-i}^{(k)} + F_{n-k-1}^{(k)}$. Of course, sequence $F_n^{(1)}$ is just a sequence of Fibonacci numbers. The second, upper Fibonacci sequence is $F_n^{(2)}$ = 1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, … 114, … (Krcadinac 2006)

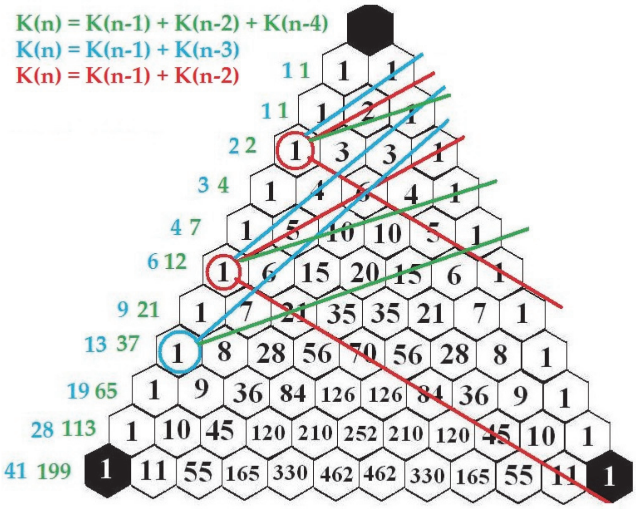

Figure 157 Some generalized Fibonacci sequences

Figure 157 shows two of the mentioned sequences on Pascal's triangle. The red lines stand for Fibonacci sequence and the green lines for Gardner's High-Phi.

9.5.2 N-dimensional generalizations

After extending the game into multidimensional game boards, it seemed reasonable to see what happens to the sequences there. An interesting case was the rook in a three-dimensional space. The related sequence was discovered in 1965 by a German mathematician Ernst Jacobsthal, and it was named after him.

Microcontrollers, which are small computers used by embedded systems in different home appliances, and other computers use conditional instructions to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This turns out to be useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits, ..., which are exactly the Jacobsthal numbers. (by Hirst, C. "Hopscotch--Multiple Bit Testing." May 15, 2006). (Weisstein 2015)

We continued our research into dimensions higher than three to see what could be found after the Jacobsthal numbers. Our next result was the sequence F(n) = F(n-1) + 3F(n-2) = 1, 4, 7, 19, 40… This sequence was counted from a 4-dimensional Pascal's Pyramid from the same inclination as the Fibonacci numbers from Pascal's triangle and Jacobsthal numbers from Pascal's pyramid. Martin Erik Horn published that same sequence in 2007 after writing in 2003 about Pascal's Pyramids (which we discussed in Chapter 8).

The Pascal Plane, which consists of binomial coefficients, can be generalized into the Pascal Space by using trinomial coefficients. Then the Pascal's Pyramid can be constructed by adding up three appropriate neighboring numbers and writing the result beneath them. The next step is to increase the dimension again by using quatronomial coefficients, which fill the four-dimensional Pascal Hyper-Space. Five Pascal's Hyper-Pyramids can be found. The three-dimensional hyper-surfaces of these four-dimensional hyper-pyramids consist of Pascal's Pyramids. This procedure can be continued infinitely many times. The multinomial coefficients live in an n-dimensional Pascal Hyper-Space, and, with the help of n + 1, Pascal's Hyper-Pyramids can be constructed. These n-dimensional hyper-pyramids possess (n–1)-dimensional hyper-surfaces which look like Pascal's Pyramids minus one dimension. (Horn 2003)

Mathematics is an astonishing subject. It is hard to know what the basis of mathematics is. And yet, there are mathematical constructions, which are fascinatingly beautiful and deeply impressive. They seem to reflect something of the hidden truth. This also applies to Fibonacci numbers of higher orders. They can be constructed by adding appropriate numbers of Fibonacci hyper-pyramids corresponding to the appropriate multinomial coefficients. For example, in four-dimensional Pascal space there is

$$\frac{n!}{m_1!m_2!m_3!(n-m_1-m_2-m_3)!} = 1, 4, 7, 19, 40, 97, 217, \text{etc. (Horn 2007)}$$

Our next step, in future research, is to combine the two previous generalizations. This means changing, also in the n-dimensional simplexes, the directions where the sequences are counted.

9.5.3 Fibonacci polynomials and some of their extensions

The Fibonacci polynomials are a polynomial sequence which can be considered as a generalization of the Fibonacci numbers. The first Fibonacci polynomials are:

$F_0(x) = 0$
$F_1(x) = 1$
$F_2(x) = x$
$F_3(x) = x^2 + 1$
$F_4(x) = x^3 + 2x$
$F_5(x) = x^4 + 3x^2 + 1$
$F_6(x) = x^5 + 4x^3 + 3x$
…
In general, when x = 1: $F_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$ (Benjamin & Quinn 1999).

If x = 1, we get the basic Fibonacci sequence. If x = 2, we get the above-mentioned Gardner's High-Phi sequence (green lines in Figure 157). The numbers of the Jacobsthal sequence have been constructed from Pascal's pyramid in the same way as the Fibonacci numbers from Pascal's triangle. If the operation that Martin Horn carried out above is repeated in the four-dimensional Pascal's pyramid, we get the sequence he presented: 1, 4, 7, 19, 40, 97, … In this research, we continued this process to higher dimensions, and Table 8 lists these "n-Fibonacci" numbers up to the dimension of n = 11.

Table 8 The first eleven "n-Fibonacci" sequences

N	2	3	4	5	6	7	8	9	10	11
F(0)	1	1	1	1	1	1	1	1	1	1
F(1)	1	1	1	1	1	1	1	1	1	1
F(2)	2	3	4	5	6	7	8	9	10	11
F(3)	3	5	7	9	11	13	15	17	19	21
F(4)	5	11	19	29	41	55	71	89	109	131
F(5)	8	21	40	65	96	133	176	225	280	341
F(6)	13	43	97	181	301	463	673	937	1261	1651
F(7)	21	85	217	441	781	1261	1905	2737	3781	5061
F(8)	34	171	508	1165	2286	4039	6616	10233	15130	21571
F(9)	55	341	1159	2929	6191	11605	19951	32129	49156	72181
F(10)	89	683	2683	7589	17621	35839	66263	113993	185329	287891
F(11)	144	1365	6160	19305	48576					

These sequences have been known earlier. For example, Sburlati wrote about some of them in 2002 and in 2016, and J.V. Leyendekkers and A.G. Shannon presented a table similar to ours above. There exists very extensive literature on generalized Fibonacci sequences, with interesting applications to number theory, where such sequences are treated as a particular case of a more general class of sequences of numbers. (Sburlati 2002)

Various characteristics of the ordinary Fibonacci and Lucas sequences, many known for centuries, have been compared and associated with generalized sequences related to the Golden Ratio. German mathematician Johannes Kepler showed that the ratio of consecutive Fibonacci numbers converges to the Golden Ratio. This is also the case for the members of the Golden Ratio Family associated with generalized Fibonacci sequences. We have sought an analogue for the other Golden Ratio Fibonacci numbers, Fn(a). (Leyendekkers & Shannon 2016)

We wrote those sequences above for the following polynomials:

$F(0) = 1$

$F(1) = 1$

$F(2) = 1 + x$

$F(3) = 1 + 2x$

$F(4) = 1 + 3x + x^2$

$F(5) = 1 + 4x + 3x^2$

$F(6) = 1 + 5x + 6x^2 + x^3$

$F(7) = 1 + 6x + 10x^2 + 4x^3$

$F(8) = 1 + 7x + 15x^2 + 10x^3 + x^4$

$F(9) = 1 + 8x + 21x^2 + 20 x^3 + 5x^4$

In general: F(k) = $\sum_{i=0}^{\lceil k/2 \rceil} \binom{k-i}{i} x^i$, where k is the level in Pascal's multi-simplex.

There might also be some connections with the primes to be found. That was the reason for coloring primes in red in Table 8. For example, F(5) and F(7) can never be primes because F(5) = 1 + 4x + $3x^2$ = (3n+1)(n+1) and F(7) = = 1 + 6x + $10x^2$ + $4x^3$ = (2n+1)($2n^2$+4n+1).

9.5.4 Summary

The next step is to combine the two above-mentioned generalizations. The simplest way is to calculate the Jacobsthal numbers in a three-dimensional model. This sequence begins with 1, 1, 1, 5, 13, 25, 57, 111, ... We have not yet found this sequence in the literature, so it might be a new one. The next Jacobsthal sequence can be found in a four-dimensional Pascal's pyramid, and its first numbers are 1, 1, 1, 10, 28, 55, 172, ...

9.6 Other concluding remarks

In this chapter we present two two topics, in which we have made less research. However, they might open totally new fields to find new results.

9.6.1 Induced cycles on Pascal's polytopes

This section provides some highlights on the induced cycles of Pascal's Polytopes. The issue came out in connection with Section 8.2. "Induced Cycles of the Pascal's Polytopes" was presented by us in the 21th Workshop on Cycles and Colourings (2012).

The extension of Pascal's arithmetical triangle to dimension N is called Pascal's Tetrahedron or Pascal's Pyramid and the extension to the fourth dimension is known as Pascal's pentachoron. In general, we refer to them as Pascal's polytopes. The concept of an induced or chordless cycle means an induced subgraph in an undirected graph G, where no two non-consecutive vertices of which are adjacent. Induced cycles are also called chordless cycles or graph holes. The problem of monitoring long induced cycles in hypercubes is known as the coil-in-the-box problem and the problem of monitoring the long induced paths in hypercubes is known as the snake-in-the-box problem. Our aim is to extend this research to simplex-models. In Figure 158, there are a few examples of induced cycles in 2-, and 3-simplex models.

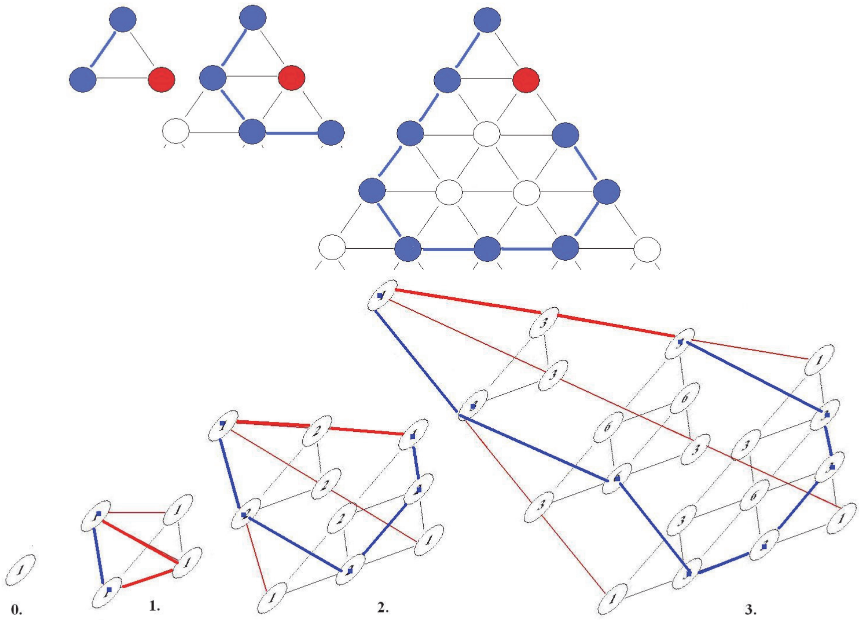

Figure 158 Some induced cycles on Pascal's triangle and pyramid

9.6.2 Notes on a domination number with some chess variations

Domination problems and independent sets is another field where a lot more research on various chess boards could be done. We clarify this with some background. The study of domination in graphs originated around 1850 with problems of placing the minimum number of queens or other chess pieces on an n x n chess board so as to cover/dominate every square. Nearly all of the earliest domination citations concern problems of placing various chess pieces on a board. The earliest theorems concerning dominating sets were given by Berge, in 1958. (Hedetniemi & Laskar 1988)

However, at least on planar chess boards – not only on rectangular, but also hexagonal and triangular boards – there are already lots of results. In a hexagonal tessellation of the plane, a knight attacks 12 hexagons and every free hexagon is attacked by at most 6 independent knights. Thus an independent set of knights can cover at most one-third of the plane. Several chess pieces are proposed. The independence numbers for almost all pieces have been considered now (Figure 159). Some small gaps are still open, however, the queens keeping their independence numbers secret. The queens independence numbers are not known. (Bode, Harborth & Harborth 2003, Bode, Harborth & Kultan 2006, Bode & Harborth 2002)

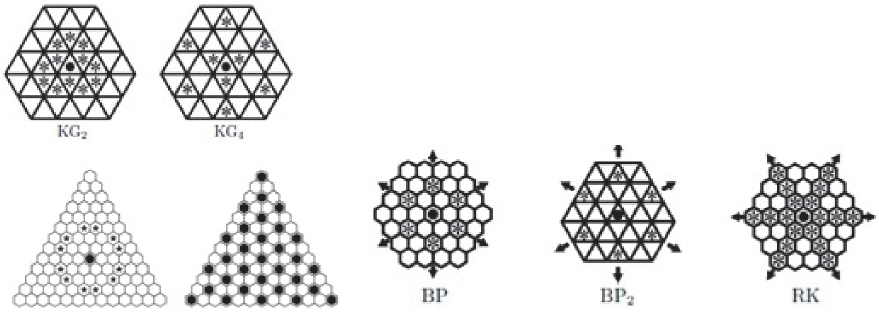

Figure 159 Domination on hexagonal and triangular boards

Previous research has mostly been made on planar chess boards and traditional chess pieces. It seems to be possible to generalize these to problems on different chess boards with different pieces.

10 SUMMARY

In this research *a mathematical, symmetric n-player game model, based on chess,* was discussed in Chapter 8. Before the creation of the model, several basic issues, such as the game theory (Chapter 2) and tiling (Chapter 3) had to be explored. Due to the earlier results mentioned in Chapter 6 and Chapter 7, the evolution of board games and chess (Chapters 4 and Chapter 5) is also included. During this research a number of spin-off results and observations were discovered. Some of them may be found in Chapter 9 and are meant to be used in further research. These topics include a *generalization of the Euler-Poincare characteristic*, concerning the genus (number of holes) in different Euclidean dimensions (9.3), *the differences between the even and odd Euclidean dimensions* (9.4) and *generalizations of Fibonacci sequences* (9.5). In addition to these topics, notes on various themes are included. For example, t*he number of a graph*, started by Frank Harary (9.1), is not a new idea, however, to our knowledge, it has not been presented in detail so far. *Strategy networks for small games* (9.2) with its leaning towards cognitive science were primarily introduced to explore decision-making processes of different people. Its extension to traditional chess is hardly possible. The topics of the last two sub-sections, *induced cycles* (9.6.1) and *domination* (9.6.2), serve as a basis for expanding this research to new areas. In addition, the main chapters offer some new observations. The chapter aimed at the early history of gaming offers two different hypotheses related to the birth of the *Hawaiian game konane* (4.1.5.1). Was the game created locally or imported? Either answer is interesting. If the game was imported, then the genesis of such games dates back thousands of years. If the game was invented locally, it is telling us about our way of thinking and about the creation of the same type of game structures in different parts of the world. The second comment in the same chapter is related to the famous *Cretan Phaistos disc* (4.3.2). This study supports the theory that this was a game. The theory is based on an ancient Iranian game, which seems to be the missing link between two other ancient games in Egypt and Mesopotamia. A minor observation concerns *an extension of Pascal's rule* (8.3), which is so simple that it is difficult to believe it is a new rule. Nevertheless, it has not been found in the literature.

YHTEENVETO (FINNISH SUMMARY)

Tutkimuksessa on rakennettu *matemaattinen, shakkiin pohjautuva n-pelaajan symmetrinen pelimalli.* Mallia käsitellään 8. luvussa. Symmetrisyydellä tarkoitetaan tässä pelaajien asemaa toisiinsa nähden. Siirtojärjestys luonnollisesti rikkoo symmetrisyyden, mutta siihenkin on löydettävissä ratkaisu. Motiivi pelimallin rakentamiseen lähti siitä, että monen pelaajan pelien kohdalla on muuttuvia tekijöitä niin paljon, että niiden kohdalla optimaalisten strategioiden löytäminen matemaattisesti on liki mahdotonta. Paras tapa löytää parhaat toimintatavat on simulointi. Tämän vuoksi oli mielekästä kehittää malli, jota voi hyödyntää peliteoreettisissa asetelmissa simuloimalla. Kun malli on kerran rakennettu, niin sitä voidaan soveltaa monella tavalla malliin pohjautuvien laskennallisten algoritmien avulla. Shakki on tässä vain perusrakenne, joka antaa selkeän pohjan. Eri pelaajien painotusarvoa voidaan muuttaa samoin kuin pelaajien nappuloiden painotusarvoja, jolloin mallilla voi peilata erilaisia tosielämän tilanteita. Peli, ja erityisesti shakki, antaa mielikuvan lautapelistä, mutta pelien kohdalla on kyse useiden eri toimijoiden keskinäisestä vuorovaikutuksesta. Peli, voi olla osa politiikan pelikenttää, metsän ekologiaa tai sään ennustamista.

Ennen mallin luomista oli kuitenkin selvitettävä sen taustalla olevia tekijöitä, joista peliteorian (luku 2) läpikäynti oli luonnollista. Tilan jakamista osiin (luku 3) eri tavoin oli tarpeen käsitellä pelimallin rakenteen vuoksi. Lautapelien ja shakin evoluutiovaiheet (luvut 4 ja 5) toimivat pohjana tekijän omille kehitelmille luvuissa 6 ja 7.

Tutkimuksen aikana syntyi lukuisia huomioita ja sivutuloksia, joista suurin osa on selkeyden vuoksi siirretty 9. lukuun työn loppuun jatkotutkimusaiheiksi. Tällaisia ovat *Eulerin-Poincaren kaavan yleistys* aukkojen määriä eri euklidisissa ulottuvuuksissa kuvaavan genus-muuttujan osalta (9.3), huomio *parillisten ja parittomien ulottuvuuksien eroista* (9.4), *Fibonaccin sarjojen yleistykset* (9.5.)

Näiden lisäksi on erilaisia huomioita. Aikanaan Frank Hararyn kanssa aloitettu *verkkojen nimeäminen yksikäsitteisesti lukujen avulla* (9.1) on ollut esillä muuallakin aiemmin lähinnä mainintana, mutta vastaavaa konkreettista esitystä on ollut vaikea löytää. *Pienten pelien strategiaverkot* (9.2) on tämän tutkielman aikana syntynyt kehitelmä, joka on suunnattu ensisijaisesti kognitiivisen tieteen puolelle tavoitteena tutkia eri ihmisten päätöksentekoprosesseja. Sen laajentaminen perinteisen shakin tasolle tuskin on mahdollista. Kahdessa viimeisessä aliluvussa esitellyt aiheet, *virittävät kehät* (9.6.1) ja *hallitsevuusluku* (9.6.2), antavat pohjan näiden alueiden tutkimuksen viemiselle uusille alueille.

Edellisten lisäksi on päätekstin luvuissa myös joitain uusia huomioita. Pelien varhaishistoriaa käsittelevässä luvussa on käsitelty *havaijilaisen Konanepelin syntyä* (4.1.5.1). Onko peli syntynyt paikallisesti vai onko tullut asutuksen mukana? Kumpikin vastaus on kiehtova. Jos peli on tuotu, niin se kertoo siitä, että tällaisten pelien synty juontaa tuhansien vuosien päähän. Jos peli on keksitty paikallisesti, niin se kertoo ajattelutapamme sisällä olevasta ominaisuudesta luoda samantyyppisiä pelirakenteita eri puolilla maailmaa. Toinen tähän lukuun liittyvä huomio perustuu kuuluisaan *kreetalaiseen Faistoksen kiekkoon* (4.3.2). Tässä tutkielmassa esitetään teoria, että kyseessä on peli ja huomio pohjautuu Iranista löytyneeseen muinaiseen peliin, joka näyttäisi olevan yhdistävä lenkki kahden muun muinaisen pelin, egyptiläisen ja mesopotamialaisen, välillä. Pienempi havainto on *Pascalin säännön laajennus* (8.3), joka on niin yksinkertainen, että se ei voi olla uusi. Yrityksistä huolimatta sitä ei kuitenkaan löytynyt kirjallisuudesta.

Koko tutkimuksen päätavoite ja tulos oli kuitenkin luoda symmetrinen n-pelaajan strategiapeli, koska tiettävästi ei ole olemassa yhtään yksinkertaista matemaattista ratkaisua, jolla voisi mallintaa n-pelaajan strategiapelejä.

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APPENDIX 1 – EMAILS ABOUT TRICHESS

This is how the whole process did start with Frank Harary.

1.
From fnh@NMSU.Edu Sat May 15 22:48:54 1993
Subject: Re: Golden Knight
----
and another perhaps fascinating chessy problem, not a game:

On a bxb board, what is the largest no. of squares that a knight can
visit without any square occurring more than once?

i.e. What is the longest path length in the Kt(b)-graph ??
ooooops, not length but 1 + length = no. of vertices 1L in a longest path.

b: 3 4 5 6 7 8 9 ....

1L: 8 15 25 ?? ?? 64 ??

This one question MIGHT be fully solvable.

2.
Dear Frank,

Yet some comments about the Hamiltonian Knights.

Here you got 35.

"22 25 34 16 32 27
35 15 23 26 06 17
24 21 07 33 28 31
14 01 29 18 10 05
20 08 03 12 30 xx
02 13 19 09 04 11"

Did you get my fax on Monday (24th)? There was drawn
a hamiltonian path in KT6. It was made by computer.
When I compared that solution to your one, it seems
that you missed in sixth step, which should have gone
where now is 34th. The best strategy seems to be such
that it is not good to go in the centre if not necessary.

Here is the "fax"-solution:

25 18 07 32 23 16
08 31 24 17 06 33
19 26 09 30 15 22
10 01 12 21 34 05
27 20 03 36 29 14
02 11 28 13 04 35

I also counted with same strategy an hour ago another solution:

01 16 29 22 07 14
28 21 08 15 32 23
09 02 17 30 13 06
18 27 20 33 24 31
03 10 38 26 05 12
36 19 04 11 34 25

"For b=7, I can only get 47 do I'm missing 2 squares."

I also got only 47. Very interesting. I try tomorrow also
a computer solution for my homeputer b=7 is too much, (s)he
doesn't stop counting.

By the way I found again a funny small result. The number of edges in KTb-graphs, when b=3,4,5,6,7,8 is 8,24,48,80,120,168, which means that e(KTb) = e(8xe(K(b-1)) or in other words: the number of edges in KTb is the number of edges in complete graph of b-1 vertice multiplied by 8.

It was also funny to see that the number of edges in KT8 is same like the number of vertices in that chessboard I told you in the morning.

3.
Subject: 48 and other stories

Dear Frank,

You were right in KT7 I got 48 vertice. I counted by hand
for the silly program didn't stop counting if there were
no Hamiltonian path!

Here:

01 32 13 22 03 24 47
12 21 02 xx 48 15 04

31 38 33 14 23 46 25
20 11 30 45 26 05 16
37 34 39 08 29 42 27
10 19 36 41 44 17 06
35 40 09 18 07 28 43

Hope it is right,
bye, Jorma

4.
From fnh@NMSU.Edu Mon May 31 20:22:53 1993
Subject: Re: 48 and other stories

DJ,
I believe 48 is correct; interesting! This is a nice question for 'mathematical recreation'.

5.
palikka:/home/tukki/tko/jaek% mail fnh@NMSU.Edu
Subject: KT9, etc.

Dear Frank,

" Conjecture; On Kt-bxb-board, there is always a Kt path either on all b^2 squares or on b^2 - 1 squares. Do you agree? "

Interesting. It would also be interesting to find out the order when there are all the squares and when -1 squares.

Now it is like this:
4: -1
5: all
6: all
7: -1
8: all
9: all, for

The path of KT9:

01 30 17 44 03 32 19 46 05
56 43 02 31 18 45 04 33 20
29 16 57 80 75 70 59 06 47
42 55 74 69 58 65 76 21 34
15 28 79 64 81 60 71 48 07
54 41 68 73 62 77 66 35 22

27 14 63 78 67 72 61 08 49
40 53 12 25 38 51 10 23 36
13 26 39 52 11 24 37 50 09

This is very nice and easy path, you just follow the outer circle.

And also KT13 has all the squares, for the chessboard with a hole KT13-KT9 has a path! And because of the same reason the KTKTK KT9 is solvable for KT9-KT5 has a path and KT5 is solvable.

Now I got one common result, which I explain tomorrow,

Result: All the KTb, when b 4n+1 and n = 1,2,3,... have a path where are all the squares and just once.

6.
From fnh@NMSU.Edu Tue Jun 1 20:19:05 1993
Subject: Re: KT9, etc.

DJ,
I greatly enjoyed your KT9 path!

7.
From fnh@NMSU.Edu Tue Jun 1 20:34 EET 1993
Date: Tue, 1 Jun 93 11:30:42 MDT

DJ,

Here is some important information from Schwenk:
Subject: RE: Knight-hamiltonian square boards

For the square bxb boards, only b=2 and 4 (for even b's) fail to have a ham cycle. Of course all the odd boards cannot have a ham cycle since the graph is bipartite and the order is odd. I haven't cataloged the ham paths.

For rectangular boards, ham cycles on an mxn board, with m <= n, exist unless:
(1) m and n are both odd
(2) m = 1 or 2 or 4
(3) m = 3 and n = 4 or 6 or 8.

8.
Thu Jun 3 00:40 EET 1993

Dear Frank,

Yesterday I told you that it is always possible to make the Knight's Tour on bxb -board, in such way that Knight visits exactly once in every square, if b = 4n+1, where n = 1, 2, 3,...

A. Schwenk, told in his mail, that on bxb -board there can always be found the Hamiltonian path if b is even number except the cases when b = 2 or 4. This follows that also if b = 4n or 4n+2 the result above is valid. So there is left only one case: b = 4n-1, n=1,2,3,...
(So these b = 4n + {-1,0,1,2} cover the boards b = 3,4,5,6,..)

Your conjecture said that if it was not possible to get all the squares then there should be left more than 1 square in any case. Since yesterday there were three 1 square cases, namely b=3, 4 and 7.

Now I would like to change this conjecture to a theorem, which says that on every bxb -board there can be found a path (every square just once), except if b = 3 and 4 (and of course b = 2). The case b = 3 is trivial and also the case b = 4 (which you counted to 15) is not difficult to prove. This can be seen clearly for example in that picture, I sent you by fax, where the hexagonal faced version of KT4 was embedded in the torus.

Here I give you KT7 with a path of 49 squares:
(so it slowly grew from 47 to 48 and 49)

31	42	21	02	33	44	23
20	01	32	43	22	03	34
41	30	09	12	15	24	45
08	19	14	47	10	35	04
29	40	11	16	13	46	25
18	07	38	27	48	05	36
39	28	17	06	37	26	49

And here yet the case b=11 (I've on same paper also KT15, but my fingers denied to type when they saw it)

1	20	39	112	3	22	41	14	5	24	43
38	111	2	21	40	113	4	23	42	115	6
19	56	87	88	77	58	89	100	79	44	25

110	37	76	57	88	99	78	59	90	7	116
55	18	97	86	65	68	71	80	101	26	45
36	109	64	75	70	103	66	91	60	117	8
17	54	85	96	67	72	69	102	81	46	27
108	35	74	63	94	83	104	61	92	9	118
53	16	95	84	73	62	93	82	121	28	47
34	107	14	51	32	105	12	49	30	119	10
15	52	33	106	13	50	31	120	11	48	29

When you make KT15 you only has to change 121 to 119 then change the labelling by adding 24 to each number and then change 225 where originally was 119. Start from B2 and after one circle you are in KT11.

Some explanations:

I only explain some main ideas of the proof. If you have a "chessroad" bx2 and b>=2 then you need 4 different paths to fill all the squares. If you make a "holy" chessboard: (b+2)x(b+2) - bxb it consists four this kind of chessroads with four paths. In every corner there are squares, that belong to some of these paths. And as a graph the NW-corner looks like this:

```
Path I:  east <--- B4 --- A2 --- C1 ---> south
          |       |
          |       |
Path II: south <-- C2 --- A1 --- B3 ---> east
          |       |
          |       |
Path III: east <--- A3 --- B1 --- D2 ---> south


Path IV: south <-- D1 --- B2 --- A4 ---> east
```

The idea of the proof is to follow these circular chessroads from outer circle towards the centre. In case b= 4m+1 this is enough, but in case b=4m-1 you start from the outer circle, go to the centre and then back to outer circle. In this second period you have to jump back to the inner circle always once. For example in KT11: 103 and 121.
The reason why it is necessary to follow the paths on chessroads like I've done can be explained in natural way. Paradoxal is only the fact that the to find the system how to follow the paths is most complicated in cases where b is even and such already proved!

Perhaps my explanation was a bit messy, but it is made too fast
and it is pretty difficult to draw good pictures in e-mail.

9.
Date: Fri, 23 Jul 93 00:25:28 +0200

Hello Frank,

I hope to send you the Knight-draft on the beginning of
next week, perhaps Tuesday. I did like your fascinating
way to remind about it.

Tomorrow more. By the way I'm now developing a new idea
(yes, yes, I leave it now for few days), a chess without
board!

10.
From fnh@NMSU.Edu Fri Jul 23 02:41 EET 1993
Subject: bored-less chess

dj,

I lk fwd to hearing about this nu type of chess, to see if it extends to other games! Such a paper wd not be boring.

11.
Subject: Alichess and ironwheels

Dear Frank,

About hexagonal tri-chess. Now I know at least 3 of them, all invented individually in different times and almost all on hexagonal boards:

a) Austrian Siegmund Wellisch in 1912 on 91-hexagonal board (no bishops). In this game horse is moving like in my first 46-chess, which I found wrong and then developed 87-chess.

b) Polish Glinski in 1949 made on same board a 2-person chess (they have started to play world championships from the year 1983! it is called simply "hexagonal chess" or Polish chess). Glinski had got exactly the same movements with buttons like I have!

c) Russian Safran made (in 50's?) a 2-person hexagonal chess on 70-board. This game didn't come popular,

d) Hexagonal trichess on 96-board (who?, when? where? don't know, only I know the reference)

e) In German was sold in 60's some trichess on triagonal board! By my opinion that game isn't anymore chess. I don't know more about it, and

f) Australian Englishman Patton made in 1975 a hexagonal trichess on 217 board!! He had same movements like Glinski (and me). By the way it is not difficult to find out where these boards have got their sights:

If a hexagon is surrounded by 5 levels of hexagons then we have 91 board, if a hexagon is surrounded by 8 levels of hexagons then we have 217-board. I also know how to develope the 96-board.

But I belive that it takes for a while to find out how I got to 87-board.

And this is what makes by my opinion my chess different. It is the only hexagonal chess which is created on same basis like traditional 64-chess.

I didn't try to invent a strange game; it came out in natural way!

12.

Date: Thu, 5 Aug 93 16:05:36 +0200

Dear Frank,

I didn't yet send the chess-draft, but I send it soon.
Then I send the Slovak-abstract.
And after these two I send trichess to M.G., in spite
of the fact that I would like to do it at first. This
is what they call self-discipline.
I've played now one 2 hours game with two amateur players
trichess. In two weeks also with the chessmaster. By my
opinion it seems, that the difference between t.ch. and
ordinary chess is something like between Canadian and
European icehockey. In trichess the situations change
very fast.

APPENDIX 2 - THE RULES OF THE ASYMMETRIC FOUR-HANDED CHESS

Chess pieces. At the beginning of the game, each player has nine either black, white, red or yellow pieces. Each has one king, one queen, one rook, one knight, one bishop, and four pawns.

Aim. The game's aim, opening procedure, and rules for moves are the same as in normal chess, except for the modifications described later.

A. The player who makes the first checkmate, to one of the opponents, will be the winner ("first checkmate to win").
B. The player whose king has been checkmated, will be out from the game, and the last player left on the board is the winner ("the last checkmate wins"). The player whose king is threatened and has no way to escape, can also resign and finish the game.

If method B is used in tournaments, then the game can be re-activated by giving two points to the winner, namely the last one to effect a checkmate, and one point to the players who made the first checkmates before it. If both of these are checkmates made by the same player, then that player gets four points. If one of the players resigns, then the winner gets all three points. Hence the game can end with the points distributed as: (4, 0, 0, 0), (3, 1, 0, 0), (2, 1, 1, 0) or (1, 1, 1, 1) among the players. The last one is the case, when all four players end up with a draw.

The Moves. Each piece can move from the point of departure, within the framework of the predetermined rules. A piece cannot move to a position that is in the area of another piece on its own side nor to a place that belongs to the area of one of the opponent's pieces. If the area of the moved piece intersects the area of one of the opponent's pieces, the latter is 'taken' and removed from the area of the game. Only the knight and the bishop may move over the areas of other pieces. The area of a piece may not intersect the boundary of the area of the game.
Because there are three opponents instead of one, the pawn can move to different directions against other opponents and capture in three directions. The pawn is coronated at the two opposite sides of the board.

Defeat. This rule is relevant only if the game has been agreed to be played with method B, as discussed earlier in this section. The players whose kings have been checkmated, have lost the game. Similarly, in case of resignation, the resigned player has lost the game. The player who has lost first is eliminated from the game, and that player's *pieces will remain "powerless" on the board*. "Powerless" means that they are not threatening any other piece but that they, including the king, can be captured from the board. So the game can end in two different ways, but the players must agree about this before they start the game. This rule is necessary to avoid a situation where the strategic positions would change on the board.

Victory. The last player left on board wins the game.

Draw between two players. If only two players are left, then the game between these players ends to a draw just like in traditional chess.

Draw between three or four players. The game can end in a draw also between all four players if the king of the player who has to do the next move, is not in check but this player cannot make any acceptable move. The game can end in a draw (stalemate) also between three players if one player is already in checkmate.

Bridge chess. The game can be played the way Trichess is played: all the players against each other. The game works well as a social game. Another way is to play it in two-person teams, either adjacent or opposing players on the same side. In this case, communication is prohibited during the game. This kind of game can be referred to as bridge chess after bridge, a well-known card game.