

Homeomorphisms of finite distortion: from the unit ball to
cusp domains in \mathbb{R}^3

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Abstract

We study the homeomorphisms of finite distortion from the unit ball onto cusp domains in \mathbb{R}^3 . Based on some works of Juhani Takkinen and Pekka Koskela in \mathbb{R}^2 , we are interested in the class of homeomorphism of finite distortion with the distortion function K_f is locally exponential integrable, which means there exists some constant $\lambda > 0$ such that $\exp(\lambda K_f(x))$ is locally L^1 -integrable. This class of homeomorphisms share many similar topological and geometric properties with quasiconformal mappings.

The origin of the problem can come back to Riemann mapping theorem, which characterize the domains in the complex plane \mathbb{C} that can be obtained as a conformal image of the unit ball $B^2(0, 1)$. But for higher dimensional case, this kind of Riemann mapping problem is pretty difficult, even for quasiconformal mappings, it is still open. Then we choose the special cusp domain in \mathbb{R}^3 to study, maybe it will give us some inspiration in how to do this kind of problems.

Key words and phases: Homeomorphism of finite distortion, Riemann mapping theorem, cusp domain, locally exponential integrable.

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1 Introduction

In this thesis, we are interested in homeomorphisms of finite distortion from the unit ball $B^3(0, 1) \subset \mathbb{R}^3$ onto a cusp domain $\Omega_s \subset \mathbb{R}^3$. The cusp domain Ω_s is the set:

$$(1.1) \quad \Omega_s = B^3(0, \sqrt{2}) \setminus \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1, \sqrt{x_2^2 + x_3^2} \leq x_1^{1+s} \right\}.$$

See Figure 1 below.

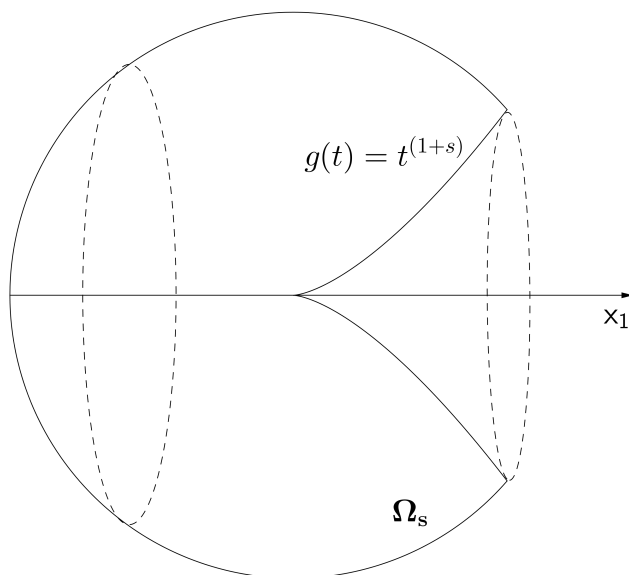


Figure 1: Cusp domain Ω_s

Definition 1.1. We say that a homeomorphism $f : \Omega \rightarrow f(\Omega) \subset \mathbb{R}^3$ on an open set $\Omega \subset \mathbb{R}^3$ has finite distortion if $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$ and there is a function $K : \Omega \rightarrow [1, \infty]$ with $K(x) < \infty$ almost everywhere such that

$$(1.2) \quad |Df(x)|^3 \leq K(x)J_f(x),$$

for almost all $x \in \Omega$. Here $|Df(x)| := \sup\{\|Df(x)e\| : e \in S^2(0, 1)\}$ and $S^2(0, 1)$ is the unit sphere in \mathbb{R}^3 .

If there is a constant $K \geq 1$ such that $K_f(x) \leq K$ for a.e. $x \in \Omega$, we call f a K -quasiconformal mapping. This is the so called analytic definition of quasiconformal mappings. We will give the geometric and metric definitions of quasiconformal mappings. For equivalence of these definitions in the Euclidean case, see [16].

For a homeomorphism of finite distortion we define the optimal distortion function as

$$(1.3) \quad K_f(x) := \begin{cases} \frac{|Df(x)|^3}{J_f(x)} & \text{for all } x \in \{J_f > 0\}, \\ 1 & \text{for all } x \in \{J_f = 0\}, \end{cases}$$

From now on, we only use the optimal distortion function.

In fact, there are different distortion functions. We call $K_f(x)$ as the outer distortion function of f . And sometimes, we also use the notation $K_O^f(x)$ for the outer distortion function. The optimal inner distortion function of f (f has finite distortion a.e.) is defined by

$$(1.4) \quad K_I^f(x) := \begin{cases} \frac{|D^\#f(x)|^3}{(J_f(x))^2} & \text{for all } x \in \{J_f > 0\}, \\ 1 & \text{for all } x \in \{J_f = 0\}, \end{cases}$$

Here $D^\#f(x)$ is the transpose of the cofactor matrix of $Df(x)$, i.e. the matrix of the 2×2 subdeterminants. We call $K_I^f(x)$ the inner distortion function of f . These distortion functions coincide for $n = 2$ but in general they are different for $n \geq 3$. And by a simple computation, we have the following double inequality

$$(K_I^f(x))^{\frac{1}{2}} \leq K_f(x) \leq (K_I^f(x))^2,$$

see [7, Chapter 7].

Based on [16, Page 62] and [10, Page 81-82], we know that there does not exist a quasiconformal mapping from the unit ball $B^3(0, 1) \subset \mathbb{R}^3$ onto the cusp domain Ω_s for any $s > 0$. The fact that quasiconformal mappings preserve n -capacity up to a constant plays a fundamental role in the proof of this result. For the convince of the reader, we will give the proof of such nonexistence result. We follow the argument in [10]. Here the capacity is a way to estimate the distance and the size of two sets in Euclidean space. We will give the exact definition of capacity in Section 2. From the definition, we see that homeomorphisms of finite distortion are generalizations of the class of quasiconformal mappings. Thus it is natural to ask whether there exists a homeomorphism of finite distortion from the unit ball $B^3(0, 1) \subset \mathbb{R}^3$ onto the cusp domain Ω_s or not? More precisely, we hope that the distortion function of the homeomorphism of finite distortion is locally exponentially integrable. We say $K_f(x)$ is locally exponentially integrable means that there exists a constant $\lambda > 0$ such that $\exp(\lambda K_f(x)) \in L_{loc}^1(B^3(0, 1))$. This class of homeomorphisms of finite distortion share many topological and geometrical properties with quasiconformal mappings.

For this kind of Riemann mapping problem for homeomorphisms of finite distortion, we obtain the following three results. Both existence and non-existence are contained. First, we describe the existence result:

Theorem 1.2. *For every $s > 0$, there exists a homeomorphism of finite distortion f from $B^3(0, 1)$ onto the cusp domain Ω_s such that its distortion function K_f satisfies*

$$\exp(\lambda K_f^\gamma(x)) \in L_{loc}^1(B^3(0, 1)),$$

for all $0 < \lambda$ and all $0 < \gamma < 1$.

To prove the result above, we will construct a suitable homeomorphism from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s .

We will also obtain the following two non-existence results:

Theorem 1.3. *For any $s > 0$, there does not exist a homeomorphism of finite distortion f from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s such that its distortion function K_f satisfies*

$$(1.5) \quad J_f(x) \exp(\lambda K_f^{\frac{1}{2}}(x)) \in L_{loc}^1(B^3(0, 1)),$$

for any $\lambda > 0$.

Another one reads as following.

Theorem 1.4. *For any $s > 0$, there does not exist a homeomorphism of finite distortion from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s such that its distortion function K_f satisfies*

$$\exp(\lambda K_f^\gamma) \in L_{loc}^1(B^3(0, 1)),$$

for any $\lambda > 0$ and $\gamma > \frac{3}{2}$.

The method of estimating the capacity has been used in the proofs of Theorem 1.3 and Theorem 1.4. For proving Theorem 1.4, we need the sharp modulus of continuity of homeomorphisms of finite distortion. These concepts will be explained.

From Theorem 1.2 and Theorem 1.4 above, the case of $\gamma \in [1, \frac{3}{2}]$ is still open. And we have the following conjecture.

Conjecture 1.5. *For any $s > 0$, there does not exist a homeomorphism of finite distortion from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s such that its distortion function K_f satisfies*

$$\exp(\lambda K_f^\gamma) \in L_{loc}^1(B^3(0, 1)),$$

for any $\lambda > 0$ and $\gamma > 1$.

For the case $\gamma = 1$, we believe there would exist such a homeomorphism $f : B^3 \rightarrow \Omega_s$ with $\exp(\lambda K_f(x)) \in L_{loc}^1(B^3)$ for sufficiently small $\lambda > 0$. And for large $\lambda > 0$, there does not exist a satisfying homeomorphism.

2 History

Our final goal is to give a geometric characterization of the image of the unit ball B^3 under a homeomorphism of finite distortion with a locally exponentially integrable distortion function. This is the homeomorphism of finite distortion version of the Riemann mapping theorem. Unfortunately, even a quasiconformal version of such problem is not known. The class of homeomorphisms of finite distortion is a generalization of the class of quasiconformal mappings; and the class of quasiconformal mappings is a generalization of the class of conformal mappings. Next, we return to conformal mappings.

2.1 Conformal mappings

Conformal mappings play a fundamental role in complex analysis. Let \mathbb{C} denote the complex plane, a complex-valued function $f : \mathbb{C} \rightarrow \mathbb{C}$ has a derivative at $z_0 \in \mathbb{C}$, if the limit

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Let us give the definition of conformal mappings in the complex plane.

Definition 2.1. *A complex-valued function $f(z)$, defined on an open set $\Omega \subset \mathbb{C}$, is said to be conformal in Ω if it is a homeomorphism and it has a derivative at each point of Ω .*

Conformal mappings enjoy many interesting properties. One of the most important one is the Riemann Mapping Theorem. Intuitively, a domain $U \subset \mathbb{C}$ is said to be simply connected if it is path-connected and there is no hole inside. Path-connected means that for every two points $x, y \in U$, we can find a curve in U with endpoints x and y . We call $\gamma \subset U$ a curve, if we can find a continuous mapping from $[0, 1]$ onto γ . If there is a conformal mapping from a domain $U \subset \mathbb{C}$ onto another domain $V \subset \mathbb{C}$, then we say that V is analytically isomorphic to U . Indeed the inverse of a conformal mapping is also conformal, so we can say U is also analytically isomorphic to V .

Theorem 2.2. *Let U be a simply connected open set which is not the whole plane. Then U is analytically isomorphic to the unit disc in the plane. More precisely, given $z_0 \in U$, there exists a conformal isomorphism*

$$f : U \rightarrow B^2(0, 1)$$

from U onto the unit disc, such that $f(z_0) = 0$. Such an isomorphism is uniquely determined up to a rotation, i.e. multiplication by $e^{i\theta}$ for some real θ , and is therefore uniquely determined by the additional condition

$$f'(z_0) > 0.$$

A proof of this result can be found in almost any complex analysis textbook, see e.g. [1, Theorem 1, Chapter 6].

From the theorem, we know that every simply connected open set which is not the whole plane is conformally equivalent with the unit disc in the plane. Since the inverse of a conformal mapping and the composition of conformal mappings are still conformal mappings, two planar simply connected domains are always conformally equivalent. However in the higher dimensional spaces, this is not correct.

A conformal mapping $f : \Omega \rightarrow f(\Omega)$ is a 1-quasiconformal; that is $|Df(x)|^2 = J_f(x)$ for every $x \in \Omega$. We use 1-quasiconformal mapping to extend the class of conformal mapping to $\mathbb{R}^n (n \geq 3)$. By a very famous theorem of Gehring, we know that every 1-quasiconformal mapping in high dimension (larger than 2) is simply the Möbius transformation. Recall that a Möbius transformation is a finite composition of reflections with respect to spheres and hyperplanes.

Theorem 2.3. *Let $\Omega, \Omega' \subset \mathbb{R}^n, n \geq 3$, be domains and $f : \Omega \rightarrow \Omega'$ be a 1-quasiconformal mapping. Then f is the restriction of a Möbius transformation to Ω .*

Proof. See e.g. P. Koskela's lecture notes [10]. □

We know that Möbius transformations transfer the unit ball to a ball or a half space, and for every ball or half space, we can find a Möbius transformation which transfers it to the unit ball. So in higher-dimension space, if a domain Ω is 1-quasiconformally equivalent to the unit ball, then Ω must be a ball or a half space.

Next, we discuss some basic results about quasiconformal mappings.

2.2 Quasiconformal mappings

There are several definitions of quasiconformal mappings, e.g. analytic (Just let $K_f(x)$ be uniformly bounded from above, see Definition 1.1), metric and geometric. J. Väisälä proved all these definitions are equivalent in \mathbb{R}^3 , see [16]. In this thesis, we just use the analytic definition of quasiconformal mappings. For those readers who are interested in the other definitions of quasiconformal mappings, we refer to Väisälä's book [16].

We need the modulus of a curve family. Curve family means the family whose elements are locally rectifiable curves. A curve in \mathbb{R}^3 is a continuous map γ of an interval $I \subset \mathbb{R}$ into \mathbb{R}^3 . We usually abuse terminology and call γ both the map and the image $\gamma(I)$. If $I = [a, b]$ is a closed interval, then the *length* of a curve $\gamma : I \rightarrow \mathbb{R}^3$ is

$$l(\gamma) = \text{length}(\gamma) = \sup \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i+1})|,$$

where the supremum is taken over all sequences $a = t_1 \leq t_2 \leq \dots \leq t_n \leq t_{n+1} = b$. If I is not closed, then we define the length of γ to be the supremum of the lengths of all closed subcurves of γ . A curve γ is rectifiable if its length is finite, and it is locally rectifiable if all its closed subcurves are rectifiable.

We will not distinguish among open, closed, or half-open intervals when they are domains of a curve; such distinction makes no difference in the forthcoming discussion. Notice, however, that if $\gamma : I \rightarrow \mathbb{R}^3$ is rectifiable and I is not closed, then γ has a unique extension to a rectifiable curve defined at the endpoints of I , where “endpoints” should be understood in the generalized sense if I is unbounded. (Strictly speaking, this extension takes values in the completion of \mathbb{R}^3 , but we ignore such issues here.)

Any rectifiable curve γ factors

$$(2.1) \quad \gamma = \gamma_s \circ S_\gamma,$$

where $S_\gamma : I \rightarrow [0, l(\gamma)]$ is the associated length function and $\gamma_s : [0, l(\gamma)] \rightarrow \mathbb{R}^3$ is the unique 1-Lipschitz continuous map such that the factorization in equation (2.1) holds. The curve γ_s is the arclength parametrization of γ .

If γ is a rectifiable curve in \mathbb{R}^3 , the line integral over γ of a Borel function $\rho : \mathbb{R}^3 \rightarrow [0, \infty]$ is

$$\int_\gamma \rho ds = \int_0^{l(\gamma)} \rho \circ \gamma_s(t) dt.$$

If γ is only locally rectifiable, we set

$$\int_\gamma \rho ds = \sup \int_{\gamma'} \rho ds,$$

where the supremum is taken over all rectifiable subcurves γ' of γ . If γ is not locally rectifiable, no line integrals are defined. Now we can give the definition of the modulus of curve family.

Definition 2.4. *Suppose that Γ is a curve family in \mathbb{R}^3 . That is, the elements of Γ are curves in \mathbb{R}^3 . We denote by $F(\Gamma)$ the set of all nonnegative Borel functions $\rho : \mathbb{R}^3 \rightarrow [0, \infty]$ such that*

$$\int_\gamma \rho ds \geq 1,$$

for every locally rectifiable curve $\gamma \in \Gamma$. For each $p \geq 1$ we set

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^3} \rho^p dm.$$

If $F(\Gamma) = \emptyset$, we define $M_p(\Gamma) = \infty$.

Here and what follows, we use the notation $M(\Gamma) := M_3(\Gamma)$ for a curve family Γ . Suppose that $f : D \rightarrow D'$ is a homeomorphism, where D and D' are domains in \mathbb{R}^3 . Consider a curve family Γ in D and its image family $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$. If f is conformal, then we can prove $M(\Gamma') = M(\Gamma)$. This fact can be generalized for quasiconformal mappings, see [16].

Corollary 2.5. *Suppose that $f : D \rightarrow D'$ is a K -quasiconformal mapping, where D and D' are domains in \mathbb{R}^3 . Then there exists a constant K_1 which just depends on K and the dimension 3, such that*

$$(2.2) \quad \frac{M(\Gamma)}{K_1} \leq M(\Gamma') \leq K_1 M(\Gamma)$$

for every curve family Γ in D .

In fact, we can use inequality (2.2) to define quasiconformal mappings. Let $f : D \rightarrow f(D)$ be a homeomorphism. If for every curve family Γ in D , the inequality (2.2) is satisfied, then we can prove that f is a quasiconformal mapping. It is the so-called geometric definition of quasiconformal mappings.

Corollary 2.6. *Let $f : D \rightarrow f(D)$ be a K -quasiconformal mapping, where D is a domain in \mathbb{R}^3 . Let $x \in D$ and $r > 0$. Define*

$$L_f(x, r) := \sup\{|f(x) - f(y)| : y \in D, |x - y| \leq r\},$$

$$l_f(x, r) := \inf\{|f(x) - f(y)| : y \in D, |x - y| \geq r\},$$

and

$$H_f(x, r) := \frac{L_f(x, r)}{l_f(x, r)}.$$

Then there exists a constant $H < \infty$ which just depends on K and the dimension 3, such that

$$H_f(x) := \limsup_{r \rightarrow 0} H_f(x, r) \leq H$$

for all $x \in D$.

We can also use Corollary (2.6) to define quasiconformal mappings. If a homeomorphism $f : D \rightarrow f(D)$ satisfies the result in the last corollary, then we can prove it is a quasiconformal mapping. That is the so-called metric definition of quasiconformal mappings. Using this definition we can extend the definition of quasiconformal mappings to general metric spaces.

As we emphasized before, it is a difficult problem to characterize the domains in Euclidean n -space that can be obtained as a quasiconformal image of the the unit ball $B^n(0, 1)$. It is a type of n -dimensional Riemann mapping problem. Gehring [3] reduced this problem to look at the boundary of domain. He showed that if a

quasiconformal mapping exists from a neighborhood of the boundary of a domain onto a neighborhood of $S^{n-1}(0, 1)$ in $B^n(0, 1)$, then a quasiconformal mapping exists between the domain and $B^n(0, 1)$. But unlike the Riemann mapping theorem (which solves the problem for $n = 2$), no conditions pertaining solely to the boundary have been discovered which guarantee a domain to be quasiconformally equivalent to $B^n(0, 1)$.

2.3 Homeomorphisms of finite distortion

The class of homeomorphisms of finite distortion is a generalization of the class of quasiconformal mappings, see Definition 1.1. Notice that when $K_f(x) \in L^\infty(\Omega)$, we recover the class of quasiconformal mappings. In this thesis, we will study homeomorphisms of finite distortion from the unit ball $B^3(0, 1)$ onto the cusp domain defined in (1.1).

3 Relative methods

In the proof of these results we got, we should use following several basic tools.

3.1 Sobolev functions

Theory of Sobolev functions is one of indispensable tools in some aspects of modern mathematics, such as analysis, PDE and so on. In this subsection, we give the definition and some properties of Sobolev functions. Throughout this subsection, let Ω denote an open subset of \mathbb{R}^3 . And $f \in C_0^\infty(\Omega)$ means that f is smooth and $\text{supp}(f) := \overline{\{x : f(x) \neq 0\}} \subset \Omega$ is compact.

Definition 3.1. Assume $f \in L_{loc}^1(\Omega)$, $i = 1, 2, 3$. We say $g_i \in L_{loc}^1(\Omega)$ is the weak partial derivative of f with respect to x_i in Ω if

$$(3.1) \quad \int_{\Omega} f \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} g_i \varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$.

By (3.1), we get that the weak partial derivative with respect to x_i , if it exists, is uniquely defined almost everywhere. We write

$$\frac{\partial f}{\partial x_i} \equiv g_i \quad (i = 1, 2, 3)$$

and

$$Df \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right),$$

provided the weak derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$ exist.

Definition 3.2. Let $1 \leq p \leq \infty$, the function f belongs to the Sobolev space $W^{1,p}(\Omega)$ if $f \in L^p(\Omega)$ and the weak partial derivatives $\frac{\partial f}{\partial x_i}$ exist and belong to $L^p(\Omega)$, $i = 1, 2, 3$. And we say f belongs to $W_{loc}^{1,p}(\Omega)$ if $f \in W^{1,p}(V)$ for each open set $V \subset\subset \Omega$. Here $V \subset\subset \Omega$ means that $V \subset \Omega$ and $\bar{V} \subset \Omega$. And if $f = (f_1, f_2, f_3)$ is a map from $\Omega \subset \mathbb{R}^3$ to \mathbb{R}^3 and $f_i \in W^{1,p}(\Omega)$ for every $i = 1, 2, 3$, then we say $f \in W^{1,p}(\Omega, \mathbb{R}^3)$.

If $f \in W^{1,p}(\Omega)$ we define

$$\|f\|_{W^{1,p}(\Omega)} \equiv \left(\int_{\Omega} |f|^p + |Df|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, and

$$\|f\|_{W^{1,\infty}(\Omega)} \equiv \text{ess sup}_{\Omega} (|f| + |Df|).$$

Here we denote $\text{ess sup}_{\Omega} f := \inf\{a \in \mathbb{R} : |\{x : |f(x)| > a\}| = 0\}$. We say a sequence of functions $\{f_k\}_{k=1}^{\infty} \subset W^{1,p}(\Omega)$ converges to a function $f \in W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$, if $\|f_k - f\|_{W^{1,p}(\Omega)}$ converges to 0 as k goes to infty. And $f_k \rightarrow f$ in $W_{loc}^{1,p}(\Omega)$ if $\|f_k - f\|_{W^{1,p}(V)}$ for each $V \subset\subset \Omega$. We are also interested in a subspace of $W^{1,p}(\Omega)$.

Definition 3.3. For $1 \leq p < \infty$, we define $W_0^{1,p}(\Omega)$ as the completion of $C_0^{\infty}(\Omega)$ in $W^{1,p}$ -norm.

Sobolev functions enjoy many interesting properties. We will give three propositions and for the proofs of these results, see [7, Theorem A.15].

Proposition 3.4. Let $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$, then u has a representative \tilde{u} (That means $|\{x \in \Omega : u(x) \neq \tilde{u}(x)\}| = 0$) that is absolutely continuous on almost all line segment in Ω parallel to the coordinate axes and whose (classical) partial derivatives belong to $L^p(\Omega)$.

Proposition 3.5. For every Sobolev function $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$, there exists a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset C^{\infty}(\Omega)$ so that $\varphi_j \rightarrow u$ in $L^p(\Omega)$ and $\{D\varphi_j\}_j$ is a Cauchy sequences in $L^p(\Omega)$.

Indeed, from the proof of Theorem A.15 in [7], both two propositions above are equivalent to the definition of Sobolev functions.

Proposition 3.6. Let $u \in W^{1,p}(5B)$ and let $p > 3$. Then

$$|u(x) - u(y)| \leq C(p)|x - y|^{1-\frac{3}{p}} \left(\int_{B(x,2|x-y|)} |Du|^p \right)^{\frac{1}{p}}$$

for all Lebesgue points $x, y \in B$ of u .

Since the topic of this thesis is homeomorphisms of finite distortion, from now on we discuss some results about such homeomorphisms.

Definition 3.7. Let $\Omega \subset \mathbb{R}^3$ be an open set, $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$. We say $f : \Omega \rightarrow \mathbb{R}$ belongs to the space $L^p \log^\alpha L(\Omega)$ if

$$\int_{\Omega} |f(x)|^p \log^\alpha(e + |f(x)|) dx < \infty.$$

We say that $f \in L^p \log^\alpha L_{loc}(\Omega)$ if $f \in L^p \log^\alpha L(V)$ for all subdomains $V \subset\subset \Omega$.

Lemma 3.8. Let $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$ be a homeomorphism of finite distortion and suppose there is $\lambda > 0$ such that $\exp(\lambda K_f) \in L_{loc}^1(\Omega)$. Then $|Df| \in L^n \log^{-1} L_{loc}(\Omega)$, and we have that f is differentiable a.e.

Proof. See [7, Lemma 2.8 and Corollary 2.25]. □

3.2 Lusin (N) and (N^{-1}) Condition

Definition 3.9. Let $\Omega \subset \mathbb{R}^3$ be open. We say that $f : \Omega \rightarrow \mathbb{R}^3$ satisfies the Lusin (N) condition if

$$\text{for each } E \subset \Omega \text{ such that } |E| = 0 \text{ we have } |f(E)| = 0.$$

From the mathematical point of view this property plays a crucial role in the change of variables formula which is an essential tool in the proof of Theorem 1.3. Now we show that Lusin (N) condition implies the change of variables formula.

Theorem 3.10. Let $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$ be an orientation-preserving homeomorphism and let η be a nonnegative Borel measurable function on \mathbb{R}^3 . Then

$$(3.2) \quad \int_{\Omega} \eta(f(x)) J_f(x) dx \leq \int_{f(\Omega)} \eta(y) dy.$$

and if f satisfy Lusin (N) condition, then there is an equality above.

Proof. See [7, Theorem A.35]. □

And we can see that Sobolev homeomorphisms are pretty good in this sense.

Theorem 3.11. Let $\Omega \subset \mathbb{R}^3$ be open and let $f \in W_{loc}^{1,3}(\Omega, \mathbb{R}^3)$ be a homeomorphism. Then the continuous representative of f satisfies the Lusin (N) condition.

Proof. See [7, Theorem 4.5]. □

Now using this theorem, we can prove that for the homeomorphism of finite distortion whose distortion function satisfies the equation (1.5) enjoys the Lusin (N) condition.

Theorem 3.12. *Let $f : B^3(0, 1) \rightarrow f(B^3(0, 1))$ be a homeomorphism of finite distortion with the distortion function satisfies (1.5), then f satisfies the Lusin (N) condition.*

Proof. Let M be a compact subset of $B^3(0, 1)$. By the distortion inequalities (1.3), (1.5) and Hölder's inequality, we can get

$$\begin{aligned} \int_M |Df(x)|^3 dx &\leq \int_M K_f(x) J_f(x) dx \\ &\leq \left(\int_M K_f^2(x) J_f(x) dx \right)^{\frac{1}{2}} \cdot \left(\int_M J_f(x) dx \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

By (1.5), we know that $J_f(x) \exp(\lambda K_f^{\frac{1}{2}}(x)) \in L^1_{loc}(B^3(0, 1))$, then we can get that $(\int_M K_f^2(x) J_f(x) dx)^{\frac{1}{2}} < \infty$ immediately. Since f is a homeomorphism which maps a compact set to a compact set, then we have $(\int_M J_f(x) dx)^{\frac{1}{2}} < |f(M)|^{\frac{1}{2}} < \infty$. Then by Theorem 3.11, we know $f \in W^{1,3}_{loc}(\Omega, \Omega')$ and f satisfies the Lusin (N) condition. \square

In many applications it is also important to know when the preimages of null sets are null sets.

Definition 3.13. *Let $\Omega \subset \mathbb{R}^3$ be open. We say that $f : \Omega \rightarrow \mathbb{R}^3$ satisfies the Lusin (N^{-1}) condition if*

$$\text{for each } E \subset f(\Omega) \text{ such that } |E| = 0 \text{ we have } |f^{-1}(E)| = 0.$$

The next theorem shows that, for the validity of the (N^{-1}) condition, it is enough to assume that the distortion satisfies $K_f \in L^{\frac{1}{2}}_{loc}$, provided f is a homeomorphism of finite distortion.

Theorem 3.14. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^3)$ be a homeomorphism of finite distortion with $K_f^{\frac{1}{2}} \in L^1_{loc}(\Omega)$. Then $J_f(x) > 0$ a.e. in Ω and hence f satisfies the Lusin (N^{-1}) condition.*

Proof. See [7, Theorem 4.13]. It gives a more general result. \square

3.3 Capacity estimates

First, let us give the definition of p -capacity between two sets E and F in a domain Ω .

Definition 3.15. Let E and F be two sets in a domain $\Omega \subset \mathbb{R}^3$, let $A(E, F; \Omega) := \{u \in W_{loc}^{1,1}(\Omega) : u \geq 1 \text{ on } F \text{ and } u \leq 0 \text{ on } E\}$. For every $p > 0$ we define the p -capacity between E and F with respect to Ω by

$$Cap_p(E, F; \Omega) := \inf_{u \in A(E, F; \Omega)} \int_{\Omega} |Du|^p dx.$$

For us the most interesting one is the 3-capacity, since it is conformally invariant. So we also call it *conformal capacity* (variational 3-capacity, 3-capacity). And by Proposition 3.5, $W_{loc}^{1,1}(\Omega) \cap C^\infty(\Omega)$ is dense in $W_{loc}^{1,1}(\Omega)$ in $W^{1,p}$ -norm, so we can also take the admissible function in the class $W_{loc}^{1,1}(\Omega) \cap C^\infty(\Omega)$.

For K -quasiconformal mappings $f : \Omega \subset \mathbb{R}^3 \rightarrow f(\Omega) \subset \mathbb{R}^3$, we have

$$(3.3) \quad \frac{1}{K} Cap_3(E, F; f(\Omega)) \leq Cap_n(f^{-1}(E), f^{-1}(F); \Omega) \leq K Cap_3(E, F; f(\Omega)).$$

For the proof of (3.3), see the more general result Lemma 3.18.

For 3-capacity, basic estimates are:

(i) If $E \subset E'$, $F \subset F'$ and $\Omega \subset \Omega'$, then

$$(3.4) \quad Cap_3(E, F; \Omega) \leq Cap_3(E', F'; \Omega').$$

(ii) If $\overline{B}(x, r) \subset B(x, R) \subset \Omega$, then

$$(3.5) \quad Cap_3(\overline{B}(x, r), S^2(x, R); \Omega) = Cap_3(\overline{B}(x, r), S^2(x, R); B(x, R)) \\ \leq \frac{\omega_2}{\left(\log \frac{R}{r}\right)^2}.$$

Here ω_2 is the (2)-dimensional volume of the unit sphere $S^2(0, 1)$. In fact, the inequality can also be reserved. Indeed if $u \in A(\overline{B}(x, r), S^2(x, R); B(x, R)) \cap C^\infty(B(x, R))$, then the fundamental theorem of calculus and Hölder's inequality give

$$1 \leq \int_r^R |Du(tw)| dt \\ \leq \int_r^R |Du(tw)| t^{\frac{2}{3} - \frac{2}{3}} dt \\ \leq \left(\int_r^R \frac{dt}{t} \right)^{\frac{2}{3}} \left(\int_r^R |Du(tw)|^3 t^2 dt \right)^{\frac{1}{3}}$$

for every $w \in S^2(0, 1)$. The desired inequality follows by raising the both sides of this inequality to the power 3 and integrating over $S^2(0, 1)$ with respect to w .

(iii) If $E, F \subset B(x, r)$ are continua with

$$\frac{\min\{\text{diam } E, \text{diam } F\}}{r} \geq \delta_1 > 0,$$

then

$$(3.6) \quad \text{Cap}_3(E, F; B(x, r)) \geq \delta(\delta_1) > 0.$$

According to the definition of capacity, the estimates (i) and (ii) are obvious. For (iii), we can find the proof of a more general result in [5].

Now we estimate of p -capacity, $2 < p < 3$, between the two continuum in a 3-dimension ball. The proof of the result is based on the Sobolev imbedding theorem on spheres.

First for a L^1 -integrable function ω , we denote $\int_A \omega(x) dx := \frac{1}{|A|} \int_A \omega(x) dx$, here A is a measurable set.

Lemma 3.16. *Let $p > 2$ and $u \in W^{1,p}(B^3(0, R))$. Then up to a measure-zero set in $B^3(0, R)$, for almost every $t \in (0, R)$ and almost every $x, y \in S^2(0, t)$ we have*

$$(3.7) \quad |u(x) - u(y)| \leq Ct \left(\int_{S^2(0,t)} |Du|^p \right)^{\frac{1}{p}},$$

where $C = C(p)$ and $S^2(0, t)$ is the 2-dimensional sphere centered at the origin with radius t .

Proof. See [7, Lemma 2.19]. □

Lemma 3.17. *Let E and F be two continuum in a ball $B^3(a, r)$, and assume that there exists constants $0 < A < B < 1$ such that $S^2(a, t) \cap E \neq \emptyset$ and $S^2(a, t) \cap F \neq \emptyset$ for every $t \in [Ar, Br]$. Let $p \in (2, 3)$ be fixed, suppose that $u \in W^{1,p}(B^3(a, r), \mathbb{R})$ is continuous and satisfies: $u \equiv 0$ on E , $u \equiv 1$ on F and $0 \leq u(x) \leq 1$. Then there is a constant $C(A, B, p) > 0$ such that.*

$$\int_{B^3(a,r)} |Du|^p dx \geq C(A, B, p) r^{3-p}.$$

Proof. By the Fubini's theorem, $u \in W^{1,p}(S^2(a, t))$ for a.e. $t \in [Ar, Br]$. Then by inequality (3.7), for a.e. $t \in [Ar, Br]$ and almost all $x \in E \cap S^2(a, t)$ and $y \in F \cap S^2(a, t)$, we have

$$\begin{aligned} 1 \leq |u(x) - u(y)| &\leq C|x - y|^{1-\frac{2}{p}} \left(\int_{S^2(a,t)} |Du(x)|^p d\delta \right)^{\frac{1}{p}} \\ &\leq Ct^{1-\frac{2}{p}} \left(\int_{S^2(a,t)} |Du(x)|^p d\delta \right)^{\frac{1}{p}}. \end{aligned}$$

Here $d\delta$ means we integral with respect to the area measure of sphere. Thus we have

$$\int_{S^2(a,t)} |Du(x)|^p d\delta \geq Ct^{2-p} \quad \text{for a.e. } t \in [Ar, Br],$$

which yields the desired conclusion

$$\int_{B^3(a,r)} |Du(x)|^p dx \geq C_1 \int_{[Ar,Br]} t^{2-p} dt \geq C_2 r^{3-p}$$

with $C_2 = \frac{C_1(B^{3-p} - A^{3-p})}{3-p}$. □

3.4 K_f and $\frac{1}{K_f}$ - inequalities

We give the so-called K_f and $\frac{1}{K_f}$ -inequalities in the following lemma, and the proofs are based on the change of variables formula. For a L^1 -integrable function $\omega \geq 0$ almost everywhere, we denote the weighted capacity by

$$(3.8) \quad \text{Cap}_3^\omega(E, F; \Omega) = \inf_{\mu \in A(E, F; \Omega)} \int_{\Omega} |D\mu(x)|^3 \omega(x) dx.$$

Here $\Omega \subset \mathbb{R}^3$ is a domain, and E, F are compact subsets of Ω .

Lemma 3.18. *Let f be a homeomorphism of finite distortion from Ω to Ω' , and let E and F be two continuum in Ω . Then we have*

$$\text{Cap}_3^{1/K_f(x)}(E, F; \Omega) \leq \text{Cap}_3(f(E), f(F); \Omega'),$$

and

$$\text{Cap}_3(E, F; \Omega) \leq \text{Cap}_3^{K_f(f^{-1}(y))}(f(E), f(F); \Omega').$$

Proof. Since $f = (f_1, f_2, f_3) \in W_{loc}^{1,1}(\Omega, \Omega')$, by Proposition 3.4, we know that f_i , $i = 1, 2, 3$, is absolutely continuous on almost all line segment in Ω parallel to the coordinate axes. Let $\mu \in A(f(E), f(F); \Omega') \cap C^\infty(\Omega')$ be an admissible function, then $\mu \circ f$ is absolutely continuous on almost all line segment in Ω parallel to the coordinate axes and (classical) partial derivatives exists. Even we have

$$\frac{\partial(\mu \circ f)(x)}{\partial x_i} = \sum_{j=1}^3 \frac{\partial \mu(f(x))}{\partial x_j} \cdot \frac{\partial f_j(x)}{\partial x_i}, \quad i = 1, 2, 3.$$

Then

$$D(\mu \circ f)(x) := \left(\frac{\partial \mu(f(x))}{\partial x_1}, \frac{\partial \mu(f(x))}{\partial x_2}, \frac{\partial \mu(f(x))}{\partial x_3} \right) = D\mu(f(x)) \cdot Df(x).$$

and $|D\mu \circ f(x)| \leq |D\mu(f(x))| \cdot |Df(x)|$. By the argument above we get that $\mu \circ f(x)$ satisfies Proposition 3.4, then we know $\mu \circ f(x) \in W_{loc}^{1,1}(\Omega)$ and $\mu \circ f(x) \in A(E, F; \Omega)$. Then by distortion inequality (1.2) and change of variables formula (3.2), we have

$$\text{Cap}_3^{1/K_f(x)}(E, F; \Omega) \leq \int_{\Omega} \frac{|D\mu \circ f(x)|^3}{K_f(x)} dx$$

$$\begin{aligned}
&\leq \int_{\Omega} |D\mu(f(x))|^3 \frac{|Df(x)|^3}{K_f(x)} dx \\
&\leq \int_{\Omega} |D\mu(f(x))|^3 J_f(x) dx \\
&\leq \int_{\Omega'} |D\mu(y)|^3 dy,
\end{aligned}$$

and

$$\begin{aligned}
Cap_3(E, F; \Omega) &\leq \int_{\Omega} |D\mu \circ f(x)|^3 dx \\
&\leq \int_{\Omega} |D\mu(f(x))|^3 |Df(x)|^3 dx \\
&\leq \int_{\Omega} |D\mu(f(x))|^3 K_f(x) J_f(x) dx \\
&\leq \int_{\Omega'} |D\mu(y)|^3 K_f(f^{-1}(y)) dy
\end{aligned}$$

Since μ is arbitrary, we get the desired result. \square

3.5 Orlicz functions

In this subsection, we introduce some results on Orlicz functions. Our results have the standard assumptions that

$$(3.9) \quad \exp(\mathcal{A}(K_f(x))) \in L^1_{loc}(B^3(0, 1))$$

for some special Orlicz function \mathcal{A} . Let us give the definition of Orlicz functions.

Definition 3.19. *We call a strictly increasing function $\mathcal{A} : [0, \infty) \rightarrow [0, \infty)$ with $\mathcal{A}(0) = 0$ and $\lim_{t \rightarrow \infty} \mathcal{A}(t) = \infty$ an Orlicz function.*

Indeed someone always assume that Orlicz functions are also differentiable. In this thesis, we will always assume our Orlicz function \mathcal{A} satisfies several conditions, for example, we assume that \mathcal{A} satisfies

$$(3.10) \quad \int_1^{\infty} \frac{\mathcal{A}'(s)}{s} ds = \frac{1}{\beta} \int_0^{[C/\exp\{\mathcal{A}(1)\}]^{1/\beta}} \frac{1}{t\mathcal{A}^{-1}(\log C/t^\beta)} dt = \infty$$

for all $C, \beta > 0$. We wish to warn the reader that the conditions (3.9) and (3.10) do not imply $K_f(x)$ to be even locally integrable and thus an additional technical assumption on \mathcal{A} must be posed. To fill up this gap, we assume that \mathcal{A} satisfies also the following condition:

$$(3.11) \quad \exists t_0 \in (0, \infty) : \mathcal{A}'(t)t \rightarrow \infty \quad \forall t \geq t_0.$$

It was proven in [13] that, under these assumptions on the distortion function, a mapping f of finite distortion is continuous. It was also shown in [13] that assumption (3.10) is sharp for continuity.

In this section, we associate with \mathcal{A} two other Orlicz functions:

$$(3.12) \quad \begin{aligned} \Phi(t) &= t \exp(\mathcal{A}(t)), \\ P(s) &= \frac{s}{\Phi^{-1}(s)} - 1, \quad s > 0, \quad \text{and } P(0) = 0 \end{aligned}$$

We notice that Φ is strictly increasing, since \mathcal{A} is strictly increasing. Therefore the inverse function Φ^{-1} is well defined. We immediately have

$$(3.13) \quad P(\Phi(t)) = \exp(\mathcal{A}(t)) - 1.$$

Then for the Orlicz functions above, we have the following proposition from [13] and three lemmata from [11].

Proposition 3.20. *Assume that \mathcal{A} is an Orlicz function satisfying (3.10) and (3.11). Then for the associated Orlicz function P defined in (3.12), we have the pointwise inequality*

$$(3.14) \quad P(KJ) \leq J + \exp(\mathcal{A}(K)) - 1$$

for all $K, J \geq 0$, where the Orlicz function P satisfies the integrability condition

$$(3.15) \quad \int_1^\infty \frac{P(s)}{s^2} ds = \infty$$

and also for every $0 < \epsilon$, there exists a constant $s_0 > 0$ such that we have

$$(3.16) \quad (s^{-1}P(s))' \leq 0 \leq (s^{\epsilon-1}P(s))'$$

for all $s \geq s_0$.

Proof. For the proof of (3.14) and (3.15), we just use the assumption (3.10).

Indeed, by the change of variables $s = \Phi(t)$, (3.12) and (3.10) we obtain

$$\begin{aligned} \int_{\Phi(1)}^\infty \frac{P(s) + 1}{s^2} ds &= \int_1^\infty \frac{(P(\Phi(t)) + 1)\Phi'(t)}{\Phi(t)^2} dt \\ &= \int_1^\infty \frac{\Phi'(t)}{t\Phi(t)} dt \\ &= \int_1^\infty \frac{(1 + t\mathcal{A}'(t)) \exp(\mathcal{A}(t))}{t^2 \exp(\mathcal{A}(t))} dt \\ &= \int_1^\infty \left(\frac{1}{t^2} + \frac{\mathcal{A}'(t)}{t} \right) dt = \infty. \end{aligned}$$

This proves (3.15). Regarding (3.14), we distinguish two cases; naturally we may assume $K \neq 0 \neq J$. If $KJ \leq \Phi(K)$, then by (3.13)

$$P(KJ) \leq P(\Phi(K)) = \exp(\mathcal{A}(K)) - 1.$$

If $KJ \geq \Phi(K)$, then

$$P(KJ) = \frac{KJ}{\Phi^{-1}(KJ)} - 1 \leq \frac{KJ}{K} - 1 = J - 1.$$

This proves (3.14). Regarding (3.16), we just need to use (3.11), we divide the proof into two cases; when $\epsilon > 1$, it is obviously that $s^{\epsilon-1}P(s)$ is increasing as the product of two positive increasing functions. For $0 < \epsilon < 1$ we define $h(s) = s^{\epsilon-1}P(s)$ and $h_1(s) := h(s) + s^{\epsilon-1}$. By (3.13)

$$h_1(\Phi(t)) = \Phi(t)^{\epsilon-1}(1 + P(\Phi(t))) = t^{\epsilon-1} \exp(\epsilon \mathcal{A}(t)).$$

Hence

$$(h_1(\Phi(t)))' = t^{\epsilon-2} \exp(\epsilon \mathcal{A}(t))[\epsilon \mathcal{A}'(t) - (1 - \epsilon)].$$

By (3.11) we find a t_0 such that $h_1(\Phi(t))$ increases for $t > t_0$. We conclude that $h(s) = s^{\epsilon-1}(P(s) + 1) - s^{\epsilon-1}$ is increasing on (s_0, ∞) , where $s_0 = \Phi(t_0)$.

We define $H(s) = s^{-1}P(s)$, then $H(\Phi(t)) = \frac{1}{t} - \frac{1}{\Phi(t)}$. Then $(H(\Phi(t)))' = \frac{-1}{t^2}(1 - \frac{1}{\exp(\mathcal{A}(t))}) + \frac{\mathcal{A}'(t)}{t \exp(\mathcal{A}(t))}$ and it is always negative. So $H(\Phi(t))$ is decreasing. And since $\Phi(t)$ is an Orlicz function, so $H(s) = s^{-1}P(s)$ is also decreasing. \square

Lemma 3.21. *Assume that \mathcal{A} is an Orlicz function satisfying (3.11), and let $p \in [1, \infty)$. Then there exists $t_2 := t_2(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \rightarrow t^{-p} \exp(\mathcal{A}(t))$$

is increasing on (t_2, ∞) .

Proof. The claim follows from the identity

$$\frac{d}{dt} t^{-p} \exp(\mathcal{A}(t)) = t^{-p-1} \exp(\mathcal{A}(t))[\mathcal{A}'(t)t - p].$$

\square

Lemma 3.22. *Assume that \mathcal{A} is an Orlicz function satisfying (3.11), and let $p \in [1, \infty)$. Then there exists $t_3 := t_3(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \rightarrow \frac{t}{\mathcal{A}^{-1}(\log t^p)}$$

is increasing on (t_3, ∞) .

Proof. Because

$$\frac{d}{dt} \exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-1} = \exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-2} \left(\frac{1}{p} \mathcal{A}'(t)t - 1\right),$$

by Lemma 3.21, we find a number $\tilde{t}_0(p, \mathcal{A})$ such that $\exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-1}$ is increasing for $t > \tilde{t}_0$. We conclude that

$$\frac{t}{\mathcal{A}^{-1}(\log t^p)} = \frac{\exp\left(\frac{\mathcal{A}(\mathcal{A}^{-1}(p \log t))}{p}\right)}{\mathcal{A}^{-1}(p \log t)}$$

is increasing on (t_3, ∞) , where $t_3 = \exp(\mathcal{A}(\tilde{t}_0)/p)$. \square

Lemma 3.23. *Assume that \mathcal{A} is an Orlicz function satisfying (3.11), and let $p \in [1, \infty)$. Then there exists $t_4 := t_4(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \rightarrow \exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right)$$

is convex on (t_4, ∞) .

Proof. Because

$$\frac{d}{dt} \exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right) = \frac{1}{p} \exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right) \mathcal{A}'\left(t^{\frac{1}{p}}\right) t^{\frac{1}{p}-1},$$

it suffices to show that the function

$$t \rightarrow \frac{\exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right)}{t}$$

is increasing for large values of t . This holds by Lemma 3.21, and so the claim follows. \square

3.6 Weakly monotone functions

As the beginning of this subsection, we first introduce the definition and some results about monotone functions.

Definition 3.24. *Let u be a continuous function in a domain $D \subset \mathbb{R}^3$ and let u have boundary values at each point of ∂D . We say that u is monotone in D if*

$$\sup_{\partial \Delta} u = \sup_{\Delta} u \quad \text{and} \quad \inf_{\partial \Delta} u = \inf_{\Delta} u$$

for each domain $\Delta \subset D$.

We have the following proposition for monotone functions.

Proposition 3.25. *A function u is monotone in $D \subset \mathbb{R}^3$ if and only if there exists no domain $\Delta \subset D$ such that u is constant on $\partial\Delta$ without being constant in Δ .*

Proof. First, we prove the necessity. If there is a domain $\Delta \subset D$ such that $u \equiv C$ (Here C is a constant.) on $\partial\Delta$ without being constant in Δ . Then, there must be a point $x \in D$ such that $u(x) < C$ or $u(x) > C$. It contradicts the fact that u is monotone in D .

For the sufficiency, assume u is not monotone in D , then there exists a domain $\Delta \subset D$ such that we have $\sup_{\partial\Delta} u < \sup_{\Delta} u$ or $\inf_{\partial\Delta} u > \inf_{\Delta} u$. We consider the first case $\sup_{\partial\Delta} u < \sup_{\Delta} u$, for another case the proof is similar. By the topological property of D , there exists a point $x_1 \in \Delta$ such that $C_1 := u(x_1) = \sup_{\Delta} u$ (here C_1 can be ∞) and $x_2 \in \partial\Delta$ such that $C_2 := u(x_2) = \sup_{\partial\Delta} u$, then we have $C_2 < C_1$. For every $C_3 \in (C_2, C_1)$, by the continuity of u , We consider the domain $\Delta_{C_3} \subset \Delta$ with the boundary $\{x \in D : u(x) \equiv C_3\} \subset D$. Obviously $x_1 \in \Delta_{C_3}$, then we know $C_1 = \sup_{\Delta_{C_3}} u > C_3 = \sup_{\partial\Delta_{C_3}} u$. So we have proven that if u is not monotone in D , we can find a domain which is a subset of D such that u is constant on the boundary of the domain without being constant in the domain itself. \square

The class of weakly monotone functions is a generalization of monotone functions. For a function $u : \Omega \rightarrow \mathbb{R}$, we define $u^+(x) := \max\{u(x), 0\}$ for $x \in \Omega$.

Definition 3.26. *A real valued function $u \in W^{1,1}(\Omega)$ is said to be weakly monotone if for every ball $B \subset \Omega$ and all constants $m \leq M$ such that*

$$(3.17) \quad \varphi = (u - M)^+ - (m - u)^+ \in W_0^{1,1}(B),$$

we have

$$(3.18) \quad m \leq u(x) \leq M$$

for almost every $x \in B$.

Now we collect results which enable us to derive regularity properties of a mapping of finite distortion from integrability of its differential. Let us consider a class $X(\Omega) \subset L^{n-1}(\Omega)$ of measurable functions on Ω satisfying the following two conditions:

(X-1) $J_f(\cdot) \in L_{loc}^1(\Omega)$ and $\det Df = \text{Det } Df$ provided $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, $|Df| \in X(\Omega)$ and $J_f(\cdot) \geq 0$ almost everywhere.

(X-2) If $g, h \geq 0$ are measurable, $g \leq ch$ for some $0 < c < \infty$ and $h \in X(\Omega)$, then $g \in X(\Omega)$.

Here the statement $\det Df = \text{Det } Df$ means that

$$(3.19) \quad \int_{\Omega} \varphi J_f(x) dx = - \int_{\Omega} f_i J(x, f_1, \dots, f_{i-1}, \varphi, f_{i+1}, \dots, f_n) dx$$

for each $i = 1, \dots, n$ and for all $\varphi \in C_0^\infty(\Omega)$.

Proposition 3.27. *For every mapping $f \in C^2(\Omega, \mathbb{R}^n)$ the equation (3.19) is satisfied for each $i = 1, \dots, n$ and for all $\varphi \in C_0^\infty(\Omega)$. Moreover, the same extends to hold for every $f \in W^{1,n}(\Omega, \mathbb{R}^n)$.*

Proof. See [7, Proposition 2.10]. □

Proposition 3.28. *Let X be a space of measurable functions satisfying (X-1) and (X-2). Let $f = (f_1, \dots, f_n) \in W^{1,n-1}(\Omega)$ be a mapping of finite distortion with $|Df| \in X(\Omega)$. Then the coordinate functions of f are weakly monotone.*

Proof. We follow the standard idea as in [8, Section 4]. Let us consider a ball $B \subset\subset \Omega$. We prove e.g. that if $f_1 \leq M$ on ∂B in the sence of traces, i.e. the positive part of $f_1 - M$ belongs to $W_0^{1,1}(B)$, then $f_1 \leq M$ a.e. in B . We consider the truncated function $\tilde{f}_1 = \min(f_1, M)$ and the mapping $\tilde{f} = (\tilde{f}_1, f_2, \dots, f_n)$. Notice that, by (X-2), $|D\tilde{f}| \in X(\Omega)$. Let φ be a smooth test function with compact support in Ω such that $\varphi = 1$ on B . Since f_1 differs from \tilde{f}_1 only on B where $D\varphi = 0$, we have $f_1 D\varphi = \tilde{f}_1 D\varphi$, and thus

$$\begin{aligned} \int_{\Omega} \varphi J_f(x) dx &= - \int_{\Omega} f_1 J(x, \varphi, f_2, \dots, f_n) dx \\ &= - \int_{\Omega} \tilde{f}_1 J(x, \varphi, f_2, \dots, f_n) dx \\ &= \int_{\Omega} \varphi J_{\tilde{f}}(x) dx. \end{aligned}$$

Hence, if we set $E = \{\tilde{f} \neq f\}$, we have

$$\int_E J_f(x) dx = \int_E J_{\tilde{f}}(x) dx = 0.$$

Since $J_{\tilde{f}}(x) \geq 0$, it follows that $J_f = 0$ a.e. on E and thus, as f is a mapping of finite distortion, $Df = 0$ a.e. in E . It follows that $D(f_1 - \tilde{f}_1) = 0$ a.e. in Ω which yields that $f_1 = \tilde{f}_1 \leq M$ a.e. in B . □

Based on the proposition above, we have the following corollary.

Corollary 3.29. *If there exists an Orlicz function $\mathcal{A} : [0, \infty) \rightarrow [0, \infty)$ satisfying (3.10) and (3.11), and the distortion function $K_f(x)$ satisfies (3.9). Then f satisfies the assumptions of Proposition 3.28. That means that the coordinate functions of f are weakly monotone.*

Proof. See [13]. □

The next proposition comes from [12, Corollary 1.3].

Proposition 3.30. *Let Φ be an Orlicz-function that satisfies*

$$(\Phi - 1) \int_1^\infty \frac{\Phi(t)}{t^{1+n}} dt = \infty.$$

($\Phi - 2$) *There is $p \in (n - 1, n)$ such that $t \mapsto t^{-p}\Phi(t)$ increases for large values of t .*

Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $J_f(x) \geq 0$ a.e. $x \in \Omega$, and assume that

$$\int_\Omega \Phi(|Df(x)|) dx < \infty.$$

Then $\det Df \in L^1_{loc}(\Omega)$ and $\det Df = \text{Det} Df$.

Proof. See [12]. □

There is a particularly elegant geometric approach to the continuity estimate of monotone functions. The idea goes back to the oscillation lemma by F.W. Gering [4]. While many interesting implications of Gehring's lemma have been discussed in the literature, the fact that one can use it for weakly monotone functions seems to be less familiar. It is surprising that the usual convolution procedure with mollifiers of Dirac distribution has little effect on the monotonicity of functions. Consequently, we take the time here to state and give a rigorous proof of this fact. Let $u \in W^{1,p}(B(a, R))$ be a Sobolev function in a ball $B(a, R)$. Fix a nonnegative $\chi \in C_0^\infty(B)$ supported in the unit ball such that $\int_B \chi(y) dy = 1$. The mollifiers $\chi_j(y) = j^n \chi(jy)$, $j = 1, \dots, n, \dots$ give rise to the sequence $u_j \in C^\infty(\mathbb{R}^n)$ defined by

$$(3.20) \quad u_j(x) = \chi_j * u(x) = \int_B u(y) \chi_j(x - y) dy.$$

It is well known that $\{u_j\}$ converges to u in $W^{1,p}_{loc}(B(a, R))$ and $u_j(x_0) \rightarrow u(x_0)$, $u_j(y_0) \rightarrow u(y_0)$ at the Lebesgue points $x_0, y_0 \in B(a, R)$ of u . For a locally integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we say $x_0 \in \mathbb{R}^n$ is a Lebesgue point of f , if we have

$$\lim_{r \rightarrow 0^+} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |f(y) - f(x_0)| dy = 0.$$

Here by $|B(x_0, r)|$ we denote the volume of the ball $B(x_0, r)$. There is a precise statement concerning monotonicity, see book [7, Lemma 2.18].

Lemma 3.31. *Let $u \in W^{1,p}(B(a, R))$ be weakly monotone in the ball $B(a, R)$ and x_0, y_0 be Lebesgue points in $B(a, r)$, $r < R$. For each $\delta > 0$ there exists N such that*

$$(3.21) \quad |u_j(x_0) - u_j(y_0)| \leq \text{osc}(u_j, \partial B(a, t)) + 2\delta$$

for all $j \geq N$ and every $t \in [r, R]$.

Proof. We claim that the estimates

$$(3.22) \quad u_j(x_0) \leq \max\{u_j(x) : x \in \partial B(a, t)\} + \delta$$

and

$$(3.23) \quad u_j(y_0) \geq \min\{u_j(y) : y \in \partial B(a, t)\} - \delta$$

hold for all $r \leq t \leq R$, whenever j is sufficiently large. We only need to show the first inequality. The second inequality follows by applying the first one to the function $-u$ and to the point y_0 in place of x_0 . Suppose, to the contrary, that there exists a sequence $\{j_k\}$ and radii $r \leq t_k \leq R$, $k = 1, 2, 3, \dots$, such that

$$u_{j_k}(x_0) > \max\{u_{j_k}(x) : x \in \partial B(a, t_k)\} + \delta.$$

Without loss of generality we may assume that $\{t_k\}$ converges to some number $t \in [r, R]$ and that $|t - t_k| < \frac{t}{2}$. Define

$$v_{j_k}(x) := u_{j_k}(x) - u_{j_k}(x_0) + \delta \quad \text{for } x \in B(a, t_k).$$

Since $v_{j_k}(x) \leq 0$ for all $x \in S^{n-1}(a, t_k)$, we conclude that $(v_{j_k})^+ \in W_0^{1,p}(B(a, t_k))$. Let us define

$$\tilde{v}_{j_k}(x) = v_{j_k} \left(a + (x - a) \frac{t_k}{t} \right) \quad \text{for } x \in B(a, t).$$

It is easy to see that $(\tilde{v}_{j_k})^+ \in W_0^{1,p}(B(a, t))$. By (3.20), we know that $v_{j_k}(x) \rightarrow u(x) - u(x_0) + \delta$ in $W^{1,p}$ because x_0 is a Lebesgue point of u . Therefore

$$\left\| \tilde{v}_{j_k}(x) - \left(u \left(a + (x - a) \frac{t_k}{t} \right) - u(x_0) + \delta \right) \right\|_{W^{1,p}} \rightarrow 0.$$

It is not difficult to show that

$$\left\| u(x) - u \left(a + (x - a) \frac{t_k}{t} \right) \right\|_{W^{1,p}} \rightarrow 0.$$

Indeed, this is easy for C^1 -functions and for general u it follows by approximation. Therefore we obtain $\tilde{v}_{j_k} u - u(x_0) + \delta$ in $W^{1,p}(B(a, R))$. This implies however that $(u - u(x_0) + \delta)^+ \in W_0^{1,p}(B(a, t))$.

As u is weakly monotone it follows that $u(x) \leq u(x_0) - \delta$ for almost every $x \in B(a, t)$, then we have $\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} u(y) dy < u(x_0)$ for all $0 < r < t - |a - x_0|$. But this is impossible since x_0 is a Lebesgue point of u in $B(a, r) \subset B(a, t)$.

Now inequalities (3.22) and (3.23) imply

$$u_j(x_0) - u_j(y_0) \leq \text{osc}(u_j, \partial B(a, t)) + 2\delta.$$

One may interchange x_0 with y_0 to conclude with inequality (3.21). \square

By Fubini's theorem we observe that the function $t \rightarrow \int_{\partial B(a,t)} |Du|^p$ belongs to $L^1_{loc}(0, R)$. Consequently, its Lebesgue points form a set of full linear measure on the interval $(0, R)$. With these preliminaries, we can now prove the following variant of the oscillation lemma.

Lemma 3.32. *Let $u \in W^{1,p}(B)$, $n - 1 < p < n$, be weakly monotone in a ball $B = B(a, r)$ and x_0, y_0 be Lebesgue points of u in $B(a, r)$, $r < R$. Then*

$$(3.24) \quad |u(x_0) - u(y_0)| \leq C(p, n)t \left(\int_{\partial B(a,t)} |Du|^p \right)^{\frac{1}{p}}$$

for almost every $t \in (r, R)$.

Proof. We apply Sobolev's inequality on spheres (3.7) to a function $u_j \in C^\infty(B)$ at (3.21) to get:

$$\begin{aligned} |u_j(x_0) - u_j(y_0)| &\leq \text{osc}(u_j, \partial B(a, t)) + 2\delta \\ &\leq C(p, n)t \left(\int_{\partial B(a,t)} |Du_j|^p \right)^{\frac{1}{p}} + 2\delta \end{aligned}$$

for all $r \leq t \leq R$. Here $C(p, n)$ is a constant depends on p and n . Fix a Lebesgue point $r < t_0 < R$ of the function $t \rightarrow \int_{\partial B(a,t)} |Du|^p$. For sufficiently small $\varepsilon > 0$, integrate over the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$, we obtain

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \left(\frac{|u_j(x_0) - u_j(y_0)| - 2\delta}{C(p, n)t} \right)^p \omega_{n-1} t^{n-1} dt \leq \int_{t_0-\varepsilon \leq |x| \leq t_0+\varepsilon} |Du_j|^p.$$

Now we can pass to the limit as $j \rightarrow \infty$, because u_j is just convolution of u . Because δ can be sufficiently small, it is also legitimate to take $\delta = 0$ in this limit inequality.

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{|u(x_0) - u(y_0)|^p \omega_{n-1} t^{n-1}}{C^p(p, n)t^p} dt \leq \int_{t_0-\varepsilon}^{t_0+\varepsilon} \left(\int_{\partial B(a,t_0)} |Du|^p \right).$$

Divide by 2ε and let ε go to zero to obtain

$$\frac{|u(x_0) - u(y_0)|^p \omega_{n-1} t_0^{n-1}}{C^p(p, n)t_0^p} \leq \int_{\partial B(a,t_0)} |Du|^p.$$

This means that

$$|u(x_0) - u(y_0)| \leq C(p, n)t_0 \left(\int_{\partial B(a,t_0)} |Du|^p \right)^{\frac{1}{p}}$$

as desired. \square

3.7 Modulus of continuity

Next we study the modulus of continuity under the assumption $\exp(\mathcal{A}(K_f)) \in L^1_{loc}(\Omega)$ for \mathcal{A} which satisfies (3.10) and (3.11). The result comes from [15].

Let \mathcal{A} be an Orlicz function satisfying the integrability condition (3.10), $n \in \{2, 3, 4, \dots\}$, $K > 0$, and $\beta > 0$. We introduce the strictly increasing function $\alpha(r) = \alpha_{\mathcal{A}, K, n, \beta}(r)$ defined for $0 < r^n < nK/\omega_{n-1}$ by the formula

$$(3.25) \quad \alpha_{\mathcal{A}, K, n, \beta}(r) = \sup \left\{ t \in (0, \frac{r}{2}) : \int_t^{r/2} \frac{1}{s\mathcal{A}^{-1}(\log nK/\omega_{n-1}s^n)} ds \geq \beta \right\}.$$

Now we can formulate our theorem.

Theorem 3.33. *Assume that an Orlicz function \mathcal{A} satisfies both (3.10) and (3.11). Let $f : \Omega \rightarrow \mathbb{R}^n$ be an orientation-preserving (which means $J_f(x) \geq 0$ for a.e. $x \in \Omega$) homeomorphism of finite distortion whose distortion function satisfies*

$$(3.26) \quad K = \int_B \exp\{\mathcal{A}(K_f(x))\} dx < \infty,$$

where $B = B(x_0, R) \subset\subset \Omega$. Then

$$(3.27) \quad |f(x) - f(x_0)| \leq C_{\mathcal{A}, K}(n, \beta) \left(\int_B J_f(z) dz \right)^{1/n} \cdot \exp \left\{ - \int_{\alpha^{-1}(|x-x_0|)}^{R/80} \frac{dt}{t\mathcal{A}^{-1}(\log C_{\mathcal{A}, n}(nK/\omega_{n-1}t^n))} \right\}$$

whenever $x \in B(x_0, \alpha(R/80))$.

We split the proof of Theorem 3.33 into two parts, Lemma 3.34 and Lemma 3.36. Lemma 3.34 is proved in [11, Lemma 4.2] and Lemma 3.36 is proved in [15, Lemma 2].

Lemma 3.34. *Under the hypotheses of Theorem 3.33, we have*

$$(3.28) \quad |f(x) - f(y)|^n \int_r^{R/2} \frac{dt}{t\mathcal{A}^{-1}(\log nK/\omega_{n-1}t^n)} \leq C_{\mathcal{A}, K}(n) \int_{B(x_0, R)} J_f(z) dz$$

whenever $x, y \in B(x_0, r) \subset B(x_0, R/2)$ are Lebesgue points of f .

Proof. Combining the distortion inequality $|Df(x)|^n \leq K_f(x)J_f(x)$ with Proposition 3.20, we obtain that $P(|Df|^n) \in L^1_{loc}(\Omega)$, where $\int_1^\infty \frac{P(s)}{s^2} ds = \infty$ and for every $\epsilon > 0$, there exists a constant t_1 such that the function $t \rightarrow t^{\epsilon-1}P(t)$ is increasing for

all $t \geq t_1$. Using Proposition 3.28 and Proposition 3.30, we conclude that the coordinate functions of f are weakly monotone (see Definition 3.26). This is based on the fact that $J_f(\cdot)$ coincides with the distributional Jacobian, i.e., (3.19) holds. Let $p = n - \frac{1}{2}$, $r < R/2$ and $x, y \in B(x_0, r)$ be Lebesgue points. Then $|Df| \in L^p_{loc}(\Omega)$. Using Lemma 3.32, which holds for mappings whose coordinate functions are weakly monotone, we have the estimate

$$(3.29) \quad \frac{|f(x) - f(y)|}{C(n, p)t} \leq \left(\int_{\partial B(x_0, t)} |Df|^p \right)^{\frac{1}{p}}$$

for almost every $t \in [r, R]$. Write $B_s = B(x_0, s)$ and $A_i = B_{2^i r} \setminus B_{2^{i-1} r}$, for all $i \in \{1, 2, \dots\}$. Define

$$G_i = \left\{ t \in [2^{i-1} r, 2^i r] : \int_{\partial B_t} \exp(\mathcal{A}(K_f(x))) dx \leq \frac{3}{2^{i-1} r} \int_{A_i} \exp(\mathcal{A}(K_f(x))) dx \right\}$$

for all $i \in \{1, 2, \dots\} \cap [1, \log_2 \frac{R}{r}] = I$. Because $2r \leq R$, we have $I \neq \emptyset$. Using Fubini's theorem, we conclude that

$$|G_i| \geq \frac{2^{i-1}}{2} r,$$

for all $i \in I$. Combining the distortion inequality and Hölder's inequality with the inequality (3.29), we have that

$$(3.30) \quad \frac{|f(x) - f(y)|^n}{C(n, p)t^n} \leq \left(\int_{\partial B_t} |K_f(x)|^{\frac{p}{n-p}} \right)^{\frac{n-p}{p}} \int_{\partial B_t} J_f(x) dx$$

for almost every $t \in [r, R]$. By Lemma 3.21, we find a number $t_2 := t_2(\frac{p}{n-p}, \mathcal{A})$ such that the function $t \rightarrow t^{-\frac{p}{n-p}} \exp(\mathcal{A}(t))$ is increasing on (t_2, ∞) . Let $t \in [r, R]$ be such that $\int_{\partial B_t} \exp(\mathcal{A}(K_f(x))) dx < \infty$, and pick τ so that

$$\exp(\mathcal{A}(\tau)) = \int_{\partial B_t} \exp(\mathcal{A}(K_f(x))) dx.$$

Write $\lambda = \max\{\tau, t_2\}$. Then we estimate

$$\begin{aligned} \int_{\partial B_t} |K_f(x)|^{\frac{p}{n-p}} dx &\leq \frac{1}{|\partial B_t|} \int_{\partial B_t \cap \{|K_f(x)| > \lambda\}} |K_f(x)|^{\frac{p}{n-p}} dx \\ &\quad + \frac{1}{|\partial B_t|} \int_{\partial B_t \cap \{|K_f(x)| \leq \lambda\}} |K_f(x)|^{\frac{p}{n-p}} dx \\ &\leq \frac{\lambda^{\frac{p}{n-p}}}{\exp(\mathcal{A}(\lambda))} \int_{\partial B_t} \exp(\mathcal{A}(K_f(x))) dx + \lambda^{\frac{p}{n-p}} \leq 2\lambda^{\frac{p}{n-p}} \end{aligned}$$

It follows that

$$(3.31) \quad \frac{|f(x) - f(y)|^n}{t} \leq C(p, n)\lambda \int_{\partial B_t} J_f(x) dx,$$

and use Jensen's inequality we get

$$\begin{aligned} \frac{|f(x) - f(y)|^n}{t} &\leq C(p, n, \mathcal{A})\mathcal{A}^{-1} \left(\log \left(\int_{\partial B_t} \exp(\mathcal{A}(K_f(x))) dx \right) \right) \\ &\quad \cdot \int_{\partial B_t} J_f(x) dx \end{aligned}$$

for almost every $t \in [r, R]$. Fix $i \in I$. For almost every $t \in G_i$, we have that

$$(3.32) \quad \frac{|f(x) - f(y)|^n}{t} \leq C(p, n, \mathcal{A})\mathcal{A}^{-1} \left(\log \left(\frac{6K}{\omega_{n-1} t^{n-1} 2^i r} \right) \right) \int_{\partial B_t} J_f(x) dx.$$

Here the constant K comes from (3.26). Integrating this estimate over the set G_i with respect to t , we arrive at

$$(3.33) \quad |f(x) - f(y)|^n \int_{G_i} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{6nK}{\omega_{n-1} t^n} \right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J_f(x) dx.$$

By Lemma 3.22, we fix $t_3 = t_3(n, \mathcal{A}) \geq 1$ so that the function $h(t) = (t \mathcal{A}^{-1} \left(\log \left(\frac{1}{t^n} \right) \right))^{-1}$ is decreasing on $\left(0, \frac{1}{t_3}\right)$. Then $t \rightarrow h\left(\frac{t}{t_3} \sqrt[n]{\frac{\omega_{n-1}}{6nK}}\right)$ is decreasing on $\left(0, \sqrt[n]{\frac{6nK}{\omega_{n-1}}}\right) \supset (0, R)$. Combining this with the estimate $|G_i| \geq \frac{2^{i-1}}{2}r$, we conclude that

$$|f(x) - f(y)|^n \int_{2^{i-2}3r}^{2^i r} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{t_3^n nK}{\omega_{n-1} t^n} \right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J_f(x) dx.$$

Because

$$(3.34) \quad \begin{aligned} \int_{2^{i-2}3r}^{2^i r} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{t_3^n nK}{\omega_{n-1} t^n} \right)} &= \int_{2^{i-1}r}^{2^{i-2}3r} \frac{ds}{(s + 2^{i-2}r) \mathcal{A}^{-1} \left(\log \left(\frac{t_3^n nK}{\omega_{n-1} (s+2^{i-2}r)^n} \right) \right)} \\ &\geq \int_{2^{i-1}r}^{2^{i-2}3r} \frac{1}{3} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{t_3^n nK}{\omega_{n-1} t^n} \right)}, \end{aligned}$$

we obtain the estimate

$$(3.35) \quad |f(x) - f(y)|^n \int_{2^{i-1}r}^{2^i r} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{t_3^n nK}{\omega_{n-1} t^n} \right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J_f(x) dx.$$

Summing over the set I , we arrive at

$$(3.36) \quad |f(x) - f(y)|^n \int_r^{R/2} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{t_3^n nK}{\omega_{n-1}t^n}\right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J_f(x) dx.$$

In the case $t \in \left(0, \sqrt[n]{\frac{nK}{\omega_{n-1}t_3}}\right)$, we have that

$$(3.37) \quad \begin{aligned} \frac{1}{t\mathcal{A}^{-1}\left(\log \frac{t_3^n nK}{\omega_{n-1}t^n}\right)} &= \frac{1}{t_3} \frac{t_3}{t\mathcal{A}^{-1}\left(\log \frac{t_3^n nK}{\omega_{n-1}t^n}\right)} \\ &\geq \frac{1}{t_3} \frac{1}{t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right)}. \end{aligned}$$

Here we used the fact that the function $s \rightarrow \frac{s}{\mathcal{A}^{-1}(\log s^n)}$ is increasing on (t_3, ∞) . Furthermore

$$(3.38) \quad \sup \left\{ \frac{t\mathcal{A}^{-1}\left(\log \frac{t_3^n nK}{\omega_{n-1}t^n}\right)}{t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right)} : \frac{nK}{\omega_{n-1}t_3} \leq t^n \leq R^n \leq \frac{6nK}{\omega_{n-1}} \right\} \leq C_{\mathcal{A},K}(n),$$

and so also in the case $R^n \geq t^n \geq \frac{nK}{\omega_{n-1}t_3}$ we have the inequality

$$(3.39) \quad t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right) \geq C_{\mathcal{A},K} t\mathcal{A}^{-1}\left(\log \frac{t_3^n nK}{\omega_{n-1}t^n}\right).$$

Combining the inequality (3.36) with the estimates (3.37) and (3.39), we complete the proof of Lemma 3.34. \square

Inequality (3.28) together with the following Lemma 3.36 gives us the desired modulus of continuity. And for the proof of the following Lemma 3.36, we need a crucial tool that is the following integral-type isoperimetric inequality:

$$(3.40) \quad \int_{B(x_0, s)} J_f(x) dx \leq \left(\int_{\partial B(x_0, s)} |Df|^{n-1} d\delta \right)^{\frac{n}{n-1}}$$

for almost every $0 < s < \text{dist}(x_0, \partial\Omega)$, where $d\delta$ is the area measure of the sphere $\partial B(x_0, s)$.

Let us give a theorem from [14, Theorem 1.1].

Theorem 3.35. *If f satisfy the assumptions of Theorem 3.33. Then f satisfies the isoperimetric inequality*

$$(3.41) \quad \left| \int_{B_r} J_f(x) dx \right| \leq I(n) \left(\int_{\partial B_r} |D^\# f(x)| dx \right)^{\frac{n}{n-1}}$$

with $I(n) = (n \sqrt[n-1]{\omega_{n-1}})^{-1}$, for every $x_0 \in \Omega$ and almost every radius $r \in (0, \text{dist}(x_0, \partial\Omega))$.

Proof. First, we want to show that under the assumptions of Theorem 3.33, we can get $f \in W_{loc}^{1, \frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$. By (3.11) we have $\lim_{t \rightarrow \infty} \frac{\mathcal{A}(t)}{\log t} = \infty$, which in turn implies that $\exp(\mathcal{A}(t))$ dominates t^p for every $p \in [1, \infty)$. Therefore, $K_f(x) \in L^p(\Omega)$ for $1 \leq p < \infty$. Then by the definition of distortion function and the Hölder inequality, for every $B(x, R) \subset \subset \Omega$, we have

$$\begin{aligned} \int_{B(x,R)} |Df(x)|^{\frac{n^2}{n+1}} dx &\leq \int_{B(x,R)} (K_f(x) J_f(x))^{\frac{n}{n+1}} dx \\ &\leq \left(\int_{B(x,R)} K_f^n(x) dx \right)^{\frac{1}{n+1}} \left(\int_{B(x,R)} J_f(x) dx \right)^{\frac{n}{n+1}} < \infty. \end{aligned}$$

The last inequality follows from the discussion above and the fact that f is a homeomorphism of finite distortion such that $\int_{B(x,R)} J_f(x) dx \leq |f(B(x, R))| < \infty$.

Let $B_R = B(x_0, R) \subset \Omega$ be a ball such that $\bar{B}_R \subset \Omega$. We approximate f in $W^{1, \frac{n^2}{n+1}}(B_R, \mathbb{R}^n)$ by mappings $f^i \in C^\infty(B_R, \mathbb{R}^n)$. Since the function $|D^\# f^i|$ converge to $|D^\# f|$ in $L^1(B_R)$ (observe that the cofactor matrix is made up from $(n-1)$ -subdeterminants of the differential matrix and $\frac{n^2}{n+1} \geq n-1$), we find by Fubini's theorem that $|D^\# f^i|$ converges to $|D^\# f|$ in $L^1(\partial B_r)$ for almost every radius $r \in (0, R)$. Fix $r \in (0, R)$ so that the functions $|D^\# f^i|$ converges to $|D^\# f|$ in $L^1(\partial B_r)$. Pick $0 < \epsilon < \frac{r}{2}$. We take a convolution approximation u_t^ϵ to the characteristic function $\chi_{B_{r-\epsilon}}$ of the ball $B_{r-\epsilon}$ by using the standard modifiers Φ_t (see [9, Formula (4.6)]) where t is chosen to be so small that $u_t^\epsilon \in C_0^\infty(B_r)$. Then $0 \leq u_t^\epsilon \leq 1$ and by the isoperimetric inequality for smooth functions, we have

$$(3.42) \quad \int_{B_r} u_t^\epsilon(x) J_{f^i}(x) dx \leq \int_{B_r} J_{f^i}(x) dx \leq I(n) \left(\int_{\partial B_r} |D^\# f^i(x)| dx \right)^{\frac{n}{n-1}}.$$

Applying Stokes' theorem for the smooth mapping f^i we find that

$$(3.43) \quad \int_{B_r} u_t^\epsilon(x) J_{f^i}(x) dx = - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, \dots, f_n^i) dx.$$

The telescoping decomposition of the Jacobian (cf. [9, Chapter 8]) leads the equation

$$\begin{aligned} \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, \dots, f_n) &- \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, \dots, f_n^i) dx \\ &= \int_{B_r} (f_1(x) - f_1^i(x)) J(x, u_t^\epsilon, f_2, \dots, f_n) dx \\ (3.44) \quad &+ \sum_{k=2}^n \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2^i, \dots, f_{k-1}^i, f_k - f_k^i, f_{k+1}, \dots, f_n) dx. \end{aligned}$$

Combining Hadamard's inequality with Hölder's inequality we find that

$$\begin{aligned}
& \left| \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, \dots, f_n) - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, \dots, f_n^i) dx \right| \\
& \leq \int_{B_r} |f_1 - f_1^i| |Du_t^\epsilon| |Df|^{n-1} \\
& \quad + \sum_{k=2}^n \int_{B_r} |f_1| |Du_t^\epsilon| |Df^i|^{k-2} |Df - Df^i| |Df|^{n-k} \\
& \leq |Du_t^\epsilon|_{L^\infty(B_r)} \left(\int_{B_r} |f_1 - f_1^i|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} \\
& \leq C(n) |Du_t^\epsilon|_{L^\infty(B_r)} \left(|f_1|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} (|Df^i| + |Df|)^{\frac{n^2}{n+1}} \right)^{\frac{n^2-n-2}{n^2}} \\
& \quad \cdot \left(\int_{B_r} |Df - Df^i|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}}.
\end{aligned} \tag{3.45}$$

By the Sobolev-Poincaré inequality we see that the right hand side of inequality (3.45) tends to zero as i goes to ∞ . Combining this with inequality (3.42) and equation (3.43) we find that

$$- \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, \dots, f_n) dx \leq I(n) \left(\int_{\partial B_r} |D^\# f(x)| dx \right)^{\frac{n}{n-1}}. \tag{3.46}$$

Applying the assumptions $u_t^\epsilon \in C_0^\infty(B_r)$ and (3.19) we conclude that

$$\int_{B_r} u_t^\epsilon(x) J_f(x) dx \leq I(n) \left(\int_{\partial B_r} |D^\# f(x)| dx \right)^{\frac{n}{n-1}}. \tag{3.47}$$

Since $u_t^\epsilon(x) J_f(x) \leq \chi_{B_r}(x) J_f(x)$ and $J(\cdot, f) \in L_{loc}^1(\Omega)$, we can use the Dominated convergence theorem. Let first $t \rightarrow 0$ and then $\epsilon \rightarrow 0$, the claim follows. \square

It is easy to obtain (3.40) from (3.41).

Lemma 3.36. *Under the hypotheses of Theorem 3.33, we have*

$$\begin{aligned}
\int_{B(x_0, r)} J_f(x) dx & \leq \exp \left\{ -n \int_r^{R/e^3} \frac{dt}{t \mathcal{A}^{-1}(\log C_{\mathcal{A}, n}(\epsilon) (nK/\omega_{n-1} t^n))} \right\} \\
& \cdot \int_{B(x_0, R)} J_f(x) dx
\end{aligned} \tag{3.48}$$

whenever $r \in (0, R/e^3)$.

Proof. Under the assumptions of Theorem 3.33, the assumptions of Theorem 3.35 are fulfilled and hence (3.40) holds; see [13] and also [12].

Write $B_s = B(x_0, s)$. The distortion $|Df(x)|^n \leq K_f(x)J_f(x)$ together with Hölder's inequality applied to the right-hand side of (3.40) yields

$$(3.49) \quad \int_{B_s} J_f(x) dx \leq \left(\int_{\partial B_s} K_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}} \int_{\partial B_s} J_f(x) d\delta.$$

Hence, the following elementary differential equation is satisfied:

$$(3.50) \quad \frac{d}{ds} \left(\log \left(\int_{B_s} J_f(x) dx \right) \right) \geq \frac{n}{s \left(\int_{\partial B_s} K_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}}.$$

By the assumption (3.11), it is easy to prove that there exists a $\tau_0 = \tau_0(n, \mathcal{A}) > 0$ such that the functions $\tau \rightarrow \exp(\mathcal{A}(\tau))$ and $\tau \rightarrow \exp\left(\mathcal{A}\left(\tau^{\frac{1}{n-1}}\right)\right)$ are convex on (τ_0, ∞) ; see Lemma 3.23. We define an auxiliary distortion function

$$(3.51) \quad \tilde{K}_f(x) := \begin{cases} K_f(x), & \text{if } K_f(x) > \tau_0, \\ \tau_0, & \text{if } K_f(x) \leq \tau_0. \end{cases}$$

The preceding differential equation gets a slightly weaker form

$$(3.52) \quad \frac{d}{ds} \left(\log \left(\int_{B_s} J_f(x) dx \right) \right) \geq \frac{n}{s \left(\int_{\partial B_s} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}}.$$

The desired decay estimate (3.48) on the integrals of Jacobian of f over balls then follows if we can show that

$$(3.53) \quad \int_r^R \frac{ds}{s \left(\int_{\partial B_s} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}} \geq \int_r^{R/e^3} \frac{dt}{\mathcal{A}^{-1}(\log(nC_{\mathcal{A},K}/\omega_{n-1}t^n))}.$$

Toward this end, let i_R and i_r be integers such that $\log R - 1 < i_R \leq \log R$ and $\log r \leq i_r < \log r + 1$. We have

$$(3.54) \quad \int_r^R \frac{ds}{s \left(\int_{\partial B_s} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}} \geq \sum_{i=i_r}^{i_R-1} \int_{e^i}^{e^{i+1}} \frac{ds}{s \left(\int_{\partial B_s} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}}.$$

We estimate each integral in the right-hand side of (3.54) in the following way. Fix $i \in \{i_r, i_r + 1, \dots, i_R - 1\}$. Changing the variable by setting $s = e^t$, we have

$$(3.55) \quad \int_{e^i}^{e^{i+1}} \frac{ds}{s \left(\int_{\partial B_s} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}} = \int_i^{i+1} \frac{dt}{\left(\int_{\partial B_{e^t}} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}}.$$

Since the function $\tau \rightarrow 1/\tau$ defined on $(0, \infty)$ is convex, the Jensen's inequality yields

$$(3.56) \quad \int_i^{i+1} \frac{dt}{\left(\int_{\partial B_t} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}}} \geq \left[\int_i^{i+1} \left(\int_{\partial B_t} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}} dt \right]^{-1}.$$

Recall that the functions $\tau \rightarrow \exp\left(\mathcal{A}\left(\tau^{\frac{1}{n-1}}\right)\right)$ and $\tau \rightarrow \exp(\mathcal{A}(\tau))$ are convex on (τ_0, ∞) . We apply the Jensen's inequality twice to obtain that

$$(3.57) \quad \begin{aligned} \int_i^{i+1} \left(\int_{\partial B_{e^t}} \tilde{K}_f(x)^{n-1} d\delta \right)^{\frac{1}{n-1}} dt &\leq \int_i^{i+1} \mathcal{A}^{-1} \left(\log \int_{\partial B_{e^t}} \exp(\mathcal{A}(\tilde{K}_f(x))) d\delta \right) dt \\ &\leq \mathcal{A}^{-1} \left(\log \int_i^{i+1} \int_{\partial B_{e^t}} \exp(\mathcal{A}(\tilde{K}_f(x))) d\delta dt \right) \\ &= \mathcal{A}^{-1} \left(\log \int_{e^i}^{e^{i+1}} \frac{1}{s} \int_{\partial B_s} \exp(\mathcal{A}(\tilde{K}_f(x))) d\delta ds \right). \end{aligned}$$

We made a change of variable in the last step. Now an easy computation gives

$$(3.58) \quad \int_{e^i}^{e^{i+1}} \left(\frac{1}{s} \int_{\partial B_s} \exp(\mathcal{A}(\tilde{K}_f(x))) d\delta \right) ds \leq \frac{e^{\tau_0} K}{\omega_{n-1} e^{ni}}.$$

Here K comes from (3.26). Combining inequalities (3.54)-(3.58), we conclude that

$$(3.59) \quad \begin{aligned} \int_r^R \frac{ds}{s \left(\int_{\partial B_s} \tilde{K}_f^{n-1} d\delta \right)^{\frac{1}{n-1}}} &\geq \sum_{i=i_r}^{i_R-1} \left[\mathcal{A}^{-1} \left(\log \left(\frac{e^{\tau_0} K}{\omega_{n-1} e^{ni}} \right) \right) \right]^{-1} \\ &\geq \int_{i_r}^{i_R-2} \left[\mathcal{A}^{-1} \left(\log \left(\frac{e^{\tau_0} K}{\omega_{n-1} e^{ns}} \right) \right) \right]^{-1} ds \\ &\geq \int_r^{R/e^3} \left[t \mathcal{A}^{-1} \left(\log \left(\frac{e^{\tau_0} K}{\omega_{n-1} t^n} \right) \right) \right]^{-1} dt, \end{aligned}$$

which proves (3.53). \square

Now we complete the proof of Theorem 3.33 as follows.

Given $x \in B(x_0, \alpha(\frac{R}{80}))$, we consider the ball $B(x_0, r)$, $r = \alpha_{\mathcal{A}, n, K, \beta}^{-1}(|x - x_0|)$. By Lemma 3.34, we have the estimate

$$(3.60) \quad |f(x) - f(y)| \leq C_{\mathcal{A}, K}(n, \beta) \left(\int_{B(x_0, r)} J_f(z) dz \right)^{\frac{1}{n}}.$$

Using Lemma 3.36, we further obtain the estimate

$$(3.61) \quad \int_{B(x_0, r)} J_f(z) dz \leq 2 \exp \left(-n \int_{2r}^{R/e^3} \frac{dt}{t \mathcal{A}^{-1} \left(\log \left(C_{\mathcal{A}, n}(\epsilon) \frac{nK}{\omega_{n-1} t^n} \right) \right)} \right) \cdot \int_{B(x_0, R)} J_f(x) dx.$$

Combining the inequality (3.60) with the estimate (3.61), we finally obtain the desired modulus of continuity (3.27).

Remark 3.37. *The integral in (3.27) of Theorem 3.33 can be taken from $|x - y|$ to $R/80$ when $|x - y|$ is sufficiently small. To see this, notice that the ratio of this integral, taken from $|x - x_0|$ to $\alpha^{-1}(|x - x_0|)$, with the corresponding integral from $|x - x_0|$ to $R/80$ tends to zero when $|x - x_0|$ tends to zero. Then we will obtain the following inequality*

$$(3.62) \quad |f(x) - f(x_0)| \leq C_{\mathcal{A}, K}(n, \beta) \left(\int_B J_f(z) dz \right)^{1/n} \cdot \exp \left\{ (\epsilon - 1) \int_{|x - x_0|}^{R/80} \frac{dt}{t \mathcal{A}^{-1} \left(\log C_{\mathcal{A}, n}(nK/\omega_{n-1} t^n) \right)} \right\}$$

where ϵ tends to zero as $|x - x_0|$ tends to zero.

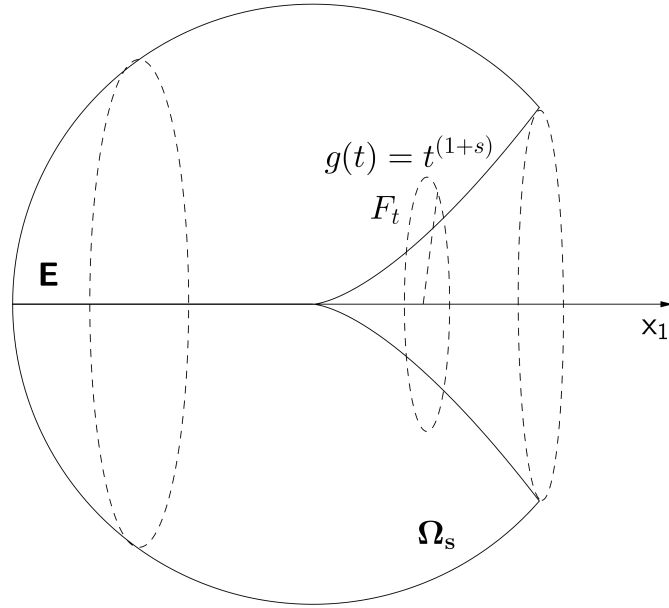
4 Main results

Base on the tools above, we are ready to give our results about the homeomorphisms of finite distortion from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s . The non-existence of quasiconformal mappings has been understood before, and the other three results are new.

4.1 Non-existence of quasiconformal mappings

In this subsection, we show that there does not exist a quasiconformal mapping from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s . We can find the result in [16] and [10]. The proofs are based on the fact that quasiconformal mappings preserve n -capacity.

Theorem 4.1. *For any $s > 0$, there does not exist a quasiconformal mapping f from the unit ball $B^3(0, 1)$ onto the cusp domain Ω_s .*

Figure 2: Cusp domain Ω_s

Proof. Suppose there is a quasiconformal mapping such that $f(B^3(0, 1)) = \Omega_s$. Pick a circle F_t of radius $2t^{1+s}$ around the cusp at the level $x_1 = t$. Let $E = \{(x, 0, 0) : -1 \leq x \leq 0\}$ on x_1 -axis. See Figure 2 above. Since $F_t \subset \overline{B^3}((t, 0, 0), 2t^{1+s})$ and $E \subset B^3((t, 0, 0), t)^C$, then by (3.4) and (3.5), we have

$$\begin{aligned} \text{Cap}_3(E, F_t; \Omega_s) &\leq \text{Cap}_3(\overline{B}((t, 0, 0), 2t^{1+s}), S^2((t, 0, 0), t); \Omega_s) \\ &\leq \frac{\omega_2}{(\log \frac{t}{2t^{1+s}})^2} \rightarrow 0 \quad \text{when } t \rightarrow 0. \end{aligned}$$

Because f is a K -quasiconformal mapping for some $K > 0$, by (3.3) it follows that

$$(4.1) \quad \text{Cap}_3(f^{-1}(E), f^{-1}(F_t); B^3(0, 1)) \leq K \text{Cap}_3(E, F_t; \Omega_s) \rightarrow 0$$

as $t \rightarrow 0$. But, on the other hand, since the curve $f^{-1}(E)$ passes through the topological loop $f^{-1}(F_t)$ for $t > 0$ small enough, we have

$$\frac{\min \{ \text{diam } f^{-1}(E), \text{diam } f^{-1}(F_t) \}}{d(f^{-1}(E), f^{-1}(F_t))} \geq 10^{-6}$$

for all small enough t , and thus by (3.6)

$$\text{Cap}_3(f^{-1}(E), f^{-1}(F_t); B^3(0, 1)) \geq \delta(3, 10^{-6}) > 0.$$

It is a contradiction with the inequality (4.1), so the claim follows. \square

4.2 Proof of Theorem 1.2

In this subsection, we construct an example which satisfies the assumption $\exp(\lambda K_f^\gamma) \in L_{loc}^1(B^3(0, 1))$ for all $0 < \gamma < 1$ and $\lambda > 0$. Let us start from a lemma which shows the distortion function of the composition of two mappings is less than or equivalent to the product of the corresponding distortion functions.

Lemma 4.2. *Let $f_1, f_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be two homeomorphisms of finite distortion. Assume they are differentiable almost everywhere and $K_{f_1}^{\frac{1}{2}}(x) \in L_{loc}^1$, then for almost every $x \in \mathbb{R}^3$ we have*

$$K_{f_2 \circ f_1}(x) \leq K_{f_2}(f_1(x)) \cdot K_{f_1}(x).$$

Proof. Since f_1 and f_2 are homeomorphisms of finite distortion, we know $f_1(x), f_2(x) \in W_{loc}^{1,1}(\mathbb{R}^3, \mathbb{R}^3)$. By Theorem 3.14, we know $\{x \in \mathbb{R}^3 : f_1 \text{ is differentiable at } x \text{ and } f_2 \text{ is differentiable at } f_1(x)\}$ is full-measure. For such $x \in \mathbb{R}^3$ such that f_1 is differentiable at x and f_2 is differentiable at $f_1(x)$, if $J_{f_1}(x) = 0$ or $J_{f_2}(f_1(x)) = 0$ is satisfied, then since $D(f_2 \circ f_1)(x) = D(f_2)(f_1(x)) \cdot D(f_1)(x)$, we get $J_{f_2 \circ f_1}(x) = 0$. By the definition of optimal distortion function and that every distortion function is not less than 1, we get $1 = K_{f_2 \circ f_1}(x) \leq K_{f_2}(f_1(x)) \cdot K_{f_1}(x)$.

For $x \in \mathbb{R}^3$, if both $J_{f_1}(x) > 0$ and $J_{f_2}(f_1(x)) > 0$, we have

$$\begin{aligned} K_{f_2 \circ f_1}(x) &= \frac{|D(f_2 \circ f_1)(x)|^3}{J_{f_2 \circ f_1}(x)} \\ &= \sup_{|v| \leq 1} \frac{\|D(f_2 \circ f_1)(x)v\|^3}{J_{f_2 \circ f_1}(x)} \\ &= \sup_{|v| \leq 1} \frac{\|D(f_2)(f_1(x)) \frac{D(f_1)(x)}{|D(f_1)(x)|} v\|^3}{J_{f_2}(f_1(x))} \cdot \frac{|D(f_1)(x)|^3}{J_{f_1}(x)} \\ &\leq \frac{|D(f_2)(f_1(x))|^3}{J_{f_2}(f_1(x))} \cdot \frac{|D(f_1)(x)|^3}{J_{f_1}(x)} \\ &= K_{f_2}(f_1(x)) \cdot K_{f_1}(x) \end{aligned}$$

as desired. □

Now we begin the proof of Theorem 1.2

Proof of Theorem 1.2. Let $0 < \gamma < 1$ be fixed. Our goal is to construct a homeomorphism $f : B^3(0, 1) \rightarrow \Omega_s$ of finite distortion such that $\exp(\lambda K_f^\gamma) \in L_{loc}^1(B^3(0, 1))$ for all $\lambda > 0$. We divide the proof into five steps.

Step 1: First, for the convenience of the compositions below, we enlarge the ball $B^3(0, 1)$ to $B^3(0, \sqrt{2})$. We define the mapping $f_0 : B^3(0, 1) \rightarrow B^3(0, \sqrt{2})$ by $f_0(x, y, z) = (\sqrt{2}x, \sqrt{2}y, \sqrt{2}z)$. Obviously, it is a conformal mapping, so $K_{f_0}(x) \equiv 1$.

Step 2: we construct a homeomorphism f_1 , which maps the ball $B(0, \sqrt{2})$ onto the following Lipschitz domain,

$$\Delta_l := B(0, \sqrt{2}) \setminus \left(\left\{ (x_1, x_2, x_3) : x_1 \geq \frac{\sqrt{10}}{5} \right\} \cup \left\{ (x_1, x_2, x_3) : x_1 \geq 0, \sqrt{x_2^2 + x_3^2} \leq x_1 \right\} \right).$$

Then we construct the homeomorphism discussed above from $B(0, \sqrt{2})$ to Δ_l in cylindrical coordinates by:

$$f_1(x, r, \theta) := \begin{cases} \left(\frac{\sqrt{2-r^2}+r}{2\sqrt{2-r^2}}x + \frac{r-\sqrt{2-r^2}}{2}, r, \theta \right), & \text{if } 0 \leq r < \frac{\sqrt{10}}{5} \\ \left(\frac{\sqrt{2-r^2}+\sqrt{10}/5}{2\sqrt{2-r^2}}x + \frac{\sqrt{10}/5-\sqrt{2-r^2}}{2}, r, \theta \right), & \text{if } \frac{\sqrt{10}}{5} \leq r < \frac{2\sqrt{10}}{5} \\ (x, r, \theta), & \text{if } \frac{2\sqrt{10}}{5} \leq r < \sqrt{2}. \end{cases}$$

And by a simple computation, we find that f_1 is a quasiconformal mapping.

Step 3: First, we define a subdomain of the cusp domain Ω_s by

$$\Omega_l := \Omega_s \setminus \left\{ (x_1, x_2, x_3) : x_1 \geq \frac{\sqrt{10}}{5} \right\}.$$

As we can see Ω_l has the same cusp at origin as the cusp domain Ω_s . Then we will give a homeomorphism of finite distortion f_2 from Δ_l onto Ω_l and prove its distortion function satisfies $\exp(\lambda K_{f_2}^\gamma) \in L_{loc}^1(\Delta_l)$ for all $0 < \gamma < 1$ and $\lambda > 0$. For the convenience of computation, we divide our argument into two parts:

First, using the cylindrical coordinates and we define f_2 on the domain $D_1 := \Delta_l \cap \{(x, r, \theta) : 0 \leq x < \frac{\sqrt{10}}{5} \text{ and } x < r < 2x\}$ by

$$f_2(x, r, \theta) = \left(g(x), \frac{2g(x) - g^{1+s}(x)}{x}r + 2g^{1+s}(x) - 2g(x), \theta \right),$$

where $g(x) = \frac{\frac{\sqrt{10}}{5} \log \log \left(\frac{5}{\sqrt{10}} \right)}{\log \log \left(\frac{1}{x} \right)}$, and we set $\hat{C} := \frac{\sqrt{10}}{5} \log \log \left(\frac{5}{\sqrt{10}} \right)$. We define a linear function $L_x(r) : (x, 2x) \rightarrow (g^{1+s}(x), 2g(x))$ by

$$L_x(r) = \frac{2g(x) - g^{1+s}(x)}{x}r + 2g^{1+s}(x) - 2g(x).$$

Now we can write $f_2(x, r, \theta) = (g(x), L_x(r), \theta)$ for $(x, r, \theta) \in D_1 \subset \Delta_l$.

For the left part $D_2 := \Delta_l \setminus D_1$, we use the spherical coordinates and define the mapping f_2 on D_2 by

$$f_2(r, \theta, \phi) = \left(\sqrt{5}g \left(\frac{r}{\sqrt{5}} \right), \theta, \phi \right).$$

Now let us prove that $\exp(\lambda K_{f_2}^\gamma(x)) \in L_{loc}^1(\Delta_l)$ for all $\lambda > 0$. We need to figure out the differential matrix of f_2 , and give an upper bound to distortion function. According to two subdomain D_1 and D_2 , we have and $|\overline{D_1} \cap \overline{D_2}| = 0$, then we divide the full processing into two cases.

Case 1 ($a \in D_1$): To compute $K_{f_2}(a)$, here we employ the cylindrical coordinates $a = (x, r, \theta) \in D_1$. We choose a local cylindrical coordinate system at every $a \in D_1 \subset \Omega_l$ by setting $e_1^a = (1, 0, 0)$, $e_2^a = (0, \cos \theta, \sin \theta)$ and $e_3^a = (0, -\sin \theta, \cos \theta)$. Notice that in Cartesian coordinate system in \mathbb{R}^3 , e_1^a points to the x -direction, both e_2^a and e_3^a point to the directions which are perpendicular to the x -direction and lie in the plane $\{(0, y, z) \in \mathbb{R}^3\}$ and $\det \left(\frac{D(e_1^a, e_2^a, e_3^a)}{D(x_1, x_2, x_3)} \right) = 1$. If we restrict to the Cartesian plane $\{(0, y, z) \in \mathbb{R}^3\}$, we find that the vector e_2^a points to the radial direction and e_3^a is perpendicular to it, so it points to the angular direction. To this basis on the preimage side we associate a similar basis on the image side, denoting it by $(e_1^{f_2(a)}, e_2^{f_2(a)}, e_3^{f_2(a)})$. We now represent the differential matrix of f_2 at the point a by using bases (e_1^a, e_2^a, e_3^a) and $(e_1^{f_2(a)}, e_2^{f_2(a)}, e_3^{f_2(a)})$ which we will refer to E_a and $E_{f_2(a)}$, respectively. The resulting differential matrix is

$$(4.2) \quad D_{f_2}(x, r, \theta) = \begin{pmatrix} \frac{d}{dx}g(x) & 0 & 0 \\ \frac{d}{dx}L_x(r) & \frac{d}{dr}L_x(r) & 0 \\ 0 & 0 & \frac{L_x(r)}{r} \frac{d}{d\theta}\theta \end{pmatrix}$$

$$= \begin{pmatrix} g'(x) & 0 & 0 \\ \alpha & \frac{2g(x)-g^{1+s}(x)}{x} & 0 \\ 0 & 0 & \frac{2g(x)-g^{1+s}(x)}{x} + \frac{2g^{1+s}(x)-g(x)}{r} \end{pmatrix}$$

where $\alpha = \frac{(2g'(x)-(1+s)g^s(x)g'(x))x-(2g(x)-g^{1+s}(x))}{x^2}r + (2(1+s)g^s(x)g'(x) - 2g'(x))$. The elements inside the matrix are computed as follows. For the convenience, we call the three parts of $f_2(x, r, \theta) = (g(x), L_x(r), \theta)$ as Cartesian part, radial part and angular part, respectively.

As the Cartesian part of f_2 depends only on x , the partial derivative of the first component of f_2 in the basis $E_{f_2(a)}$ in the e_1^a -direction is $\frac{d}{dx}g(x)$. The Cartesian part does not depend on r and θ , and so the partial derivatives of the first component of f_2 in the basis $E_{f_2(a)}$ in the e_2^a and e_3^a -direction are zeros.

Next we observe that for $\epsilon > 0$ the change in the radial part, and thus in the $e_2^{f_2(a)}$ -direction, is $L_{x+\epsilon}(r) - L_x(r)$ when the change to the e_1^a -direction is ϵ . Thus the partial derivative of the second component of f_2 in the basis $E_{f_2(a)}$ in the e_1^a -direction is

$$\lim_{\epsilon \rightarrow 0} \frac{L_{x+\epsilon}(r) - L_x(r)}{\epsilon} = \frac{d}{dx}L_x(r).$$

Similarly, the partial derivative of the second component of f_2 in the basis $E_{f_2(a)}$ in e_2^a -direction is

$$\lim_{\epsilon \rightarrow 0} \frac{L_x(r + \epsilon) - L_x(r)}{\epsilon} = \frac{d}{dr} L_x(r).$$

And obviously, $L_x(r)$ does not depend on θ , so $\frac{d}{d\theta} L_x(r) = 0$.

For the third component of f_2 , since we observe that the angular part does not depend on x and r , so the partial derivatives of the third component of f_2 in the basis $E_{f_2(a)}$ in the e_1^a and e_2^a -directions are zeros. As discussed above, the partial derivative of the third component of f_2 in the basis $E_{f_2(a)}$ in the e_3^a -direction is

$$\lim_{\epsilon \rightarrow 0} \frac{L_x(r)((\theta + \epsilon) - \theta)}{r\epsilon} = \frac{L_x(r)}{r} \frac{d}{d\theta} \theta.$$

To estimate $K_{f_2}(x)$ from 4.2 we use the following known result (see [16]) which states that for a linear bijection $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the distortion K of A satisfies

$$(4.3) \quad K \leq \frac{(\sum_{i,j=1}^3 a_{i,j}^2)^{3/2}}{\det(A)}.$$

Here $(a_{i,j})$ is the matrix of A , and it gives us a simple upper estimate for K . And we give the relative estimates below.

The determinant of the differential matrix is

$$\begin{aligned} \det(D_{f_2}(x, r, \theta)) &= \left(\frac{\hat{C}}{x \log \frac{1}{x} (\log \log \frac{1}{x})^2} \right) \left(\frac{2g(x) - g^{1+s}(x)}{x} \right) \\ &\quad \cdot \left(\frac{2g(x) - g^{1+s}(x)}{x} + \frac{2g^{1+s}(x) - 2g(x)}{r} \right) \\ &= \left(\frac{\hat{C}}{x \log \frac{1}{x} (\log \log \frac{1}{x})^2} \right) \cdot \left(\frac{2\hat{C}}{x \log \log \frac{1}{x}} - \frac{\hat{C}^{1+s}}{x (\log \log \frac{1}{x})^{1+s}} \right) \\ &\quad \cdot \left(\frac{2\hat{C}}{x \log \log \frac{1}{x}} - \frac{\hat{C}^{1+s}}{x (\log \log \frac{1}{x})^{1+s}} + \frac{2\hat{C}^{1+s}}{r (\log \log \frac{1}{x})^{1+s}} - \frac{2\hat{C}}{r \log \log \frac{1}{x}} \right) \\ &\geq \frac{\hat{C}^{3+s}}{x^3 \log \frac{1}{x} (\log \log \frac{1}{x})^{4+s}}, \end{aligned}$$

the last inequality follows from $x < r < 2x$ and $g^s(x) \leq 1$. And by a simple calculation, we have

$$\left(\frac{d}{dx} g(x) \right)^2 = (g'(x))^2 = \frac{\hat{C}^2}{x^2 (\log \frac{1}{x})^2 (\log \log \frac{1}{x})^4};$$

$$\begin{aligned} \left(\frac{d}{dr}L_x(r)\right)^2 &= \left(\frac{2g(x) - g^{1+s}(x)}{x}\right)^2 = \left(\frac{\frac{2\hat{C}}{\log \log \frac{1}{x}} - \left(\frac{\hat{C}}{\log \log \frac{1}{x}}\right)^{1+s}}{x}\right)^2 \\ &\leq \frac{\hat{C}^2}{x^2 (\log \log \frac{1}{x})^2}; \end{aligned}$$

$$\begin{aligned} \left(\frac{L_x(r)}{r}\right)^2 &= \left(\frac{2g(x) - g^{1+s}(x)}{x} + \frac{2(g^{1+s}(x) - g(x))}{r}\right)^2 \\ &= \left(\frac{\frac{2\hat{C}}{\log \log \frac{1}{x}} - \frac{\hat{C}^{1+s}}{(\log \log \frac{1}{x})^{1+s}}}{x} + \frac{2\left(\frac{\hat{C}^{1+s}}{(\log \log \frac{1}{x})^{1+s}} - \frac{\hat{C}}{\log \log \frac{1}{x}}\right)}{r}\right)^2 \\ &\leq \left(\frac{\hat{C}}{x \log \log \frac{1}{x}} + \frac{\hat{C}^{1+s}}{x (\log \log \frac{1}{x})^{1+s}}\right)^2 \\ &= \frac{\hat{C}^2}{x^2 (\log \log \frac{1}{x})^2} + \frac{2\hat{C}^{2+s}}{x^2 (\log \log \frac{1}{x})^{2+s}} + \frac{\hat{C}^{2+2s}}{x^2 (\log \log \frac{1}{x})^{2+2s}}; \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}L_x(r) = \alpha &= \frac{(2g'(x) - (1+s)g^s(x)g'(x))x - (2g(x) - g^{1+s}(x))}{x^2}r \\ &\quad + (2(1+s)g^s(x)g'(x) - 2g'(x)) \\ &= \left(\frac{2rg'(x)}{x} + \frac{rg^{1+s}(x)}{x^2} + 2(1+s)g^s(x)g'(x)\right) \\ &\quad - \left(\frac{r(1+s)g^s(x)g'(x)}{x} + \frac{2rg(x)}{x^2} + 2g'(x)\right) \\ &=: I_1 - I_2. \end{aligned}$$

Now it is obvious that $\alpha^2 \leq I_1^2 + I_2^2$. Let us estimate I_1^2 and I_2^2 .

$$\begin{aligned} I_1^2 &= \left(\frac{2rg'(x)}{x} + \frac{rg^{1+s}(x)}{x^2} + 2(1+s)g^s(x)g'(x)\right)^2 \\ &= \left(\frac{2\hat{C}r}{x^2 \log \frac{1}{x} (\log \log \frac{1}{x})^2} + \frac{\hat{C}^{1+s}r}{x^2 (\log \log \frac{1}{x})^{1+s}} + \frac{2(1+s)\hat{C}^{1+s}}{x \log \frac{1}{x} (\log \log \frac{1}{x})^{2+s}}\right)^2 \\ &\leq \frac{C_1(s, \hat{C})}{x^2 (\log \log \frac{1}{x})^{2+2s}}. \end{aligned}$$

The last inequality follows $x < r < 2x$ and $0 \leq x < \frac{\sqrt{10}}{5}$. And similarly

$$\begin{aligned} I_2^2 &= \left(\frac{r(1+s)g^s(x)g'(x)}{x} + \frac{2rg(x)}{x^2} + 2g'(x) \right)^2 \\ &= \left(\frac{(1+s)\hat{C}^{1+s}r}{x^2 \log \frac{1}{x} (\log \log \frac{1}{x})^{2+s}} + \frac{2\hat{C}r}{x^2 \log \log \frac{1}{x}} + \frac{2\hat{C}}{x \log \frac{1}{x} (\log \log \frac{1}{x})^2} \right)^2 \\ &\leq \frac{C_2(s, \hat{C})}{x^2 (\log \log \frac{1}{x})^2}. \end{aligned}$$

Then by the computations above, we can estimate the distortion function from above by

$$\begin{aligned} K_{f_2}(x, r, \theta) &\leq \frac{\left(\left(\frac{d}{dx} g(x) \right)^2 + \left(\frac{d}{dr} L_x(r) \right)^2 + \left(\frac{d}{dx} L_x(r) \right)^2 + \left(\frac{L_x(r)}{r} \right)^2 \right)^{\frac{3}{2}}}{\det(D_{f_2})} \\ &\leq C_3(s, \hat{C}) \log \frac{1}{x} \left(\log \log \frac{1}{x} \right)^{1+s}. \end{aligned}$$

then we get $\exp(\lambda K_{f_2}^\gamma) \in L^1(D_1)$, for all $0 < \gamma < 1$ and $\lambda > 0$.

Case 2 ($a \in D_2$): The ideas are similar to Case 1. Here we attach a local coordinate system at every point $a \in D_2$ by setting $e_1^a = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta)$, $e_2^a = (\sin \theta \cos \phi, \sin \theta \sin \phi, -\cos \theta)$ and $e_3^a = (\sin \phi, -\cos \phi, 0)$ and we have $\det \left(\frac{D(e_1^a, e_2^a, e_3^a)}{D(x_1, x_2, x_3)} \right) = 1$. To this basis on the preimage side we associate a similar basis on the image side, denoting by $(e_1^{f_2(a)}, e_2^{f_2(a)}, e_3^{f_2(a)})$. The result differential matrix is

$$D_{f_2}(r, \theta, \phi) = \begin{pmatrix} \frac{\sqrt{5}\hat{C}}{r \log \left(\frac{\sqrt{5}}{r} \right) (\log \log \left(\frac{\sqrt{5}}{r} \right))} & 0 & 0 \\ 0 & \frac{\sqrt{5}\hat{C}}{r \log \log \left(\frac{\sqrt{5}}{r} \right)} & 0 \\ 0 & 0 & \frac{\sqrt{5}\hat{C}}{r \log \log \left(\frac{\sqrt{5}}{r} \right)} \end{pmatrix}.$$

Then by a simple computation, we know that the distortion function is

$$K_{f_2}(x, r, \theta) = \log \frac{\sqrt{5}}{r} \left(\log \log \frac{\sqrt{5}}{r} \right)^2$$

Then we get $\exp(\lambda K_{f_2}^\gamma) \in L^1(D_2)$ for all $0 < \gamma < 1$ and $\lambda > 0$.

Step 4: We give the homeomorphism from Ω_l to Ω_s by

$$f_3(x, r, \theta) := \begin{cases} \left(\frac{5r^{1+s}}{\sqrt{10}} x, r, \theta \right), & \text{if } 0 \leq x < \frac{\sqrt{10}}{5} \text{ and } \left(\frac{\sqrt{10}}{5} \right)^{1+s} < r < 1, \\ \left(\frac{5\sqrt{2-r^2}}{\sqrt{10}} x, r, \theta \right), & \text{if } 0 \leq x < \frac{\sqrt{10}}{5} \text{ and } 1 \leq r < \frac{2\sqrt{10}}{5}, \\ (x, r, \theta), & \text{elsewhere.} \end{cases}$$

By a simple computation, we know it is a quasiconformal mapping.

Step 5: By the result above, we know $f(x) = f_3 \circ f_2 \circ f_1 \circ f_0(x)$ is a homeomorphism of finite distortion between the ball $B(0, 1)$ and cusp domain Ω_s . It is easy to check f_i ($i = 0, 1, 2, 3$) is differentiable almost everywhere and satisfy Lusin (N^{-1}) condition. By Lemma 4.2, we get $K_f(x) \leq K_{f_3}(f_2(f_1(f_0(x)))) \cdot K_{f_2}(f_1(f_0(x))) \cdot K_{f_1}(f_0(x)) \cdot K_{f_0}(x)$. Since f_0 is conformal, both f_1 and f_3 are quasiconformal and K_{f_2} satisfies $\exp(\lambda K_{f_2}^\gamma) \in L_{loc}^1(\Delta_l)$ for all $0 < \gamma < 1$ and $\lambda > 0$, then we get that $K_f(x)$ satisfies $\exp(\lambda K_f^\gamma) \in L_{loc}^1(B(0, 1))$ for all $0 < \gamma < 1$ and $\lambda > 0$ as deserved. \square

4.3 Proof of Theorem 1.3

Proof of Theorem 1.3. Let us assume there is such a homeomorphism of finite distortion f with K_f satisfies the inequality (1.5), then by Lemma 3.18, we have

$$Cap_3(f^{-1}(E), f^{-1}(F_t), B^3(0, 1)) \leq Cap_3^{K_f(f^{-1}(y))}(E, F_t, \Delta_s).$$

Since

$$\frac{\min\{\text{diam } f^{-1}(E), \text{diam } f^{-1}(F_t)\}}{d(f^{-1}(E), f^{-1}(F_t))} \geq 10^{-6}$$

for all t , and following (3.6), we have

$$Cap_3(f^{-1}(E), f^{-1}(F_t), B^3(0, 1)) \geq \delta(3, 10^{-6}) > 0.$$

for the estimates above please refer the proof of Theorem 4.1. Next, we will show that $Cap_3^{K_f(f^{-1}(y))}(E, F_t, \Delta_s) \rightarrow 0$ as $t \rightarrow 0$, and this gives us a desired contradiction.

We write $\omega(y) = K_f(f^{-1}(y))$ and define

$$v(r) = \int_{t^{1+s}}^r \frac{ds}{\left(\int_{S^2((t,0,0),s)} \omega d\delta\right)^{\frac{1}{2}}},$$

for $t^{1+s} \leq r \leq t$ and further define $u : B((t, 0, 0), t) \setminus \overline{B((t, 0, 0), t^{1+s})} \rightarrow \mathbb{R}$ by setting

$$u(x) = 1 - \frac{v(|x|)}{v(t)}.$$

We extend u in an obvious way to the exterior of $B((t, 0, 0), t) \setminus \overline{B((t, 0, 0), t^{1+s})}$ and we obtain a Lipschitz function (Since $K_f \geq 1$, and also $\omega \geq 1$ and it is easy to check that v is Lipschitz). We conclude by the Fubini theorem that

$$\int_{B((t,0,0),t)} |Du(x)|^3 \omega(x) dx \leq \int_{t^{1+s}}^t \int_{S^2((t,0,0),s)} \left(\frac{1}{v(t) \left(\int_{S^2(s)} \omega d\sigma\right)^{\frac{1}{2}}} \right)^3 \omega d\sigma ds$$

$$\leq v(t)^{-3} \int_{t^{1+s}}^t \frac{ds}{\left(\int_{S^2((t,0,0),s)} \omega d\sigma \right)^{\frac{1}{2}}} = v(t)^{-2}.$$

Hence it suffices to show $v(t) \rightarrow \infty$ as $t \rightarrow 0$.

Next, pick integers i_t and $i_{t^{1+s}}$ so that $\log t - 1 < i_t \leq \log t$ and $\log t^{1+s} \leq i_{t^{1+s}} < \log t^{1+s} + 1$. Then

$$v(t) \geq \sum_{i=i_{t^{1+s}}}^{i_t-1} \int_{e^i}^{e^{i+1}} \frac{ds}{\left(\int_{S^2((t,0,0),s)} \omega d\sigma \right)^{\frac{1}{2}}}.$$

Now, a change of variable, convexity of $t \rightarrow \frac{1}{t}$ and Jensen's inequality show that

$$\begin{aligned} \int_{e^i}^{e^{i+1}} \frac{ds}{\left(\int_{S^2((t,0,0),s)} \omega d\sigma \right)^{\frac{1}{2}}} &= \int_{e^i}^{e^{i+1}} \frac{ds}{s \left(\omega_2 \int_{S^2((t,0,0),s)} \omega d\sigma \right)^{\frac{1}{2}}} \\ &= \int_i^{i+1} \frac{dr}{\left(\omega_2 \int_{S^2((t,0,0),e^r)} \omega d\sigma \right)^{\frac{1}{2}}} \\ &\geq \left(\int_i^{i+1} \left(\omega_2 \int_{S^2((t,0,0),e^r)} \omega d\sigma \right)^{\frac{1}{2}} dr \right)^{-1}, \end{aligned}$$

for each $i_{t^{1+s}} \leq i \leq i_t - 1$. Applying Jensen's inequality again, for the convex functions $r \rightarrow \exp(r)$ and $r \rightarrow \max\{e, \exp(r^{\frac{1}{2}})\}$, we see that

$$\begin{aligned} \int_i^{i+1} \left(\omega_2 \int_{S^2((t,0,0),e^r)} \omega d\sigma \right)^{\frac{1}{2}} dr &\leq \frac{\omega_2^{\frac{1}{2}}}{\lambda} \log \left(\int_i^{i+1} \exp \left(\left(\int_{S^2((t,0,0),e^r)} \lambda^2 \omega d\sigma \right)^{\frac{1}{2}} \right) dr \right) \\ &\leq \frac{\omega_2^{\frac{1}{2}}}{\lambda} \log \left(\int_i^{i+1} \int_{S^2((t,0,0),e^r)} \exp(\lambda \widehat{\omega}^{\frac{1}{2}}) d\sigma dr \right), \end{aligned}$$

where $\widehat{\omega} = \max\{1, \omega\}$. An easy computation shows that

$$\begin{aligned} \int_i^{i+1} \int_{S^2(e^r)} \exp(\lambda \widehat{\omega}^{\frac{1}{2}}) d\sigma dr &= \int_{e^i}^{e^{i+1}} \frac{1}{s} \int_{S^2((t,0,0),s)} \exp(\lambda \widehat{\omega}^{\frac{1}{2}}) d\sigma ds \\ &\leq \frac{1}{\omega_2 e^{3i}} \int_{e^i}^{e^{i+1}} \int_{S^2((t,0,0),s)} \exp(\lambda \widehat{\omega}^{\frac{1}{2}}) d\sigma ds \\ &\leq \frac{CI}{\omega_2 e^{3i}}, \end{aligned}$$

where $C = 1 + \exp(\lambda)$ and $I = \int_{B((t,0,0),t)} \exp(\lambda K_f^{\frac{1}{2}}(f^{-1}(y)))$, by (1.5) and Theorem 3.10, we know $I < \infty$. It is easy to see $\frac{CI}{\omega_2 e^{3i}} \gtrsim 2$ for all $i \leq i_t - 1$. Combining the inequalities above, we conclude that

$$\begin{aligned} v(t) &\geq \sum_{i=i_{t1+s}}^{i_t-1} \int_{e^i}^{e^{i+1}} \frac{ds}{\left(\int_{S^2(s)} \omega d\sigma\right)^{\frac{1}{2}}} \\ &\geq \frac{\lambda}{\omega_2^{\frac{1}{2}}} \sum_{i=i_{t1+s}}^{i_t-1} \log^{-1} \left(\frac{CI}{\omega_2 e^{3i}} \right) \\ &\geq \frac{\lambda}{\omega_2^{\frac{1}{2}}} \int_{t^{1+s}}^{t/e^3} \frac{dr}{r \log \left(\frac{CI}{\omega_2 r^3} \right)} \end{aligned}$$

and we get $v(t) \rightarrow \infty$ as $t \rightarrow 0$. Hence we know $Cap_3^\omega(E, F_t, \Delta_s)$ goes to 0, as t goes to 0.

For every admissible function ρ , we have

$$\int_{\Omega_s} \rho^3(y) K_f(f^{-1}(y)) dy = \int_{\Omega_s} \rho^3 \omega(y) dy,$$

and since $Cap_3^\omega(E, F_t, \Omega_s)$ goes to 0, as t goes to 0, we have $Cap_3^{K_f(f^{-1}(y))}$ goes to 0 as t goes to 0, then we get a contradiction and we finish the proof. \square

4.4 Proof of Theorem 1.4

Proof of Theorem 1.4. First, let us assume there exists such a homeomorphism. Pick a circle F_t of radius $2t^{1+s}$ around the cusp at the level $x_1 = t$ and let $E = \{(x, 0, 0) : -1 \leq x \leq 0\}$, see Figure 2. Then by the same argument as in the Proof of Theorem 1.3, we have

$$\begin{aligned} Cap_3(E, F_t; \Omega_s) &\leq Cap_3(\bar{B}((t, 0, 0), 2t^{1+s}), S^2((t, 0, 0), t); \Omega_s) \\ &\leq \frac{\omega_2}{\left(\log \frac{t}{2t^{1+s}}\right)^2} \rightarrow 0 \text{ when } t \rightarrow 0. \end{aligned}$$

We denote $E' = f^{-1}(E)$ and $F_t' = f^{-1}(F_t)$. Then E' is a curve passing through the topological circle F_t' . Therefore, for every t close enough to 0, we can find $x_0^t \in F_t'$ and $x_1^t \in E'$ such that $r_2 = \text{dist}(x_0^t, x_1^t) = \text{dist}(E', F_t')$. We define $r_1 = \max\{\text{dist}(x_0^t, x) : x \in F_t'\}$, and it is not difficult to check that $r_1 \geq \sqrt{2}r_2$ and

r_1, r_2 tend to zero as t tends to zero. So for every $r \in [r_2, \sqrt{2}r_2]$, $E' \cap S(x_0^t, r) \neq \emptyset \neq F'_t \cap S(x_0^t, r)$. Then by the Lemma 3.17, we get

$$(4.4) \quad C(p, 3)r_2^{3-p} \leq \int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} |Du|^p dx,$$

for every $u \in A(E', F'_t; B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0, 1)) \cap C^\infty(B(0, 1))$. Then for such admissible function u , using the Hölder's inequality and Jensen's inequality for the convex function $t \rightarrow \max\{\exp\left(\frac{p-\gamma(3-p)}{\gamma(3-p)}\right), \exp\left(t^{\frac{\gamma(3-p)}{p}}\right)\}$ we have

$$\begin{aligned} C(p, 3) &\leq r_2^p \int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \frac{|Du|^p}{\lambda^{\frac{p}{3\gamma}} K_f^{\frac{p}{3}}(x)} dx \\ &\leq \left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \frac{|Du(x)|^3}{\lambda^{\frac{1}{\gamma}} K_f(x)} dx \right)^{\frac{p}{3}} \cdot \left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \lambda^{\frac{p}{\gamma(3-p)}} K_f^{\frac{p}{3-p}}(x) dx \right)^{\frac{3-p}{3}} \\ &\leq \left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \frac{|Du(x)|^3}{\lambda^{\frac{1}{\gamma}} K_f(x)} dx \right)^{\frac{p}{3}} \\ &\quad \cdot \log^{\frac{p}{3\gamma}} \left(\exp \left(\left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \lambda^{\frac{p}{\gamma(3-p)}} K_f^{\frac{p}{3-p}}(x) dx \right)^{\frac{\gamma(3-p)}{p}} \right) \right) \\ &\leq \left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \frac{|Du(x)|^3}{\lambda^{\frac{1}{\gamma}} K_f(x)} dx \right)^{\frac{p}{3}} \cdot \log^{\frac{p}{3\gamma}} \left(\int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \exp \left(\lambda \hat{K}_f^\gamma(x) \right) dx \right), \end{aligned}$$

where $\hat{K}_f(x) = \max\left\{\left(\frac{\gamma-1}{\lambda}\right), K_f(x)\right\}$. Then for every admissible function u , we get

$$(4.5) \quad \frac{C(p, 3, \lambda, \gamma)}{\log^{\frac{1}{\gamma}}\left(\frac{1}{r_2^n}\right)} \leq \int_{B^3(x_0^t, \sqrt{2}r_2) \cap B^3(0,1)} \frac{|Du(x)|^3}{K_f(x)} dx.$$

Because f is a homeomorphism of finite distortion with the distortion function $K_f(x)$, by Lemma 3.18

$$Cap_3^{1/K_f(x)}(f^{-1}(E), f^{-1}(F_t); B^3(0, 1)) \leq Cap_3(E, F_t, \Omega_s),$$

And by the discussion above we get

$$(4.6) \quad \frac{C(p, 3, \lambda, \gamma)}{\log^{\frac{1}{\gamma}}\left(\frac{1}{r_2^n}\right)} \leq Cap_3^{1/K_f(x)}(f^{-1}(E), f^{-1}(F_t); B^3(0, 1)).$$

By the discussion above, we can obtain

$$(4.7) \quad Cap_3(E, F_t; \Omega_s) \leq \frac{\omega_2}{\left(\log \frac{1}{2t^s}\right)^2} \leq \frac{C}{\left(\log \frac{1}{|f(x)-f(x_0)|}\right)^2}.$$

we take $\mathcal{A}(t) = \lambda t^\gamma$, and by the modulus of continuity (exactly, by inequality(3.62)), we get

$$\begin{aligned}
Cap_3(E, F_t; \Omega_s) &\leq \frac{C}{\left(\log \frac{1}{|f(x)-f(x_0)|}\right)^2} \\
&\leq \frac{C}{\left(\log \frac{1}{C_{\mathcal{A},K}(3,\beta)\left(\int_B J_f(z)dz\right)^{\frac{1}{3}}}\exp\left((\epsilon-1)\int_{|x-x_0|}^{1/4} \frac{1}{t\mathcal{A}^{-1}\left(\log \frac{3C_{\mathcal{A},3K}}{\omega_2 t^3}\right)}\right)\right)^2} \\
&\leq \frac{C}{\left(\log \frac{1}{C_{\mathcal{A},K}(3,\beta)\left(\int_B J_f(z)dz\right)^{\frac{1}{3}}} + \frac{1-\epsilon}{\lambda^{\frac{1}{\gamma}}}\int_{|x-x_0|}^{1/4} \frac{dt}{t \log^{\frac{1}{\gamma}} \frac{3C_{\mathcal{A},3K}}{\omega_2 t^3}}\right)^2} \\
&\leq \frac{C}{\left(\frac{\gamma(1-\epsilon)}{\lambda^{\frac{1}{\gamma}}(3\gamma-3)}\left(\log^{1-\frac{1}{\gamma}}\left(\frac{3C_{\mathcal{A},3K}}{\omega_2|x-x_0|^3}\right) - \log^{1-\frac{1}{\gamma}}\frac{192C_{\mathcal{A},3K}}{\omega_2}\right)\right)^2} \\
(4.8) \quad &\leq \frac{C'}{\left(\log^{1-\frac{1}{\gamma}}\frac{C_1}{|x-x_0|}\right)^2},
\end{aligned}$$

where $K = \int_B \exp(\lambda K_f^\gamma) dx < \infty$ and $|x - x_0| = r_2$.

Then by the inequalities (4.5), (4.8) and Lemma 3.18, we can get

$$(4.9) \quad \frac{C''}{\log^{\frac{1}{\gamma}}\left(\frac{1}{r_2}\right)} \leq \frac{C'}{\log^{2-\frac{2}{\gamma}}\frac{C_1}{r_2}}.$$

Since $\gamma > \frac{3}{2}$, we have $\frac{1}{\gamma} < 2 - \frac{2}{\gamma}$, which contradicts the above inequality (4.9) as r_2 tends to zero. Thus we get the desired result. \square

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