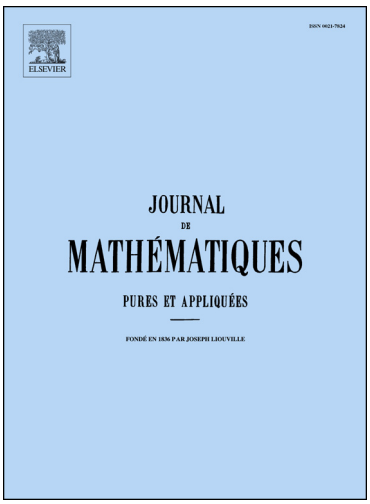

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GENERALIZED HARNACK INEQUALITY FOR SEMILINEAR ELLIPTIC EQUATIONS

VESA JULIN

Abstract. This paper is concerned with semilinear equations in divergence form

$$\operatorname{div}(A(x)Du) = f(u)$$

where $f : \mathbb{R} \to [0, \infty)$ is nondecreasing. We introduce a sharp Harnack type inequality for nonnegative solutions which is a quantified version of the condition for strong maximum principle found by Vazquez and Pucci-Serrin in [30, 24] and is closely related to the classical Keller-Osserman condition [15, 22] for the existence of entire solutions.

Dans cet article on s'intéresse à des équations semi-linéaires sous forme divergence du type:

$$\operatorname{div}(A(x)Du) = f(u),$$

où $f : \mathbb{R} \mapsto [0, \infty)$ est une fonction croissante. On démontre une inégalité optimale de type Harnack pour les solutions positives. Cette inégalité représente une version "quantifiée" de la condition établie par Vazquez et Pucci-Serrin dans [30, 24] pour la validité du principe du maximum fort et elle est étroitement liée à la condition de Keller-Osserman [15, 22] pour l'existence de solutions entières.

1. Introduction

In this paper we study nonnegative solutions of the equation

$$\operatorname{div}(A(x)Du) = f(u). \tag{1.1}$$

The coefficient matrix $A(x)$ is assumed to be symmetric, measurable and to satisfy the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$$

for every $\xi \in \mathbb{R}^n$, where $0 < \lambda \leq \Lambda$. The function $f : \mathbb{R} \to [0, \infty)$ is assumed to be nonnegative and nondecreasing. Note that we allow f to have jump discontinuity. Throughout the paper we denote the integral function of f by F, i.e.,

$$F(t) = \int_0^t f(s)\, ds.$$

The goal of this paper is to prove a general Harnack inequality for (1.1). To be more precise we seek to answer the following simple question in a quantitative way. If u is a nonnegative solution of (1.1) in B_2, then does the value $\inf_{B_1} u$ control $\sup_{B_1} u$? I would like to stress that this paper does not concern regularity of solutions of (1.1). In fact, the regularity for (1.1) is well understood. Indeed, since f is nonnegative, any solution of (1.1) is a weak subsolution of the corresponding linear equation. Therefore by De Giorgi theorem [8] nonnegative solutions

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are locally bounded and by the De Giorgi-Nash-Moser theorem they are Hölder continuous. However, the point is that both the L^∞-bound and the Hölder norm depend on the L^2-norm of the solution and it is not clear how one can bound the L^2-norm by knowing the value of the solution only at one point.

The main result reads as follows.

Theorem 1.1. *Let $u \in W^{1,2}(B_2)$ be a nonnegative solution of (1.1). Denote $M = \sup_{B_1} u$ and $m = \inf_{B_1} u$. There is a constant C which depends only on the ellipticity constants λ, Λ and the dimension n such that*

$$\int_m^M \frac{dt}{\sqrt{F(t)} + t} \leq C. \tag{1.2}$$

In particular, C is independent of the solution u and of the function f.

Theorem 1.1 is in the same spirit as [14] for nondivergence form equations with nonhomogeneous gradient drift term. The above result is stronger and more complete than the main result in [14], because we do not need any regularity nor growth assumptions on f, and most importantly, the constant in Theorem 1.1 is independent of f. In particular, the estimate is stable under scaling.

Harnack inequality for linear elliptic equations by Moser [21] is one of the most important results in the theory of elliptic partial differential equations. There are numerous generalizations of this theorem from which the most relevant for us is by Di Benedetto-Trudinger [10] who proved the Harnack inequality for quasiminimizers of integral functionals. Harnack inequality for general quasilinear equations has been considered e.g. by Serrin [28] (see also [18] for the Hölder continuity). This result has the disadvantage that the constant will depend on the solution itself. In Theorem 1.1 the inequality (1.2) is not in the classical form but its structure depends on the scaling of the equation. This has the advantage that the constant is then independent of the solution. Theorem 1.1 is more similar to Pucci-Serrin [25] who introduced a Harnack inequality in $\mathbb{R}^2$ for quasilinear equations similar to (1.1) where the operator is allowed to be nonlinear but not to have dependence on x. Compared to [25] the advantage of Theorem 1.1 is that the dependence on every parameter in (1.2) is explicit and we do not need any further assumptions on f other than the monotonicity. This makes the result more general and the esimate (1.2) more stable. Moreover, the result holds in any dimension and the operator is allowed to have merely bounded coefficients. In particular, it is not possible to prove Theorem 1.1 by a comparison argument.

The drawback of (1.2) is that it is in implicit form and it may be difficult to write excplicitly the relation between the maximum and the minimum. On the other hand (1.2) is a natural generalization of the classical Harnack inequality for equations of type (1.1). We illustrate this by giving a complete answer to the following problems. In the following $u \in W^{1,2}(B_2)$ is a nonnegative solution of (1.1) with $M = \sup_{B_1} u$ and $m = \inf_{B_1} u$.

(i) **Strong minimum principle**. If u is zero at one point in B_1, is it zero everywhere?
(ii) **Boundedness**. Is u bounded in B_1 by a constant which depends only on $u(0)$?
(iii) **Local Boundedness**. Is there a radius $r > 0$ such that u is bounded in B_r by a constant which depends only on $u(0)$?

The problem (i) has been considered by Vazquez [30] and by Pucci-Serrin and their collaborators [23, 24, 26, 27] and we know that the strong minimum principle holds if f is positive

and

$$\int_0^1 \frac{dt}{\sqrt{F(t)}} = \infty. \tag{1.3}$$

In fact, this condition is also necessary [3, 9]. Theorem 1.1 is in accordance with this and it provides a quantification of the strong minimum principle. By this we mean that if

$$\int_0^1 \frac{dt}{\sqrt{F(t)}+t} = \infty,$$

then there is a continuous, increasing function $\Phi : [0,1] \to [0,1]$ such that for $M \leq 1$ it holds $m \geq \Phi(M)$, (see also [25]). In order to quantify the strong minimum principle we need to replace the function $\sqrt{F(t)}$ in (1.3) by $\sqrt{F(t)}+t$. This is natural since in the linear case $f \equiv 0$ the estimate (1.2) then reduces to the classical Harnack inequality.

Similarly we find an answer to (ii). If

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}+t} = \infty,$$

then there is a continuous, increasing function $\Psi : [1,\infty) \to [1,\infty)$ such that for $m \geq 1$ it holds $M \leq \Psi(m)$, (see also [25]). This condition is very similar to the Keller-Osserman condition [15, 22] which states that if f is positive and u is a nonnegative and nontrivial solution of (1.1) in the whole $\mathbb{R}^n$ then necessarily

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty. \tag{1.4}$$

For the proof of this see [17] or Theorem 1.2 below. The above condition is sharp for the uniform boundedness of the solutions. Indeed if f does not satisfy (1.4) then there exists a sequence of nonnegative solutions (u_k) of (1.1) such that $u_k(0) \leq 1$ and $\sup_{B_1} u_k \to \infty$ as $k \to \infty$. We leave this to the reader.

Finally to answer (iii) we find that if u and f are as in Theorem 1.1, then we may always find a radius $r > 0$ such that $\sup_{B_r} u$ is uniformly bounded by a constant which depends on the value $u(0)$. Indeed, since the constant in Theorem 1.1 does not depend on u and f, we obtain by a simple scaling argument that for $M_r = \sup_{B_r} u$ it holds

$$\int_{u(0)}^{M_r} \frac{dt}{r\sqrt{F(t)}+t} \leq C.$$

Note that when $r \to 0$ the above estimate converges to the classical Harnack inequality. Therefore when $r > 0$ is small enough we have that $M_r < \infty$. This result can be seen as a weak counterpart of the short time existence result for the one dimensional initial value problem

$$y'' = f(y), \qquad y(0) = y_0 \quad \text{and} \quad y'(0) = y_0'.$$

Note that if f satisfies (1.4) the above initial value problem has a solution in the whole $\mathbb{R}$.

The statement of Theorem 1.1 is sharp which can be seen already in dimension one (see [22, Remark 1]). The assumption $f \geq 0$ is also necessary, i.e., Theorem 1.1 is not true for equations

$$-\Delta u = f(u)$$

where f is nonnegative and monotone. This can be seen by a simple example which we give at the end of the paper. One reason for this is that the above equation is the Euler equation of the nonconvex functional

$$\int_{B_2} \frac{1}{2}|Du|^2\,dx - F(u)\,dx$$

and criticality alone is not enough to prove the optimal L^∞- bound for minimizers (see [4] and the references therein). On the other hand Theorem 1.1 could still hold without the monotonicity assumption on f.

The proof of Theorem 1.1 is rather long and has several stages, and therefore we give its outline here. In this paper we develop further the ideas from [14]. The main difficulty is to overcome the lack of regularity and growth condition on f, and to avoid the constant C to depend on f. In order to do this we will revisit the proof of the classical Harnack inequality by Di Benedetto-Trudinger [10] in order to have a more suitable and sharper version which allows us to treat the nonhomogeneous case.

To overcome the lack of regularity and to have the constant independent of f, we use the fact that the equation is in divergence form and integrate it locally over the level sets of the solution (as in [20, 29]). This will give us precise information how fast the level sets locally decay. Similar method has been used to study global regularity for solutions of elliptic equations e.g. in [7]. Here we use it to prove local estimates.

The first observation is that we have a good local estimate on the decay rate of the level set $\{u \geq t\}$ when we are in a ball $B(x,r)$ whose radius is small $r \simeq t/\sqrt{F(t)}$ and the density of the level set in the ball

$$\sigma_t(x) = \frac{|\{u \geq t\} \cap B(x,r)|}{|B(x,r)|}$$

is not close to one or zero (Lemma 3.3). The second observation is a measure theoretical lemma (Lemma 3.4) which states that the measure of the set where the density σ_t is between $1/5$ and $4/5$ (which can be thought to be the "boundary" of the level set $\{u \geq t\}$) is related to the measure of the set where the density is larger than $4/5$ (which corresponds to the "interior" of the level set $\{u \geq t\}$). This is of course very much related to the isoperimetric inequality. We combine these two lemmas and obtain the following sharp estimate (see Proposition (3.5)) for the decay rate of the level set $\mu(t) = |\{u \geq t\} \cap B_2|$,

$$-\mu'(t) \geq c \min\Big\{ \frac{1}{\sqrt{F(t)}} \mu(t)^{\frac{n-1}{n}}, \frac{1}{t}\mu(t) \Big\}, \tag{1.5}$$

for almost every $t > 0$ for which $\mu(t) \leq |B_2|/2$. When $\mu(t) \geq |B_2|/2$ we have a similar estimate for $\eta(t) = |\{u < t\} \cap B_2|$. Note that (1.5) has two parts on the right hand side. The first one is the nonhomogeneous estimate and the second is the homogeneous one. In the sublinear case $f(t) \leq t$ we may integrate (1.5) and conclude that the solution is L^ε-intergable. In the nonhomogeneous case the inequality (1.5) may oscillate between the two estimates.

Another issue in the proof is to overcome the fact that we do not have any growth condition on f. In [14] it was assumed that the nonhomogeneity is of type $f(t) = g(t)t$, where g is a slowly increasing function. The proof was based on the idea that under this assumption any unbounded supersolution blows up as the fundamental solution of the linear equation. This is certainly not true in our case. To solve this problem we study more closely subsolutions of (1.1) and prove an estimate (Lemma 4.3) which roughly speaking quantifies the fact that if f is very large then solutions of (1.1) will grow fast compared to the solutions of the corresponding

linear equation. We iterate Lemma 4.3 and obtain the following lower bound for subsolutions. Similar result is obtained also in [17].

Theorem 1.2. *Let $u \in W^{1,2}(B(x_0, 2R))$ be a continuous and nonnegative subsolution of*

$$\operatorname{div}(A(x)Du) \geq f(u)$$

and denote $M = \sup_{B(x_0,R)} u$ and $m = \inf_{B(x_0,R)} u$. If $u(x_0) > 0$ then it holds

$$\int_{m/4}^{M} \frac{dt}{\sqrt{F(t)}} \geq c\, R$$

for a constant $c > 0$ which is independent of u and f.

In fact, we will not need this result in the proof of Theorem 1.1 but only Lemma 4.3. However since Theorem 1.2 follows rather easily from Lemma 4.3 we choose to state it. At the end of Section 4 in Corollary 4.4 we show that Theorem 1.2 implies the Keller-Osserman condition for entire solutions of (1.1). This result has been proved in [17]. This result is also known for wide class of nondivergence form operators [5, 6, 12].

Let us now briefly give a rough version of the proof of Theorem 1.1. We integrate (1.5) (and its counterpart for η) and conclude that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\mu(t_\varepsilon) = |\{u \geq t_\varepsilon\} \cap B_2| \leq \varepsilon$ and

$$\int_{m}^{t_\varepsilon} \frac{dt}{\sqrt{F(t)} + t} \leq C_\varepsilon.$$

For simplicity assume that for every $t > t_\varepsilon$ we have the nonhomogeneous (the first) estimate in (1.5). Then integrating (1.5) gives

$$c \int_{t_\varepsilon}^{\infty} \frac{dt}{\sqrt{F(t)}} \leq -\int_{t_\varepsilon}^{\infty} \mu^{\frac{1}{n}-1}(t)\mu'(t)\, dt = \frac{1}{n}\mu^{\frac{1}{n}}(t_\varepsilon) \leq \frac{1}{n}\varepsilon^{\frac{1}{n}}.$$

Finally we conclude that it has to hold $M = \max_{B_1} u \leq 4t_\varepsilon$. Indeed, otherwise Theorem 1.2 would imply for $M_{3/2} = \max_{B_{3/2}} u$ that

$$\int_{t_\varepsilon}^{M_{3/2}} \frac{dt}{\sqrt{F(t)}} \geq \frac{c}{2}$$

which is a contradiction when ε is small.

The paper is organized as follows. In the next section we recall basic results from measure theory. In Section 3 we prove estimates for nonnegative supersolutions of (1.1). The main result of that section is Proposition 3.5. In Section 4 we prove estimates for subsolutions of (1.1) and prove Theorem 1.2 and show in Corollary 4.4 how it implies the Keller-Osserman condition for entire solutions. Finally in Section 5 we give the proof of Theorem 1.1.

2. Preliminaries

Throughout the paper we denote by $B(x, r)$ the open ball centred at x with radius r. In the case $x = 0$ we simply write $B(0, r) = B_r$.

Let $U \subset \mathbb{R}^n$ be an open set. A set $E \subset \mathbb{R}^n$ has finite perimeter in U if

$$P(E, U) := \sup\left\{ \int_E \operatorname{div} \varphi\, dx \,:\, \varphi \in C_0^1(U, \mathbb{R}^n),\ ||\varphi||_\infty \leq 1 \right\} < \infty.$$

Here $P(E,U)$ is the perimeter of E in U. Sometimes we call E a set of finite perimeter if it is clear from the context which is the reference domain U. Let E be a set of finite perimeter in U. The reduced boundary of E is denoted by $\partial^* E$. It is smaller than the topological boundary which we denote by ∂E. For any open set $V \subset U$ it holds $P(E,V) = \int_{\partial^* E \cap V} d\mathcal{H}^{n-1}$, where $\mathcal{H}^{n-1}$ is the standard $(n-1)$-dimensional Hausdorff measure. Moreover the Gauss-Green formula holds

$$\int_E \operatorname{div} X \, dx = \int_{\partial^* E} \langle X, \nu \rangle \, d\mathcal{H}^{n-1}$$

for every $X \in W_0^{1,\infty}(U, \mathbb{R}^n)$. Here ν is the outer unit normal of E which exists on $\partial^* E$. For an introduction to the theory of sets of finite perimeter we refer to [2] and [19]. All the following results can be found in these books.

The most important result from geometric measure theory for us is the relative isoperimetric inequality. It states that for every set of finite perimeter E in the ball B_R the following inequality holds

$$P(E, B_R) \geq c \min \left\{ |E \cap B_R|^{\frac{n-1}{n}}, |B_R \setminus E|^{\frac{n-1}{n}} \right\}$$

for a constant c which depends on the dimension. The proof of the main result is mostly based on this inequality.

We recall the coarea formula for Lipschitz functions. Let g be Lipschitz continuous in an open set U and let $h \in L^1(U)$ be nonnegative. Then it holds

$$\int_U h(x) |Dg(x)| \, dx = \int_{-\infty}^{\infty} \left(\int_{\{g=t\} \cap U} h(x) d\mathcal{H}^{n-1}(x) \right) dt.$$

The formula still holds if g is locally Lipschitz continuous and $h|Dg| \in L^1(U)$. From the coarea formula one deduces immediately that almost every level set $\{g > t\}$ of a Lipschitz function is a set of finite perimeter. In the case $g \in C^\infty(U)$ the level sets are even more regular, since by the Morse-Sard Lemma the image of the critical set $K = \{x \in U : |Dg(x)| = 0\}$ has measure zero $|g(K)| = 0$. In particular, almost every level set of a smooth function has smooth boundary.

Throughout the paper we denote the sublevel sets of a measurable function $u : B_2 \to \mathbb{R}$ by

$$E_t := \{x \in B_2 : u(x) < t\}$$

and the superlevel sets by

$$A_t := \{x \in B_2 : u(x) \geq t\}.$$

Moreover we denote $\mu(t) = |A_t|$ and $\eta(t) = |E_t|$. In Section 3 we estimate the differential of $\mu(t)$ (and $\eta(t)$) when u is nonnegative supersolution of the equation (1.1). The differential of $\mu(\cdot)$ is a measure and we denote its absolutely continuous part by μ'. To avoid pathological situations (see [1]) we regularize the equation in order to work with smooth functions. Then by the result from [1] we may write μ' as

$$\mu'(t) = -\int_{\{u=t\} \cap B_2} \frac{1}{|Du|} \, d\mathcal{H}^{n-1}$$

for almost every t.

Let us turn our attention to elliptic equations in divergence form. As an introduction to the topic we refer to [13].

Definition 2.1. A function $u \in W^{1,2}(U)$ is a supersolution of (1.1) in an open set U if

$$\int_U \langle A(x)Du, D\varphi\rangle \, dx \geq -\int_U f(u)\varphi \, dx \qquad \text{for all nonnegative } \varphi \in W_0^{1,2}(U).$$

A function $u \in W^{1,2}(U)$ is a subsolution of (1.1) in Ω if

$$\int_U \langle A(x)Du, D\varphi\rangle \, dx \leq -\int_U f(u)\varphi \, dx \qquad \text{for all nonnegative } \varphi \in W_0^{1,2}(U).$$

Finally $u \in W^{1,2}(U)$ is a solution of (1.1) in U if it is both super- and subsolution.

As we already mentioned we will regularize the equation (1.1) and work with solutions of

$$\operatorname{div}(A_\varepsilon(x)Dv) = f_\varepsilon(v) \tag{2.1}$$

where A_ε is smooth, symmetric and elliptic, and f_ε is positive, increasing and smooth. Moreover we choose A_ε and f_ε such that $A_\varepsilon \to A$ in L^1 and $f_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R})$ such that $f_{\varepsilon_2} \geq f_{\varepsilon_1}$ for $\varepsilon_2 > \varepsilon_1$. Let $u \in W^{1,2}(B_2)$ be a nonnegative solution of (1.1) in B_2. For every $\varepsilon > 0$ we define u_ε to be the solution of (2.1) having the same boundary values as u, i.e., $u_\varepsilon - u \in W_0^{1,2}(B_2)$. Since (u_ε) are locally uniformly Hölder continuous we have that $u_\varepsilon \to u$ uniformly in B_1 and (denote $F_\varepsilon(t) = \int_0^t f_\varepsilon(s)\, ds$, $M_\varepsilon = \sup_{B_1} u_\varepsilon$ and $m_\varepsilon = \inf_{B_1} u_\varepsilon$)

$$\lim_{\varepsilon\to 0} \int_{m_\varepsilon}^{M_\varepsilon} \frac{dt}{\sqrt{F_\varepsilon(t)} + t} = \int_m^M \frac{dt}{\sqrt{F(t)} + t}$$

by monotone convergence. Hence, we may assume that the solution u in Theorem 1.1 is smooth, i.e., $u \in C^\infty(B_2)$.

Finally for De Giorgi iteration we recall the following lemma which can be found e.g. in [13, Lemma 7.1].

Lemma 2.2. *Let (x_i) be a sequence of positive numbers such that*

$$x_{i+1} \leq C_0 B^i x_i^{1+\frac{1}{n}}$$

with $C_0 > 0$ and $B > 1$. If $x_0 \leq C_0^{-n} B^{-n^2}$ then

$$\lim_{i\to\infty} x_i = 0.$$

3. Estimates for supersolutions

In this section we study nonnegative supersolutions of (1.1). We begin by proving a standard Caccioppoli inequality and its variant, which involves a boundary term. As we mentioned in the previous section we prefer to work with smooth functions. Recall that $E_t = \{u < t\} \cap B_2$.

Proposition 3.1. *Assume that $u \in C^\infty(B_R)$ is a nonnegative supersolution of*

$$\operatorname{div}(A(x)Du) \leq f(u). \tag{3.1}$$

Then for $r < R$ it holds

$$\int_{E_t\cap B_r} |Du|^2 \, dx \leq C\left(\frac{t^2}{(R-r)^2} + F(t)\right)|E_t \cap B_R|$$

for a constant depending on λ and Λ. Moreover for almost every $t > 0$ we have

$$\int_{\{u=t\}\cap B_r} |Du| \, d\mathcal{H}^{n-1} \leq C\left(\frac{t}{(R-r)^2} + \frac{F(t)}{t}\right)|E_t \cap B_R|.$$

Proof. Let us fix $t > 0$ and choose a testfunction $\varphi(x) = (t - u(x))_+ \zeta^2(x)$, where $\zeta \in C_0^1(B_R)$ is a standard cut-off function such that $\zeta(x) = 1$ in B_r and $|D\zeta| \leq \frac{2}{R-r}$. We get after using the ellipticity and Young's inequality that

$$\int_{E_t \cap B_r} |Du|^2 \, dx \leq \frac{C}{(R-r)^2} \int_{B_R} (t-u)_+^2 \, dx + \int_{B_R} f(u)(t-u)_+ \, dx.$$

The standard Caccioppoli inequality follows from

$$\begin{aligned} \int_{B_R} f(u)(t-u)_+ \, dx &= \int_{B_R} f(u) \int_0^t \chi_{(t-u)>s} \, ds dx = \int_{B_R} f(u) \int_0^t \chi_{(u<l)} \, dl \, dx \\ &= \int_0^t \int_{B_R} f(u) \chi_{(u<s)} \, dx \, ds \leq \int_0^t \int_{B_R} f(s) \chi_{(u<s)} \, dx \, ds \\ &\leq \left(\int_0^t f(s) \, ds \right) |\{u < t\} \cap B_R| \\ &= F(t) |E_t \cap B_R|. \end{aligned} \tag{3.2}$$

Here χ_E denotes the characteristic function of a set E.

By Morse-Sard Lemma it holds $|Du| > 0$ on $\{u = t\}$ for almost every $t > 0$. Let us prove that the second statement holds for every such t. We integrate the equation (3.1) over the set $E_t \cap B_\rho$ and get

$$\int_{\{u=t\} \cap B_\rho} \langle A(x) Du, \frac{Du}{|Du|} \rangle \, d\mathcal{H}^{n-1} \leq - \int_{\partial B_\rho \cap E_t} \langle A(x) Du, \frac{x}{|x|} \rangle \, d\mathcal{H}^{n-1} + \int_{E_t \cap B_\rho} f(u) \, dx.$$

For a rigorous argument see [29] and note that (see [11])

$$\frac{d}{dt} \int_{\{u<t\} \cap B_\rho} \langle A(x) Du, Du \rangle \, dx = \int_{\{u=t\} \cap B_\rho} \langle A(x) Du, \frac{Du}{|Du|} \rangle \, d\mathcal{H}^{n-1}.$$

By the ellipticity we have

$$\int_{\{u=t\} \cap B_\rho} |Du| \, d\mathcal{H}^{n-1} \leq C \int_{\partial B_\rho \cap E_t} |Du| \, d\mathcal{H}^{n-1} + C \int_{E_t \cap B_\rho} f(u) \, dx.$$

Choose $\hat{\rho} = \frac{r+R}{2}$ and integrate the previous inequality with respect to ρ over $(r, \hat{\rho})$ and get

$$(R-r) \int_{\{u=t\} \cap B_r} |Du| \, d\mathcal{H}^{n-1} \leq C \int_{B_{\hat{\rho}} \cap E_t} |Du| \, dx + C(R-r) \int_{E_t \cap B_R} f(u) \, dx. \tag{3.3}$$

By the previous Caccioppoli inequality we have

$$\begin{aligned} \int_{E_t \cap B_{\hat{\rho}}} |Du| \, dx &\leq |E_t \cap B_{\hat{\rho}}|^{\frac{1}{2}} \left(\int_{E_t \cap B_{\hat{\rho}}} |Du|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C \left(\frac{t}{(R-r)} + \sqrt{F(t)} \right) |E_t \cap B_R|. \end{aligned}$$

Arguing as in (3.2) we get

$$\int_{E_t \cap B_R} f(u) \, dx \leq \frac{1}{t} \int_{E_t \cap B_R} f(u)(t-u)_+ \, dx \leq \frac{F(t)}{t} |E_t \cap B_R|.$$

Therefore from (3.3) we get by Young's inequality

$$\int_{\{u=t\}\cap B_r} |Du|\, d\mathcal{H}^{n-1} \le C\left(\frac{t}{(R-r)^2} + \frac{\sqrt{F(t)}}{R-r} + \frac{F(t)}{t}\right)|E_t \cap B_R| \le C\left(\frac{t}{(R-r)^2} + \frac{F(t)}{t}\right)|E_t \cap B_R|.$$

□

Next we observe that if the measure of the sublevel set $|E_t \cap B_2|$ is very small compared to the ratio $t/\sqrt{F(t)}$, then we may apply the estimates from the linear theory. The next result follows from the previous Caccioppoli inequality together with the standard De Giorgi iteration. The argument is almost exactly the same as [13, Lemma 7.4] and therefore we write only the outline of the proof.

Lemma 3.2. *Assume that $u \in C^\infty(B_2)$ is a nonnegative supersolution of*

$$\operatorname{div}(A(x)Du) \le f(u).$$

There is $\delta_0 > 0$ such that if for some $t > 0$ it holds

$$|E_t \cap B_2| \le \delta_0 \left(\frac{t}{\sqrt{F(t)} + t}\right)^n$$

then one has

$$\inf_{B_1} u \ge \frac{t}{2}.$$

Proof. Let $t > 0$ satisfy the assumption of the lemma. For $0 < h < k \le t$ we define

$$v = \begin{cases} 0, & \text{if } u \ge k \\ k - u, & \text{if } h < u < k \\ k - h, & \text{if } u \le h. \end{cases}$$

By possibly decreasing the value of δ_0 we may assume that $|E_t \cap B_2| \le \frac{1}{2}|B_2|$. Using the Caccioppoli inequality (Proposition 3.1) and arguing as in [13, Lemma 7.4] we conclude that for every $1 \le \rho < R \le 2$ and $0 < h < k \le t$ it holds

$$(k-h)|E_h \cap B_\rho|^{\frac{n-1}{n}} \le C\left(\frac{k}{R-\rho} + \sqrt{F(k)}\right)|E_k \cap B_R|.$$

Define $r_i = (1 + 2^{-i})$ and $k_i = \frac{t}{2}(1 + 2^{-i})$ and apply the above inequality for $R = r_i$, $\rho = r_{i+1}$, $h = k_{i+1}$ and $k = k_i$. We set $x_i = |E_{k_i} \cap B_{r_i}|$ and obtain

$$x_{i+1} \le C_0\left(1 + \frac{\sqrt{F(t)}}{t}\right) 4^i x_i^{1+\frac{1}{n}}.$$

We choose $\delta_0 > 0$ such that

$$\delta_0 \le C_0^{-n} 4^{-n^2}.$$

By the assumption on t we have

$$x_0 = |E_t \cap B_2| \le \delta_0 \left(\frac{t}{\sqrt{F(t)} + t}\right)^n$$

and we conclude from Lemma 2.2 that $\lim_{i\to\infty} x_i = 0$. In other words $\inf_{B_1} u \ge t/2$. □

We turn our attention to Proposition 3.5 which is the main result of this section. To that aim we will need two lemmas. In the first lemma we use the Caccioppoli inequality to study the local decay rate of the level sets E_t. Due to the nonhomogeneity of the equation we do this only in small balls $B(x,r)$ with radius $r \leq t/\sqrt{F(t)}$. Due to the relative isoperimetric inequality the estimate depends whether the density $|E_t \cap B(x,r)|/|B_r|$ is close to one or close to zero.

Lemma 3.3. *Assume that $u \in C^\infty(B_2)$ is a nonnegative supersolution of*

$$\operatorname{div}(A(x)Du) \leq f(u).$$

Then for almost every $t > 0$ it holds

$$\int_{\{u=t\}\cap B(x_0,2r)} \frac{1}{|Du|}\,\mathcal{H}^{n-1} \geq \frac{c}{t}\,\min\Big\{\big|E_t \cap B(x_0,r)\big|, \big|B(x_0,r)\setminus E_t\big|\Big\}$$

whenever $B(x_0,2r) \subset B_2$ is such that

$$r \leq \frac{t}{\sqrt{F(t)}}.$$

Proof. Without loss of generality we may assume that $x_0 = 0$. We may also assume that $|E_t \cap B(x_0,r)| \leq |B(x_0,r)\setminus E_t|$ since in the other case the proof is similar. Moreover by Morse-Sard Lemma we may assume that $|Du| > 0$ on $\{u=t\}\cap B_2$.

Let us first assume that

$$|E_t \cap B_r| < 2^{-n-2}|B_r|. \tag{3.4}$$

Let us fix $x_i \in E_t \cap B_r$. For x_i we define a radius R_i such that

$$R_i := \inf\big\{\rho > 0 : |B(x_i,\rho)\cap E_t| \leq (1-2^{-n-1})|B(x_i,\rho)|\big\}.$$

Since $x_i \in E_t$ and E_t is an open set we have that $R_i > 0$. Since the point x_i is in B_r, a simple argument gives

$$|B(x_i,r)\setminus B_r| \leq |B_r| - |B_{r/2}| = (1-2^{-n})|B_r|.$$

Therefore we deduce from the assumption (3.4) that

$$\begin{aligned}|B(x_i,r)\cap E_t| &\leq |B(x_i,r)\setminus B_r| + |E_t\cap B_r|\\ &< (1-2^{-n})|B(x_i,r)| + 2^{-n-2}|B(x_i,r)|\\ &\leq (1-2^{-n-1})|B(x_i,r)|.\end{aligned}$$

Thus we conclude that $R_i \leq r$. Hence we obtain a family of balls $\bar B(x_i,R_i)$ which cover $E_t\cap B_r$ and satisfy $0 < R_i \leq r$. By the Besikovitch covering theorem [19, Corollary 5.2] we may choose a countable disjoint subfamily, say $\mathcal{F}$, such that $\xi(n)\sum_{i\in\mathcal{F}}|B(x_i,R_i)| \geq |E_t\cap B_r|$. Moreover by the definition of R_i it holds

$$|E_t\cap B(x_i,R_i)| = (1-2^{-n-1})|B(x_i,R_i)| = cR_i^n.$$

In particular, the density $|E_t\cap B(x_i,R_i)|/|B_{R_i}|$ is bounded away from zero and one.

Let us fix a ball $B(x_i,R_i)$ in the Besikovitch cover. We use Proposition 3.1 in $B(x_i,R_i)$ and obtain

$$\begin{aligned}\int_{\{u=t\}\cap B(x_i,R_i)} |Du|\,d\mathcal{H}^{n-1} &\leq C\left(\frac{t}{r^2}+\frac{F(t)}{t}\right)|E_t\cap B(x_i,2R_i)|\\ &\leq Ct\,R_i^{n-2},\end{aligned} \tag{3.5}$$

where the last inequality follows from $R_i \le r \le t/\sqrt{F(t)}$. On the other hand the relative isoperimetric inequality yields

$$\begin{aligned}\int_{\{u=t\}\cap B(x_i,R_i)} |Du|\, d\mathcal{H}^{n-1} &\ge c\left(\int_{\{u=t\}\cap B(x_i,R_i)} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}\right)^{-1} P(E_t, B(x_i,R_i))^2 \\ &\ge c\left(\int_{\{u=t\}\cap B(x_i,R_i)} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}\right)^{-1} |E_t\cap B(x_i,R_i)|^{\frac{2(n-1)}{n}} \\ &\ge c\left(\int_{\{u=t\}\cap B(x_i,R_i)} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}\right)^{-1} R_i^{2n-2}.\end{aligned} \tag{3.6}$$

Therefore (3.5) and (3.6) give

$$\int_{\{u=t\}\cap B(x_i,R_i)} \frac{1}{|Du|}\, d\mathcal{H}^{n-1} \ge \frac{c}{t} R_i^n \ge \frac{c}{t}\,|B(x_i,R_i)| \tag{3.7}$$

for every ball $B(x_i,R_i)\subset B_{2r}$ in the cover. Summing (3.7) gives

$$\begin{aligned}\int_{\{u=t\}\cap B_{2r}} \frac{1}{|Du|}\, d\mathcal{H}^{n-1} &\ge \sum_i \int_{\{u=t\}\cap B(x_i,R_i)} \frac{1}{|Du|}\, d\mathcal{H}^{n-1} \\ &\ge \frac{c}{t}\sum_i |B(x_i,R_i)| \\ &\ge \frac{c}{\xi(n)t}\,|E_t\cap B_r|\end{aligned}$$

and the claim follows.

Let us assume next

$$|E_t\cap B_r| \ge 2^{-n-2}|B_r|. \tag{3.8}$$

In this case we do not need any covering argument. Since we assumed that $|E_t\cap B_r| \le |B_r\setminus E_t|$ we have

$$|E_t\cap B_r| \le \frac{|B_r|}{2}.$$

We use Proposition 3.1 in B_r and argue as above to conclude that

$$\int_{\{u=t\}\cap B_r} |Du|\, d\mathcal{H}^{n-1} \le C\left(\frac{t}{r^2}+\frac{F(t)}{t}\right)|E_t\cap B_{2r}| \le Ct\, r^{n-2},$$

where we used the fact that $r\le t/\sqrt{F(t)}$. We use the relative isoperimetric inequality and (3.8), argue as in (3.6) and get

$$\int_{\{u=t\}\cap B_r} |Du|\, d\mathcal{H}^{n-1} \ge c\left(\int_{\{u=t\}\cap B_r} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}\right)^{-1} r^{2n-2}.$$

Hence the claim follows. □

Next we prove a measure theoretical lemma which is related to the relative isoperimetric inequality. To this aim we denote for every $\delta\in(0,1]$ the truncated distance function to the boundary ∂B_2 by

$$d_\delta(x) := c\min\{(2-|x|),\delta\}, \tag{3.9}$$

where $c \in (0,1]$ is a number which we choose later. For every measurerable set $A \subset B_2$ we define a density function $\sigma_A : B_2 \to \mathbb{R}$ by

$$\sigma_A(x) := \frac{|A \cap B(x, d_\delta(x))|}{|B_{d_\delta(x)}|}. \tag{3.10}$$

Although it is not apparent from the notation, the function σ_A depends on δ. The point of the following lemma is to study the size of the set where the density function of a given set A takes values between $1/5$ and $4/5$, i.e., away from zero and one. Heuristically one may think that the set $\{\sigma_A = 1/2\}$ corresponds to the boundary of A and that $\{1/5 < \sigma_A < 4/5\}$ forms a layer around it. The thickness of this layer depends on the Lipschitz constant of σ_A which can be estimated by a simple geometrical argument as follows.

Fix $x \in B_2$ and a unit vector e. When $h > 0$ is small we may estimate

$$|\sigma_A(x + he) - \sigma_A(x)| \leq \frac{|B(x, d_\delta(x) + h) \setminus B(x, d_\delta(x))|}{|B(x, d_\delta(x)/2)|} \leq \frac{Ch}{d_\delta(x)}$$

where C depends on the dimension. Therefore σ_A is locally Lipschitz continuous and its gradient can be estimated by

$$|D\sigma_A(x)| \leq \frac{C}{d_\delta(x)} \tag{3.11}$$

for almost every $x \in B_2$.

Lemma 3.4. *Suppose $A \subset B_2$ is a measurable set such that $|A| \leq \frac{1}{2}|B_2|$ and let $\delta \in (0,1]$. Let $d_\delta(\cdot)$ be the truncated distance (3.9) and let σ_A be the density function (3.10). There is a constant c_n which depends on the dimension such that*

$$\big|\{1/5 < \sigma_A \leq 4/5\}\big| \geq c_n \min\big\{\delta \big|\{\sigma_A > 4/5\}\big|^{\frac{n-1}{n}}, \big|\{\sigma_A > 4/5\}\big|\big\}.$$

It is not difficult to see that if A is a smooth set then by letting $\delta \to 0$ the previous lemma reduces to the relative isoperimetric inequality.

Proof. Let us fix $\delta \in (0,1]$. Throughout the proof we denote for every $s \in [0,1]$ the superlevel sets of σ_A by

$$A^s := \{x \in B_2 \,:\, \sigma_A > s\}.$$

We may write σ_A as a convolution

$$\sigma_A(x) = \frac{1}{|B_{d_\delta(x)}|} \int_{B_2} \chi_A(y) \chi_{B_{d_\delta(x)}}(x - y)\, dy.$$

Therefore by Fubini's theorem we have

$$\int_{B_2} \sigma_A(x)\, dx = \int_{B_2} \chi_A(y) \left(\int_{B_2} \frac{1}{|B_{d_\delta(x)}|} \chi_{B_{d_\delta(x)}}(x - y)\, dx \right) dy = |A|.$$

Hence it holds

$$|A| = \int_{B_2} \sigma_A(x)\, dx \geq \int_{A^{3/5}} \sigma_A(x)\, dx \geq \frac{3}{5}|A^{3/5}|.$$

Since $|A| \leq \frac{1}{2}|B_2|$ we have

$$|A^s| \leq \frac{5}{6}|B_2| \tag{3.12}$$

for every $3/5 \leq s \leq 4/5$.

Let us define

$$\delta_n := \frac{\delta}{2n-1} \qquad \text{and} \qquad R_n := 2 - \frac{\delta_n}{2}. \tag{3.13}$$

We divide the proof in two cases. Let us first assume that there exists $\hat{s} \in (7/10, 4/5)$ such that

$$|A^{\hat{s}} \cap B_{R_n}| \geq \frac{1}{8}|A^{\hat{s}}|. \tag{3.14}$$

This means that part of the level set $A^{\hat{s}} = \{\sigma_A > \hat{s}\}$ is away from the boundary ∂B_2. We observe that for every $x \in B_{R_n}$ it holds $d_\delta(x) \geq c\delta$ for a dimensional constant $c > 0$. By possibly decreasing δ we deduce from (3.12) that $|A^s \cap B_{R_n}| \leq \frac{6}{7}|B_{R_n}|$ for every $3/5 \leq s \leq 4/5$. Then it follows from (3.11), from the coarea formula and from the relative isoperimetric inequality that

$$\begin{aligned}
|\{1/5 < \sigma_A \leq 4/5\}| &\geq c \int_{(A^{1/5} \setminus A^{4/5}) \cap B_{R_n}} d_\delta(x) |D\sigma_A(x)|\, dx \\
&\geq c\delta \int_{3/5}^{\hat{s}} \int_{\{\sigma_A = s\} \cap B_{R_n}} d\mathcal{H}^{n-1}(x)\, ds \\
&= c\delta \int_{3/5}^{\hat{s}} P(A^s, B_{R_n})\, ds \\
&\geq c\delta \int_{3/5}^{\hat{s}} |A^s \cap B_{R_n}|^{\frac{n-1}{n}}\, ds \\
&\geq c\delta |A^{\hat{s}} \cap B_{R_n}|^{\frac{n-1}{n}}.
\end{aligned}$$

Hence the claim follows from (3.14) since

$$|A^{\hat{s}} \cap B_{R_n}| \geq \frac{1}{8}|A^{\hat{s}}| \geq \frac{1}{8}|A^{4/5}| = \frac{1}{8}|\{\sigma_A > 4/5\}|.$$

Let us next assume that for every $s \in (7/10, 4/5)$ it holds

$$|A^s \cap B_{R_n}| \leq \frac{1}{8}|A^s|. \tag{3.15}$$

This means that large part of the level set A^s is close to the boundary ∂B_2.

We denote the reduced boundary of A^s by $\partial^* A^s$. Let us first show that for almost every $s \in (7/10, 4/5)$ it holds

$$\int_{\partial^* A^s \cap B_2} d_\delta(x)\, d\mathcal{H}^{n-1} \geq c|A^s|. \tag{3.16}$$

To this aim fix $s \in (7/10, 4/5)$ such that A^s has locally finite perimeter in B_2. We deduce from the definition of R_n (3.13), from (3.15) and from the coarea formula that there is $\rho_n \in (2 - \delta_n, R_n)$ such that

$$\begin{aligned}
\frac{1}{8}|A^s| \geq |A^s \cap B_{R_n}| &\geq |A^s \cap (B_{R_n} \setminus B_{2-\delta_n})| \\
&= \int_{R_n}^{2-\frac{\delta_n}{2}} \mathcal{H}^{n-1}(\partial B_\rho \cap A^s) d\rho \\
&= \frac{\delta_n}{2} \mathcal{H}^{n-1}(\partial B_{\rho_n} \cap A^s).
\end{aligned} \tag{3.17}$$

Choose a vector field $X(x) = (|x| - 2)\frac{x}{|x|}$ and observe that by the definition of δ_n (3.13) it holds

$$(\operatorname{div} X)(x) = n - \frac{2(n-1)}{|x|} \geq \frac{1}{2}$$

for every $|x| \geq 2 - \delta_n$. Therefore the Gauss-Green formula and (3.17) yield

$$\begin{aligned}
\frac{1}{2}|A^s \setminus B_{\rho_n}| &\leq \int_{A^s \setminus B_{\rho_n}} \operatorname{div}(X)\, dx \\
&= \int_{\partial^* A^s \cap (B_2 \setminus B_{\rho_n})} (|x| - 2)\langle \frac{x}{|x|}, \nu \rangle \, \mathcal{H}^{n-1} + \int_{\partial B_{\rho_n} \cap A^s} (2 - |x|)\mathcal{H}^{n-1} \\
&\leq C \int_{\partial^* A^s \cap (B_2 \setminus B_{\rho_n})} d_\delta(x)\, \mathcal{H}^{n-1} + \delta_n \mathcal{H}^{n-1}(\partial B_{\rho_n} \cap A^s) \\
&\leq C \int_{\partial^* A^s \cap B_2} d_\delta(x)\, \mathcal{H}^{n-1} + \frac{1}{4}|A^s|.
\end{aligned}$$

The inequality (3.16) then follows from (3.15) as follows

$$|A^s \setminus B_{\rho_n}| \geq |A^s \setminus B_{R_n}| \geq \frac{7}{8}|A^s|.$$

Finally we use (3.11), the coarea formula and (3.16) to conclude

$$\begin{aligned}
|\{1/5 < \sigma_A \leq 4/5\}| &\geq c \int_{A^{7/10} \setminus A^{4/5}} d_\delta(x) |\nabla \sigma_A(x)|\, dx \\
&= c \int_{7/10}^{4/5} \left(\int_{\{\sigma_A = s\} \cap B_2} d_\delta(x)\, d\mathcal{H}^{n-1}(x) \right) ds \\
&= c \int_{7/10}^{4/5} \left(\int_{\partial^* A^s \cap B_2} d_\delta(x)\, d\mathcal{H}^{n-1}(x) \right) ds \\
&\geq c \int_{7/10}^{4/5} |A^s|\, ds \\
&\geq c|A^{4/5}|.
\end{aligned}$$

□

We use the two previous lemmas to estimate the decay rate of the level sets.

Proposition 3.5. *Assume that $u \in C^\infty(B_2)$ is a nonnegative supersolution of*

$$\operatorname{div}(A(x)Du) \leq f(u).$$

Denote $\mu(t) = |A_t \cap B_2|$ and $\eta(t) = |E_t \cap B_2|$ where $E_t = \{u < t\}$ and $A_t = \{u \geq t\}$. Then for almost every $t > 0$ with $\mu(t) \leq \frac{|B_2|}{2}$ it holds

$$-\mu'(t) \geq c \min\Big\{ \frac{1}{\sqrt{F(t)}} \mu(t)^{\frac{n-1}{n}}, \frac{1}{t}\mu(t) \Big\}$$

and for almost every $t > 0$ with $\eta(t) \leq \frac{|B_2|}{2}$ it holds

$$\eta'(t) \geq c \min\Big\{ \frac{1}{\sqrt{F(t)}} \eta(t)^{\frac{n-1}{n}}, \frac{1}{t}\eta(t) \Big\}.$$

Here μ' is the absolutely continuous part of the differential of μ.

Proof. We will only prove the first inequality since the second follows from a similar argument. By Morse-Sard Lemma for almost every $t > 0$ it holds that $|Du| > 0$ on $\{u = t\} \cap B_2$ and by [1] we may write

$$-\mu'(t) = \int_{\{u=t\}\cap B_2} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}.$$

Let us fix $t > 0$ for which this holds. Let us choose

$$\delta = \frac{t}{\sqrt{F(t)}},$$

define the truncated distance by

$$d_\delta(x) := \frac{1}{100}\min\{(2 - |x|), \delta\},$$

and denote the density function by

$$\sigma_{A_t}(x) := \frac{|A_t \cap B(x, d_\delta(x))|}{|B_{d_\delta(x)}|}.$$

Let us divide the proof in two cases and assume first that

$$\big|\{\sigma_{A_t} > 4/5\}\big| \le \frac{1}{2}|A_t|.$$

This means that there is a large set where the density of A_t is low. In particular, it holds

$$\big|A_t \cap \{\sigma_{A_t} \le 4/5\}\big| \ge \frac{1}{2}|A_t|$$

For every $x_i \in A_t \cap \{\sigma_{A_t}(x) \le 4/5\}$ we choose a ball $B(x_i, d_\delta(x_i))$ and thus obtain a covering of $A_t \cap \{\sigma_{A_t} \le 4/5\}$. By the Besicovitch covering theorem [19, Corollary 5.2] we may choose a countable disjoint subfamily, say $\mathcal{F}$, such that

$$\xi(n)\sum_{i\in\mathcal{F}} \big|\,(A_t \cap \{\sigma_{A_t} \le 4/5\}) \cap B(x_i, d_\delta(x_i))\big| \ge \big|A_t \cap \{\sigma_{A_t} \le 4/5\}\big| \ge \frac{1}{2}|A_t|. \tag{3.18}$$

We observe that if $B(x_i, d_\delta(x_i))$ and $B(x_j, d_\delta(x_j))$ are two balls from $\mathcal{F}$ such that the enlarged balls $B(x_i, 2d_\delta(x_i))$ and $B(x_j, 2d_\delta(x_j))$ overlap each other, then their radii are comparable,

$$\frac{d_\delta(x_i)}{3} \le d_\delta(x_j) \le 3d_\delta(x_i).$$

Therefore since the balls in $\mathcal{F}$ are disjoint it follows that the enlarged balls $B(x_i, 2d_\delta(x_i))$ intersect every point in B_2 only finitely many times, say, $\eta(n)$ times. In particular, we have

$$\eta(n)\int_{\{u=t\}\cap B_2} \frac{1}{|Du|}\, d\mathcal{H}^{n-1} \ge \sum_{i\in\mathcal{F}} \int_{\{u=t\}\cap B(x_i, 2d_\delta(x_i))} \frac{1}{|Du|}\, d\mathcal{H}^{n-1}. \tag{3.19}$$

Let $B(x_i, d_\delta(x_i)) \in \mathcal{F}$. Since $x_i \in \{\sigma_{A_t} \le 4/5\}$ we have

$$\big|B(x_i, d_\delta(x_i)) \setminus E_t\big| = \big|A_t \cap B(x_i, d_\delta(x_i))\big| \le \frac{4}{5}\big|B(x_i, d_\delta(x_i))\big|.$$

Therefore

$$\big|E_t \cap B(x_i, d_\delta(x_i))\big| \ge \frac{1}{4}\big|B(x_i, d_\delta(x_i)) \setminus E_t\big|$$

and Lemma 3.3 gives

$$\int_{\{u=t\}\cap B(x_i,2d_\delta(x_i))} \frac{1}{|Du|}\,\mathcal{H}^{n-1} \geq \frac{c}{t}\left|B(x_i,d_\delta(x_i))\setminus E_t\right| = \frac{c}{t}\left|A_t\cap B(x_i,d_\delta(x_i))\right|.$$

Summing the above inequality and using (3.18) and (3.19) yield

$$\begin{aligned}\eta(n)\int_{\{u=t\}\cap B_2}\frac{1}{|Du|}\,d\mathcal{H}^{n-1} &\geq \sum_{i\in\mathcal{F}}\int_{\{u=t\}\cap B(x_i,2d_\delta(x_i))}\frac{1}{|Du|}\,d\mathcal{H}^{n-1}\\ &\geq \frac{c}{t}\sum_{i\in\mathcal{F}}\left|A_t\cap B(x_i,d_\delta(x_i))\right|\\ &\geq \frac{c}{\xi(n)t}|A_t|.\end{aligned}$$

In other words $-\mu'(t)\geq \frac{c}{t}\mu(t)$ and the claim follows in this case.

Let us next assume that

$$\left|\{\sigma_{A_t}>4/5\}\right|\geq\frac{1}{2}|A_t|. \tag{3.20}$$

This means that there is a large set where the density of A_t is close to one. In this case we do not cover the set $\{\sigma_{A_t}\leq 4/5\}$ but only the part where the density σ_{A_t} is between $1/5$ and $4/5$. Then we estimate the measure of this set using Lemma 3.4 .

For every $x_i\in\{1/5<\sigma_{A_t}\leq 4/5\}$ we choose a ball $B(x_i,d_\delta(x_i))$ and thus obtain a covering of $\{1/5<\sigma_{A_t}\leq 4/5\}$. By the Besicovitch covering theorem we may choose a countable disjoint subfamily $\mathcal{F}$ such that

$$\left|\{1/5<\sigma_{A_t}\leq 4/5\}\right|\leq\xi(n)\sum_{i\in\mathcal{F}}\left|B(x_i,d_\delta(x_i))\right|. \tag{3.21}$$

Moreover as in (3.19) we have

$$\eta(n)\int_{\{u=t\}\cap B_2}\frac{1}{|Du|}\,d\mathcal{H}^{n-1}\geq\sum_{i\in\mathcal{F}}\int_{\{u=t\}\cap B(x_i,2d_\delta(x_i))}\frac{1}{|Du|}\,d\mathcal{H}^{n-1}. \tag{3.22}$$

Let $B(x_i,d_\delta(x_i))\in\mathcal{F}$. Since $x_i\in\{1/5<\sigma_{A_t}\leq 4/5\}$ we have

$$\frac{1}{5}\leq\frac{\left|B(x_i,d_\delta(x_i))\setminus E_t\right|}{\left|B_{d_\delta(x_i)}\right|}=\frac{\left|A_t\cap B(x_i,d_\delta(x_i))\right|}{\left|B_{d_\delta(x_i)}\right|}\leq\frac{4}{5}.$$

Therefore Lemma 3.3 gives

$$\int_{\{u=t\}\cap B(x_i,2d_\delta(x_i))}\frac{1}{|Du|}\,\mathcal{H}^{n-1}\geq\frac{c}{t}\left|B(x_i,d_\delta(x_i))\right|.$$

Summing the above inequality and using (3.21) and (3.22) yield

$$\begin{aligned}\eta(n)\int_{\{u=t\}\cap B_2}\frac{1}{|Du|}\,d\mathcal{H}^{n-1} &\geq \sum_{i\in\mathcal{F}}\int_{\{u=t\}\cap B(x_i,2d_\delta(x_i))}\frac{1}{|Du|}\,d\mathcal{H}^{n-1}\\ &\geq \frac{c}{t}\sum_{i\in\mathcal{F}}\left|B(x_i,d_\delta(x_i))\right|\\ &\geq \frac{c}{\xi(n)t}\left|\{1/5<\sigma_{A_t}\leq 4/5\}\right|.\end{aligned}$$

Finally we use Lemma 3.4 to conclude that

$$\big|\{1/5 < \sigma_{A_t} \le 4/5\}\big| \ge c \min\big\{\delta\,\big|\{\sigma_{A_t} > 4/5\}\big|^{\frac{n-1}{n}}, \big|\{\sigma_{A_t} > 4/5\}\big|\big\}.$$

Therefore by the two previous inequalities and by (3.20) we get

$$-\mu'(t) = \int_{\{u=t\}\cap B_2} \frac{1}{|Du|}\, d\mathcal{H}^{n-1} \ge \frac{c}{t} \min\big\{\delta\,\mu(t)^{\frac{n-1}{n}}, \mu(t)\big\}.$$

The result follows from

$$\delta = \frac{t}{\sqrt{F(t)}}.$$

□

4. Estimates for subsolutions

In this section we prove estimates for subsolutions of the equation (1.1). The most important result of this section for Theorem 1.1 is Lemma 4.3. Similar but slightly different result is proved in [16]. The proof is fairly standard. Instead of using a capacity argument [16] we give a short proof based on De Giorgi iteration. After Lemma 4.3 we give the proof of Theorem 1.2 and Corollary 4.4.

We begin by recalling the following standard result for subsolutions of the linear equation

$$\operatorname{div}(A(x)Du) = 0. \tag{4.1}$$

Lemma 4.1. *Let $u \in W^{1,2}(B(x_0, 2r))$ be a subsolution of (4.1) and denote $M_r := \sup_{B(x_0,r)} u$. There is $\varepsilon_0 > 0$ such that if*

$$\big|\{u \ge u(x_0)/2\} \cap B(x_0, r)\big| \le \varepsilon_0 |B_r|,$$

then $M_r \ge 4u(x_0)$.

Proof. Let us recall that by De Giorgi theorem any subsolution v of (4.1) is locally bounded and satisfies

$$\sup_{B(x_0,r/2)} v \le C r^{-\frac{n}{2}} ||v||_{L^2(B(x_0,r))}.$$

The result follows by applying this to $v = (u - u(x_0)/2)_+$ which is a subsolution of (4.1). □

We recall the notation $A_t = \{u \ge t\}$. We have the following Caccioppoli inequality for subsolutions of (1.1).

Lemma 4.2. *Let $u \in W^{1,2}(B(x_0, 2r))$ be a subsolution of*

$$\operatorname{div}(A(x)Du) \ge f(u)$$

and denote $M_r := \sup_{B(x_0,r)} u$. Then for every $\rho < r$ and $t < M_r$ it holds

$$\int_{B(x_0,\rho)} (u-t)_+^2\, dx \le \frac{C}{(r-\rho)^2}\, |A_t \cap B_r|^{\frac{1}{n}} \left(\frac{M_r}{\sqrt{F(t)}}\right) \int_{B(x_0,r)} (u-t)_+^2\, dx.$$

Proof. Without loss of generality we may assume that $x_0 = 0$. We use testfunction $\varphi = (u-t)_+\zeta^2$ in (1.1) where $\zeta \in C_0^\infty(B_r)$ is a standard cut-off function such that $\zeta = 1$ in B_ρ and $|D\zeta| \le \frac{2}{r-\rho}$. This gives

$$\int_{B_r} |D((u-t)_+\zeta)|^2\, dx \le C \int_{B_r} (u-t)_+^2 |D\zeta|^2\, dx - \int_{B_r} f(u)(u-t)_+\zeta^2\, dx.$$

Since f is nondecreasing we have for every $x \in A_t \cap B_r$ that

$$f(u(x)) \geq f(t) \geq \frac{F(t)}{t} \geq \frac{F(t)}{M_r^2}(u(x) - t).$$

Therefore we have

$$\int_{B_r} f(u)(u-t)_+\zeta^2 \, dx \geq \frac{F(t)}{M_r^2} \int_{B_r} (u-t)_+^2 \zeta^2 \, dx.$$

We denote $w = (u-t)_+\zeta \in W_0^{1,2}(B_r)$ and obtain by Young's and Sobolev inequalities

$$\begin{aligned}
\int_{B_r} |D((u-t)_+\zeta)|^2 \, dx + \frac{F(t)}{M_r^2} \int_{B_r} (u-t)_+^2 \zeta^2 \, dx & \\
&\geq c\frac{\sqrt{F(t)}}{M_r} \int_{B_r} |wDw| \, dx \\
&= c\frac{\sqrt{F(t)}}{M_r} \int_{B_r} \frac{1}{2}|D(w^2)| \, dx \\
&\geq c\frac{\sqrt{F(t)}}{M_r} \left(\int_{B_r} w^{\frac{2n}{n-1}} \, dx \right)^{\frac{n-1}{n}}.
\end{aligned}$$

The result follows from Jensen's inequality

$$\left(\int_{B_\rho} (u-t)_+^{\frac{2n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \geq |A_t \cap B_\rho|^{-\frac{1}{n}} \int_{B_\rho} (u-t)_+^2 \, dx.$$

□

By the standard De Giorgi iteration we obtain the estimate we need.

Lemma 4.3. *Let $u \in W^{1,2}(B(x_0, 2r))$ be a subsolution of*

$$\operatorname{div}(A(x)Du) \geq f(u)$$

and denote $M_r := \sup_{B(x_0,r)} u$. If $0 < M_r \leq 2u(x_0)$, then

$$\frac{u(x_0)}{\sqrt{F\left(\frac{u(x_0)}{2}\right)}} \geq cr.$$

Proof. By rescaling and translating we may assume that u is a subsolution of

$$\operatorname{div}(A(x)Du) \geq r^2 f(u)$$

in B_2 and $0 < M_1 \leq 2u(0)$. We need to show that

$$\Lambda_0 := \frac{u(0)}{\sqrt{r^2 F\left(\frac{u(0)}{2}\right)}} \geq c > 0.$$

Let τ, ρ be such that $1/2 < \rho < \tau < 1$ and h, k such that $u(0)/2 \leq h < k \leq u(0)$. Then it follows from the assumption $M_1 \leq 2u(0)$ and Lemma 4.2 that

$$\int_{A_k \cap B_\rho} (u-k)_+^2 \, dx \leq C\Lambda_0 \, |A_k \cap B_\tau|^{\frac{1}{n}} \frac{1}{(\tau-\rho)^2} \int_{A_h \cap B_\tau} (u-h)_+^2 \, dx.$$

Since

$$|A_k \cap B_\tau| \le \frac{1}{(k-h)^2} \int_{A_h \cap B_\tau} (u-h)_+^2 \, dx$$

one gets

$$\int_{A_k \cap B_\rho} (u-k)_+^2 \, dx \le C\Lambda_0 \, \frac{1}{(k-h)^{\frac{2}{n}}} \frac{1}{(\tau-\rho)^2} \left(\int_{A_h \cap B_\tau} (u-h)_+^2 \, dx \right)^{1+\frac{1}{n}}. \tag{4.2}$$

Define $r_i = \frac{1}{2}(1+2^{-i})$ and $k_i = u(0)\left(\frac{3}{4} - 4^{-1-i}\right)$. Write (4.2) for $h = k_i$, $k = k_{i+1}$, $\tau = r_i$ and $\rho = r_{i+1}$ and set

$$x_i = M_1^{-2} \int_{A_{k_i} \cap B_{r_i}} (u-k_i)_+^2 \, dx.$$

Since $M_1 \le 2u(0)$ we obtain

$$x_{i+1} \le C_0 C_1^i \Lambda_0 \, x_i^{1+\frac{1}{n}}$$

for $C_0 > 1$ and $C_1 > 1$. Let us show that it holds

$$\Lambda_0 \ge |B_1|^{-\frac{1}{n}} C_0^{-1} C_1^{-n}. \tag{4.3}$$

Indeed, we argue by contradiction and assume that (4.3) does not hold. This implies

$$x_0 \le M_1^{-2} \int_{B_1} u^2 \, dx \le |B_1| \le C_0^{-n} C_1^{-n^2} \Lambda_0^{-n}.$$

It follows from Lemma 2.2 that $\lim_{i\to\infty} x_i = 0$. This means that

$$\sup_{B_{1/2}} u \le \frac{3}{4} u(0)$$

which is a contradiction. Therefore we have (4.3) and the claim follows. □

Proof of Theorem 1.2. Without loss of generality we may assume that $x_0 = 0$. Let us denote $M_0 = u(0)$ and $R_0 = 0$. We define radii $R_k \in (0, R]$, where $k = 1, \dots, K$, such that the corresponding maxima $M_k := \sup_{B_{R_k}} u$ satisfy

$$M_k = 2M_{k-1}$$

for every $k = 1, \dots, K-1$ and $R_K = R$ and $\sup_{B(x_0,R)} u = M_K \le 2M_{K-1}$. Denote also $r_k = R_k - R_{k-1}$ for $i = 1, \dots, K$. For notational reasons we also define $M_{-1} = M_0/2$ and $M_{-2} = M_0/4$.

Let us fix $k \in 1, \dots, K$. Let $x_{k-1} \in \partial B_{R_{k-1}}$ be a point such that $u(x_{k-1}) = M_{k-1}$. Note that because of the maximum principle the maximum is attained on the boundary. Since $\sup_{B(x_{k-1}, r_k)} u \le M_k = 2M_{k-1}$ Lemma 4.3 yields

$$\frac{M_{k-1}}{\sqrt{F(M_{k-1}/2)}} \ge c r_k.$$

Note that $M_{k-1} = 4M_{k-3}$ and $M_{k-1}/2 = M_{k-2}$. Therefore by the monotonicity of F it holds

$$\int_{M_{k-3}}^{M_{k-2}} \frac{t}{\sqrt{F(t)}} \ge \frac{M_{k-3}}{\sqrt{F(M_{k-2})}} \ge \frac{1}{4} \frac{M_{k-1}}{\sqrt{F(M_{k-1}/2)}} \ge c r_k.$$

Summing the above inequality over $k = 1, \dots, K$ yields

$$\int_{M_{-2}}^{M_{K-2}} \frac{dt}{\sqrt{F(t)}} \geq c \sum_{k=1}^{K} r_k = c\, R.$$

Since $M_{-2} = u(0)/4 \geq m/4$ and $M_{K-2} \leq M$ one has

$$\int_{M_{-2}}^{M_{K-2}} \frac{dt}{\sqrt{F(t)}} \leq \int_{m/4}^{M} \frac{dt}{\sqrt{F(t)}}.$$

The result follows from the previous two inequalities. □

Let us briefly discuss about the previous theorem. At the first glance the statement of Theorem 1.2 might seem unsatisfactory. First, one could think that the assumption $u(x_0) > 0$ in Theorem 1.2 is unnecessary. However, it is easy to see that it can not be removed. Second, one can not reduce the interval of integration from $[m/4, M]$ to $[m, M]$, i.e., the estimate

$$\int_{m}^{M} \frac{dt}{\sqrt{F(t)}} \geq c\, R \tag{4.4}$$

is not true. To see this choose f such that $f(t) = 0$ for $t \in [0, 1]$ and $f(t) = 1$ for $t > 1$. Construct a one dimensional solution of $u'' = f(u)$ in $(0, 1)$ by $u(x) = 1$ for $0 < x \leq 1 - \varepsilon$ and $u(x) = \frac{1}{2}(x - 1 + \varepsilon)^2 + 1$ for $1 - \varepsilon < x < 1$. This solution does not satisfy the estimate (4.4).

Corollary 4.4. *Suppose that there exists a continuous subsolution $u \in W^{1,2}_{loc}(\mathbb{R}^n)$ of*

$$\operatorname{div}(A(x)Du) \geq f(u)$$

in $\mathbb{R}^n$ which is not constant. Then necessarily

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{F(t)}} = \infty.$$

Proof. Since u is not constant there exists a point x_0 such that $u(x_0) > \inf_{B(x_0,1)} u$. Without loss of generality we may assume that $x_0 = 0$. Let us fix a large radius $R > 1$ and denote $m_{2R} = \inf_{B_{2R}} u$ and $M_R = \sup_{B_R} u$. We define $v = u - m_{2R}$ which is a nonnegative subsolution of

$$\operatorname{div}(A(x)Dv) \geq \tilde{f}(v)$$

in B_{2R} where $\tilde{f}(t) = f(t + m_{2R})$. Denote also $\tilde{M} = \sup_{B_R} v = M_R - m_{2R}$ and $\tilde{m} = \inf_{B_R} v = m_R - m_{2R}$. Since $v(0) = u(0) - m_{2R} > 0$ we deduce from Theorem 1.2 that

$$\int_{\tilde{m}/4}^{\tilde{M}} \frac{dt}{\sqrt{\tilde{F}(t)}} \geq c\, R.$$

Since

$$\int_{\tilde{m}/4}^{\tilde{M}} \frac{dt}{\sqrt{\tilde{F}(t)}} \leq \int_{m_{2R}}^{M_R} \frac{dt}{\sqrt{F(t)}}$$

the result follows by letting $R \to \infty$. □

5. Proof of the Main Theorem

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let u be a nonnegative solution of

$$\operatorname{div}(A(x)Du) = f(u)$$

in B_2. As we discussed in Section 2 we may assume that $u \in C^\infty(B_2)$. Let us once more recall our notation $E_t = \{u < t\}$, $A_t = \{u \geq t\}$, $\eta(t) = |E_t \cap B_2|$ and $\mu(t) = |A_t \cap B_2|$. Let $t_0 > 0$ be such that $\eta(t_0) = \frac{|B_2|}{2}$ or, if such number does not exist, the supremum over the numbers t for which $\eta(t) \leq \frac{|B_2|}{2}$.

Let $\delta_0 > 0$ be from Lemma 3.2. Let us first show that there exists C, which depends on δ_0, such that either the integral is bounded

$$\int_0^{t_0} \frac{ds}{\sqrt{F(s)} + s} \leq C$$

or there exists $t_\delta \in (0, t_0)$ such that

$$\eta(t_\delta) \leq \delta_0 \left(\frac{t_\delta}{\sqrt{F(t_\delta)} + t_\delta} \right)^n \qquad \text{and} \qquad \int_{t_\delta}^{t_0} \frac{ds}{\sqrt{F(s)} + s} \leq C. \tag{5.1}$$

Indeed, it follows from Proposition 3.5 that for every $t \leq t_0$ for which the first inequality in (5.1) does not hold, i.e.,

$$\eta(t) \geq \delta_0 \left(\frac{t}{\sqrt{F(t)} + t} \right)^n$$

we have

$$\eta'(t) \geq c \frac{\eta^{1-\frac{1}{n}}(t)}{\sqrt{F(t)} + t}$$

for a constant which depends on δ_0. In other words

$$\frac{d}{dt}(\eta^{\frac{1}{n}}(t)) \geq \frac{c}{\sqrt{F(t)} + t}.$$

We integrate the above inequality over (t, t_0) and get

$$|B_2| \geq \eta^{\frac{1}{n}}(t_0) - \eta^{\frac{1}{n}}(t) \geq c \int_t^{t_0} \frac{ds}{\sqrt{F(s)} + s}.$$

Therefore if

$$\int_0^{t_0} \frac{ds}{\sqrt{F(s)} + s} \geq \frac{2|B_2|}{c}$$

we conclude that there exists $t_\delta \in (0, t_0)$ for which (5.1) holds. Since t_δ satisfies (5.1) we deduce from Lemma 3.2 that

$$m = \inf_{B_1} u \geq \frac{t_\delta}{2}.$$

Therefore we have

$$\int_m^{t_0} \frac{ds}{\sqrt{F(s)} + s} \leq \int_{t_\delta/2}^{t_\delta} \frac{ds}{s} + \int_{t_\delta}^{t_0} \frac{ds}{\sqrt{F(s)} + s} \leq C \tag{5.2}$$

for a constant C which is independent of u and f.

We need yet to show that

$$\int_{t_0}^{M} \frac{ds}{\sqrt{F(s)}+s} \leq C$$

where $M = \sup_{B_1} u$. To that aim let $\varepsilon > 0$ be a small number which we choose later. Let $t_\varepsilon \geq t_0$ be the first value of u such that

$$|\{u \geq t_\varepsilon\} \cap B_2| \leq \varepsilon,$$

or, to be more precise,

$$t_\varepsilon := \sup\{t > 0 : \mu(t) > \varepsilon\}.$$

Proposition 3.5 implies that for every $t \in (t_0, t_\varepsilon)$ it holds

$$-\mu'(t) \geq c \frac{\mu^{1-\frac{1}{n}}(t)}{\sqrt{F(t)}+t}$$

for a constant c which depends on ε. In other words

$$-\frac{d}{dt}(\mu^{\frac{1}{n}}(t)) \geq \frac{c}{\sqrt{F(t)}+t}.$$

Therefore we have

$$|B_2| \geq \mu^{\frac{1}{n}}(t_0) - \mu^{\frac{1}{n}}(t_\varepsilon) \geq c \int_{t_0}^{t_\varepsilon} \frac{ds}{\sqrt{F(s)}+s}.$$

We will show that when $\varepsilon > 0$ is chosen small enough it holds $M \leq 2t_\varepsilon$. This will imply

$$\int_{t_0}^{M} \frac{ds}{\sqrt{F(s)}+s} \leq \int_{t_0}^{t_\varepsilon} \frac{ds}{\sqrt{F(s)}+s} + \int_{t_\varepsilon}^{2t_\varepsilon} \frac{ds}{s} \leq C.$$

The result then follows from the previous estimate and from (5.2).

To prove $M \leq 2t_\varepsilon$ we argue by contradiction and assume that $M > 2t_\varepsilon$. Recall that by the assumption on t_ε it holds $|\{u \geq M/2\} \cap B_2| \leq \varepsilon$. We note that by the maximum principle the maximum of u in any ball $\bar{B}_R$ is obtained on the boundary. Therefore when ε is small we conclude from Lemma 4.1 that

$$\sup_{B_{\frac{5}{4}}} u \geq 4M \qquad \text{and} \qquad \sup_{B_{\frac{7}{4}}} u \geq 4 \sup_{B_{\frac{3}{2}}} u. \tag{5.3}$$

To prove (5.3) choose $x_0 \in \partial B_1$ such that $u(x_0) = M$ and $y_0 \in \partial B_{\frac{3}{2}}$ such that $u(y_0) = \sup_{B_{\frac{3}{2}}} u$ and apply Lemma 4.1 in $B(x_0, 1/4)$ and in $B(y_0, 1/4)$.

We choose radii $1 = R_0 < R_1 < R_2 < \dots$ such that the corresponding maxima

$$M_k := \sup_{B_{R_k}} u$$

satisfy

$$M_k = 2M_{k-1}.$$

Here we use notation M_0 for $M = \sup_{B_1} u$. We continue to do this until the first time we find R_K such that $R_K \geq 3/2$. It follows from (5.3) that $K \geq 3$ and that $R_K \leq 7/4 < 2$. In particular, R_K is well defined. Denote $r_k = R_k - R_{k-1}$. We also deduce from (5.3) that $R_2 \leq 5/4$ and therefore

$$r_1 + r_2 = R_2 - R_0 \leq \frac{1}{4}. \tag{5.4}$$

Let us fix $k = 3, \dots, K$. I claim that for small c it holds

$$\mu^{\frac{1}{n}}(t) \geq cr_k \tag{5.5}$$

for every $t \in (M_{k-3}, M_{k-2}]$. Note that it is enough to show (5.5) for $t = M_{k-2}$. We argue by contradiction and assume that $\mu^{\frac{1}{n}}(M_{k-2}) \leq cr_k$. This can be written as

$$\mu(M_{k-2}) \leq \varepsilon_1 |B_{r_k}|$$

for small $\varepsilon_1 > 0$. Since $M_{k-1} = 2M_{k-2}$ this can be again written as

$$|\{u \geq M_{k-1}/2\} \cap B_2| \leq \varepsilon_1 |B_{r_k}|.$$

Let $x_{k-1} \in \partial B_{R_{k-1}}$ be such that $u(x_{k-1}) = M_{k-1}$. When ε_1 is small it follows from Lemma 4.1 that

$$\sup_{B(x_{k-1}, r_k)} u \geq 4M_{k-1}.$$

However since $M_k \geq \sup_{B(x_{k-1}, r_k)} u$ this contradicts the fact that M_k was chosen such that $M_k = 2M_{k-1}$. Hence we have (5.5). Note also that it follows from

$$0 < \sup_{B(x_{k-1}, r_k)} u \leq M_k = 2M_{k-1} = 2u(x_{k-1})$$

and from Lemma 4.3 that

$$\frac{M_{k-1}}{\sqrt{F(M_{k-1}/2)}} \geq cr_k.$$

For every $t \in (M_{k-3}, M_{k-2}]$ it holds $M_{k-1} = 4M_{k-3} < 4t$ and $M_{k-1}/2 = M_{k-2} \geq t$. Therefore by the monotonicity of F we have

$$\frac{t}{\sqrt{F(t)}} \geq cr_k \tag{5.6}$$

for every $t \in (M_{k-3}, M_{k-2}]$.

Let us fix $k = 3, \dots, K$. We deduce from Proposition 3.5 and from (5.5) and (5.6) that for every $t \in (M_{k-3}, M_{k-2}]$ we have the estimate

$$-\mu'(t) \geq \frac{c}{t} \mu^{1-\frac{1}{n}}(t) r_k.$$

In other words

$$-\frac{d}{dt}(\mu^{\frac{1}{n}}(t)) \geq \frac{c}{t} r_k.$$

We integrate this over (M_{k-3}, M_{k-2}) and use the fact that $M_{k-2} = 2M_{k-3}$ to conclude

$$\mu^{\frac{1}{n}}(M_{k-3}) - \mu^{\frac{1}{n}}(M_{k-2}) \geq cr_k.$$

Recall that $M_0 = M = \sup_{B_1} u$. Summing the previous inequality over $k = 3, \dots, K$ gives

$$\mu^{\frac{1}{n}}(M) \geq \sum_{k=3}^{K} \mu^{\frac{1}{n}}(M_{k-3}) - \mu^{\frac{1}{n}}(M_{k-2}) \geq c \sum_{k=3}^{K} r_k. \tag{5.7}$$

Recall that $r_k = R_k - R_{k-1}$, $R_0 = 1$ and $R_K \geq 3/2$. Therefore $\sum_{k=1}^{K} r_k = R_K - R_0 \geq \frac{1}{2}$. Thus we deduce from (5.4) that

$$\sum_{k=3}^{K} r_k \geq \frac{1}{4}.$$

Therefore (5.7) yields

$$\mu^{\frac{1}{n}}(M) \geq c.$$

However, by the choice of t_ε we have

$$\mu(M) = |\{u \geq M\} \cap B_2| \leq |\{u \geq t_\varepsilon\} \cap B_2| \leq \varepsilon$$

which is a contradiction when $\varepsilon > 0$ is small. □

At the end let us show why Theorem 1.1 does not hold for equation

$$-\Delta u = f(u) \tag{5.8}$$

when $n \geq 2$. If Thereom 1.1 would be true for nonnegative solutions of (5.8) then there would be C_0 such that

$$\int_m^M \frac{1}{\sqrt{F(t)} + t} \leq C_0. \tag{5.9}$$

Let $\varphi \geq 0$ be the fundamental solution of the Laplace equation with $\inf_{B_1} \varphi = 1$ and singularity at the origin. Since φ is unbounded we find a radius $r > 0$ such that for the value of φ on ∂B_r, denote it by T, it holds

$$\int_1^T \frac{1}{t} \geq 2C_0. \tag{5.10}$$

We define $u \in C^{1,1}(B_2)$ such that $u(x) = \varphi(x)$ for $x \in B_2 \setminus \bar{B}_r$ and $u(x) = a(r^2 - |x|^2) + T$ for $x \in B_r$. Here $a > 0$ is chosen such that $u \in C^{1,1}(B_2)$. Then u is a solution of (5.8) for some f which satisfies $f(t) = 0$ for $t \in (0, T)$. Therefore (5.9) can not hold because of (5.10).

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