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244

Marjaana Nokka

A Posteriori Error Estimates for Variational Problems in the Theory of Viscous Fluids

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ABSTRACT

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The papers included in the thesis are focused on functional type a posteriori error estimates for the Stokes problem, the Stokes problem with friction type boundary conditions, the Oseen problem, and the anti-plane Bingham problem. In the summary of the thesis we consider only the Oseen problem. The papers present and justify special forms of these estimates which are suitable for the approximations generated by the Uzawa algorithm. The estimates are of two main types. Estimates of the first type use exact solutions obtained on the steps of the Uzawa algorithm. They show how errors encompassed in Uzawa approximations behave and have mainly theoretical meaning. Estimates of the second type operate only with approximations (e.g. finite element solutions). Therefore, they are fully computable. In the thesis it is shown that estimates of this type indeed provide realistic evaluation of errors for finite element approximations of problems associated with viscous incompressible fluids.

Keywords: functional a posteriori error estimates, Stokes problem, Oseen problem, Bingham problem, nonlinear boundary conditions, Uzawa algorithm

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Marjaana Nokka
Jyväskylä, September 2016

CONTENTS

LIST OF INCLUDED ARTICLES

PI I. Anjam, M. Nokka, and S. Repin. On a posteriori error bounds for approximations of the generalized Stokes problem generated by the Uzawa algorithm. *Russian J. Numer. Anal. Math. Modelling, **27**(4), 321–338*, 2012.

PII P. Neittaanmäki, M. Nokka, and S. Repin. A posteriori error bounds for approximations of the Stokes problem with friction type boundary conditions. *Reports of the Department of Mathematical Information Technology, University of Jyväskylä, Series B. Scientific Computing, No. B. 5/2016*, 2016.

PIII M. Nokka and S. Repin. A posteriori error bounds for approximations of the Oseen problem and applications to Uzawa iteration algorithm. *Comput. Methods Appl. Math. **14**(3)*, 2014.

PIV M. Nokka and S. Repin. Error estimates of Uzawa iteration method for a class of Bingham fluids. *Mathematical Modeling And Optimization of Complex Structures*, 2015.

1 INTRODUCTION

Various phenomena appearing in natural sciences are modeled with the help of partial differential equations. Typical examples are presented by diffusion type problems, electromagnetic problems, continuum media, etc. Other typical applications include medicine, meteorology, and aerodynamics. In the thesis, we focus our attention on mathematical models of viscous incompressible fluids, which belong to the class of most important and interesting models in continuum mechanics.

Only in very rare cases partial differential equations can be solved exactly. In the majority of cases, only approximations (numerical solutions) are indeed available. Here, the most known methods are the finite element method (FEM) (there is a large amount of literature devoted to this subject, see, e.g., [10, 17, 49]), the finite difference method (FDM) (a consequent exposition is presented in, e.g., [50]) and the finite volume method (FVM) (see, e.g., [27]). By using numerical methods, we find an approximate solution, which always contain errors of different origin.

There exist two main approaches to analysis of these errors. The first approach is to measure the error *a priori*, which means that we determine the error before finding the approximation of the exact solution. A priori analysis forms the first stage of studying. Typical a priori estimate is derived under rather demanding conditions (higher regularity of the exact solution, certain regularity of the mesh, Galergin orthogonality), which are difficult to guarantee in real life computations. This approach provides a general presentation on properties of approximations. It is difficult to use if we wish to get a realistic estimate of the error encompassed in concrete numerical solution. Moreover, a priori estimates do not suggest an information on local distribution of errors, what is very important for modern mesh-adaptive algorithms.

For these reasons, other error control methods (known as *a posteriori* methods) started to attract serious attention. A posteriori type error estimate targets to get computable and realistic estimates of the accuracy of particular solution. There are several types of a posteriori error indicators, namely, the explicit residual method [3, 4, 6, 48, 49], equilibration based methods [11] and gradient averaging type methods [76, 77]. In this thesis we use and investigate another group of estimates called *a posteriori estimates of the functional type*, (see [51, 66], and references cited therein). A posteriori estimates of this type are independent of the numerical method that has been used to compute the approximate solution. The estimates contain only global constants, which are independent of the mesh used to discretize the domain.

The main goal of the research is to investigate how a posteriori estimates of functional type can be used in the context of iteration Uzawa type methods, which are often used in computer modeling of viscous problems. For this purpose, we deduced special modifications of the estimates adapted to the structure of Uzawa approximations and obtained easily computable error majorants for the Stokes, Stokes with nonlinear boundary conditions, Oseen and Bingham problems.

Author's contribution to included articles:

[PI] This paper is concerned with new type of functional a posteriori error estimates adapted to approximations generated by the Uzawa method. The proofs of the error estimates presented here are based on a posteriori error estimates for the generalized Stokes problem as they are presented in [71]. The mathematical results were done in close collaboration with co-authors I. Anjam and S. Repin.

[PII] This paper is concerned with a posteriori error estimates for the stationary Stokes problem with nonlinear friction type boundary conditions. The mathematical results were done in close collaboration with co-authors P. Neittaanmäki and S. Repin.

[PIII] The focus of this paper is to derive error estimates for the stationary Oseen problem. Also in this paper the estimates are applied to the Uzawa method. The numerical experiments confirming the results in the paper were carried out by the author, using The FEniCS Project [28]. The mathematical results were carried out in close collaboration with co-author S. Repin.

[PIV] In this paper, we focus on bounds of error for variational problems in the theory of visco-plastic fluids. These bounds of error have two forms. The first form includes global constants (such as the constant in Friedrichs inequality) and the second one is based on decomposition of the domain into a collection of subdomains, and uses local constants. The mathematical results were carried out in close collaboration with co-author S. Repin.

2 MATHEMATICAL BACKGROUND

In this chapter, we introduce the notation and basic mathematical knowledge that we need for the results presented in this thesis.

By $\mathbb{R}^d$ we denote the space of d-dimensional real valued vectors, and by $\mathbb{M}^{d\times d}$ the space of real valued second order tensors ($d \geq 2$). The inner products of $a, b \in \mathbb{R}^d$ and $\tau, \sigma \in \mathbb{M}^{d\times d}$ are defined by

$$a \cdot b := \sum_{i=1}^{d} a_i b_i \quad \text{and} \quad \tau : \sigma := \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_{ij} \sigma_{ij},$$

respectively. They generate the norms $|a| = \sqrt{a \cdot a}$ and $|\sigma| = \sqrt{\sigma : \sigma}$, respectively. Throughout the thesis $\Omega \subset \mathbb{R}^d$ denotes a connected bounded domain with Lipschitz continuous boundary $\partial\Omega$ and $\bar{\Omega}$ is the closure of Ω. These requirements for Ω will not be repeated, and additional requirements will be separately emphasized if necessary.

2.1 Function spaces

By V we denote a Banach space with the norm $\| \cdot \|_V$. The respective dual space is V^*, it consists of linear functions of V and is equipped with norm

$$\|f\|_{V^*} := \sup_{v \in V, v \neq 0} \frac{f(v)}{\|v\|_V}.$$

If V is Hilbert space, then the norm $\| \cdot \|_V$ is generated by the scalar product $(\cdot, \cdot)^{1/2}$. The so-called duality product $\langle \cdot, \cdot \rangle_{V^* \times V} : V^* \times V \to \mathbb{R}$ is defines the value of a linear functional v^* on the element v

$$\langle v^*, v \rangle_{V^* \times V} := v^*(v), \;\; \forall v \in V.$$

The Lebesgue-measurable functions v with finite norm

$$\|v\|_{L^p} := \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{1/p}$$

form a separable Banach space that is denoted by $L^p(\Omega),\ p \in [1, +\infty[$. The case $p = \infty$ defines the space of essentially bounded functions and the norm is defined as

$$\|v\|_{L^\infty} := \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|.$$

Our main interest is the Hilbert space of square-integrable functions $L^2(\Omega)$ equipped with the norm

$$\| \cdot \| := (\cdot, \cdot)^{1/2}_{L^2(\Omega)}$$

induced by

$$(u, v)_{L^2(\Omega)} = (u, v) := \int_\Omega uv \, \mathrm{d}x, \ \ \forall u, v \in L^2(\Omega).$$

The space of scalar valued square summable functions with zero mean is denoted by $\widetilde{L}_2(\Omega)$. The αth generalized derivative of $u \in L^2(\Omega)$ is denoted by $w = D^\alpha u \in L^2(\Omega)$ and is defined as a function satisfying the integral identity

$$\int_\Omega wv \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u D^\alpha v \, \mathrm{d}x, \ \ \forall v \in C_0^\infty(\Omega).$$

The Banach space $W_p^k(\Omega)$, $p \in [1, +\infty[$, and $k \in \mathbb{N}$ is called the Sobolev space

$$W_p^k(\Omega) := \{ v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega), |\alpha| \leq k \}$$

and is equipped with the norm

$$\|v\|_{W_p^k} := \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p} \right)^{1/p}.$$

Often the Hilbert spaces with $p = 2$ are denoted as $H^k(\Omega) = W_2^k(\Omega)$. In this thesis, we use spaces

$$H^1(\Omega) := \left\{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega, \mathbb{R}^d) \right\}, \text{ and} \tag{1}$$

$$H(\mathrm{div}, \Omega) := \left\{ v \in L^2(\Omega, \mathbb{R}^d) \mid \mathrm{div} v \in L^2(\Omega, \mathbb{R}^d) \right\}. \tag{2}$$

Also, we use the space H^{-1}, which is defined as the dual space of $H_0^1(\Omega)$, i.e.

$$H^{-1}(\Omega) := (H_0^1(\Omega))^*.$$

The space $H^{-1}(\Omega)$ is equipped with the norm

$$\|v\|_{-1} := \sup_{w \in H_0^1(\Omega)} \frac{| < v, w > |}{\|\nabla w\|}.$$

$\mathbb{S}(\Omega)$ denotes the subspace of $V := H^1(\Omega)$ that consists of solenoidal (divergence free) functions and $V_0(\Omega)$ denotes the subspace of V that consists of the functions vanishing on the Dirichlét part of the boundary.

Important differential operators applied in the thesis are listed below:

– The gradient of a scalar valued function v is a vector

$$\nabla v := \left[\frac{\partial w}{\partial x_i}\right]_{i=1}^{d} \in \mathbb{R}^d.$$

– Gradient of a vector valued function $[w]_{i=1}^{d}$ is a tensor valued function

$$\nabla w := \left[\frac{\partial w_i}{\partial x_j}\right]_{i,j=1}^{d} \in \mathbb{M}^{d\times d}.$$

– For vector and tensor valued functions the divergence is defined by the relations

$$\mathrm{div} w := \sum_{i=1}^{d} \frac{\partial w_i}{\partial x_i}$$

and

$$\mathrm{Div} \tau := \left[\sum_{i=1}^{d} \frac{\tau_{ij}}{\partial x_i}\right]_{j=1}^{d},$$

respectively.

2.2 Inequalities

In the thesis, we use the following inequalities.

– For $a, b \in \mathbb{R}$ and for any $\alpha > 0$ the general Young inequality reads as follows:

$$ab \leq \frac{1}{p}(\alpha a)^p + \frac{1}{p'}\left(\frac{b}{\alpha}\right)^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{3}$$

– For any functional $\mathcal{F} : V \to \mathbb{R}$ and its convex conjugate $\mathcal{F}^* : V^* \to \mathbb{R}$, the generalized Fenchel inequality reads

$$< v^*, v >_{V^*\times V} \leq \mathcal{F}^*(v^*) + \mathcal{F}(v), \ \ \forall v^* \in V^*, \forall v \in V. \tag{4}$$

– When Young and Fenchel inequalities are used in combination, then we sometimes use the term Young-Fenchel inequality.

– For all integrable functions f, g the Hölder inequality reads

$$\int_\Omega fg \, \mathrm{d}x \leq \|f\|_{L^p} \|g\|_{L^{p'}}, \ \ \forall f \in L^p(\Omega), \forall g \in L^{p'}(\Omega), \ \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{5}$$

– As a special case of the Hölder inequality we have the Cauchy-Schwarz inequality
$$\int_{\Omega} fg \, dx \leq \|f\| \, \|g\| \quad \forall f, g \in L^2(\Omega). \tag{6}$$
– The Friedrichs inequality
$$\|w\| \leq C_F \|\nabla w\|, \ \ \forall w \in \mathring{H}^1(\Omega), \tag{7}$$
where $C_F > 0$ is a constant, independent of the function w.
– The classic trace inequality reads
$$\|\phi\|_\Gamma \leq C_\Gamma \|\nabla \phi\| \ \ \forall \phi \in H^1(\Omega). \tag{8}$$

Lemma 2.2.1 *[43, 44] Let Ω be a bounded domain with Lipschitz continuous boundary. There exists a constant $C_{LBB} > 0$ depending only on the domain Ω, such that for any function $g \in \widetilde{L}_2(\Omega)$ there is a function $v \in V_0$ satisfying the condition* $\mathrm{div} v = g$ *and*
$$\|\nabla v\| \leq C_{LBB}^{-1} \|g\|.$$

Here the constant C_{LBB} is the constant in the well-known Ladyzhenskaya-Babuška-Brezzi (LBB) condition.

Above lemma leads to important condition in theory of incompressible fluids. The so called Inf-Sup (or Ladyzhenskaya-Babuška-Brezzi (LBB)) condition reads: There exists a constant $C_{LBB} > 0$ such that
$$\inf_{q \in \widetilde{L}_2(\Omega),\, q \neq 0} \ \sup_{w \in V,\, w \neq 0} \frac{\int_{\Omega} q \, \mathrm{div} w \, dx}{\|q\| \|\nabla w\|} \geq C_{LBB}. \tag{9}$$

The LBB condition and its discrete analogs are in proving the convergence and stability of numerical methods in problems related to the theory of viscous incompressible fluids. The LBB condition can also be justified by the Necas inequality (for domains with Lipschitz boundaries [12]). For estimates of C_{LBB} for various domains, see, e.g., [24, 56, 58].

3 BASIC MATHEMATICAL MODELS OF VISCOUS FLUIDS

It is commonly accepted that evolution of a generalized Newtonian fluid is described by the Navier-Stokes differential equation of motion

$$u_t - \mathrm{Div}\sigma + (u \cdot \nabla u) = f - \nabla p \quad \text{in } \Omega, \tag{10}$$

incompressibility condition

$$\mathrm{div} u = 0 \quad \text{in } \Omega, \tag{11}$$

and the differential inclusion

$$\sigma \in \partial \pi(\nabla u) \tag{12}$$

that reflects mechanical properties of the fluid. In (12) π denotes the dissipative potential of the fluid and ∂ denotes the subdifferential. In addition, the system (10)-(12) should be supplied with boundary and initial conditions. See, e.g., [25] for mathematical details.

3.1 The generalized Stokes problem

The Stokes model is one of the simplest models in the theory of viscous incompressible fluids. The generalized formulations of the Stokes problem are often motivated by semi-discrete formulations of evolutionary problems. In these formulations $u(x,t)$ is represented as a sequence of approximations $u^n(x) := u(x, t_n)$, where t_n, $n = 0, 1, \ldots, N$ are values of the time variable t. Here we consider a scheme

$$\frac{u^n - u^{n-1}}{t_n - t_{n-1}} - \mathrm{Div}(\nu \nabla u^n) + \mathrm{Div}(u^{n-1} \otimes u^{n-1}) = f - \nabla p \text{ in } \Omega$$

$$\mathrm{div} u^n = 0 \text{ in } \Omega.$$

This scheme differs from the Stokes problem by the term μu, where we have $\mu = \frac{1}{t_n - t_{n-1}}$.

We consider the (stationary) generalized Stokes problem described as

$$-\mathrm{Div}(\nu\nabla u) + \mu u = f - \nabla p \ \text{ in } \Omega, \tag{13}$$

$$\mathrm{div} u = 0 \ \text{ in } \Omega, \tag{14}$$

$$u = u_D \ \text{ on } \Gamma_D, \tag{15}$$

$$\sigma n = F \ \text{ on } \Gamma, \tag{16}$$

where $\nu > 0$ is the viscosity, n denotes the unit outward normal vector to the boundary, p is the pressure, $f \in L^2(\Omega, \mathbb{R}^d)$, and $\sigma = \nu\nabla u$ is the stress tensor. The function u_D defines the Dirichlét boundary condition and satisfies

$$\int_{\partial\Omega} u_D \cdot n \, \mathrm{d}S = 0.$$

The system (13)-(16) describes a slow motion of an incompressible fluid. We assume that

$$0 < \underline{\nu} \leq \nu(x) \leq \bar{\nu}; \ \ \forall x \in \bar{\Omega}$$

and

$$0 \leq \underline{\mu} \leq \mu(x) \leq \bar{\mu}; \ \ \forall x \in \bar{\Omega}.$$

The generalized solution of (13)-(15) is a vector valued function $u \in V_0 + u_D$ satisfying the integral identity

$$\int_\Omega (\nu\nabla u : \nabla w + \mu u \cdot w) \, \mathrm{d}x = \int_\Omega (f \cdot w + p \mathrm{div} w) \, \mathrm{d}x, \ \ \forall w \in V_0(\Omega). \tag{17}$$

Notice, that u can also be defined as a divergence free (solenoidal) vector function satisfying

$$\int_\Omega (\nu\nabla u : \nabla w + \mu u \cdot w) \, \mathrm{d}x = \int_\Omega f \cdot w \, \mathrm{d}x, \ \ \forall w \in \mathbb{S}_0(\Omega).$$

3.1.1 Nonlinear boundary conditions

In the second paper, we consider the system (13)-(15) with the nonlinear boundary condition on Γ:

$$u_n = 0, \quad -\sigma_t \in g\partial|u_t| \text{ on } \Gamma, \tag{18}$$

where $g \geq 0$ is a constant and u_n, u_t are the normal and tangential components of u and

$$\sigma_n := \sigma_n n, \ \ \sigma_t := \sigma_{nn} n, \ \ \sigma_{nn} := \sigma_n \cdot n = \sigma_{ij} n_i n_j.$$

This boundary condition is equivalent to a friction type condition

$$|\sigma_n| \leq g, \ \ \sigma_n u_n + g|u_n| = 0 \text{ on } \Gamma.$$

The generalized solution $u \in \mathbb{S}_0(\Omega) + u_D$ of (13)-(15),(18) can be defined by the variational inequality (see [31])

$$a(u, v - u) + \int_\Gamma (j(v_t) - j(u_t)) \, \mathrm{d}S \geq (f, v - u) \ \ \forall v \in \mathbb{S}_0(\Omega), \tag{19}$$

where $j(\eta) := g|\eta|$ for $\eta \in H^{1/2}(\Gamma)$.

3.2 The Oseen problem

The Oseen problem can be viewed as a linearization of the Navier-Stokes problem at a neighborhood of a constant velocity field $a \in \mathbb{S}(\Omega)$. The Oseen system is often used in quantative analysis of the Navier-Stokes problem. The classical formulation of the stationary Oseen problem is to find the velocity field $u \in \mathbb{S}(\Omega) + u_D$ and the pressure function $p \in \widetilde{L}_2(\Omega)$, which satisfy the relations

$$-\mathrm{Div}(\nu\nabla u) + \mathrm{Div}(a \otimes u) = f - \nabla p \ \text{ in } \Omega, \tag{20}$$

$$\mathrm{div} u = 0 \ \text{ in } \Omega, \tag{21}$$

$$u = u_D \ \text{ on } \partial\Omega, \tag{22}$$

where a, u_D, and f are given vector valued functions. It is assumed that

$$\int_{\partial\Omega} u_D \cdot n \,\mathrm{d}x = 0, \tag{23}$$

ν is a positive bounded function for all $x \in \bar{\Omega}$, and $a \in \mathbb{S}(\Omega)$ is a bounded vector function. The generalized solution of (20)-(23) is a function $u \in \mathbb{S}(\Omega) + u_D$ such that

$$\int\limits_{\Omega} (\nu\nabla u : \nabla w - (a \otimes u) : \nabla w) \,\mathrm{d}x = \int\limits_{\Omega} f \cdot w \,\mathrm{d}x \ \ \forall w \in \mathbb{S}(\Omega). \tag{24}$$

Existence and uniqueness of generalized solution to Oseen problem is well established (see, e.g., [43])

3.3 The Bingham model

Models of fluids with nonlinear viscosity are commonly used in natural sciences and engineering applications [18, 32, 33]. Physically interesting models are described by the dissipative potential in the form

$$\pi(\varepsilon) = \frac{1}{m}\nu|\varepsilon|^m + k_*|\varepsilon|, \tag{25}$$

where $m > 1$ is associated with the energy growth at infinity, ν is the viscosity parameter, and $k_* \geq 0$ is the plasticity parameter. This type of models are called Bingham models. The most known Bingham fluid model is described by the potential

$$\pi(\varepsilon) = \frac{1}{2}\nu|\varepsilon|^2 + k_*|\varepsilon|, \quad k_* > 0. \tag{26}$$

This type of models are often used for computer simulation of the blood flow, see, e.g., [18, 20, 29].

In the thesis, we consider a stationary anti-plane flow of Bingham type in a long domain $\Omega\times]0,L[$, with the potential

$$\pi(\varepsilon) = \frac{1}{2}A\varepsilon : \varepsilon + k_*|\varepsilon|.$$

Here $k_* > 0$ and A is a symmetric matrix satisfying the condition

$$c_1^2|\xi|^2 \leq A\xi\cdot\xi \leq c_2^2|\xi|^2 \quad \forall\xi\in\mathbb{R}^d, c_2 \geq c_1.$$

In this case (see [25, 35]), the problem can be presented as a variational inequality: Find the velocity $u \in V_0 + u_D$ such that

$$a(u, w-u) + \int\limits_\Omega (j(\nabla w) - j(\nabla u))\,\mathrm{d}x \geq \ell(w-u) \quad \forall w \in V_0 + u_D, \tag{27}$$

where $u_D \in V$ is defines Dirichlet type boundary conditions, $a : V\times V \to \mathbb{R}$ is a bilinear V-elliptic form

$$a(v,w) := \int\limits_\Omega A\nabla v\cdot\nabla w\,\mathrm{d}x,$$

and $j : \mathbb{R}^d \to \mathbb{R}$ is a convex continuous function, and is defined as

$$j(\nabla w) = k_*|\nabla w|.$$

The inequality (27) is equivalent to the following variational problem: Find $u \in V_0 + u_D$ such that

$$J(u) = \inf_{w\in V_0+u_D} J(w), \tag{28}$$

where

$$J(w) := \frac{1}{2}a(w,w) + \int\limits_\Omega j(\nabla w)\,\mathrm{d}x - \ell(w).$$

Due to well known results in the theory of variational calculus, the problem (28) associated with a strictly convex and lower semicontinuous functional is uniquely solvable (see, e.g., [25]).

3.4 The Uzawa algorithm

Uzawa type methods form a well-known and easily implemented class of iterative methods for solving saddle-point problems that arise, for example, in fluid dynamics. Other applications include constrained optimization, linear elasticity, and economics (see a survey of the use of the Uzawa algorithm in various saddle point problems [7]). Originally, the classical Uzawa algorithm was proposed in [2]. The classical Uzawa algorithm also many modifications and generalizations, that include inexact Uzawa algorithm (see, e.g., [13, 15, 26, 41, 45, 46, 59, 78]) and augmented Lagrangian (see, e.g., [8, 36, 47]).

3.4.1 Uzawa algorithm for Stokes and Oseen problems

In the case of generalized Stokes problem and Oseen problem, the Uzawa algorithm is often used to avoid difficulties with exact satisfaction of the divergence-free condition. It can be used in the following form:

1. Set $k = 0$ and $p^0 \in \widetilde{L}_2(\Omega)$.
2. Find $u^k \in V_0(\Omega) + u_D$ such that

$$\int_\Omega (\nu \nabla u^k : \nabla w + \mu u^k \cdot w - (a \otimes u^k) : \nabla w)\, dx = \int_\Omega (f \cdot w + p^k \mathrm{div} w)\, dx \quad \forall w \in V_0. \tag{29}$$

3. Find
$$p^{k+1} = p^k - \rho \mathrm{div} u^k, \text{ where } \rho \in (0, \bar{\rho}). \tag{30}$$
4. Set $k = k + 1$ and go to step 2.

In the above algorithm, equation (29) corresponds to generalized Stokes problem by choosing $a = 0$ and to Oseen problem by choosing $\mu = 0$. It is well known (see, e.g., [73]) that approximations generated by the Uzawa algorithm converge in the sense that

$$u^k \longrightarrow u \in V(\Omega, \mathbb{R}^d),$$
$$p^k \longrightarrow p \text{ weakly in } L^2(\Omega),$$

as $k \to \infty$, provided that

$$0 < \rho < \bar{\rho} := \begin{cases} 2\underline{\nu} & \text{if } \mu \equiv 0 \\ 2 \min \left\{ \underline{\mu}, \underline{\nu} \right\} & \text{otherwise.} \end{cases}$$

3.4.2 Uzawa algorithm for the anti-plane Bingham flow problem

For the anti-plane Bingham problem (27), numerical approximations can be constructed by different methods. Many of them are discussed in [14, 34, 35, 38] (see also publications cited therein). In the thesis, we consider the simplest form of the Uzawa method.

1. Set $k = 0$. Define ρ satisfying (34) and
$$\lambda^0 \in K := \{\lambda \in L^\infty(\Omega) \;:\; |\lambda| \leq 1\}. \tag{31}$$

2. Find $u^{k+1} \in V_0$ as a generalized solution of the problem
$$\int_\Omega (A \nabla u^{k+1} \cdot \nabla w + k_* \lambda^k \cdot \nabla w)\, dx = \int_\Omega f w\, dx, \;\; \forall w \in V_0. \tag{32}$$

3. Find
$$\lambda^{k+1} = \Pi(\lambda^k + \rho \nabla u^{k+1}), \tag{33}$$
where $\Pi : L^\infty \to K$ is the projection operator on the set K
$$\Pi\lambda(x) = \frac{\lambda(x)}{\max(1, |\lambda(x)|)}, \text{ a.e. in } \Omega.$$

4. Set $k = k + 1$ and go to step 2.

It is well known (see, e.g., [22]) that approximations generated by algorithm described above converge to the exact solution as $k \to \infty$ in the sense that
$$u^k \longrightarrow u \text{ in } V$$
provided that
$$0 < \rho < \bar{\rho} := \frac{2c_1}{k_*^2}. \tag{34}$$
The algorithms (29)-(30) and (32)-(33) generate infinite sequences $\{u^k\}$. In practical computations, we append them with certain stopping criteria that follow from a posteriori estimates that control the accuracy of $\{u^k\}$.

4 MAIN RESULTS

In the field of computational hydrodynamics, the error control problems have gained a lot of interest, especially in solving problems related to incompressibility condition.

In the framework of residual method, a posteriori error estimates for the Stokes problem has been studied in numerous paper (see, for example [1, 5, 16, 21, 39, 42, 55, 57, 74, 75]). Computable error estimates, that do not contain constants arising for solution methods or domain discrezation, we studied in [61, 64]. For elliptic type problems of the divergent type, estimates were obtained in [60, 61, 64]. These estimates utilize duality theory in the calculus of variations (see [51]). Another method, based on transformations of the integral identities, was presented in [63, 64, 65, 66].

Functional type a posteriori error estimates for Stokes problem were derived in [63, 65]. The practical efficiency of these estimates was studied in [37]. For nonlinear boundary conditions, the error estimates were studied in [52, 72]. In [9, 30, 71] similar type of estimates were derived for classes of generalized Newtonian fluids. For considered type of Bingham fluids, a posteriori error estimates were studied in [62, 66, 68].

In the thesis, we focused on finding explicitly computable error estimates that are realistic and generated by Uzawa iterations. These estimates take into account the approximation errors that are caused by the dicsretization of the boundary value problem.

4.1 Upper bounds generated by the Uzawa algorithm for Oseen problem

Guaranteed upper bounds of the distance to the exact solution of the Oseen problem were derived in [66, 69]. Following the methods presented there, we obtain certain form of these estimates, which are suitable for the Uzawa algorithm. The respective proofs and more details are presented in the paper [PIII]. Below, we explain the principal ideas with the paradigm of one simplest case.

Theorem 4.1.1 *Let u be the exact solution of the problem* (20)-(22) *with pure Dirichlét boundary conditions. Let $v \in V_0(\Omega) + u_D$. Then for any $q \in \widetilde{L}_2(\Omega)$ and $\tau \in \Sigma(\Omega)$ we have*

$$\| u - v \| \leq \underline{\nu}^{-1/2} \|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v,\tau,q)\|_\Sigma \\ + (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\mathrm{div} v\| := M_\oplus(v,\tau,q). \quad (35)$$

where

$$r(\tau) := f + \mathrm{Div}\tau, \qquad C_\Omega = C_{F\Omega}\|\nu^{-1/2} a\|_{\infty,\Omega}, \\ d(v,\tau,q) := \tau - \nu\nabla v + a \otimes v + \mathbb{I}q,$$

and

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - \tau : \nabla w)\, \mathrm{d}x}{\|\nabla w\|_\Sigma}.$$

It is easy to see that the error majorant consists of residuals of two basic equations forming the Oseen equation and a penalty term that penalizes violations of the divergence free condition.

Now we want to use (35) and deduce a computable and realistic estimate of $u^k - u$. We use Theorem 4.1.1 with

$$v = u^k, \quad (36)$$
$$q = p^k, \quad (37)$$
$$\tau = \nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k. \quad (38)$$

In this case,

$$d(v,\tau,q) := \tau - \nu\nabla u^k + a \otimes u^k + \mathbb{I}p^k = 0$$

and in view of (29),

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - (\nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k) : \nabla w)\, \mathrm{d}x}{\|\nabla w\|_\Sigma} = 0. \quad (39)$$

Thus we arrive at the following result.

Theorem 4.1.2 *Let u^k be the exact solution computed on the step k of the Uzawa algorithm. Then*

$$\| u - u^k \| \leq (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\mathrm{div} u^k\| := M_\oplus^{Uz}(u^k). \quad (40)$$

This estimate shows the principal behavior of the error associated with the Uzawa solutions. It is simply the value of $\|\mathrm{div} u^k\|$ with the factor depending on viscosity and constant $C_{F\Omega}$

4.2 Advanced upper bounds

If we would have the function u^k (exact solution of the boundary value problem in the first step of the Uzawa method), then the estimate in Theorem 4.1.2 would give a complete answer to the question stated. However, in practice, the problem is solved on a certain mesh $\mathcal{T}_h$ (h is considered as a characteristic size of cell of the mesh). In order to take the approximation error into account, we need an advanced form of the estimate.

Let $V_{0h}(\Omega, \mathbb{R}^d)$ and $\widetilde{L}_{2h}(\Omega)$ be a finite dimensional subspaces of $V_0(\Omega)$ and $\widetilde{L}_2(\Omega)$, respectively. It is assumed, that these spaces are constructed so that the corresponding numerical problem is stable and satisfies the discrete LBB-condition. Let $u_h^k \in V_{0h} + u_D$ be an approximation of u^k and $p_h^k \in \widetilde{L}_{2h}(\Omega)$ be an approximation of p^k. We have

Theorem 4.2.1 *For any $\eta \in \Sigma(\Omega, \mathrm{Div})$,*

$$\| u - u_h^k \| \leq \mathcal{E}^h(u_h^k, p_h^k, \eta) + M_{\oplus}^{Uz}(u_h^k), \tag{41}$$

where the first term

$$\mathcal{E}(u_h^k, p_h^k, \eta) = \underline{\nu}^{-1/2}(C_{F\Omega}\|r(\eta)\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma)$$

is related to the approximation error and the second term presents the error associated with the Uzawa method.

We see that the both terms are indeed computable and the numerical applicability of this technology was verified. For an arbitrary domain Ω, the constant κ_Ω may be difficult to define. However, for plane domains that are star-shaped with respect to a ball, good upper bounds have been derived, see [19]. For arbitrary domains, it was shown (see [66, 67]) that computation of the distance to the set of divergence free fields can be performed by ideas of domain decomposition using constants for local (simple) subdomains.

In [PIII] we presented the results from numerical computations to test the majorants. The computations were performed with the help of the FEniCS Project open source software [28]. Indeed, we see that the estimates provide guaranteed upper bounds of error and the bounds reflect correctly the decrease of the corresponding errors and also indicate the moment when the mesh adaptation is required.

YHTEENVETO (FINNISH SUMMARY)

A posteriori virhearvioita variaatio-ongelmille viskoottisten nesteiden tapauksessa

Matemaattiset mallit luovat tärkeän tavan tulkita luonnonilmiöitä. Erityisesti virtausmekaniikka on tärkeässä roolissa fysikaalisia ilmiöitä mallinnettaessa. Virtauksen mallintaminen on tärkeää esimerkiksi lääketieteessä, meteorologiassa, aerodynamiikassa ja putkistosuunnittelussa. Virtauksen mallintamiseen käytettävät yhtälöt ovat yleensä osittaisdifferentiaaliyhtälöitä.

Vaikka osittaisdifferentiaaliyhtälöiden tutkimus on edennyt viime vuosisadalla hyvinkin nopeasti, on tarkkojen ratkaisujen löytäminen vaikeaa, yleensä jopa mahdotonta. Tapauksissa, joissa analyyttinen ratkaisu voidaan löytää kohtuullisella vaivalla, on yleensä tehty huomattavia yksinkertaistuksia, tai yhtälö on alkujaankin keinotekoinen. Näiden yksinkertaistusten myötä menetetään mallinnuksen tarkkuus. Jotta edellämainituilta yksinkertaistuksilta vältyttäisiin, on osittaisdifferentiaaliyhtälöille kehitetty monia numeerisia ratkaisumenetelmiä, jotka tuottavat osittaisdifferentiaaliyhtälölle numeerisen ratkaisun. Numeerisen ratkaisuun sisältyy aina jonkunlainen virhe, ja näinollen on olennaista tietää kuinka suuri tämä virhe on.

Tässä väitöskirjassa keskitytään tutkimaan funktionaalisia a posteriori -virhearvioita. Väitöskirjaan sisällytetyissä artikkeleissa käsitellään virhearvioita ajasta riippumattomille Stokesin, Oseenin ja Binghamin yhtälöille. Kolmessa julkaisussa sovelletaan homogenisiä reunaehtoja ja numeerisia ratkaisuja tuotetaan Uzawan algoritmin avulla. Uzawan algoritmin erityispiirteitä hyödyntäen virhearvioille saadaan hyvinkin yksinkertainen muoto. Neljännessä julkaisussa mahdollistetaan ratkaisun epälineaarinen käytös alueen reunalla Stokesin yhtälön tapauksessa. Esitetyt virhearviot edustavat kahta eri tyyppiä. Ensimmäisen tyypin virhearviot ottavat huomioon Uzawan algoritmin erityispiirteet ja niillä on lähinnä teoreettinen merkitys. Toisen typpiset virhearviot ottavat huomioon myös approksimaatiovirheen, mikä tekee niistä täysin laskettavissa olevia.

Tässä väitöskirjassa esitellyt virhearviot tarjoavat aina todellisen ylärajan virheelle. Virhearviot ovat myös täysin laskettavissa, eli kaikki esiintyvät muuttujat ovat tiedossa. Tässä työssä tehdyt numeeriset kokeet suoritettiin FEniCSin Python-rajapintaa hyödyntäen.

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ON A POSTERIORI ERROR BOUNDS FOR APPROXIMATIONS OF THE GENERALIZED STOKES PROBLEM GENERATED BY THE UZAWA ALGORITHM

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On *a posteriori* error bounds for approximations of the generalized Stokes problem generated by the Uzawa algorithm

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Abstract — In this paper, we derive computable *a posteriori* error bounds for approximations computed by the Uzawa algorithm for the generalized Stokes problem. We show that for each Uzawa iteration both the velocity error and the pressure error are bounded from above by a constant multiplied by the L_2-norm of the divergence of the velocity. The derivation of the estimates essentially uses *a posteriori* estimates of the functional type for the Stokes problem.

1. Introduction

Let $\Omega \in \mathbb{R}^n$ be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$. Henceforth, we use the space of vector valued functions

$$V(\Omega,\mathbb{R}^n) := W_2^1(\Omega,\mathbb{R}^n)$$

and two spaces of tensor-valued functions

$$\Sigma(\Omega) := L_2(\Omega,\mathbb{M}^{n\times n})$$
$$\Sigma(\mathrm{Div},\Omega) := \{w \in \Sigma(\Omega) \mid \mathrm{Div}\, w \in L_2(\Omega,\mathbb{R}^n)\}$$

where $\mathbb{M}^{n\times n}$ is the space of symmetric $n\times n$-matrices (tensors). The scalar product of tensors is denoted by two dots (:), and the L_2 norm of Σ is denoted by $\|\cdot\|_\Sigma$. The L_2 norm of scalar and vector valued functions is denoted by $\|\cdot\|$.

By $\mathring{S}(\Omega)$ we denote the closure of smooth solenoidal functions w with compact supports in Ω with respect to the norm $\|\nabla w\|_\Sigma$. Let $V_0(\Omega,\mathbb{R}^n)$ denote the subspace

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of $V(\Omega,\mathbb{R}^n)$ that consists of functions with zero traces on $\partial\Omega$. The space of scalar valued square summable functions with zero mean is denoted by $\widetilde{L}_2(\Omega,\mathbb{R})$.

The classical statement of the generalized Stokes problem consists of finding a velocity field $u \in \mathring{S}(\Omega) + u_D$ and pressure $p \in \widetilde{L}_2(\Omega)$ which satisfy the relations

$$-\mathrm{Div}\,(\nu\nabla u) + \mu u + \nabla p = f \quad \text{in } \Omega \tag{1.1}$$

$$\mathrm{div}\, u = 0 \quad \text{in } \Omega \tag{1.2}$$

$$u = u_D \quad \text{on } \partial\Omega \tag{1.3}$$

where $f \in L_2(\Omega,\mathbb{R}^n)$, and

$$\int_{\partial\Omega} u_D \cdot n \,\mathrm{d}x = 0.$$

Here and later on n denotes the outward unit normal vector to $\partial\Omega$, and we assume that the material parameters ν and μ belong to the space $L_\infty(\Omega,\mathbb{R})$, and

$$0 < \underline{\nu} \leqslant \nu(x) \leqslant \overline{\nu}, \quad \forall x \in \overline{\Omega}$$
$$0 \leqslant \underline{\mu} \leqslant \mu(x) \leqslant \overline{\mu}, \quad \forall x \in \overline{\Omega}.$$

The generalized solution of (1.1)–(1.3) is a function $u \in \mathring{S}(\Omega) + u_D$ such that

$$\int_\Omega (\nu\nabla u : \nabla w + \mu u \cdot w)\,\mathrm{d}x = \int_\Omega f \cdot w \,\mathrm{d}x \quad \forall w \in \mathring{S}(\Omega). \tag{1.4}$$

It is well known that u can be defined as the first component of the saddle point problem generated by any of the Lagrangians

$$L(v,q) := \int_\Omega \left(\frac{1}{2}\nu|\nabla v|^2 + \frac{1}{2}\mu|v|^2 - q\,\mathrm{div}\, v - f\cdot v\right)\mathrm{d}x$$
$$L_A(v,q) := \int_\Omega \left(\frac{1}{2}\nu|\nabla v|^2 + \frac{1}{2}\mu|v|^2 + \frac{1}{2}\lambda\,|\mathrm{div}\, v|^2 - q\,\mathrm{div}\, v - f\cdot v\right)\mathrm{d}x.$$

The quantity in L_A is called the augmented Lagrangian (in which $\lambda \in \mathbb{R}_+$). We have

$$L(v,p) \leqslant L(u,p) \leqslant L(u,q) \qquad \forall v \in V_0 + u_D,\ q \in L_2$$
$$L_A(v,p) \leqslant L_A(u,p) \leqslant L_A(u,q) \quad \forall v \in V_0 + u_D,\ q \in L_2.$$

From the right-hand side inequalities we see that $\int_\Omega (p-q)\mathrm{div}\, u \,\mathrm{d}x = 0$ for all $q \in L_2$, from which we conclude that $\mathrm{div}\, u = 0$. From the left-hand side inequalities it follows that for any solenoidal v we have $J(v) > J(u)$, where

$$J(v) := \int_\Omega \left(\frac{1}{2}\nu|\nabla v|^2 + \frac{1}{2}\mu|v|^2 - f\cdot v\right)\mathrm{d}x.$$

Indeed, the exact solution of the problems

$$\inf_{v\in V_0+u_D}\sup_{q\in L_2} L(v,q), \qquad \inf_{v\in V_0+u_D}\sup_{q\in L_2} L_A(v,q)$$

is (u,p). For a detailed exposition of this subject, we refer to [4].

Finding approximations of (u,p) can be performed by the Uzawa algorithm presented below.

Algorithm 1.1 (Uzawa algorithm).

1: Set k=0 and $\rho\in\mathbb{R}_+$. Make initial guess for $p^k\in\widetilde{L}_2$.

2: Find u^k by minimizing the Lagrangian $L(v,p^k)$ or $L_A(v,p^k)$ w.r.t. v, i.e., by solving either (1.5) or (1.6), respectively.

For the Lagrangian L, we have the problem: Find $u^k\in V_0+u_D$ such that:

$$\int_\Omega\Big(\nu\nabla u^k:\nabla w+\mu u^k\cdot w\Big)\,\mathrm{d}x=\int_\Omega\Big(f\cdot w+p^k\operatorname{div}w\Big)\,\mathrm{d}x\quad\forall w\in V_0. \tag{1.5}$$

For the Lagrangian L_A, we have the problem: Find $u^k\in V_0+u_D$ such that:

$$\int_\Omega\Big(\nu\nabla u^k:\nabla w+\mu u^k\cdot w+\lambda\operatorname{div}u^k\operatorname{div}w\Big)\,\mathrm{d}x$$
$$=\int_\Omega\Big(f\cdot w+p^k\operatorname{div}w\Big)\,\mathrm{d}x\quad\forall w\in V_0. \tag{1.6}$$

3: Find

$$p^{k+1}=(p^k-\rho\operatorname{div}u^k)\in\widetilde{L}_2. \tag{1.7}$$

4: Set $k=k+1$ and go to step 2.

Our goal is to deduce computable bounds of the difference between u^k and the exact solution u in terms of the energy norms

$$|\!|\!|\, w\,|\!|\!|^2:=\int_\Omega\big(\nu|\nabla w|^2+\mu|w|^2\big)\,\mathrm{d}x$$

and

$$]\!|\!|\, w\,|\!|\!|_\lambda^2:=\int_\Omega\big(\nu|\nabla w|^2+\mu|w|^2+\lambda|\operatorname{div}w|^2\big)\,\mathrm{d}x.$$

Theorem 1.1. *The Uzawa algorithm (Algorithm* 1.1*) converges, i.e.,*

$$u^k \xrightarrow{k\to\infty} u \qquad \textit{strongly in } V(\Omega,\mathbb{R}^n)$$
$$p^k \xrightarrow{k\to\infty} p \qquad \textit{weakly in } L_2(\Omega)$$

provided that

$$0<\rho<2\min(\underline{\nu},\underline{\mu}) \tag{1.8}$$

and $p^0\in\widetilde{L}_2(\Omega)$*. If* $\mu\equiv0$*, the condition is* $0<\rho<2\underline{\nu}$*. These conditions are the same for both* (1.5) *and* (1.6)*.*

Proof. The proof is based on well known arguments (see, e.g., [13]). However, for the convenience of the reader, we present the proof for the generalized Stokes problem, in the case of (1.5).

The exact solution of the generalized Stokes problem satisfies the relation

$$\int_\Omega (\nu\nabla u : \nabla w + \mu u \cdot w)\,\mathrm{d}x = \int_\Omega (f\cdot w + p\,\mathrm{div}\,w)\,\mathrm{d}x \quad \forall w \in V_0(\Omega). \tag{1.9}$$

We set $w = u^k - u$ and subtract (1.9) from (1.5), which gives

$$|||\, u^k - u \,|||^2 = \int_\Omega (p^k - p)\mathrm{div}(u^k - u)\mathrm{d}x.$$

Let $v^k := u^k - u$ and $q^k := p^k - p$. Then we rewrite this relation in the form

$$|||\, v^k \,|||^2 = \int_\Omega q^k \mathrm{div}\, v^k \,\mathrm{d}x. \tag{1.10}$$

On the other hand, (1.7) is equivalent to

$$\int_\Omega (p^{k+1} - p^k)\phi \,\mathrm{d}x + \rho \int_\Omega \mathrm{div}\, u^k \phi \,\mathrm{d}x = 0 \quad \forall \phi \in L_2(\Omega).$$

By setting $\phi = p^{k+1} - p$ we obtain

$$\int_\Omega (p^{k+1} - p^k)(p^{k+1} - p)\mathrm{d}x + \rho \int_\Omega \mathrm{div}\, u^k (p^{k+1} - p)\mathrm{d}x = 0$$

which is equivalent to

$$\int_\Omega (q^{k+1} - q^k) q^{k+1} \,\mathrm{d}x + \rho \int_\Omega \mathrm{div}\, v^k q^{k+1} \,\mathrm{d}x = 0$$

and

$$\|q^{k+1}\|^2 - \|q^k\|^2 + \|q^{k+1} - q^k\|^2 = -2\rho \int_\Omega \mathrm{div}\, v^k q^{k+1} \,\mathrm{d}x. \tag{1.11}$$

By combining (1.10) and (1.11), we obtain

$$\begin{aligned}
\|q^{k+1}\|^2 - \|q^k\|^2 + \|q^{k+1} - q^k\|^2 + 2\rho \,|||\, v^k \,|||& \\
&= -2\rho \int_\Omega \mathrm{div}\, v^k (q^{k+1} - q^k)\mathrm{d}x \\
&\leqslant 2\rho \|\mathrm{div}\, v^k\| \, \|q^{k+1} - q^k\| \\
&\leqslant \delta^{-1}\rho^2 \|\mathrm{div}\, v^k\|^2 + \delta \|q^{k+1} - q^k\|^2 \\
&\leqslant \delta^{-1}\rho^2 \left(\|\nabla v^k\|^2_\Sigma + \|v^k\|^2\right) + \delta \|q^{k+1} - q^k\|^2
\end{aligned} \tag{1.12}$$

where $\delta \in (0,1)$. Note that

$$|\!|\!| v^k |\!|\!|^2 \geqslant \underline{\nu}\|\nabla v^k\|_\Sigma^2 + \underline{\mu}\|v^k\|^2 \ \geqslant \ \min(\underline{\nu},\underline{\mu})\left(\|\nabla v^k\|_\Sigma^2 + \|v^k\|^2\right)$$

and, therefore, (1.12) implies the estimates

$$\|q^{k+1}\|^2 - \|q^k\|^2 + (1-\delta)\|q^{k+1}-q^k\|^2 \\ + \rho\left(2\min(\underline{\nu},\underline{\mu}) - \delta^{-1}\rho\right)\left(\|\nabla v^k\|_\Sigma^2 + \|v^k\|^2\right) \leqslant 0. \tag{1.13}$$

Now, we sum inequalities (1.13) for $k = 0,\dots,N$ and find that

$$\|q^{N+1}\|^2 + (1-\delta)\sum_{k=0}^{N}\|q^{k+1}-q^k\|^2 \\ + \rho\left(2\min(\underline{\nu},\underline{\mu}) - \delta^{-1}\rho\right)\sum_{k=0}^{N}\left(\|\nabla v^k\|_\Sigma^2 + \|v^k\|^2\right) \leqslant \|q^0\|. \tag{1.14}$$

Because of condition (1.8), there exists a $\delta_* \in (0,1)$ such that

$$2\min(\underline{\nu},\underline{\mu}) - \delta_*^{-1}\rho > 0.$$

We set $\delta = \delta_*$ in (1.14), and see that

$$\|\nabla v^k\|_\Sigma^2 + \|v^k\|^2 = \|\nabla(u^k - u)\|_\Sigma^2 + \|u^k - u\|^2 \ \stackrel{k\to\infty}{\longrightarrow} \ 0.$$

Also, we see that $\|q^k\| = \|p^k - p\|$ is bounded in $L_2(\Omega)$, so $\|p^k\|$ is bounded in $L_2(\Omega)$. We also observe from (1.14), that

$$\|q^{k+1} - q^k\|^2 = \|p^{k+1} - p^k\|^2 \ \stackrel{k\to\infty}{\longrightarrow} \ 0$$

so we can extract from p^k a subsequence $p^{k'}$, which converges to some element p^* weakly in $L_2(\Omega)$. The equation (1.5) gives in the limit

$$\int_\Omega (\nu\nabla u : \nabla w + \mu u\cdot w)\,\mathrm{d}x = \int_\Omega (f\cdot w + p^*\operatorname{div} w)\,\mathrm{d}x \quad \forall w \in V_0$$

and by comparison to (1.9) we find that

$$\int_\Omega (p - p^*)\operatorname{div} w\,\mathrm{d}x = 0 \quad \forall w \in V_0$$

which means that $p^* = p + c$, where $c \in \mathbb{R}$. In other words, the sequence $p^{k'}$ converges weakly to p in $\widetilde{L}_2(\Omega)$ However, if $p^0 \in \widetilde{L}_2$, then it is easy to see from (1.7) that $p^k \in \widetilde{L}_2$ with all k. From this we make the conclusion that the sequence $p^{k'}$ converges weakly to p in $L_2(\Omega)$.

2. Error estimates for exact solutions generated by the Uzawa algorithm

In this section, we show that the errors of approximations generated by the Uzawa algorithm are controlled by the L_2-norm of the divergence of the velocity. First, we compare approximations computed on two consequent iterations and establish the following result.

Theorem 2.1. *Let* (u^k, p^k) *and* (u^{k+1}, p^{k+1}) *be the solutions of two consecutive iterations of the Uzawa algorithm. Then, for both* (1.5) *and* (1.6) *we have*

$$|\!|\!|\, u^{k+1} - u^k \,|\!|\!| \;\leqslant\; \sqrt{\underline{\nu}}^{-1} \rho \|\mathrm{div}\, u^k\| \tag{2.1}$$

$$\|p^{k+1} - p^k\| \;=\; \rho \|\mathrm{div}\, u^k\|. \tag{2.2}$$

In addition, for (1.6) *we also have*

$$]\!|\!|\, u^{k+1} - u^k \,|\!|\![_\lambda \leqslant\; \sqrt{\underline{\nu}}^{-1} \rho \|\mathrm{div}\, u^k\|. \tag{2.3}$$

Proof. The equation for pressure (2.2) follows directly from (1.7). By subtracting the kth equation (1.5) from the $(k+1)$th equation, we obtain

$$\int_\Omega \nu \nabla (u^{k+1} - u^k) : \nabla w + \mu (u^{k+1} - u^k) \cdot w \, \mathrm{d}x = \int_\Omega (p^{k+1} - p^k) \mathrm{div}\, w \, \mathrm{d}x.$$

Since

$$\|\mathrm{div}\, w\| \leqslant \|\nabla w\|_\Sigma \leqslant \sqrt{\underline{\nu}}^{-1} \|\sqrt{\nu} \nabla w\|_\Sigma \leqslant \; \sqrt{\underline{\nu}}^{-1} \; |\!|\!|\, w \,|\!|\!|$$

we can estimate the right-hand side with

$$\begin{aligned} \int_\Omega (p^{k+1} - p^k) \mathrm{div}\, w \, \mathrm{d}x \; &\leqslant \|p^{k+1} - p^k\| \; \|\mathrm{div}\, w\| \\ &\leqslant \sqrt{\underline{\nu}}^{-1} \|p^{k+1} - p^k\| \; |\!|\!|\, w \,|\!|\!| \, . \end{aligned}$$

By choosing $w = u^{k+1} - u^k$, we obtain

$$|\!|\!|\, u^{k+1} - u^k \,|\!|\!|^2 \leqslant \; \sqrt{\underline{\nu}}^{-1} \|p^{k+1} - p^k\| \; |\!|\!|\, u^{k+1} - u^k \,|\!|\!| \, .$$

By (2.2) we obtain the estimate for velocity (2.1). The estimate (2.3) is obtained with exactly the same arguments applied for the augmented form (1.6). Since $|\!|\!|\, w \,|\!|\!| \leqslant]\!|\!|\, w \,|\!|\![_\lambda$ for all $\lambda \in \mathbb{R}_+$, we see by (2.3), that the estimate (2.1) holds also for approximations calculated by (1.6).

Henceforth, we will use functional *a posteriori* error estimates for the Stokes problem derived in [11, 12]. For a consequent exposition of the theory of functional *a posteriori* error estimates we refer the reader to [8, 10].

The following lemma is essential in deriving our main results.

Lemma 2.1. *Let Ω be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Then there exists a positive constant $C_{\rm LBB}$ depending on the domain Ω such that for any function $g \in \widetilde{L}_2(\Omega)$ there is a function $v \in V_0$ satisfying the condition* $\operatorname{div} v = g$, *and*

$$\|\nabla v\|_\Sigma \;\leqslant\; C_{\rm LBB}^{-1}\|g\|.$$

Here $C_{\rm LBB}$ is the constant in the well-known Ladyzhenskaya-Babuška-Brezzi (LBB) condition (*see, e.g.,* [1,2]). *See proof in* [6,7].

For some simple domains the constant $C_{\rm LBB}$, or the bounds for it, are known (see, e.g., [3,5,9]).

Lemma 2.1 implies an important corollary. Let $v \in V_0$, and $\operatorname{div} v = g$. Then there exists a function $v_g \in V_0$ such that $\operatorname{div}(v - v_g) = 0$, and

$$\|\nabla v_g\|_\Sigma \;\leqslant\; C_{\rm LBB}^{-1}\|g\| \;=\; C_{\rm LBB}^{-1}\|\operatorname{div} v\|.$$

This means that there exists a solenoidal field $v_0 = (v - v_g) \in \mathring{S}(\Omega)$ such that

$$\|\nabla(v - v_0)\|_\Sigma \;\leqslant\; C_{\rm LBB}^{-1}\|\operatorname{div} v\|.$$

A similar estimate holds for $v \in V_0 + u_D$. Indeed, for $v - u_D$ we can find a solenoidal field $v_0 \in \mathring{S}(\Omega)$ such that

$$\|\nabla(v - u_D - v_0)\|_\Sigma \;\leqslant\; C_{\rm LBB}^{-1}\|\operatorname{div}(v - u_D)\| \;\leqslant\; C_{\rm LBB}^{-1}\|\operatorname{div} v\|.$$

Thus, we can find a function $w_0 \in \mathring{S}(\Omega) + u_D$ such that

$$\|\nabla(v - w_0)\|_\Sigma \;\leqslant\; C_{\rm LBB}^{-1}\|\operatorname{div} v\|. \tag{2.4}$$

With the help of (2.4) we can now derive our main results. We show that the errors of u^k and p^k generated on the iteration k of the Uzawa algorithm are both estimated from above by the L_2-norm of the divergence of u^k multiplied by a constant depending on $C_{\rm LBB}$. The proofs are based on the derivation of functional *a posteriori* error estimates for the generalized Stokes problem as they are presented in [12].

Theorem 2.2. *Let u^k be the exact solution computed on the iteration k of the Uzawa algorithm. Then, for solutions calculated by* (1.5) *or* (1.6)*, we have*

$$|\!|\!|\, u - u^k \,|\!|\!| \leqslant 2C\|\operatorname{div} u^k\| \tag{2.5}$$

where

$$C := C_{\rm LBB}^{-1}\sqrt{C_{\rm F}^2\overline{\mu} + \overline{\nu}}. \tag{2.6}$$

Here $C_{\rm F}$ is the constant in the Friedrichs inequality

$$\|w\| \;\leqslant\; C_{\rm F}\|\nabla w\|_\Sigma$$

and $C_{\rm LBB}$ is the constant in the LBB-*condition.*

Proof. Let $u_0 \in \mathring{S}(\Omega) + u_D$ be such that, by using (2.4), we have

$$\|\nabla(u^k - u_0)\|_\Sigma \leqslant C_{\mathrm{LBB}}^{-1} \|\operatorname{div} u^k\|. \tag{2.7}$$

Let the pair (u^k, p^k) be an approximation of the saddle point computed on the iteration k. We can now write

$$|\!|\!| \, u - u^k \, |\!|\!| \leqslant |\!|\!| \, u - u_0 \, |\!|\!| + |\!|\!| \, u_0 - u^k \, |\!|\!| \, . \tag{2.8}$$

First, we estimate from above the first term on the right-hand side of (2.8). Let $w \in \mathring{S}$. By subtracting the integral $\int_\Omega (\nu \nabla u_0 : \nabla w + \mu u_0 \cdot w)\,\mathrm{d}x$ from both sides of (1.4) we obtain

$$\int_\Omega (\nu \nabla (u - u_0) : \nabla w + \mu (u - u_0) \cdot w)\,\mathrm{d}x \\ = \int_\Omega ((f - \mu u_0) \cdot w - \nu \nabla u_0 : \nabla w)\,\mathrm{d}x. \tag{2.9}$$

It is easy to see that

$$\int_\Omega (\operatorname{Div} \tau \cdot w + \tau : \nabla w)\,\mathrm{d}x = 0 \quad \forall \tau \in \Sigma(\operatorname{Div}, \Omega), \;\; w \in V_0(\Omega) \tag{2.10}$$

and

$$\int_\Omega (\nabla q \cdot w + q \operatorname{div} w)\,\mathrm{d}x = 0 \quad \forall q \in W_2^1(\Omega, \mathbb{R}), \;\; w \in V_0(\Omega).. \tag{2.11}$$

By adding (2.10) and (2.11) to the right-hand side of (2.9), we rewrite it in the form

$$\int_\Omega ((f - \mu u_0 + \operatorname{Div} \tau - \nabla q) \cdot w + (\tau - \nu \nabla u_0) : \nabla w)\,\mathrm{d}x \tag{2.12}$$

which is equivalent to

$$\int_\Omega \Big(\Big(f - \mu u^k + \operatorname{Div} \tau - \nabla q \Big) \cdot w + \Big(\tau - \nu \nabla u^k \Big) : \nabla w \Big)\,\mathrm{d}x \\ + \int_\Omega \Big(\nu \nabla (u^k - u_0) : \nabla w + \mu (u^k - u_0) \cdot w \Big)\,\mathrm{d}x. \tag{2.13}$$

Let us choose $\tau = \nu \nabla u^k$ and $q = p^k$. In view of (1.5), we see that that the first integral of (2.13) vanishes. Indeed,

$$\int_\Omega \Big(\Big(f - \mu u^k + \operatorname{Div} \nu \nabla u^k - \nabla p^k \Big) \cdot w \; + \Big(\nu \nabla u^k - \nu \nabla u^k \Big) : \nabla w \Big)\,\mathrm{d}x \\ = \int_\Omega \Big(f \cdot w + p^k \operatorname{div} w - \nu \nabla u^k : \nabla w - \mu u^k \cdot w \Big)\,\mathrm{d}x = 0. \tag{2.14}$$

Since w is a function from $\mathring{S}$, the same conclusion is also true if u^k has been calculated by (1.6). We combine (2.9) with (2.12)–(2.14), and arrive at the relation

$$\int_\Omega (\nu\nabla(u-u_0) : \nabla w + \nu(u-u_0)\cdot w)\,\mathrm{d}x = \int_\Omega \Big(\nu\nabla(u^k-u_0) : \nabla w + \mu(u^k-u_0)\cdot w\Big)\,\mathrm{d}x. \tag{2.15}$$

The right-hand side of (2.15) can be estimated from above as follows:

$$\begin{aligned}\int_\Omega \big(\nu\nabla(u^k - u_0) : \nabla w + \mu(u^k-u_0)\cdot w\big)\mathrm{d}x &= \int_\Omega \Big(\sqrt{\nu}\nabla(u^k-u_0) : \sqrt{\nu}\nabla w + \sqrt{\mu}(u^k-u_0)\cdot\sqrt{\mu}w\Big)\,\mathrm{d}x \\ &\leqslant \|\sqrt{\nu}\nabla(u^k-u_0)\|_\Sigma\|\sqrt{\nu}\nabla w\|_\Sigma + \|\sqrt{\mu}(u^k-u_0)\|\;\|\sqrt{\mu}w\| \\ &\leqslant \;|\!|\!|\; u^k-u_0 \;|\!|\!|\;|\!|\!|\; w \;|\!|\!| \end{aligned} \tag{2.16}$$

where we have used the Cauchy–Schwarz inequality. We set $w = u - u_0$, and find that

$$|\!|\!|\; u-u_0 \;|\!|\!| \leqslant |\!|\!|\; u^k-u_0 \;|\!|\!|\;. \tag{2.17}$$

Note that for all $w \in V$ we have

$$\begin{aligned}|\!|\!|\; w \;|\!|\!|^2 \;&=\; \|\sqrt{\nu}\nabla w\|_\Sigma^2 + \|\sqrt{\mu}w\|^2 \\ &\leqslant\; \overline{\nu}\|\nabla w\|_\Sigma^2 + \overline{\mu}\|w\|^2 \\ &\leqslant\; \overline{\nu}\|\nabla w\|_\Sigma^2 + C_{\mathrm{F}}^2\overline{\mu}\|\nabla w\|_\Sigma^2 \\ &\leqslant\; \big(C_{\mathrm{F}}^2\overline{\mu}+\overline{\nu}\big)\,\|\nabla w\|_\Sigma^2.\end{aligned} \tag{2.18}$$

We substitute (2.17) into (2.8), and use (2.18) with $w = u - u_0$, and obtain

$$\begin{aligned}|\!|\!|\; u-u^k \;|\!|\!| \;&\leqslant\; 2\;|\!|\!|\; u_0-u^k \;|\!|\!| \\ &\leqslant\; 2\sqrt{C_{\mathrm{F}}^2\overline{\mu}+\overline{\nu}}\,\|\nabla(u_0-u^k)\|_\Sigma.\end{aligned} \tag{2.19}$$

Now, (2.7) and (2.19) imply the estimate

$$|\!|\!|\; u-u^k \;|\!|\!| \leqslant 2C_{\mathrm{LBB}}^{-1}\sqrt{C_{\mathrm{F}}^2\overline{\mu}+\overline{\nu}}\,\|\mathrm{div}\,u^k\| = 2C\|\mathrm{div}\,u^k\|$$

where C is defined in (2.6).

In order to prove a similar estimate for the pressure, we also need Lemma 2.1. Let $q \in \widetilde{L}_2$ be an approximation of the exact pressure p. Then $(p-q) \in \widetilde{L}_2$ and there exists a function $\overline{w} \in V_0$ such that

$$\mathrm{div}(\overline{w}) = p - q \tag{2.20}$$

and

$$\|\nabla \overline{w}\|_\Sigma \leqslant C_{\mathrm{LBB}}^{-1}\|p-q\|. \tag{2.21}$$

Theorem 2.3. *Let p^k be the function computed on the iteration k of the Uzawa algorithm. Then,*

$$\|p-p^k\| \;\leqslant\; \mathbb{C}\|\operatorname{div} u^k\| \tag{2.22}$$

where $\mathbb{C}=2C^2$ for (1.5)*, and $\mathbb{C}=2C^2+\lambda$ for* (1.6).

Proof. We use (2.20) for $q=p^k$ and obtain

$$\|p-p^k\|^2=\int_\Omega \operatorname{div}\overline{w}(p-p^k)\mathrm{d}x=\int_\Omega \operatorname{div}\overline{w}\,p+\nabla p^k\cdot\overline{w}\,\mathrm{d}x.$$

Multiplying (1.1) by $\overline{w}$ and integrating over Ω, we obtain

$$\int_\Omega \operatorname{div}\overline{w}\,p\;\mathrm{d}x \;=\int_\Omega (\nu\nabla u:\nabla\overline{w}+\mu u\cdot\overline{w}-f\cdot\overline{w})\,\mathrm{d}x.$$

In view of this relation, we have

$$\|p-p^k\|^2=\int_\Omega\Big(\nu\nabla u:\nabla\overline{w}+\mu u\cdot\overline{w}-f\cdot\overline{w}+\nabla p^k\cdot\overline{w}\Big)\,\mathrm{d}x.$$

We use (2.10) with $w=\overline{w}$, and arrive at the relation

$$\begin{aligned}\|p-p^k\|^2=&\int_\Omega\Big(\Big(-f+\mu u^k-\operatorname{Div}\tau+\nabla p^k\Big)\cdot\overline{w}+\Big(\nu\nabla u^k-\tau\Big):\nabla\overline{w}\Big)\,\mathrm{d}x\\&+\int_\Omega\Big(\nu\nabla(u-u^k):\nabla\overline{w}+\mu(u-u^k)\cdot\overline{w}\Big)\,\mathrm{d}x.\end{aligned} \tag{2.23}$$

As before, we choose $\tau=\nu\nabla u^k$, and observe that the first integral is zero. By estimating the latter integral with the help of the same arguments as in (2.16), we find that

$$\|p-p^k\|^2\;\leqslant |\!|\!|\, u-u^k \,|\!|\!|\;|\!|\!|\,\overline{w}\,|\!|\!|\;. \tag{2.24}$$

By (2.18) and (2.21), we obtain

$$\begin{aligned}|\!|\!|\,\overline{w}\,|\!|\!|^2\;&\leqslant\;\left(C_{\mathrm{F}}^2\overline{\mu}+\overline{\nu}\right)\|\nabla\overline{w}\|_\Sigma^2\\&\leqslant\;C_{\mathrm{LBB}}^{-2}\left(C_{\mathrm{F}}^2\overline{\mu}+\overline{\nu}\right)\|p-p^k\|^2\\&=\;C^2\|p-p^k\|^2\end{aligned} \tag{2.25}$$

where C is defined in (2.6). Substituting (2.25) into (2.24) results in the estimate

$$\|p-p^k\|\;\leqslant\;C\;|\!|\!|\,u-u^k\,|\!|\!|\;.$$

Now, we apply Theorem 2.2 and deduce (2.22).

In the case of (1.6), we add

$$\int_\Omega \lambda \operatorname{div}(u^k - u^k)\operatorname{div}\overline{w}\,\mathrm{d}x = 0$$

to (2.23) and obtain

$$\begin{aligned}\|p-p^k\|^2 = &\int_\Omega \Big(\Big(-f+\mu u^k - \operatorname{Div}\tau + \nabla p^k\Big)\cdot\overline{w} + \lambda \operatorname{div} u^k \operatorname{div}\overline{w}\Big)\,\mathrm{d}x \\ &+ \int_\Omega \Big(\nu\nabla u^k - \tau\Big) : \nabla\overline{w}\,\mathrm{d}x \\ &+ \int_\Omega \Big(\nu\nabla(u-u^k) : \nabla\overline{w} + \mu(u-u^k)\cdot\overline{w} - \lambda \operatorname{div} u^k \operatorname{div}\overline{w}\Big)\,\mathrm{d}x.\end{aligned}$$

Again, we choose $\tau = \nu\nabla u^k$, and see from (1.6) that the first and second integrals are zero. By estimating the latter integral with same arguments as in (2.16), we obtain

$$\|p-p^k\|^2 \leqslant \|\!|\, u-u^k \,|\!\|\ \|\!|\, \overline{w} \,|\!\| + \lambda\|\operatorname{div} u^k\|\ \|\operatorname{div}\overline{w}\|. \tag{2.26}$$

Recall that $\operatorname{div}\overline{w} = p - p^k$. Now, (2.25) and (2.26) imply the estimate

$$\|p-p^k\| \leqslant C\,\|\!|\, u-u^k \,|\!\| + \lambda\|\operatorname{div} u^k\|.$$

Applying Theorem 2.2 results in (2.22).

By Theorems 2.2 and 2.3, we easily conclude the following statement.

Remark 2.1. The classical Stokes problem corresponds to the case where $\mu \equiv 0$ and ν is a constant. Let (u^k, p^k) be the exact solution computed on the iteration k of the Uzawa algorithm, for the Stokes problem. Then, for velocity we have (for both cases (1.5) and (1.6))

$$\|\nabla(u-u^k)\| \leqslant 2C_{\mathrm{LBB}}^{-1}\|\operatorname{div} u^k\|.$$

For the pressure we have

$$\|p-p^k\| \leqslant \tilde{\mathbb{C}}\|\operatorname{div} u^k\|$$

where $\tilde{\mathbb{C}} = 2C_{\mathrm{LBB}}^{-2}\nu$ for (1.5) and $\tilde{\mathbb{C}} = 2C_{\mathrm{LBB}}^{-2}\nu + \lambda$ for (1.6).

3. Computable error estimates for approximations generated by the Uzawa algorithm

Let $\mathscr{T}_h$ be a mesh having the characteristic size h, and let the spaces $V_{0h}(\Omega,\mathbb{R}^n)$ and $Q_h(\Omega)$ be finite dimensional subspaces of $V_0(\Omega,\mathbb{R}^n)$ and $\widetilde{L}_2(\Omega)$, respectively. We assume that for all $v_h \in V_{0h} + u_D$ it holds that $\operatorname{div} v_h \in Q_h$. We also assume that the spaces are constructed so that they satisfy the discrete LBB-condition, i.e, for any $q_h \in Q_h$ with zero mean, there exists $v_h \in V_{0h}$ such that

$$\operatorname{div} v_h = q_h$$

and

$$\|\nabla v_h\|_\Sigma \leqslant c\|q_h\|$$

where the positive constant c does not depend on h.

Let $u_h^k \in V_{0h} + u_D$ be an approximation of u^k calculated on the mesh $\mathscr{T}_h$. We need to combine the error of the pure Uzawa algorithm with the approximation error. Below we present the corresponding results, where we set $p^k = p_h^k \in Q_h$ on the iteration k, and understand u^k as satisfying (1.5), or (1.6), with the chosen p_h^k. Then, the pair (u^k, p_h^k) can be viewed as the exact pair associated with the Uzawa algorithm on iteration k.

Our first goal is to derive fully computable error majorants $M_\oplus^k$ and $M_\oplus^{k,\lambda}$ for approximate solutions (e.g., u_h^k) of the problems generated at the first step of Uzawa algorithm by the Lagrangians L and L_A, respectively. In order to make the quality of the majorants robust with respect to small or large values of the material functions ν or μ, we apply the same method that was suggested in [12] for the generalized Stokes problem.

Later we combine these estimates with the estimates of the difference between u and u^k and obtain estimates applicable for approximate solutions computed within the framework of finite dimensional approximations.

First, we prove the following result for the problem generated by the Lagrangian L.

Theorem 3.1. *Let (u^k, p_h^k) be the exact solution on the iteration k of the Uzawa algorithm. Then, for the solutions calculated by* (1.5)*, and for an approximation $u_h^k \in V_{0h} + u_D$ we have*

$$|\!|\!| u^k - u_h^k |\!|\!|^2 \leqslant M_\oplus^k(u_h^k, p_h^k, \tau, \beta) \quad \forall \tau \in H(\operatorname{Div}, \Omega), \;\; \beta \in \mathbb{R}_+$$

where

$$M_\oplus^k(u_h^k, p_h^k, \tau, \beta) := \int_\Omega H_1(\nu,\mu,\beta) r^2(u_h^k, \tau)\mathrm{d}x + H_2(\beta)\|\sqrt{\nu}^{-1} d(u_h^k, p_h^k, \tau)\|_\Sigma^2$$

and

$$H_1(\nu,\mu,\beta) := \frac{C_\mathrm{F}^2(1+\beta)}{\underline{\nu} + C_\mathrm{F}^2(1+\beta)\mu} \tag{3.1}$$

$$H_2(\beta) := 1 + \beta^{-1} \tag{3.2}$$

$$r(u_h^k, \tau) := f - \mu u_h^k + \operatorname{Div}\tau \tag{3.3}$$

$$d(u_h^k, p_h^k, \tau) := \tau - \nu\nabla u_h^k + \mathbb{I} p_h^k. \tag{3.4}$$

Here $\mathbb{I}$ denotes the unit tensor.

Proof. By equation (1.5) we have

$$\int_\Omega \Big(\nu\nabla u^k : \nabla w + \mu u^k\cdot w\Big)\,\mathrm{d}x \;=\; \int_\Omega \Big(f\cdot w + p_h^k \operatorname{div} w\Big)\,\mathrm{d}x.$$

We subtract the integral $\int_\Omega \big(\nu\nabla u_h^k : \nabla w + \mu u_h^k\cdot w\big)\,\mathrm{d}x$ from both sides of the above equation, and obtain

$$\int_\Omega \nu\nabla(u^k - u_h^k) : \nabla w + \mu(u^k-u_h^k)\cdot w\,\mathrm{d}x$$
$$= \int_\Omega \Big((f-\mu u_h^k)\cdot w - \nu\nabla u_h^k : \nabla w + p_h^k \operatorname{div} w\Big)\,\mathrm{d}x. \qquad (3.5)$$

By adding (2.10) to the right-hand side of (3.5) we have

$$\int_\Omega \Big(\nu\nabla(u^k - u_h^k) : \nabla w + \mu(u^k-u_h^k)\cdot w\Big)\,\mathrm{d}x$$
$$= \int_\Omega \Big((f-\mu u_h^k + \operatorname{Div}\tau)\cdot w + (\tau - \nu\nabla u_h^k + \mathbb{I}p_h^k) : \nabla w\Big)\,\mathrm{d}x$$
$$= \int_\Omega \Big(r(u_h^k,\tau)\cdot w + d(u_h^k, p_h^k,\tau) : \nabla w\Big)\,\mathrm{d}x \qquad (3.6)$$

where we have used the notation (3.3) and (3.4). Note that

$$\int_\Omega r\cdot w\,\mathrm{d}x = \int_\Omega \Big(\sqrt{\mu}^{-1}\alpha r\cdot\sqrt{\mu}w + (1-\alpha)r\cdot w\Big)\,\mathrm{d}x$$
$$\leqslant \|\sqrt{\mu}^{-1}\alpha r\|\,\|\sqrt{\mu}w\| + \|(1-\alpha)r\|\,\|w\|$$
$$\leqslant \|\sqrt{\mu}^{-1}\alpha r\|\,\|\sqrt{\mu}w\| + C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\|\,\|\sqrt{\nu}\nabla w\|_\Sigma \qquad (3.7)$$

where $0\leqslant\alpha(x)\leqslant 1$. Also, we have

$$\int_\Omega d : \nabla w\,\mathrm{d}x \;\leqslant\; \|\sqrt{\nu}^{-1}d\|_\Sigma\|\sqrt{\nu}\nabla w\|_\Sigma. \qquad (3.8)$$

By (3.7) and (3.8) the right-hand side of (3.6) becomes

$$\Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\| + \|\sqrt{\nu}^{-1}d\|_\Sigma\Big)\|\sqrt{\nu}\nabla w\|_\Sigma + \|\sqrt{\mu}^{-1}\alpha r\|\,\|\sqrt{\mu}w\|$$
$$\leqslant \sqrt{\Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\| + \|\sqrt{\nu}^{-1}d\|_\Sigma\Big)^2 + \|\sqrt{\mu}^{-1}\alpha r\|^2}\;|\!|\!|\, w \,|\!|\!|\,. \qquad (3.9)$$

We set $w = u^k - u_h^k$, use (3.6) and (3.9), and obtain

$$|\!|\!|\, u^k - u_h^k \,|\!|\!|^2 \leqslant \Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\| + \|\sqrt{\nu}^{-1}d\|_\Sigma\Big)^2 + \|\sqrt{\mu}^{-1}\alpha r\|^2$$
$$\leqslant (1+\beta)C_{\mathrm{F}}^2\underline{\nu}^{-1}\|(1-\alpha)r\|^2$$
$$+\;(1+\beta^{-1})\|\sqrt{\nu}^{-1}d\|_\Sigma^2 \;+\; \|\sqrt{\mu}^{-1}\alpha r\|^2. \qquad (3.10)$$

It is easy to see that the optimal value of α is defined by the relation

$$\alpha = \frac{C_{\mathrm{F}}^2(1+\beta)\mu}{\underline{\nu}+C_{\mathrm{F}}^2(1+\beta)\mu} \tag{3.11}$$

so that (3.10) implies the estimate

$$\begin{aligned}|\!|\!| u^k - u_h^k |\!|\!|^2 &\leqslant \int_\Omega \frac{C_{\mathrm{F}}^2(1+\beta)}{\underline{\nu}+C_{\mathrm{F}}^2(1+\beta)\mu} r^2\,\mathrm{d}x + (1+\beta^{-1})\|\sqrt{\nu}^{-1} d\|_\Sigma^2 \\ &= \int_\Omega H_1 r^2\,\mathrm{d}x + H_2\|\sqrt{\nu}^{-1} d\|_\Sigma^2\end{aligned}$$

where we have used the notation (3.1) and (3.2).

Remark 3.1. It is easy to see that the upper bound $M_\oplus^k$ is sharp. Indeed, by setting $\tau = \nu\nabla u^k - \mathbb{I}p_h^k$, and letting β tend to infinity, we get the exact error in the energy norm $|\!|\!| \cdot |\!|\!|$.

A similar estimate can be derived for the problem generated by the augmented Lagrangian L_A.

Theorem 3.2. *Let (u^k, p_h^k) be the exact solution on the iteration k of the Uzawa algorithm. Then, for the solutions calculated by* (1.6)*, and for an approximation $u_h^k \in V_{0h} + u_D$ we have*

$$|\!|\!| u^k - u_h^k |\!|\!|^2 \leqslant]\!]\!] u^k - u_h^k |\!|\!|[_\lambda^2 \leqslant M_\oplus^{k,\lambda}(u_h^k, p_h^k, \tau, \beta) \quad \forall \tau \in H(\mathrm{Div}, \Omega),\ \ \beta \in \mathbb{R}_+$$

where

$$M_\oplus^{k,\lambda}(u_h^k, p_h^k, \tau, \beta) := \int_\Omega H_1(\nu, \mu, \beta) r^2(u_h^k, \tau)\mathrm{d}x + H_2(\beta)\|\sqrt{\nu}^{-1} d^\lambda(u_h^k, p_h^k, \tau)\|_\Sigma^2.$$

The quantities H_1, H_2, and r are defined in (3.1)–(3.3)*, and*

$$d^\lambda(u_h^k, p_h^k, \tau) := \tau - \nu\nabla u_h^k + \mathbb{I}(p_h^k - \lambda \operatorname{div} u_h^k). \tag{3.12}$$

Proof. By (1.6), we have

$$\int_\Omega \Big(\nu\nabla u^k : \nabla w + \mu u^k \cdot w + \lambda \operatorname{div} u^k \operatorname{div} w\Big)\,\mathrm{d}x = \int_\Omega \Big(f\cdot w + p_h^k \operatorname{div} w\Big)\,\mathrm{d}x.$$

We subtract the integral $\int_\Omega \big(\nu\nabla u_h^k : \nabla w + \mu u_h^k \cdot w + \lambda \operatorname{div} u_h^k \operatorname{div} w\big)\,\mathrm{d}x$ from both sides of the above equation, and use (2.10), and obtain

$$\begin{aligned}
&\int_\Omega \Big(\nu\nabla(u^k-u_h^k):\nabla w+\mu(u^k-u_h^k)\cdot w+\lambda\,\mathrm{div}(u^k-u_h^k)\mathrm{div}\,w\Big)\,\mathrm{d}x\\
&\quad=\int_\Omega \Big((f-\mu u_h^k)\cdot w-\nu\nabla u_h^k:\nabla w+(p_h^k-\lambda\,\mathrm{div}\,u_h^k)\mathrm{div}\,w\Big)\,\mathrm{d}x\\
&\quad=\int_\Omega \Big((f-\mu u_h^k+\mathrm{Div}\,\tau)\cdot w+\Big(\tau-\nu\nabla u_h^k+\mathbb{I}(p_h^k-\lambda\,\mathrm{div}\,u_h^k)\Big):\nabla w\Big)\,\mathrm{d}x\\
&\quad=\int_\Omega \Big(r(u_h^k,\tau)\cdot w+d^\lambda(u_h^k,p_h^k,\tau):\nabla w\Big)\,\mathrm{d}x \qquad (3.13)
\end{aligned}$$

where we have used the notation (3.3) and (3.12). By the same arguments as in (3.7) and (3.8), we represent the right-hand side of (3.13) in the form

$$\begin{aligned}
&\Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\|+\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma\Big)\,\|\sqrt{\nu}\nabla w\|_\Sigma+\|\sqrt{\mu}^{-1}\alpha r\|\,\|\sqrt{\mu}w\|\\
&\leqslant\sqrt{\Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\|+\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma\Big)^2+\|\sqrt{\mu}^{-1}\alpha r\|^2}\;]\!|\!|\, w\,|\!|\!|_\lambda \qquad (3.14)
\end{aligned}$$

since $|\!|\!|\, w\,|\!|\!| \leqslant\,]\!|\!|\, w\,|\!|\!|_\lambda$. By choosing $w=u^k-u_h^k$, (3.13) and (3.14) give

$$\begin{aligned}
]\!|\!|\, u^k-u_h^k\,|\!|\!|_\lambda^2 &\leqslant \Big(C_{\mathrm{F}}\sqrt{\underline{\nu}}^{-1}\|(1-\alpha)r\|+\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma\Big)^2+\|\sqrt{\mu}^{-1}\alpha r\|^2\\
&\leqslant(1+\beta)C_{\mathrm{F}}^2\underline{\nu}^{-1}\|(1-\alpha)r\|^2\\
&\quad+\;(1+\beta^{-1})\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma^2\;+\;\|\sqrt{\mu}^{-1}\alpha r\|^2.
\end{aligned}$$

Again, we see that the optimal value of α is given by the relation (3.11), and obtain

$$\begin{aligned}
]\!|\!|\, u^k-u_h^k\,|\!|\!|_\lambda^2 &\leqslant \int_\Omega \frac{C_{\mathrm{F}}^2(1+\beta)}{\underline{\nu}+C_{\mathrm{F}}^2(1+\beta)\mu}r^2\,\mathrm{d}x+(1+\beta^{-1})\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma^2\\
&=\int_\Omega H_1r^2\,\mathrm{d}x+H_2\|\sqrt{\nu}^{-1}d^\lambda\|_\Sigma^2
\end{aligned}$$

where we have used the notation (3.1) and (3.2).

Finally, by using Theorems 2.2, 3.1, and 3.2 we obtain the final result.

Theorem 3.3. *Let u be the exact velocity, (u^k,p_h^k) be the exact solution calculated on the iteration k of the Uzawa algorithm, and $u_h^k\in V_{0h}+u_D$ be an approximation of the velocity calculated on this iteration. For* (1.5) *we have*

$$|\!|\!|\, u-u_h^k\,|\!|\!| \leqslant \mathbf{M}_\oplus^k(u_h^k,p_h^k,\tau,\beta)\quad \forall\tau\in H(\mathrm{Div},\Omega),\;\beta\in\mathbb{R}_+$$

and for (1.6) *we have*

$$|\!|\!|\, u-u_h^k\,|\!|\!| \leqslant \mathbf{M}_\oplus^{k,\lambda}(u_h^k,p_h^k,\tau,\beta)\quad \forall\tau\in H(\mathrm{Div},\Omega),\;\beta\in\mathbb{R}_+$$

where

$$\mathbf{M}^k_{\oplus}(u^k_h, p^k_h, \tau, \beta) := 2C\|\operatorname{div} u^k_h\| + (2C\sqrt{\underline{\nu}}^{-1} + 1)\sqrt{M^k_{\oplus}(u^k_h, p^k_h, \tau, \beta)}$$
$$\mathbf{M}^{k,\lambda}_{\oplus}(u^k_h, p^k_h, \tau, \beta) := 2C\|\operatorname{div} u^k_h\| + (2C\sqrt{\underline{\nu}}^{-1} + 1)\sqrt{M^{k,\lambda}_{\oplus}(u^k_h, p^k_h, \tau, \beta)}$$

with C defined in (2.6)*.*

Proof. It is clear that

$$|||\, u - u^k_h \,||| \leqslant \ |||\, u - u^k \,||| + |||\, u^k - u^k_h \,|||\ .$$

By Theorem 2.2 we have

$$\begin{aligned}
|||\, u - u^k_h \,||| &\leqslant 2C\|\operatorname{div} u^k\| + |||\, u^k - u^k_h \,||| \\
&\leqslant 2C\|\operatorname{div} u^k_h\| + 2C\|\operatorname{div}(u^k - u^k_h)\| + |||\, u^k - u^k_h \,||| \\
&\leqslant 2C\|\operatorname{div} u^k_h\| + 2C\sqrt{\underline{\nu}}^{-1}\|\sqrt{\nu}\nabla(u^k - u^k_h)\| + |||\, u^k - u^k_h \,||| \\
&\leqslant 2C\|\operatorname{div} u^k_h\| + (2C\sqrt{\underline{\nu}}^{-1} + 1)\ |||\, u^k - u^k_h \,|||\ .
\end{aligned}$$

Using the upper bounds presented in Theorems 3.1 and 3.2 for the two cases (1.5) and (1.6), respectively, we arrive at the result.

Finally, we note that estimates for the pressure follows from the above derived estimates. The exact pressure in the Uzawa algorithm is calculated by (1.7), i.e.,

$$p^{k+1} = (p^k_h - \rho \operatorname{div} u^k) \in \widetilde{L}_2(\Omega) \tag{3.15}$$

and an approximation of it is calculated within the framework of the selected finite dimensional subspaces, i.e.,

$$p^{k+1}_h = (p^k_h - \rho \operatorname{div} u^k_h) \in Q_h(\Omega). \tag{3.16}$$

Theorem 3.4. *Let* (u^k, p^k_h) *be the exact solution calculated on the iteration* k *of the Uzawa algorithm, and* $u^k_h \in V_{0h} + u_D$ *be an approximation of the velocity calculated on this iteration. Now, we apply the estimates presented in Theorems* 3.1 *and* 3.2*, and obtain for* (1.5):

$$\|p^{k+1} - p^{k+1}_h\| \leqslant \rho\sqrt{\underline{\nu}}^{-1}\sqrt{M^k_{\oplus}(u^k_h, p^k_h, \tau, \beta)} \quad \forall \tau \in H(\mathrm{Div}, \Omega),\ \beta \in \mathbb{R}_+$$

and for (1.6)

$$\|p^{k+1} - p^{k+1}_h\| \leqslant \rho\sqrt{\underline{\nu}}^{-1}\sqrt{M^{k,\lambda}_{\oplus}(u^k_h, p^k_h, \tau, \beta)} \quad \forall \tau \in H(\mathrm{Div}, \Omega),\ \ \beta \in \mathbb{R}_+.$$

Proof. Indeed, from (3.15) and (3.16) we find that

$$\begin{aligned}\|p^{k+1} - p_h^{k+1}\| &= \rho \|\mathrm{div}(u^k - u_h^k)\| \\ &\leqslant \rho \sqrt{\underline{\nu}}^{-1} \|\sqrt{\nu}\nabla(u^k - u_h^k)\| \\ &\leqslant \rho \sqrt{\underline{\nu}}^{-1} \,|\!|\!|\, u^k - u_h^k \,|\!|\!|\, .\end{aligned}$$

Applying the error bounds presented in Theorems 3.1 and 3.2 completes the proof.

This paper is focused on theoretical analysis of *a posteriori* error bounds for approximations computed by the Uzawa algorithm. However, it is worth adding some comments on the practical applications of the above derived error majorants. The majorants contain the function $\tau \in H(\mathrm{Div}, \Omega)$ and a positive parameter β, which in general can be taken arbitrary. Getting sharp estimates requires a proper selection of them. Finding an optimal β leads to a one-dimensional optimization problem which is easy solvable. The reconstruction of the stress tensor τ based upon computed functions u_h^k and p_h^k provides a reasonable first guess. A better selection can be performed by methods that have been developed and tested for various elliptic problems (see, e.g., [8, 10, 14] and the references cited therein). A systematical study of computational questions in the context of above derived estimates will be exposed in a separate paper, which is now in preparation.

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PII

A POSTERIORI ERROR BOUNDS FOR APPROXIMATIONS OF THE STOKES PROBLEM WITH FRICTION TYPE BOUNDARY CONDITIONS

by

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A Posteriori Error Bounds for Approximations of the Stokes Problem with Friction Type Boundary Conditions

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Abstract

In the paper, we derive computable and fully guaranteed estimates of the difference between exact solutions (velocity) of Stokes problems with nonlinear (friction type) boundary conditions and vector functions from the admissible energy space. The estimates are valid for any function satisfying the main boundary conditions and possessing first generalized derivatives. The estimates can be used for a posteriori error control of numerical solutions obtained by various numerical methods.

1 Introduction

Stokes type equations of viscous incompressible fluid supplied with nonlinear "slip" and "leak" boundary conditions are often used for simulation of blood flow in veins affected by sclerosis and in modelling of avalanche of water and rocks. These mathematical models were introduced by Fujita in [9]. In [10] Fujita proved existence and uniqueness of weak solutions to Stokes problem with nonlinear slip and leak boundary conditions. Applications to oil flow beneath or over sand layers are studied by Kawarada, Fujita and Saito [16], and Kawarada and Saito [17]. In this paper, we are concerned with friction type boundary conditions, which are suitable, when modeling some fragile state of the surface, that allows the fluid to slip on the surface, but as long as the the pushing force is below a threshold, the fluid does not slip. Similar boundary conditions are often used in mathematical models of solid mechanics. Numerical methods for such type nonlinear problems are well developed (see, e.g., [8, 12, 14, 13, 27, 34]).

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In this paper, we address a different problem, which arises when a numerical solution has been already found and we need to estimate how accurately it represents the respective exact solution. For the considered class of problems, we deduce such type estimates (error majorants). The majorants include only known functions (approximations of the velocity, stress, and pressure fields) and global constants associated with certain functional inequalities (Friedrichs, Poincaré, inf sup). Estimates of these constants can be found by known methods discussed in the paper. Therefore, the majorants are fully computable. Moreover, they are nonnegative and vanish if and only if approximate solutions coincide with the exact ones. Another important property of the majorants is their universality: they are valid for any approximation satisfying the main boundary conditions and having first generalized derivatives. Hence they do not depend on special features of approximations (e.g., Galerkin orthogonality, extra regularity) or properties of a numerical method used. A posteriori error majorants of this type has been derived and tested for a wide spectrum of boundary value and initial boundary value problems (see [30, 20] and references cited therein).

1.1 Notation and basic equations

Let Ω be a bounded domain in $\mathbb{R}^d,\ d = 2,3$ with Lipshitz boundary $\partial\Omega$, which is composed of two disjoint measurable parts Γ and Γ_D. Throughout the paper, we use the following notation: n denotes the outward unit normal vector to $\partial\Omega$,

$$\widetilde{L}_2(\Omega) := \left\{ \phi \in L^2(\Omega) \ \mid \ \{\,\phi\,\}_\Omega := \frac{1}{|\Omega|}\int_\Omega \phi\, dx \right\},$$

$\mathbb{S}(\Omega)$ denotes the subspace of $V := H^1(\Omega, \mathbb{R}^d)$ that consists of solenoidal (divergence free) functions, $V_0(\Omega)$ denotes the subspace of $V(\Omega)$ that consists of the functions vanishing on the Dirichlet part of the boundary Γ_D.

Also, we use spaces of tensor valued functions $\Sigma(\Omega) := L^2(\Omega, \mathbb{M}^{d\times d})$, where $\mathbb{M}^{d\times d}$ is the space of $d \times d$-matrices (tensors). $\mathbb{I}$ denotes the unit element of $\mathbb{M}^{d\times d}$. Since no confusion may arise, we denote L^2-norm of scalar and vector valued functions by $\|\cdot\|$, which is associated with the corresponding inner product $(\ ,\ )$. The scalar product of tensors is denoted by two dots (:). Since no confusion may arise, the norm of this space containing tensor valued functions is denoted by the same symbol, i.e., $\|\tau\|^2 := \int_\Omega |\tau|^2\, dx$. By div and Div, we denote the divergence of vector and tensor fields, respectively, and introduce the Hilbert spaces

$$H(\mathrm{div}, \Omega) := \{w \in \Sigma(\Omega) \mid \mathrm{div}\, w \in L^2(\Omega)\},$$
$$H(\mathrm{Div}, \Omega) := \{\tau \in \Sigma(\Omega) \mid \mathrm{Div}\, \tau \in L^2(\Omega, \mathbb{R}^d)\}.$$

Henceforth, we use the bilinear form

$$a(u, v) := \int\limits_\Omega \nu \nabla u : \nabla v \,\mathrm{d}x,$$

where ν is a positive constant (viscosity). In general, ν can be a strictly positive bounded function with values in $[\underline{\nu}, \bar{\nu}]$. The forms are defined for u, v and $p \in L^2(\Omega)$. The subindex 0 denotes subspaces of functions vanishing on Γ, i.e., $V_0 := \{v \in V \mid v = 0 \text{ on } \Gamma\}$. We assume that Γ has positive $d-1$ measure, so that for the functions in V_0 the norms

$$\|\!|v|\!\|^2 := \int\limits_\Omega \nu \nabla v : \nabla v \,\mathrm{d}x \quad \text{and} \quad \|\!|\eta|\!\|_*^2 := \int\limits_\Omega \nu^{-1} \eta \cdot \eta \,\mathrm{d}x$$

are equivalent to the original norm of V. Analogously, $\mathbb{S}_0(\Omega)$ denotes the subspace of $\mathbb{S}(\Omega)$ consisting of functions vanishing on Γ.

The classical Stokes problem consists of finding a velocity field $u \in \mathbb{S}_0(\Omega) + u_D$ and a pressure field $p \in \widetilde{L}_2(\Omega)$ which satisfy the relations

$$-\operatorname{Div}(\nu \nabla u) + \nabla p = f \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{2}$$

$$u = u_D \qquad \text{on } \Gamma_D, \tag{3}$$

$$\sigma n = F \qquad \text{on } \Gamma, \tag{4}$$

where $f \in L_2(\Omega, \mathbb{R}^d)$, n denotes the unit outward normal vector to the boundary, u_D is a given divergence free function, and $\sigma = \nu \nabla u$ is the stress tensor.

It is well known (see, e.g., [18, 19]) that the problem (1)–(4) possesses a unique generalised solution, which satisfies the integral identity

$$a(u, v) = \int\limits_\Omega f \cdot w \,\mathrm{d}x + \int\limits_\Gamma F w \, ds \qquad \forall w \in \mathbb{S}_0(\Omega). \tag{5}$$

Approximation theory and various numerical methods for the Stokes problem and other problems related to models of viscous incompressible fluids are well studied (see, e.g., [11, 34, 8]). In this paper, we are concerned with a posteriori error estimation methods, which form another essential part of quantitative analysis of boundary value problems. In this concise introduction we have no space to present a consequent overview of the literature related to this actively studied problem (the reader can find it in monographs [30, 20] and other publications cited therein). We recall only some previous results related to Stokes type problems, which are closely related to the method used in the paper.

Let $v \in V_0 + u_D$ be a vector valued function considered as an approximation of u. In general, a posteriori analysis of a problem is aimed to deduce fully guaranteed two sided bounds of the error $e = u - v$ in terms of the natural energy norm that depends on the problem data and known approximation v (i.e., the estimates must be indeed computable). To be of practical relevance, the estimates must possess other properties. They must be continuous with respect to basic variables, vanish at the exact solution, and do not generate significant gaps between the error and the estimate. For the Stokes problem, such estimates has been derived in [29, 33, 30].

One of them (error majorant) has the following form: let $v \in V_0$ and $\{ \operatorname{div} v \}_\Omega = 0$. Then,

$$\|\nu\nabla(u-v)\| \leq M(v,q,\tau) := \|\nu\nabla v - \tau - \mathbb{I}q\| \\ + C_F\|\operatorname{Div}\tau + f\| + C_\Gamma\|\tau_t - F\| + 2\nu\mathbb{C}_\Omega\|\operatorname{div} v\|. \quad (6)$$

Here $q \in \widetilde{L}^2(\Omega)$ is considered as an approximation of the exact pressure p and τ as an approximation of the exact stress σ. The majorant involve three constants. The first constant comes from the Friedrichs inequality

$$\|w\|_\Gamma \leq C_F\|\nabla v\| \qquad \forall v \in V_0, \quad (7)$$

C_Γ is a constant in the trace inequality

$$\|\phi\|_\Gamma \leq C_\Gamma\|\nabla\varphi\| \qquad \varphi \in H^1(\Omega), \quad (8)$$

and $\mathbb{C}_\Omega$ is the constant in Babuška–Aziz–Ladyzhenskaya–Solonnikov lemma (also called Ladyzhenskaya-Babuška-Brezzi (LBB) condition; see [3, 2, 4, 19]).

Lemma 1. *For any function* $g \in \widetilde{L}_2(\Omega)$ *there exists a function* $v \in V$ *vanishing at the boundary such that* $\operatorname{div} v = g$*, and*

$$\|\nabla v\| \leq \mathbb{C}_\Omega\|g\|,$$

where $\mathbb{C}_\Omega$ *is a positive constant depending on the shape of* Ω*.*

Lemma 1 implies the estimate

$$\inf_{\substack{v_0 \in \mathbb{S}, \\ v_0 = 0 \text{ on } \partial\Omega}} \|\nabla(v - v_0)\| \leq \mathbb{C}_\Omega\|\operatorname{div}\ v\|. \quad (9)$$

In general, the constants C_F, C_Γ, and $\mathbb{C}_\Omega$ are unknown and for domains with complicated boundaries getting guaranteed and realistic majorants of them may be a difficult task (especially for $\mathbb{C}_\Omega$). In [30], it was shown that using ideas of decomposition and Poincaré inequality, we can avoid difficulties related to C_F and in [] it was shown that in a posteriori estimates $\mathbb{C}_\Omega$ can be replaced by a collection of local constants associated with simple subdomains covering Ω (this result is based on a modified version of Lemma 1. Estimates of the constant $\mathbb{C}_\Omega$ has been studied by several authors (mostly for $d = 2$; see [6, 7, 15, 5]...). Thus, for relatively simple domains (e.g., triangles, rectangles) we have explicit estimates of $\mathbb{C}_\Omega$ and, therefore, difficulties related to this constant can be overtaken.

The trace constant C_Γ can be excluded if $\tau_t = F$ on Γ. However, this condition (as well as other more general conditions considered below) may be difficult to exactly satisfy if either F or Γ are complicated. To avoid these difficulties, we apply a version of Poincaré inequality (the so-called "sloshing" inequality), which reads

$$\|\varphi\|_\Gamma \leq \widetilde{C}_\Gamma(\Omega)\|\nabla\varphi\|_\Omega \quad (10)$$

for any $\varphi \in H^1(\Omega)$ such that $\{ \varphi \}_\Gamma = 0$. It is clear, that if we know the constant $\widetilde{C}_\Gamma(\Omega')$ for some "simple" subdomain $\Omega' \subset \Omega$ (which boundary contains Γ), then we can use it for the whole Ω. For such domains as parallelepipeds, triangles, tetrahedrons C_Γ has been found analytically or numerically (see [22, 21] and references therein). In the last paper, sharp bounds of $\widetilde{C}_\Gamma$ has been found for arbitrary simplexes. Obviously the inequality (8) for scalar valued functions yields similar inequality for vector valued functions, i.e.,

$$\|v\|_\Gamma^2 = \widetilde{C}_\Gamma^2(\Omega)\|\nabla v\|^2, \tag{11}$$

provided that $\{ v_i \}_\Gamma = 0$ for $i \in 1, \dots, d$.

1.2 Nonlinear boundary conditions

In this paper, we consider the system (1)–(3) with the following nonlinear boundary conditions on Γ:

$$u_n = 0, \quad -\sigma_t \in g\partial|u_t| \text{ on } \Gamma, \tag{12}$$

where $g \geq 0$ is a constant (in general, a scalar valued function), $u_n := u \cdot n$ and $u_t := u - nu_n$ (normal and tangential components of the velocity). Also, we define

$$\sigma_n := \sigma_n n, \quad \sigma_t := \sigma_n - \sigma_{nn} n, \quad \sigma_{nn} := \sigma_n \cdot n = \sigma_{ij} n_i n_j.$$

Since

$$\partial|z| = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0,\ z \in \mathbb{R}^d, \\ \zeta \in \mathbb{R}^d, \quad |\zeta| \leq 1 & \text{if } z = 0,\ z \in \mathbb{R}^d, \end{cases}$$

the condition (12) is equivalent to

$$|\sigma_n| \leq g,\ \sigma_n u_n + g|u_n| = 0 \quad \text{on } \Gamma.$$

Now we need to redefine the set $\mathbb{S}_0$ that contains the divergence free functions satisfying homogeneous main boundary conditions, i.e.,

$$\mathbb{S}_0 := \{v \in \mathbb{S} \mid v = 0 \text{ on } \Gamma_D,\ v_n = 0 \text{ on } \Gamma\}.$$

The generalized solution $u \in \mathbb{S}_0 + u_D$ of (1)–(3), (12) is defined by the variational inequality [9]

$$a(u, v - u) + \int_\Gamma (j(v_t) - j(u_t))\, \mathrm{d}S \geq (f, v - u) \quad \forall v \in \mathbb{S}_0, \tag{13}$$

where $j(\zeta) := g|\zeta|$ for $\eta \in H^{1/2}(\Gamma)$. Hence, if $|\zeta^*| \leq g$, then

$$D_j(\zeta, \zeta^*) = \int_\Gamma (j(\zeta) + j(\zeta^*) - \zeta^* \cdot \zeta)\, \mathrm{d}S = \int_\Gamma (g|\zeta| - \zeta^* \cdot \zeta)\, \mathrm{d}S.$$

Our analysis is based on functional type a posteriori estimates. For a consequent exposition of the theory of functional a posteriori error estimates we address the reader to [23, 30]. A posteriori estimates for elasticity problems with nonlinear boundary conditions were obtained in [24] by methods of duality theory and convex analysis. For the Stokes problem, first a posteriori estimates of the functional type were established in [28, 33]. Later, different form of these estimates has been studied in [30]. In [1], the estimates were adapted to approximations generated by the Uzawa method. In [26], a posteriori error majorants were derived for Bingham fluids.

A significant modification of the estimates was presented in [32] (for the Stokes problem) and in [25] (for the Oseen problem). In these papers, new majorants were derived, where violations of the divergence free condition are estimated by terms containing local constants $\mathbb{C}_{\Omega_i}$ for subdomains forming the domain Ω. This "localization method" is used in this paper for similar purposes. Using these ideas, we obtain new estimates that provide guaranteed error bounds for the problem with nonlinear boundary conditions of friction type.

2 Error majorant for solenoidal approximations

Let $v \in \mathbb{S}_0 + u_D$. Since $(u - v)$ vanishes on Γ_D, (13) implies

$$a(u - v, v - u) + \int_\Gamma (j(v_t) - j(u_t))\, \mathrm{d}S \geq (f, v - u) - a(v, v - u), \tag{14}$$

wherefrom

$$\begin{aligned} \| u - v \|^2 \leq \int_\Omega (\nu \nabla v - \tau - \mathbb{I} q) : \nabla(v - u)\, \mathrm{d}x \\ + \int_\Gamma (j(v_t) - j(u_t))\, \mathrm{d}S - \int_\Omega (f \cdot (v - u) - \tau : \nabla(v - u))\, \mathrm{d}x. \end{aligned} \tag{15}$$

Here τ is any tensor function in Σ, $q \in L_2(\Omega)$ and $\mathbb{I}$ is the unit tensor. In view of the Young-Fenchel inequality, for any $\eta \in L^2(\Gamma, \mathbb{R}^d)$, we have

$$\int_\Gamma -j(u_t)\, \mathrm{d}S \leq \int_\Gamma (j^*(\eta) - \eta \cdot u_t)\, \mathrm{d}S, \tag{16}$$

where j^* is the functional conjugate to j. Notice that η can be viewed as an image of the true normal stress $\sigma \cdot n$. In general, j^* is defined on a wider set $H^{-1/2}$, however, we will always operate with tensors having summable normal traces on Γ and restrict admissible arguments of j^* accordingly. From (15) and (16), it follows that

$$\begin{aligned}\|u - v\|^2 \leq \int_\Omega (\nu\nabla v - \tau - \mathbb{I}q) : \nabla(v-u)\,\mathrm{d}x \\ + \int_\Gamma \left(j(v_t) + j^*(\eta) - \eta\cdot v_t\right)\,\mathrm{d}S \\ + \int_\Gamma \eta\cdot(v_t - u_t)\,\mathrm{d}S - \int_\Omega \left(f\cdot(v-u) - \tau : \nabla(v-u)\right)\,\mathrm{d}x. \quad (17)\end{aligned}$$

Notice that for any η and τ

$$\mathcal{L}_{\eta,\tau}(w) := \int_\Omega (\tau : \nabla w - f\cdot w)dx + \int_\Gamma \eta\cdot w_t\,\mathrm{d}S$$

is a linear continuous functional on

$$V_0 := \left\{ w \in H^1(\Omega, \mathbb{R}^d) \mid w = 0 \text{ on } \Gamma_D,\ w_n = 0 \text{ on } \Gamma \right\}.$$

Define

$$|\!|\!| \mathcal{L}_{\eta,\tau} |\!|\!| := \sup_{w\in V_0} \frac{|\mathcal{L}_{\eta,\tau}(w)|}{\|\nabla w\|}.$$

The set $\operatorname{Ker} \mathcal{L}_{\eta,\tau}$ contains tensor-valued functions that satisfy (in a generalized sense) the equilibrium equation $\operatorname{Div} \tau + f$ and the condition $-\tau_t = \eta$, which states the identity between two functions serving as images of the true normal stress σn. Then,

$$\begin{aligned}\int_\Omega (\nu\nabla v - \tau - \mathbb{I}q) : \nabla(v-u)\,\mathrm{d}x + |\mathcal{L}_{\eta,\tau}(v-u)| \\ \leq \frac{\alpha}{2}\|u-v\|^2 + \frac{1}{2\alpha}\left(\|\nu\nabla v - \tau - \mathbb{I}q\|_* + \frac{1}{\sqrt{\underline{\nu}}}|\!|\!|\mathcal{L}_{\eta,\tau}|\!|\!|\right)^2,\end{aligned}$$

and we find that for any $\alpha < 2$

$$\begin{aligned}\frac{1}{2}\|u-v\|^2 \leq M^1_\oplus(v,\tau,\eta,q,\alpha) =: \frac{1}{2-\alpha}D_j(v_t,\eta) \\ + \frac{1}{2\alpha(2-\alpha)}\left(\|d(v,q,\tau)\|_* + \frac{1}{\sqrt{\underline{\nu}}}|\!|\!|\mathcal{L}_{\eta,\tau}|\!|\!|\right)^2, \quad (18)\end{aligned}$$

where

$$D_j(v_t,\eta) := \int_\Gamma \left(j(v_t) + j^*(\eta) - \eta\cdot v_t\right)\,\mathrm{d}S$$

and

$$d(v,q,\tau) := \nu\nabla v - \tau - \mathbb{I}q.$$

Assume that $M^1_\oplus(v,\tau,\eta,q,\alpha)$=0. Then,

$$\operatorname{Div}\tau + f = 0, \quad \eta = -\tau_t, \quad \eta \in \partial j(v_t), \quad \tau = \nu\nabla v - \mathbb{I}q.$$

These relations mean that $-\tau_n \in \partial j(v_t)$ and τ and q satisfy (1). Thus, the majorant vanishes if and only if v, τ, and q coincide with the exact velocity, stress, and pressure.

So far the estimate is not fully computable because of the term $[\![\mathcal{L}_{\eta,\tau}]\!]$ is defined with the help of supremum over an infinite amount of functions (however, majorants of these type can be useful for other purposes, e.g., for analysis of errors in Uzawa type methods). We can deduce a fully computable error majorant if the sets of admissible η and τ are narrowed, namely,

$$\tau \in H_\Gamma(\operatorname{Div},\Omega) := \left\{\tau \in H(\operatorname{Div},\Omega),\ \tau_t \in \widetilde{L}^2(\Omega,\mathbb{R}^d) \text{ on } \Gamma\right\}, \tag{19}$$

$$\int_\Gamma (\eta - \tau_t)\,\mathrm{d}S = 0. \tag{20}$$

It is clear that for $\tau = \sigma$ and $\eta = \sigma_t$ this condition holds, so that these restrictions do not exclude physically meaningful functions. In this case,

$$\mathcal{L}_{\eta,\tau}(w) = \int\limits_\Omega (f + \operatorname{Div}\tau)\cdot w\,\mathrm{d}x + \int\limits_\Gamma (\eta - \tau_t)\cdot w_t\ \mathrm{d}S.$$

Notice that $w_n = 0$ on Γ, so that the last integral is estimated as follows:

$$\begin{aligned}\int_\Gamma (\eta-\tau_t)\cdot w_t\,\mathrm{d}S &= \int_\Gamma (\eta-\tau_t)\cdot(w_t - \{\,w_t\,\}_\Gamma)\,\mathrm{d}S \\ &\le \|\eta-\tau_t\|_\Gamma\,\|w - \{\,w\,\}_\Gamma\| \le \widetilde{C}_\Gamma\|\eta-\tau_t\|_\Gamma\,\|\nabla w\|.\end{aligned}$$

where $\widetilde{C}_\Gamma$ is a constant in the inequality (11). Analogously,

$$\int\limits_\Omega (f + \operatorname{Div}\tau)\cdot w\,\mathrm{d}x \le C_F\|f + \operatorname{Div}\tau\|\|\nabla w\|,$$

where C_F is a constant in the Friedrichs type inequality

$$\|w\| \le C_F\|\nabla w\| \quad \forall w \in V_0. \tag{21}$$

As a result, we find that

$$[\![\mathcal{L}_{\eta,\tau}]\!] \ \le\ C_F\|f + \operatorname{Div}\tau\| + \widetilde{C}_\Gamma\|\eta - \tau\cdot n\|_\Gamma$$

and replace (18) by

$$\begin{aligned}\frac{1}{2}|\!|\!|u-v|\!|\!|^2 \le M^2_\oplus(v,\tau,\eta,q,\alpha) &:= \frac{1}{2-\alpha}D_j(v_t,\eta) \\ &+ \frac{1}{2\alpha(2-\alpha)}\left(|\!|\!|d(v,q,\tau)|\!|\!|_* + \frac{1}{\sqrt{\underline{\nu}}}(C_F\|f+\operatorname{Div}\tau\| + \widetilde{C}_\Gamma\|\eta-\tau_t\|_\Gamma)\right)^2.\end{aligned} \tag{22}$$

Now, the only remaining difficulty is that the majorant is applicable only to solenoidal approximation v.

3 Non solenoidal approximations

We begin with one auxiliary result important for extending majorant $M^2_\oplus$ to functions in $V_0 + u_D$. In [30, Chapter 6, Sect. 2.6], it was shown that if $v \in H^1(\Omega, \mathbb{R}^d)$ satisfies homogeneous Dirichlet boundary conditions on Γ_D (which is only a part of the overall boundary) and $\{\operatorname{div} v\}_\Omega = 0$, then the distance to the set of divergence free fields vanishing on Γ_D meets (9). Since in our case the mean condition is automatically satisfied, we conclude that

$$\inf_{v_0 \in \mathbb{S}_0} \|\nabla(v - v_0)\| \;\leq\; \mathbb{C}_\Omega \|\operatorname{div} v\|. \tag{23}$$

Let $v \in V_0 + u_D$. Then, for any $\beta > 0$, we have

$$\begin{aligned} |\!|\!| u - v |\!|\!|^2 &\leq (|\!|\!| u - v_0 |\!|\!| + |\!|\!| v - v_0 |\!|\!|)^2 \\ &\leq (1 + \beta) |\!|\!| u - v_0 |\!|\!|^2 + (1 + \frac{1}{\beta}) |\!|\!| v - v_0 |\!|\!|^2, \end{aligned}$$

where $v_0 \in V$ is a divergence free field vanishing on Γ. We need to estimate $M^1_\oplus(v_0, \tau, \eta, q, \alpha)$, which majorates the first norm in the right-hand side.

Consider the respective parts of the majorant. First, we notice that

$$\begin{aligned} |\!|\!| d(v_0, q, \tau) |\!|\!|_* &\leq |\!|\!| \nu \nabla v - \tau - \mathbb{I} q |\!|\!|_* + |\!|\!| \nu \nabla (v - v_0) |\!|\!|_* \\ &\leq |\!|\!| \nu \nabla v - \tau - \mathbb{I} q |\!|\!|_* + |\!|\!| \nabla (v - v_0) |\!|\!| \\ &\leq |\!|\!| \nu \nabla v - \tau - \mathbb{I} q |\!|\!|_* + \sqrt{\overline{\nu}} \|\nabla (v - v_0)\|. \end{aligned}$$

Since v_0 vanishes on the boundary

$$\begin{aligned} \int_\Gamma (j(v_{0t}) + j^*(\eta) - \eta \cdot v_{0t}) \, \mathrm{d}S &= D_j(v_t, \eta) + \int_\Gamma (\eta \cdot v_t - j(v_t)) \, \mathrm{d}S \\ &\leq D_j(v_t, \eta) + \int_\Gamma ((\eta - \tau_t) \cdot v_t + j^*(\tau_t)) \, \mathrm{d}S. \end{aligned}$$

Notice that $j^*(\tau_t) = 0$ if $|\tau_t| \leq g$ and

$$\begin{aligned} \int_\Gamma (\eta - \tau_t) \cdot v_t \, \mathrm{d}S &= \int_\Gamma (\eta - \tau_t) \cdot (v_t - v_{0t}) \, \mathrm{d}S \\ &\leq \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla (v_t - v_{0t})\| \leq \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla (v - v_0)\|. \end{aligned}$$

We find that

$$\begin{aligned} M^1_\oplus(v_0, \tau, \eta, q, \alpha, \beta) \;\leq\; & \frac{1}{2 - \alpha} D_j(v_t, \eta) + \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla (v - v_0)\| \\ & + \frac{1}{2\alpha(2 - \alpha)} \Big(R(v, q, \tau, \eta) + \sqrt{\overline{\nu}} \|\nabla (v - v_0)\| \Big)^2, \end{aligned}$$

where

$$R(v,q,\tau,\eta) := \|d(v,q,\tau)\|_* + \frac{1}{\sqrt{\underline{\nu}}}\left(C_F\|f+\operatorname{Div}\tau\| + \widetilde{C}_\Gamma\|\eta-\tau_t\|_\Gamma\right).$$

In view of Lemma 1, there exists v_0 such that

$$\|\nabla(v-v_0)\| \;\leq\; \mathbb{C}_\Omega\|\operatorname{div} v\|.$$

Hence, we can estimate all terms containing the respective norms. As a result, we arrive at the estimate.

$$\begin{aligned}\|u-v\|^2 \leq \frac{1+\beta}{2-\alpha}\Bigg(& D_j(v_t,\eta) + \frac{1}{2\alpha}R^2(v,q,\tau,\eta) \\ & + \left(\frac{\sqrt{\overline{\nu}}}{\alpha}R(v,q,\tau,\eta) + (2-\alpha)\widetilde{C}_\Gamma\|\eta-\tau_t\|\right)\mathbb{C}_\Omega\|\operatorname{div} v\| \\ & + \frac{4\alpha-2\alpha^2+\beta}{2\alpha}\overline{\nu}\mathbb{C}_\Omega^2\|\operatorname{div} v\|^2\Bigg). \qquad (24)\end{aligned}$$

This estimate holds for $\alpha \in (0,2)$, $\beta > 0$, and τ and η must satisfy additional conditions

$$|\tau_t| \leq g, \quad |\eta| \leq g.$$

We see that the right-hand side of (24) contains terms of three types. First, these are the terms without a penalty for violations of the divergence free condition (in the majority of cases these "zero order" terms will contain the main part of the approximation error). The second term contains $\|\operatorname{div} v\|$. It adds the major correction associated with not exact satisfaction of the divergence free condition. Finally, the third term contains "second order" corrections related to violations of the divergence free condition.

Remark 1. Finding $\mathbb{C}_\Omega$ (or a close lower bound of this quantity) for an arbitrary domain Ω is not an easy task. To overcome this difficulty, we use the following result from [31].

Lemma 2. *Let Ω be represented as a set of nonintersecting convex subdomains Ω_i, $i = 1,\ldots,N$, i.e., $\bar{\Omega} = \sum_{i=1}^N \bar{\Omega}_i$.*

For any function

$$v \in V_N^{\operatorname{div}} := \{w \in V_0 \mid \{\operatorname{div} v\}_{\Omega_i} = 0\}$$

there exists $v_0 \in \mathbb{S}_0$ such that

$$\|\nabla(v-v_0)\|^2 \leq d_N^2(v) := \sum_{i=1}^N \mathbb{C}_{\Omega_i}^2\|\operatorname{div} v\|^2. \qquad (25)$$

By means of this lemma and arguments analogous to those used above, we obtain another estimate

$$\|\!|u - v|\!\|^2 \le \frac{1+\beta}{2-\alpha}\Big(D_j(v_t,\eta) + \frac{1}{2\alpha} R^2(v,q,\tau,\eta) + \left(\frac{\sqrt{\overline{\nu}}}{\alpha} R(v,q,\tau,\eta) + (2-\alpha)\widetilde{C}_\Gamma \|\eta - \tau_t\| \right) d_N(v) + \frac{4\alpha - 2\alpha^2 + \beta}{2\alpha} \overline{\nu} d_N^2(v) \Big). \quad (26)$$

The estimate (26) can be viewed as a generalisation of (6) to the case of friction type boundary conditions, where the trace constant is replaced by the constant $\widetilde{C}_\Gamma$ and $\mathbb{C}_\Omega$ is replaced by a collection of local constants $\mathbb{C}_{\Omega_i}$ for subdomains (which may be easier to find or to estimate from above). Therefore, this estimate may be more convenient numerically than the estimates based on global constants.

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PIII

A POSTERIORI ERROR BOUNDS FOR APPROXIMATIONS OF THE OSEEN PROBLEM AND APPLICATIONS TO UZAWA ITERATION ALGORITHM

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Research Article

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A Posteriori Error Bounds for Approximations of the Oseen Problem and Applications to the Uzawa Iteration Algorithm

Abstract: We derive computable bounds of deviations from the exact solution of the stationary Oseen problem. They are applied to approximations generated by the Uzawa iteration method. Also, we derive an advanced form of the estimate, which takes into account approximation errors arising due to discretization of the boundary value problem, generated by the main step of the Uzawa method. Numerical tests confirm our theoretical results and show practical applicability of the estimates.

Keywords: Oseen Problem, Estimates of Deviations from Exact Solutions, Uzawa Iteration Method

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1 Introduction

We consider the stationary Oseen problem in a bounded connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$. Throughout the paper, we use the following notation: n denotes the outward unit normal vector to the boundary $\partial\Omega$; the space of scalar valued square summable functions with zero mean is denoted by $\tilde{L}^2(\Omega)$; $S_0(\Omega)$ denotes the closure of smooth solenoidal functions with compact supports in Ω with respect to the norm of $V(\Omega) := H^1(\Omega, \mathbb{R}^d)$; and $V_0(\Omega)$ denotes the subspace of $V(\Omega)$ that consists of the functions with zero traces on $\partial\Omega$.

Also, we use spaces of tensor valued functions $\Sigma(\Omega) := L^2(\Omega, \mathbb{M}^{d\times d})$, where $\mathbb{M}^{d\times d}$ is the space of $d \times d$-matrices (tensors). $\mathbb{I}$ denotes the unit element of $\mathbb{M}^{d\times d}$. The L^2-norms of scalar and vector valued functions are denoted by $\|\cdot\|$ and the corresponding inner products are denoted by $(\ ,\ )$. The scalar product of tensors is denoted by two dots (:), and the norm of Σ is denoted by $\|\cdot\|_\Sigma$. By div and Div, we denote the divergence of vector and tensor fields, respectively. Finally, we introduce the Hilbert space

$$\Sigma(\mathrm{Div}, \Omega) := \{w \in \Sigma(\Omega) \mid \mathrm{Div}\, w \in L^2(\Omega, \mathbb{R}^d)\},$$

which can be viewed as a tensor analogous to the vector space $H(\Omega, \mathrm{div})$ containing L^2 vector functions with square summable divergence.

The classical formulation of the stationary Oseen problem is to find the velocity field $u \in S_0(\Omega) + u_D$ and the pressure function $p \in \tilde{L}^2(\Omega)$, which satisfy the relations

$$-\mathrm{Div}(\nu\nabla u) + \mathrm{Div}(a \otimes u) = f - \nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\mathrm{div}\, u = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = u_D \quad \text{on } \partial\Omega, \tag{1.3}$$

where a, u_D, and f are given vector valued functions. It is assumed that

$$\int_{\partial\Omega} u_D \cdot n \, \mathrm{d}x = 0, \tag{1.4}$$

that the viscosity ν is a positive bounded function, i.e., $0 < \underline{\nu} \le \nu(x) \le \bar{\nu}$ for all $x \in \overline{\Omega}$, and that $a \in S_0(\Omega)$ is a bounded vector function.

The generalized solution of (1.1)–(1.4) is a function $u \in S_0(\Omega) + u_D$ such that

$$\int_\Omega \big(\nu\nabla u : \nabla w - (a \otimes u) : \nabla w\big)\,dx = \int_\Omega f \cdot w\,dx \quad \text{for all } w \in S_0(\Omega). \tag{1.5}$$

Existence and uniqueness of generalized solutions to the Stokes and Oseen problems are well established (see, e.g., [9]). In essence, the corresponding results are based on the following lemma.

Lemma 1.1. *For any function $g \in \tilde{L}^2(\Omega)$, there exists a function $v \in V_0(\Omega)$ satisfying the condition $\operatorname{div} v = g$ such that*

$$\|\nabla v\|_\Sigma \le \kappa_\Omega \|g\|.$$

Here κ_Ω is a positive constant depending only on the domain Ω.

We note that the constant inverse to κ_Ω arises in the so-called Ladyzhenskaya–Babuška–Brezzi (LBB) condition (see, e.g., [2, 4]), which can be viewed as a different form of Lemma 1.1. Also, these results guarantee boundedness of the energy norm of the exact solution, namely, $|||u||| := \|\nu^{1/2}\nabla u\|_\Sigma \le c$, where the constant c depends on the problem data and on the constant $C_{F\Omega}$ in the Friedrichs type inequality

$$\|w\| \le C_{F\Omega}\|\nabla w\|_\Sigma \quad \text{for all } w \in V_0.$$

The constants $C_{F\Omega}$ and κ_Ω play an important role in our analysis because they control distances between a vector valued function and the set of solenoidal fields evaluated in different norms (see [12, 15–17]). In particular, Lemma 1.1 implies an important corollary: for any $v \in V_0(\Omega)$ there exists $v_0 \in S_0(\Omega)$ such that

$$\|\nabla(v - v_0)\|_\Sigma \le \kappa_\Omega \|\operatorname{div} v\|. \tag{1.6}$$

A similar estimate holds for $v \in V_0(\Omega) + u_D$ with some $v_0 \in S_0(\Omega) + u_D$.

If functions vanish on the whole boundary, then a guaranteed upper bound of $C_{F\Omega}$ is easy to find. For some domains the constant $C_{\mathrm{LBB}} = \kappa_\Omega^{-1}$ or computable bounds for it can be found if the field satisfies some additional requirements (see, e.g., [5, 8, 13, 18, 20]).

In [16, 17], guaranteed and fully computable bounds of the distance between the exact solution of the stationary Stokes problem and any function in $V_0(\Omega)+u_D$ were derived by transformations of integral relations similar to (1.5). If the function compared with u is an approximation, then these estimates yield robust and efficient a posteriori error bounds (for the Stokes problem, they were numerically tested in [6, 7], see also [11]). In [20], analogous estimates were derived for the generalized Stokes problem. In Section 2 of the present paper, we use the same ideas in order to derive estimates of the distance to the exact solution of (1.1)–(1.4). We obtain estimates for the velocity, pressure, and stress fields. In Section 3, similar estimates are derived for the combined error norm, which encompasses errors of approximations related to all fields. In Section 4, the estimates are applied to approximations generated by the Uzawa algorithm. Section 5 contains results of numerical tests, which confirm practical applicability and efficiency of the estimates.

2 Estimates of Deviations from the Exact Velocity Field

Theorem 2.1. *Let $v \in V_0(\Omega) + u_D$. Then for all $q \in \tilde{L}^2(\Omega)$ and $\tau \in \Sigma(\Omega)$ we have*

$$|||u - v||| \le \underline{\nu}^{-1/2}\|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v,\tau,q)\|_\Sigma + (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\operatorname{div} v\| := M_\oplus(v,\tau,q), \tag{2.1}$$

where

$$r(\tau) := f + \operatorname{Div}\tau, \quad C_\Omega = C_{F\Omega}\|\nu^{-1/2} a\|_{\infty,\Omega}, \quad d(v,\tau,q) := \tau - \nu\nabla v + a \otimes v + \mathbb{I}q,$$

and

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - \tau : \nabla w)\,dx}{\|\nabla w\|_\Sigma}.$$

Proof. For any $v_0 \in S_0(\Omega) + u_D$, we have

$$|||u - v||| \le |||u - v_0||| + |||v_0 - v|||. \tag{2.2}$$

First, we estimate from above the first term of the right-hand side of (2.2). Let $w \in S_0(\Omega)$. By subtracting the integral

$$\int_\Omega \big(\nu \nabla v_0 : \nabla w - (a \otimes v_0) : \nabla w\big)\,\mathrm{d}x$$

from both sides of (1.5), we obtain

$$\int_\Omega \big(\nu \nabla (u - v_0) : \nabla w - (a \otimes (u - v_0)) : \nabla w\big)\,\mathrm{d}x = \int_\Omega \big(f \cdot w - \nu \nabla v_0 : \nabla w + (a \otimes v_0) : \nabla w\big)\,\mathrm{d}x. \tag{2.3}$$

For any $\tau \in \Sigma(\Omega)$ and $q \in \tilde{L}^2(\Omega)$ we rewrite the right-hand side and estimate it as follows:

$$\begin{aligned}
&\int_\Omega \big(f \cdot w - \tau : \nabla w + (\tau - \nu \nabla v_0 + a \otimes v_0 + \mathbb{I} q) : \nabla w\big)\,\mathrm{d}x \\
&= \int_\Omega \big(r(\tau) \cdot w + d(v_0, \tau, q) : \nabla w\big)\,\mathrm{d}x \\
&\le \|r(\tau)\|_{-1,\Omega} \|\nabla w\|_\Sigma + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \|\nu^{1/2} \nabla w\|_\Sigma \\
&\le \big(\underline{\nu}^{-1/2} \|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma\big) \|\nu^{1/2} \nabla w\|_\Sigma.
\end{aligned} \tag{2.4}$$

Set $w = u - v_0$. Since

$$\int_\Omega (a \otimes (u - v_0)) : \nabla (u - v_0)\,\mathrm{d}x = 0,$$

the estimates (2.4) and (2.3) yield

$$|||u - v_0||| \le \underline{\nu}^{-1/2} \|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma. \tag{2.5}$$

Now, we estimate the second term in the right-hand side of (2.5). We have

$$\|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \le \|\nu^{1/2} \nabla (v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma + \|\nu^{-1/2} a \otimes (v_0 - v)\|_\Sigma.$$

Note that

$$\|\nu^{-1/2} a \otimes (v_0 - v)\|_\Sigma \le \|\nu^{-1/2} a\|_{\infty,\Omega} \|v_0 - v\|,$$

where

$$\|\nu^{-1/2} a\|_{\infty,\Omega} := \max_{i=1,\dots,d} \sup_{x \in \Omega} \{\nu^{-1/2} a_i\}.$$

We find that

$$\|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \le (\bar{\nu}^{1/2} + C_\Omega) \|\nabla (v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma.$$

Hence,

$$|||u - v||| \le (2\bar{\nu}^{1/2} + C_\Omega) \|\nabla (v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma + \underline{\nu}^{-1/2} \|r(\tau)\|_{-1,\Omega}.$$

In view of (1.6), we finally obtain (2.1). □

Remark 2.2. If $\tau \in \Sigma(\mathrm{Div}, \Omega)$, then it is easy to show that

$$\|r(\tau)\|_{-1,\Omega} \le C_{F\Omega} \|r(\tau)\|.$$

In this case, (2.1) is reduced to the majorant derived for the Oseen problem in [17].

3 Estimate of Deviations from the Exact Pressure and Stress Fields

Let $q \in \tilde{L}^2(\Omega)$ be a function considered as an approximation of the exact pressure p. Then $(p-q) \in \tilde{L}^2(\Omega)$ and due to Lemma 1.1 there exists a function $\bar{w} \in V_0(\Omega)$ such that

$$\operatorname{div}(\bar{w}) = p - q \tag{3.1}$$

and

$$\|\nabla \bar{w}\|_\Sigma \le \kappa_\Omega \|p - q\|. \tag{3.2}$$

As in the case of the Stokes problem (see [16, 17]), this fact allows us to deduce computable majorants of $\|p-q\|$.

Theorem 3.1. *Let $q \in \tilde{L}^2(\Omega)$. Then for all $\tau \in \Sigma(\Omega)$*

$$\frac{1}{\kappa_\Omega}\|p - q\| \le C_{a,\nu}|||u - v||| + \|d(v,\tau,q)\|_\Sigma + \|r(\tau)\|_{-1,\Omega}, \tag{3.3}$$

where $C_{a,\nu} := (\bar{\nu}^{1/2} + \underline{\nu}^{-1/2} C_{F\Omega}\|a\|_{\infty,\Omega})$ and $|||u - v|||$ is estimated by (2.1).

Proof. From (3.1) we have

$$\|p - q\|^2 = \int_\Omega \operatorname{div} \bar{w}(p - q)\,\mathrm{d}x = \int_\Omega (\operatorname{div} \bar{w}\, p - q\mathbb{I} : \nabla \bar{w})\,\mathrm{d}x. \tag{3.4}$$

Multiplying (1.1) by $\bar{w}$ and integrating over Ω, we obtain

$$\int_\Omega \operatorname{div} \bar{w}\, p\,\mathrm{d}x = \int_\Omega \big(\nu\nabla u : \nabla\bar{w} - (a \otimes u) : \nabla \bar{w} - f \cdot \bar{w}\big)\,\mathrm{d}x. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\|p - q\|^2 \le \int_\Omega \big((\nu\nabla v - \tau - a \otimes v - \mathbb{I}q) : \nabla\bar{w}\big)\,\mathrm{d}x + \|r(\tau)\|_{-1,\Omega}\|\nabla \bar{w}\| + \int_\Omega \big(\nu\nabla(u - v) : \nabla\bar{w} - a \otimes (u - v) : \nabla\bar{w}\big)\,\mathrm{d}x.$$

Here

$$\begin{aligned}\int_\Omega \big(\nu\nabla(u - v) : \nabla\bar{w} - a \otimes (u - v) : \nabla\bar{w}\big)\,\mathrm{d}x &\le \bar{\nu}^{1/2}|||u - v|||\|\nabla\bar{w}\|_\Sigma + \|a\|_{\infty,\Omega}\|u - v\|\|\nabla\bar{w}\|_\Sigma \\ &\le \kappa_\Omega\big(\bar{\nu}^{1/2} + \underline{\nu}^{-1/2} C_{F\Omega}\|a\|_{\infty,\Omega}\big)|||u - v|||\|p - q\|\end{aligned}$$

and in view of (3.2) we have

$$\int_\Omega d(v,\tau,q) : \nabla\bar{w}\,\mathrm{d}x \le \|d(v,\tau,q)\|_\Sigma\|\nabla\bar{w}\|_\Sigma \le \kappa_\Omega\|d(v,\tau,q)\|_\Sigma\|p - q\|.$$

Thus, we arrive at the estimate (3.3). □

The exact solution generates the tensor

$$\sigma := \nu\nabla u - a \otimes u - p\mathbb{I}.$$

Assume that $v \in V_0(\Omega) + u_D$, $\eta \in \Sigma(\Omega)$, and $q \in \tilde{L}^2(\Omega)$ approximate u, σ, and p, respectively. Then,

$$\begin{aligned}\|\eta - \sigma\|_\Sigma = \|\eta - \nu\nabla u + a \otimes u + p\mathbb{I}\|_\Sigma &\le \|\eta - \nu\nabla v + a \otimes v + q\mathbb{I}\|_\Sigma + \|\nu\nabla(u - v)\|_\Sigma + \|a\|_{\infty,\Omega}\|u - v\| + \sqrt{d}\|p - q\| \\ &\le \|d(v,\eta,q)\|_\Sigma + C_{a,\nu}|||u - v||| + \sqrt{d}\|p - q\|.\end{aligned}$$

By (3.3) we obtain

$$\|\eta - \sigma\|_\Sigma \le \sqrt{d}\kappa_\Omega\|d(v,\tau,q)\|_\Sigma + \|d(v,\eta,q)\|_\Sigma + \sqrt{d}\kappa_\Omega\|r(\tau)\|_{-1,\Omega} + (1 + \sqrt{d}\kappa_\Omega)C_{a,\nu}M_\oplus(v,\tau,q), \tag{3.6}$$

where $C_{a,\nu}$ is defined in Theorem 3.1.

Remark 3.2. If we choose $\eta = \nu\nabla v - a \otimes v - q\mathbb{I}$, then $\|d(v, \eta, q)\|_\Sigma = 0$.

Also, we can measure the error in terms of the norm of the product space

$$W := (V_0(\Omega) + u_D) \times \tilde{L}^2(\Omega) \times \Sigma(\Omega),$$

which is

$$\|(v, q, \eta)\|_W := |||v||| + \|q\| + \|\eta\|_\Sigma.$$

Combining the estimates (2.1), (3.3) and (3.6) we find that

$$\|(u - v, p - q, \eta - \sigma)\|_W \leq c_\oplus M_\oplus(v, \tau, q) + \|d(v, \eta, q)\|_\Sigma,$$

where

$$c_\oplus := 1 + (\kappa_\Omega + \sqrt{d}\kappa_\Omega)(C_{a,\nu} + \max\{1, C_{F\Omega}\}).$$

4 Error Estimates for Approximate Solutions Generated by the Uzawa Algorithm

Uzawa type algorithms are commonly used for solving various saddle point problems (see, e.g., the survey article [3]). They are widely used in numerical analysis of incompressible media. In our case, the algorithm can be used in the following form:

(1) Set $k = 0$ and $p^0 \in \tilde{L}^2(\Omega)$.

(2) Find $u^k \in V_0(\Omega) + u_D$ such that

$$\int_\Omega (\nu\nabla u^k : \nabla w - (a \otimes u^k) : \nabla w)\,dx = \int_\Omega (f \cdot w + p^k \operatorname{div} w)\,dx \quad \text{for all } w \in V_0. \tag{4.1}$$

(3) Find

$$p^{k+1} = p^k - \rho \operatorname{div} u^k, \quad \text{where } \rho \in (0, \bar{\rho}). \tag{4.2}$$

(4) Set $k = k + 1$ and go to step (2).

It is well known (see, e.g., [21]) that approximations generated by the Uzawa algorithm converge (as $k \to \infty$) in the sense that

$$u^k \to u \quad \text{in } V(\Omega, \mathbb{R}^d), \quad p^k \to p \quad \text{weakly in } L^2(\Omega)$$

provided that

$$0 < \rho < \bar{\rho} := 2\underline{\nu}.$$

Our first goal is to deduce computable and realistic estimates of $u^k - u$ and $p^k - p$ in terms of the respective norms.

For this purpose, we use results of previous sections. We set

$$v = u^k, \quad q = p^k, \quad \tau = \nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k.$$

In this case,

$$d(v, \tau, q) := \tau - \nu\nabla u^k + a \otimes u^k + \mathbb{I}p^k = 0$$

and in view of (4.1),

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - (\nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k) : \nabla w)\,dx}{\|\nabla w\|_\Sigma} = 0.$$

We use the estimate (2.1) and arrive at the following result.

Theorem 4.1. *Let u^k be the exact solution computed in the k-th step of the Uzawa algorithm. Then*

$$|||u - u^k||| \leq (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\operatorname{div} u^k\| := M_\oplus^{Uz}(u^k) \tag{4.3}$$

and

$$\|p - p^k\| \leq \kappa_\Omega C_{a,\nu} M_\oplus^{Uz}(u^k). \tag{4.4}$$

Remark 4.2. Since

$$|||u^k - u|||^2 \geq \frac{1}{d}\underline{\nu}\|\operatorname{div} u^k\|^2 =: M_\ominus^{Uz}(u^k),$$

we find that

$$M_\ominus^{Uz}(u^k) \leq |||u - u^k||| \leq M_\oplus^{Uz}(u^k). \tag{4.5}$$

This means that the efficiency index of the majorant is bounded by an explicitly computable constant, namely

$$I^{\text{eff}}(M_\oplus^{Uz}(u^k)) := \frac{M_\oplus^{Uz}(u^k)}{|||u - u^k|||} \leq I_\oplus^{\text{eff}}, \tag{4.6}$$

where

$$I_\oplus^{\text{eff}} := \frac{M_\oplus^{Uz}(u^k)}{M_\ominus^{Uz}(u^k)} \leq \frac{2\bar{\nu}^{1/2} + C_{F\Omega}\|\nu^{-1/2}a\|_{\infty,\Omega}}{\underline{\nu}^{1/2}}\sqrt{d}\kappa_\Omega.$$

We note that $\kappa_\Omega \geq \frac{1}{\sqrt{d}}$, so that $I_\oplus^{\text{eff}} \geq 1$ (which of cause also follows directly from (4.5) and (4.6)).

In particular, for the Stokes problem with constant ν the ratio is smaller than $2\sqrt{d}\kappa_\Omega$. The estimate (4.5) shows that the quantity $\|\operatorname{div} u^k\|$ reliably controls convergence of u^k to u in V.

The estimates (4.3) and (4.4) are of theoretical relevance. In practice, the problem (4.1) is solved numerically on a certain mesh $\mathcal{T}_h$, whose cells have the characteristic size h. For this case, we need an advanced form of the error majorant, which is derived below.

Let $V_{0h}(\Omega, \mathbb{R}^d)$ and $\tilde{L}_h^2(\Omega)$ be finite dimensional subspaces of $V_0(\Omega)$ and $\tilde{L}^2(\Omega)$, respectively. We also assume that the spaces are constructed so that the corresponding numerical problem is stable and satisfies the discrete LBB-condition.

Let $u_h^k \in V_{0h} + u_D$ be an approximation of u^k calculated in the k-th step (4.1) of the Uzawa algorithm and $p_h^k, p_h^{k+1} \in \tilde{L}_h^2(\Omega)$ be approximations of the pressure related to step (4.2). Our goal is to derive a fully computable error majorant for the pair (u_h^k, p_h^k) generated in step k.

Theorem 4.3. *For any $\eta \in \Sigma(\Omega, \operatorname{Div})$,*

$$|||u - u_h^k||| \leq \mathcal{E}^h(u_h^k, p_h^k, \eta) + M_\oplus^{Uz}(u_h^k) := M_\oplus^{Uz}(u_h^k, p_h^k, \eta),$$

where the first term

$$\mathcal{E}(u_h^k, p_h^k, \eta) = \underline{\nu}^{-1/2}(C_{F\Omega}\|r(\eta)\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma)$$

is related to the approximation error and the second term presents the error associated with the Uzawa method.

Analogously,

$$\frac{1}{\kappa_\Omega}\|p - p_h^k\| \leq (1 + C_{a,\nu})\mathcal{E}(u_h^k, p_h^k, \eta) + C_{a,\nu}M_\oplus^{Uz}(u_h^k).$$

Proof. We set

$$v = u_h^k, \quad q = p_h^k, \quad \tau = \nu\nabla u_h^k - a \otimes u_h^k - \mathbb{I}p_h^k$$

and use the estimate (2.1). In this case, $d(u_h^k, \tau, p_h^k) = 0$ and

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - (\nu\nabla u_h^k - a \otimes u_h^k - \mathbb{I}p_h^k) : \nabla w)\,\mathrm{d}x}{\|\nabla w\|_\Sigma}.$$

Let $\eta \in \Sigma(\operatorname{Div}, \Omega)$. Then,

$$\|r(\eta)\|_{-1,\Omega} = \sup_{w \in V_0(\Omega)} \frac{\int_\Omega ((f + \operatorname{Div}\eta) \cdot w - (d(u_h^k, \eta, p_h^k)) : \nabla w)\,\mathrm{d}x}{\|\nabla w\|_\Sigma} \leq C_{F\Omega}\|f + \operatorname{Div}\eta\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma.$$

Hence, we arrive at estimate (4.3). Now, we use (3.3) and find that

$$\frac{1}{\kappa_\Omega}\|p - p_h\| \le C_{a,\nu}|||u - u_h^k||| + C_{F\Omega}\|f + \operatorname{Div}\eta\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma. \qquad \square$$

Remark 4.4. Analogously to (4.5), we find that

$$M_\ominus^{Uz}(u_h^k) \le |||u - u_h^k||| \le M_\oplus^{Uz}(u_h^k) + \mathcal{E}(u_h^k, p_h^k, \eta), \tag{4.7}$$

which means that the guaranteed efficiency index of the error majorant is subject to similar estimates, namely

$$I^{\text{eff}}\big(M_\oplus^{Uz}(u_h^k)\big) \le I_\oplus^{\text{eff}} + I_k^{\text{eff}}, \tag{4.8}$$

where the second term

$$I_k^{\text{eff}} := \frac{\mathcal{E}(u_h^k, p_h^k, \eta)}{|||u - u_h^k|||}$$

represents the efficiency index associated with the approximation error.

We end up this section with a short comment on practical applications of the estimate (4.7). First we note that it has the form which is natural to expect. It is clear that the quality of error estimation related to solving the boundary value problem (4.1) by means of a certain numerical method should enter the estimate and increase the overall value of the majorant (cf. (4.8)). In the numerical tests presented below, we indeed observed this effect. In these examples, the function η was defined by means of very simple (and very cheap) reconstructions of the numerical stress (based on local averaging) and, therefore, the term $\mathcal{E}(u_h^k, p_h^k, \eta)$ made a considerable contribution to the overall error bound. Nevertheless, the majorant correctly reflects the decreasing of the error in the process of the Uzawa iterations. Certainly more sophisticated stress reconstruction procedures (e.g., global minimization) would lead to much better results (see a consequent discussion of the corresponding methods in [11]). However, even if the approximation error would be defined sharply, for sufficiently large k the right-hand side of (4.7) will not decrease because the mesh $\mathcal{T}_h$ is too coarse for getting approximations with a required accuracy. In practice, this "saturation" phenomenon is easily detected by comparing the values of two terms forming the majorant (in our tests this phenomenon was observed). This means that fully reliable computations based on the Uzawa type methods require "modeling-discretization" adaptive algorithms in the spirit of, e.g., [19].

5 Numerical Experiments

Below we present results from numerical computations performed to test the majorants and minorants. Approximations for model problem were calculated with MINI-elements [1] for the velocity field, linear triangular elements for the pressure field, and linear Raviart–Thomas elements [14] for the stress field.

We consider the Oseen problem with $\nu = 1$ in $\Omega = [0,1] \times [0,1]$ and homogeneous Dirichlet boundary conditions. The exact velocity

$$u(x,y) = \begin{pmatrix} 20x^2y(2y-1)(x-1)^2(y-1) \\ -20xy^2(2x-1)(x-1)(y-1)^2 \end{pmatrix}$$

and the pressure $p(x,y) = 2x - 1$ generate the right-hand side of the equation. The iterations were started with $p^0 = 0$ in Ω. Computations were performed with the help of FEniCS Project open source software [10]. Uniform refinements of the mesh were performed if the majorant for the velocity field shows that practically the error does not decrease (if the absolute value of difference between the values computed for two consecutive iterations was less than 10%). At the very beginning we had 512 elements. At every refinement one triangle element was divided into four similarly shaped triangles (so that we had 2048 degrees of freedom after the first refinement, and then 8192 after the second refinement). The algorithm was stopped after the third mesh refinement.

k	$\frac{M_{\oplus}(v,\tau,q)}{\|v\|}$	$\frac{\|\operatorname{div} v\|}{\|v\|}$	$\frac{\|d(v,\tau,q)\|}{\|v\|}$	$\frac{\|r(\tau)\|}{\|v\|}$	$\frac{\|(u-v,p-q,\eta-\sigma)\|_W}{\|(v,q,\eta)\|_W}$	$\frac{c_{\oplus}M_{\oplus}(v,\tau,q)}{\|(v,q,\eta)\|_W}$
6	0.814861	0.108033	0.186701	6.14041×10^{-5}	0.18877	3.3947
9	0.394441	0.0533815	0.0840590	3.76671×10^{-6}	0.0921263	1.61059
12	0.197608	0.0266837	0.0424575	2.34323×10^{-7}	0.0509854	0.798602

Table 1. Components of the majorant, $a = (1, 0)$.

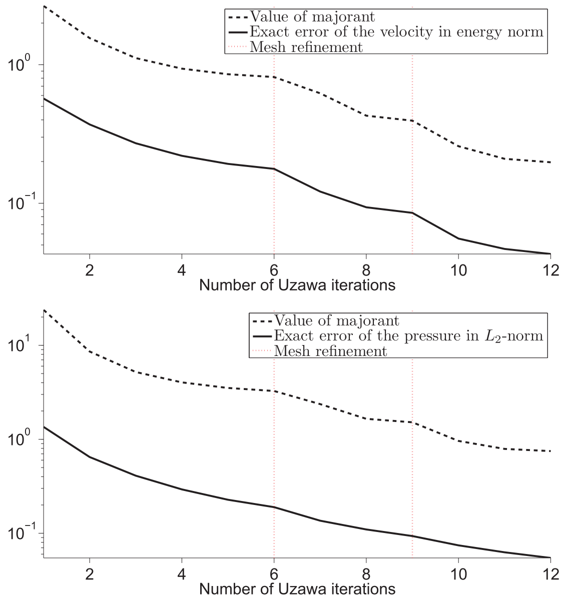

Figure 1. Behavior of the majorants, $a = (1, 0)$.

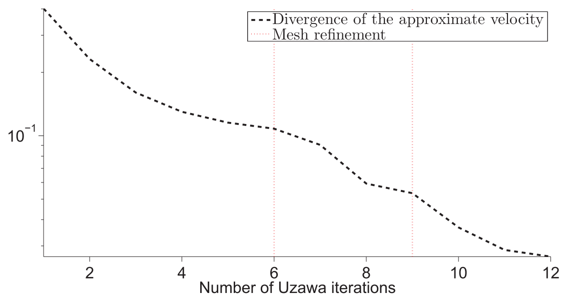

Figure 2. Divergence of the approximate velocity, $a = (1, 0)$.

k	$\frac{M_\oplus(v,\tau,q)}{\\|v\\|}$	$\frac{\\|\mathrm{div}\, v\\|}{\\|v\\|}$	$\frac{\\|d(v,\tau,q)\\|}{\\|v\\|}$	$\frac{\\|r(\tau)\\|}{\\|v\\|}$	$\frac{\\|(u-v,p-q,\eta-\sigma)\\|_W}{\\|(v,q,\eta)\\|_W}$	$\frac{c_\oplus M_\oplus(v,\tau,q)}{\\|(v,q,\eta)\\|_W}$
6	0.816497	0.108253	0.187053	6.13097×10^{-5}	0.187327	3.36203
9	0.394957	0.0534651	0.0840883	3.76496×10^{-6}	0.0913371	1.59401
12	0.197734	0.0267046	0.0424624	2.34283×10^{-7}	0.0505014	0.78988

Table 2. Components of the majorant, $a = (1, 1)$.

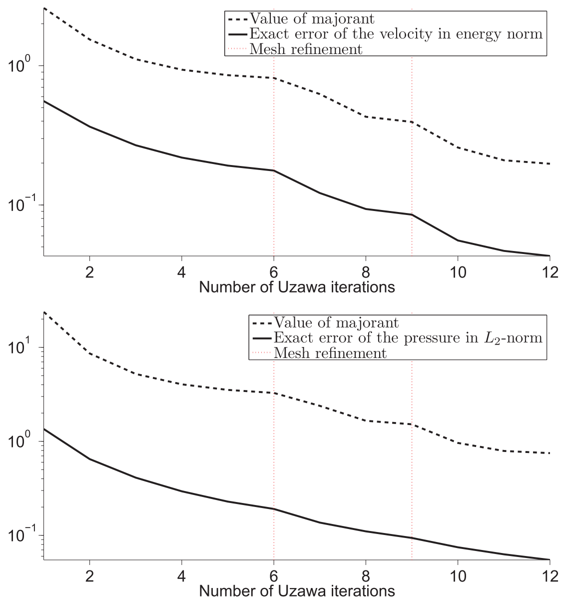

Figure 3. Behavior of the majorants, $a = (1, 1)$.

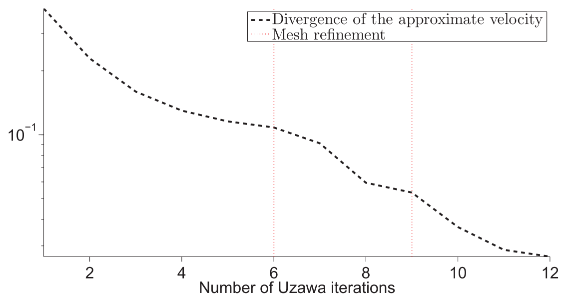

Figure 4. Divergence of the approximate velocity, $a = (1, 1)$.

k	$\frac{M_{\oplus}(v,\tau,q)}{\|v\|}$	$\frac{\|\mathrm{div}\, v\|}{\|v\|}$	$\frac{\|d(v,\tau,q)\|}{\|v\|}$	$\frac{\|r(\tau)\|}{\|v\|}$	$\frac{\|(u-v,p-q,\eta-\sigma)\|_W}{\|(v,q,\eta)\|_W}$	$\frac{c_{\oplus}M_{\oplus}(v,\tau,q)}{\|(v,q,\eta)\|_W}$
5	0.78383	0.114972	0.182948	6.12371×10^{-5}	0.219877	3.03241
8	0.372967	0.0553391	0.0837532	3.76496×10^{-6}	0.102283	1.39201
11	0.185204	0.0273769	0.0421268	2.34320×10^{-7}	0.0542261	0.680833

Table 3. Components of the majorant, $a = (0, 0)$.

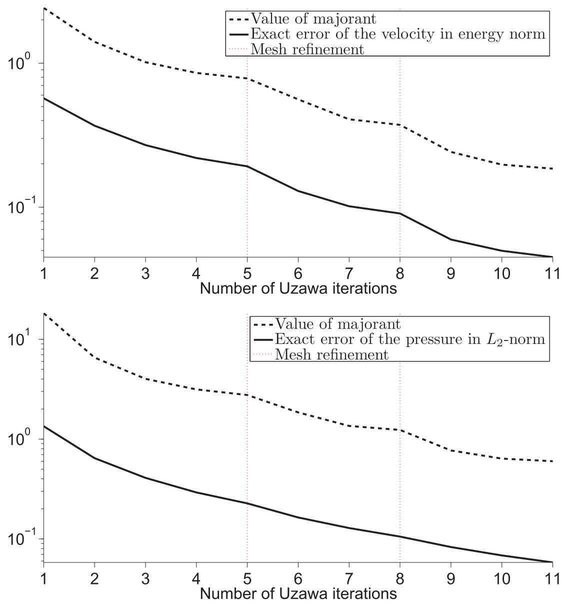

Figure 5. Behavior of the majorants, $a = (0, 0)$.

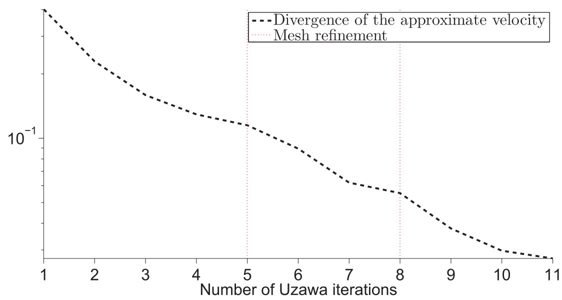

Figure 6. Divergence of the approximate velocity, $a = (0, 0)$.

We tested the algorithm for different a. Below, we focus attention on three examples, which present typical results. We set $a = (1, 0)$, $a = (1, 1)$ and $a = (0, 0)$ (this case corresponds to the Stokes problem). Values for majorants and exact errors are shown in Figure 1 for $a = (1, 0)$, Figure 3 for $a = (1, 1)$ and Figure 5 for $a = (0, 0)$. In Figures 2, 4 and 6 we show how the norm of the divergence decreases in the process of Uzawa iterations. For the velocity field, errors are calculated in the energy norm and for the pressure in the L^2-norm. Values of the majorants and exact errors for the velocity and pressure are normalized with the norms $|\!|\!|v|\!|\!|$ and $\|q\|$, respectively. Dotted vertical lines mark the iterations after which mesh refinements were done. In the examples, the "free" function τ was computed by minimization of the majorant on the same mesh that was used for the velocity field. Also, we can compute guaranteed bounds on the errors in terms of stresses and the combined primal-dual norm (see Table 1 for $a = (1, 0)$, Table 2 for $a = (1, 1)$ and Table 3 for $a = (0, 0)$). We see that the estimates indeed provide guaranteed upper bounds of errors in the functions computed by means of the Uzawa iterations. These bounds correctly reflect decrease of the corresponding errors and indicate the moment when adaptation of the mesh is required.

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PIV

ERROR ESTIMATES OF UZAWA ITERATION METHOD FOR A CLASS OF BINGHAM FLUIDS

by

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