Author(s): Cavalletti, Fabio; Rajala, Tapio

Title: Tangent Lines and Lipschitz Differentiability Spaces

Year: 2016

Version:

Please cite the original version:

Cavalletti, F., & Rajala, T. (2016). Tangent Lines and Lipschitz Differentiability Spaces. Analysis and Geometry in Metric Spaces, 4(1). https://doi.org/10.1515/agms-2016-0004

DE GRUYTER OPEN Anal. Geom. Metr. Spaces 2016; 4:85–103

Research Article **Open Access**

Fabio Cavalletti* and Tapio Rajala

Tangent Lines and Lipschitz Differentiability Spaces

DOI 10.1515/agms-2016-0004
Received August 21, 2015; accepted April 15, 2016

Abstract: We study the existence of tangent lines, i.e. subsets of the tangent space isometric to the real line, in tangent spaces of metric spaces. We first revisit the almost everywhere metric differentiability of Lipschitz continuous curves. We then show that any blow-up done at a point of metric differentiability and of density one for the domain of the curve gives a tangent line.
Metric differentiability enjoys a Borel measurability property and this will permit us to use it in the framework of Lipschitz differentiability spaces. We show that any tangent space of a Lipschitz differentiability space contains at least n distinct tangent lines, obtained as the blow-up of n Lipschitz curves, where n is the dimension of the local measurable chart. Under additional assumptions on the space, such as curvature lower bounds, these n distinct tangent lines span an n-dimensional part of the tangent space.

Keywords: metric geometry; Lipschitz differentiability spaces; tangent of metric spaces; Ricci curvature

MSC: 51F99, 53B99

1 Introduction

During the past few years there has been growing interest towards studying the infinitesimal structure of “nice” metric measure spaces. One class of nice metric measure spaces is formed by the ones in which Lipschitz functions are differentiable almost everywhere with respect to Lipschitz charts covering the space. The study of such spaces originates from the work of Cheeger [10] and the spaces are now often called Lipschitz differentiability spaces (following Bate [7]). Cheeger proved that a doubling condition on the reference measure and the validity of a local Poincaré inequality (as defined by Heinonen and Koskela [13]) are sufficient for the space to be a Lipschitz differentiability space. Although there are quite wild examples of doubling metric measure spaces supporting a local Poincaré inequality [9, 24, 36], these assumptions still have strong geometric implications, [10, 22, 37]. In particular, there are lots of rectifiable curves joining any two points in such a space.

A general Lipschitz differentiability space might not contain any rectifiable curve besides the trivial one. However, they always contain sufficiently many broken curves in different directions so that the reference measure can be expressed by independent Alberti representations that completely characterize derivatives of Lipschitz functions, see the work of Bate [7]. On the other hand, when we perform a Gromov-Hausdorff blow-up of a broken bi-Lipschitz curve $\gamma\colon \mathrm{Dom}(\gamma) \to X$ of the metric space X at a density point of the domain $\mathrm{Dom}(\gamma)$ the broken curve approaches, after passing to a subsequence, a limit curve defined on the whole $\mathbb{R}$.

We first define metric differentiability, see Definition 3.3, and then prove that, at points of metric differentiability, this limit curve is a line-segment, see Proposition 3.10. By a result of Kirchheim [21], we observe that

***Corresponding Author: Fabio Cavalletti:** University of Pavia, Dipartimento di Matematica, Pavia, Italy, E-mail: fabio.cavalletti@unipv.it
Tapio Rajala: University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyvaskyla, Finland, E-mail: tapio.m.rajala@jyu.fi

metric derivative coincides with the metric speed at almost every point of Dom (γ). Therefore we deduce that a Lipschitz curve γ is metrically differentiable at almost every point (see also Proposition 3.8 for an alternative proof of this fact). Thus broken bi-Lipschitz curves always converge to a line-segment at almost every point of their domain.

Given an n-dimensional Lipschitz chart on a Lipschitz differentiability space we know from the work of Bate [7] that there exist n independent Alberti representations. Using the measurability of the metric differential, Lemma 4.1, one can deduce that (see Proposition 4.3) at almost every point the blow-up will give n distinct tangent lines. If one also assumes the Lipschitz differentiability space to be doubling, then one can find n distinct tangent lines at every point of the tangent space. Note that $\mathrm{Tan}(X, \mathsf{d}, \bar{x})$ will denote the collection of all the tangent cones obtained by blow-up of the space (X, d) at the point $\bar{x}$.

Theorem 1.1 (Theorem 4.5). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling Lipschitz differentiability space and (U, φ) be an n-dimensional chart. Then for $\mathfrak{m}$-almost every $\bar{x} \in U$, there exist $v_1, \dots, v_n \in \mathbb{R}^n$ linearly independent such that for any element $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ and for each $z \in X_\infty$ there exist $\iota_1^z, \dots, \iota_n^z : \mathbb{R} \to X_\infty$ so that*

i) $\iota_j^z(0) = z$, *for any* $j = 1, \dots, n$;

ii) $\mathsf{d}_\infty(\iota_j^z(t), \iota_j^z(s)) = |t - s|$, *for any* $j = 1, \dots, n$, *for all* $s, t \in \mathbb{R}$;

iii) $\mathsf{d}_\infty(\iota_j^z(t), \iota_k^z(t)) \geq C|t| \cdot |v_j - v_k|$, *for any* $j, k = 1, \dots, n$, *for all* $t \in \mathbb{R}$;

for some positive constant $C = C(z)$. For each $z \in X_\infty$, each line ι_i^z is obtained as the blow-up of a Lipschitz curve, with the blow-up depending on z.

The question is then how and what kind of subspace of the tangent space these tangent lines form. Since the Heisenberg group is a Lipschitz differentiability space and purely 2-unrectifiable [4], we know that the tangent lines do not always span an n-rectifiable set. However under the additional assumption that the space is Ahlfors n-regular with n being the dimension of the chart, at almost every point there is a tangent space bi-Lipschitz equivalent to $\mathbb{R}^n$, see [12]. We are interested in finding other conditions that would provide information on the tangents.

Our considerations originate from the study of another class of nice metric measure spaces - namely of those with Ricci curvature lower bounds. There are many notions of Ricci curvature lower bounds on metric measure spaces. For the most strict one, the $\mathsf{RCD}^*(K, N)$ spaces (defined in [1, 3, 5, 14]), it is known that they infinitesimally look like Euclidean spaces, [16, 27]. Moreover, the tangents in an $\mathsf{RCD}^*(K, N)$ space are almost everywhere spanned by the tangent lines obtained from the Lipschitz charts as described above, see Section 5 for details. Thus the infinitesimal structure of $\mathsf{RCD}^*(K, N)$ spaces is already well understood.

We would like to understand the structure of spaces with Ricci curvature lower bounds with the more general definitions. Most of the definitions are known to imply a doubling condition on the measure and a local Poincaré inequality. Thus these spaces are Lipschitz differentiability spaces and Theorem 1.1 holds. One line of investigation is to continue from the proof in [16]. There the fact that $\mathsf{RCD}^*(K, N)$ spaces have at least one Euclidean tangent space was proven following the idea of Preiss [30] (and its adaptation to metric spaces by Le Donne [25]) of iterated tangents. The proof essentially used only the fact that the tangent spaces split off any part that is isometric to $\mathbb{R}$.

Taking into consideration also the Lipschitz charts, the splitting of tangents property (defined in Section 5) implies the existence of $\mathbb{R}^n$ in each of the tangents at almost every point, where n is again the dimension of the chart.

Theorem 1.2 (Theorem 5.1). *Suppose that $(X, \mathsf{d}, \mathfrak{m})$ is a doubling Lipschitz differentiability space with the splitting of tangents property. Let (U, φ) be an n-dimensional chart of $(X, \mathsf{d}, \mathfrak{m})$. Then for $\mathfrak{m}$-a.e. $\bar{x} \in U$ any $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ is of the form*

$$(X_\infty^d \times \mathbb{R}^d, \mathsf{d}_\infty^d \times |\cdot|, (\bar{x}_\infty^d, 0)),$$

with $d \geq n$.

For the more general $\mathsf{CD}(K, N)$ spaces (see [26, 38, 39] for the definitions) isometric splitting of tangents is impossible since already $\mathbb{R}^n$ with any norm and the Lebesgue measure satisfies $\mathsf{CD}(0, n)$. On the other hand, Ohta has recently shown that a version of splitting theorem holds for Finsler manifolds [29]. Such weaker versions might be enough to give some information on the infinitesimal structure. For example, if the existence of a tangent line would always imply that the tangent could be written to be bi-Lipschitz equivalent to a product $\mathbb{R} \times Y$ for some metric space Y, the n dimensional Lipschitz chart could result in a piece of the tangent bi-Lipschitz equivalent to $\mathbb{R}^n$.

Let us note that for the even more general notion $\mathsf{MCP}(K, N)$ of Ricci curvature lower bound (see [28, 39] for the definitions) the above splitting result does not hold even in a topological sense [20]. Moreover, it is not known if a local Poincaré inequality holds in $\mathsf{MCP}(K, N)$ spaces without the non-branching assumption, and hence we do not know if $\mathsf{MCP}(K, N)$ spaces are Lipschitz differentiability spaces. Even more, it is known that for example the Heisenberg group satisfies the $\mathsf{MCP}(K, N)$ condition, see [18]. Thus the tangent lines cannot bi-Lipschitz span a part of the tangent.

The paper is organized as follows. In Section 2 we recall the notions of pointed measured Gromov-Hausdorff convergence, tangent functions and Lipschitz differentiability spaces. In Section 3 we define the notion of metric differentiability that we will use in this paper and show, using an identity proved by Kirchheim in [21], that the metric derivative agrees almost everywhere with the metric speed. We also show that at almost every point of a bi-Lipschitz curve the blow-up will be a tangent line. In Section 4 we consider the blow-ups in a Lipschitz differentiability space showing that we have n independent tangent lines at almost every point. In the final section, Section 5, following the ideas of David and Schioppa [12, 34], we prove that if tangents split off tangent lines then the n independent tangent lines in a Lipschitz differentiability space span a Euclidean $\mathbb{R}^n$ in the tangent.

2 Preliminaries

A metric measure space is a triple $(X, \mathsf{d}, \mathfrak{m})$ where (X, d) is a complete and separable metric space and $\mathfrak{m}$ a positive Borel measure that is also finite on bounded sets. As the main object of our study will be proper spaces, i.e. metric spaces such that each bounded closed set is also compact, we directly incorporate in the definition of metric measure space also the properness assumption. Consequently $\mathfrak{m}$ will be a positive Radon measure.

We list here two general properties of metric measure spaces that we will consider during the paper. The metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *(uniformly locally) doubling* if for each $R > 0$ there exists $C(R) > 0$ such that

$$0 < \mathfrak{m}(B_{2r}(x)) \le C(R)\, \mathfrak{m}(B_r(x)), \qquad \text{for every } x \in X,\ r \le R.$$

With no loss in generality, the function C can be taken non-decreasing. Moreover a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ supports a *local p-Poincaré inequality* for some $p \ge 1$ if every ball in X has positive and finite measure and for every $g \in \mathrm{Lip}(X, \mathsf{d}) := \{l \colon X \to \mathbb{R} \mid l \text{ is Lipschitz}\}$,

$$⨍_B |g(x) - g_B|\, d\mathfrak{m}(x) \le Lr \left(⨍_{B_{rL}(x_0)} |Dg|^p(x)\, d\mathfrak{m}(x) \right)^{1/p},$$

for some positive constant L, where $B = B_r(x_0)$ and $g_B = ⨍_B g(x)\, d\mathfrak{m}(x)$. Here for $g \in \mathrm{Lip}(X, \mathsf{d})$ we also adopt the following notation:

$$|Dg|(x) := \limsup_{\substack{y \to x, \\ y \ne x}} \frac{\mathsf{d}(g(y), g(x))}{\mathsf{d}(y, x)}.$$

2.1 Convergence of metric measure spaces

The standard notion of topology on equivalence classes of pointed, proper, separable metric spaces is the one induced by the pointed Gromov-Hausdorff convergence, pGH-convergence in brief. This convergence can be characterized in many equivalent ways. We will adopt the one with ε-isometries.

A map $f\colon (X, \mathsf{d}_X) \to (Y, \mathsf{d}_Y)$ between compact metric spaces is called an *ε-isometry* provided

(i) it almost preserves distances: for all $z, w \in X$,

$$|\mathsf{d}_X(z, w) - \mathsf{d}_Y(f(z), f(w))| \le \varepsilon;$$

(ii) it is almost surjective:

$$\forall\, y \in Y,\ \exists\, x \in X: \quad \mathsf{d}_Y(f(x), y) \le \varepsilon.$$

In order to deal with possibly non-compact spaces, it is customary to fix a distinguished point $\bar{x} \in X$ and to consider ε-isometries defined on an increasing family of balls centered in $\bar{x}$. When a distinguished point is fixed, we use $(X, \mathsf{d}, \bar{x})$ to denote the pointed metric space.

Definition 2.1. A sequence $\{(X_i, \mathsf{d}_i, \bar{x}_i)\}_{i\in\mathbb{N}}$ of pointed, proper, complete metric spaces converges to a pointed, proper, complete metric space $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ in pointed Gromov-Hausdorff, and write

$$(X_i, \mathsf{d}_i, \bar{x}_i) \longrightarrow (X_\infty, \mathsf{d}_\infty, \bar{x}_\infty), \qquad pGH,$$

if and only if there exist sequences of positive real numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}, \{R_i\}_{i\in\mathbb{N}}$ with $\varepsilon_i \to 0$, $R_i \to \infty$ and a sequence of ε_i-isometries,

$$f_i\colon B^{X_i}_{R_i}(\bar{x}_i) \longrightarrow B^{X_\infty}_{R_i}(\bar{x}_\infty), \qquad f_i(\bar{x}_i) = \bar{x}_\infty,$$

where $B^{X_i}_{R_i}(\bar{x}_i)$ is the ball in X_i, centered in $\bar{x}$ and of radius R_i.

We also consider pointed metric measure spaces: a quadruple $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ where $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space and $\bar{x} \in X$ a distinguished point.

Definition 2.2. A sequence $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i)\}_{i\in\mathbb{N}}$ of pointed metric measure spaces converges in the pointed measured Gromov-Hausdorff sense to a pointed metric measure space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i) \longrightarrow (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty), \qquad pmGH,$$

if and only if there exist sequences of positive real numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}, \{R_i\}_{i\in\mathbb{N}}$ with $\varepsilon_i \to 0$, $R_i \to \infty$ and a sequence of ε_i-isometries,

$$f_i\colon B^{X_i}_{R_i}(\bar{x}_i) \longrightarrow B^{X_\infty}_{R_i}(\bar{x}_\infty), \qquad f_i(\bar{x}_i) = \bar{x}_\infty,$$

such that

$$\lim_{i\to\infty} \int_{X_\infty} \varphi(z)\, d(f_{i\,\sharp}\mathfrak{m}_i)(z) = \int_{X_\infty} \varphi(z)\, d\mathfrak{m}_\infty(z), \qquad \forall\, \varphi \in C_b(X_\infty),$$

where $C_b(X_\infty)$ stands for the space of continuous and bounded functions with compact support in X_∞.

Both, the pGH-convergence and the $pmGH$-convergence can be used to define and study (measured) tangent spaces.

If (X, d) is a metric space and $\bar{x} \in X$ is a distinguished point, then any limit point in the pGH-convergence of any sequence of the form $\{(X, \mathsf{d}/r_i, \bar{x})\}_{i\in\mathbb{N}}$, with $r_i \to 0$, is a tangent space of (X, d) at $\bar{x}$. We use $\mathrm{Tan}(X, \mathsf{d}, \bar{x})$ to denote the set of all possible tangent spaces of (X, d) at $\bar{x}$.

If $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space and $\bar{x} \in \mathrm{supp}(\mathfrak{m})$ is a distinguished point, for any $r > 0$, the rescaled and normalized pointed metric measure space is defined as follows:

$$\left(X, \frac{1}{r}\mathsf{d}, \mathfrak{m}^{\bar{x}}_r, \bar{x}\right), \qquad \mathfrak{m}^{\bar{x}}_r := \left(\int_{B_r(\bar{x})} 1 - \frac{1}{r}\mathsf{d}(\bar{x}, z)\, d\mathfrak{m}(z)\right)^{-1} \mathfrak{m}.$$

Then a limit point in the $pmGH$-convergence of the sequence $\{(X, \mathsf{d}/r_i, \mathfrak{m}^{\bar{x}}_{r_i}, \bar{x})\}_{i\in\mathbb{N}}$ is a measured tangent space of $(X, \mathsf{d}, \mathfrak{m})$ at $\bar{x}$ and to denote the set of all possible measured tangent spaces of $(X, \mathsf{d}, \mathfrak{m})$ at $\bar{x}$ we use $\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$.

It is worth noticing that, thanks to compactness properties of the collection of uniformly doubling metric measure spaces (see [40], Theorem 27.32 and [17], Lemma 3.32), $\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is always non empty, provided $(X, \mathsf{d}, \mathfrak{m})$ is doubling.

2.2 Tangent functions

Here we recall a few objects and related results presented in [10] and in [19].

If (X, d_X) and (Y, d_Y) are metric spaces and $f: X \to Y$ is an ε-isometry, then there exists a (4ε)-isometry $f': Y \to X$ so that for all $x \in X$ and $y \in Y$ it holds

$$\mathsf{d}_X(f' \circ f(x), x) \le 3\varepsilon, \qquad \mathsf{d}_Y(f \circ f'(y), y) \le \varepsilon.$$

Such a map is usually called an *ε-inverse* of f and accordingly we will often adopt the notation f^{-1} to denote it.

Consider now any element $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ and a sequence of $r_i \to 0$ such that

$$\left(X, \frac{1}{r_i}\mathsf{d}, \mathfrak{m}^{\bar{x}}_r, \bar{x}\right) \longrightarrow (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty), \qquad pmGH.$$

Then to any Lipschitz function $g: X \to \mathbb{R}$ we can associate a sequence of rescaled functions centered at $\bar{x}$:

$$g_i(x) := \frac{g(x) - g(\bar{x})}{r_i}.$$

If g is L-Lipschitz in (X, d), then so is g_i in $(X, \mathsf{d}/r_i)$. With this in mind, we say that $u_g: X_\infty \to \mathbb{R}$ is a *compatible tangent function of g at $\bar{x}$* if

$$\lim_{i\to\infty} g_i(f_i^{-1}(z)) = \lim_{i\to\infty} \frac{g(f_i^{-1}(z)) - g(\bar{x})}{r_i} = u_g(z), \qquad \forall\, z \in X_\infty,$$

where f_i^{-1} is any ε_i-inverse of the approximate isometry f_i given by the $pmGH$ convergence of $(X, \mathsf{d}/r_i, \mathfrak{m}^{\bar{x}}_r, \bar{x})$ to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$. The term *compatible* is used to underline that we used the same scaling for the distance and the function g.

Remark 2.3. The definition of u_g does not depend on the choice of the sequence of the ε_i-inverses. Since f_i is almost surjective, for any $z \in X_\infty$ and $i \in \mathbb{N}$ sufficiently large, there exists $x_i \in X$ such that

$$\mathsf{d}_\infty(f_i(x_i), z) \le \varepsilon_i.$$

One then easily observes that $|g_i(f_i^{-1}(z)) - g_i(f_i^{-1} \circ f_i(x_i))| \to 0$. If f_i^{-1} and $\hat{f}_i^{-1}$ are two distinct ε_i-inverses of f_i, it follows, by the triangle inequality that

$$\lim_{i\to\infty} \frac{1}{r_i}\mathsf{d}(f_i^{-1} \circ f_i(x_i), \hat{f}_i^{-1} \circ f_i(x_i)) = 0,$$

and since g is Lipschitz, it follows that $g_i(f_i^{-1}(z))$ and $g_i(\hat{f}_i^{-1}(z))$ have the same limit.

Concerning the existence of compatible tangent functions, the following compactness result holds.

Lemma 2.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space and a sequence $r_i \to 0$ such that*

$$\left(X, \frac{1}{r_i}\mathsf{d}, \mathfrak{m}^{\bar{x}}_r, \bar{x}\right) \longrightarrow (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x}),$$

where the convergence is in the $pmGH$ sense. Fix also a countable collection $\mathcal{F}$ of uniformly Lipschitz functions defined on X. Then possibly choosing a subsequence of $\{r_i\}_{i\in\mathbb{N}}$, for each $g \in \mathcal{F}$ there exists u_g a compatible tangent function of g at $\bar{x}$.

The proof of Lemma 2.4 follows from a standard use of Ascoli-Arzela Theorem. See [23] for details. As one might expect, tangent functions of Lipschitz functions enjoy a generalized notion of linearity. It has different names according to different authors. Here we follow [10] and say that tangent functions to Lipschitz functions, wherever they exists, are *generalized linear*, see Definition 8.1 of [10]. The terminology used is justified by the fact that being generalized linear on a Euclidean space is the same as being linear in the usual sense, see again [10], Theorem 8.11.

2.3 Lipschitz differentiability spaces

Under fairly general assumptions on the structure of the metric measure space, it is proved in [10] that the space of germs of Lipschitz functions has finite dimension in the following sense.

Definition 2.5. Let (X, d) be a metric space and $n \in \mathbb{N}$. A Borel set $U \subset X$ and a Lipschitz function $\varphi \colon X \to \mathbb{R}^n$ form a *chart of dimension* n, (U, φ), and a function $g \colon X \to \mathbb{R}$ is *differentiable at* $x_0 \in U$ *with respect to* (U, φ) if there exists a unique $Dg(x_0) \in \mathbb{R}^n$ such that

$$\limsup_{x \to x_0} \frac{|g(x) - g(x_0) - Dg(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\mathsf{d}(x, x_0)} = 0.$$

Furthermore a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is called a *Lipschitz differentiability space* if there exists a countable decomposition of X into charts such that any Lipschitz function $g \colon X \to \mathbb{R}$ is differentiable at $\mathfrak{m}$-almost every point of every chart.

A celebrated result by Cheeger [10] on Lipschitz differentiability spaces can be summarized by the following

Theorem 2.6. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a doubling metric measure space supporting a p-Poincaré inequality with constant* $L \geq 1$ *for some* $p \geq 1$. *Then* $(X, \mathsf{d}, \mathfrak{m})$ *is a Lipschitz differentiability space.*

Subsequently in [7] a finer analysis on curves, and their possible directions with respect to a given chart, was carried out. Here we report only the main statement. We use $\Gamma(X)$ to denote the set of bi-Lipschitz (onto their image) maps

$$\gamma \colon \mathrm{Dom}\,(\gamma) \to X,$$

with $\mathrm{Dom}\,(\gamma) \subset \mathbb{R}$ non-empty and compact.

Theorem 2.7 ([7], Theorem 6.6, Corollary 6.7)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a Lipschitz differentiability space and* (U, φ) *an n-dimensional chart. Then for* $\mathfrak{m}$*-a.e.* $x \in U$, *there exist* $\gamma_1^x, \dots, \gamma_n^x \in \Gamma(X)$ *such that:*

i) $(\gamma_i^x)^{-1}(x) = 0$ *is a point of density one of* $(\gamma_i^x)^{-1}(U)$ *for each* $i = 1, \dots, n$;

ii) $\{(\varphi \circ \gamma_i^x)'(0)\}_{i=1,\dots,n}$ *are linearly independent.*

Moreover, for any such γ_i^x, *for any Lipschitz* $g \colon X \to \mathbb{R}$ *and* $\mathfrak{m}$*-a.e.* $x \in U$, *the gradient of g at x with respect to* φ *and* $\gamma_1^x, \dots, \gamma_n^x$ *equals* $Dg(x)$, *that is*

$$\left(g \circ \gamma_i^x\right)'(0) = Dg(x) \cdot \left(\varphi \circ \gamma_i^x\right)'(0), \quad \mathfrak{m} - a.e.\, x \in U,$$

for $i = 1, \dots, n$.

Hence not only the space of germs of Lipschitz functions has locally finite dimension but also each Lipschitz function is locally described in terms of directional derivative with respect to a family of bi-Lipschitz curves.

Here it is worth mentioning Keith's results on coordinate functions: in [19] it is proved that the role of the coordinate map φ in chart (U, φ) can be played by distance functions from a suitable set. We report here Theorem 2.7 of [19].

Theorem 2.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric measure space admitting a p-Poincaré inequality with $\mathfrak{m}$ doubling. Then there exists a measurable differentiable structure $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ such that each $\varphi_i : U_i \to \mathbb{R}^{d(i)}$ is of the form*

$$\varphi_i(z) = \left(\mathsf{d}(z, x_1), \dots, \mathsf{d}(z, x_{d(i)})\right),$$

for some $x_1, \dots, x_{d(i)} \in X$.

For a generalization of the previous result to doubling differentiability spaces, see Corollary 6.31 of [35].

2.4 Geodesics in product spaces

If (X, d_X) and (Y, d_Y) are two metric spaces, we can consider the product distance d_{XY} defined by

$$\mathsf{d}_{XY} := \sqrt{\mathsf{d}_X^2 + \mathsf{d}_Y^2}.$$

Then $(X \times Y, \mathsf{d}_{XY})$ is again a metric space. We recall an easy lemma on geodesics in product spaces. By a geodesic in a metric space (X, d_X) we mean a map $\gamma\colon [0, 1] \to X$ satisfying $\mathsf{d}_X(\gamma_s, \gamma_t) = |t - s|\mathsf{d}_X(\gamma_0, \gamma_1)$ for all $s, t \in [0, 1]$, where we use the usual abbreviation $\gamma_t = \gamma(t)$.

Lemma 2.9. *A curve $[0, 1] \ni t \mapsto (\gamma_t^1, \gamma_t^2) \in (X \times Y, \mathsf{d}_{XY})$ is a geodesic if and only if γ^1 is a geodesic in (X, d_X) and γ^2 is a geodesic in (Y, d_Y).*

Proof. It is immediate that if γ^1 and γ^2 are geodesics, then also (γ_t^1, γ_t^2) is a geodesic. So, let us show the other direction. We start with the easy inequality: for a, b, c, d positive real numbers,

$$(a^2 + b^2)(c^2 + d^2) \geq (bd + ac)^2. \tag{2.1}$$

Then let $[0, 1] \ni t \mapsto (\gamma_t^1, \gamma_t^2) \in X \times Y$ be a geodesic and suppose by contradiction that γ^1 is not. For ease of notation, we can assume that

$$\mathsf{d}_X(\gamma_{-s}^1, \gamma_s^1) < \mathsf{d}_X(\gamma_0^1, \gamma_{-s}^1) + \mathsf{d}_X(\gamma_0^1, \gamma_s^1), \tag{2.2}$$

for some $s > 0$. By the fact that (γ_t^1, γ_t^2) is a geodesic we have

$$\mathsf{d}_X^2(\gamma_{-s}^1, \gamma_s^1) + \mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_s^2) = \left(\sqrt{\mathsf{d}_X^2(\gamma_{-s}^1, \gamma_0^1) + \mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_0^2)} + \sqrt{\mathsf{d}_X^2(\gamma_0^1, \gamma_s^1) + \mathsf{d}_Y^2(\gamma_0^2, \gamma_s^2)}\right)^2.$$

Expanding the squares and using (2.2), we obtain that

$$\begin{aligned}\mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_s^2) > \mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_0^2) + \mathsf{d}_Y^2(\gamma_0^2, \gamma_s^2) + 2\sqrt{\mathsf{d}_X^2(\gamma_{-s}^1, \gamma_0^1) + \mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_0^2)} \cdot \sqrt{\mathsf{d}_X^2(\gamma_0^1, \gamma_s^1) + \mathsf{d}_Y^2(\gamma_0^2, \gamma_s^2)} \\ - 2\mathsf{d}_X(\gamma_{-s}^1, \gamma_0^1)\mathsf{d}_X(\gamma_0^1, \gamma_s^1).\end{aligned}$$

We can now use the inequality (2.1) to get

$$\mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_s^2) > \mathsf{d}_Y^2(\gamma_{-s}^2, \gamma_0^2) + \mathsf{d}_Y^2(\gamma_0^2, \gamma_s^2) + 2\mathsf{d}_Y(\gamma_{-s}^2, \gamma_0^2)\mathsf{d}_Y(\gamma_0^2, \gamma_s^2),$$

violating the triangle inequality. The claim follows. □

3 Tangent lines

Let us start this section by recalling a result from [33], Theorem 7.10: a more general version of Lebesgue Differentiation Theorem. Here and in the sequel $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Definition 3.1. Fix $x \in \mathbb{R}^d$ and a sequence of Borel sets $\{E_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^d$. We say that $\{E_i\}_{i\in\mathbb{N}}$ *shrinks nicely to* x provided there exist $r_i > 0$ and $\alpha > 0$ such that for each $i \in \mathbb{N}$ we have

$$E_i \subset B_{r_i}(x) \qquad \text{and} \qquad \mathcal{L}^d(E_i) \geq \alpha \mathcal{L}^d(B_{r_i}(x)).$$

For the nicely shrinking sets we have the following general version of Lebesgue Differentiation Theorem.

Theorem 3.2. *Let $f \in L^1(\mathbb{R}^d, \mathbb{R})$ be any function. Associate to each $x \in \mathbb{R}^d$ a sequence $\{E_i(x)\}_{i\in\mathbb{N}}$ of sets nicely shrinking to x. Then*

$$f(x) = \lim_{i\to\infty} \frac{1}{\mathcal{L}^d(E_i(x))} \int_{E_i(x)} f(y)dy,$$

for every Lebesgue point x of f. In particular it holds for $\mathcal{L}^d$-almost every x.

Consider now (X, d) a complete, and separable metric space and note that for the next statement we do not need to assume (X, d) to be proper.

Definition 3.3. Let $\gamma\colon \mathrm{Dom}(\gamma) \to X$ be any curve. We say that γ is *metric differentiable* at $t \in \mathrm{Dom}(\gamma)$ provided the following limit

$$\lim_{\substack{s,\tau\to 0\\ nicely}} \frac{\mathsf{d}(\gamma_{t+s}, \gamma_{t+\tau})}{|s-\tau|}$$

exists for any sequence of s and τ, where with *nicely* we ask for the interval with boundary formed by $t+s$ and $t+\tau$ to shrink nicely to t. In case the limit exists, we denote it with $|d\gamma|(t)$.

Remark 3.4. By definition, the existence of $|d\gamma|$ is a priori a more demanding property compared to existence of metric speed $|\dot\gamma|$, for its definition see [6]. Actually the two notions are different. Consider for instance the curve $\gamma\colon [-1, 1] \to \mathbb{R}^2$ defined by $\gamma(t) := (t, t)$ for $t \geq 0$ and $\gamma(t) := (t, -t)$ for $t \leq 0$. Then the metric speed always exists and is 1, while $|d\gamma|$ does not exists for $t = 0$. The converse trivially holds. For curves with values in a Euclidean space, at any point of differentiability, $|d\gamma|(t_0)$ coincides with the modulus of the derivative.

Remark 3.5. Another notion of differentiability for maps with values in metric spaces was introduced by Kirchheim in [21]: for any $g\colon \mathbb{R}^n \to (X, \mathsf{d})$ consider the following quantity

$$MD(g, x)(u) := \lim_{r\searrow 0} \frac{1}{r}\mathsf{d}(g(x+ru), g(x))$$

for all $x, u \in \mathbb{R}^n$, whenever the limit exists. In Theorem 2 of [21] it is proved that for Lipschitz functions g, MD exists almost everywhere, with respect to Lebesgue measure, and at almost every point where it exists, it is a seminorm.

Theorem 3.6 ([21]). *Let $g\colon \mathbb{R}^n \to X$ be Lipschitz. Then, for almost every $x \in \mathbb{R}^n$, $MD(g, x)(\cdot)$ is a seminorm on $\mathbb{R}^n$ and*

$$\mathsf{d}(g(z), g(y)) - MD(g, x)(z-y) = o(|z-x| + |z-y|). \tag{3.1}$$

In the case of Lipschitz curves ($n = 1$) the quantity MD coincides with the metric speed and at any point where it exists it is also a seminorm. As the objective of this paper is the study of tangent lines, (3.1) is the relevant identity. It is straightforward to observe that if (3.1) holds at $t \in \mathrm{Dom}(\gamma)$ then t is a point of metric differentiability and $|d\gamma|(t) = MD(\gamma, t)(1)$. Also the converse implication holds. We include here a short proof for the reader's convenience.

Lemma 3.7. *Suppose a Lipschitz curve $\gamma : [-c, c] \to X$ is metric differentiable at 0. Then*

$$\mathsf{d}(\gamma_t, \gamma_s) - |d\gamma|(0) \cdot |t-s| = o(|t| + |t-s|).$$

Proof. Denote the Lipschitz constant of γ by L. Let $\epsilon > 0$. From the metric differentiability there exists $r_\epsilon > 0$ such that if $\epsilon|t| < |t-s| < r_\epsilon$, we have

$$\mathsf{d}(\gamma_t, \gamma_s) - |d\gamma|(0) \cdot |t-s| \le \epsilon|t-s|.$$

On the other hand, if $0 < |t-s| < \epsilon|t|$, we have from the Lipschitz-continuity

$$\mathsf{d}(\gamma_t, \gamma_s) - |d\gamma|(0) \cdot |t-s| \le 2L|t-s| < 2L\epsilon|t|.$$

The claim follows by combining the estimates. □

In this paper we prefer to analyze the properties of $|d\gamma|$ rather than (3.1).

Taking advantage of Theorem 3.2, it is fairly easy to obtain the almost everywhere existence of $|d\gamma|$.

Proposition 3.8. *Let $\gamma\colon \mathrm{Dom}(\gamma) \to X$ be a Lipschitz curve. Then metric differentiability holds $\mathcal{L}^1$-a.e. in $\mathrm{Dom}(\gamma)$.*

The proof can be obtained already from what was said in Remark 3.5. However, we present here an alternative proof obtained following the ideas of the proof of existence of the metric speed for $\mathcal{L}^1$-a.e. $t \in [0, 1]$, see [6], Theorem 4.1.6.

Proof. **Step 1.**
Consider $\Lambda := \gamma(\mathrm{Dom}(\gamma))$. By continuity of γ, the set Λ is compact and we can consider a dense sequence $\{x_n\} \subset \Lambda$. We define a sequence of Lipschitz functions as follows:

$$\mathrm{Dom}(\gamma) \ni t \mapsto \varphi_n(t) := \mathsf{d}(\gamma_t, x_n).$$

The Lipschitz constant of φ_n is bounded from above by the Lipschitz constant of γ. For each $n \in \mathbb{N}$ we denote with $\hat{\varphi}_n$ a Lipschitz extension of φ_n. We can assume $\hat{\varphi}_n$ to be defined on an interval, say on (a, b), containing $\mathrm{Dom}(\gamma)$. By Rademacher's theorem, each $\hat{\varphi}_n$ is differentiable $\mathcal{L}^1$-a.e. and therefore we can define the following map

$$p(t) := \sup_{n\in\mathbb{N}} |\dot{\hat{\varphi}}_n(t)|,$$

at least for almost every $t \in (a, b)$.

Step 2.
For the rest of the proof we fix $t \in \mathrm{Dom}(\gamma)$ which is a point of differentiability of all φ_n and a Lebesgue-point of p. We also fix two sequences $s_m, \tau_m \to 0$ so that the interval $(t + s_m, t + \tau_m)$ shrinks nicely to t. For ease of notation, $s = s_m$, $\tau = \tau_m$ with $s + s_m, t + \tau_m \in \mathrm{Dom}(\gamma)$. Then for any $n \in \mathbb{N}$ we have

$$\frac{\mathsf{d}(\gamma_{t+s}, \gamma_{t+\tau})}{|\tau-s|} \ge \frac{|\varphi_n(\gamma_{t+s}) - \varphi_n(\gamma_{t+\tau})|}{|\tau-s|} = \frac{|\hat{\varphi}_n(\gamma_{t+s}) - \hat{\varphi}_n(\gamma_{t+\tau})|}{|\tau-s|},$$

and therefore

$$\liminf_{s,\tau\to0} \frac{\mathsf{d}(\gamma_{t+s}, \gamma_{t+\tau})}{|\tau-s|} \ge |\dot{\hat{\varphi}}_n(t)|.$$

We can take the supremum over all n without changing the left hand side of the previous inequality and obtaining on the right hand side $p(t)$.

Step 3.
Since $\{x_n\}_{n\in\mathbb{N}}$ is a dense sequence

$$\begin{aligned}
\mathsf{d}(\gamma_{t+s}, \gamma_{t+\tau}) &= \sup_{n\in\mathbb{N}} |\mathsf{d}(\gamma_{t+s}, x_n) - \mathsf{d}(x_n, \gamma_{t+\tau})| \\
&= \sup_{n\in\mathbb{N}} |\varphi_n(t+s) - \varphi_n(t+\tau)| \\
&\le \sup_{n\in\mathbb{N}} \int_{(t+s,t+\tau)} |\dot{\hat{\varphi}}_n(\sigma)| d\mathcal{L}^1(\sigma) \\
&\le \int_{(t+s,t+\tau)} p(\sigma) d\mathcal{L}^1(\sigma).
\end{aligned}$$

By assumption t is a Lebesgue-point of p, then

$$\limsup_{s,\tau\to 0} \frac{\mathsf{d}(\gamma_{t+s},\gamma_{t+\tau})}{|\tau - s|} \leq \limsup_{s,\tau\to 0} \frac{1}{|\tau - s|}\int_{t+s}^{t+\tau} p(\sigma) d\mathcal{L}^1(\sigma) = p(t),$$

where the last identity follows from Theorem 3.2. For $\mathcal{L}^1$-a.e. $t \in \mathrm{Dom}(\gamma)$:

$$p(t) \leq \liminf_{\substack{s,\tau\to 0\\ nicely}} \frac{\mathsf{d}(\gamma_{t+s},\gamma_{t+\tau})}{|\tau - s|} \leq \limsup_{\substack{s,\tau\to 0\\ nicely}} \frac{\mathsf{d}(\gamma_{t+s},\gamma_{t+\tau})}{|\tau - s|} \leq p(t),$$

where the first inequality follows from **Step 2.**, and the claim follows. □

3.1 Existence of Tangent lines

From Proposition 3.8 one can prove that at each point of metric differentiability the blow up of the Lipschitz curve is a tangent line. Note that we use the properness assumption of the base space (X, d) in the proof.

Lemma 3.9. *Let (X, d) be a complete, proper and separable metric space. Fix $\bar{x} \in X$ and assume the existence of a pointed, proper, complete and separable metric space $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$. Let $\gamma \in \Gamma(X)$ be such that*

$$Dom\,(\gamma) = [-c, c], \qquad \gamma_0 = \bar{x}, \qquad |d\gamma|(0) > 0.$$

Then $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ contains an isometric copy of $\mathbb{R}$, in brief a line, which is a limit of γ.

Proof. By assumption there exists a sequence of positive real numbers $\{r_i\}_{i\in\mathbb{N}}$ with $r_i \to 0$ such that

$$\left(X, \frac{1}{r_i}\mathsf{d}, \bar{x}\right) \to (X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \qquad \text{in the } pGH\text{-convergence.}$$

Consider the sequence of approximate isometries $f_i : X \to X_\infty$ associated to the convergence. For any two real numbers δ, η and $i \in \mathbb{N}$ sufficiently large it holds:

$$\left| \frac{1}{r_i}\mathsf{d}(\gamma_{r_i\delta}, \gamma_{r_i\eta}) - \mathsf{d}_\infty(f_i(\gamma_{r_i\delta}), f_i(\gamma_{r_i\eta})) \right| \leq \varepsilon_i.$$

Thanks to metric differentiability

$$\lim_{i\to\infty} \mathsf{d}_\infty(f_i(\gamma_{r_i\delta}), f_i(\gamma_{r_i\eta})) = |\delta - \eta||d\gamma|(0).$$

Since the limit space is proper, using a diagonal argument, we have convergence of $\{f_i(\gamma_{r_i\delta})\}_{i\in\mathbb{N}}$ for all rational numbers δ. By density there exists a curve $z\colon \mathbb{R} \to X_\infty$ such that

$$\mathsf{d}_\infty(z_\eta, z_\delta) = |\eta - \delta| \cdot |d\gamma|(0).$$

It follows that $z(\mathbb{R})$ is isometric to $\mathbb{R}$. □

In Lemma 3.9 we have not assumed any length structure on the metric space (X, d). Hence the assumption $\mathrm{Dom}\,(\gamma) = [-c, c]$ could sound a bit restrictive. In what follows we consider $\gamma \in \Gamma(X)$ with a more general domain.

Proposition 3.10. *Let (X, d) be a complete, proper and separable metric space. Fix $\bar{x} \in X$ and assume the existence of a pointed, proper, complete and separable metric space $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$. Let $\gamma \in \Gamma(X)$ be such that*

$$\gamma_0 = \bar{x}, \qquad 0 \text{ is a point of density one of } Dom\,(\gamma), \qquad |d\gamma|(0) > 0.$$

Then $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ contains a line, which is a limit of γ.

Proof. Let us consider fixed sequences of positive numbers $\varepsilon_i \to 0$, $R_i \to \infty$ and

$$f_i\colon B^{X_i}_{R_i}(\bar{x}) \to B^{X_\infty}_{R_i}(\bar{x}_\infty), \qquad \varepsilon_i-\text{isometry}, \qquad f_i(\bar{x}) = \bar{x}_\infty,$$

where $B^{X_i}_{R_i}(\bar{x})$ is the ball in $(X, \mathsf{d}/r_i)$, centered in $\bar{x}$ and of radius R_i.

Step 1.

Denote with $I := \mathrm{Dom}\,(\gamma)$ and for any positive r we consider $I_r := \{x \in \mathbb{R} : xr \in I\}$. Consider any sequence $\varrho_i \to 0$, then for each n define

$$I(n) := \bigcup_{i \geq n} I_{\varrho_i}.$$

The set $\cap_{n\in\mathbb{N}} I(n)$ is formed by all real numbers δ such that there exists a subsequence ϱ_{i_k} so that $\varrho_{i_k}\delta \in \mathrm{Dom}\,(\gamma)$ for all $k \in \mathbb{N}$. To underline its dependence on $\{\varrho_i\}_{i\in\mathbb{N}}$, we also use the following notation

$$I(\{\varrho_i\}) := \bigcap_{n\in\mathbb{N}} I(n).$$

We observe that for each $n \in \mathbb{N}$

$$\mathcal{L}^1\left(\mathbb{R} \setminus I(n)\right) = 0.$$

Indeed for any $M > 0$, and $j \in \mathbb{N}$, $j \geq n$

$$\left((-M, M) \setminus \bigcup_{i\geq n} I_{\varrho_i}\right) \subset \left((-M, M) \setminus I_{\varrho_j}\right).$$

Then since 0 has density one in I,

$$\lim_{i\to\infty} \frac{\mathcal{L}^1(\{\delta \in \mathbb{R} : |\delta| \leq \varrho_i M,\ \delta \notin I\})}{2\varrho_i M} = 0.$$

Since

$$\mathcal{L}^1(\{\delta \in \mathbb{R} : |\delta| \leq \varrho_i M,\ \delta \notin I\}) = \varrho_i \mathcal{L}^1(\{\delta \in \mathbb{R} : |\delta| \leq M,\ \varrho_i\delta \notin I\}),$$

it follows that

$$\lim_{j\to\infty} \mathcal{L}^1\left((-M, M) \setminus I_{\varrho_j}\right) = \lim_{j\to\infty} \mathcal{L}^1(\left\{\delta \in \mathbb{R} : |\delta| \leq M,\ \varrho_j\delta \notin I\right\}) = 0.$$

Therefore for any $M \in \mathbb{R}$

$$\mathcal{L}^1\left((-M, M) \setminus I(n)\right) = 0,$$

consequently $\mathcal{L}^1\left(\mathbb{R} \setminus I(n)\right) = 0$ for all $n \in \mathbb{N}$, and finally also

$$\mathcal{L}^1\left(\mathbb{R} \setminus I(\{\varrho_i\})\right) = 0, \tag{3.2}$$

holds.

Step 2.

Consider now the sequence of radii $\{r_i\}$ for which the pointed Gromov-Hausdorff convergence holds. Fix also a sequence $\eta_n \to 0$, an enumeration of all rational numbers $\{q_m\}_{m\in\mathbb{N}}$, a bijection $\mathbb{N} \ni h \to (n(h), m(h)) \in \mathbb{N}\times\mathbb{N}$ and the associated family of open balls in $\mathbb{R}$:

$$B_h := B_{\eta_{n(h)}}(q_{m(h)}).$$

Then from (3.2), $B_1 \cap I(\{r_i\}) \neq \emptyset$. Hence by definition of $I(\{r_i\})$, there exists $t_1 \in B_1$ for which there exists a subsequence of $\{r_i\}_{i\in\mathbb{N}}$, say $\{r_{i_1(k)}\}_{k\in\mathbb{N}}$ so that

$$t_1 r_{i_1(k)} \in I = \mathrm{Dom}\,(\gamma), \quad \forall\, k \in \mathbb{N}.$$

In particular we can consider the sequence

$$\{f_{i_1(k)}(\gamma_{t_1 r_{i_1(k)}})\}_{k\in\mathbb{N}} \subset X_\infty,$$

where f_i is an ε_i-isometry from pointed Gromov-Hausdorff convergence. Then since the aforementioned sequence stays in a bounded neighborhood of $\bar{x}_\infty$ and $(X_\infty, \mathsf{d}_\infty)$ is proper, there exists a subsequence of $\{t_1 r_{i_1(k)}\}_{k\in\mathbb{N}}$ still denoted by $\{t_1 r_{i_1(k)}\}_{k\in\mathbb{N}}$, such that

$$z_1 = \lim_{k\to\infty} f_{i_1(k)}(\gamma_{t_1 r_{i_1(k)}}),$$

for some $z_1 \in X_\infty$.

We repeat the construction now with $h = 2$. Again from (3.2), $B_2 \cap I(\{r_{i_1(k)}\}) \neq \emptyset$ and therefore there exist $t_2 \in B_2$ and a subsequence of $\{r_{i_1(k)}\}_{k\in\mathbb{N}}$, call it $\{r_{i_2(k)}\}_{k\in\mathbb{N}}$ for which $t_2 r_{i_2(k)} \in I$ and

$$z_2 = \lim_{k\to\infty} f_{i_2(k)}(\gamma_{t_2 r_{i_2(k)}}),$$

for some $z_2 \in X_\infty$.

Thanks to (3.2), we can repeat the same argument for any h and with a diagonal argument we infer the existence of sequences $\{t_h\}_{h\in\mathbb{N}}$ and $\{r_{i_k}\}_{k\in\mathbb{N}}$, such that for any h, for all sufficiently large k we have

$$r_{i_k} t_h \in I, \qquad z_h = \lim_{k\to\infty} f_{i_k}(\gamma_{t_h r_{i_k}}).$$

Step 3.
For $n, m \in \mathbb{N}$ we have:

$$\mathsf{d}_\infty(z_n, z_m) = \lim_{k\to\infty} \mathsf{d}_\infty(f_{i_k}(\gamma_{t_n r_{i_k}}), f_{i_k}(\gamma_{t_m r_{i_k}})) = \lim_{k\to\infty} \frac{1}{r_{i_k}} \mathsf{d}(\gamma_{t_n r_{i_k}}, \gamma_{t_m r_{i_k}}).$$

Since $|d\gamma|(0) > 0$, and $t_n r_{i_k}$ and $t_m r_{i_k}$ converge to 0 nicely, we have

$$\mathsf{d}_\infty(z_n, z_m) = |t_n - t_m||d\gamma|(0).$$

Define therefore the curve:

$$\gamma^\infty : \{t_h\}_{h\in\mathbb{N}} \to X_\infty, \qquad \gamma^\infty_{t_h} := z_h.$$

Hence we have

$$\mathsf{d}_\infty(\gamma^\infty_{t_n}, \gamma^\infty_{t_m}) = |t_n - t_m| \cdot |d\gamma|(0).$$

Now observe that the set of points $\{t_h\}_{h\in\mathbb{N}}$ is dense in $\mathbb{R}$, indeed for each $h \in \mathbb{N}$ the inclusion $t_h \in B_h$ holds. It follows that γ^∞ can be extended by continuity to any $s \in \mathbb{R}$. So we have proved the existence of

$$\gamma^\infty : \mathbb{R} \to X_\infty, \qquad \mathsf{d}_\infty(\gamma^\infty_t, \gamma^\infty_s) = |t - s| \cdot |d\gamma|(0).$$

The claim follows. □

Remark 3.11. The constructions done in the previous proof can be done simultaneously for finitely many curves. In particular suppose we have $\gamma^1, \dots, \gamma^n \in \Gamma(X)$ such that $\gamma^j_0 = \bar{x}$, 0 is a point of density one of $\mathrm{Dom}\,(\gamma^j)$ and $|d\gamma^j|(0) > 0$, for $j = 1, \dots, n$. Then there exists a dense countable set of times $\{t_h\}$ and a subsequence i_k such that:

$$z^j_h = \lim_{k\to\infty} f_{i_k}(\gamma^j_{t_h r_{i_k}}), \quad \mathsf{d}_\infty(z^j_h, z^j_\eta) = |t_h - t_\eta| \cdot |d\gamma^j|(0),$$

for any $h, \eta \in \mathbb{N}$ and $j = 1, \dots, n$.

3.2 Change of base point

It is also possible to extend Proposition 3.10 to any other point of the tangent space that is, if $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ is a pointed tangent space of $(X, \mathsf{d}, \bar{x})$ and $z_\infty \in X_\infty$, then one can find a tangent line passing through z_∞, obtained as the blow-up of the same curve.

This can be obtained using the fact that for almost every $\bar{x}$ and for every $z_\infty \in X_\infty$ also $(X_\infty, \mathsf{d}_\infty, z_\infty)$ is a pointed tangent space, provided the ambient measure $\mathfrak{m}$ is doubling. This has been proved by Preiss in [30] in the Euclidean framework and adapted to the metric space case by Le Donne in [25].

Theorem 3.12 ([25], Theorem 1.1)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space. Then for $\mathfrak{m}$-a.e. $\bar{x} \in X$, for all $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$, and for all $z_\infty \in X_\infty$ we have*

$$(X_\infty, \mathsf{d}_\infty, z_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x}).$$

Combining Proposition 3.10 and Theorem 3.12 we have

Corollary 3.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space, $\bar{x} \in X$ outside the exceptional set of Theorem 3.12 and $\gamma \in \Gamma(X)$ such that*

$$\gamma_0 = \bar{x}, \qquad 0 \textit{ is a point of density one of } \mathrm{Dom}(\gamma), \qquad |d\gamma|(0) > 0.$$

Then for any $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ and any $z_\infty \in X_\infty$ there exists a line, which is a limit of γ, passing through z_∞.

4 Tangent lines in Lipschitz differentiability spaces

In order to apply metric differentiability to Lipschitz differentiability spaces, a Borel regularity with respect to a precise Polish space is needed. We therefore recall a few definitions from [7] that will be needed only in this section.

For a metric space (X, d) define $H(X)$ to be the collection of non-empty compact subsets of $\mathbb{R} \times X$ with the Hausdorff metric, so that $H(X)$ is complete and separable. Moreover identify $\Gamma(X)$ with its isometric image in $H(X)$ via the map $\gamma \to \mathrm{graph}(\gamma)$ and consider

$$A(X) := \left\{(x, \gamma) \in X \times \Gamma(X) \,:\, \exists\, t \in \mathrm{Dom}(\gamma),\ x = \gamma_t\right\}.$$

One can show (Lemma 2.7, [7]) that $\Gamma(X)$ is a Borel subset of $H(X)$ and $A(X)$ is a Borel subset of $X \times H(X)$. Modifying Lemma 2.8 of [7] we obtain

Lemma 4.1. *Let (X, d) be a complete and separable metric space. The map $F\colon A(X) \to \mathbb{R} \cup \{\infty\}$ defined as*

$$F(x, \gamma) := \begin{cases} |d\gamma|(\gamma^{-1}(x)) & \textit{if it exists} \\ \infty & \textit{otherwise} \end{cases} \tag{4.1}$$

is Borel.

Proof. The proof is a slight modification of the proof of Lemma 2.8 of [7].
Let $q, \delta, \epsilon > 0$ and $\alpha \in (0, 1]$. The set of $(\gamma_{t_0}, \gamma) \in A(X)$ with

$$\left| d(\gamma_{t_0+t}, \gamma_{t_0+s}) - q|t - s| \right| \le \epsilon |t - s|,$$

for all t, s with $t_0 + t, t_0 + s \in \mathrm{Dom}(\gamma)$ and $|t|, |s| \le \delta$ and $|t - s| \ge \alpha \max\{t, s\}$, is closed. After taking suitable countable intersection and unions as in [7], the set where F exists and belongs to some open subset of $\mathbb{R}$ is Borel and the claim follows. □

Now we obtain the following improved version of Theorem 2.7, stated in Section 2.3.

Proposition 4.2. *Let (U, φ) be an n-dimensional chart in a Lipschitz differentiability space $(X, \mathsf{d}, \mathfrak{m})$. Then for almost every $x \in U$, there exists $\gamma_1^x, \ldots, \gamma_n^x \in \Gamma(X)$ such that:*

i) $(\gamma_i^x)^{-1}(x) = 0$ *is a point of density one of* $(\gamma_i^x)^{-1}(U)$ *for each* $i = 1, \ldots, n$*;*

ii) the metric differential in 0 *exists and* $|d\gamma_i^x|(0) > 0$ *for each* $i = 1, \ldots, n$*;*

iii) $\{(\varphi \circ \gamma_i^x)'(0)\}_{i=1,\ldots,n}$ *are linearly independent.*

Moreover, for any such γ_i^x, for any Lipschitz $g\colon X \to \mathbb{R}$ and almost every $x \in U$, the gradient of g at x with respect to φ and $\gamma_1^x, \dots, \gamma_n^x$ equals $Dg(x)$, that is

$$\left(g \circ \gamma_i^x\right)'(0) = Dg(x_0) \cdot \left(\varphi \circ \gamma_i^x\right)'(0),$$

for $i = 1, \dots, n$.

Even though the proof of Proposition 4.2 contains no novelty with respect to Theorem 2.7, we included it here for reader's convenience.

Proof. By Theorem 6.6 of [7] we have the existence of a countable decomposition $U = \cup_j U_j$ of U into sets with n φ-independent Alberti representations (whose definition can be found in Section 2 of [7]). We consider for $k = 1, \dots, n$ the Borel function $F_k : A(X) \to \mathbb{R} \cup \{\infty\}$ defined by

$$F_k(x, \gamma) := \begin{cases} (\varphi^k \circ \gamma)'(\gamma^{-1}(x)) & \text{if it exists} \\ \infty & \text{otherwise,} \end{cases}$$

where φ^k is the k-th component of the coordinate map $\varphi : U \to \mathbb{R}^n$. Moreover, we define F_0 to be the function F considered in Lemma 4.1.

For each $k = 0, \dots, n$ all the assumption of Proposition 2.9 of [7] are satisfied. The case $k = 0$ follows from Proposition 3.8 and Lemma 4.1, while the case $k \geq 1$ from Lemma 2.8 of [7]. Then we can repeat the same argument for the n φ-independent Alberti representations on each U_j.

Hence for each $j \in \mathbb{N}$ there exists $V_j \subset U_j$ with $\mathfrak{m}(U_j \setminus V_j) = 0$ such that for each $x \in V_j$ there exist $\gamma_1^x, \dots, \gamma_n^x \in \Gamma(X)$ such that

i) $(\gamma_i^x)^{-1}(x) = 0$ is a point of density one of $(\gamma_i^x)^{-1}(V_j)$;
ii) the metric differential in 0 exists and $|d\gamma_i^x|(0) > 0$;
iii) $(\varphi \circ \gamma_i^x)'(0)$ are linearly independent,

for $i = 1, \dots, n$ and for each $k = 0, \dots, n$ the map $x \mapsto F_k(x, \gamma_i^x)$ is measurable. Since $V_j \subset U$, i) implies that $(\gamma_i^x)^{-1}(x) = 0$ is a point of density one of $(\gamma_i^x)^{-1}(U)$. This proves the first part of the statement. The second part just follows from Theorem 2.7. □

We can now use the previous result to obtain the following

Proposition 4.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a Lipschitz differentiability space and (U, φ) be an n-dimensional chart. Then for $\mathfrak{m}$-almost every $\bar{x} \in U$, any element $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ contains n disjoint (neglecting $\bar{x}_\infty$) isometric copies of $\mathbb{R}$, obtained as limits of Lipschitz curves.*

Proof. Take any pointed metric measure space $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ and the corresponding sequence of dilations $r_i > 0$, with $r_i \to 0$.

The existence of n isometric copies of $\mathbb{R}$ follows straightforwardly from Definition 2.2, Proposition 3.10 and Proposition 4.2. It only remains to prove that the copies are disjoint. To this end we consider a chart (U, φ) with $\bar{x} \in U$ and use Remark 3.11: there exists a dense sequence $\{t_h\}_{h \in \mathbb{N}} \subset \mathbb{R}$ and a subsequence i_k such that:

$$z_h^j = \lim_{k \to \infty} f_{i_k} \circ \gamma^j_{t_h r_{i_k}}, \quad \mathsf{d}_\infty(z_h^j, z_\eta^j) = |t_h - t_\eta| \cdot |d\gamma^j|(0),$$

for $j = 1, \dots, n$, where f_{i_k} is the sequence of approximate isometries and γ^j are given by Proposition 4.2. Recall that the closure in d_∞ of each $\{z_h^j : h \in \mathbb{N}\}$ forms the isometric copies of $\mathbb{R}$ in X_∞. Via a reparametrization, without loss of generality, we may also assume that $|d\gamma^j|(0) = 1$ for all $j = 1, \dots, n$.

- 10.1515/agms-2016-0004

Now we just observe that for $j, l = 1, \dots, n$

$$\begin{aligned}\mathsf{d}_\infty(z_h^j, z_h^l) &= \lim_{k\to\infty} \mathsf{d}_\infty(f_{i_k} \circ \gamma^j_{t_h r_{i_k}}, f_{i_k} \circ \gamma^l_{t_h r_{i_k}}) \\ &= \lim_{k\to\infty} \frac{1}{r_{i_k}} \mathsf{d}(\gamma^j_{t_h r_{i_k}}, \gamma^l_{t_h r_{i_k}}) \\ &\geq \frac{1}{L} \lim_{k\to\infty} \frac{1}{r_{i_k}} |\varphi \circ \gamma^j_{t_h r_{i_k}} - \varphi \circ \gamma^l_{t_h r_{i_k}}| \\ &\geq \frac{t_h}{L} | \left(\varphi \circ \gamma^j\right)'(0) - \left(\varphi \circ \gamma^l\right)'(0)|,\end{aligned}$$

where L is the Lipschitz constant of φ. Therefore we have proved that

$$\mathsf{d}_\infty(z_h^j, z_h^l) \geq \frac{t_h}{L} | \left(\varphi \circ \gamma^j\right)'(0) - \left(\varphi \circ \gamma^l\right)'(0)|, \tag{4.2}$$

that implies, by linear independence, that $\mathsf{d}_\infty(z_h^j, z_h^l) > 0$, for all $h \in \mathbb{N}$. Since intersection for different times is not possible (at time 0 they start from the same point, with the same speed), the claim follows. □

We summarize the disjointness property of the isometric embeddings of $\mathbb{R}$.

Corollary 4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a Lipschitz differentiability space and (U, φ) be an n-dimensional chart. Then for $\mathfrak{m}$-almost every $\bar{x} \in U$, there exist $v_1, \dots, v_n \in \mathbb{R}^n$ linearly independent such that for any element $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ there exist $\iota_1, \dots, \iota_n : \mathbb{R} \to X_\infty$ so that*

i) $\iota_j(0) = \bar{x}_\infty$*, for any* $j = 1, \dots, n$;

ii) $\mathsf{d}_\infty(\iota_j(t), \iota_j(s)) = |t - s|$*, for any* $j = 1, \dots, n$*, for all* $s, t \in \mathbb{R}$;

iii) $\mathsf{d}_\infty(\iota_j(t), \iota_k(t)) \geq C|t| \cdot |v_j - v_k|$*, for any* $j, k = 1, \dots, n$*, for all* $t \in \mathbb{R}$;

for some positive constant C. Each of the ι_i is obtained as the limit of a Lipschitz curve.

If the Lipschitz differentiability space is also doubling, one can argue as in Corollary 3.13 to obtain information on lines through any point of the tangent space.

Theorem 4.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling Lipschitz differentiability space and (U, φ) be an n-dimensional chart. Then for $\mathfrak{m}$-almost every $\bar{x} \in U$, there exist $v_1, \dots, v_n \in \mathbb{R}^n$ linearly independent such that for any element $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ and for each $z \in X_\infty$ there exist $\iota_1^z, \dots, \iota_n^z : \mathbb{R} \to X_\infty$ so that*

i) $\iota_j^z(0) = z$*, for any* $j = 1, \dots, n$;

ii) $\mathsf{d}_\infty(\iota_j^z(t), \iota_j^z(s)) = |t - s|$*, for any* $j = 1, \dots, n$*, for all* $s, t \in \mathbb{R}$;

iii) $\mathsf{d}_\infty(\iota_j^z(t), \iota_k^z(t)) \geq C|t| \cdot |v_j - v_k|$*, for any* $j, k = 1, \dots, n$*, for all* $t \in \mathbb{R}$;

for some positive constant $C = C(z)$. For each $z \in X_\infty$, each line ι_i^z is obtained as the blow-up of a Lipschitz curve, with the blow-up depending on z.

5 Tangent lines in spaces with splitting tangents

As stated in Theorem 2.6, doubling metric measure spaces supporting a local p-Poincaré inequality are Lipschitz differentiability spaces and in particular Corollary 4.4 applies. For this class of more regular metric measure spaces, results on the structure of tangent spaces were already available. For instance in [10], Theorem 8.5, existence of integral curves for tangent functions was proved. This in turn implies the existence of sufficiently many geodesic lines in the tangent space. But no explicit relation between geodesic lines in the tangent space and curves on the metric measure space was shown to exist. Therefore Corollary 4.4 brings new information also on the structure of tangent spaces for doubling metric measure space supporting a local p-Poincaré inequality.

In this last section we show that if n is the dimension of a chart of the measurable differentiable structure of $(X, \mathsf{d}, \mathfrak{m})$ seen as a Lipschitz differentiability space and if d is the dimension of a Euclidean tangent space at x, then $n \leq d$ at $\mathfrak{m}$-a.e. point of X.

More precisely, we are interested in a special class of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ having the *splitting of tangents* property: if

$$(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$$

for some $\bar{x} \in X$ and if X_∞ contains an isometric copy of $\mathbb{R}$ going through $\bar{x}_\infty$, then $(X_\infty, \mathsf{d}_\infty)$ is isometric to

$$(\mathbb{R} \times Y, |\cdot| \times \mathsf{d}_Y)$$

where (Y, d_Y) is a metric space, and the same property holds for Y. In particular if $\iota : \mathbb{R} \to X_\infty$ parametrizes the isometric copy of $\mathbb{R}$ contained in X_∞, then there exists an isometry

$$h : (X_\infty, \mathsf{d}_\infty) \to (\mathbb{R} \times Y, |\cdot| \times \mathsf{d}_Y),$$

such that $h \circ \iota(\mathbb{R}) \subset \mathbb{R} \times \{\bar{y}\}$, for some $\bar{y} \in Y$. The same property has to hold for the metric space (Y, d_Y).

We obtain the following result.

Theorem 5.1. *Suppose that* $(X, \mathsf{d}, \mathfrak{m})$ *is a doubling Lipschitz differentiability space with the splitting of tangents property. Let* (U, φ) *be an* n*-dimensional chart of* $(X, \mathsf{d}, \mathfrak{m})$*. Then for* $\mathfrak{m}$*-a.e.* $\bar{x} \in U$ *any* $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ *is of the form*

$$(X_\infty^d \times \mathbb{R}^d, \mathsf{d}_\infty^d \times |\cdot|, (\bar{x}_\infty^d, 0)),$$

with $d \geq n$*.*

Compare Theorem 5.1 to the result from [16], that can be rephrased as

Theorem 5.2. *Suppose that* $(X, \mathsf{d}, \mathfrak{m})$ *is a geodesic doubling metric measure space with the splitting of tangents property. Then at* $\mathfrak{m}$*-a.e. point in X there exists a Euclidean tangent space.*

Theorem 5.2 was formulated in [16] for $\mathsf{RCD}^*(K, N)$ spaces (metric measure spaces with Riemannian Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension from above by N), for which any tangent is an $\mathsf{RCD}^*(0, N)$ space having the splitting property, as was shown by Gigli [15]. Theorem 5.1 now shows that taking into account the fact that $\mathsf{RCD}^*(K, N)$ spaces are doubling and support a local Poincaré inequality [31, 32], we immediately have that any tangent space contains an $\mathbb{R}^n$ part with dimension at least the dimension of the chart. For a comprehensive treatise on the above mentioned family of spaces we refer to [26, 38, 39] for the defintion of $\mathsf{CD}(K, N)$ and to [2, 3] for the infinite dimensional Riemannian version. Finally $\mathsf{RCD}^*(K, N)$ with $N \in \mathbb{R}$ has been introduced independently in [5] and [14].

For $\mathsf{RCD}^*(K, N)$ spaces more can be said on the relation of the charts and the tangent spaces than the conclusion of Theorem 5.1. A recent result by Mondino and Naber in [27] states that for $(X, \mathsf{d}, \mathfrak{m})$ verifying $\mathsf{RCD}^*(K, N)$, at $\mathfrak{m}$-a.e. $x \in X$ there exists a unique tangent space and it is isomorphic, in the sense of metric measure spaces, to $(\mathbb{R}^d, |\cdot|, \mathcal{L}^d)$, with d varying measurably in x. Moreover, they proved the following theorem.

Theorem 5.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}^*(K, N)$ *space for some* $K, N \in \mathbb{R}$ *with* $N > 1$*. Then there exists a countable collection* $\{R_j\}_{j\in\mathbb{N}}$ *of* $\mathfrak{m}$*-measurable subsets of X, covering X up to an* $\mathfrak{m}$*-negligible set, such that each* R_j *is bi-Lipschitz to a measurable subset of* $\mathbb{R}^{k_j}$*, for some* $1 \leq k_j \leq N$*,* k_j *possibly depending on j.*

However, if a complete Lipschitz differentiability space is bi-Lipschitz embeddable into some Euclidean space $\mathbb{R}^k$, then at almost every point all the tangent spaces are bi-Lipschitz equivalent to $\mathbb{R}^n$, where $n \leq k$ is the dimension of the chart given by the Lipschitz differentiability, see Corollary 8.1 in [12]. Moreover (see [7]) any positive measure subset of a Lipschitz differentiability space is itself a Lipschitz differentiability space. Since from inner regularity any measurable subsets of positive measure can be approximated, up to a set of measure zero, from inside with an increasing family of compact sets (obtaining therefore also the completeness), combining these two result with Theorem 5.3 we get that for $\mathsf{RCD}^*(K,N)$ spaces at almost every point the tangent is $\mathbb{R}^n$ where the n is the dimension of the chart. Let us note that it is still unknown if in this context the dimension n of the tangent (and the chart) depends on the point.

We prove Theorem 5.1, which is valid without the bi-Lipschitz embeddability to $\mathbb{R}^n$.

Proof of Theorem 5.1. **Step 1.**
By Corollary 4.4 any element $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ has n distinct isometric copies of $\mathbb{R}$:

$$\iota_j : \mathbb{R} \to X_\infty, \qquad \iota_j(0) = \bar{x}_\infty, \qquad j = 1, \dots, n,$$

and each ι_j is the blow-up of has a corresponding Lipschitz curve γ^j, see Proposition 4.3. By the splitting property, there exists an isometry

$$h_1 : (X_\infty, \mathsf{d}_\infty) \longrightarrow (X^1_\infty \times \mathbb{R}, \mathsf{d}^1_\infty \times |\cdot|), \qquad h_1(\bar{x}_\infty) = (\bar{x}^1_\infty, 0),$$

with $h_1(\iota_1(\mathbb{R})) = \{(\bar{x}^1_\infty, t) : t \in \mathbb{R}\}$. Since the n geodesics are all disjoint, composing isometries and applying Lemma 2.9 we deduce the existence of $n-1$ geodesics, again denoted with

$$\iota_j : (\mathbb{R}, |\cdot|) \to (X^1_\infty, \mathsf{d}^1_\infty), \qquad j = 2, \dots, n.$$

By Lemma 2.9 we can also deduce that $\iota_2(\mathbb{R}), \dots, \iota_n(\mathbb{R})$ are all disjoint and we can use again the splitting property to rule out another isometric copy of $\mathbb{R}$.

The same reasoning cannot be repeated to obtain a splitting of the form $X_\infty \sim X^n_\infty \times \mathbb{R}^n$. It might be the case that for some $j = 3, \dots, n$, $\iota_j(\mathbb{R})$ is already contained in the Euclidean component of the tangent space, and therefore the projection in the purely metric component of X_∞ could be the constant geodesic, not producing a new component to rule out via the splitting property.

Step 2.
Consider the n-dimensional chart (U, φ) with $\varphi : U \to \mathbb{R}^n$ Lipschitz and any $\bar{x} \in U$ such that Corollary 3.13 applies. Fix also $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty) \in \mathrm{Tan}(X, \mathsf{d}, \bar{x})$ and u_φ, the tangent function of φ at $\bar{x}$. Note that, possibly passing to subsequences, u_φ is well-defined.

Repeating the argument of **Step 1.** changing the reference point (see [25], Theorem 1.1), we have the following: for some $d \in \mathbb{N}$ all the possible splittings obtained from the lines of Theorem 4.5 give a decomposition of the following type: $X_\infty = X^d_\infty \times \mathbb{R}^d$, where the identity holds in the sense of metric spaces, and $\mathbb{R}^d$ is equipped with the Euclidean distance.

Step 3.
We now show that $d \geq n$. Consider the sequence $\{r_i\}_{i \in \mathbb{N}}$ producing $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ as the tangent space and the $3\varepsilon_i$-isometries f_i and f_i^{-1}. Let $z \in X_\infty$ be any point, then by definition

$$u_\varphi(z) = \lim_{i \to \infty} \frac{\varphi(f_i^{-1}(z)) - \varphi(\bar{x})}{r_i}.$$

As observed in the proof of Proposition 4.3, after a suitable reparametrization with unit speed, there exists a sequence of times $\{t_i\}_{i \in \mathbb{N}}$ with $t_i \to 1$ as $i \to \infty$ such that $\mathsf{d}_\infty(\iota_j(1), f_i(\gamma^j_{t_i r_i})) \to 0$, for each $j = 1, \dots, n$. We pose $z = \iota_j(1)$ and observe that

$$\begin{aligned}
\frac{1}{r_i} |\varphi(\gamma^j_{t_i r_i}) - \varphi(f_i^{-1}(z))| &\leq L \frac{1}{r_i} \mathsf{d}(\gamma^j_{t_i r_i}, f_i^{-1}(z)) \\
&\leq L \left(\varepsilon_i + \mathsf{d}_\infty(f_i(\gamma^j_{t_i r_i}), f_i(f_i^{-1}(z))) \right) \\
&\leq L \left(\varepsilon_i + \mathsf{d}_\infty(f_i(\gamma^j_{t_i r_i}), z) + \mathsf{d}_\infty(z, f_i(f_i^{-1}(z))) \right) \\
&\leq C\varepsilon_i.
\end{aligned}$$

It therefore follows that

$$u_\varphi(\iota_j(1)) = \lim_{i\to\infty} \frac{\varphi(\gamma^j_{t_i r_i}) - \varphi(\bar{x})}{r_i} = (\varphi\circ\gamma^j)'(0).$$

Using a different t_i converging to some other real number, it is easy to observe that $s \mapsto u_\varphi(\iota_j(s))$ is linear and

$$Span\left\{u_\varphi(\iota_1(1)),\dots,u_\varphi(\iota_n(1))\right\} = \mathbb{R}^n.$$

Thanks to Proposition 3.1 of [12], the same argument works for any $z \in X_\infty$. We can therefore consider the isometries $\iota^z_1,\dots,\iota^z_n$ such that $\iota^z_j(0) = z$ for $j = 1,\dots,n$ such that $s \mapsto u_\varphi(\iota^z_j(s))$ is linear, for any $z \in X_\infty$ and $j = 1,\dots,n$.

Finally, we consider $\bar{u}_\varphi$, the restriction of u_φ to $\left\{\bar{x}^d_\infty\right\} \times \mathbb{R}^d \to \mathbb{R}^n$. The claim can now be proven via showing that $\bar{u}_\varphi$ is a quotient map (again we refer to [12] for the relative definition). This can be obtained repeating verbatim the proof of Corollary 5.1 of [12] and using the linearity of $s \mapsto u_\varphi(\iota^z_j(s))$, together with Theorem 4.5. □

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure, *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 4661–4701.

[2] L. Ambrosio, N Gigli and G. Savaré, Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds, *Annals of Probab.* **43** (2015), no. 1, 339–404.

[3] L. Ambrosio, N Gigli and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, *Duke Math. J.* **163** (2014), 1405–1490.

[4] L. Ambrosio and B. Kirchheim, Rectifiable sets in metric and Banach spaces, *Math. Ann.* **318** (2000), 527–555.

[5] L. Ambrosio, A. Mondino and G. Savaré, Nonlinear diffusion equations and curvature conditions in metric measure spaces, *preprint* arXiv:1509.07273.

[6] L. Ambrosio and P. Tilli, Topics on Analysis in Metric Spaces. *Oxford University press*, Oxford Lecture Series in Mathematics and Its Applications, 2004.

[7] D. Bate, Structure of measures in Lipschitz differentiability spaces, *Journal Amer. Math. Soc.* **28** (2015), 421–482.

[8] D. Bate and S. Li, Characterizations of rectifiable metric measure spaces, *preprint*, arXiv:1409.4242.

[9] M. Bourdon and H. Pajot, Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings, *Proc. Amer. Math. Soc.* **127** (1999), no. 8, 2315–2324.

[10] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* **9** (1999), 428–517.

[11] J. Cheeger and B. Kleiner and A. Schioppa, Infinitesimal structure of differentiability spaces, and metric differentiation, *preprint* arXiv:1503.07348.

[12] G.C. David, Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces, *Geom. Funct. Anal.* **25** (2015), no. 2, 553–579.

[13] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.

[14] M Erbar, Kuwada and K.T. Sturm, On the Equivalence of the Entropic Curvature-Dimension Condition and Bochner's Inequality on Metric Measure Space, *Invent. Math.* **201** (2015), no. 3, 993 – 1071.

[15] N. Gigli, The splitting theorem in non-smooth context, *preprint*, arXiv:1302.5555.

[16] N. Gigli, A. Mondino and T. Rajala, Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below, *J. Reine Angew. Math.* **705** (2015), 233–244.

[17] N. Gigli, A. Mondino and G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, *to appear in Proc. London Math. Soc.* doi: 10.1112/plms/pdv047.

[18] N. Juillet, Geometric inequalities and generalized Ricci bounds in the Heisenberg group, *Int. Math. Res. Notices* **2009** (2009), 2347–2373.

[19] S. Keith, Measurable Differentiable Structures and the Poincaré Inequality, *Indiana Univ. Math. J.* **53** (2004), 1127–1150.

[20] C. Ketterer and T. Rajala, Failure of topological rigidity results for the measure contraction property, *Potential Anal.* **42** (2015), no. 3, 645–655.

[21] B. Kirchheim, Rectifiable metric space: local structure and regularity of the Hausdorff measure, *Proc. Am. Math. Soc.* **121** (1994), 113–123.

[22] R. Korte, Geometric implications of the Poincaré inequality, *Result. Math.* **50** (2007), 93–107.

[23] B. Kleiner and J. Mackay, Differentiable structures on metric measure spaces: A Primer *preprint*, arXiv:1108.1324.

[24] T. Laakso, Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.* **10** (2000), 111–123.

[25] E. Le Donne, Metric spaces with unique tangents, *Ann. Acad. Sci. Fenn. Math.* **36** (2011), 683–694.
[26] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* (2) **169** (2009), 903–991.
[27] A. Mondino and A. Naber, Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds I, *preprint*, arXiv:1405.2222.
[28] S.-I. Ohta, On the measure contraction property of metric measure spaces, *Comment. Math. Helv.* **82** (2007), 805–828.
[29] S.-I. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature, *J. Reine Angew. Math.* **700** (2015), 155–174.
[30] D. Preiss, Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities, *Ann. of Math.* **125** (1987), 537–643.
[31] T. Rajala, Local Poincaré inequalities from stable curvature conditions on metric spaces, *Calc. Var. Partial Differential Equations* **44** (2012), 477–494.
[32] T. Rajala, Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm, *J. Funct. Anal.* **263** (2012), 896–924.
[33] W. Rudin, Real and complex analysis, *McGraw-Hill Book Company*, International Edition 1987.
[34] A. Schioppa Derivations and Alberti representations, *preprint*, arXiv:1311.2439.
[35] A. Schioppa On the relationship between derivations and measurable differentiable structures, *Ann. Acad. Sci. Fenn. Math.*, 39 (2014), no. 1, 275–304.
[36] S. Semmes, Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities, *Selecta Math.* **2** (1996), 155–295.
[37] S. Semmes, Some novel types of fractal geometry, *Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press*, New York, 2001.
[38] K.T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* **196** (2006), 65–131.
[39] K.T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* **196** (2006), 133–177.
[40] C. Villani, Optimal transport. Old and new, *Grundlehren der Mathematischen Wissenschaften*, **338**, Springer-Verlag, Berlin, (2009).