Dyadic norms for trace spaces

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Master's thesis in mathematics

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Spring 2016

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Abstract

Let $D \subset \mathbb{R}^2$ be the unit disk. The fractional Sobolev space $W^{1-1/p,p}(S^1)$ is the trace space of $W^{1,p}(D)$ for $p > 1$, that is, there exists a unique continuous linear mapping $\mathcal{T}$ from $W^{1,p}(D)$ into $W^{1-1/p,p}(S^1)$ such that $\mathcal{T}u = u|_{S^1}$ for all $u \in C^\infty(\overline{D})$, and there exists a continuous linear mapping E from $W^{1-1/p,p}(S^1)$ into $W^{1,p}(D)$ such that $Eg|_{S^1} = g$ for all $g \in W^{1-1/p,p}(S^1)$.

We would like to use the dyadic energy $\mathcal{E}(g; p, \lambda)$ obtained via the summation of the differences between the averages associated with a dyadic decomposition to characterize the trace of some Sobolev space. After modifying the energy $\mathcal{E}(g; p, \lambda)$ to $\mathbf{E}(g; p, \lambda)$ and $\mathbf{E}(g; \Phi)$ with $\Phi(t) = t^p \log^\lambda(e + t)$, we define the Banach spaces $T^\Phi(S^1)$ and $\widetilde{T}^\Phi(S^1)$ with norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^*$, respectively. Then we prove that $T^\Phi(S^1)$ is the trace space of the weighted Orlicz-Sobolev space $W^{1,\Phi}_{p-2}(D)$ and that $\widetilde{T}^\Phi(S^1)$ is the trace space of another weighted Orlicz-Sobolev space $W^{1,p}_{w_\Phi}(D)$. Moreover, we show that $T^\Phi(S^1)$ and $\widetilde{T}^\Phi(S^1)$ coincide as sets, but $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$ do not. Hence, this is an example of two different Banach spaces that have the same trace space.

To verify the results above, for the extension part, we use a Whitney-type decomposition of D and an associated partition of unity to define the extension operator. Then the operator is shown to be continuous and linear via a series of calculations. For the trace part, we first show that $T^\Phi(S^1)$ and $\widetilde{T}^\Phi(S^1)$ are Banach spaces and that $C^\infty(\overline{D})$ is dense in $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$. Then we prove that the restriction operator for functions in $C^\infty(\overline{D})$ is continuous and linear via a series of calculations. Using the density properties of $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$ and the completeness of $T^\Phi(S^1)$ and $\widetilde{T}^\Phi(S^1)$, we finally give the continuous linear trace operators on $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$ which coincide with the restriction operators for functions in $C^\infty(\overline{D})$.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor Pekka Koskela, who gave me the golden chance to do the wonderful project on this topic and who always assists me whenever I need his help not only in mathematics but also in other things.

The Department of Mathematics and Statistics at our university provides an excellent research environment. It is really a nice experience to study and work here. I thank all of my colleagues in our department, especially, Yi Zhang, Zheng Zhu and Shirsho Mukherjee, for their kind help.

Finally, I thank my dear family and friends.

Jyväskylä, May 2016

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, i.e., an open connected subset of $\mathbb{R}^n$.

Definition 1.1. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $i \in \{1, 2, \cdots, n\}$. We say $g_i \in L^1_{\mathrm{loc}}(\Omega)$ is the *weak partial derivative* (*distributional derivative*) of f with respect to x_i in Ω if

$$\int_\Omega u\, \phi_{x_i}\, dx = -\int_\Omega g_i\, \phi\, dx$$

for all $\phi \in C_c^\infty(\Omega)$. Then $Du := (g_1, \cdots, g_n)$ is the *weak derivative* or *weak gradient* of u.

Then we introduce the *Sobolev space* $W^{1,p}(\Omega)$ with $p \in [1, \infty]$ as the set of all functions $u \in L^p(\Omega)$ whose weak derivative Du belongs to the space $L^p(\Omega)$. The Sobolev space $W^{1,p}(\Omega)$ is a Banach space, i.e., complete normed vector space with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |u(x)|^p\, dx\right)^{1/p} + \left(\int_\Omega |Du(x)|^p\, dx\right)^{1/p}$$

for $1 \le p < \infty$, and

$$\|u\|_{W^{1,\infty}(\Omega)} := \operatorname*{ess\,sup}_\Omega (|u| + |Du|).$$

We would like to characterize the *trace* of a Sobolev function $u \in W^{1,p}(\Omega)$, namely, the restriction of u to the boundary $\partial\Omega$. It would be interesting to find a Banach space $X(\partial\Omega, \|\cdot\|)$ characterizing the trace space such that the trace operator becomes a bounded linear operator $\mathcal{T} : W^{1,p}(\Omega) \to X(\partial\Omega, \|\cdot\|)$. Also there is a converse problem, namely, the problem of extension. Given the Banach space $X(\partial\Omega, \|\cdot\|)$, we would like to find a bounded linear operator $E : X(\partial\Omega, \|\cdot\|) \to W^{1,p}(\Omega)$ such that $Eg|_{\partial\Omega} = g$ for all $g \in X(\partial\Omega, \|\cdot\|)$.

For technical reasons, we only consider the case that Ω is the unit disk $D \subset \mathbb{R}^2$. Then we have $\partial\Omega = S^1$. For the Sobolev space $W^{1,p}(D)$ with $1 \le p < \infty$, we have the density property of smooth functions, i.e., $C^\infty(\overline{D})$ is dense in $W^{1,p}(D)$ (see [6, Theorem 4.3]). Here $C^\infty(\overline{D})$ is the set of all functions $u = u(x)$ infinitely differentiable in D, whose derivatives $D^\alpha u$ are bounded and uniformly continuous. Now $u|_{S^1}$ is well-defined for a function $u \in C^\infty(\overline{D})$. Assume that there exists a Banach space $X(S^1, \|\cdot\|)$ such that the trace operator $\mathcal{T} : C^\infty(\overline{D}) \to X(S^1, \|\cdot\|)$ with $\mathcal{T}u = u|_{S^1}$ is a bounded linear operator, i.e., there exists a constant $C > 0$ such that for all $u \in C^\infty(\overline{D})$ we have $\|\mathcal{T}u\|_{X(S^1)} \le C\|u\|_{W^{1,p}(D)}$. Then we can define the trace operator $\mathcal{T} : W^{1,p}(D) \to X(\partial\Omega, \|\cdot\|)$ with $\mathcal{T}u$ the limit of $\mathcal{T}u_k$ in the norm sense, where $u_k \in C^\infty(\overline{D})$ converge to u in $W^{1,p}(D)$ as $k \to \infty$.

Using the idea above, one can prove that the trace space of $W^{1,p}(D)$ is the fractional Sobolev space $W^{1-1/p,p}(S^1)$ for $p > 1$, see [2, Theorem 6.8.13 and Theorem 6.9.2]. Indeed, there exists a unique continuous linear mapping $\mathcal{T}$ from $W^{1,p}(D)$ into

$W^{1-1/p,p}(S^1)$ such that $\mathcal{T}u = u|_{S^1}$ for all $u \in C^\infty(\overline{D})$, and there exists a continuous linear mapping E from $W^{1-1/p,p}(S^1)$ into $W^{1,p}(D)$ such that $Eg|_{S^1} = g$ for all $g \in W^{1-1/p,p}(S^1)$. Here the fractional Sobolev space $W^{1-1/p,p}(S^1)$ consists of all the functions $g \in L^p(S^1)$ such that

$$\|g\|^p_{W^{1-1/p,p}(S^1)} = \|g\|^p_{L^p(S^1)} + \int_{S^1}\int_{S^1} \frac{|g(x)-g(y)|^p}{|x-y|^p}\, d\mathcal{H}^1_x d\mathcal{H}^1_y < \infty.$$

In the paper [5], one finds a dyadic version for the energy of $g : S^1 \to \mathbb{R}$, given by

$$\mathcal{E}(g;p,\lambda) := \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|^p, \tag{1.1}$$

where $p > 1$ and $\lambda \in \mathbb{R}$. Here, $\{I_{i,j} : i \in \mathbb{N}, j = 1, \cdots, 2^i\}$ is a dyadic decomposition of S^1, such that for a fixed $i \in \mathbb{N}$, $\{I_{i,j} : j = 1, \cdots, 2^i\}$ is a family of arcs of length $2\pi/2^i$ with $\bigcup_j I_{i,j} = S^1$. The next generation is constructed in such a way that for each $j \in \{1, \cdots, 2^{i+1}\}$, there exists a unique number $k \in \{1, \cdots, 2^i\}$, satisfying $I_{i+1,j} \subset I_{i,k}$. We denote this parent of $I_{i+1,j}$ by $\widehat{I}_{i+1,j}$ and set $\widehat{I}_{1,j} = S^1$ for $j = 1, 2$. By g_A, $A \subset S^1$, we denote the mean value $g_A = \fint_A g\, d\mathcal{H}^1 = \frac{1}{\mathcal{H}^1(A)}\int_A g\, d\mathcal{H}^1$. For more details, see Section 6.

From Remark 6.4 (also see [5, Theorem 3.1]), we have a sufficient condition for $\mathcal{E}(g;p,\lambda) < \infty$. If $g : S^1 \to \mathbb{R}$ is bounded and

$$\int_{S^1}\int_{S^1} \frac{|g(x)-g(y)|^p}{|x-y|^2} \log^\lambda\left(e + \frac{|g(x)-g(y)|}{|x-y|}\right) d\mathcal{H}^1_x d\mathcal{H}^1_y < \infty, \tag{1.2}$$

then $\mathcal{E}(g;p,\lambda) < \infty$.

When $p = 2$ and $\lambda = 0$, we have

$$\mathcal{E}(g;2,0) := \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|^2.$$

From Example 6.1, we obtain a function g such that $\mathcal{E}(g;2,0) < \infty$ but (1.2) is not satisfied, i.e.,

$$\int_{S^1}\int_{S^1} \frac{|g(x)-g(y)|^2}{|x-y|^2} d\mathcal{H}^1_x d\mathcal{H}^1_y = \infty.$$

So it is not possible to characterize $W^{\frac{1}{2},2}(D)$ by using the energy $\mathcal{E}(g;2,0)$. We consider another energy, given by

$$\mathbf{E}(g;2,0) := \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k} |g_{I_{i,j}} - g_{I_k}|^2; \tag{1.3}$$

where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$. Here $I_{i,0} := I_{i,2^i}$ and $I_{i,2^i+1} := I_{i,1}$. For the new energy $\mathbf{E}(g;2,0)$, we have the following theorem:

Theorem 1.1. *(i) For any function $u \in C^\infty(\overline{D})$, we have $\mathbf{E}(u|_{S^1};2,0) < \infty$. More precisely, there exists a constant $C > 0$ such that for all $u \in C^\infty(\overline{D})$, we have*

$$\|u|_{S^1}\|^2_{L^2(S^1)} + \mathbf{E}(u|_{S^1};2,0) \le C\|u\|^2_{W^{1,2}(D)}.$$

(ii) There exists a constant $C > 0$ such that for any function $g \in L^2(S^1)$ with $\mathbf{E}(g;2,0) < \infty$, we can find a a function $u \in W^{1,2}(D)$ which satisfies $u|_{S^1} = g$ and

$$\|u\|^2_{W^{1,2}(D)} \le C\left(\|g\|^2_{L^2(S^1)} + \mathbf{E}(g;2,0)\right).$$

Here, we write $u|_{S^1} = g$ if for a.e. $x = e^{i\theta} \in S^1$, when $\{x_n = r_n e^{i\theta}\}_{n=1}^\infty$ with $x_n \in D$ and $r_n \to 1$, we have $\lim_{n\to\infty} u(x_n) = g(x)$.

For a general $p > 1$, $\mathbf{E}(g;p,0)$ is the correct energy for the trace of a suitable weighted Sobolev space. For details about weighted Sobolev spaces, see Section 2. The fractional Sobolev space $W^{\frac{1}{p},p}$ is the trace space of the weighted Sobolev space $W^{1,p}_{p-2}(D)$ (see [3, Theorem 2.10]). Here, $W^{\frac{1}{p},p}(S^1)$ consists of all the functions g such that

$$\|g\|^p_{W^{\frac{1}{p},p}(S^1)} = \|g\|^p_{L^p(S^1)} + \int_{S^1}\int_{S^1} \frac{|g(x)-g(y)|^p}{|x-y|^2}\, d\mathcal{H}^1_x d\mathcal{H}^1_y < \infty.$$

In order to deal with the general case $p > 1$ and $\lambda \in \mathbb{R}$, we modify the energy (1.3) by setting

$$\mathbf{E}(g;p,\lambda) := \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p \tag{1.4}$$

and

$$\mathbf{E}(g;\Phi) := \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k 2^{-ip} \Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right), \tag{1.5}$$

where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$ and $\Phi(t) = t^p \log^\lambda(e+t)$. It is easy to see that the energy (1.3) is a special case of (1.4). Moreover, the following proposition gives us the connection between the above two energies.

Proposition 1.1. *Let $g : S^1 \to \mathbb{R}$, $g \in L^\Phi(S^1)$ for $\Phi(t) = t^p \log^\lambda(e+t)$, where $1 < p < \infty$ and $\lambda \in \mathbb{R}$. Then $\mathbf{E}(g;p,\lambda) < \infty$ is equivalent to $\mathbf{E}(g;\Phi) < \infty$.*

For $p>1$ and general $\lambda\in\mathbb{R}$, Sobolev spaces and weighted Sobolev spaces do not suffice for us, We need to consider Orlicz-Sobolev spaces and weighted Orlicz-Sobolev spaces, for more details see Section 4. Here, we introduce the weighted Orlicz-Sobolev space $W^{1,\Phi}_{p-2}(D)$ as follows:

$$W^{1,\Phi}_{p-2}(D)=\{u\in L^{\Phi}(D):\int_D \Phi(s|Du(x)|)w(x)\,dx<\infty \quad \text{for some} \ \ s>0\},$$

where $w(x)=\operatorname{dist}(x,S^1)^{p-2}$. The space $W^{1,\Phi}_{p-2}(D)$ is a Banach space and $C^\infty(\overline{D})$ is dense in it with the norm

$$\begin{aligned}\|u\|_{W^{1,\Phi}_{p-2}(D)} &= \|u\|_{L^{\Phi}(D)}+\|Du\|_{L^{\Phi}_{p-2}(D)}\\ &=\inf\left\{k:\int_D\Phi\left(\frac{|u|}{k}\right)\,dx\le 1\right\}+\inf\left\{k:\int_D\Phi\left(\frac{|Du|}{k}\right)w(x)\,dx\le 1\right\},\end{aligned}$$

where Du is the weak derivative of u. For the density property, see Section 5.

Our next result is about extensions.

Theorem 1.2. *Let $g\in L^{\Phi}(S^1)$ for $\Phi(t)=t^p\log^{\lambda}(e+t)$, where $1<p<\infty$ and $\lambda\in\mathbb{R}$. If $\mathbf{E}(g;p,\lambda)<\infty$, then there exists a function $u\in W^{1,\Phi}_{p-2}(D)$ such that $u|_{S^1}=g$.*

In the process of verifying the above results, we actually found the trace space of $W^{1,\Phi}_{p-2}(D)$, which is much stronger than the above results. Define the space $T^{\Phi}(S^1)$ by setting:

$$T^{\Phi}(S^1):=\{g\in L^{\Phi}(S^1):\|g\|_{\Phi}<\infty\},$$

where

$$\|g\|_{\Phi}=\|g\|_{L^{\Phi}(S^1)}+\|g\|_{\mathbf{E}_{\Phi}}$$

and

$$\|g\|_{\mathbf{E}_{\Phi}}=\inf\left\{k>0,\mathbf{E}\left(\frac{g}{k};\Phi\right)\le 1\right\}.$$

Then $T^{\Phi}(S^1)$ is the trace space of $W^{1,\Phi}_{p-2}(D)$. This is given by the following theorem.

Theorem 1.3. *(i) There exists an unique continuous linear mapping $\mathcal{T}$ from $W^{1,\Phi}_{p-2}(D)$ into $T^{\Phi}(S^1)$ such that $\mathcal{T}u=u|_{S^1}$ for all $u\in C^\infty(\overline{D})$.*

(ii) There exists a continuous linear mapping E from $T^{\Phi}(S^1)$ into $W^{1,\Phi}_{p-2}(D)$ such that $Eg|_{S^1}=g$.

In the process of verifying Theorem 1.3 and Proposition 1.1, we obtained a similar result. Define $\widetilde{T}^{\Phi}(S^1)$ as follows:

$$\widetilde{T}^{\Phi}(S^1):=\{g\in L^{\Phi}(S^1):\|g\|^*_{\Phi}<\infty\},$$

where

$$\|g\|_{\Phi}^{*} = \|g\|_{L^{\Phi}(S^1)} + \|g\|_{\mathbf{E}_\Phi}^{*}$$

and

$$\|g\|_{\mathbf{E}_\Phi}^{*} = (\mathbf{E}(g, p, \lambda))^{1/p}.$$

Then $\widetilde{T}^{\Phi}(S^1)$ is the trace space of $W^{1,p}_{w_\Phi}(D)$ where w_Φ is the weight in Example 3.2. This result is given by the following theorem.

Theorem 1.4. *(i) There exists an unique continuous linear mapping $\mathcal{T}^*$ from $W^{1,p}_{w_\Phi}(D)$ into $\widetilde{T}^{\Phi}(S^1)$ such that $\mathcal{T}^* u = u|_{S^1}$ for all $u \in C^\infty(\overline{D})$.*

(ii) There exists a continuous linear mapping E^* from $\widetilde{T}^{\Phi}(S^1)$ into $W^{1,p}_{w_\Phi}(D)$ such that $E^* g|_{S^1} = g$.

For more details about $W^{1,p}_{w_\Phi}(D)$ see Example 4.4. Moreover, $C^\infty(\overline{D})$ is also dense in $W^{1,p}_{w_\Phi}(D)$; see Section 5.

Remark 1.1. From Proposition 1.1, we know that $T^{\Phi}(S^1)$ and $\widetilde{T}^{\Phi}(S^1)$ are equal as sets, but the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^*$ are not equivalent. Moreover, $W^{1,\Phi}_{p-2}(D) \neq W^{1,p}_{w_\Phi}(D)$, see Remark 4.2. Hence we have two different Banach spaces which have the same trace space.

Remark 1.2. It is easy to see that Theorem 1.1 is a special case of Theorem 1.3 or of Theorem 1.4 with $p = 2$ and $\lambda = 0$. Moreover, we can obtain Theorem 1.2 directly via the proof of Theorem 1.3. Hence we only need to give the proofs of Theorem 1.3 and Theorem1.4.

We have not been able to find the results contained in Proposition 1.1, Theorem 1.3 and Theorem 1.4 in the literature.

This thesis is organized as follows. Section 2 mainly introduces the weighted Sobolev spaces. In section 3, we introduce A_p-weights and maximal functions with respect to the measure coming from an A_p-weight. In Section 4, we recall Orlicz and Orlicz-Sobolev spaces, and give the definitions of the spaces $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$. In section 5, we mainly give the proof of the density properties of $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$. We discuss the three different energies and their connections in Section 6. In Section 8, 9 and 10 we give the proofs of Proposition 1.1, Theorem 1.3 and Theorem 1.4, respectively.

Finally, we make some conventions about the notation. We denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. The formula $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{R}$ the set of real numbers. For any locally integrable function u and measurable set E of positive measure with respect to a measure μ, we denote $u_E = \fint_E u\, d\mu$ the average of u over E, namely, $\fint_E u\, d\mu = \frac{1}{\mu(E)} \int_E u\, d\mu$.

2 Weighted Sobolev spaces

Definition 2.1. Let $p \in [1, \infty)$ and suppose that Ω is a nonempty open subset of $\mathbb{R}^n$. Let $w : \Omega \to (0, \infty)$ be a given weight function, i.e a measurable function which is positive almost every where in Ω. The weighted Sobolev space $W_w^{1,p}(\Omega)$ is defined to be the set of all functions $u \in L_w^p(\Omega)$ whose distributional derivatives Du belongs to the weighted Lebesgue space $L_w^p(\Omega)$, i.e.,

$$\|Du\|_{L_w^p(\Omega)} = \left(\int_\Omega |Du(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

Now we can define the norm by setting

$$\|u\|_{W_w^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p w(x) \, dx + \int_\Omega |Du(x)|^p w(x) \, dx \right)^{1/p}.$$

Let $w = \rho^r$ with $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, $r > -1$. Let $k \in \mathbb{N}$ and $1 < p < \infty$. Following the argument above, we define

$$W_r^{1,p}(\Omega) = \left\{ u : \int_\Omega |u(x)|^p \rho(x)^r \, dx + \int_\Omega |Du(x)|^p \rho(x)^r \, dx < \infty \right\}.$$

We also define another weighted Sobolev space by setting

$$\mathcal{W}_r^{1,p}(\Omega) = \left\{ u : \|u\|_{L^p(\Omega)}^p + \int_\Omega |Du(x)|^p \rho(x)^r \, dx < \infty \right\}$$

If our domain is a bounded Lipschitz domain and r is in an appropriate range, then $\mathcal{W}_r^{k,p}(\Omega)$ is no different from $W_r^{k,p}(\Omega)$. This statement follows from the following lemma which can be found in [3, Lemma 2.4].

Lemma 2.1. *Let $-1 < r < p$ and Ω be a bounded Lipschitz domain. Then $\mathcal{W}_r^{1,p}(\Omega) = W_r^{1,p}(\Omega)$.*

Remark 2.1. When we abolish the restriction of r, the above two definitions of weighted Sobolev spaces are not necessarily equal, i.e., $W \neq \mathcal{W}$. For example, let $\Omega = (-1, 1)$, $p = 4$, $r = 8$ and $u(x) = \operatorname{dist}(x, \partial\Omega)^{-1/2}$. Then we claim that $u \in W_r^{1,p}(\Omega)$ and $u \notin \mathcal{W}_r^{1,p}(\Omega)$. It is sufficient to consider the case when $x \in [0, 1)$. Then $u(x) = (1-x)^{-1/2}$ and $u'(x) = \frac{1}{2}(1-x)^{-3/2}$ when $x \in [0, 1)$. Since

$$\int_0^1 |u(x)|^p \rho^r \, dx + \int_0^1 |u'(x)|^p \rho^r \, dx = \int_0^1 (1-x)^6 + \frac{1}{16}(1-x)^2 \, dx < \infty$$

and

$$\int_0^1 |u(x)|^p \, dx = \int_0^1 (1-x)^{-2} \, dx = \infty,$$

the claim follows. Hence $W \neq \mathcal{W}$.

3 A_p-weights and Maximal functions

Definition 3.1. Let w be a locally integrable nonnegative function on $\mathbb{R}^n$ and assume that $0 < w < \infty$ almost everywhere. We say that w belongs to the *Muckenhoupt class* A_p, $1 < p < \infty$, or that w is an A_p*-weight*, if there is a constant $c_{p,w}$ such that

$$\fint_B w\,dx \le c_{p,w} \left(\fint_B w^{1/(1-p)}\,dx \right)^{1-p} \tag{3.1}$$

for all balls B in $\mathbb{R}^n$.

Let μ stand for the measure whose Radon-Nikodym derivative w is,

$$\mu(E) = \int_E w\,dx.$$

According to the following lemma, μ is a doubling Radon measure (see [4, Corollary 15.7]).

Lemma 3.1. *If $w \in A_p$, then μ is a doubling measure; that is,*

$$\mu(2B) \le C\mu(B)$$

for all balls B in $\mathbb{R}^n$, where $C = 2^{np}c_{p,w}$.

Proof. We have

$$\begin{aligned}
|B| &= \int_B w^{1/p} w^{-1/p}\,dx \le \left(\int_B w\,dx \right)^{1/p} \left(\int_B w^{1/(1-p)}\,dx \right)^{(p-1)/p} \\
&\le \mu(B)^{1/p} \left(\int_{2B} w^{1/(1-p)}\,dx \right)^{(p-1)/p} \\
&= \mu(B)^{1/p} \left(\fint_{2B} w^{1/(1-p)}\,dx \right)^{(p-1)/p} |2B|^{(p-1)/p} \\
&\le c_{p,w}^{1/p} \mu(B)^{1/p} \left(\fint_{2B} w\,dx \right)^{-1/p} |2B|^{(p-1)/p} \\
&= c_{p,w}^{1/p} \left(\frac{\mu(B)}{\mu(2B)} \right)^{1/p} |2B|,
\end{aligned}$$

where we use the Hölder inequality and the definition of A_p-weights.

Hence we get that

$$c_{p,w} \frac{\mu(B)}{\mu(2B)} \ge \left(\frac{|B|}{|2B|} \right)^{1/p} = 2^{-np}.$$

Thus, we have

$$\mu(2B) \le 2^{np} c_{p,w} \mu(B)$$

and the claim follows. $\square$

We assume that each of the following Radon measures μ comes from an A_p-weight, i.e.,

$$\mu(E) = \int_E w\, dx.$$

Hence μ is a doubling Radon measure which is absolutely continuous with respect to Lebesgue measure.

Example 3.1. Define $w : \mathbb{R}^2 \to (0, \infty)$ with $w(x) = \operatorname{dist}(x, S^1)^{p-2}$ where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ and $p > 1$. Then w is an A_p-weight.

Proof. To prove that w is an A_p-weight, it suffices to check the condition (3.1). We divide this into two cases: (i) $\operatorname{dist}(B, S^1) \ge \frac{1}{2} \operatorname{diam}(B)$; (ii) $\operatorname{dist}(B, S^1) < \frac{1}{2} \operatorname{diam}(B)$.

For the case (i), $\forall\ x \in B$, we have $\operatorname{dist}(B, S^1) \le \operatorname{dist}(x, S^1) < 3 \operatorname{dist}(B, S^1)$. Let $\operatorname{dist}(B, S^1) = d$. We have

$$\min\{1, 3^{p-2}\} d^{p-2} \le w(x) \le \max\{1, 3^{p-2}\} d^{p-2}, \quad \forall\ x \in B.$$

Then we have

$$\fint_B w(x)\, dx \sim d^{p-2},$$

and

$$\left(\fint_B w^{1/(1-p)}\, dx \right)^{1-p} \sim d^{p-2}.$$

Hence the condition (3.1) is satisfied as desired.

For the case (ii), we divide it into two subcases: subcase (1), $\operatorname{diam}(B) \le \frac{2}{3}$; subcase (2), $\operatorname{diam}(B) > \frac{2}{3}$.

For the subcase (1), we have $\operatorname{dist}(B, S^1) < \frac{1}{2} \operatorname{diam}(B)$ and $\operatorname{diam}(B) \le \frac{2}{3}$. We can find a new ball $B_r = B(x_0, r)$ whose center x_0 is on S^1 with $r = \frac{2}{3} \operatorname{diam}(B) \le 1$ such that $B \subset B_r$. Let $E = \{x \in \mathbb{R}^2, \operatorname{dist}(x, S^1) < r\}$. Then $B_r \subset E$. Now let $\mathcal{F}$ be the maximal collection with

$$\mathcal{F} = \{B(x_i, r) : x_i \in S^1, B(x_i, r) \cap B(x_j, r) = \emptyset \ \text{ for } \ i \ne j, i, j \in \mathbb{N}\}.$$

Since $r \le 1$, from geometry, we have $\#\mathcal{F} \sim \frac{1}{r}$. For any $B(x_i, r) \in \mathcal{F}$, we have

$$\int_{B_r} w(x)\, dx = \int_{B(x_i, r)} w(x)\, dx \lesssim \frac{1}{1/r} \int_E w(x)\, dx.$$

Hence

$$\begin{aligned}
\fint_B w(x)\,dx &\lesssim \frac{1}{r^2}\int_{B_r} w(x)\,dx \lesssim \frac{1}{r}\int_E w(x)\,dx \\
&= 2\pi\frac{1}{r}\int_0^r t^{p-2}(1-t)\,dt + 2\pi\frac{1}{r}\int_0^r t^{p-2}(1+t)\,dt \\
&= 4\pi\frac{1}{p-1}r^{p-2} \sim r^{p-2},
\end{aligned}$$

and

$$\begin{aligned}
\fint_B w^{1/(1-p)}\,dx &\lesssim \frac{1}{r}\int_E w^{1/(1-p)}\,dx \\
&= 2\pi\frac{1}{r}\int_0^r t^{\frac{p-2}{1-p}}(1-t)\,dt + 2\pi\frac{1}{r}\int_0^r t^{\frac{p-2}{1-p}}(1+t)\,dt \\
&= 4\pi(p-1)r^{\frac{p-2}{1-p}} \sim r^{\frac{p-2}{1-p}}.
\end{aligned}$$

Since $1-p<0$, we have

$$\left(\fint_B w^{1/(1-p)}\,dx\right)^{1-p} \gtrsim \left(r^{\frac{p-2}{1-p}}\right)^{1-p} = r^{p-2}.$$

Thus, the condition (3.1) is satisfied as desired.

For the subcase (2), we have $\operatorname{dist}(B,S^1) < \frac{1}{2}\operatorname{diam}(B)$ and $\operatorname{diam}(B) > \frac{2}{3}$. Let $F = B(0,1+\ell)$ with $\ell = \frac{3}{2}\operatorname{diam}(B) > 1$. Then we have $B\subset F$ and $\ell^p \gtrsim 1$. Hence we have the estimate:

$$\begin{aligned}
\fint_B w(x)\,dx &\lesssim \frac{1}{\ell^2}\int_F w(x)\,dx = \frac{1}{\ell^2}\int_{B(0,1)} w(x)\,dx + \frac{1}{\ell^2}\int_{F\setminus B(0,1)} w(x)\,dx \\
&= \frac{2\pi}{p(p-1)}\ell^{-2} + \frac{2\pi}{p-1}\ell^{p-3} + \frac{2\pi}{p}\ell^{p-2} \lesssim \ell^{p-2}
\end{aligned}$$

and

$$\begin{aligned}
\fint_B w^{1/(1-p)}\,dx &\lesssim \frac{1}{\ell^2}\int_F w^{1/(1-p)}\,dx = \frac{1}{\ell^2}\int_{B(0,1)} w^{1/(1-p)}\,dx + \frac{1}{\ell^2}\int_{F\setminus B(0,1)} w^{1/(1-p)}\,dx \\
&= \frac{2\pi(p-1)^2}{p}\ell^{-2} + 2\pi(p-1)\ell^{\frac{1}{p-1}-2} + \frac{2\pi(p-1)}{p}\ell^{\frac{p}{p-1}-2} \\
&\lesssim \ell^{\frac{p}{p-1}-2} = \ell^{\frac{p-2}{1-p}}.
\end{aligned}$$

Using the same argument as in subcase (1), we get the condition (3.1). Hence w is an A_p-weight. $\square$

Remark 3.1. From the proof above, we can get a more general result. Let $p > 1$ and the weight $w_r : \mathbb{R}^2 \to (0, \infty)$ be $w_r(x) = \operatorname{dist}(x, S^1)^r$ where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. Then w_r is an A_p-weight provided that $-1 < r < p - 1$.

Example 3.2. Define $w : \mathbb{R}^2 \to (0, \infty)$ by setting

$$w(x) = \begin{cases} \operatorname{dist}(x, S^1)^{p-2} \log^{\lambda}\left(\frac{4}{\operatorname{dist}(x, S^1)}\right), & 0 \le |x| \le 2 \\ \log^{\lambda}(4), & |x| > 2. \end{cases}$$

where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, $p > 1$ and $\lambda \in \mathbb{R}$. Then w is an A_p-weight.

Proof. Using the same idea as in the proof of Example 3.1, we consider case (i) and case (ii). Case (i) is obvious, and for the case (ii), we first consider subcase (1). The main point is to compute

$$\int_0^r t^{\alpha} \log^{\lambda}(4/t)\, dt$$

for $1 > r > 0$ and $\alpha > -1$.

Using Integration by Parts, we have

$$r^{\alpha+1} \log^{\lambda}(4/r) = \int_0^r (\alpha + 1) t^{\alpha} \log^{\lambda}(4/t)\, dt - \int_0^r \lambda t^{\alpha} \log^{\lambda-1}(4/t)\, dt$$

When $0 < r \le 4 \exp(-\frac{2|\lambda|}{\alpha+1})$, for $0 < t \le r$, we have

$$\left| \frac{\lambda}{(\alpha + 1) \log(4/t)} \right| \le \frac{1}{2}.$$

When $1 > r > 4 \exp(-\frac{2|\lambda|}{\alpha+1})$, for $4 \exp(-\frac{2|\lambda|}{\alpha+1}) \le t \le r$, we have $r \sim t$. Hence we obtain

$$\frac{1}{\alpha + 1} r^{\alpha+1} \log^{\lambda}(4/r) \sim \int_0^r t^{\alpha} \log^{\lambda}(4/t)\, dt, \tag{3.2}$$

for all $0 < r \le 1$. Using the above computation, we can easily see that the claim is satisfied in subcase (1).

For the subcase (2), notice that $w(x)$ is a constant when $|x| > 2$. It is easy to see that the claim is also satisfied in subcase (2) via a simple computation.

Hence w is an A_p-weight. □

For a measurable function f on $\mathbb{R}^n$, we define the Hardy-Littlewood maximal function of f with respect to μ by setting

$$M_{\mu} f(x) = \sup \fint_B |f|\, d\mu = \sup \frac{1}{\mu(B)} \int_B |f|\, d\mu,$$

where the supremum is taken over all open balls B that contain x. Then we have an important inequality which asserts that the maximal operator maps $L^s(\mathbb{R}^n; \mu)$ continuously into itself for $s > 1$.

Lemma 3.2. 1) : *If $f \in L^1(\mathbb{R}^n;\mu)$ and $t > 0$, then*

$$\mu(\{M_\mu f > t\}) \le \frac{C}{t}\int_{\{M_\mu f > t\}} |f|\,d\mu \le \frac{C}{t}\int_{\mathbb{R}^n} |f|\,d\mu,$$

where C depends only on n, p and the A_p-constant $c_{p,w}$.
2) : *If $f \in L^s(\mathbb{R}^n;\mu)$, $1 < s < \infty$, then we have that*

$$\int_{\mathbb{R}^n} |M_\mu f|^s\,d\mu \le C\int_{\mathbb{R}^n} |f|^s\,d\mu,$$

where C depends only on n, p, s and the A_p-constant $c_{p,w}$.

Proof. 1): We may assume that $M := \int_{\{M_\mu f > t\}} |f|\,d\mu < \infty$. For each compact subset $E \subset \{M_\mu f > t\}$ and for any $x \in E$, there is a open ball B such that $x \in B$ and

$$\fint_B |f|\,d\mu > t.$$

Then we have

$$\mu(B) < t^{-1}\int_B |f|\,d\mu.$$

If $y \in B$, then $M_\mu f(y) > t$ and thus $B \subset \{M_\mu f > t\}$. So

$$\mu(B) < t^{-1}\int_B |f|\,d\mu \le \frac{1}{t}\int_{\{M_\mu f > t\}\cap B} |f|\,d\mu.$$

Since E is compact, we can select a finite subset $\{B_j : 1 \le j \le m\}$ from $\{B_x : x \in E\}$ such that $E \subset \cup_{j=1}^m B_j$. Now $\sup\{\operatorname{diam}(B_j)\}$ is bounded. Hence we may use the $5r$-covering lemma to find pairwise disjoint balls $B_1, B_2, \cdots, B_k$ as above so that $E \subset \cup_{j=1}^m B_j \subset \bigcup_{j=1} 5B_j$. Then using the doubling property of μ, i.e., Lemma 3.1, we have

$$\mu(E) \le \sum_j \mu(5B_j) \le C\sum_j \mu(B_j) \le \frac{C}{t}\int_{\{M_\mu f > t\}} |f|\,d\mu \le \frac{C}{t}\int_{\mathbb{R}^n} |f|\,d\mu,$$

where C depends only on n, p and the A_p-constant $A_{p,w}$. We take the supremum over all such compact sets $E \subset \{M_\mu f > t\}$, and the conclusion 1) is proved.

2) Recall the Cavalieri principle:

$$\begin{aligned}\int |u|^p\,d\mu &= p\int\int_0^{|v(x)|} t^{p-1}\,dt\,d\mu\\ &= p\int\int_0^\infty t^{p-1}\chi_{\{|v|>t\}}\,dt\,d\mu\end{aligned}$$

$$= p \int_0^\infty t^{p-1} \mu(\{|v| > t\})\, dt.$$

Fix $t > 0$ and define

$$f_t(x) = \begin{cases} |f(x)|, & |f(x)| > t/2 \\ 0, & |f(x)| \le t/2. \end{cases}$$

Then we have that

$$|f(x)| \le |f_t(x)| + t/2$$

and

$$M_\mu f(x) \le M_\mu f_t(x) + t/2.$$

Thus,

$$\{x : M_\mu f(x) > t\} \subset \{M_\mu f_t(x) > t/2\}.$$

By the Cavalieri principle, part 1) of this lemma and the Fubini theorem, we obtain the estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} |M_\mu f(x)|^p \, d\mu &= p \int_0^\infty t^{p-1} \mu(\{|M_\mu f(x)| > t\})\, dt \\
&\le p \int_0^\infty t^{p-1} \mu(\{|M_\mu f_t(x)| > t/2\})\, dt \\
&\le Cp \int_0^\infty t^{p-1} \frac{1}{t} \int_{\mathbb{R}^n} |f_t| \, d\mu \, dt \\
&\le Cp \int_0^\infty t^{p-2} \int_{\{|f(x)|>t/2\}} |f| \, d\mu \, dt \\
&\le Cp \int_0^\infty t^{p-2} \int_{\mathbb{R}} |f| \chi_{\{|f(x)|>t/2\}} \, d\mu \, dt \\
&= Cp \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2} \, dt \, d\mu \\
&= C' \int_{\mathbb{R}^n} |f(x)| \, \mu,
\end{aligned}$$

where C' depends on n, p, s and the A_p-constant $c_{p,w}$. $\square$

The definition and the doubling property of A_p-weights are from the monograph [4]. The ideas for the maximal function and Lemma 3.2 are from the lecture notes [10], but we generalize the Lebesgue measure to a Radon measure which comes from an A_p-weight. Example 3.1 and Example 3.2 were verified by us.

4 Orlicz and Orlicz-Sobolev spaces

In this section and the following sections, we assume that the Radon measure μ comes from an A_p-weight, i.e., there exist w_μ which is an A_p-weight such that

$$\mu(E) = \int_E w_\mu \, dx.$$

Hence μ is a doubling Radon measure which is absolutely continuous with respect to Lebesgue measure.

Definition 4.1. We say that $\Phi : [0, \infty) \to [0, \infty)$ is a *Young function* if

$$\Phi(t) = \int_0^t \varphi(s) \, ds, \quad t \geq 0,$$

where the real-valued function φ defined on $[0, \infty)$ has the following properties:
(i) $\varphi(0) = 0$;
(ii) $\varphi(s) > 0$ for $s > 0$;
(iii) φ is right continuous at any point $s \geq 0$;
(iv) φ is nondecreasing on $(0, \infty)$;
(v) $\lim_{s\to\infty} \varphi(s) = \infty$.

The following properties of Young function can be easily checked.

Lemma 4.1. *A Young function Φ is continuous, nonnegative, strictly increasing and convex on $[0, \infty)$. Moreover,*

$$\Phi(0) = 0, \quad \lim_{t\to\infty} \Phi(t) = \infty;$$

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty;$$

$$\Phi(\alpha t) \leq \alpha\Phi(t) \quad for \ \ \alpha \in [0, 1] \ \ and \ \ t \geq 0;$$

$$\Phi(\beta t) \geq \beta\Phi(t) \quad for \ \ \beta > 1 \quad and \ \ t \geq 0.$$

Since Φ is convex, it satisfies the following *Jensen's inequality.*

Lemma 4.2. *Let Φ be convex on $\mathbb{R}$.*
(i) Let $t_1, \cdots, t_n \in \mathbb{R}$ and let $\alpha_1, \cdots, \alpha_n$ be positive numbers. Then

$$\Phi\left(\frac{\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}\right) \leq \frac{\alpha_1 \Phi(t_1) + \alpha_2 \Phi(t_2) + \cdots + \alpha_n \Phi(t_n)}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}.$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\alpha = \alpha(x)$ be defined and positive almost everywhere on Ω. Then

$$\Phi\left(\frac{\int_\Omega u(x)\alpha(x)\,dx}{\int_\Omega \alpha(x)\,dx}\right) \le \frac{\int_\Omega \Phi(u(x))\alpha(x)\,dx}{\int_\Omega \alpha(x)\,dx}$$

for every nonnegative function u provided all the integrals in the above inequality are meaningful.

Then (i) is called Jensen's inequality and (ii) is called Jensen's integral inequality.

For any Young function Φ, we can define the complementary function.

Definition 4.2. Let Φ be a Young function generated by the function φ, i.e.,

$$\Phi(t) = \int_0^t \varphi(s)\,ds.$$

We put

$$\psi(t) = \sup_{\varphi(s)\le t} s, \qquad t \ge 0,$$

and

$$\Psi(t) = \int_0^t \psi(s)\,ds.$$

It is easy to check that Ψ is also a Young function. The function Ψ is called the *complementary function* to Φ. We call Φ, Ψ a *pair of complementary Young functions.*

We now introduce Young's inequality.

Lemma 4.3. *Let Φ, Ψ be a pair of complementary Young functions. Then for all $a, b \in [0, \infty)$, we have that*

$$ab \le \Phi(a) + \Psi(b).$$

Equality holds if and only if

$$b = \varphi(a) \quad \text{or} \quad a = \psi(b).$$

Moreover, if $u(x)$ and $v(x)$ are measurable functions on Ω, we get

$$\int_\Omega |u \cdot v|\,d\mu \le \int_\Omega \Phi(|u|)\,d\mu + \int_\Omega \Psi(|v|)\,d\mu.$$

Equality occurs if

$$|v(x)| = \varphi(|u(x)|) \quad \text{or} \quad |u(x)| = \psi(|v(x)|).$$

There is a special class of Young functions which is very important.

Definition 4.3. A Young function Φ is said to be *doubling* if there exists a constant $k > 0$ such that

$$\Phi(2t) \le k\Phi(t) \quad \text{for all} \quad t \ge 0.$$

Combining convexity and the doubling property, we can easily get the following property:

Proposition 4.1. *If a Young function Φ is doubling, then for any constant $c > 0$, there exist $c_1, c_2 > 0$ such that*

$$c_1\Phi(t) \le \Phi(ct) \le c_1\Phi(t) \quad \text{for all} \quad t \ge 0,$$

where c_1 and c_2 depend only on c and the doubling constant k.

Now we introduce an ordering in the class of Young functions.

Definition 4.4. Let Φ_1, Φ_2 be two Young functions. If there exist two constants $k > 0$ and $t \ge T$ such that

$$\Phi_1(t) \le \Phi_2(ct) \quad \text{for} \quad t \ge T,$$

we write

$$\Phi_1 \prec \Phi_2.$$

Remark 4.1. If we have $\Phi_1 \prec \Phi_2$, then their complementary functions Ψ_1, Ψ_2 satisfy $\Psi_2 \prec \Psi_1$.

Example 4.1. Let $p > 1$. Then the function $\Phi(t) = t^p/p$ is a Young function and the complementary function is $\Psi(t) = t^q/q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, Φ satisfies the doubling condition, where we can put $k = 2^p$.

Let $\Phi(t) = t^p \log^\lambda(e + t)$, where $1 < p < \infty$ and $\lambda \in \mathbb{R}$. Then

$$t^{p-\epsilon} \prec \Phi(t) \prec t^{p+\epsilon},$$

for $p > 1$ and $0 < \epsilon < p - 1$.

Next, we introduce Orlicz spaces.

Definition 4.5. Let Φ be a Young function and u be a measurable function defined almost everywhere on $\Omega \subset \mathbb{R}^n$. The space

$$L^\Phi(\Omega; \mu) := \{u \in L^1_{\mathrm{loc}}(\Omega; \mu) : \int_\Omega \Phi(s|u(x)|)\, d\mu < \infty \quad \text{for some} \quad s > 0\}$$

is called an *Orlicz space*.

Let Ψ be the complementary Young function of Φ. We define the Orlicz norm of $u \in L^{\Phi}(\Omega;\mu)$ by setting

$$\|u\|_{L^{\Phi}(\Omega;\mu)} = \sup_{v} \int_{\Omega} |u(x)v(x)|\, d\mu,$$

where the supremum is taken over all the measurable functions v such that

$$\int_{\Omega} \Psi(|v(x)|)\, d\mu \leq 1.$$

Since the above norm requires the knowledge of the expression for the complementary Young function Ψ, we define another norm which is expressed only in terms of Φ. Define the *Luxemburg norm* of $u \in L^{\Phi}(\Omega;\mu)$ by setting

$$\|u\|^{L}_{L^{\Phi}(\Omega;\mu)} = \inf\left\{k > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{k}\right)\, d\mu \leq 1\right\}.$$

The following proposition tells us that the two norms $\|\cdot\|_{L^{\Phi}(\Omega;\mu)}$ and $\|\cdot\|^{L}_{L^{\Phi}(\Omega;\mu)}$ are equivalent (see [2, Theorem 3.8.5]).

Proposition 4.2. *For each $u \in L^{\Phi}(\Omega;\mu)$,*

$$\|u\|^{L}_{L^{\Phi}(\Omega;\mu)} \leq \|u\|_{L^{\Phi}(\Omega;\mu)} \leq 2\|u\|^{L}_{L^{\Phi}(\Omega;\mu)}.$$

To prove the above proposition, we need the following lemma:

Lemma 4.4. *Let Φ be a Young function and let $u \in L^{\Phi}(\Omega;\mu)$ be such that $\|u\|_{L^{\Phi}(\Omega;\mu)} \neq 0$. Then we have*

$$\int_{\Omega} \Phi\left(\frac{|u|}{\|u\|_{L^{\Phi}(\Omega;\mu)}}\right)\, d\mu \leq 1. \tag{4.1}$$

Proof. For $u \in L^{\Phi}(\Omega;\mu)$, we claim that

$$\int_{\Omega} |u(x)v(x)|\, d\mu \leq \begin{cases} \|u\|_{L^{\Phi}(\Omega;\mu)} & \text{for } \int_{\Omega} \Psi(|v|)\, d\mu \leq 1, \\ \|u\|_{L^{\Phi}(\Omega;\mu)} \int_{\Omega} \Psi(|v|)\, d\mu & \text{for } \int_{\Omega} \Psi(|v|)\, d\mu > 1. \end{cases} \tag{4.2}$$

The first part of inequality (4.2) follows from the definition of the Orlicz norm. For the second part, we use the convexity of Ψ, i.e., $\Psi(\alpha t) \leq \alpha\Psi(t)$ for $t \geq 0$ and $\alpha \in [0,1]$. Hence we obtain that

$$\int_{\Omega} \Psi\left(\frac{|v(x)|}{\int_{\Omega} \Psi(|v|)\, d\mu}\right)\, d\mu \leq \frac{1}{\int_{\Omega} \Psi(|v|)\, d\mu} \int_{\Omega} \Psi(|v(x)|)\, d\mu = 1.$$

By the definition of the Orlicz norm,

$$\int_\Omega |u(x)| \frac{|v(x)|}{\int_\Omega \Psi(|v|)\, d\mu}\, d\mu \le \|u\|_{L^\Phi(\Omega;\mu)}$$

which gives the proof of the second part of inequality (4.2).

Let us first suppose that $u \in L^\Phi(\Omega;\mu)$ is bounded and that $u(x) = 0$ for $x \in \Omega \backslash \Omega_0$ with $\mu(\Omega_0) < \infty$. If we take

$$v(x) = \varphi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right),$$

then also the functions $\Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right)$ and $\Psi(|v(x)|)$ are bounded and integrable over Ω_0; furthermore, they belong to $L^1(\Omega;\mu)$ because they are zero outside of Ω_0.

Using the Young's inequality in Lemma 4.3, and we check that the equality occurs, i.e.,

$$\int_\Omega \frac{1}{\|u\|_{L^\Phi(\Omega;\mu)}} |u(x)v(x)|\, d\mu = \int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu + \int_\Omega \Psi(|v(x)|)\, d\mu.$$

Then using inequality 4.2 and we get that

$$\max\left(\int_\Omega \Psi(|v(x)|)\, d\mu, 1\right) \ge \int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu + \int_\Omega \Psi(|v(x)|)\, d\mu.$$

If $\int_\Omega \Psi(|v(x)|)\, d\mu > 1$, then necessarily

$$\int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu = 0.$$

If $\int_\Omega \Psi(|v(x)|)\, d\mu \le 1$, then

$$\int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu \le 1.$$

Hence we proved inequality (4.1) when u is bounded.

Now, let $u \in L^\Phi(\Omega;\mu)$ be arbitrary. We pick a sequence of subsets $\Omega_n \subset \Omega$, $n \in \mathbb{N}$ such that $\Omega_n \subset \Omega_{n+1}$, $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n=1}^\infty \Omega_n$. Then we define the functions u_n, $n \in \mathbb{N}$ by

$$u_n(x) = \begin{cases} u(x) & \text{for } \ x \in \Omega_n \ \text{ and } \ |u(x)| \le n, \\ n & \text{for } \ x \in \Omega_n \ \text{ and } \ |u(x)| > n, \\ 0 & \text{for } \ x \in \Omega \setminus \Omega_n. \end{cases}$$

It follows From the first part of the proof that (4.1) holds for every function u_n, i.e.,

$$\int_\Omega \Phi\left(\frac{|u_n(x)|}{\|u_n\|_{L^\Phi(\Omega;\mu)}}\right) d\mu \leq 1.$$

Moreover, as $|u_n(x)| \leq |u(x)|$ for all $x \in \Omega$, we can easily check that $\|u_n\|_{L^\Phi(\Omega;\mu)} \leq \|u\|_{L^\Phi(\Omega;\mu)}$ for all $n \in \mathbb{N}$. Thus, we have

$$\frac{|u_n(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}} \leq \frac{|u_n(x)|}{\|u_n\|_{L^\Phi(\Omega;\mu)}} \quad \text{and} \quad \Phi\left(\frac{|u_n(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) \leq \Phi\left(\frac{|u_n(x)|}{\|u_n\|_{L^\Phi(\Omega;\mu)}}\right).$$

Consequently,

$$\int_\Omega \Phi\left(\frac{|u_n(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu \leq 1.$$

The sequences $\{|u_n(x)|\}_{n=1}^\infty$ and $\left\{\Phi\left(\frac{|u_n(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right)\right\}_{n=1}^\infty$ are nondecreasing. Hence, using the Monotone Convergence Theorem, we get that

$$\int_\Omega \Phi\left(\frac{|u|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu = \lim_{n\to\infty} \int_\Omega \Phi\left(\frac{|u_n(x)|}{\|u\|_{L^\Phi(\Omega;\mu)}}\right) d\mu \leq 1,$$

and inequality (4.1) follows. $\square$

Proof of Proposition 4.2. Using Lemma 4.4, it follows that if $u \in L^\Phi(\Omega;\mu)$, we have

$$\|u\|^L_{L^\Phi(\Omega;\mu)} \leq \|u\|_{L^\Phi(\Omega;\mu)}.$$

For the other inequality, define $w = u/\|u\|^L_{L^\Phi(\Omega;\mu)}$. Then we have

$$\|w\|_{L^\Phi(\Omega;\mu)} = \sup_v \int_\Omega |u(x)v(x)|\, d\mu \leq \int_\Omega \Phi(|w|)\, d\mu + 1,$$

where we used the Young inequality in Lemma 4.3 and that the supremum is taken over all the measurable functions such that

$$\int_\Omega \Psi(|v(x)|)\, d\mu \leq 1.$$

Now, let us estimate $\int_\Omega \Phi(|w|)\, d\mu$. From the definition of Luxemburg norm, if we let k tend to $\|u\|^L_{L^\Phi(\Omega;\mu)}$ in

$$\int_\Omega \Phi\left(\frac{|u|}{k}\right) d\mu \leq 1,$$

Fatou's lemma gives us

$$\int_\Omega \Phi(|w|)\, d\mu = \int_\Omega \Phi\left(\frac{|u|}{\|u\|^L_{L^\Phi(\Omega;\mu)}}\right) d\mu \leq \liminf_k \int_\Omega \Phi\left(\frac{|u|}{k}\right) d\mu \leq 1.$$

Hence we have

$$\|w\|_{L^\Phi(\Omega;\mu)} \leq 2$$

and the other inequality follows. □

Example 4.2. (i): When $\Phi(t) = t^p$, $p \geq 1$, we have $L^\Phi(\Omega;\mu) = L^p(\Omega;\mu)$ and $\|\cdot\|^L_{L^\Phi(\Omega;\mu)} = \|\cdot\|_{L^p(\Omega,\mu)}$.

(ii): When $\mu = \mathcal{L}^n$, i.e., μ is the Lebesgue measure, we denote $L^\Phi(\Omega,\mu)$ by $L^\Phi(\Omega)$.

(iii): If w is an A_p-weight such that $\mu(A) = \int_A w\,dx$ for any measurable set A, then we denote $L^\Phi(\Omega,\mu)$ by $L^\Phi_w(\Omega)$ with

$$\begin{aligned}\|u\|^L_{L^\Phi(\Omega;\mu)} &= \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|u|}{k}\right)\,d\mu \leq 1\right\}\\ &= \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|u|}{k}\right) w(x)\,dx \leq 1\right\} = \|u\|^L_{L^\Phi_w(\Omega)}.\end{aligned}$$

For the Lebesgue L^p-spaces, we have the Hölder inequality, i.e., if $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_\Omega |u(x)v(x)|\,dx \leq \|u\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)}.$$

The following lemma provides an analogous inequality for Olicz spaces.

Lemma 4.5. *Let Φ, Ψ be a pair of complementary Young functions. If $u \in L^\Phi(\Omega)$ and $v \in L^\Psi(\Omega)$, the $u \cdot v \in L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)|\,dx \leq \|u\|_{L^\Phi(\Omega)} \cdot \|v\|_{L^\Psi(\Omega)}.$$

Proof. For $\|v\|_{L^\Psi(\Omega)} = 0$, inequality is obvious. If $\|v\|_{L^\Psi(\Omega)} \neq 0$, we apply Lemma 4.4 for the Young function Ψ, obtaining

$$\int_\Omega \Psi\left(\frac{v}{\|v\|_{L^\Psi(\Omega)}}\right)\,dx \leq 1.$$

Then the inequality follows from the definition of the Orlicz norm of u:

$$\int_\Omega |u(x)v(x)|\,dx = \|v\|_{L^\Psi(\Omega)} \int_\Omega \left|u(x)\frac{v}{\|v\|_{L^\Psi(\Omega)}}\right|\,dx \leq \|u\|_{L^\Phi(\Omega)} \cdot \|v\|_{L^\Psi(\Omega)}.$$

□

For the Lebesgue spaces $L^p(\Omega)$ and $L^q(\Omega)$, if $\mu(\Omega) < \infty$ and $1 \leq p < q$, then

$$L^q(\Omega) \subset L^p(\Omega).$$

We can also obtain a similar result for the Orlicz spaces L^{Φ_1} and L^{Φ_2} making use of the ordering $\prec$ introduced in Definition 4.4. We have the following lemma (see [2, Theorem 3.17.1 and Theorem 3.17.5]):

Lemma 4.6. *Let Φ_1, Φ_2 be two Young functions and $\mu(\Omega) < \infty$. If $\Phi_2 \prec \Phi_1$, then there exists a constant $k > 0$ such that*

$$\|u\|_{L^{\Phi_2}(\Omega,\mu)} \leq k\|u\|_{L^{\Phi_1}(\Omega,\mu)}$$

for all $u \in L^{\Phi_1}(\Omega, \mu)$.

Proof. Suppose $\Phi_2 \prec \Phi_1$ holds. If we denote by Ψ_1, Ψ_2 the complementary functions to Φ_1, Φ_2, respectively, then according to Remark 4.1 we have $\Psi_1 \prec \Psi_2$, i.e., there exist $C > 0$ and $T \geq 0$ such that

$$\Psi_1(t) \leq \Psi_2(Ct) \quad \text{for} \quad t \geq T,$$

or, equivalently,

$$\Psi_1(t/C) \leq \Psi_2(t) \quad \text{for} \quad t \geq CT.$$

Since

$$\Psi_1(t/C) \leq \Psi_1(T) \quad \text{for} \quad t \leq CT,$$

we have that

$$\Psi_1(t/C) \leq \Psi_1(T) + \Psi_2(t) \quad \text{for} \quad t \geq 0.$$

Now, let v satisfy $\int_\Omega \Psi_2(|v|)\, d\mu \leq 1$. Then we have

$$\int_\Omega \Psi_1(|v|/C)\, d\mu \leq \Psi_1(T)\mu(\Omega) + \int_\Omega \Psi_2(|v(x)|) \leq 1 + \Psi_1(T)\mu(\Omega) < \infty.$$

If we denote $\alpha = (\Psi_1(T)\mu(\Omega) + 1)^{-1} \leq 1$ and $k = C/\alpha$, then we conclude from the convexity of Ψ_2 that

$$\int_\Omega \Psi_1(|v|/k)\, d\mu = \int_\Omega \Psi_1\left(\alpha\frac{|v(x)|}{C}\right) d\mu \leq \alpha \int_\Omega \Psi_1\left(\frac{|v(x)|}{C}\right) d\mu \leq \alpha\alpha^{-1} = 1.$$

Thus, we have that $\int_\Omega \Psi_2(|v|)\, d\mu \leq 1$ implies $\int_\Omega \Psi_2(|v|/k)\, d\mu \leq 1$. Hence our claim follows from the definition of the Orlicz space:

$$\|u\|_{L^{\Phi_2}(\Omega,\mu)} = \sup_v \int_\Omega |u(x)v(x)|\, d\mu = k \sup_v \int_\Omega \left|u(x)\frac{v(x)}{k}\right| d\mu$$

$$\begin{aligned}
&\leq k \sup_{v/k} \int_\Omega \left| u(x) \frac{v(x)}{k} \right| d\mu = k \sup_{w} \int_\Omega |u(x)w(x)|\, d\mu \\
&= k\|u\|_{L^{\Phi_1}(\Omega,\mu)},
\end{aligned}$$

where the supremum of v is taken over all v such that $\int_\Omega \Psi_2(|v|)\, d\mu \leq 1$ and the supremum of v/k is taken over all v/k such that $\int_\Omega \Psi_2(|v|/k)\, d\mu \leq 1$. $\square$

Next, we will introduce Orlicz-Sobolev spaces. Here, we consider the case when $\mu = \mathcal{L}^n$, i.e., μ is the Lebesgue measure.

Definition 4.6. Let Φ be a Young function and suppose that Ω is a nonempty open subset of $\mathbb{R}^n$. The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is defined to be the set of all functions $u \in L^\Phi(\Omega)$ whose distributional derivative Du also belongs to the space $L^\Phi(\Omega)$. Then $W^{1,\Phi}(\Omega)$ is the linear set

$$\{u \in L^\Phi(\Omega) : Du \in L^\Phi(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,\Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \|Du\|_{L^\Phi(\Omega)}.$$

Similarly, we can also give the definition of weighted Orlicz-Sobolev spaces.

Definition 4.7. Let Φ be a Young function and suppose that Ω is a nonempty open subset of $\mathbb{R}^n$. The weighted Orlicz-Sobolev space with weight w, $W^{1,\Phi}_w(\Omega)$ is the linear set

$$\{u \in L^\Phi(\Omega) : Du \in L^\Phi_w(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,\Phi}_w(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \|Du\|_{L^\Phi_w(\Omega)}.$$

Example 4.3. Let Ω be the unit disk D and $w(x) = \operatorname{dist}(x, S^1)^{p-2}$. We have that $w(x)$ is an A_p-weight. Let μ stand for the measure whose Radon-Nikodym derivative $w(x)$ is,

$$\mu(E) = \int_E w(x)\, dx.$$

Then we can also define the Orlicz space $L^\Phi(D, \mu)$ with respect to measure μ, equipped with the norm

$$\|u\|_{L^\Phi(D,\mu)} = \inf\{k > 0 : \int_D \Phi\left(\frac{|u|}{k}\right) d\mu \leq 1\}.$$

Then $L^\Phi(D, \mu) = L^\Phi_{p-2}(D)$. The definition of $W^{1,\Phi}_{p-2}(D)$ is obtained via Definition 4.7.

Example 4.4. Let Ω be the unit disk D and $w_\Phi(x)$ be the weight in Example 3.2 which is an A_p-weight. Then we can define a new weighted Orlicz-Sobolev-type space $W^{1,p}_{w_\Phi}(D)$ by setting:

$$W^{1,p}_{w_\Phi}(D) = \left\{ u \in L^\Phi(D) : \int_D |Du|^p w_\Phi(x)\, dx < \infty \right\}$$

with norm

$$\|u\|_{W^{1,p}_{w_\Phi}(D)} = \|u\|_{L^\Phi(D)} + \left(\int_D |Du|^p w_\Phi(x)\, dx \right)^{1/p},$$

where $\Phi(t) = t^p \log^\lambda(e+t)$ with $p > 1$ and $\lambda \in \mathbb{R}$.

We give an example to show that $W^{1,\Phi}_{p-2}(D) \neq W^{1,p}_{w_\Phi}(D)$ for $\Phi(t) = t^p \log^\lambda(e+t)$ with $p > 1$ and $\lambda \in \mathbb{R}$.

Remark 4.2. When $\lambda \neq 0$, we have $W^{1,\Phi}_{p-2}(D) \neq W^{1,p}_{w_\Phi}(D)$.

Define a function u by setting

$$u(x) = \int_{|x|}^1 t^{-2/p} \log^{-\frac{1}{p} - \frac{\lambda}{2p}} \left(e + \frac{1}{t} \right) dt.$$

Then

$$|\nabla u(x)| = |x|^{-2/p} \log^{-\frac{1}{p} - \frac{\lambda}{2p}} \left(e + \frac{1}{|x|} \right).$$

Moreover, we have the estimate

$$|u(x)| \lesssim \int_{|x|}^1 t^{-\frac{2}{p} - \epsilon}\, dt \lesssim (1 + |x|^{\frac{p-2}{p} - \epsilon}),$$

for some ϵ small enough. For $p > 1$ and ϵ small enough, we check that

$$\int_D \Phi(|u(x)|)\, dx \lesssim \int_D (\Phi(1) + \Phi(|x|^{\frac{p-2}{p} - \epsilon})\, dx < \infty,$$

and hence $u \in L^\Phi(D)$.

Now, we divide the unit disk D into two parts: $D_1 = \{x \in D : |x| \leq \frac{1}{2}\}$ and $D_2 = \{x \in D : \frac{1}{2} < |x| < 1\}$. Then we have $|\nabla u(x)| \sim 1$ for $x \in D_2$ and $w_\Phi(x) \sim 1 \sim \operatorname{dist}(x, S^1)^{p-2} = w(x)$ for $x \in D_1$.

For $\lambda > 0$, we have

$$\int_D \Phi(|\nabla u(x)|) w(x)\, dx \geq \int_{D_1} \Phi(|\nabla u(x)|) w(x)\, dx \sim \int_0^{\frac{1}{2}} t^{-1} \log^{-1 + \frac{\lambda}{2}} (e + t^{-1})\, dt = \infty$$

and

$$\begin{aligned}\int_D |\nabla u(x)|^p w_\Phi(x)\,dx &= \int_{D_1} |\nabla u(x)|^p w_\Phi(x)\,dx + \int_{D_2} |\nabla u(x)|^p w_\Phi(x)\,dx \\ &\sim \int_{D_2} w(x)\,dx + \int_{D_1} |x|^{-2}\log^{-1-\lambda/2}(e+|x|^{-1})\,dx \\ &\le C + \int_0^{\frac{1}{2}} t^{-1}\log^{-1-\lambda/2}(e+t^{-1})\,dx < \infty.\end{aligned}$$

Hence, $u \in W^{1,p}_{w_\Phi}(D)$ but $u \notin W^{1,\Phi}_{p-2}(D)$.

For $\lambda < 0$, using a similar argument, we have

$$\int_D |\nabla u|^p w_\Phi(x)\,dx \ge \int_{D_1} |\nabla u|^p w_\Phi(x)\,dx \sim \int_0^{\frac{1}{2}} t^{-1}\log^{-1-\lambda/2}(e+t^{-1})\,dt = \infty$$

and

$$\begin{aligned}\int_D \Phi(|\nabla u|)w(x)\,dx &= \int_{D_1} \Phi(|\nabla u|)w(x)\,dx + \int_{D_2} \Phi(|\nabla u|)w(x)\,dx \\ &\sim \int_{D_1} w(x)\,dx + \int_{D_2} |\nabla u|^p \log^\lambda(e+|\nabla u|)\,dx \\ &\lesssim C + \int_0^{\frac{1}{2}} t^{-1}\log^{-1+\frac{\lambda}{2}}(e+t^{-1})\,dt < \infty.\end{aligned}$$

Hence, $u \in W^{1,\Phi}_{p-2}(D)$ but $u \notin W^{1,p}_{w_\Phi}(D)$.

Most of the results in this section are from the monograph [2] except for Remark 4.2 which is due to us. However, we have generalized the results from the Lebesgue measure to a Radon measure which comes from an A_p-weight.

5 Mean continuity and Density Property

first , we discuss mean continuity in $L^p(\Omega)$ for $1 \le p < \infty$.

Definition 5.1. Let $1 \le p < \infty$ and $u \in L^p(\Omega)$. Then the function u is said to be *p-mean continuous* if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|u_h - u\|_{L^p(\Omega)} < \epsilon$$

for $h \in \mathbb{R}^n$ with $|h| < \delta$, where

$$u_h(x) = \begin{cases} u(x+h) & \textit{if} \ \ x \in \Omega \ \ \textit{and} \ \ x+h \in \Omega, \\ 0 & \textit{otherwise} \ \ \textit{in} \ \ \mathbb{R}^n. \end{cases}$$

A basic result for p-mean continuity is the following lemma (see [2, Theorem 2.4.2]):

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then any function $u \in L^p(\Omega)$ is p-mean continuous.*

Proof. Let $\epsilon > 0$. Using the absolute continuity of integration, we have that there exist an $\eta > 0$ such that for each $E \subset \Omega$ for which $\mathcal{L}^n(E) < 4\eta$, we have

$$\left(\int_E |u(x)|^p \, dx\right)^{1/p} < \epsilon. \tag{5.1}$$

For this η, there exists a ϱ such that $\mathcal{L}^n(H_\varrho) < \eta$, where

$$H_\varrho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \varrho\}.$$

Put $\Omega_\varrho = \Omega \setminus H_\varrho$. Clearly f is a measurable on Ω_ϱ and thus Luzin's theorem implies the existence of a closed set $F_\eta^1 \subset \Omega_\varrho$ such that the restriction of the function u to F_η^1 is continuous, $\mathcal{L}^n(\Omega_\varrho \setminus F_\eta^1) < \eta$ and thus

$$\mathcal{L}^n(\Omega \setminus F_\eta^1) < 2\eta.$$

Since F is closed and bounded, it is compact. Then f is uniformly continuous on F_η^1. Hence there exists a $\delta \in (0, \varrho)$ such that

$$|f(x+h) - f(x)| \leq \frac{\epsilon}{(\mathcal{L}^n(\Omega))^{1/p}} \tag{5.2}$$

for all $x, x+h \in F_\eta^1$ with $|h| < \delta$.

Let $|h| < \delta$, $F_\eta^2 = \{x \in \Omega : x+h \in F_\eta^1\}$. Then $\mathcal{L}^n(F_\eta^2) = \mathcal{L}^n(F_\eta^1)$, according to the translation invariance of Lebesgue measure. Hence $\mathcal{L}^n(\Omega \setminus F_\eta^2) < \eta$. Let $F_\eta = F_\eta^1 \cap F_\eta^2$. Then we have

$$\mathcal{L}^n(\Omega \setminus F_\eta) \leq \mathcal{L}^n(\Omega \setminus F_\eta^1) + \mathcal{L}^n(\Omega \setminus F_\eta^2) < 4\eta.$$

Using inequality (5.1) we obtain

$$\left(\int_{\Omega\setminus F_\eta} |f(x+h)| \, dx\right)^{1/p} + \left(\int_{\Omega\setminus F_\eta} |f(x)| \, dx\right)^{1/p} < 2\epsilon,$$

and since inequality (5.2) holds for arbitrary $x \in F_\eta$, we have

$$\left(\int_{F_\eta} |f(x+h) - f(x)| \, dx\right)^{1/p} < \epsilon.$$

Hence

$$\left(\int_\Omega |f(x+h)-f(x)|\,dx\right)^{1/p} \leq \left(\int_{\Omega\setminus F_\eta} |f(x+h)|\,dx\right)^{1/p} + \left(\int_{\Omega\setminus F_\eta} |f(x)|\,dx\right)^{1/p} + \left(\int_{F_\eta} |f(x+h)-f(x)|\,dx\right)^{1/p} < 3\epsilon.$$

Then the claim follows. □

For Orlicz spaces, we have the following definition of Φ-mean continuity.

Definition 5.2. Let $u \in L^\Phi(\Omega;\mu)$. Then the function u is said to be Φ-*mean continuous* if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|u_h - u\|_{L^\Phi(\Omega;\mu)} < \epsilon$$

for $h \in \mathbb{R}^n$ with $|h| < \delta$, where

$$u_h(x) = \begin{cases} u(x+h) & \textit{if} \;\; x \in \Omega \;\; \textit{and} \;\; x+h \in \Omega, \\ 0 & \textit{otherwise} \;\; \textit{in} \;\; \mathbb{R}^n. \end{cases}$$

Using the same idea as in the proof of Lemma 5.1, we can get the following result of Φ-mean continuity. The main point in the proof below is to take care of the measure μ which is not as good as Lebesgue measure.

Lemma 5.2. *Let $u \in L^\Phi(\Omega,\mu)$ with Φ doubling, $\Omega \subset \mathbb{R}^n$ bounded. Then u is Φ-mean continuous.*

Proof. Let $u \in L^\Phi(\Omega;\mu)$ and $\epsilon > 0$. Let us first prove that there exists a $\delta = \delta(\epsilon)$ such that for any $h \in \mathbb{R}^n$ with $|h| < \delta$ we have

$$\int_\Omega \Phi(|u_h - u|)\,d\mu < \epsilon. \tag{5.3}$$

Since $u \in L^\Phi(\Omega;\mu)$, there exists $s > 0$ such that

$$\int_\Omega \Phi(s|u|)\,d\mu < \infty.$$

Since Φ is doubling, $\Phi(s|u|) \sim \Phi(|u|)$ according to Proposition 4.2. Hence

$$\int_\Omega \Phi(|u|)\,d\mu < \infty,$$

i.e., $\Phi(|u|)$ is integrable. Using the absolute continuity of integration, we have that there exist an $\eta > 0$ such that for each $E \subset \Omega$ for which $\mu(E) < 5\eta$, we have

$$\int_E \Phi(|u|)\, d\mu < \epsilon.$$

For this η, since μ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^n$, there exists a ϱ such that $\mu(H_\varrho) < \eta$, where

$$H_\varrho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \varrho\}$$

Put $\Omega_\varrho = \Omega \setminus H_\varrho$. Clearly f is measurable on Ω_ϱ and thus Luzin's theorem implies the existence of a closed set $F_\eta^1 \subset \Omega_\varrho$ such that the restriction of the function u to F_η^1 is continuous, $\mu(\Omega_\varrho \setminus F_\eta^1) < \eta$ and thus

$$\mu(\Omega \setminus F_\eta^1) < 2\eta.$$

Let $|h| < \varrho$ be small, $F_\eta^2 = \{x \in \Omega : x + h \in F_\eta^1\}$. Assume that w is the Radon-Nikodym derivative of μ. Then $w \in L^1(\Omega)$. Using 1-mean continuity of w, we have

$$\int_{F_\eta^1} |w(x+h) - w(x)|\, dx \to 0, \quad \text{as} \quad |h| \to 0.$$

Moreover, since F is closed and bounded, it is compact. Then f is uniformly continuous on F_η^1. Hence there exists a $\delta \in (0, \varrho)$ small enough such that

$$|f(x+h) - f(x)| \leq \Phi^{-1}\left(\frac{\epsilon}{\mu(\Omega)}\right)$$

for all $x, x + h \in F_\eta^1$, and

$$\int_{F_\eta^1} |w(x+h) - w(x)|\, dx < \eta$$

for all $|h| < \delta$. Hence $|\mu(F_\eta^1) - \mu(F_\eta^2)| \leq \eta$. Then $\mu(\Omega \setminus F_\eta^2) < 3\eta$. Letting $F_\eta = F_\eta^1 \cap F_\eta^2$, we get that

$$\mu(\Omega \setminus F_\eta) \leq \mu(\Omega \setminus F_\eta^1) + \mu(\Omega \setminus F_\eta^2) < 5\eta.$$

Using the same argument as the one in the proof of Lemma 5.1, we have

$$\begin{aligned}
&\int_\Omega \Phi(|f(x+h) - f(x)|)\, d\mu \\
&\leq \int_{\Omega \setminus F_\eta} \Phi(|f(x+h)| + |f(x)|)\, d\mu + \int_{F_\eta} \Phi(|f(x+h) - f(x)|)\, d\mu
\end{aligned}$$

$$\begin{aligned}
&\le C \int_{\Omega\setminus F_\eta} \Phi(|f(x+h)|)\, d\mu + C \int_{\Omega\setminus F_\eta} \Phi(+|f(x)|)\, d\mu + \epsilon \\
&\le (2C+1)\epsilon.
\end{aligned}$$

Here we used the doubling property of Φ, and C is a constant that depends only on the doubling constant. Hence we get the claim (5.3).

Now, let us prove that u is Φ-mean continuous. Let $k = \epsilon/2$ be fixed, and let us denote $v = u/k$. Then $v_h = u_h/k$, $v \in L^\Phi(\Omega;\mu)$. By applying claim (5.3) to v, we get that there exists a $\delta = \delta(\epsilon)$ such that for $h \in \mathbb{R}^n$ with $|h| < \delta$, one has

$$\int_\Omega \Phi(|v_h - v|)\, d\mu = \int_\Omega \Phi(|v_h - v|/k)\, d\mu \le 1.$$

From the definition of Luxemberg norm, we have

$$|u_h - u|^L_{L^\Phi(\Omega;\mu)} < \frac{1}{2}\epsilon.$$

From Proposition 4.2, we immediately obtain

$$|u_h - u|_{L^\Phi(\Omega;\mu)} \le 2|u_h - u|^L_{L^\Phi(\Omega;\mu)} < \epsilon.$$

Hence u is Φ-mean continuous. □

Denote by $\mathcal{S}$ the set of all functions η satisfying

$$\begin{gathered}
\eta \in C_c^\infty(\mathbb{R}^n), \\
\eta(x) \ge 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \\
\textstyle\int_{\mathbb{R}^n} \eta(x)\, dx = 1, \\
\operatorname{supp}(\eta) = \{x \in \mathbb{R}^n; |x| \le 1\}.
\end{gathered} \tag{5.4}$$

$\mathcal{S}$ is not an empty set, since we have the following classical example of a function in $\mathcal{S}$.

Example 5.1. Define a C_c^∞-function $\eta : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\eta(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where the constant $c > 0$ is adjusted so that

$$\int_{\mathbb{R}^n} \eta(x)\, dx = 1.$$

Then it is easy to check that $\eta \in \mathcal{S}$. Write

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}) \quad (\epsilon > 0, x \in \mathbb{R}^n);$$

η_ϵ is called the standard mollifier.

Now, for any $\epsilon > 0$, write

$$\Omega_\epsilon := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

Given $f \in L^1_{\mathrm{loc}}(\Omega)$, define

$$f^\epsilon := \eta_\epsilon * f;$$

that is,

$$f^\epsilon(x) := \int_\Omega \eta_\epsilon(x-y) f(y)\, dy = \int_{B(x,\epsilon)} \eta_\epsilon(y) f(x-y)\, dy$$

for $x \in \Omega_\epsilon$.

Mollification provides us with a systematic technique for approximating Orlicz-functions by C^∞-functions.

Lemma 5.3. *(i): If $f \in L^1_{\mathrm{loc}}(\Omega)$, then for each $\epsilon > 0$, $f^\epsilon \in C^\infty(\Omega_\epsilon)$.*

(ii): Let Φ be doubling. If $f \in L^\Phi(\Omega;\mu) \cap L^1_{\mathrm{loc}}(\Omega)$ and Ω is bounded, then

$$f^\epsilon \to f \quad in \quad L^\Phi(V;\mu),$$

whenever $V \subset \Omega$ is compact.

(iii): Let Φ be doubling. If $f \in W^{1,\Phi}(\Omega;\mu) \cap W^{1,1}_{\mathrm{loc}}(\Omega)$ and Ω is bounded, then

$$f^\epsilon_{x_i} = \eta_\epsilon * f_{x_i}, \quad (i = 1, \cdots, n) \quad on \quad U_\epsilon$$

and

$$f^\epsilon \to f \quad in \quad W^{1,\Phi}(V;\mu),$$

whenever $V \subset \Omega$ is compact.

Proof. (i): Fix any point on $x \in \Omega_\epsilon$ and choose $i \in \{1, \cdots, n\}$. We let e_i denote the i-th coordinate vector $(0, \cdots, 1, \cdots, 0)$. Then for $|h|$ small enough, $x + he_i \in \Omega_\epsilon$, and thus

$$\begin{aligned}
&\frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} \\
=&\frac{1}{\epsilon^n}\int_\Omega \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)\, dy \\
=&\frac{1}{\epsilon^n}\int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)\, dy
\end{aligned}$$

for some $V \subset\subset \Omega$. The difference quotient converges as $h \to 0$ to

$$\frac{1}{\epsilon}\eta_{x_i}\left(\frac{x-y}{\epsilon}\right) = \epsilon^n \eta_{\epsilon, x_i}(x-y)$$

for each $y \in V$. Furthermore, the absolute value of the integrand is bounded by

$$\frac{1}{\epsilon}\|D\eta\|_{L^\infty}|f| \in L^1(V).$$

Hence the Dominated Convergence Theorem implies

$$f^\epsilon_{x_i} = \lim_{h\to 0}\frac{f^\epsilon(x+he_i)-f^\epsilon(x)}{h}$$

exists and equals to

$$\int_\Omega \eta_{\epsilon,x_i}(x-y)f(y)\,dy.$$

Using a similar argument, we conclude that partial derivatives of f^ϵ of all orders exist and are continuous at each point of Ω_ϵ. Hence $f^\epsilon \in C^\infty(\Omega_\epsilon)$.

(ii) Given a compact set $V \subset \Omega$, let Ψ be the complementary function of Φ and v satisfy $\int_V \Psi(|v|)\,d\mu \le 1$. Then let $\epsilon <$ dist $(V, \partial\Omega)$. We have

$$\begin{aligned}
&\int_V |f^\epsilon(x)-f(x)||v(x)|\,d\mu\\
=&\int_V \left|\int_{B(x,\epsilon)} \eta_\epsilon(y)[f(x-y)-f(x)]\,dy\right|\cdot|v(x)|\,d\mu\\
\le&\int_V \left[\int_{B(x,\epsilon)} |\eta_\epsilon(y)|\cdot|f(x-y)-f(x)|\cdot|v(x)|\,dy\right]\,d\mu\\
=&\int_{B(x,\epsilon)} \left[\int_V |f(x-y)-f(x)|\cdot|v(x)|\,d\mu\right]\eta_\epsilon(y)\,dy\\
\le& 2\int_{B(x,\epsilon)} \|f_y(x)-f(x)\|_{L^\Phi(V;\mu)}\eta_\epsilon(y)\,dy
\end{aligned}$$

Here, we denoted $f_y(x) = f(x-y)$ and used Fubini's theorem. In the last inequality, we used the Hölder inequality and the inequality

$$\|v\|_{L^\Psi(V;\mu)} \le \int_V \Psi(|v|)\,d\mu + 1 \le 2.$$

From the definition of the Orlicz norm we obtain

$$\|f^\epsilon(x)-f(x)\|_{L^\Phi(V;\mu)} \le 2\int_{B(x,\epsilon)} \|f_y(x)-f(x)\|_{L^\Phi(V;\mu)}\eta_\epsilon(y)\,dy.$$

Using Lemma 5.2, when ϵ small enough, for any $y \in B(x,\epsilon)$ we have

$$\|f_y(x)-f(x)\|_{L^\Phi(V;\mu)} \le \|f_h(x)-f(x)\|_{L^\Phi(\Omega;\mu)} \to 0,$$

for $|h| \le \epsilon$.

Hence for any $V \subset \Omega$ compact, we have

$$f^\epsilon \to f \quad in \quad L^\Phi(V;\mu).$$

(iii) As computed in (i), we have

$$\begin{aligned} f^\epsilon_{x_i}(x) &= \int_\Omega \eta_{\epsilon,x_i}(x-y)f(y)\,dy = -\int_\Omega \eta_{\epsilon,y_i}(x-y)f(y)\,dy \\ &= \int_\Omega \eta_\epsilon(x-y)f_{x_i}(y)\,dy = (\eta_\epsilon * f_{x_i})(x) \end{aligned}$$

for $x \in \Omega_\epsilon$. Using (ii), we can get

$$f^\epsilon \to f \quad in \quad W^{1,\Phi}(V;\mu)$$

immediately, where $V \subset \Omega$ is compact. $\square$

When $\Phi(t) = t^p \log^\lambda(e+t)$ for $p > 1$, $\lambda \in \mathbb{R}$, we can get the following density property.

Proposition 5.1. *Let $D \subset \mathbb{R}^2$ be the unit disk. Then $C^\infty(\overline{D})$ is dense in $W^{1,\Phi}_{p-2}(D)$ for $\Phi(t) = t^p \log^\lambda(e+t)$ and $p > 1$, $\lambda \in \mathbb{R}$. Moreover, $C^\infty(\overline{D})$ is also dense in $W^{1,p}_{w_\Phi}(D)$. Here $W^{1,\Phi}_{p-2}(D)$ and $W^{1,p}_{w_\Phi}(D)$ are the spaces in Example 4.3 and 4.4.*

Proof. First, let us check $W^{1,\Phi}_{p-2}(D) \subset W^{1,1}(D)$. Since for $u \in W^{1,\Phi}_{p-2}(D)$, we have $u \in L^\Phi(D)$, and it follows that $u \in L^1(D)$. Now it is enough to prove that the distributional derivative $Du \in L^\Phi_{p-2}(D)$ is in $L^1(D)$. Let $w(x) = \operatorname{dist}(x, S^1)^{p-2}$. We have the estimate

$$\begin{aligned} \int_D |Du(x)|\,dx &= \int_D |Du(x)| w(x)^{1/(p-\epsilon)} w(x)^{-1/(p-\epsilon)}\,dx \\ &\le \left(\int_D |Du(x)|^{p-\epsilon} w(x)\,dx\right)^{1/(p-\epsilon)} \left(\int_D w(x)^{-1/(p-\epsilon-1)}\,dx\right)^{\frac{p-\epsilon-1}{p-\epsilon}}. \end{aligned}$$

When ϵ is small enough, $\int_D w(x)^{-1/(p-\epsilon-1)}\,dx < \infty$. Using Lemma 4.6 and Example 4.1, we know that

$$\left(\int_D |Du(x)|^{p-\epsilon} w(x)\,dx\right)^{1/(p-\epsilon)} \le K\|Du\|_{L^\Phi_{p-2}} < \infty.$$

Hence $|Du(x)| \in L^1(D)$ and we get $W^{1,\Phi}_{p-2}(D) \subset W^{1,1}(D)$.

Let $R_1 = \{x \in \mathbb{R}_2 : \frac{1}{2} < |x| < 1\}$ and $R_2 = \{x \in \mathbb{R}_2 : 1 < |x| < 2\}$. Define a map $F : R_1 \to R_2$ by setting

$$F(x) = \frac{x}{|x|^2}.$$

Then we know that $F^{-1}=F$ and that F is a bi-Lipschitz map between R_1 and R_2.

For any function $u\in W^{1,\Phi}_{p-2}(D)$, we extend u to $\widetilde{u}$ which is defined on $D'=\{x\in\mathbb{R}_2 : |x|<2\}$ by setting $\widetilde{u}(x)=u(x)$ for $x\in D$ and $\widetilde{u}(x)=u(F(x))$ for $x\in R_2$. Since $u\in W^{1,1}(D)$, there exists a function v with $v(x)=u(x)$ for a.e $x\in D$ such that the functions $g(t,\theta)=v(t\cos\theta,t\sin\theta)$ are absolutely continuous on almost all lines $L=\{te^{i\theta} : t\in(0,1),\theta\in[0,2\pi)\}$. In addition, $g'(t,\theta)\in L^1(D)$. Hence $g(1,\theta)=g(\frac{1}{2},\theta)+\int_{\frac{1}{2}}^1 g'(t,\theta)\,dt$ exists a.e. $\theta\in[0,2\pi)$. Then we can define $\widetilde{u}(\cos\theta,\sin\theta)=g(1,\theta)$. Define $\widetilde{g}(t,\theta)=v(t\cos\theta,t\sin\theta)$ for $0\le t\le 1$, $\theta\in[0,2\pi)$; $\widetilde{g}(t,\theta)=v(F(t\cos\theta,t\cos\theta))$ for $1<t<2$, $\theta\in[0,2\pi)$. Then we can check that $\widetilde{g}(t,\theta)=\widetilde{u}(t\cos\theta,t\sin\theta)$ almost everywhere. Moreover, since F is bi-Lipschitz, we can check that for a.e. $\theta\in[0,2\pi)$, $\tilde{g}(t,\theta)$ is absolutely continuous on almost all lines $L_\theta=\{te^{i\theta} : t\in(0,2)\}$; and that for a.e. $t\in(0,2)$, $\tilde{g}(t,\theta)$ is absolutely continuous on almost all lines $L_t=\{te^{i\theta} : \theta\in[0,2\pi)\}$; and that $|\nabla\tilde{g}|\in L^1(D')$. Hence $\widetilde{u}\in W^{1,1}(D')$.

Next, let us show that $\widetilde{u}\in W^{1,\Phi}_{p-2}(D')$. Since F is bi-Lipschitz map, we have

$$\int_{R_2}\Phi(|\widetilde{u}(x)|)\,dx\lesssim\int_{R_1}\Phi(|u(F(x))|)\,dx$$

and $|D\widetilde{u}(x)|\sim|Du(F^{-1}(x))|$ for $x\in R_2$. Hence, we have

$$\int_{R_2}\Phi(|D\widetilde{u}(x)|)w(x)\,dx\lesssim\int_{R_1}\Phi(|Du(F(x))|)w(F(x))\,dx,$$

where we used $w(x)\sim w(F(x))$ which is from

$$1-|x|\sim\frac{1-|x|}{|x|}=1-\frac{1}{|x|},\quad\text{for all } x\in R_1.$$

Then we have

$$\int_{D'}\Phi(|\widetilde{u}(x)|)\,dx+\int_{D'}\Phi(|D\widetilde{u}(x)|)w(x)\,dx\lesssim\int_{D}\Phi(|u(x)|)\,dx+\int_{D}\Phi(|Du(x)|)w(x)\,dx<\infty,$$

which means that $\widetilde{u}\in W^{1,\Phi}_{p-2}(D')$.

By Lemma 5.3, for a compact set $V\supset D$, we can find a sequence $f_i\in C^\infty(V)$ with $f_i\to\widetilde{u}$ in $W^{1,\Phi}_{p-2}(V)$. Now, define $u_i=f_i|_D$. Then $u_i\in C^\infty(\overline{D})$ and $u_i\to\widetilde{u}|_D=u$ in $W^{1,\Phi}_{p-2}(D)$. Hence, we get the density property for $W^{1,\Phi}_{p-2}(D)$. Using a similar proof as above, we can also get the density property for $W^{1,p}_{w_\Phi}(D)$. Thus the proof is complete. $\square$

The results regarding p-mean continuity and Φ-mean continuity are from the monograph [2]. The ideas for the density properties, i.e., Lemma 5.3 and Proposition 5.1 are from the monograph [6], but we generalized the results from Sobolev spaces to weighted Orlicz-Sobolev spaces.

6 Energies and the trace spaces

We begin by recalling the definitions of our dyadic decomposition and the three energies. Fix a dyadic decomposition of S^1, such that for a fixed $i \in \mathbb{N}$, $\{I_{i,j} : j = 1, \cdots, 2^i\}$ is a family of arcs of length $2\pi/2^i$ with $\bigcup_j I_{i,j} = S^1$. The next generation is constructed in such a way that for each $j \in \{1, \cdots, 2^{i+1}\}$, there exists a unique number $k \in \{1, \cdots, 2^i\}$, satisfying $I_{i+1,j} \subset I_{i,k}$. We denote this parent of $I_{i+1,j}$ by $\widehat{I}_{i+1,j}$ and $\widehat{I}_{1,j} = S^1$ for $j = 1, 2$. By g_A, $A \subset S^1$, we denote the mean value $g_A = ⨍_A g \, d\mathcal{H}^1 = \frac{1}{\mathcal{H}^1(A)} \int_A g \, d\mathcal{H}^1$.

Then we define three energies as follows:

$$\mathcal{E}(g; p, \lambda) := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|^p; \tag{6.1}$$

$$\mathbf{E}(g; p, \lambda) := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \sum_{k} |g_{I_{i,j}} - g_{I_k}|^p; \tag{6.2}$$

$$\mathbf{E}(g; \Phi) := \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_{k} 2^{-ip} \Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \right) \tag{6.3}$$

where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$. Here $I_{i,0} = I_{i,2^i}$ and $I_{i,2^i+1} = I_{i,1}$.

Remark 6.1. Let Φ be doubling. In the definition of (6.2) and (6.3), we can replace $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$ with $I_k \in \{I_{i,j+1}, I_{i,j-1}\}$. we claim that the new energies are equivalent to the original ones.

First, we know that $\widehat{I}_{i,j}$ is equal to $I_{i,j} \bigcup I_{i,j+1}$ or $I_{i,j} \bigcup I_{i,j-1}$. If $\widehat{I}_{i,j} = I_{i,j} \bigcup I_{i,j+1}$, then we have

$$\begin{aligned} g_{\widehat{I}_{i,j}} = ⨍_{\widehat{I}_{i,j}} g \, d\mathcal{H}^1 &= \frac{1}{\mathcal{H}^1(\widehat{I}_{i,j})} \left(\int_{I_{i,j}} g \, d\mathcal{H}^1 + \int_{I_{i,j+1}} g \, d\mathcal{H}^1 \right) \\ &= \frac{1}{2\mathcal{H}^1(I_{i,j})} \int_{I_{i,j}} g \, d\mathcal{H}^1 + \frac{1}{2\mathcal{H}^1(I_{i,j+1})} \int_{I_{i,j+1}} g \, d\mathcal{H}^1 \\ &= \frac{1}{2} ⨍_{\widehat{I}_{i,j}} g \, d\mathcal{H}^1 + \frac{1}{2} ⨍_{\widehat{I}_{i,j+1}} g \, d\mathcal{H}^1 = \frac{1}{2}(g_{I_{i,j}} + g_{I_{i,j+1}}). \end{aligned}$$

Hence we obtain

$$\left| g_{I_{i,j}} - g_{\widehat{I}_{i,j}} \right| = \frac{1}{2} \left| g_{I_{i,j}} - g_{I_{i,j+1}} \right| .$$

Similarly, if $\widehat{I}_{i,j} = I_{i,j} \bigcup I_{i,j-1}$, we obtain

$$\left| g_{I_{i,j}} - g_{\widehat{I}_{i,j}} \right| = \frac{1}{2} \left| g_{I_{i,j}} - g_{I_{i,j-1}} \right|.$$

Since Φ is doubling, we have

$$\left| g_{I_{i,j}} - g_{\widehat{I}_{i,j}} \right|^p \lesssim \left| g_{I_{i,j}} - g_{I_{i,j+1}} \right|^p + \left| g_{I_{i,j}} - g_{I_{i,j-1}} \right|^p$$

and

$$\Phi\left(\frac{|g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|}{2^{-i}} \right) \lesssim \Phi\left(\frac{|g_{I_{i,j}} - g_{I_{i,j-1}}|}{2^{-i}} \right) + \Phi\left(\frac{|g_{I_{i,j}} - g_{I_{i,j+1}}|}{2^{-i}} \right).$$

Then the claim follows.

The above remark tells us that there is no difference if I_k is only from the set $\{I_{i,j+1}, I_{i,j-1}\}$. Then let us consider the connections between these three energies.

Proposition 1.1 gives us the connection between $\mathbf{E}(g;p,\lambda)$ and $\mathbf{E}(g;\Phi)$. Now, let us consider $\mathcal{E}(g;p,\lambda)$ and $\mathbf{E}(g;p,\lambda)$. It is obvious that $\mathcal{E}(g;p,\lambda) \leq \mathbf{E}(g;p,\lambda)$. From the proof of [5, Theorem 3.1], we obtain a sufficient condition for $\mathbf{E}(g;p,\lambda) < \infty$.

Remark 6.2. Let $g : S^1 \to \mathbb{R}$, be bounded. Let $\lambda \in \mathbb{R}$ and $p > 1$. If

$$\int_{S^1}\int_{S^1} \frac{|g(x)-g(y)|^p}{|x-y|^2} \log^\lambda\left(e + \frac{|g(x)-g(y)|}{|x-y|} \right) d\mathcal{H}^1_x d\mathcal{H}^1_y < \infty, \tag{6.4}$$

then $\mathcal{E}(g;p,\lambda) \leq \mathbf{E}(g;p,\lambda) < \infty$. Moreover, if the function g is Hölder continuous with exponent $\alpha > \frac{1}{p}$, i.e., there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x-y|^\alpha$$

for any $x, y \in S^1$, then the condition (6.4) is satisfied, and hence $\mathcal{E}(g;p,\lambda) \leq \mathbf{E}(g;p,\lambda) < \infty$.

The following example tells us that there exists a function g such that $\mathcal{E}(g;p,\lambda) < \infty$ but $\mathbf{E}(g;p,\lambda) = \infty$ for a fixed dyadic decomposition.

Example 6.1. Let $\lambda \geq -1$. Fix a dyadic decomposition $\{I_{i,j} : i \in \mathbb{N}, j = 1, 2, \cdots, 2^i\}$. Define

$$g(x) = \chi_{I_{1,1}} = \begin{cases} 1, & x \in I_{1,1}, \\ 0, & x \in I_{1,2}. \end{cases}$$

Then, we have

$$\mathcal{E}(g;p,\lambda) = \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|^p$$

$$= |g_{I_{1,1}} - g_{I_{0,1}}|^p + |g_{I_{1,2}} - g_{I_{0,1}}|^p = 2^{1-p}.$$

and

$$\begin{aligned}\mathbf{E}(g;p,\lambda) &\geq \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{I_{i,j+1}}|^p \\ &\geq \sum_{i=1}^{\infty} 2\, i^{\lambda} = \infty.\end{aligned}$$

Moreover, we can also see that condition (6.4) is not satisfied.

If we do not fix the dyadic decomposition, we could consider the supremum of the energies over all the dyadic decompositions of S^1. Then for the fixed function g in the above example, we have

$$\sup\{\mathcal{E}(g;p,\lambda)\} = \sup\{\mathbf{E}(g;p,\lambda)\} = \infty,$$

where the supremum is taken over all the dyadic decompositions of S^1. Hence we have the following open question.

Question 6.1. If we have $\sup\{\mathcal{E}(g;p,\lambda)\} < \infty$, do we have $\sup\{\mathbf{E}(g;p,\lambda)\} < \infty$? Here the supremum is taken over all the dyadic decompositions of S^1.

Now, let us focus on the spaces $T^{\Phi}(S^1)$ and $\widetilde{T}^{\Phi}(S^1)$ defined via

$$T^{\Phi}(S^1) := \{g \in L^{\Phi}(S^1) : \|g\|_{\Phi} < \infty\}; \quad \widetilde{T}^{\Phi}(S^1) := \{g \in L^{\Phi}(S^1) : \|g\|_{\Phi}^{*} < \infty\},$$

where

$$\|g\|_{\Phi} = \|g\|_{L^{\Phi}(S^1)} + \|g\|_{\mathbf{E}_{\Phi}}; \quad \|g\|_{\Phi}^{*} = \|g\|_{L^{\Phi}(S^1)} + \|g\|_{\mathbf{E}_{\Phi}}^{*},$$

and

$$\|g\|_{\mathbf{E}_{\Phi}} = \inf\left\{k > 0, \mathbf{E}\left(\frac{g}{k};\Phi\right) \leq 1\right\}; \quad \|g\|_{\mathbf{E}_{\Phi}}^{*} = \left(\mathbf{E}(g,p,\lambda)\right)^{1/p}.$$

The following lemma tells us that $T^{\Phi}(S^1)$ and $\widetilde{T}^{\Phi}(S^1)$ are Banach spaces.

Lemma 6.1. *$\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\Phi}^{*}$ are well-defined norms and $T^{\Phi}(S^1)$ and $\widetilde{T}^{\Phi}(S^1)$ are Banach spaces, i.e., complete normed vector spaces.*

Before the proof, we first give the following lemma (see [7, Theorem 1.3.9]).

Lemma 6.2. *A normed space X is a Banach space if and only if each absolutely convergent series in X converges, i.e., for each sequence $(x_n)_{n=1}^{\infty} \subset X$, if $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in X.*

Proof of Lemma 6.1. The first issue is to check that $\|\cdot\|_\Phi$ is a norm.

(i) $\|g\|_\Phi \geq 0$ is obvious. If $g = 0$, then $\|g\|_\Phi = 0$. If $\|g\|_\Phi = 0$, then $\|g\|_{L^\Phi(S^1)} = 0$. Hence $g = 0$. Thus, $\|g\|_\Phi = 0 \Leftrightarrow g = 0$.

(ii) For any $\alpha \in \mathbb{R}$, $\|\alpha g\|_\Phi = |\alpha|\|g\|_\Phi$ is obvious from the definition.

(iii) In order to prove the triangle inequality, it suffices to prove the triangle inequality of $\|\cdot\|_{\mathbf{E}_\Phi}$. Assume g, h satisfy $\|g\|_{\mathbf{E}_\Phi} = k_1$ and $\|h\|_{\mathbf{E}_\Phi} = k_2$. If $k_1k_2 = 0$, using (i), we have that

$$\|g + h\|_{\mathbf{E}_\Phi} = k_1 + k_2,$$

which satisfies the triangle inequality. If $k_1k_2 \neq 0$, it suffices to prove that

$$\|g + h\|_{\mathbf{E}_\Phi} \leq k_1 + k_2.$$

Using Jensen's inequality in Lemma 4.2, we obtain

$$\begin{aligned}
\mathbf{E}(\frac{g+h}{k_1+k_2};\Phi) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k} 2^{-ip}\Phi\left(\frac{|(g+h)_{I_{i,j}} - (g+h)_{I_k}|}{(k_1+k_2)2^{-i}}\right) \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k} 2^{-ip}\Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}| + |h_{I_{i,j}} - h_{I_k}|}{(k_1+k_2)2^{-i}}\right) \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k} 2^{-ip}\left(\frac{k_1}{k_1+k_2}\Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{k_1 2^{-i}}\right) + \frac{k_2}{k_1+k_2}\Phi\left(\frac{|h_{I_{i,j}} - h_{I_k}|}{k_2 2^{-i}}\right)\right) \\
&\leq \frac{k_1}{k_1+k_2} + \frac{k_2}{k_1+k_2} \leq 1.
\end{aligned}$$

Hence $\|g + h\|_{\mathbf{E}_\Phi} \leq k_1 + k_2 = \|h\|_{\mathbf{E}_\Phi} + \|g\|_{\mathbf{E}_\Phi}$. Thus, $\|\cdot\|_\Phi$ is a norm.

In order to prove that $T^\Phi(S^1)$ is a Banach space, from Lemma 6.2, it suffices to show that for any sequence $(g_n)_{n=1}^\infty \subset T^\Phi(S^1)$, if $\sum_{n=1}^\infty \|g_n\|_\Phi < \infty$, then $\sum_{n=1}^\infty g_n$ converges in $T^\Phi(S^1)$. Since $\sum_{n=1}^\infty \|g_n\|_\Phi < \infty$ implies $\sum_{n=1}^\infty \|g_n\|_{L^\Phi} < \infty$ and $L^\Phi(S^1)$ is a Banach space, then $\sum_{n=1}^\infty g_n$ converges in $L^\Phi(S^1)$ by using the above lemma again. Moreover,

$$\left\|\sum_{n=1}^\infty g_n - \sum_{n=1}^m g_n\right\|_\Phi \leq \sum_{n=m+1}^\infty \|g_n\|_\Phi \to 0, \quad \text{as} \quad m \to \infty.$$

Hence $\sum_{n=1}^\infty g_n$ converges in $T^\Phi(S^1)$.

Using a similar argument, we can prove the same results for $\|\cdot\|_\Phi^*$ and $\widetilde{T}^\Phi(S^1)$. Thus, the proof is finished. $\square$

The energy (6.1) is from the paper [5]. The modified energies (6.2) and (6.3) may be new. Most of the results in this section have been obtained by ourselves, except for the ones that we gave references for.

7 Whitney-type decomposition of unit disk and associated partition of unity

Let $\{Q_{i,j} : i \in \mathbb{N}, j = 1, 2, \cdots, 2^i\}$ be the set of all $Q_{i,j}$ such that

$$Q_{i,j} = \{re^{i\theta} : 1 - \frac{1}{2^{i-1}} \leq r \leq 1 - \frac{1}{2^i}, \frac{(j-1)\pi}{2^{i-1}} \leq \theta \leq \frac{j\pi}{2^{i-1}}.\}$$

for $i \in \mathbb{N}, j = 1, 2, \cdots, 2^i$. If $P : B(0,1) \to S^1$ is the radial projection map, then $P(Q_{i,j}) = I_{i,j}$ for all $i \in \mathbb{N}, j = 1, 2, \cdots, 2^i$, where $\{I_{i,j} : i \in \mathbb{N}, j = 1, 2, \cdots, 2^i\}$ is a dyadic decomposition of S^1.

Now, $\{Q_{i,j} : i \in \mathbb{N}, j = 1, 2, \cdots, 2^i\}$ is a *Whitney-type decomposition* of unit disk $D = B(0,1)$, since we have that

(1) $D = \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{2^i} Q_{i,j}$;

(2) $Q_{i,j}$ are pairwise almost disjoint;

(3) there exist c_1 and c_2 such that

$$c_1 \operatorname{diam}(Q_{i,j}) \leq \operatorname{dist}(Q_{i,j}, D^C) = \operatorname{dist}(Q_{i,j}, S^1) \leq c_2 \operatorname{diam}(Q_{i,j});$$

(4) there exist a constant C such that for any $Q_{i,j}$ with center $x_{i,j}$, we have

$$B(x_{i,j}, C^{-1} \operatorname{diam}(Q_{i,j})) \subset Q_{i,j} \subset B(x_{i,j}, C \operatorname{diam}(Q_{i,j})). \tag{7.1}$$

Associated to this decomposition, there exists a *partition of unity*, that is, there exists a family of smooth functions $\{\varphi_{i,j}\}_{i,j\in\mathbb{N}}$ such that

(i) $\mathrm{supp}(\varphi_{i,j}) \subset \frac{5}{4}Q_{i,j} := \{re^{i\theta} : 1 - \frac{5}{4\cdot 2^{i-1}} \leq r \leq 1 - \frac{3}{4\cdot 2^i}, \frac{(j-\frac{5}{4})\pi}{2^{i-1}} \leq \theta \leq \frac{(j+\frac{1}{4})\pi}{2^{i-1}}.\}$

(ii) There exists a constant $L > 0$ such that $|\nabla\varphi_{i,j}| \leq \frac{L}{\operatorname{diam}(Q_{i,j})} \sim 2^i$.

(iii) $\sum_{i,j} \varphi_{i,j}(x) = \chi_D$.

For the existence of such functions $\{\varphi_{i,j}\}_{i,j\in\mathbb{N}}$, see [9, pp. 168-171]. For any $x \in D$, let $\mathcal{J}_x$ be the collection of all (i,j) such that $x \in \frac{5}{4}Q_{i,j}$. Then from the properties of Whitney-type decomposition, we know that

$$\#\mathcal{J}_x \lesssim 1, \quad \forall x \in D.$$

More precisely, if $x \in Q_{i,j}$, then $\{I_k, k = (i', j') \in \mathcal{J}_x\} \subset \{I_{i,j}, I_{i,j-1}, I_{i,j+1}, \widehat{I}_{i,j}, \widehat{I}_{i,j-1}, \widehat{I}_{i,j+1}\} \cup \{I_k, \widehat{I}_k = I_{i,j} \text{ or } I_{i,j-1} \text{ or } I_{i,j+1}\}$. Hence $\#\mathcal{J}_x \leq 12$.

8 Proof of Proposition 1.1

Proof. When $\lambda = 0$, there is nothing to prove since $\mathbf{E}(g; p, \lambda) = \mathbf{E}(g; \Phi)$.

When $\lambda > 0$, first we estimate the logarithmic term from above. Since $g \in L^\Phi$, we have

$$\int_{S^1} |g|^p \, d\mathcal{H}_x^1 \leq \int_{S^1} |g|^p \log^\lambda(e + |g|) \, d\mathcal{H}_x^1 < \infty.$$

Hence $\|g\|_{L^p} < \infty$.

Then, using the Hölder inequality and $\mathcal{H}^1(I_{i,j}) \sim \mathcal{H}^1(I_k) \sim 2^{-i}$, we have

$$\begin{aligned} |g_{I_{i,j}} - g_{I_k}| &\leq |g_{I_{i,j}}| + |g_{I_k}| = |\fint_{I_{i,j}} g\, d\mathcal{H}^1_x| + |\fint_{I_k} g\, d\mathcal{H}^1_x| \\ &\leq \fint_{I_{i,j}} |g|\, d\mathcal{H}^1_x + \fint_{I_k} |g|\, d\mathcal{H}^1_x \\ &\leq \left(\fint_{I_{i,j}} |g|^p\, d\mathcal{H}^1_x \right)^{1/p} + \left(\fint_{I_k} |g|^p\, d\mathcal{H}^1_x \right)^{1/p} \\ &\lesssim 2^{i/p} \|g\|_{L^p(S^1)}. \end{aligned}$$

Hence $\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \lesssim 2^{(1+1/p)i} \|g\|_{L^p(S^1)}$. Moreover, we get

$$\log^\lambda \left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \right) \lesssim \log^\lambda \left(e + 2^{(1+1/p)i} \|g\|_{L^p(S^1)} \right) \leq C i^\lambda,$$

where $C = C(\|g\|_{L^p(S^1)}, p, \lambda)$. Now, we can estimate $\mathbf{E}(g; \Phi)$ as follows:

$$\begin{aligned} \mathbf{E}(g; \Phi) &= \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k 2^{-ip} \Phi \left(\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \right) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p \log^\lambda \left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \right) \\ &\leq C \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p i^\lambda = C\mathbf{E}(g; p, \lambda), \end{aligned}$$

where $C = C(\|g\|_{L^p(S^1)}, p, \lambda)$.

In order to estimate the logarithmic term from below, we define

$$\chi(i,k) = \begin{cases} 1, & \text{if } \ |g_{I_{i,j}} - g_{I_k}| > 2^{-\frac{2i}{p+1}} \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

Then we have

$$\begin{aligned} \mathbf{E}(g; p, \lambda) &= \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p \\ &= \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} \sum_k \chi(i,k) |g_{I_{i,j}} - g_{I_k}|^p \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \sum_{k} (1-\chi(i,k)) |g_{I_{i,j}} - g_{I_k}|^p \\
&=: P_1 + P_2.
\end{aligned}
$$

If $|g_{I_{i,j}} - g_{I_k}| > 2^{-\frac{2i}{p+1}}$, since $p > 1$ and $\lambda > 0$, we have

$$
\log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) > \log^{\lambda}\left(e + 2^{\frac{p-1}{p+1}i}\right) \geq C i^{\lambda},
$$

where $C = C(\lambda, p)$. Hence we have

$$
P_1 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_{k} |g_{I_{i,j}} - g_{I_k}|^p \log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) = C\mathbf{E}(g; \Phi).
$$

For P_2, since for $|g_{I_{i,j}} - g_{I_k}| \leq 2^{-\frac{2i}{p+1}}$,

$$
\log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \leq \log^{\lambda}\left(e + 2^{\frac{p-1}{p+1}i}\right) \leq C i^{\lambda},
$$

we have that

$$
P_2 \lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} (1-\chi(i,k)) 2^{-\frac{2pi}{p+1}} \cdot i^{\lambda} \leq \sum_{i=1}^{\infty} i^{\lambda} 2^{-\frac{2pi}{p+1}} \cdot 2^i = \sum_{i=1}^{\infty} i^{\lambda} 2^{\frac{1-p}{1+p}i} \leq C.
$$

Therefore, we obtain $P_1 + P_2 \leq C\mathbf{E}(g; \Phi) + C$.

Hence we obtain that for $\lambda > 0$ we have

$$
\frac{1}{C}\mathbf{E}(g; \Phi) \leq \mathbf{E}(g; p, \lambda) \leq C\mathbf{E}(g; \Phi) + C \tag{8.2}
$$

where C depends on $\|g\|_{L^p(S^1)}, p$ and λ.

When $\lambda < 0$, in order to estimate the logarithmic term from above, using definition (8.1), we obtain the estimate

$$
\begin{aligned}
\mathbf{E}(g; \Phi) &= \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_{k} |g_{I_{i,j}} - g_{I_k}|^p \log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_{k} \chi(i,k) |g_{I_{i,j}} - g_{I_k}|^p \log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_{k} (1-\chi(i,k)) |g_{I_{i,j}} - g_{I_k}|^p \log^{\lambda}\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right)
\end{aligned}
$$

$$=: P_1' + P_2'.$$

If $|g_{I_{i,j}} - g_{I_k}| > 2^{-\frac{2i}{p+1}}$, since $p > 1$ and $\lambda < 0$, we have

$$\log^\lambda\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) < \log^\lambda\left(e + 2^{\frac{p-1}{p+1}i}\right) \le Ci^\lambda,$$

where $C = C(\lambda, p)$. Hence we have

$$P_1' \le C \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p i^\lambda = C\mathbf{E}(g; p, \lambda).$$

For P_2' we use the definition of χ and $\log^\lambda(e + t) \le 1$ to obtain

$$P_2' \lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} (1 - \chi(i,k)) 2^{-\frac{2pi}{p+1}} \cdot 1 \le \sum_{i=1}^{\infty} 2^{-\frac{2pi}{p+1}} \cdot 2^i = \sum_{i=1}^{\infty} 2^{\frac{1-p}{1+p}i} \le C.$$

Therefore, we obtain $P_1' + P_2' \le C\mathbf{E}(g; p, \lambda) + C$.

Next, we estimate the logarithmic term from below. Since $g \in L^\Phi$, using Lemma 4.6 and Example 4.1, we know that $g \in L^{p-\epsilon}$ for $0 < \epsilon < p - 1$. Fix ϵ.

Using the same argument as in the case $\lambda > 0$, we can get that

$$\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}} \lesssim 2^{(1+\frac{1}{p-\epsilon})i} \|g\|_{L^{p-\epsilon}(S^1)}.$$

Hence we have

$$\log^\lambda\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \gtrsim \log^\lambda\left(e + 2^{(1+\frac{1}{p-\epsilon})i} \|g\|_{L^{p-\epsilon}(S^1)}\right) \ge Ci^\lambda,$$

where $C = C(\|g\|_{L^{p-\epsilon}(S^1)}, p, \lambda)$. Now we get the estimate via

$$\begin{aligned}
\mathbf{E}(g; p, \lambda) &= \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p \\
&\le C \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \sum_k \chi(i,k) |g_{I_{i,j}} - g_{I_k}|^p \log^\lambda\left(e + \frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \\
&\le C\mathbf{E}(g; \Phi).
\end{aligned}$$

Hence we obtain that for $\lambda < 0$, we have

$$\frac{1}{C}\mathbf{E}(g; p, \lambda) \le \mathbf{E}(g; \Phi) \le C\mathbf{E}(g; p, \lambda) + C \tag{8.3}$$

where C depends on $\|g\|_{L^{p-\epsilon}(S^1)}, p$ and λ.

Combining the inequalities (8.2) and (8.3) and the finiteness of $\|g\|_{L^p(S^1)}$and $\|g\|_{L^{p-\epsilon}(S^1)}$ with respect to $\lambda > 0$ and $\lambda < 0$, we obtain that $\mathbf{E}(g; p, \lambda) < \infty$ is equivalent to $\mathbf{E}(g; \Phi) < \infty$. □

9 Proof of Theorem 1.3

For the proof, we need the following lemma (see [2, Theorem 6.4.1 and 6.4.2]):

Lemma 9.1. *Let $D \subset \mathbb{R}^2$ be the unit disk. For $1 \le p < 2$, $q = (2p-p)/(2-p)$ or $p \ge 2$, $q \ge 1$, there exists a unique continuous linear mapping $\mathcal{R} : W^{1,p}(D) \to L^q(S^1)$ such that $\mathcal{R}u = u|_{S^1}$ for all $u \in C^\infty(\overline{D})$.*

Proof of Theorem 1.3 (i). First, we prove that for any function $u \in C^\infty(\overline{D})$, we have

$$\|u|_{S^1}\|_\Phi \le C\|u\|_{W^{1,\Phi}_{p-2}(D)}$$

where C is a constant independent of u.

Fix $u \in C^\infty(\overline{D})$, and let $g = u|_{S^1}$. Let $x \in I_{i,j}$ and $y \in I_k$, where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$. Then

$$|g(x) - g(y)| \le \int_{\gamma_{x,y}} |\nabla u| ds$$

where $\gamma_{x,y}$ is an arc which is a part of a circle with $\gamma_{x,y} \perp S^1$ at x and y. Since $\cup_{y \in I_k} \gamma_{x,y} \subset CQ_{i,j} \cap D$ where $C = 2\pi$, using the Fubini Theorem, we obtain

$$\begin{aligned}\int_{I_k} |g(x) - g(y)|\, d\mathcal{H}^1_y &\le \int_{I_k} \int_{\gamma_{x,y}} |\nabla u| ds d\mathcal{H}^1_y \\ &\le \int_{CQ_{i,j} \cap D} |\nabla u(z)| dz,\end{aligned}$$

where $\{Q_{i,j}\}$ is the Whitney decomposition from Section 7.

If $\ell = \mathcal{H}^1(I_{i,j}) \sim \mathcal{H}^1(I_k) \sim \operatorname{diam} Q_{i,j} \sim 2^{-i}$, then $|CQ_{i,j} \cap D| \sim \ell^2$. We obtain

$$\begin{aligned}|g_{I_{i,j}} - g_{I_k}| &= |⨍_{I_{i,j}} g(x)\, d\mathcal{H}^1_x - ⨍_{I_k} g(y)\, d\mathcal{H}^1_y| \\ &\le ⨍_{I_{i,j}} ⨍_{I_k} |g(x) - g(y)|\, d\mathcal{H}^1_y d\mathcal{H}^1_x \\ &\lesssim \frac{1}{\ell} \int_{CQ_{i,j} \cap D} |\nabla u(z)| dz.\end{aligned}$$

Let $\Phi(t) = t^p \log^\lambda(e+t)$ and $\Psi(t) = t^r \log^{\lambda r/p}(e+t)$ with $\max\{1, p-1\} < r < p$ where $\lambda \in \mathbb{R}$. Then $\Phi(t) = \Psi^{p/r}(t)$ and both Φ, Ψ are doubling. Fix ϵ with $\max\{1, p-1\} < \epsilon < r$. Then $\Psi^{\frac{1}{\epsilon}}$ is also a Young function and doubling. Recall that $\rho(z) = d(z, S^1)$. Using Jensen's integral inequality in Lemma 4.2, we obtain

$$\Psi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell}\right) \lesssim \left(\Psi^{\frac{1}{\epsilon}}\left(⨍_{CQ_{i,j} \cap D} |\nabla u(z)| dz\right)\right)^\epsilon$$

$$
\begin{aligned}
&\leq \left(\fint_{CQ_{i,j}\cap D} \Psi^{\frac{1}{\epsilon}}(|\nabla u(z)|)dz \right)^{\epsilon} \\
&= \left(\fint_{CQ_{i,j}\cap D} \Psi^{\frac{1}{\epsilon}}(|\nabla u(z)|)\rho(z)^{\frac{p-2}{\epsilon}}\rho(z)^{\frac{2-p}{\epsilon}} dz \right)^{\epsilon} \\
&\leq \left(\fint_{CQ_{i,j}\cap D} \Psi(|\nabla u(z)|)\rho(z)^{p-2}dz \right) \left(\fint_{CQ_{i,j}\cap D} \rho(z)^{\frac{2-p}{\epsilon}\cdot\frac{\epsilon}{\epsilon-1}} dz \right)^{\frac{\epsilon-1}{\epsilon}\cdot\epsilon}
\end{aligned}
$$

where we use the Hölder inequality in the last inequality. Then we can estimate that

$$
\begin{aligned}
\left(\fint_{CQ_{i,j}\cap D} \rho(z)^{\frac{2-p}{\epsilon}\cdot\frac{\epsilon}{\epsilon-1}} dz \right)^{\frac{\epsilon-1}{\epsilon}\cdot\epsilon} &= \left(\fint_{CQ_{i,j}\cap D} \rho(z)^{\frac{2-p}{\epsilon-1}} dz \right)^{\epsilon-1} \\
&\lesssim \left(\frac{1}{\ell^2} \int_0^{C\ell} \int_0^{C\ell} t^{\frac{2-p}{\epsilon-1}}\, dtdx \right)^{\epsilon-1} \\
&\lesssim \ell^{2-p},
\end{aligned}
$$

since $\frac{2-p}{\epsilon-1} > -1$ for $\max\{1, p-1\} < \epsilon < r$.

Let $\mu(E) = \int_E \rho(z)^{p-2}\, dz$ for any measurable set $E \subset \mathbb{R}^2$. We obtain

$$
\begin{aligned}
\Psi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell} \right) &\lesssim \ell^{-p} \int_{CQ_{i,j}\cap D} \Psi(|\nabla u(z)|)\rho(z)^{p-2}dz \\
&= \ell^{-p} \int_{CQ_{i,j}\cap D} \Psi(|\nabla u(z)|)\, d\mu.
\end{aligned}
$$

Since $\Phi = \Psi^{p/r}$, we have that

$$
\begin{aligned}
\ell^p \Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell} \right) &= \left(\ell^r \Psi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell} \right) \right)^{p/r} \lesssim \ell^{p(1-\frac{p}{r})} \left(\int_{CQ_{i,j}\cap D} \Psi(|\nabla u(z)|)\, d\mu \right)^{p/r} \\
&\leq \ell^{p(1-\frac{p}{r})} \left(\int_{CQ_{i,j}\cap D} \Psi^{\frac{q}{r}}(|\nabla u(z)|) d\mu \right)^{p/q} (\mu(CQ_{i,j}\cap D))^{\frac{q-r}{q}\cdot\frac{p}{r}}
\end{aligned}
$$

where $r < q < p$ and we used the Hölder inequality in the last inequality.

From the condition (7.1) in Section 7, we know that for any $Q_{i,j}$, there exists a ball $B_{i,j}$ such that $B_{i,j} \subset Q_{i,j} \subset CB_{i,j}$, where C is independent of $Q_{i,j}$. Moreover, since μ is doubling, we have that

$$
\mu(CQ_{i,j}\cap D) = \int_{CQ_{i,j}\cap D} \rho(z)^{p-2}\, dz \sim \ell^p \sim \mu(CQ_{i,j}) \sim \mu(Q_{i,j}) \sim \mu(B_{i,j}) \sim \mu(CB_{i,j}).
$$

Hence we obtain

$$\begin{aligned}
\ell^p \Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell}\right) &\lesssim \ell^{p(1-\frac{p}{r})} \cdot \ell^{\frac{p(q-r)p}{qr}} \left(\int_{CQ_{i,j}\cap D} \Psi^{\frac{q}{r}}(|\nabla u(z)|)d\mu\right)^{p/q} \\
&\lesssim \ell^p \left(\fint_{CQ_{i,j}\cap D} \Psi^{\frac{q}{r}}(|\nabla u(z)|)d\mu\right)^{p/q} \\
&\lesssim \ell^p \left(\fint_{CB_{i,j}} \Psi^{\frac{q}{r}}(|\nabla u(z)|)\chi_B(z)d\mu\right)^{p/q} \\
&\lesssim \ell^p \left(\fint_{B_{i,j}} M_\mu(F)(z)d\mu\right)^{p/q} \lesssim \int_{B_{i,j}} M_\mu^{\frac{p}{q}}(F)(z)d\mu \\
&\le \int_{Q_{i,j}} M_\mu^{\frac{p}{q}}(F)(z)d\mu.
\end{aligned}$$

Here

$$F(z) = \Psi^{\frac{q}{r}}(|\nabla u(z)|)\chi_D(z)$$

and

$$M_\mu(F)(z) = \sup \fint_B |F(y)|\, d\mu,$$

where the supremum is taken over all ope balls B that contain z.

Since

$$2^{-ip}\Psi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \sim \ell^p \Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{\ell}\right),$$

and $\rho(z)^{p-2}$ is an A_p-weight (see Example 3.1), using Lemma 3.2, we can estimate $\mathbf{E}(g;\Phi)$ as follows:

$$\begin{aligned}
\mathbf{E}(g;\Phi) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_k 2^{-ip}\Phi\left(\frac{|g_{I_{i,j}} - g_{I_k}|}{2^{-i}}\right) \\
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \int_{Q_{i,j}} M_\mu^{\frac{p}{q}}(F)(z)d\mu \\
&\le \int_{\mathbb{R}^2} M_\mu^{\frac{p}{q}}(F)(z)d\mu \lesssim \int_{\mathbb{R}^2} F^{\frac{p}{q}}\, d\mu \\
&= \int_D \Psi^{\frac{p}{r}}(|\nabla u(z)|)\, d\mu = \int_D \Phi(|\nabla u(z)|)\rho(z)^{p-2}\, dz.
\end{aligned}$$

In conclusion, there exists a constant C independent of u and g such that

$$\mathbf{E}(g;\Phi) \le C \int_D \Phi(|\nabla u(z)|)\rho(z)^{p-2}\, dz. \tag{9.1}$$

Next, we use the above inequality to prove that

$$\|g\|_{\mathbf{E}_\Phi} \le \max\{1, C\}\|\nabla u\|_{L^\Phi_{p-2}(D)}. \tag{9.2}$$

Assume $\|\nabla u\|_{L^\Phi_{p-2}(D)} = t < \infty$. From the definition of norm $\|\cdot\|_{L^\Phi_{p-2}(D)}$, we have

$$\int_D \Phi\left(\frac{|\nabla(u)(z)|}{t}\right)\rho(z)^{p-2}\,dz \le 1.$$

If $C \le 1$, then (9.1) applied to g/t gives that

$$\mathbf{E}\left(\frac{g}{t};\Phi\right) \le \int_D \Phi\left(\left|\nabla\left(\frac{u(z)}{t}\right)\right|\right)\rho(z)^{p-2}\,dz = \int_D \Phi\left(\frac{|\nabla(u)(z)|}{t}\right)\rho(z)^{p-2}\,dz \le 1,$$

Hence $\|g\|_{\mathbf{E}_\Phi} \le t$. If $C > 1$, then

$$\mathbf{E}\left(\frac{g}{t};\Phi\right) \le C\int_D \Phi\left(\frac{|\nabla(u)(z)|}{t}\right)\rho(z)^{p-2}\,dz \le C.$$

Using the convexity of Φ, we obtain

$$1 \ge \frac{1}{C}\mathbf{E}\left(\frac{g}{t};\Phi\right) \ge \mathbf{E}\left(\frac{g}{Ct};\Phi\right),$$

and hence $\|g\|_{\mathbf{E}_\Phi} \le Ct$. Thus we get inequality (9.2).

Now let us prove that

$$\|g\|_{L^\Phi(S^1)} \lesssim \|u\|_{W^{1,\Phi}_{p-2}(D)}. \tag{9.3}$$

Because of Lemma 4.6 and Lemma 9.1, we expect that there exist $p-1 > \delta > 0$ and $q > 1$ such that

$$\|g\|_{L^\Phi(S^1)} \lesssim \|g\|_{L^{p+\delta}(S^1)} \lesssim \|u\|_{W^{1,\frac{p-\delta}{q}}(D)} = \|u\|_{L^{\frac{p-\delta}{q}}} + \|\nabla u\|_{L^{\frac{p-\delta}{q}}} =: H_3 + H_4.$$

Assume this for a moment. We estimate H_3 and H_4. For H_3, using the Hölder inequality and Lemma 4.6, we obtain

$$H_3 \lesssim \|u\|_{L^{p-\delta}(D)} \lesssim \|u\|_{L^\Phi(D)}.$$

For H_4, we also use the Hölder inequality and Lemma 4.6. If $\frac{2-p}{q-1} > -1$, we get

$$\begin{aligned} H_4 &= \left(\int_D |\nabla u|^{\frac{p-\delta}{q}}\rho(z)^{\frac{p-2}{q}}\rho(z)^{\frac{2-p}{q}}\,dz\right)^{\frac{q}{p-\delta}} \\ &\le \left(\int_D |\nabla u|^{p-\delta}\rho(z)^{p-2}\,dz\right)^{\frac{1}{p-\delta}}\left(\int_D \rho(z)^{\frac{2-p}{q}\frac{q}{q-1}}\right)^{\frac{q-1}{p-\delta}} \end{aligned}$$

$$
\begin{aligned}
&= \|\nabla u\|_{L^{p-\delta}_{p-2}(D)} \left(\int_D \rho(z)^{\frac{2-p}{q-1}} \right)^{\frac{q-1}{p-\delta}} \\
&\lesssim \|\nabla u\|_{L^{p-\delta}_{p-2}(D)} \lesssim \|\nabla u\|_{L^{\Phi}_{p-2}(D)}.
\end{aligned}
$$

Hence, what remains is to check the existence of δ and q as above, which is equivalent to both of the following systems of inequalities

$$
\begin{cases}
p-1 > \delta > 0, q > 1 \\
p \geq 2 \\
(p-\delta)/q \geq 1 \\
\frac{2-p}{q-1} > -1.
\end{cases}
$$

and

$$
\begin{cases}
p-1 > \delta > 0, q > 1 \\
1 < p < 2 \\
(p-\delta)/q \geq 1 \\
p+\delta > \frac{(p-\delta)/q}{2-(p-\delta)/q} \\
\frac{2-p}{q-1} > -1
\end{cases}
$$

to have solutions for δ and q. It is easy to check that the above systems of inequalities have solutions for δ and q. Hence we obtain

$$
\|g\|_{L^{\Phi}(S^1)} \lesssim H_3 + H_4 \lesssim \|\nabla u\|_{L^{\Phi}_{p-2}(D)} + \|\nabla u\|_{L^{\Phi}_{p-2}(D)} = \|u\|_{W^{1,\Phi}_{p-2}(D)},
$$

which gives inequality (9.3).

Together with inequalities (9.2) and (9.3), we have that there exists a constant $C > 0$ independent of u such that

$$
\|u|_{S^1}\|_{\Phi} \leq C\|u\|_{W^{1,\Phi}_{p-2}(D)} \tag{9.4}
$$

for all $u \in C^\infty(\overline{D})$. Using the density property from Proposition 5.1, for every $u \in W^{1,\Phi}_{p-2}(D)$, we have a Cauchy sequence u_i such that $u_i \in C^\infty(\overline{D})$ and $\|u_i - u\|_{W^{1,\Phi}_{p-2}(D)} \to 0$ as $i \to \infty$. Then, from the norm inequality (9.4), we obtain that $u_i|_{S^1}$ is a Cauchy sequence in $T^\Phi(S^1)$, and hence we can define $\mathcal{T}u$ as the limit of $u_i|_{S^1}$ under the norm $\|\cdot\|_\Phi$. Since $T^\Phi(S^1)$ is a Banach space, the limit exists and is unique. Hence we get the existence and uniqueness of the mapping $\mathcal{T}$. Thus the proof is completed. □

Before proving Theorem 1.3 (ii), we give the following lemma:

Lemma 9.2. *Using the Whitney decomposition and the associated partition of unity in Section 7, for any $g \in T^\Phi(S^1)$, we define Eg as*

$$
Eg = u(x) = \textstyle\sum_{k \in \mathcal{J}_x} \varphi_k(x) g_{I_k}, \quad x \in D. \tag{9.5}
$$

where $\{\varphi_k(x)\}$ is the partition of unity in Section 7. Then Eg can be extended to the boundary S^1 with $Eg|_{S^1} = g$ for a.e. x, i.e., for a.e. $x = e^{i\theta} \in S^1$, when $\{x_n = r_n e^{i\theta}\}_{n=1}^{\infty}$ with $x_n \in D$ and $r_n \to 1$, we have $\lim_{n\to\infty} Eg(x_n) = g(x)$.

Proof. From the argument in section 7, we know that $\#\mathcal{J}_x \le 12$. Then for x_n, we redefine $\mathcal{J}_{x_n} = \{I_k^n : 1 \le k \le 12, n \in \mathbb{N}\}$. Now, we know that for any k, we have $I_k^n \to x$ as $n \to \infty$. Let us prove that

$$\lim_{n\to\infty} g_{I_k^n} = f(x) \quad for \quad a.e. \quad x. \tag{9.6}$$

To prove this, we need the Lebesgue differentiation theorem (see [8, Theorem 1.3, Page 104]), i.e., if g is integrable on S^1, then for $x \in I$, we have

$$\lim_{\mathcal{H}^1(I)\to 0} \fint_I g(y)\, d\mathcal{H}^1_y = g(x) \quad \text{for} \quad a.e. \quad x.$$

Fix k. Then there are two cases: (1) $x \in I_k^n$; (2) $x \notin I_k^n$. For the case (1), since $g \in L^\Phi(S^1)$ is integrable on S^1, we may use the Lebesgue differentiation theorem directly to obtain condition (9.6).

For the case (2), we need more arguments. Here, we consider one of the cases; the other cases follow in the same way. For every level n, if $x \in I_{i_n,j_n}$ and $I_k^n = I_{i_n,j_n+1}$, then $x \in I_{i_n,j_n} \cup I_{i_n,j_n+1} = \widetilde{I}^n$. Moreover, we have $\mathcal{H}^1(\widetilde{I}^n) = 2\mathcal{H}^1(I_{i_n,j_n}) = 2\mathcal{H}^1(I_{i_n,j_n+1})$ and

$$\lim_{n\to\infty} \fint_{\widetilde{I}^n} g(y)\, d\mathcal{H}^1_y = g(x) \quad \text{for} \quad a.e. \quad x,$$

and

$$\lim_{n\to\infty} \fint_{I_{i_n,j_n}} g(y)\, d\mathcal{H}^1_y = g(x) \quad \text{for} \quad a.e. \quad x.$$

Hence we can get condition (9.6) after a simple calculation.

Now using (9.6), together with $\sum_{k\in\mathcal{J}_x} \varphi_k(x) = 1$ for any x_n, we get $\lim_{n\to\infty} Eg(x_n) = g(x)$ for a.e. x. Hence $Eg|_{S^1} = g$ for a.e. x. $\square$

Proof of Theorem 1.3 (ii). We define Eg as in Lemma 9.2. Then we know that $Eg|_{S^1} = g$ for a.e. x.

For $x \in D$, we have

$$|u(x)| = \left|\sum_{k\in\mathcal{J}_x} \varphi_k(x) g_{I_k}\right| \le \sum_{k\in\mathcal{J}_x} \varphi_k(x)|g_{I_k}| \le \sum_{k\in\mathcal{J}_x} \varphi_k(x)|g|_{I_k} \le \sum_{k\in\mathcal{J}_x} |g|_{I_k}.$$

For any $x \in Q_{i,j}$, and any $k \in \mathcal{J}_x$, we have $|Q_{i,j}| \sim 2^{-2i}$, $\mathcal{H}^1(I_k) \sim 2^{-i}$. Since Φ is doubling, using Jensen's inequality, we obtain

$$\int_D \Phi(|u|)\, dx = \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \int_{Q_{i,j}} \Phi(|u|)\, dx \lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} 2^{-2i} \Phi\left(\sum_{k\in\mathcal{J}_x} |g|_{I_k}\right)$$

$$
\begin{aligned}
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k\in\mathcal{J}_x} 2^{-2i}\Phi(|g|_{I_k}) \lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 2^{-2i}\Phi(|g|_{I_{i,j}})\\
&\le \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 2^{-2i} \fint_{I_{i,j}} \Phi(|g|)\,dx \lesssim \sum_{i=1}^{\infty} 2^{-i}\int_{S^1}\Phi(|g|)\,dx\\
&\le \int_{S^1}\Phi(|g|)\,dx.
\end{aligned}
$$

Using the same argument as in the proof of Theorem 1.3 (i), we get

$$
\|u\|_{L^\Phi(D)} \lesssim \|g\|_{L^\Phi(S^1)}. \tag{9.7}
$$

For any $x\in Q_{i,j}$, from the definition of u as in (9.5), we have

$$
u(x)-g_{I_{i,j}} = \sum_{k\in\mathcal{J}_x}\varphi_k(x)g_{I_k} - g_{I_{i,j}} = \sum_{k\in\mathcal{J}_x}\varphi_k(x)(g_{I_k}-g_{I_{i,j}}),
$$

since $\sum_{k\in\mathcal{J}_x}\varphi_k(x)=1$. Hence we obtain

$$
|\nabla u(x)| = \left|\nabla(u(x)-g_{I_{i,j}})\right| \le \sum_{k\in\mathcal{J}_x}|\nabla\varphi_k(x)|\left|g_{I_k}-g_{I_{i,j}}\right| \lesssim \sum_{k\in\mathcal{J}_x} 2^i\left|g_{I_k}-g_{I_{i,j}}\right|.
$$

Then since Φ is doubling, we can get the estimate

$$
\begin{aligned}
\int_D \Phi(|\nabla u|)\rho(z)^{p-2}\,dz &\lesssim \sum_{n=1}^{\infty}\sum_{j=1}^{2^i}\int_{Q_{i,j}}\Phi(|\nabla u(x)|)\rho(z)^{p-2}\,dx\\
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 2^{-2i}\Phi\left(\sum_{k\in\mathcal{J}_x}\frac{\left|g_{I_k}-g_{I_{i,j}}\right|}{2^{-i}}\right)2^{-i(p-2)}\\
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 2^{-ip}\Phi\left(\sum_{k\in\mathcal{J}_x}\frac{\left|g_{I_k}-g_{I_{i,j}}\right|}{2^{-i}}\right)\\
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k\in\mathcal{J}_x} 2^{-ip}\Phi\left(\frac{\left|g_{I_k}-g_{I_{i,j}}\right|}{2^{-i}}\right)\\
&\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{2^i}\sum_{k} 2^{-ip}\Phi\left(\frac{\left|g_{I_k}-g_{I_{i,j}}\right|}{2^{-i}}\right)
\end{aligned}
$$

where $I_k\in\{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$.

Hence we have $\int_D \Phi(|\nabla u|)\rho(z)^{p-2}\,dz \lesssim \mathbf{E}(g;\Phi)$, and using the same argument as in the proof of Theorem 1.3 (i), we obtain

$$
\|\nabla u\|_{L^{1,\Phi}_{p-2}(D)} \lesssim \|g\|_{\mathbf{E}_\Phi}. \tag{9.8}
$$

Combing inequalities (9.7) and (9.8), we get the inequality

$$\|u\|_{W^{1,\Phi}_{p-2}(D)} \le C\|g\|_{\Phi} \tag{9.9}$$

where C is a constant independent of g. Hence the extension operator is linear and continuous with $Eg|_{S^1} = g$. □

10 Proof of Theorem 1.4

Proof of Theorem 1.4 (i). Using the same idea as in proof of Theorem 1.3 (i), fix $u \in C^\infty(\overline{D})$, and let $g = u|_{S^1}$. For $i \in \mathbb{N}$ and $j = 1, 2, \cdots, 2^i$. Now $\ell = \mathcal{H}^1(I_{i,j}) \sim \mathcal{H}^1(I_k) \sim \operatorname{diam} Q_{i,j} \sim 2^{-i}$ and $|CQ_{i,j} \cap D| \sim \ell^2$. We have

$$|g_{I_{i,j}} - g_{I_k}| \lesssim \frac{1}{\ell}\int_{CQ_{i,j}\cap D} |\nabla u(z)|dz \sim \ell \fint_{CQ_{i,j}\cap D} |\nabla u(z)|dz,$$

where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$.

Fix $\max\{1, p-1\} < r < p$ and ϵ with $\max\{1, p-1\} < \epsilon < r$. Then we have

$$\begin{aligned}
|g_{I_{i,j}} - g_{I_k}|^r &\lesssim \ell^r \left(\fint_{CQ_{i,j}\cap D} |\nabla u(z)|dz\right)^r \le \ell^r \left(\fint_{CQ_{i,j}\cap D} |\nabla u(z)|^{\frac{r}{\epsilon}}dz\right)^\epsilon \\
&= \ell^r \left(\fint_{CQ_{i,j}\cap D} |\nabla u(z)|^{\frac{r}{\epsilon}}(w_\Phi)^{1/\epsilon}(w_\Phi)^{-1/\epsilon}dz\right)^\epsilon \\
&\le \ell^r \left(\fint_{CQ_{i,j}\cap D} |\nabla u(z)|^r w_\Phi\, dz\right)\left(\fint_{CQ_{i,j}\cap D} (w_\Phi)^{\frac{-1}{\epsilon}\cdot\frac{\epsilon}{\epsilon-1}}\right)^{\epsilon-1},
\end{aligned}$$

where we used the Hölder inequality in the last inequality. Moreover, using the same argument as in Example 3.2, we estimate

$$\begin{aligned}
\left(\fint_{CQ_{i,j}\cap D} (w_\Phi)^{\frac{-1}{\epsilon}\cdot\frac{\epsilon}{\epsilon-1}}\right)^{\epsilon-1} &= \left(\fint_{CQ_{i,j}\cap D} \rho(z)^{\frac{2-p}{\epsilon-1}} \log^{\frac{-\lambda}{\epsilon-1}}\left(\frac{4}{\rho(z)}\right)dz\right)^{\epsilon-1} \\
&\lesssim \left(\frac{1}{\ell^2}\int_0^{C\ell}\int_0^{C\ell} t^{\frac{2-p}{\epsilon-1}} \log^{\frac{-\lambda}{\epsilon-1}}\left(\frac{4}{t}\right) dtdx\right)^{\epsilon-1} \\
&\lesssim \ell^{2-p}\log^\lambda\left(\frac{4}{C\ell}\right) \sim \ell^{2-p}i^{-\lambda};
\end{aligned}$$

notice that $\frac{2-p}{\epsilon-1} > -1$ for $\max\{1, p-1\} < \epsilon < r$ and $\ell \sim 2^{-i}$.

Set $\mu(E) = \int_E w_\Phi(z)\, dz$ for each measurable set E. We obtain that

$$|g_{I_{i,j}} - g_{I_k}|^r \lesssim i^{-\lambda}\ell^{r-p}\int_{CQ_{i,j}\cap D} |\nabla u(z)|^r w_\Phi\, dz = i^{-\lambda}\ell^{r-p}\int_{CQ_{i,j}\cap D} |\nabla u(z)|^r\, d\mu.$$

Fix $r < q < p$. Using the Hölder inequality, we obtain that

$$
\begin{aligned}
i^\lambda |g_{I_{i,j}} - g_{I_k}|^p = i^\lambda \left(|g_{I_{i,j}} - g_{I_k}|^r\right)^{\frac{p}{r}} &\lesssim i^{\lambda(1-\frac{p}{r})} \ell^{p(1-\frac{p}{r})} \left(\int_{CQ_{i,j}\cap D} |\nabla u(z)|^r \, d\mu\right)^{\frac{p}{r}} \\
&\leq i^{\lambda(1-\frac{p}{r})} \left(\int_{CQ_{i,j}\cap D} |\nabla u(z)|^q \, d\mu\right)^{p/q} (\mu(CQ_{i,j} \cap D))^{\frac{q-r}{q}\cdot\frac{p}{r}}.
\end{aligned}
$$

From condition (7.1) in Section 7, we know that for any $Q_{i,j}$ there exists a ball $B_{i,j}$ such that $B_{i,j} \subset Q_{i,j} \subset CB_{i,j}$, where C is independent of $Q_{i,j}$. Moreover, since μ is doubling, we have that

$$
\mu(CQ_{i,j}\cap D) = \int_{CQ_{i,j}\cap D} w_\Phi(z)\, dz \sim i^\lambda \ell^p \sim \mu(CQ_{i,j}) \sim \mu(Q_{i,j}) \sim \mu(B_{i,j}) \sim \mu(CB_{i,j}).
$$

Hence we obtain that

$$
\begin{aligned}
i^\lambda |g_{I_{i,j}} - g_{I_k}|^p &\lesssim i^{\lambda(1-p/q)} \ell^{p(1-\frac{p}{q})} \left(\int_{CQ_{i,j}\cap D} |\nabla u(z)|^q \, d\mu\right)^{p/q} \\
&\leq i^\lambda \ell^p \left(⨍_{CQ_{i,j}\cap D} |\nabla u(z)|^q \, d\mu\right)^{p/q} \\
&\lesssim i^\lambda \ell^p \left(⨍_{CB_{i,j}} |\nabla u(z)|^q \chi_D(z) \, d\mu\right)^{p/q} \\
&\lesssim i^\lambda \ell^p \left(⨍_{B_{i,j}} M_\mu(G)(z) \, d\mu\right)^{p/q} \lesssim \int_{B_{i,j}} M_\mu^{\frac{p}{q}}(G)\, d\mu \\
&\leq \int_{Q_{i,j}} M_\mu^{\frac{p}{q}}(G)\, d\mu.
\end{aligned}
$$

Here

$$
G(z) = |\nabla u(z)|^q \chi_D(z)
$$

and

$$
M_\mu(G)(z) = \sup ⨍_B |F(y)|\, d\mu,
$$

where the supremum is taken over all open balls B that contain z.

Since w_Φ is an A_p-weight (see Example 3.2), using Lemma 3.2, we can estimate $\mathbf{E}(g; p, \lambda)$ as follows:

$$
\mathbf{E}(g; p, \lambda) = \sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} \sum_k |g_{I_{i,j}} - g_{I_k}|^p \lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \int_{Q_{i,j}} M_\mu^{\frac{p}{q}}(G)\, d\mu
$$

$$\leq \int_{\mathbb{R}^2} M_\mu^{\frac{p}{q}}(G)\, d\mu \lesssim \int_{\mathbb{R}^2} G^{\frac{p}{q}}\, d\mu = \int_D |\nabla u(z)|^p w_\Phi(z)\, dz.$$

Hence we have the inequality

$$\|g\|^*_{\mathbf{E}_\Phi} \lesssim \left(\int_D |\nabla u(z)|^p w_\Phi(z)\, dz\right)^{1/p}. \tag{10.1}$$

Using the same idea as in the proof of inequality (9.3), i.e., $\|g\|_{L^\Phi(S^1)} \lesssim \|u\|_{W^{1,\Phi}_{p-2}(D)}$, we can also prove that

$$\|g\|_{L^\Phi(S^1)} \lesssim \|u\|_{W^{1,p}_{w_\Phi}(D)}. \tag{10.2}$$

Combing inequalities (10.1) and (10.2), we have that there exists a constant C independent of u such that

$$\|u|_{S^1}\|^*_\Phi \leq C\|u\|_{W^{1,p}_{w_\Phi}(D)} \tag{10.3}$$

for all $u \in C^\infty(\overline{D})$. Using the density property from Proposition 5.1, for every $u \in W^{1,p}_{w_\Phi}(D)$, we have a Cauchy sequence u_i such that $u_i \in C^\infty(\overline{D})$ and $\|u_i - u\|_{W^{1,p}_{w_\Phi}(D)} \to 0$ as $i \to \infty$. Then, from the norm inequality (10.3), we obtain that $u_i|_{S^1}$ is a Cauchy sequence in $\widetilde{T}^\Phi(S^1)$. Hence we can define $\mathcal{T}^*u$ as the limit of $u_i|_{S^1}$ under the norm $\|\cdot\|^*_\Phi$. Since $\widetilde{T}^\Phi(S^1)$ is a Banach space, then the limit exists and is unique. Hence we get the existence and uniqueness of the mapping $\mathcal{T}^*$. Thus we finish the proof. □

Proof of Theorem 1.4 (ii). Using the Whitney decomposition and the associated partition of unity in section 7, for any $g \in \widetilde{T}^\Phi(S^1)$, we define E^*g by setting

$$E^*g = u(x) = \textstyle\sum_{k\in\mathcal{J}_x} \varphi_k(x) g_{I_k}, \quad x \in B. \tag{10.4}$$

Hence $E^*g|_{S^1} = g$ for a.e. x by Lemma 9.2. From the proof of Theorem 1.3 (ii), we know that

$$\|u\|_{L^\Phi(B)} \lesssim \|g\|_{L^\Phi(S^1)}. \tag{10.5}$$

Moreover, for any $x \in Q_{i,j}$, we can also get that

$$|\nabla u(x)| \lesssim \sum_{k\in\mathcal{J}_x} 2^i |g_{I_k} - g_{I_{i,j}}|.$$

Hence we have the estimate

$$\int_D |\nabla u(x)|^p w_\Phi(x)\, dx \lesssim \sum_{n=1}^{\infty}\sum_{j=1}^{2^i} \int_{Q_{i,j}} |\nabla u(x)|^p w_\Phi(x)\, dx$$

$$\lesssim \sum_{n=1}^{\infty} \sum_{j=1}^{2^i} 2^{-2i} \left(\sum_{k \in \mathcal{J}_x} 2^i |g_{I_k} - g_{I_{i,j}}| \right)^p 2^{-i(p-2)} i^{\lambda}$$
$$\lesssim \sum_{n=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \sum_{k \in \mathcal{J}_x} |g_{I_k} - g_{I_{i,j}}|^p$$
$$\lesssim \sum_{n=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \sum_{k} |g_{I_k} - g_{I_{i,j}}|^p$$

where $I_k \in \{I_{i,j+1}, I_{i,j-1}, \widehat{I}_{i,j}\}$.

Thus, we have

$$\left(\int_D |\nabla u(x)|^p w_\Phi(x)\, dx \right)^{1/p} \lesssim (\mathbf{E}(g; p, \lambda))^{1/p} = \|g\|^*_{\mathbf{E}_\Phi}.$$

Combining with inequality (10.5), we arrive at

$$\|u\|_{W^{1,p}_{w_\Phi}(D)} \leq C\|g\|^*_\Phi$$

where C is a constant independent of g. Hence the extension operator is linear and continuous with $E^* g|_{S^1} = g$. $\square$

We have not been able to find the results contained in Proposition 1.1, Theorem 1.3 and Theorem 1.4 in the literature. The proofs of these results were given by us.

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