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KOSKIMAA, Raine and FENYVESI, Kristóf

A MISSION IMPOSSIBLE? LEARNING THE LOGIC OF SPACE WITH IMPOSSIBLE FIGURES IN EXPERIENCE-BASED MATHEMATICS EDUCATION

Introduction

Creating visual illusions, paradox structures and ‘impossible’ figures through playful and artistic procedures, holds an exciting pedagogical opportunity for raising students’ interest towards mathematics and natural sciences and technical aspects of visual arts. The *Experience Workshop International Math-Art Movement* (EWM)¹ has a number of pedagogical methods, which are connected to visual learning, including visual paradoxes and perspective illusions.

EWM started in 2008 at the *Ars Geometrica Conferences* (2007–2010, Hungary) as a collaborative effort of mathematicians, artists, and teachers of mathematics and the arts. In the open network of the EWM almost two hundred scholars, artists, teachers of various subjects, craftsmen and toymakers experiment with various new educational methods and approaches to develop interactive and play-oriented combinations of mathematics and arts. EWM organizes math-art festivals, workshops, exhibitions for children and their parents, trainings and conferences for teachers and professionals interested in experience-based mathematics education.

EWM’s math-art workshops are based on the active and creative manual participation of the students. EWM’s programs include such experimental, practical workshops in playful forms that rely on mathematical connections in the arts which exceed the mathematics curricula taught in ordinary schools.

There are several visual artists, teachers, and mathematicians in the EWM’s community who work on visual paradoxes and their pedagogical implementation in the experience-based education of mathematics. There are digital games as well which employ visual illusions as a part of their game mechanics. Most of these games were not designed as an educational game, but they may be used for educational purposes, to clarify mathematical concepts behind and related to visual illusions (symmetry, perspective, isometric projection etc.), much in the same way as the EWM approaches. In this article we will briefly introduce an EWM workshop related to impossible figures and analyze which characteristics of certain digital games based on visual illusions can contribute to the pedagogic impact, and how to best take advantage of them.

Math-Art Workshops Inside and Beyond the Classroom

In EWM’s programs, the pupils can become acquainted with mathematical and artistic procedures through various math-art educational tools and art-related games developed originally by EWM members. EWM events feature programs with topics like planar and spatial tessellations; collaborative construction of complex spatial structures (e.g., 3D

¹ Homepage: www.experienceworkshop.hu, last accessed 06. 02. 2015.

projections of multidimensional objects²) with ZomeTool, 4dFrame, and other math-art toolkits. EWM has a wide selection of educational tools and a large international collection of mathematical art to develop e.g. the playful recognition of symmetries and other skills based on systems thinking; EWM also provides many ways to demonstrate non-Euclidean geometries with the help, among others, of Lénárt-spheres; Möbius strips, self-similar fractals; artworks of Escher,³ Vasarely,⁴ and so on; movement and dancing, experiments with musical instruments, etc. (cf. Figure 1).

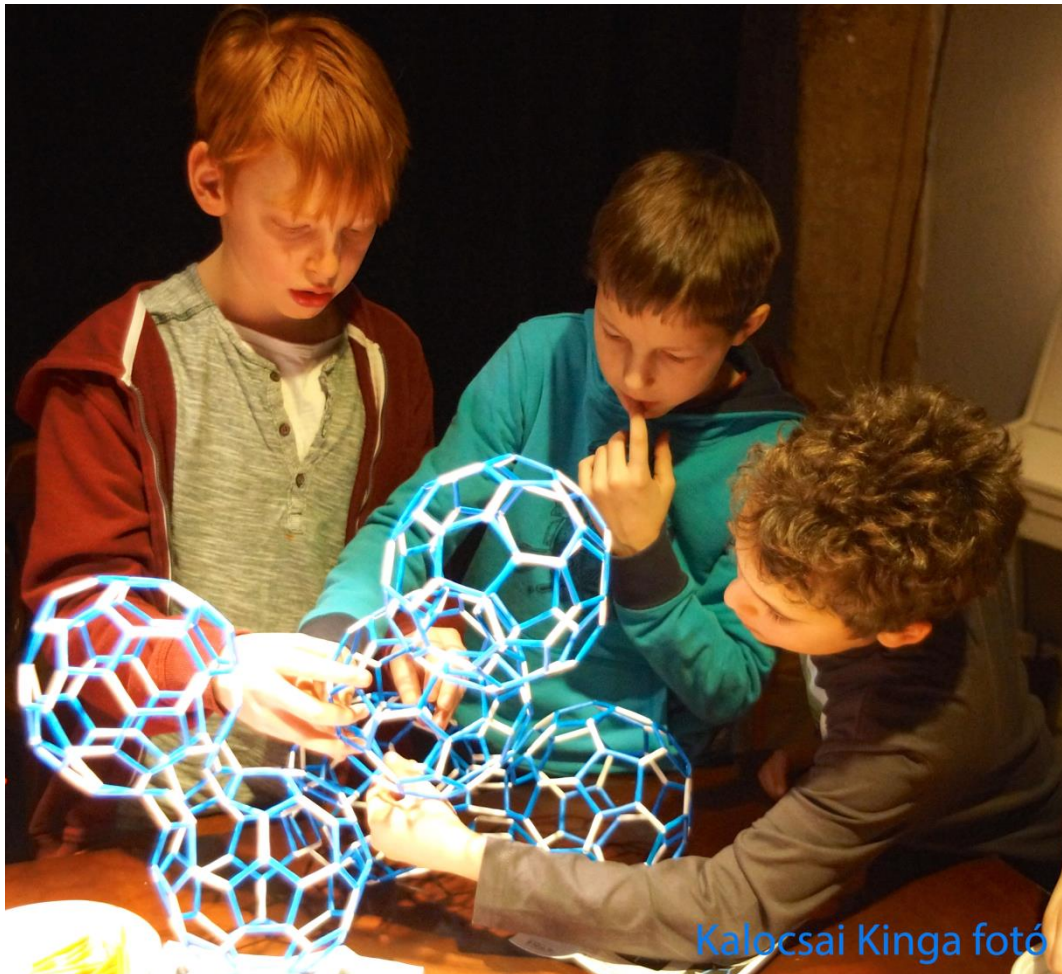


Figure 1: Collaborative work with 4Dframe kit at an Experience Workshop event.
Photo: Kinga Kalocsai

Impossible Figures and the Power of Visual Paradoxes: an Example from Experience Workshop's Repertoire

² Sándor Kabai et al., "Experience-workshop in mathematics and space education: joyful teaching program in Hungary", *43rd Lunar and Planetary Science Conference*, #2611, LPI, Houston, 2012.

³ György Darvas and Kristóf Fenyvesi, "Symmetries in Action! The Hungarian Experience Workshop Movement in the Education of Mathematics and Symmetry Studies through Arts", *Proceedings of the 12th International Conference of the Mathematics Education into the 21st Century Project*, Alan Rogerson, ed., Herceg Novi, Montenegro, 2014; György Darvas, "Preface", *Catalogue of Tamás F. Farkas*, Budapest: Demax, 2010, pp. 2–3.

⁴ Slavik Jablan and Kristóf Fenyvesi, "The Vasarely Playhouse: Look and Combine!", *Proceedings of Bridges 2011 World Conference*, Coimbra: Tessellation Publications, 2011, pp. 633–636.

According to Margo Kondratieva, paradoxes, and especially visual paradoxes, are potentially useful for teaching mathematics due to their engaging power and the effect of surprise.⁵ Kondratieva also sees visual paradoxes as highly useful in classroom as they can be easily implemented as exercises where the pupils can experiment with alternative solutions through drawing or manipulating cut-out shapes. Similarly to EWM's approach, Kondratieva emphasizes the importance of hands-on activities: "Manipulations with physical models and figures of geometrical objects allow learners to get a better understanding through reorganization of the perceived information and construction of an appropriate structural skeleton for a corresponding mental model."⁶

Even though there are clear benefits in this sort of visual experimentation, it is equally evident that the power of visual reasoning is restricted in some aspects. For example, negative and complex numbers cannot be dealt with and are excluded as topics. This limitation may, at least to some extent, be overcome when physical objects are replaced by manipulating virtual objects in digital environments. Another limitation is the reliability of visual images: we cannot necessarily always rely on our own eyes, as various well known visual illusions make this evident. Visual illusions and paradoxes, however, may be turned into means of engagement, and pedagogical tools in themselves.

The key in the visual approach is to foster an easy and fast way to try out several alternative solutions to the given problem: "the point of the exercise is to make a large number of observations, to learn how to make a picture talk to you about its properties, to retrieve the information compressed in a drawing".⁷

A special group of visual paradoxes and illusions, namely impossible figures, are apparently enjoying special interest and are receiving special attention in the EWM's community. Artists Tamás F. Farkas and István Orosz create impossible figures as a part of their artistic oeuvre, Ildikó Szabó, a mathematics teacher, develops a math-art education program based on Farkas' and Orosz's artworks, and the mathematician László Vörös carries out geometrical research connected to Farkas' and Orosz' art pieces.

Bruno Ernst defined impossible figures as figures which can be imagined or drawn, but which cannot be made in any concrete form.⁸ Their effect is based on (at least) two separate layers of illusion. As Ernst summarizes, the first layer is the illusion of spatiality: all we are really looking at is a set of lines printed on a piece of paper (flat), yet we appear to see a solid object. And the second layer is the illusion of continuity: the bars which make up an impossible tri-bar cannot meet in real space (different perspectives united in an isometric drawing), but we still try to assign a meaning.⁹ There are several noted examples of impossible figures from the fine arts, certainly the most famous ones are the Dutch artist M.

⁵ Margo Kondratieva, "Understanding mathematics through resolution of paradoxes", *Mediterranean Journal for Research in Mathematics Education*, 6 (2007), pp. 127–138; Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, pp. 3–8.

⁶ Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 4.

⁷ Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 5.

⁸ Bruno Ernst, *Adventures with Impossible Figures*, Tarquin Publications, 1986.

⁹ Bruno Ernst, *Adventures with Impossible Figures*, Tarquin Publications, 1986, pp. 10–15.

C. Escher's *Belvedere* (1958), *Ascending and Descending* (1960), and *Waterfall* (1961), and the phenomenon is equally fascinating and challenging to psychologists and mathematicians.

Impossible figures were first described scientifically by psychiatrist Lionel Penrose and his son, the later world famous mathematical physicist, Roger Penrose, in their paper: "Impossible Objects: A Special Type of Visual Illusion", published in the *British Journal of Psychology* in 1958. The paper included illustrations such as the impossible triangle and the impossible steps (Figures 2 and 3), both of which were also used by both the Swedish painter Oscar Reutersvärd and M. C. Escher in their works.

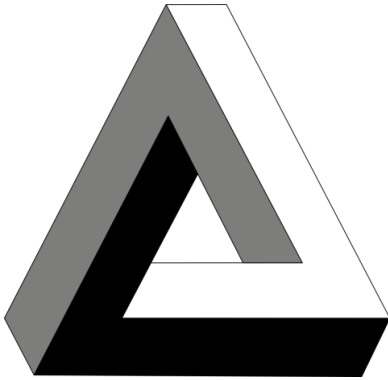


Figure 2: Penrose's impossible triangle

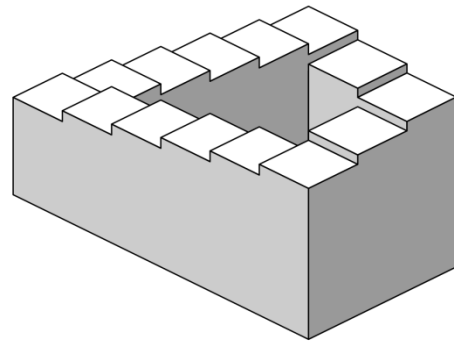


Figure 3: Penrose's impossible steps

In the case of impossible figures, a specific correspondence develops between the two- and three-dimensional space. Therefore studying or drawing these figures can play an important role in visual art studies as well as in mathematics education. Studying impossible figures not only helps in thinking creatively but it also improves depth perception. Furthermore, getting acquainted with impossible objects can open the way to understanding higher (more than 3) dimensional spaces and high-dimensional structures within them.



Figure 4: *Tamás F. Farkas's compositions with impossible figures*

At Farkas's EWM workshops, students use the artists' templates to recreate his impossible figure designs (cf. Figure 4). The templates are based on the connection between the structural properties of impossible figures and tessellations with special modules, called Necker or Koffka cubes. The Necker cube is an optical illusion of perceptual inversion first published as a rhomboid in 1832 by Swiss crystallographer Louis Albert Necker. Some decades later, the German psychologist Kurt Koffka, one of the founders of Gestalt psychology re-discovered reversible figures like the Necker Cube, as a part of his experiments on problem-solving and creativity. As EWM's leading expert of visual mathematics, Slavik Jablan writes in his seminal article "Modularity in Art", Necker or Koffka Cubes are "multi-ambiguous" objects: "they can be interpreted as three rhombuses with joint vertex, as convex or concave trihedron, or as a cube. If we accept its 'natural' 3D interpretation — a cube — then for a viewer there are three possible positions in space: upper, lower left, and lower right, having equal right to be a point of view. So, for the corresponding three directions, a Koffka cube represents a turning point. Having such multiple symmetry, it fully satisfies the conditions to be a suitable basic modular element." Jablan also calls attention to the connection between the Koffka cube and Thiery-figures (proposed at the end of 19th century) consisting of two Koffka cubes, Reuteswärd's impossible objects, the Penrose tribar (Figure 5), and artworks by Victor Vasarely (Figure 6), among other examples. All of them could be derived as modular structures from a Koffka cube, as "from Koffka cubes we could construct an infinite family of impossible figures. In the process of their growing, in every point, we have a possibility to proceed in three

directions, i.e. to choose each from six oriented ways”¹⁰ and exactly this is the underlying principle of Farkas’s impossible designs.

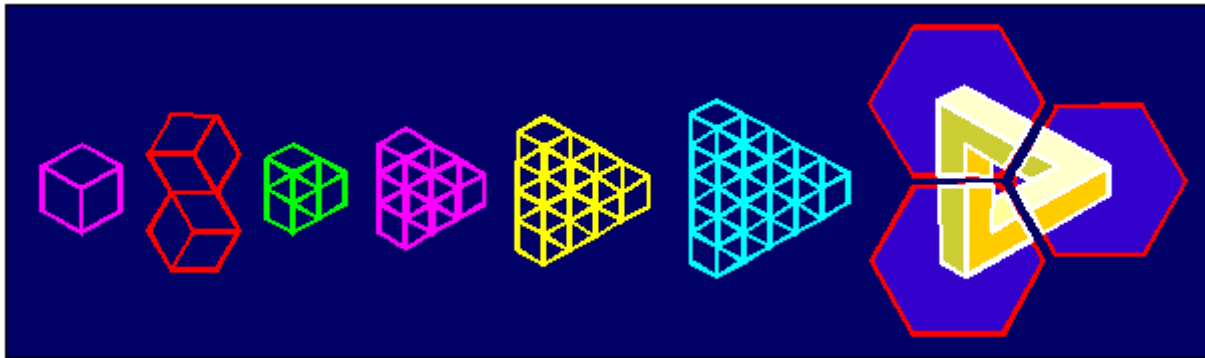


Figure 5: The “evolution” of the Penrose tri-bar from tessellated Koffka cubes.
Source: Jablan.

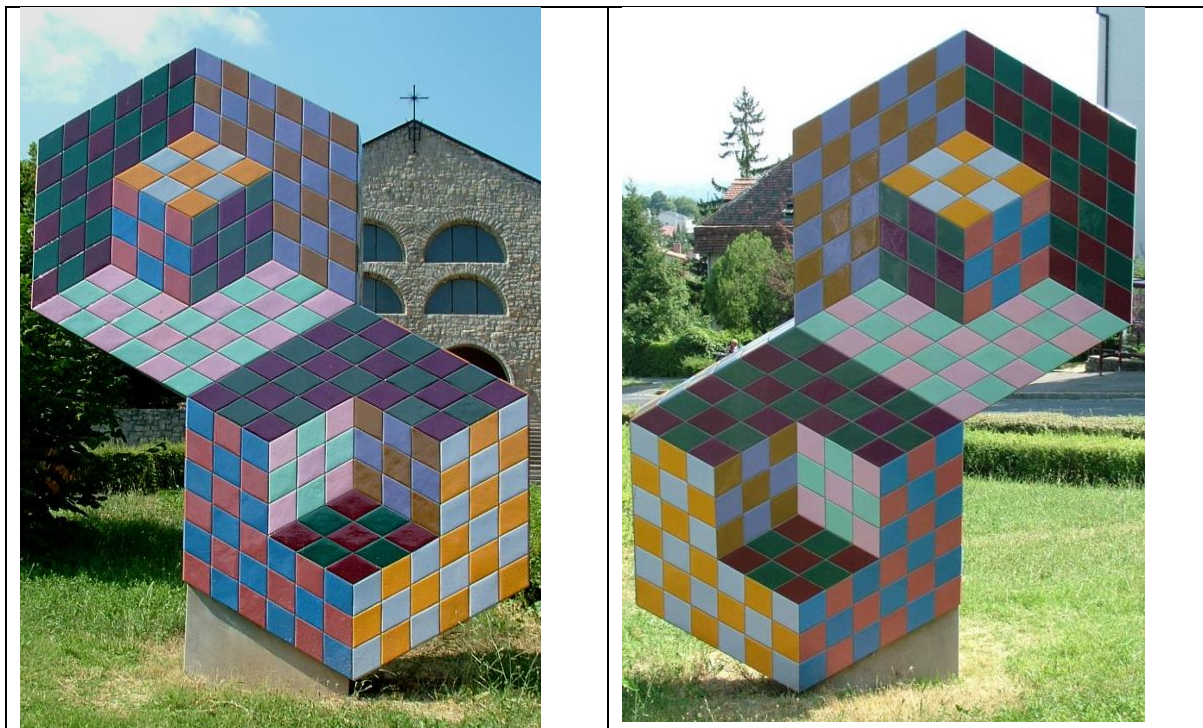


Figure 6: Koffka cubes on Victor Vasarely’s JEL sculpture.
(Pécs, Hungary, 1977)

F. Farkas’s workshop starts with the deep study of his impossible artworks and a free discussion on their gradually discovered geometrical properties. Then each student chooses a figure which they would like to re-create.

¹⁰ Slavik V. Jablan, “Modularity in Art”, *Modularity: Understanding the Development and Evolution of Natural Complex Systems*, Werner Callebaut and Diego Rasskin-Gutman, eds., Cambridge, MA: MIT Press, 2005, pp. 259–282, cf. esp. pp. 263–264.

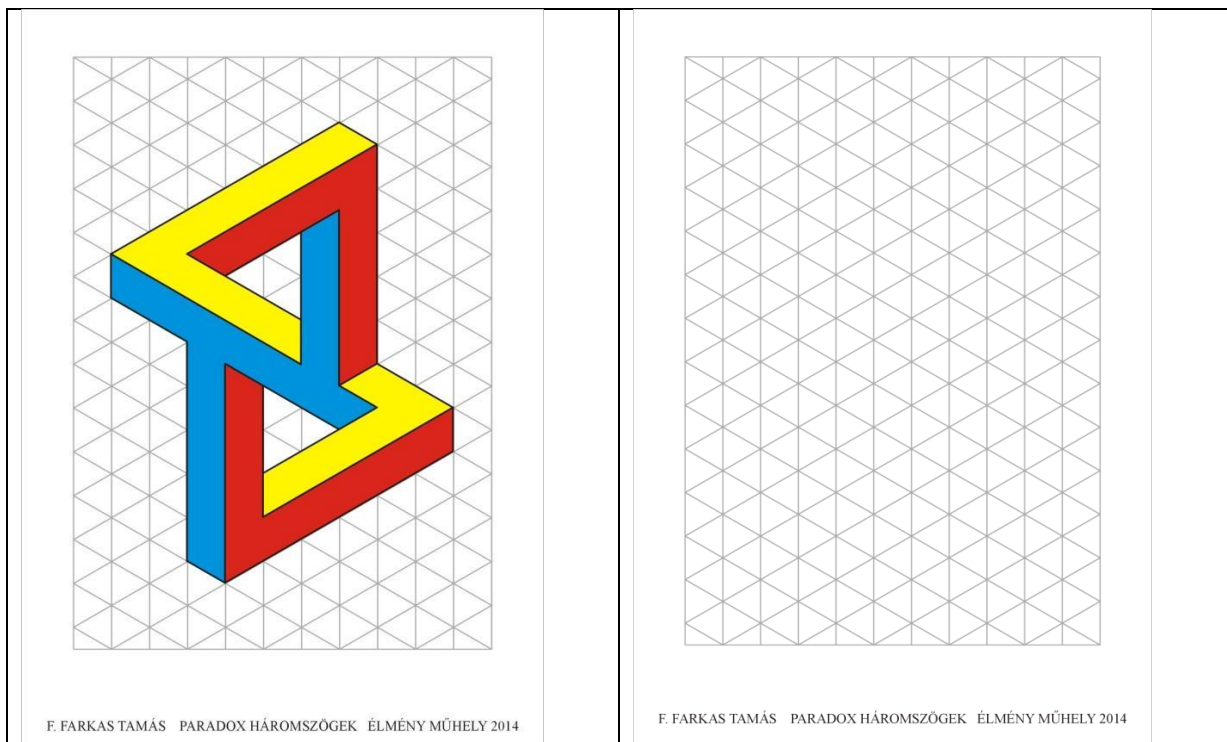


Figure 7: Tamás F. Farkas's templates for creating impossible figures with students. Template A (left), Template B (right). The two Penrose tri-bars of the composition on Template A are diagonally symmetric.

Copies of templates A and B belonging to the given artwork are printed according to the number of participating students (cf. Figure 7). The figure on template A is cut into parts along the black line bordering elements with a pair of scissors. Afterwards, the students' task is to recompose the figure on the raster net B belonging to the given form. The facilitator of the activity might draw the participants' attention to the fact that two elements of the same colour cannot border each other (cf. Figure 8). After completing a figure the participants give a verbal description of the object, defining their specific geometric features and discussing observations obtained during the construction together. The raster net B can also be used by students to draw the figure as well. After becoming familiar with the geometric features of impossible objects, students try to design their own impossible objects on the raster net B, by implementing their geometrical knowledge, developed at the workshop.



Figure 8: Building a “Koffka” pyramid as an introductory exercise in the Impossible Figures workshop with lower primary school pupils in F. Farkas Tamás’s workshop in Experience Workshop — International Movement of Experience-Centred Mathematics Education (www.experienceworkshop.hu) event at ANK School in Pécs.

Photo: Csaba József Szabó.

The experience-centered process of exploratory introduction to geometry problems related to impossible figures can be successfully supported by using Dynamic Geometry Softwares (DGS) such as the free-access GeoGebra (www.geogebra.org) to extend investigations and foster deeper understanding of impossible figures’ geometrical properties. GeoGebra is accessible, engaging, encourages students to further explore the geometrical situation, and provides opportunities for making and evaluating conjectures of geometrical results. Students can construct the image of Farkas’s impossible figures in GeoGebra and be used to study such questions as e.g. how many different shapes can be seen in the image (different colours, but same shapes not to be regarded as different)? What transformations have to be applied to re-create a figure from single modules? What kind of symmetries can you identify in each figure? etc.

Digital Games in Mathematics Education

Mathematics educational games (mathgames) are another option to introduce experiential approaches to mathematics teaching. They differ from the exercises described in the previous section in that they do not involve such hands-on connection to physical materials, but provide experiential practices through manipulation of virtual objects and environments (the similarity of educational computer games and hands-on approaches has been emphasized by, amongst others, Squire¹¹). Games are employed in mathematics classrooms

¹¹ Kurt Squire, “Changing The Game: What Happens When Video Games Enter the Classroom?”, *Innovate: Journal of Online Education*, 1(6), 2005, p. 5.

in various ways and to various extents. The main ways to employ games are, as Bragg sums it up:¹²

As reward for early finishers. This has not much to do with teaching mathematics, as the reward games are not necessarily even related to mathematics contentwise. The practice of letting pupils to play games as a reward, however, reflects the desirability of game play amongst them.

To enhance students' attitudes toward maths. This approach is directly related to the topic of this paper. There is a perceived situation of pupils not liking mathematics as a subject, and games as pedagogical means are considered to improve that attitude. The results are not necessarily always positive, neither what comes to improving the attitudes, nor the learning outcomes.

In repetitive practices such as learning arithmetical computation. One of the peculiarities of games is how they manage to create a symbolic reward system which is enough to lure the players in endless repetitive trial and error cycles. Even though the games are often highly frustrating (and in some cases even intentionally so) the prospective progression to a new game level builds motivation to go through the 'grinding'. This is exactly what is required in many computation tasks, and games like Ekapeli¹³ – a renowned Finnish education game for improving reading and calculating skills – are building on that feature. (Ekapeli is targeted especially for kids with dyscalculia, which puts extra demands for keeping up the motivation, and there is an elaborate system making the game adapt to player's individual skill level).

Mathgames, however, are used clearly less to teach problem solving and creative thinking in classrooms. As Kim and Chang have stated: "Although there is overall support for the idea that games have a positive effect on affective aspects of learning, there have been mixed research results regarding the role of games in promoting cognitive gains and academic achievement."¹⁴

There are many games, both educational as well as entertainment ones, with potential in this field. Here we discuss only three of them, *The Bridge* (by Ty Taylor and Mario Castañeda, 2013), and *The Monument Valley* (ustwogames, 2014), which are both based on Escherian visual paradoxes, and *Miegakure* (by Marc Ten Bosch, forthcoming), which takes place in a four-dimensional world. They all belong to the category of independent games, which means they are commercially distributed but produced by small teams of a few developers.

In this article we discuss only two ways to employ games for facilitating visual learning of mathematics here: (1) the use of visual paradoxes as game mechanic to facilitate proof construction in the case of *The Monument Valley* and *The Bridge*, and (2) the use of simulated game world in making a non-intuitive phenomenon such as four-dimensionality better graspable, in the case of *Miegakure*. These approaches are in line with the opinion of one of the most prominent proponents of educational use of games, James Paul Gee, who emphasizes the fundamental similarities between scientific simulations and the structure of games in general.¹⁵ Whereas digital games may lack some in the concreteness of the manipulation, they make the exploration of the situation and its specific characteristics even

¹² L. Bragg, "Students' conflicting attitudes towards games as a vehicle for learning mathematics: A methodological dilemma", *Mathematics Education Research Journal*, vol. 19, no. 1 (2007), pp. 29–44.

¹³ Ekapeli's website: <http://www.lukimat.fi/lukeminen/materiaalit/ekapeli/ekapeli-in-english-1>, last accessed 06. 02. 2015.

¹⁴ S. Kim and M. Chang, "Computer Games for the Math Achievement of Diverse Students" *Educational Technology & Society*, vol. 13, no. 3, (2010), pp. 224–232, cf. esp. p. 225.

¹⁵ James Paul Gee, "Why Are Video Games Good For Learning?", 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 3, last accessed 18.12.2014.

more easy and engaging, thus helping the pupil to build a strong understanding of the problem in much the same vein Gee is describing under his notion of “performance before competence” in regards to educational gaming.¹⁶

Visual Paradoxes as Game Mechanic

Games like *The Monument Valley* and *The Bridge* (see Figures 9 and 10) pose challenges based on visual paradoxes. The player frequently faces situations, where proceeding is apparently blocked. There are pathways abruptly ending, staircases leading to solid walls, and targets placed on such positions where no path exists. In order to proceed, the player has first to identify potential paradoxical structures. The player has to observe the game environment and decide which elements are the most promising to offer the needed scaffolding. Then, the player has to experiment with the options provided by the game interface in order to find a way to manipulate the game world successfully. It is usually considered as bad game design, if the player has to recede to the strategy of going through all available options more or less systemically, in order to eventually stumbling into the right solution, but from educational perspective even this kind of mechanical approach bears merit in helping the player to see the different aspects of the visual presentation. When the design is successful, the initial proceeding by surprises-through-mechanic-selections gives increasingly way to proceeding-through-reasoning when the player grasps the logic of the particular visual paradoxes employed in the game.

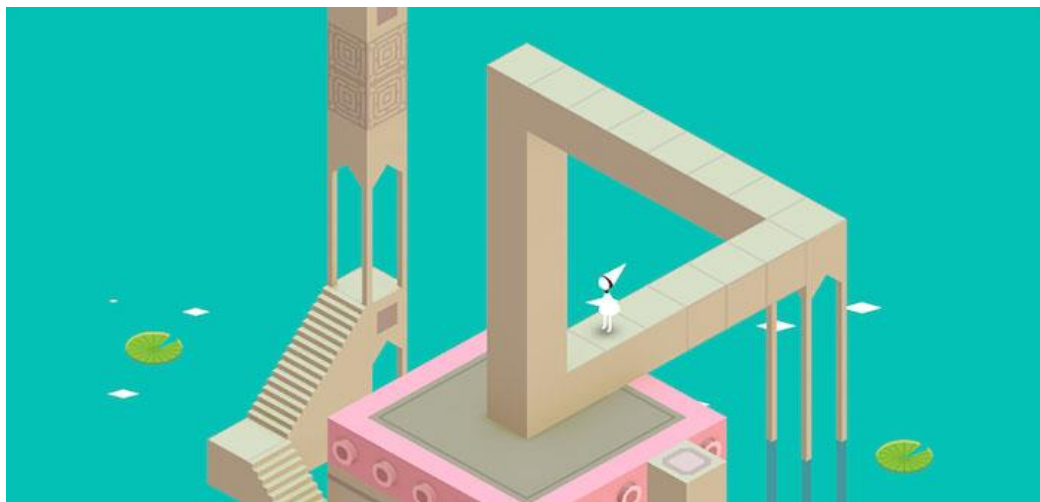


Figure 9: A scene with Penrose triangle in *The Monument Valley*. Source: Ustwo games.

When the player experiments with game environment, she builds up her understanding of the problem, or, as Kondratieva formulates it, she is “making large number of observations, making the ‘picture talk’”, as she is “search[ing] for the flaw in the initial understanding of the situation”. Intuition often helps in choosing the most promising directions in the initial phases of problem solving, but it is the very nature of visual paradoxes (as of paradoxes in general) that they are counter-intuitive. The process of going through a number of various alternatives in a systematic way, not precluding any alternatives but experimenting also with attempts that by first sight seem simply impossible,

¹⁶ James Paul Gee, “Why Are Video Games Good For Learning?”, 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 13, last accessed 18.12.2014.

bears two kinds of pedagogical potential. First, it is a way to build up an understanding of how a particular visual paradox is created, but even more importantly, it helps to build up a wholly new understanding of the world surrounding us, forcing first to reject the naturalistic assumptions and then expanding the pupil's skills to the extent that the solution can be found purely through reasoning. When this point is reached, the step necessary for deductive process is considerably eased: "...visual paradoxes helped students to develop a sense of the purpose of proofs by examining the links between the given information and the conclusion – the core of any deductive process. Their ability to understand and validate logical arguments was enhanced by the search for a flaw in the reasoning leading to a false conclusion."¹⁷

Thus, the game helps pupils to understand the particular problem, and consequently, to construct a corresponding mental model making it easier to understand how the formal proof is constructed. The game as such, however, does not teach the construction of formal proof itself, but this is still done by the teacher. The math games still face the same challenge as algebraic tiles and other such experiential materials, in that they should be properly embedded in classroom teaching in order to gain their full pedagogical potential. The games discussed in this section are not addressing any clear-cut topic in math curricula, which poses more requirements for the teachers in recognizing their usefulness in teaching specific topics. It is, then, a question of teacher education to provide capabilities to both recognize the pedagogical potential of non-educational games, and to implement them in the classroom teaching.

The game play in *The Bridge* is much more challenging than in *The Monument Valley*, and in addition to the purely logical puzzles of figuring out the visual paradoxes, the game requires also motoric skills and dexterity from the player. Using a game like *The Bridge* in classroom is clearly more challenging, in that it probably is not suitable for all pupils in any given class. On the other hand, with more engaging and demanding game play it may function better with avid digital game players.



Figure 10: A scene from *The Bridge*. Source: Ty Taylor and Mario Castañeda.

¹⁷ Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 8.

A Game for Learning the Logic of a Four-Dimensional Space

Miegakure is a game which takes place in a four-dimensional space. The graphics engine uses a four-coordinate system for each game position but, as it is not possible to present four spatial dimensions, the player sees a three dimensional projection of the four dimensional world. Moving around in a four dimensional space using three dimensional projection is tricky and highly counter-intuitive. In this sort of environment “puzzles happen naturally: they are just simple consequences of 4D space”, as the game designer Ten Bosch has stated in an interview.¹⁸

Miegakure is not primarily intended as an educational game, but there is a strong educational aspect in it, as evidenced by the designer: “Ever since people discovered the concept of a fourth dimension of space around a century and a half ago, they have tried to come up with what would be possible if space was actually four-dimensional. Walking »through« walls would be one of the simplest consequences of being able to move in 4D space. But what would it actually look like? *It's not often that watching a video-game trailer actually teaches you about real math.*”¹⁹

The higher dimensions challenge our intuitive notion of space and *Miegakure* provides a way to better make sense of such abstract concept. Even though it is not technically that difficult to make the calculations required for three dimensional projection of four dimensional space, the amount of calculations required to project areas with details in them quickly grows high. Furthermore, to really be able to investigate the four dimensional space, one has to constantly shift her perspective, resulting in even more calculations, as each shift changes the projected “three dimensional slice”. It is hard to fathom such hand-on experiment which would let the pupil experiment with three dimensional projections of four dimensional space as easily and in such an engaging way as this sort of game. *Miegakure* exemplifies how, in Gee’s words “the player must recognize problems and solve them from within the inside of the simulated world”²⁰ (cf. Figure 11).



Figure 11: A scene from *Miegakure*

Source: miegakure.com

¹⁸ Beckett Mufson, “This Man Is Using Math To Create An Impossible 4D Video Game”, The Creators Project Blog, <http://thecreatorsproject.vice.com/blog/impossible-4d-puzzle-game-defies-physics-and-lets-you-walk-through-walls> (2014), last accessed 02.12.2014.

¹⁹ Marc Ten Bosch, “New Trailer: How to walk through walls using the 4th Dimension”, Marc Ten Bosch Blog, 2014, <http://marctenbosch.com/news/2014/08>, last accessed 2.12.2014, emphasis in original.

²⁰ James Paul Gee, “Why Are Video Games Good For Learning?”, 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 3, last accessed 18.12.2014.

Miegakure shows one way how a certain mental model may be brought from purely computational to experiential mode, where it may be played and experimented with. Here again, however, teacher's role is central in explaining the logic behind the projections, even though the game includes also levels where the logic is demonstrated through analogy, using two dimensional projections of three dimensional space. Also, the game includes the shape of a modified 120-cell (Polydodecahedron), which is one of the four dimensional analogs of three dimensional Platonic solids, and the varying projections of this regular shape helps to understand the underlying logic.²¹

Conclusions

Elster and Ward, specialists of The Royal Conservatory of Music's *Learning Through the Arts* (LTTA) program in Toronto, Canada call the attention to the case of Escher "whose poor grades at school precluded a career in architecture, came to prominence in the 1950's when mathematicians recognized in his work an extraordinary visualization of mathematical principles, including plane and projective geometry, Euclidean geometry, and structure and mechanisms. Escher's work also embraces the notions of paradox and »impossible« figures, giving the viewer a means to consider not only the geometry of space but the logic of space. This is really quite extraordinary given that Escher failed mathematics in a traditional classroom, and that he did not pursue formal mathematics training after that. We are finding parallel breakthroughs in LTTA classrooms around the world."²² EWM's programs support this claim and confirm that visual learning of mathematics, especially learning mathematics through the arts can offer several benefits for every actors of the education process, including students, teachers and parents.

As successful international examples have shown, LTTA programs can "facilitate the development of analytical and problem-solving skills; stimulate natural curiosity; cultivate a broad range of thinking skills; make learning relevant for students of the many diverse cultural backgrounds that exist in today's schools; enhance teamwork; strengthen the ability to use and acquire information and to master different types of symbol systems; develop creative thinking skills and thereby access to higher order thinking skills; serve as a vehicle to help students make meaning of what they are learning"²³ and can increase parental engagement.²⁴

The creative, artistic, and playful approaches enable students to familiarize and better understand the abstractions and algebraically formulated regularities of mathematical thinking while also contributing to their skills in working with abstract notions and applying systems thinking in problem-solving and decision making. The approaches discussed here extend the regular classroom instruction in — at least — two essential ways: in their

²¹ Marc Ten Bosch, "What is that shape at the end of the trailer?", Marc Ten Bosch Blog, 2014, <http://marctenbosch.com/news/2014/04/what-is-that-shape-at-the-end-of-the-trailer-hd-screenshot>, last accessed 02.12.2014.

²² Angela Elster and Peggy Ward, "Learning Math Through the Arts", in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 173.

²³ Angela Elster and Peggy Ward, "Learning Math Through the Arts", in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 169.

²⁴ Angela Elster and Peggy Ward, "Learning Math Through the Arts", in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 171.

methods and in their thematics. By providing opportunities for the teacher to experiment with the role of a facilitator, EWM's workshops also let the students solve mathematical problems through playful participation and hands-on activities. Students and teachers, while testing their own creativity, perform such skills and abilities, which have remained latent in traditional classroom processes.

References

- [1] Homepage: www.experienceworkshop.hu, last accessed 06. 02. 2015.
- [2] Sándor Kabai et al., "Experience-workshop in mathematics and space education: joyful teaching program in Hungary", *43rd Lunar and Planetary Science Conference*, #2611, LPI, Houston, 2012.
- [3] György Darvas and Kristóf Fenyvesi, "Symmetries in Action! The Hungarian Experience Workshop Movement in the Education of Mathematics and Symmetry Studies through Arts", *Proceedings of the 12th International Conference of the Mathematics Education into the 21st Century Project*, Alan Rogerson, ed., Herceg Novi, Montenegro, 2014; György Darvas, "Preface", Catalogue of Tamás F. Farkas, Budapest: Demax, 2010, pp. 2–3.
- [4] Slavik Jablan and Kristóf Fenyvesi, "The Vasarely Playhouse: Look and Combine!", *Proceedings of Bridges 2011 World Conference*, Coimbra: Tessellation Publications, 2011, pp. 633–636.
- [5] Margo Kondratieva, "Understanding mathematics through resolution of paradoxes", *Mediterranean Journal for Research in Mathematics Education*, 6 (2007), pp. 127–138; Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, pp. 3–8.
- [6] Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 4.
- [7] Margo Kondratieva, "Geometrical Sophisms and Understanding of Mathematical Proofs", *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 5.
- [8] Bruno Ernst, *Adventures with Impossible Figures*, Tarquin Publications, 1986.
- [9] Bruno Ernst, *Adventures with Impossible Figures*, Tarquin Publications, 1986, pp. 10–15.
- [10] Slavik V. Jablan, "Modularity in Art", *Modularity: Understanding the Development and Evolution of Natural Complex Systems*, Werner Callebaut and Diego Rasskin-Gutman, eds., Cambridge, MA: MIT Press, 2005, pp. 259–282, cf. esp. pp. 263–264.
- [11] Kurt Squire, "Changing The Game: What Happens When Video Games Enter the Classroom?", *Innovate: Journal of Online Education*, 1(6), 2005, p. 5.
- [12] L. Bragg, "Students' conflicting attitudes towards games as a vehicle for learning mathematics: A methodological dilemma", *Mathematics Education Research Journal*, vol. 19, no. 1 (2007), pp. 29–44.
- [13] Ekapeli's website: <http://www.lukimat.fi/lukeminen/materiaalit/ekapeli/ekapeli-in-english-1>, last accessed 06. 02. 2015.
- [14] S. Kim and M. Chang, "Computer Games for the Math Achievement of Diverse Students" *Educational Technology & Society*, vol. 13, no. 3, (2010), pp. 224–232, cf. esp. p. 225.

- [15] James Paul Gee, “Why Are Video Games Good For Learning?”, 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 3, last accessed 18.12.2014.
- [16] James Paul Gee, “Why Are Video Games Good For Learning?”, 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 13, last accessed 18.12.2014.
- [17] Margo Kondratieva, “Geometrical Sophisms and Understanding of Mathematical Proofs”, *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, Fou-Lai Lin et al., eds., vol 2. The Department of Mathematics, National Taiwan Normal University Taipei, Taiwan, 2009, p. 8.
- [18] Beckett Mufson, “This Man Is Using Math To Create An Impossible 4D Video Game”, The Creators Project Blog, <http://thecreatorsproject.vice.com/blog/impossible-4d-puzzle-game-defies-physics-and-lets-you-walk-through-walls> (2014), last accessed 02.12.2014.
- [19] Marc Ten Bosch, “New Trailer: How to walk through walls using the 4th Dimension”, Marc Ten Bosch Blog, 2014, <http://marctenbosch.com/news/2014/08>, last accessed 2.12.2014, emphasis in original.
- [20] James Paul Gee, “Why Are Video Games Good For Learning?”, 2005, <http://www.academiccolab.org/resources/documents/MacArthur.pdf>, p. 3, last accessed 18.12.2014.
- [21] Marc Ten Bosch, “What is that shape at the end of the trailer?”, Marc Ten Bosch Blog, 2014, <http://marctenbosch.com/news/2014/04/what-is-that-shape-at-the-end-of-the-trailer-hd-screenshot>, last accessed 02.12.2014.
- [22] Angela Elster and Peggy Ward, “Learning Math Through the Arts”, in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 173.
- [23] Angela Elster and Peggy Ward, “Learning Math Through the Arts”, in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 169.
- [24] Angela Elster and Peggy Ward, “Learning Math Through the Arts”, in Michele Emmer, ed., *Mathematics and Culture V*, Berlin – Heidelberg – New York: Springer, 2007, p. 171.