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A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting

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Abstract

The aim of this paper is to introduce an enhanced incremental procedure that can be used for the numerical evaluation and reliable estimation of the limit load. A conventional incremental method of limit analysis is based on parametrization of the respective variational formulation by the loading parameter $\zeta \in (0, \zeta_{lim})$, where ζ_{lim} is generally unknown. The enhanced incremental procedure is operated in terms of an inverse mapping $\psi : \alpha \mapsto \zeta$ where the parameter α belongs to $(0, +\infty)$ and its physical meaning is work of applied forces at the equilibrium state. The function ψ is continuous, nondecreasing and its values tend to ζ_{lim} as $\alpha \to +\infty$. Reduction of the problem to a finite element subspace associated with a mesh $\mathcal{T}_h$ generates the discrete limit parameter $\zeta_{lim,h}$ and the discrete counterpart ψ_h to the function ψ. We prove pointwise convergence $\psi_h \to \psi$ and specify a class of yield functions for which $\zeta_{lim,h} \to \zeta_{lim}$. These convergence results enable to find reliable lower and upper bounds of ζ_{lim}. Numerical tests confirm computational efficiency of the suggested method.

Keywords: Variational problems with linear growth energy, incremental limit analysis, elastic-perfectly plastic problems, finite element approximation

1. Introduction

Elastic-perfectly plastic models belong among fundamental nonlinear models which are useful for estimation of yield strengths or failure zones in bodies caused by applied forces. Such models are mostly quasistatic (see, e.g., [4, 7, 13]) to catch the unloading phenomenon. Since we are only interested in monotone loading processes, this phenomenon can be neglected and the class of models based on the deformation theory of plasticity is adequate (see, e.g., [11, 12, 13, 16, 18]). The Hencky model associated with the von Mises yield criterion belongs to this class as well as other models with different yield conditions. Each model from this class leads to a static problem for a given load functional L representing the work of surface or volume forces. The problem can be formulated both in terms of stresses or displacements. These two approaches generate a couple of mutually dual problems.

The variational problem formulated in terms of stresses leads to minimization of a strictly convex, quadratic functional on the set of statically and plastically admissible stress fields. On the other hand, the stored energy functional appearing in the variational problem for displacements has only a linear growth at infinity with respect to the strain tensor or some components of this tensor. Existence of a finite limit load reflects specifics of this class of problems. Unlike other problems in continuum mechanics

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with superlinear growth of energy, exceeding of the limit load leads to absence of a solution satisfying the equilibrium equations and constitutive relations. Physically this means that under this load the body cannot exist as a consolidated object. Therefore, finding limit loads is an important problem in the theory of elasto–plastic materials and other close problems.

To introduce the limit load for the functional L the problem is usually parametrized at first. Instead of the fixed load, the set $\{\zeta L \mid \zeta \in \mathbb{R}_+\}$ of loads is considered. The limit value ζ_{lim} of the parameter is defined as a supremum of all $\zeta \geq 0$ for which the intersection of the sets of statically and plastically admissible stress fields is nonempty. In particular, no solution exists for the load ζL with $\zeta > \zeta_{lim}$.

There exist several approaches how to evaluate ζ_{lim}. The first type of methods is based on the use of a specific variational problem which characterizes directly the the limit state. It can be formulated either in terms of displacements (kinematical approach) or in terms of stresses (static approach). Both are mutually dual [3, 18]. As a computational method the static limit analysis has been used in [19], while the kinematic one in [1]. For example, the respective problem of kinematic limit analysis for the classical Hencky model with the von Mises condition reads as follows:

$$\zeta_{lim} = \inf_{\substack{v \in V,\, L(v)=1 \\ \mathrm{div}=0}} \int_\Omega |\varepsilon(v)|\, dx,$$

where $\mathbb{V}$ is a subspace of $H^1(\Omega;\mathbb{R}^3)$ of functions vanishing on the Dirichlet part of the boundary (see notation of Section 2). However, this problem is not simple for numerical analysis because it is related to a nondifferentiable functional and contains the divergence free constraint. The respective numerical approaches developed to overcome these difficulties often use saddle point formulations with augmented Lagrangians (see, e.g., [1, 3]). Other methods use techniques developed for minimization of nondifferentiable functionals.

The classical approach uses incremental techniques to enlarge ζ up to its limit value [14, 20]. The load increments have to be chosen adaptively since the value of ζ_{lim} is not known. The incremental limit analysis is usually combined with the standard finite element method and the resulting parametrized problem $(\mathcal{P}_h)_\zeta$ is then solved in terms of displacements. The main drawback of this approach is that the discrete limit value $\zeta_{lim,h}$ can overestimate ζ_{lim} and convergence of $\{\zeta_{lim,h}\}_h$ to ζ_{lim} is not guaranteed in general.

Besides ζ_{lim}, the incremental approach enables to detect other interesting thresholds on the loading path that represent global material response, namely, $\zeta_{e,h}$ - the end of elasticity and $\zeta_{prop,h}$ - the limit of proportionality. For $\zeta \leq \zeta_{e,h}$, the response is purely elastic (linear) and for $\zeta \in [\zeta_{prop,h}, \zeta_{lim,h}]$, the response is strongly nonlinear. To investigate global material response, it is necessary to introduce a quantity α depending on $\zeta < \zeta_{lim,h}$. For example, α can represent a computed displacement at a point in which the body response is the most sensitive on the applied load. Examples of such α-ζ curves are introduced, e.g., in [4, Section 7,8].

In [17], the response parameter α has been introduced for the Hencky problem and the linear simplicial ($P1$) elements as follows: $\alpha = L(u_{h,\zeta})$ where $u_{h,\zeta}$ denotes a solution of $(\mathcal{P}_h)_\zeta$ for $\zeta < \zeta_{lim,h}$. This parameter is universal for any load and geometry. Moreover, there exists a function $\psi_h : \alpha \mapsto \zeta$ that is continuous, nondecreasing and satisfying $\psi_h(\alpha) \to \zeta_{lim,h}$ as $\alpha \to +\infty$. Further, for a given value of α, a minimization problem $(\mathcal{P}_h)^\alpha$ for the stored strain energy functional subject to the constraint $L(v) = \alpha$ has been derived. Its solution coincides with a solution to problem $(\mathcal{P}_h)_\zeta$ for $\zeta = \psi_h(\alpha)$ and thus the loading process can be controlled indirectly through the parameter α. Consequently, in [2], suitable numerical methods for both problems, $(\mathcal{P}_h)_\zeta$ and $(\mathcal{P}_h)^\alpha$, have been proposed and theoretically justified. Further, the load incremental methods controlled through ζ and α have been compared there.

The aim of this paper is to get reliable estimates of ζ_{lim} using the incremental procedure. To this end, we introduce a continuous, nondecreasing function $\psi : \mathbb{R}_+ \to (0, \zeta_{lim})$ satisfying $\psi(\alpha) \to \zeta_{lim}$ as $\alpha \to +\infty$. In comparison to [2, 17], the function ψ is defined within a continuous setting of the problem and also for a general yield criterion. The derivation of ψ however is not straightforward owing to the fact that the primal formulation is not well-posed on classical Sobolev spaces. Therefore the dual formulation of the problem in terms of stresses will be used. Further, it is considered the discrete counterpart ψ_h of ψ within the $P1$ elements. In case of the von Mises yield criterion, the definition of ψ_h coincides with [2, 17]. From the computational point of view, it is crucial to show that $\lim_{h\to 0_+} \psi_h(\alpha) = \psi(\alpha)$ for any $\alpha \geq 0$ and use the estimate $\psi(\alpha) \leq \zeta_{lim} \leq \zeta_{lim,h}$. We also specify a class of yield functions for which $\zeta_{lim,h} \to \zeta_{lim}$ holds.

The paper is organized as follows: In Section 2, we introduce basic notation, define elasto-plastic problems, and recall some results concerning properties of solutions. In Section 3, the loading parameters ζ and α are introduced. Then the function $\psi : \alpha \mapsto \zeta$ is constructed and its properties are established. In Section 4, we formulate problems in terms of stresses and displacements related to a prescribed value of α. Section 5 is devoted to standard finite element discretizations of the problems and to convergence analysis. Finally, in Section 6, we present two examples with different yield functions and compute lower and upper bounds of the limit load using the suggested incremental procedure.

2. Elastic-perfectly plastic problem based on the deformation theory of plasticity

We consider an elasto-plastic body occupying a bounded domain $\Omega \subseteq \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. It is assumed that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D and Γ_N are open and disjoint sets, Γ_D has a positive surface measure. Surface tractions of density f are applied on Γ_N and the body is subject to a volume force F.

For the sake of simplicity, we assume that the material is homogeneous. Then, the generalized Hooke's law is represented by the tensor C, which does not depend on $x \in \Omega$ and satisfies the following conditions of symmetry and positivity:

$$\begin{gathered} C\eta \in \mathbb{R}^{3\times 3}_{sym} \qquad \forall \eta \in \mathbb{R}^{3\times 3}_{sym}, \\ C\eta : \xi = \eta : C\xi \qquad \forall \eta, \xi \in \mathbb{R}^{3\times 3}_{sym}, \\ \exists \delta > 0 : \quad C\eta : \eta \geq \delta(\eta : \eta) \qquad \forall \eta \in \mathbb{R}^{3\times 3}_{sym}, \end{gathered}$$

where $\mathbb{R}^{3\times 3}_{sym}$ is the space of all symmetric, (3×3) matrices and $\eta : \xi = \eta_{ij}\xi_{ij}$ denotes the scalar product on $\mathbb{R}^{3\times 3}_{sym}$.

By $S := L^2(\Omega; \mathbb{R}^{3\times 3}_{sym})$, we denote the set of symmetric tensor valued functions with square summable coefficients representing stress and strain fields. On S, we define the scalar product

$$\langle \tau, e \rangle = \int_\Omega \tau : e \,\mathrm{d}x, \quad \tau, e \in S,$$

and the respective norm $\|\tau\| = \langle \tau, \tau \rangle^{1/2}$. Also, we use equivalent norms suitable for stress (τ) and strain (e) fields, respectively:

$$\|\tau\|_{C^{-1}} := \langle C^{-1}\tau, \tau \rangle^{1/2}, \quad \|e\|_C = \langle Ce, e \rangle^{1/2}.$$

Further, let

$$\mathbb{V} := \left\{ v \in H^1(\Omega; \mathbb{R}^3) \mid \ v = 0 \text{ on } \Gamma_D \right\}$$

denote the space of kinematically admissible displacements and

$$L(v) := \int_\Omega F\cdot v \mathrm{d}x + \int_{\Gamma_N} f\cdot v \mathrm{d}s, \quad v \in \mathbb{V}$$

be the load functional. We assume that

$$(L_1) \quad F \in L^2(\Omega;\mathbb{R}^3), \;\; f \in L^2(\Gamma_N;\mathbb{R}^3),$$
$$(L_2) \quad \|F\|_{L^2(\Omega;\mathbb{R}^3)} + \|f\|_{L^2(\Gamma_N;\mathbb{R}^3)} > 0.$$

The following closed, convex sets represent statically and plastically admissible stress fields, respectively:

$$\Lambda_L := \{\tau \in S \mid \langle \tau, \varepsilon(v)\rangle = L(v) \quad \forall v \in \mathbb{V}\},$$
$$P := \{\tau \in S \mid \Phi(\tau(x)) \le \gamma \quad \text{for a. a. } x \in \Omega\}.$$

Here, $\Phi:\ \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}$ is a continuous, convex yield function such that $\Phi(0) = 0$, $\gamma > 0$ represents the initial yield stress (which is constant in Ω due to the homogeneity assumption) and $\varepsilon(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$ is the linearized strain tensor corresponding to the displacement v.

In accordance with the Haar-Karman variational principle, the actual stress is a minimizer of the variational problem:

$$(\mathcal{P}^*) \qquad \text{find } \sigma \in \Lambda_L \cap P: \quad \mathcal{I}(\sigma) \le \mathcal{I}(\tau) \quad \forall \tau \in \Lambda_L \cap P,$$

where

$$\mathcal{I}(\tau) := \frac{1}{2}\|\tau\|^2_{C^{-1}}, \quad \tau \in S.$$

Problem $(\mathcal{P}^*)$ has a unique solution if and only if $\Lambda_L \cap P \neq \emptyset$.

The corresponding dual problem is formulated in terms of displacements. It has the form:

$$(\mathcal{P}) \qquad \text{find } u \in \mathbb{V}: \quad J(u) \le J(v) \quad \forall v \in \mathbb{V},$$

where

$$J(v) := \Psi(\varepsilon(v)) - L(v), \quad v \in \mathbb{V},$$

$$\Psi(e) := \sup_{\tau \in P}\left\{\langle \tau, e\rangle - \frac{1}{2}\|\tau\|^2_{C^{-1}}\right\} = -\frac{1}{2}\|\Sigma(e)\|^2_{C^{-1}} + \langle \Sigma(e), e\rangle \qquad \forall e \in S \tag{2.1}$$

and $\Sigma: S \to S$ is defined by $\Sigma(e) = \Pi(Ce)$ for any $e \in S$. Here Π denotes the projection of S on P with respect to the scalar product $\langle C^{-1}\sigma, \tau\rangle$. In addition, Σ is the Fréchet derivative of Ψ, i.e. $\Sigma(e) = \mathbb{D}\Psi(e)$ for any $e \in S$. The functional Ψ is convex and differentiable but has only a linear growth at infinity. Therefore, existence of a solution to $(\mathcal{P})$ is not guaranteed in $\mathbb{V}$ or other Sobolev spaces.

If $\Lambda_L \cap P \neq \emptyset$ then $(\mathcal{P})$ and $(\mathcal{P}^*)$ have finite infima and the duality relation

$$\inf_{v \in \mathbb{V}} J(v) = \sup_{\tau \in \Lambda_L \cap P}\{-\mathcal{I}(\tau)\}. \tag{2.2}$$

holds. If $(\mathcal{P})$ has a solution u then it satisfies the variational equation

$$\langle \sigma, \varepsilon(v)\rangle = L(v) \quad \forall v \in \mathbb{V}, \tag{2.3}$$

where $\sigma := \Sigma(\varepsilon(u))$ is the unique solution to $(\mathcal{P}^*)$.

Remark 2.1. In the special case, $P = S$, the problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ lead to well-known primal and dual formulations of elasticity problems:

$$(\mathcal{P}_e) \qquad \text{find } u_e \in \mathbb{V}: \quad J_e(u_e) \leq J_e(v) \quad \forall v \in \mathbb{V},$$

where

$$J_e(v) := \frac{1}{2}\|\varepsilon(v)\|_C^2 - L(v), \quad v \in \mathbb{V},$$

and

$$(\mathcal{P}_e^*) \qquad \text{find } \sigma_e \in \Lambda_L: \quad \mathcal{I}(\sigma_e) \leq \mathcal{I}(\tau) \quad \forall \tau \in \Lambda_L \ \text{(Castigliano's principle)},$$

respectively. Both problems have unique solutions and $C\varepsilon(u_e) = \sigma_e$. Notice that if $C\varepsilon(u_e) \in P$ then $\Sigma(\varepsilon(u_e)) = C\varepsilon(u_e)$ and u_e also solves $(\mathcal{P})$.

3. Parametrization of the problem

Problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ are defined for a prescribed load functional L. Henceforth, we consider a one parametric family of loads ζL, where $\zeta \in \mathbb{R}_+$. Therefore, we use notation $(\mathcal{P})_\zeta$, $(\mathcal{P}^*)_\zeta$, $(\mathcal{P}_e)_\zeta$, $(\mathcal{P}_e^*)_\zeta$, $\Lambda_{\zeta L}$, and J_ζ instead of $(\mathcal{P})$, $(\mathcal{P}^*)$, $(\mathcal{P}_e)$, $(\mathcal{P}_e^*)$, Λ_L, and J, respectively.

The limit load parameter ζ_{lim} is defined by

$$\zeta_{lim} := \sup \mathcal{D}, \quad \mathcal{D} := \{\zeta \in \mathbb{R}_+ \mid \Lambda_{\zeta L} \cap P \neq \emptyset\}.$$

Notice that, in some cases, ζ_{lim} may be infinite. However, in the majority of cases, the value of ζ_{lim} is finite. From now on, we assume that

$$(L3) \qquad \zeta_{lim} > 0.$$

Problem $(\mathcal{P}^*)_\zeta$ has a unique solution for any $\zeta \in \mathcal{D}$. Depending on the definition of the yield function Φ, we may have one of the following two situations:

$$(a) \ \ \mathcal{D} = [0, \zeta_{lim}) \quad \text{or} \quad (b) \ \ \mathcal{D} = [0, \zeta_{lim}]. \tag{3.1}$$

In general, it is not known, whether $\zeta_{lim} \in \mathcal{D}$, i.e. $\Lambda_{\zeta_{lim} L} \cap P \neq \emptyset$. This is true, for example, for the von Mises or Tresca criterion (see [18]).

From the practical point of view it is very important to know the value of ζ_{lim}. The related problem of limit analysis has been considered in [3, 16, 18] and publications cited therein. This minimization problem can be solved independently of the original plasticity problem by various numerical methods (see, e.g., [1, 3]). However, solving this problem leads to rather complicated numerical procedures.

The aim of this paper is to propose and justify a robust way of finding ζ_{lim}, which is based on a different loading parameter. The first principal idea is to introduce a nonnegative function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ as follows:

$$\phi(\zeta) = \begin{cases} \mathcal{I}(\sigma(\zeta)), & \zeta \in \mathcal{D}, \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.2}$$

Here, $\sigma := \sigma(\zeta)$ denotes the unique solution to $(\mathcal{P}^*)_\zeta$. Properties of ϕ are summarized in the following lemma.

Lemma 3.1. *Let the assumptions $(L1) - (L3)$ be satisfied and let $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by (3.2). Then,*

ϕ is a nonnegative, strictly convex and increasing function in $\mathcal{D}$. Moreover,

$$\phi(\zeta_0) \leq \left(\frac{\zeta_0}{\zeta_1}\right)^2 \phi(\zeta_1) \quad \forall \zeta_0, \zeta_1 \in \mathcal{D},\ \zeta_0 < \zeta_1. \tag{3.3}$$

Proof. Let ζ_0, ζ_1 be as in (3.3) and

$$\zeta_\lambda := (1-\lambda)\zeta_0 + \lambda\zeta_1, \qquad \lambda \in [0,1].$$

Then $(1-\lambda)\sigma(\zeta_0)+\lambda\sigma(\zeta_1) \in \Lambda_{\zeta_\lambda L} \cap P$, where $\sigma(\zeta_\theta)$ denotes the solution to $(\mathcal{P}^*)_{\zeta_\theta}$, $\theta \in [0,1]$. Consequently,

$$\mathcal{I}(\sigma(\zeta_\lambda)) \leq \mathcal{I}((1-\lambda)\sigma(\zeta_0) + \lambda\sigma(\zeta_1)) \leq (1-\lambda)\mathcal{I}(\sigma(\zeta_0)) + \lambda\mathcal{I}(\sigma(\zeta_1)). \tag{3.4}$$

Notice that the strict inequality holds in (3.4) for $\lambda \in (0,1)$ as $\sigma(\zeta_0) \neq \sigma(\zeta_1)$ in view of the assumption $(L2)$. Thus, ϕ is convex on $\mathbb{R}$ and strictly convex on $\mathcal{D}$.

From the definition of the yield function Φ, it follows that $\frac{\zeta_0}{\zeta_1}\sigma(\zeta_1) \in \Lambda_{\zeta_0 L} \cap P$. Therefore, we have:

$$\phi(\zeta_0) = \mathcal{I}(\sigma(\zeta_0)) \leq \mathcal{I}\left(\frac{\zeta_0}{\zeta_1}\sigma(\zeta_1)\right) = \left(\frac{\zeta_0}{\zeta_1}\right)^2 \phi(\zeta_1).$$

Hence, (3.3) is proved and, since, $\sigma(\zeta_1) \neq 0$ we conclude that ϕ is an increasing function on $\mathcal{D}$. □

Lemma 3.2. *Let $\zeta_{lim} \notin \mathcal{D}$. Then*

$$\lim_{\zeta \to \zeta_{lim}^-} \phi(\zeta) = +\infty. \tag{3.5}$$

Proof. If $\zeta_{lim} = +\infty$ then (3.5) follows from (3.3). Let

$$\zeta_{lim} < +\infty$$

and suppose that $\lim_{\zeta \to \zeta_{lim}^-} \phi(\zeta) \in \mathbb{R}^1_+$. Then there exist sequences $\{\zeta_j\}$, $\{\sigma(\zeta_j)\}$ and an element $\bar{\sigma} \in S$ such that

$$\zeta_j \to \zeta_{lim}^-, \quad \sigma(\zeta_j) \rightharpoonup \bar{\sigma} \text{ in } S, \quad j \to +\infty.$$

In addition, $\bar{\sigma} \in \Lambda_{\zeta_{lim} L} \cap P$ which contradicts the assumption. □

Lemma 3.3. *The function ϕ defined by (3.2) is continuous in $\mathcal{D}$.*

Proof. Continuity of ϕ in $\operatorname{int}\mathcal{D}$ follows from its convexity. From (3.3), we see that

$$\lim_{\zeta_0 \to 0_+} \phi(\zeta_0) = 0 = \mathcal{I}(\sigma(0)) = \phi(0).$$

Let $\zeta_{lim} \in \mathcal{D}$ and

$$\lim_{\zeta \to \zeta_{lim}^-} \phi(\zeta) = c \in \mathbb{R}_+.$$

To show that $c = \phi(\zeta_{lim})$ we proceed as in Lemma 3.2. Let $\zeta_j \to \zeta_{lim}^-$ and $\sigma(\zeta_j) \rightharpoonup \bar{\sigma} \in \Lambda_{\zeta_{lim} L} \cap P$. Let $\tau \in \Lambda_{\zeta_{lim} L} \cap P$ be arbitrary and set $\tau_j = \frac{\zeta_j}{\zeta_{lim}}\tau \in \Lambda_{\zeta_j L} \cap P$. Then $\tau_j \to \tau$ in S and from the definition of $(\mathcal{P})^*_{\zeta_j}$ we have

$$\phi(\zeta_j) = \mathcal{I}(\sigma(\zeta_j)) \leq \mathcal{I}(\tau_j).$$

Hence,

$$\mathcal{I}(\bar{\sigma}) \leq \liminf_{j\to+\infty} \mathcal{I}(\sigma(\zeta_j)) \leq \lim_{j\to+\infty} \mathcal{I}(\tau_j) = \mathcal{I}(\tau),$$

i.e. $\bar{\sigma} = \sigma(\zeta_{lim})$ proving that $\phi(\zeta_{lim}) = \mathcal{I}(\sigma(\zeta_{lim})) \leq c$. The opposite inequality $\phi(\zeta_{lim}) \geq c$ follows from monotonicity of ϕ. □

Remark 3.1. It is worth noting that:

a) $\phi(\zeta) = \zeta^2 \mathcal{I}(\sigma_e)$ if $\zeta \in [0, \zeta_e]$, where

$$\zeta_e := \sup\{\zeta \in \mathbb{R}_+ \mid \zeta C\varepsilon(u_e) \in P\}, \quad u_e \text{ solves } (\mathcal{P}_e).$$

b) (3.3) ensures a quadratic growth of ϕ at infinity if $\zeta_{lim} = +\infty$.

Now, we introduce a new parameter α, which plays a crucial role in forthcoming analysis. We set

$$\left.\begin{array}{lll} \alpha = 0 & \text{if} & \zeta = 0, \\ \alpha \in \partial\phi(\zeta) & \text{if} & \zeta \in \mathcal{D} \setminus \{0\} \end{array}\right\} \tag{3.6}$$

From monotonicity of ϕ, it follows that $\partial\phi(\zeta) \subset (0, +\infty)$ for any $\zeta \in \mathcal{D} \setminus \{0\}$. Moreover,

$$\bigcup_{\zeta\in\mathcal{D}\setminus\{0\}} \partial\phi(\zeta) = (0, +\infty). \tag{3.7}$$

Indeed, from the definition of the subgradient of ϕ at ζ we know that $\alpha \in \partial\phi(\zeta)$ if and only if

$$\phi(\zeta) - \alpha\zeta \leq \phi(\tilde{\zeta}) - \alpha\tilde{\zeta} \quad \forall \tilde{\zeta} \in \mathcal{D}. \tag{3.8}$$

From Lemma 3.1 - 3.3 and Remark 3.1 *b)* we know that the function $\tilde{\zeta} \mapsto \phi(\tilde{\zeta}) - \alpha\tilde{\zeta}$ has a unique minimizer ζ in $\mathcal{D}$ for any $\alpha \in [0, +\infty)$ so that (3.7) holds. This fact enables us to define the function $\psi : \mathbb{R}_+ \to \mathcal{D}$, $\psi : \alpha \mapsto \zeta$, where $\zeta = \psi(\alpha) \in \mathcal{D}$ is the unique solution of (3.8) for given α. In the next theorem, we establish some useful properties of ψ.

Theorem 3.1. *Let the assumptions* $(L1) - (L3)$ *be satisfied. Then*

(i) ψ *is continuous and nondecreasing in* $\mathbb{R}_+$*;*

(ii) $\psi(\alpha) \to \zeta_{lim}$ *as* $\alpha \to +\infty$.

Proof. Let $\alpha > 0$ be given and ϕ^* be the Legendre-Fenchel transformation of ϕ. It is well known that ϕ^* is a convex function in $\mathbb{R}_+$ and $(3.6)_2$ holds if and only if $\zeta \in \partial\phi^*(\alpha)$. Since $\zeta = \psi(\alpha)$, it holds that $\partial\phi^*(\alpha)$ is singleton and $(\phi^*)'(\alpha) = \psi(\alpha) \geq 0$. Therefore, convexity and differentiability of ϕ^* in $\mathbb{R}_+$ entail that ψ is continuous and nondecreasing in $\mathbb{R}_+$ and (i) holds.

By (i), there exists $\zeta_{max} \leq \zeta_{lim}$ such that $\zeta_{max} = \lim_{\alpha\to+\infty} \psi(\alpha)$. Suppose that

$$\lim_{\alpha\to+\infty} \psi(\alpha) = \zeta_{max} < \zeta_{lim}. \tag{3.9}$$

Then $\phi(\psi(.))$ is bounded on $\mathbb{R}_+$ and

$$\lim_{\alpha\to+\infty} \psi(\alpha) = \lim_{\alpha\to+\infty} \left\{\psi(\alpha) - \frac{\phi(\psi(\alpha))}{\alpha}\right\} \overset{(3.8)}{=} \lim_{\alpha\to+\infty} \sup_{\tilde{\zeta}\in\mathcal{D}} \left\{\tilde{\zeta} - \frac{\phi(\tilde{\zeta})}{\alpha}\right\} \geq \lim_{\alpha\to+\infty} \left\{\hat{\zeta} - \frac{\phi(\hat{\zeta})}{\alpha}\right\} = \hat{\zeta}$$

holds for any $\hat{\zeta} \in \mathcal{D}$. The choice $\hat{\zeta} \in (\zeta_{max}, \zeta_{lim})$ contradicts (3.9) and thus (ii) holds. $\square$

Remark 3.2. It is easy to show that

$$\psi(\alpha) = \frac{1}{2\mathcal{I}(\sigma_e)}\alpha \qquad \forall \alpha \in [0, \alpha_e],$$

where $\alpha_e = 2\zeta_e \mathcal{I}(\sigma_e)$, σ_e solves $(\mathcal{P}_e^*)$ and ζ_e is the same as in Remark 3.1.

Figure 1 depicts three possible cases of the behaviour of ϕ, ψ for $\zeta \to \zeta_{lim}$, and $\alpha \to +\infty$, respectively.

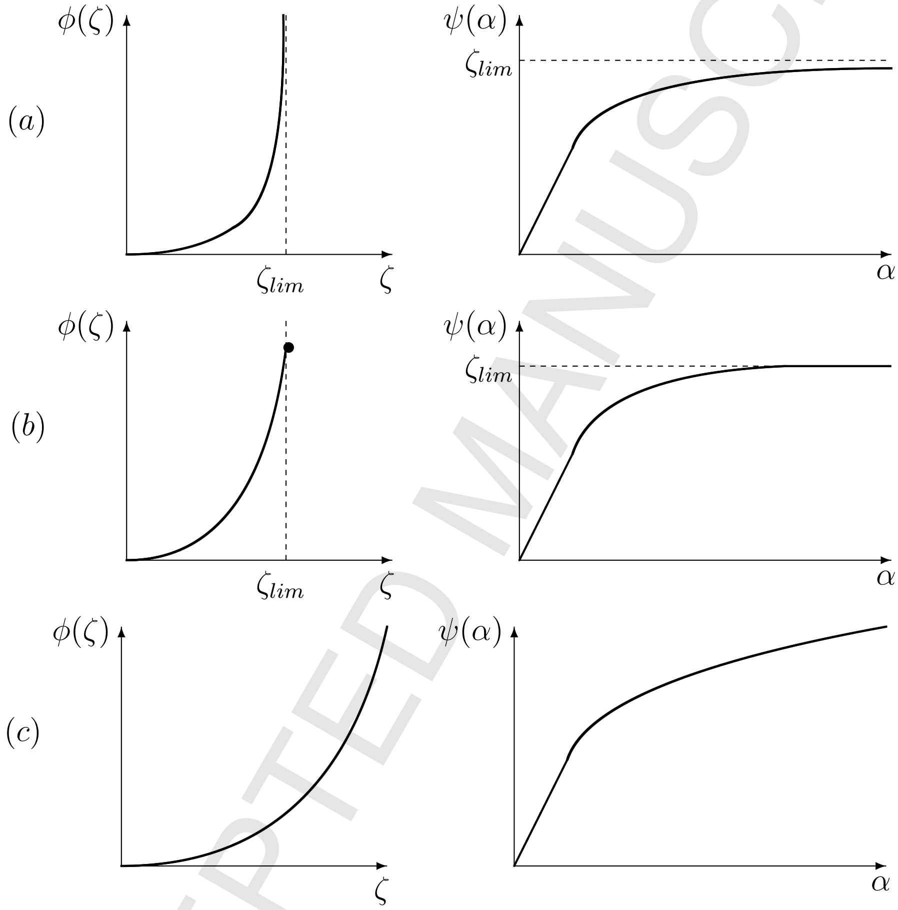

Figure 1: Graphs of ϕ and ψ: (a) $\partial\phi(\zeta_{lim}) = \emptyset$, (b) $\partial\phi(\zeta_{lim}) \neq \emptyset$, (c) $\zeta_{lim} = +\infty$.

4. Stress and displacement problems for given $\alpha \in \mathbb{R}_+$

In this section, we formulate new variational problems in terms of stresses and displacements enabling us to compute function values $\psi(\alpha)$ for $\alpha \in \mathbb{R}_+$. The parameter α will be used to control the loading process and to get the respective loading path graph$[\psi]$ for a larger class of yield functions than in [17].

To derive the formulation in terms of stresses, we introduce the following set:

$$\tilde{\Lambda}_L = \bigcup_{\tilde{\zeta} \in \mathbb{R}_+} \Lambda_{\tilde{\zeta}L} = \{\tilde{\zeta}\tau \mid \tilde{\zeta} \in \mathbb{R}_+,\ \tau \in \Lambda_L\}.$$

Clearly, $\tilde{\Lambda}_L$ is a closed, convex and non-empty subset of S and for any $\tau \in \tilde{\Lambda}_L$ there exists a unique loading parameter $\tilde{\zeta}$ such that $\tau \in \Lambda_{\tilde{\zeta}L}$ owing to $(L2)$. To stress that $\tau \in \Lambda_{\tilde{\zeta}L}$ with $\tilde{\zeta} \in \mathbb{R}_+$ and using the fact that such $\tilde{\zeta}$ is unique, we shall write $\tilde{\zeta} = \omega(\tau)$ in what follows. It is readily seen that the function

$$\omega : \tilde{\Lambda}_L \to \mathbb{R}_+$$

is concave in $\tilde{\Lambda}_L$ and satisfies the relation

$$\omega(\lambda\tau_1 + (1-\lambda)\tau_2) = \lambda\omega(\tau_1) + (1-\lambda)\omega(\tau_2) \quad \forall \tau_1, \tau_2 \in \tilde{\Lambda}_L,\ \forall \lambda \in [0,1].$$

Moreover,

$$\bigcup_{\tilde{\zeta}\in\mathcal{D}} \Lambda_{\tilde{\zeta}L} \cap P = \tilde{\Lambda}_L \cap P. \tag{4.1}$$

Let $\alpha > 0$ be given and $\zeta = \psi(\alpha)$. Then,

$$\begin{aligned}
\phi(\zeta) - \alpha\zeta &\overset{(3.8)}{=} \inf_{\tilde{\zeta}\geq 0}\{\phi(\tilde{\zeta}) - \alpha\tilde{\zeta}\} = \inf_{\tilde{\zeta}\in\mathcal{D}}\left\{\inf_{\tau\in\Lambda_{\tilde{\zeta}L}\cap P} \mathcal{I}(\tau) - \alpha\tilde{\zeta}\right\} = \\
&= \inf_{\tilde{\zeta}\in\mathcal{D}} \inf_{\tau\in\Lambda_{\tilde{\zeta}L}\cap P} \{\mathcal{I}(\tau) - \alpha\omega(\tau)\} = \inf_{\tau\in\tilde{\Lambda}_L\cap P} \{\mathcal{I}(\tau) - \alpha\omega(\tau)\}
\end{aligned}$$

using the definition of ω and (4.1).

On basis of this result we formulate the following problem in terms of stresses: given $\alpha \geq 0$,

$$(\mathcal{P}^*)^\alpha \qquad \text{find } \sigma := \sigma(\alpha) \in \tilde{\Lambda}_L \cap P: \quad \mathcal{I}(\sigma) - \alpha\omega(\sigma) \leq \mathcal{I}(\tau) - \alpha\omega(\tau) \quad \forall \tau \in \tilde{\Lambda}_L \cap P.$$

Properties of the functions $\mathcal{I}$ and ω ensure that for any $\alpha \geq 0$ problem $(\mathcal{P}^*)^\alpha$ has a unique solution σ. Moreover, $\zeta = \psi(\alpha) = \omega(\sigma)$ and σ also solves $(\mathcal{P}^*)_\zeta$. Conversely, if σ is the unique solution to $(\mathcal{P}^*)_\zeta$, $\zeta \in \mathcal{D} \setminus \{0\}$, then σ also solves $(\mathcal{P}^*)^\alpha$ for $\alpha \in \partial\phi(\zeta)$.

Now, we derive the dual problem to $(\mathcal{P}^*)^\alpha$ in terms of displacements for given $\alpha > 0$. Let $\zeta = \psi(\alpha) > 0$. Then,

$$\begin{aligned}
\phi(\zeta) - \alpha\zeta &\overset{(3.8)}{=} \inf_{\tilde{\zeta}\geq 0}\{\phi(\tilde{\zeta}) - \alpha\tilde{\zeta}\} = \inf_{\tilde{\zeta}\geq 0}\left\{\inf_{\tau\in\Lambda_{\tilde{\zeta}L}\cap P} \mathcal{I}(\tau) - \alpha\tilde{\zeta}\right\} = \\
&\overset{(2.2)}{=} \inf_{\tilde{\zeta}\geq 0}\left\{\sup_{v\in\mathbb{V}}[-J_{\tilde{\zeta}}(v)] - \alpha\tilde{\zeta}\right\} = \\
&= \inf_{\tilde{\zeta}\geq 0}\sup_{v\in\mathbb{V}} \mathcal{L}(\tilde{\zeta}, v),
\end{aligned}$$

where

$$\mathcal{L}(\tilde{\zeta}, v) = -\Psi(\varepsilon(v)) + \tilde{\zeta}(L(v) - \alpha), \quad (\tilde{\zeta}, v) \in \mathbb{R}_+ \times \mathbb{V}$$

and Ψ is defined by (2.1). From [5, Proposition VI.2.3], it follows that

$$\phi(\zeta) - \alpha\zeta = \inf_{\tilde{\zeta}\geq 0}\sup_{v\in\mathbb{V}} \mathcal{L}(\tilde{\zeta}, v) = \sup_{v\in\mathbb{V}}\inf_{\tilde{\zeta}\geq 0} \mathcal{L}(\tilde{\zeta}, v) = -\inf_{v\in\mathbb{V},\, L(v)\geq\alpha} \Psi(\varepsilon(v)). \tag{4.2}$$

Since Ψ is convex on S and $\Psi(0) = 0$ it holds:

$$\inf_{v\in\mathbb{V},\, L(v)\geq\alpha} \Psi(\varepsilon(v)) = \inf_{v\in\mathbb{V}^\alpha} \Psi(\varepsilon(v)), \tag{4.3}$$

where

$$\mathbb{V}^\alpha = \{v \in \mathbb{V} \mid \ L(v) = \alpha\}.$$

Indeed, for any $v \in \mathbb{V}$, $L(v) > \alpha$, one can set $w = \frac{\alpha}{L(v)} v$ belonging to $\mathbb{V}^\alpha$ and satisfying

$$\Psi(\varepsilon(w)) \leq \frac{\alpha}{L(v)} \Psi(\varepsilon(v)) \leq \Psi(\varepsilon(v)).$$

Therefore, the problem in terms of displacements for given $\alpha > 0$ reads as follows:

$$(\mathcal{P})^\alpha \qquad \text{find } u := u(\alpha) \in \mathbb{V}^\alpha : \qquad \Psi(\varepsilon(u)) \leq \Psi(\varepsilon(v)) \qquad \forall v \in \mathbb{V}^\alpha.$$

This and (4.2) yield

$$\inf_{v \in \mathbb{V}^\alpha} \Psi(\varepsilon(v)) = - \inf_{\tau \in \tilde{\Lambda}_L \cap P} [\mathcal{I}(\tau) - \alpha\omega(\tau)],$$

i.e., $(\mathcal{P})^\alpha$ and $(\mathcal{P}^*)^\alpha$ are mutually dual. Notice that this result can also be derived using some parts of the proof of Lemma 5.2 in [18]. Solvability of $(\mathcal{P})^\alpha$ is problematic on $\mathbb{V}$ from the same reasons as in the case of $(\mathcal{P})_\zeta$. However, this formulation is useful for numerical realization of its discretization. If we admit that $(\mathcal{P})^\alpha$ has a solution for some $\alpha > 0$ then the following result holds.

Theorem 4.1. *Suppose that there exists a solution u to $(\mathcal{P})^\alpha$, $\alpha > 0$. Then*

$$\zeta = \psi(\alpha) = \frac{1}{\alpha} \langle \Sigma(\varepsilon(u)), \varepsilon(u) \rangle. \tag{4.4}$$

In addition, u is the solution to $(\mathcal{P})_\zeta$ and $\sigma = \Sigma(\varepsilon(u))$ is the solution to problems $(\mathcal{P}^*)^\alpha$ and $(\mathcal{P}^*)_\zeta$.

Conversely, if u is a solution to $(\mathcal{P})_\zeta$ then u also solves $(\mathcal{P})^\alpha$ for $\alpha = L(u)$.

Proof. Let u be a solution to $(\mathcal{P})^\alpha$, $\alpha > 0$ and $\zeta = \psi(\alpha) > 0$. Then using (4.2), (4.3), the pair (ζ, u) is a saddle point of the Lagrangian $\mathcal{L}$:

$$\mathcal{L}(\zeta, v) \leq \mathcal{L}(\zeta, u) \leq \mathcal{L}(\tilde{\zeta}, u) \quad \forall (\tilde{\zeta}, v) \in \mathbb{R}_+ \times \mathbb{V},$$

or equivalently

$$\begin{cases} L(u) = \alpha, \ \ \zeta > 0, \\ \langle \Sigma(\varepsilon(u)), \varepsilon(v) \rangle = \zeta L(v) \quad \forall v \in \mathbb{V}, \end{cases} \tag{4.5}$$

i.e. u solves $(\mathcal{P})_\zeta$. Consequently, $\sigma = \Sigma(\varepsilon(u))$ solves $(\mathcal{P}^*)_\zeta$ and also $(\mathcal{P}^*)^\alpha$. Moreover, inserting $v = u$ into $(4.5)_2$, we obtain (4.4).

Conversely, let u be a solution to $(\mathcal{P})_\zeta$ for $\zeta \in \mathcal{D}$ and denote $\alpha := L(u)$. Then $u \in \mathbb{V}^\alpha$ and

$$\Psi(\varepsilon(u)) = J_\zeta(u) + \zeta\alpha \leq \inf_{v \in V^\alpha} J_\zeta(v) + \zeta\alpha = \inf_{v \in V^\alpha} \Psi(\varepsilon(v)).$$

Hence, u is the solution to $(\mathcal{P})^\alpha$. □

Remark 4.1. Theorem 4.1 expresses the relation between ζ and α through displacements. If u is a solution to $(\mathcal{P})_\zeta$ then $\alpha = L(u)$. Therefore, one can say that α represents work of external forces. The equality $\alpha = L(u)$ is in accordance with [2, 17].

5. Discretization and convergence analysis

5.1. Setting of discretized problems

For the sake of simplicity, we now suppose that Ω is a *polyhedral* domain. Let $\{\mathcal{T}_h\}$, $h > 0$ be a collection of regular partitions of $\overline{\Omega}$ into tetrahedrons $\triangle$ which are consistent with the decomposition of $\partial\Omega$ into Γ_D and Γ_N. Here, h is a positive mesh size parameter. With any $\mathcal{T}_h$ we associate the following finite-dimensional spaces:

$$\mathbb{V}_h = \{v_h \in C(\overline{\Omega};\mathbb{R}^3) \;\mid\; v_h|_\triangle \in P_1(\triangle;\mathbb{R}^3) \;\; \forall\triangle \in \mathcal{T}_h, \quad v_h = 0 \text{ on } \Gamma_D\},$$

$$S_h = \{\tau_h \in S \;\mid\; \tau_h|_\triangle \in P_0(\triangle;\mathbb{R}^{3\times3}_{sym}) \quad \forall\triangle \in \mathcal{T}_h\},$$

where $P_k(\triangle)$, $k \geq 0$ integer, stands for the space of all polynomials of degree less or equal k defined in $\triangle \in \mathcal{T}_h$. The spaces $\mathbb{V}_h$ and S_h are the simplest finite element approximations of $\mathbb{V}$ and S, respectively. Next we shall suppose that $\overline{\mathbb{V} \cap C^\infty(\overline{\Omega};\mathbb{R}^3)} = \mathbb{V}$. Further, define the following convex sets:

$$P_h = P \cap S_h,$$

$$\Lambda^h_{\zeta L} = \{\tau_h \in S_h \;\mid\; \langle\tau_h, \varepsilon(v_h)\rangle = \zeta L(v_h) \quad \forall v_h \in \mathbb{V}_h\}, \quad \zeta \geq 0,$$

$$\tilde{\Lambda}^h_L = \bigcup_{\zeta\in\mathbb{R}_+} \Lambda^h_{\zeta L},$$

$$\mathbb{V}^\alpha_h = \{v_h \in \mathbb{V}_h \mid L(v_h) = \alpha\}, \quad \alpha \geq 0,$$

$$\mathcal{D}_h := \{\zeta \in \mathbb{R}_+ \mid \Lambda^h_{\zeta L} \cap P_h \neq \emptyset\},$$

which are natural discretizations of P, $\Lambda_{\zeta L}$, $\tilde{\Lambda}_L$, $\mathbb{V}^\alpha$, and $\mathcal{D}$, respectively. We also consider the functions ϕ_h, ψ_h, ω_h and the limit load parameter $\zeta_{lim,h}$ with the analogous definitions and properties as their continuous counterparts.

The discrete versions of $(\mathcal{P}^*)_\zeta$, $(\mathcal{P}^*)^\alpha$, $(\mathcal{P})_\zeta$, $(\mathcal{P})^\alpha$ for given $\zeta \geq 0$ or $\alpha \geq 0$ read as follows:

$$(\mathcal{P}^*_h)_\zeta \qquad \text{find } \sigma_h := \sigma_h(\zeta) \in \Lambda^h_{\zeta L} \cap P_h : \quad \mathcal{I}(\sigma_h) \leq \mathcal{I}(\tau_h), \quad \forall\tau_h \in \Lambda^h_{\zeta L} \cap P_h,$$

$$(\mathcal{P}^*_h)^\alpha \qquad \text{find } \sigma_h := \sigma_h(\alpha) \in \tilde{\Lambda}^h_L \cap P_h : \quad \mathcal{I}(\sigma_h) - \alpha\omega_h(\sigma_h) \leq \mathcal{I}(\tau_h) - \alpha\omega_h(\tau_h), \quad \forall\tau_h \in \tilde{\Lambda}^h_L \cap P_h,$$

$$(\mathcal{P}_h)_\zeta \qquad \text{find } u_h := u_h(\zeta) \in \mathbb{V}_h : \quad J_\zeta(u_h) \leq J_\zeta(v_h), \quad \forall v_h \in \mathbb{V}_h,$$

$$(\mathcal{P}_h)^\alpha \qquad \text{find } u_h := u_h(\alpha) \in \mathbb{V}^\alpha_h : \qquad \Psi(\varepsilon(u_h)) \leq \Psi(\varepsilon(v_h)) \qquad \forall v_h \in \mathbb{V}^\alpha_h.$$

Clearly problems $(\mathcal{P}^*_h)_\zeta$ and $(\mathcal{P}^*_h)^\alpha$ have unique solutions for any $\zeta \in \mathcal{D}_h$, $\alpha \geq 0$ and $h > 0$. Further, the existence of solutions to $(\mathcal{P}_h)_\zeta$ and $(\mathcal{P}_h)^\alpha$ is guaranteed for any $\zeta \in [0, \zeta_{lim,h})$, $\alpha \geq 0$ and $h > 0$, see e.g. [6, 17]. The mutual relations among the solutions to these problems remain the same as in the continuous setting. The relation between ζ and α is defined using the functions ϕ_h and ψ_h, analogously to the continuous case: $\alpha \in \partial\phi_h(\zeta)$ if $\zeta \in \mathcal{D}_h \setminus \{0\}$, $\alpha = 0$ if $\zeta = 0$ and $\zeta = \psi_h(\alpha)$. In particular,

$$\zeta = \psi_h(\alpha) = \frac{1}{\alpha}\langle\Sigma(\varepsilon(u_h)), \varepsilon(u_h)\rangle, \tag{5.1}$$

where u_h is any solution to $(\mathcal{P}_h)^\alpha$. It is worth noticing that (5.1) enables us to express ζ elementwise:

$$\zeta = \sum_{\triangle\in\mathcal{T}_h} \zeta_\triangle, \quad \zeta_\triangle = \frac{|\triangle|}{\alpha}\Sigma(\varepsilon(u_h)|_\triangle) : \varepsilon(u_h)|_\triangle.$$

5.2. Convergence analysis

In what follows, we study convergence of $(\mathcal{P}_h^*)_\zeta$, $(\mathcal{P}_h^*)^\alpha$ and ψ_h to their continuous counterparts when the discretization parameter $h \to 0_+$. To this end we need the following well-known results [12, 8].

Lemma 5.1. *For any $v \in \mathbb{V}$ there exists a sequence $\{v_h\}$, $v_h \in \mathbb{V}_h$ such that $v_h \to v$ in $\mathbb{V}$ as $h \to +\infty$.*

Lemma 5.2. *Let $r_h : S \to S_h$ be the orthogonal projection of S on S_h with respect to the scalar product $\langle \cdot, \cdot \rangle$, i.e.,*

$$r_h\tau|_\triangle = \frac{1}{|\triangle|}\int_\triangle \tau\, dx \qquad \forall \triangle \in \mathcal{T}_h \quad \forall \tau \in S.$$

Then $r_h\tau \in P_h$ for any $\tau \in P$, $r_h\tau \in \Lambda_{\zeta L}^h$ for any $\tau \in \Lambda_{\zeta L}$, $\zeta \geq 0$ and

$$r_h\tau \to \tau \quad \text{in } S \text{ as } h \to 0+.$$

Corollary 5.1. *$\zeta_{lim,h} \geq \zeta_{lim}$ for any $h > 0$.*

Proof. It is sufficient to show that $\mathcal{D} \subset \mathcal{D}_h$ for any $h > 0$. If $\zeta \in \mathcal{D}$ then there exists $\tau \in \Lambda_{\zeta L} \cap P$. From Lemma 5.2, $r_h\tau \in \Lambda_{\zeta L}^h \cap P_h$ for any $h > 0$. Therefore, $\zeta \in \mathcal{D}_h$ for any $h > 0$. □

Lemma 5.3. *Let $\tau \in S$ and $\{\tau_h\}$, $\tau_h \in S_h$ be a sequence such that $\tau_h \in \Lambda_{\zeta L}^h \cap P_h$, $\zeta \geq 0$ and $\tau_h \rightharpoonup \tau$ (weakly) in S as $h \to 0_+$. Then $\tau \in \Lambda_{\zeta L} \cap P$.*

The following convergence result is a direct consequence of Lemmas 5.1–5.3.

Theorem 5.1. *Let $\zeta \in \mathcal{D}$ and σ_h be a solution to $(\mathcal{P}_h^*)_\zeta$, $h \to 0_+$. Then*

$$\sigma_h \to \sigma \ \text{ in } S, \ \ h \to 0_+,$$

$$\phi_h(\zeta) \to \phi(\zeta), \ \ h \to 0_+,$$

where $\sigma \in \Lambda_{\zeta L} \cap P$ is the unique solution to $(\mathcal{P}^*)_\zeta$.

To prove convergence of solutions of $(\mathcal{P}_h^*)^\alpha$ to a solution of $(\mathcal{P}^*)^\alpha$, we need some other auxilliary results.

Lemma 5.4. *For any $v \in \mathbb{V}^{\alpha=1}$, there exists a sequence $\{w_h\}$, $w_h \in \mathbb{V}_h^{\alpha=1}$ such that $w_h \to v$ in $\mathbb{V}$ as $h \to +\infty$.*

Proof. Let $v \in \mathbb{V}^{\alpha=1}$ and $\{v_h\}$, $v_h \in \mathbb{V}_h$ be a sequence such that $v_h \to v$ in $\mathbb{V}$ as $h \to +\infty$. Then, $L(v_h) \to L(v) = 1$ as $h \to 0_+$ and $w_h = \frac{1}{L(v_h)}v_h \in \mathbb{V}_h^{\alpha=1}$ has the required property. □

Lemma 5.5. *There exists a constant $c > 0$ such that for any sufficiently small $h > 0$*

$$\omega_h(\tau_h) \leq c\|\tau_h\|_{C^{-1}} \quad \forall \tau_h \in \tilde{\Lambda}_L^h.$$

Proof. Let $v \in \mathbb{V}^{\alpha=1}$ and $\epsilon > 0$ be given. Then, there exists a sequence $\{w_h\}$, $w_h \in \mathbb{V}_h^{\alpha=1}$ such that $w_h \to v$ in $\mathbb{V}$ as $h \to +\infty$. Hence,

$$\exists h_0 > 0: \quad \|\varepsilon(w_h)\|_C \leq \|\varepsilon(v)\|_C + \epsilon \quad \forall 0 < h \leq h_0$$

and using the definition of $\tilde{\Lambda}_L^h$, we obtain

$$\omega_h(\tau_h) = \omega_h(\tau_h)L(w_h) = \langle \tau_h, \varepsilon(w_h)\rangle \leq \|\tau_h\|_{C^{-1}}\|\varepsilon(w_h)\|_C \leq c\|\tau_h\|_{C^{-1}} \quad \forall 0 < h \leq h_0,\ \forall \tau_h \in \tilde{\Lambda}_L^h,$$

where $c = \|\varepsilon(v)\|_C + \epsilon$. □

Lemma 5.6. *Let $\{\tau_h\}$, $\tau_h \in \tilde{\Lambda}_L^h \cap P_h$ be such that $\tau_h \rightharpoonup \tau$ (weakly) in S and $\omega_h(\tau_h) \to \zeta$ as $h \to 0_+$. Then $\tau \in \tilde{\Lambda}_L \cap P$ and $\omega(\tau) = \zeta$.*

Proof. Since $\tau_h \rightharpoonup \tau$ and P is a closed convex set, $\tau \in P$. Let $v \in \mathbb{V}$ and $\{v_h\}$, $v_h \in \mathbb{V}_h$ be such that $v_h \to v$ in $\mathbb{V}$ as $h \to +\infty$. From the definition of $\tilde{\Lambda}_L^h$, it follows that

$$\langle \tau_h, \varepsilon(v_h)\rangle = \omega_h(\tau_h)L(v_h).$$

Passing to the limit with $h \to 0_+$, we conclude that $\tau \in \tilde{\Lambda}_L \cap P$ and $\omega(\tau) = \zeta$. □

Theorem 5.2. *Let $\alpha \geq 0$ be given and $\{\sigma_h\}$ be a sequence of solutions to $(\mathcal{P}_h^*)^\alpha$, $h > 0$. Then $\sigma_h \to \sigma$ in S, $\omega_h(\sigma_h) \to \omega(\sigma)$ and $\psi_h(\alpha) \to \psi(\alpha)$ as $h \to 0_+$, where σ is a solution to $(\mathcal{P}^*)^\alpha$.*

Proof. The proof consists of three steps.

Step 1 (Boundedness). Let $\tau \in \tilde{\Lambda}_L \cap P$ be fixed. Then $r_h\tau \in \Lambda^h_{\omega(\tau)L} \cap P_h \subset \tilde{\Lambda}_L^h \cap P_h$ and $r_h\tau \to \tau$ in S as $h \to 0_+$. From the definition of $(\mathcal{P}_h^*)^\alpha$ it follows:

$$\mathcal{I}(\sigma_h) - \alpha\omega_h(\sigma_h) \leq \mathcal{I}(r_h\tau) - \alpha\omega(\tau) \quad \forall h > 0$$

since $\omega_h(r_h\tau) = \omega(\tau)$. From this and Lemma 5.5, we obtain

$$\exists c_1 > 0, c_2 \in \mathbb{R}, h_0 > 0: \quad \frac{1}{2}\|\sigma_h\|^2_{C^{-1}} = \mathcal{I}(\sigma_h) \leq c_1\|\sigma_h\|_{C^{-1}} + c_2 \quad \forall h \in (0, h_0).$$

This implies boundedness of $\{\sigma_h\}$ and consequently boundedness of $\{\omega_h(\sigma_h)\}$.

Step 2 (Weak convergence). One can pass to subsequences $\{\sigma_{h'}\} \subset \{\sigma_h\}$ and $\{\omega_{h'}(\sigma_{h'})\} \subset \{\omega_h(\sigma_h)\}$ such that

$$\left.\begin{array}{l} \sigma_{h'} \rightharpoonup \sigma \ \text{ in } S \quad \textit{as} \quad h' \to 0_+, \\ \omega_{h'}(\sigma_{h'}) \to \zeta \quad \textit{as} \quad h' \to 0_+. \end{array}\right\} \tag{5.2}$$

From Lemma 5.6, it follows that $\sigma \in \tilde{\Lambda}_L \cap P$ and $\zeta = \omega(\sigma)$. Let $\tau \in \tilde{\Lambda}_L \cap P$ be arbitrary. Then $r_{h'}\tau \in \Lambda^{h'}_{\omega(\tau)L} \cap P_{h'} \subset \tilde{\Lambda}_L^{h'} \cap P_{h'}$, $\omega_{h'}(r_{h'}\tau) = \omega(\tau)$ and $r_{h'}\tau \to \tau$ in S as $h' \to 0_+$. Hence,

$$\mathcal{I}(\sigma) - \alpha\omega(\sigma) \leq \liminf_{h' \to 0_+}[\mathcal{I}(\sigma_{h'}) - \alpha\omega_{h'}(\sigma_{h'})] \leq \liminf_{h' \to 0_+}[\mathcal{I}(r_{h'}\tau) - \alpha\omega(r_{h'}\tau)] = \mathcal{I}(\tau) - \alpha\omega(\tau),$$

i.e., σ is the solution to $(\mathcal{P}^*)^\alpha$. Since $(\mathcal{P}^*)^\alpha$ has a unique solution, (5.2) holds for the whole sequence. Consequently,

$$\psi_h(\alpha) = \omega_h(\sigma_h) \to \omega(\sigma) = \psi(\alpha) \quad \text{as} \quad h \to 0_+.$$

Step 3 (Strong convergence). Since $r_h\sigma \in \tilde{\Lambda}_L^h \cap P_h$, $\omega_h(r_h\sigma) = \omega(\sigma)$ and $r_h\sigma \to \sigma$ in S as $h \to 0_+$, we have

$$\begin{aligned}\mathcal{I}(\sigma) &\leq \liminf_{h\to 0_+} \mathcal{I}(\sigma_h) \leq \limsup_{h\to 0_+} \mathcal{I}(\sigma_h) = \limsup_{h\to 0_+} [\mathcal{I}(\sigma_h) - \alpha\omega_h(\sigma_h)] + \alpha\omega(\sigma) \\ &\leq \lim_{h\to 0_+} [\mathcal{I}(r_h\sigma) - \alpha\omega_h(r_h\sigma)] + \alpha\omega(\sigma) = \mathcal{I}(\sigma).\end{aligned}$$

Therefore,

$$\|\sigma_h\|^2_{C^{-1}} = 2\mathcal{I}(\sigma_h) \to 2\mathcal{I}(\sigma) = \|\sigma\|^2_{C^{-1}} \quad \text{as} \quad h \to 0_+,$$

which implies strong convergence of $\{\sigma_h\}$ to σ in S. □

Remark 5.1. We summarize the properties of the functions ψ and ψ_h, $h > 0$:

a) ψ and ψ_h are nondecreasing and continuous in $\mathbb{R}_+$ for any $h > 0$;

b) $\psi(\alpha) \to \zeta_{lim}$, $\psi_h(\alpha) \to \zeta_{lim,h}$ as $\alpha \to +\infty$, for any $h > 0$;

c) $\zeta_{lim,h} \geq \zeta_{lim} \geq \psi(\alpha)$ for any $h > 0$ and $\alpha \geq 0$;

d) $\psi_h(\alpha) \to \psi(\alpha)$ as $h \to 0_+$ for any $\alpha \geq 0$.

Notice that from Remark 5.1 *b), d)* it follows that for any $\epsilon > 0$ there exists α large enough and $h_0 > 0$ small enough such that $|\psi_h(\alpha) - \zeta_{lim}| < \epsilon \;\; \forall h \leq h_0$. Direct convergence of $\zeta_{lim,h}$ to ζ_{lim} is guaranteed only for some yield functions Φ as follows from the next theorem.

Theorem 5.3. *Let the yield function Φ be coercive on $\mathbb{R}^{3\times 3}_{sym}$ and the assumptions $(L1), (L2)$ be satisfied. Then*

$$\zeta_{lim,h} \to \zeta_{lim} \quad \textit{as} \quad h \to 0_+. \tag{5.3}$$

Proof. Coerciveness of Φ ensures that the set P is bounded in $L^\infty(\Omega; \mathbb{R}^{3\times 3}_{sym})$, i.e.

$$\exists c > 0: \quad |\tau_{ij}(x)| \leq c \quad \forall \tau \in P,\ \forall i,j = 1,2,3, \text{ for a.a. } x \in \Omega. \tag{5.4}$$

Next, we show that $\{\zeta_{lim,h}\}$ is bounded. Consider a bounded sequence $\{w_h\}$, $w_h \in \mathbb{V}_h^{\alpha=1}$:

$$\exists M > 0: \quad \|\varepsilon(w_h)\|_{L^1(\Omega;\mathbb{R}^{3\times 3}_{sym})} \leq M \quad \forall h > 0. \tag{5.5}$$

The existence of such a sequence is guaranteed by Lemma 5.4. Then for any $\zeta \in \mathcal{D}_h$ and $\tau_h \in \Lambda^h_{\zeta L} \cap P_h$ it holds

$$\zeta = \zeta L(w_h) = \langle \tau_h, \varepsilon(w_h)\rangle \overset{(5.4)}{\leq} c\|\varepsilon(w_h)\|_{L^1(\Omega;\mathbb{R}^{3\times 3}_{sym})} \overset{(5.5)}{\leq} cM.$$

Hence, $\zeta_{lim,h} \leq cM < +\infty$ for any $h > 0$. In addition, from boundedness of P, it follows that $\zeta_{lim,h} \in \mathcal{D}_h$ for any $h > 0$.

Let $\{\tau_h\}$, $h > 0$, be such that $\tau_h \in \Lambda^h_{\zeta_{lim,h}L} \cap P_h$. Then $\{\tau_h\}$ is bounded in S and there exist subsequences $\{\zeta_{lim,h'}\}$ and $\{\tau_{h'}\}$, $\tau_{h'} \in \Lambda^{h'}_{\zeta_{lim,h'}L} \cap P_{h'}$ such that

$$\tau_{h'} \rightharpoonup \tau \text{ in } S, \quad \zeta_{lim,h'} \to \hat{\zeta}, \quad h' \to 0_+.$$

Clearly, $\tau \in \Lambda_{\hat{\zeta}L} \cap P$ and thus $\hat{\zeta} \in \mathcal{D}$. Therefore $\hat{\zeta} = \zeta_{lim}$ using Corollary 5.1. □

6. Numerical experiments

In order to verify the previous theoretical results, we have performed several numerical experiments with two yield functions presented below. Problem $(\mathcal{P}_h)^\alpha$ which is needed for the evaluation of $\psi_h(\alpha)$ is solved by a regularized semismooth Newton method. This method has been proposed and theoretically justified in [2, ALG3]. Each iterative step leads to a quadratic programming problem. After finding a solution $u_h := u_h(\alpha)$ of $(\mathcal{P}_h)^\alpha$, the value $\zeta = \psi_h(\alpha)$ of the load parameter is computed by (5.1).

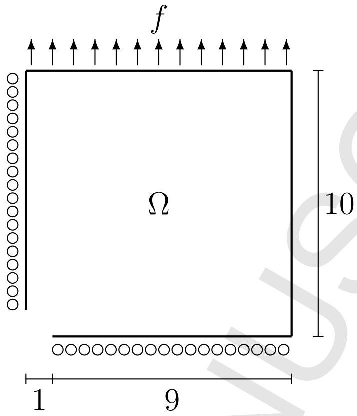

Figure 2: Geometry of the plane strain problem.

The performed experiments are related to a plain strain problem with Ω depicted in Figure 2: Ω is a quarter of the square containing the circular hole of radius 1 in its center. The constant traction of density $f = (0, 450), (0, 0)$ is applied on the upper, and the right vertical side, respectively. This load corresponds to $\zeta = 1$. On the rest of $\partial\Omega$ the symmetry boundary conditions are prescribed. We consider linear Hooke's law for a homogeneous, isotropic elastic material:

$$\tau = Ce \;\Leftrightarrow\; \tau = \lambda\,\mathrm{tr}(e)\,\iota + 2\mu e, \quad e, \tau \in \mathbb{R}^{3\times 3}_{sym}, \tag{6.1}$$

where ι is the (3×3) identity matrix, $\mathrm{tr}(e) = e_{ii}$ is the trace of e and $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$ are positive constants representing Lame's coefficients. The elastic material parameters are set as follows: $E = 206900$ (Young's modulus) and $\nu = 0.29$ (Poisson ratio).

The loading paths represented by the graph of $\psi_h : \alpha \mapsto \zeta$ are compared for seven different meshes with 1080, 2072, 3925, 10541, 23124, 41580 and 92120 nodes. The problem is implemented in MatLab.

6.1. Yield function 1

Consider the yield function

$$\Phi(\tau) = \sqrt{C^{-1}\tau : \tau}, \quad \tau \in \mathbb{R}^{3\times 3}_{sym}$$

(a similar yield function has been considered in, e.g., [15, 1]). Then

$$(\Sigma(e))\,(x) = \mathbb{D}\Psi(e)(x) = \left\{ \begin{array}{rl} Ce(x), & \sqrt{Ce(x) : e(x)} \leq \gamma, \\ \frac{\gamma}{\sqrt{Ce(x):e(x)}} Ce(x), & \sqrt{Ce(x) : e(x)} \geq \gamma, \end{array} \right. , \quad \forall e \in S, \;\; \text{for a.a. } x \in \Omega,$$

$$\Psi(e) = \frac{1}{2}\int_\Omega \left\{ Ce : e - \left[\left(\sqrt{Ce : e} - \gamma \right)^+ \right]^2 \right\} dx, \quad \forall e \in S,$$

respectively, where $(g)^+$ denotes the positive part of a function g.

From Theorem 3.1 $(i), (ii)$ we know that for any $\alpha \in (0, +\infty)$ the values $\psi(\alpha)$, $\psi_h(\alpha)$ give a lower bound of ζ_{lim}, and $\zeta_{lim,h}$, respectively. Since Φ is coercive on $\mathbb{R}^{3\times3}_{sym}$, it holds that $\zeta_{lim,h} \to \zeta_{lim}$ as $h \to 0_+$ using Theorem 5.3.

For purposes of the experiment, we choose $\gamma = 10$ and the increments $\triangle\alpha$ defined as follows: $\triangle\alpha = 20$ for $\alpha \in [0, 2000]$ and $\triangle\alpha = 100$ for $\alpha = [2000, 10000]$. The path-following procedure has been terminated if $\alpha \geq 10000$.

The comparison of the loading paths for seven different meshes is shown in Figure 3. Since the curves practically coincide the zoom is depicted in Figure 4. We see that the value $\zeta \approx 9.48$ turns out to be a suitable lower bound of ζ_{lim}. Further, one can see that $\psi_h \leq \psi_{h'}$ for $h \leq h'$. Therefore one can expect uniform convergence of $\{\psi_h\}$ to ψ on closed and bounded intervals using Dini's theorem.

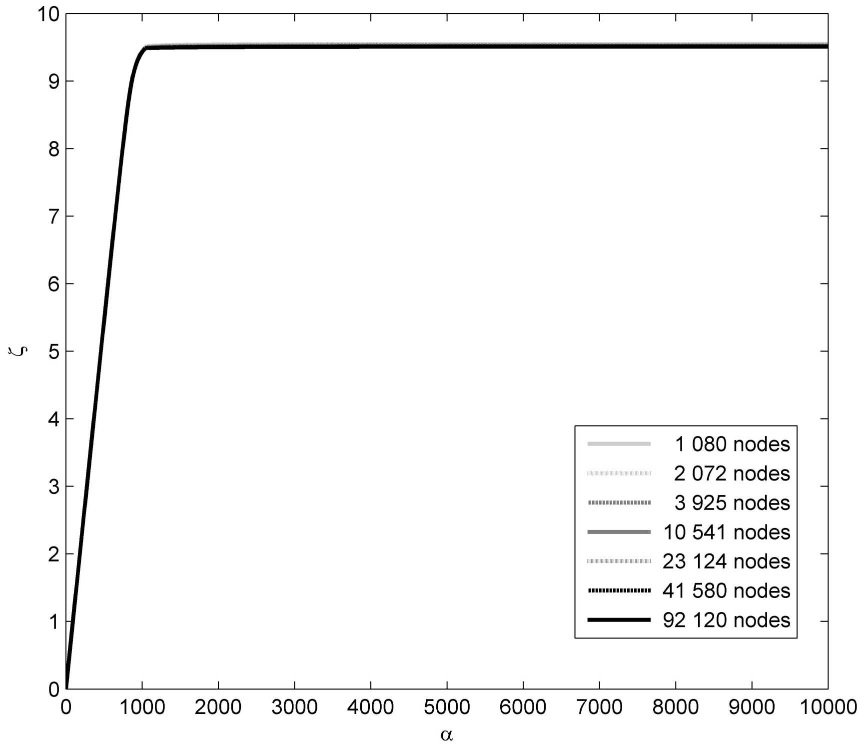

Figure 3: Loading paths up to $\alpha \in [0, 10000]$.

6.2. Yield function 2 - von Mises criterion

The von Mises criterion [18, 3, 17, 2] is suitable for an isotropic and pressure insensitive material. The corresponding yield function has the form

$$\Phi(\tau) = \tau^D : \tau^D, \quad \tau \in \mathbb{R}^{3\times3}_{sym}, \tag{6.2}$$

where $\tau^D = \tau - 1/3\,\mathrm{tr}(\tau)\iota$ is the deviatoric part of τ. If the elasticity tensor C is defined as in (6.1), then

$$\Psi(e) := \int_\Omega \left\{ \frac{1}{2}Ce : e - \frac{1}{4\mu}\left[\left(2\mu\sqrt{e^D : e^D} - \gamma\right)^+\right]^2 \right\} dx.$$

Unlike Yield function 1, Φ defined by (6.2) is not coercive on $\mathbb{R}^{3\times3}_{sym}$. Therefore convergence $\zeta_{lim,h} \to \zeta_{lim}$ as $h \to 0_+$ is not guaranteed.

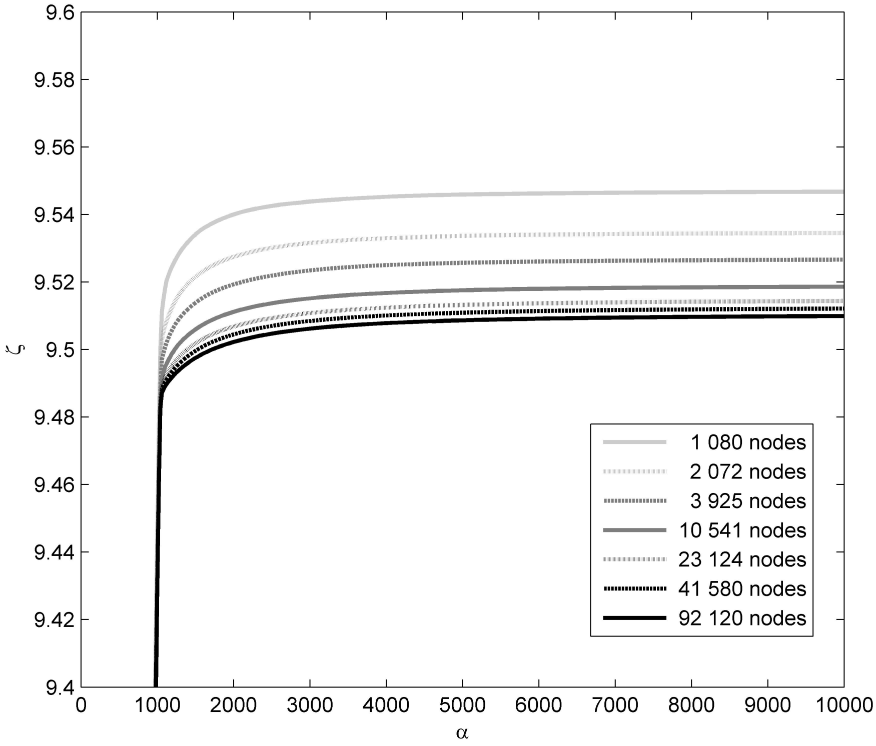

Figure 4: Loading paths for $\alpha \in [0, 10000]$ (zoom).

We choose $\gamma = 450\sqrt{2/3}$ and $\triangle\alpha = 5, 100, 1000$ for $\alpha \in [0, 300], [300, 10000], [10000, 100000]$, respectively. The comparison of the loading paths for seven different meshes is shown in in Figure 5. The curves practically coincide up to $\zeta = 1$. Therefore the value $\zeta = 1$ seems to be a reliable lower estimate of ζ_{lim}. As in the previous example, one can see that $\psi_h \leq \psi_{h'}$ for $h \leq h'$.

In Figure 6, zooms of the loading paths up to $\alpha = 100000$ for the seven meshes are displayed. We observe that the curve representing the coarsest mesh is almost constant in a vicinity of $\alpha = 100000$ and the corresponding value of ψ_h is approximately equal to 1.14 there. So one can expect that $\zeta_{lim} \in [1.00, 1.14]$. On the other hand, pointwise convergence of $\{\psi_h(\alpha)\}$ becomes slow for large values of α. Therefore, direct convergence $\zeta_{lim,h} \to \zeta_{lim}$ as $h \to 0_+$ seems to be at least problematic.

7. Conclusion

The paper deals with an enhanced incremental procedure for reliable estimation of the limit load in deformation plasticity models. This procedure is based on a continuation parameter α ranging in $(0, +\infty)$ which is dual to the standard loading parameter $\zeta \in (0, \zeta_{lim})$, where ζ_{lim} is the critical value of ζ. We have shown that there exists a continuous, nondecreasing function ψ in $(0, +\infty)$ and such that $\psi(\alpha) \to \zeta_{lim}$ if $\alpha \to +\infty$. Therefore $\psi(\alpha)$ gives a guaranteed lower bound of ζ_{lim} for any $\alpha \in (0, +\infty)$. To evaluate $\psi(\alpha)$ for given α we derived a minimization problem for the stored energy functional subject to the constraint $L(v) = \alpha$ whose solutions define the respective value $\psi(\alpha)$. The second part of the paper was devoted to a finite element discretization and convergence analysis. The main result of this part is the proof of pointwise convergence $\psi_h \to \psi$ which is crucial for finding a reliable lower bound of ζ_{lim}. Further, we specified a class of yield functions for which $\zeta_{lim,h} \to \zeta_{lim}$ as $h \to 0_+$. It is worth mentioning that elastoplastic models with bounded yield surfaces belong to this class. In the follow-up

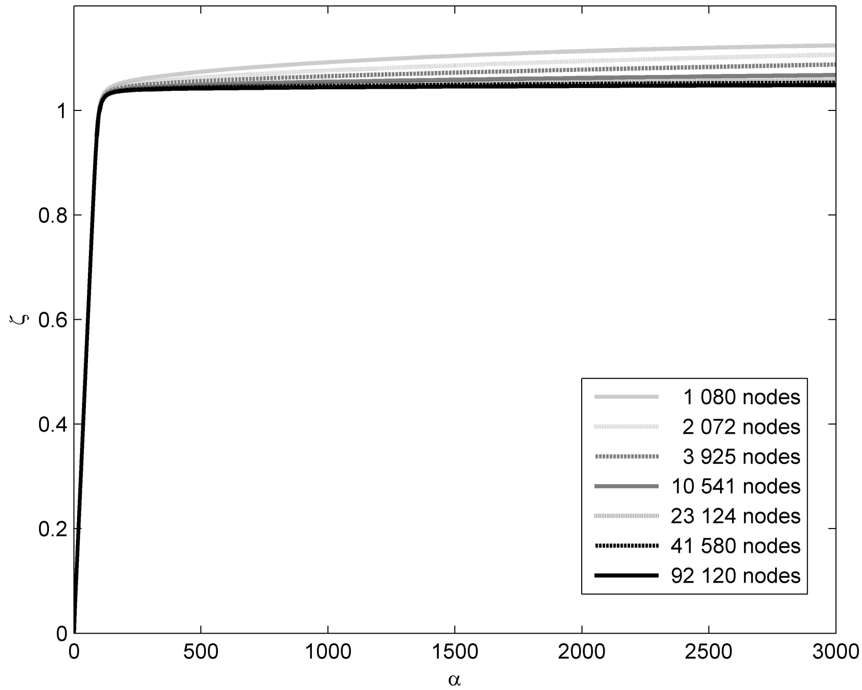

Figure 5: Loading paths up to $\alpha = 3000$.

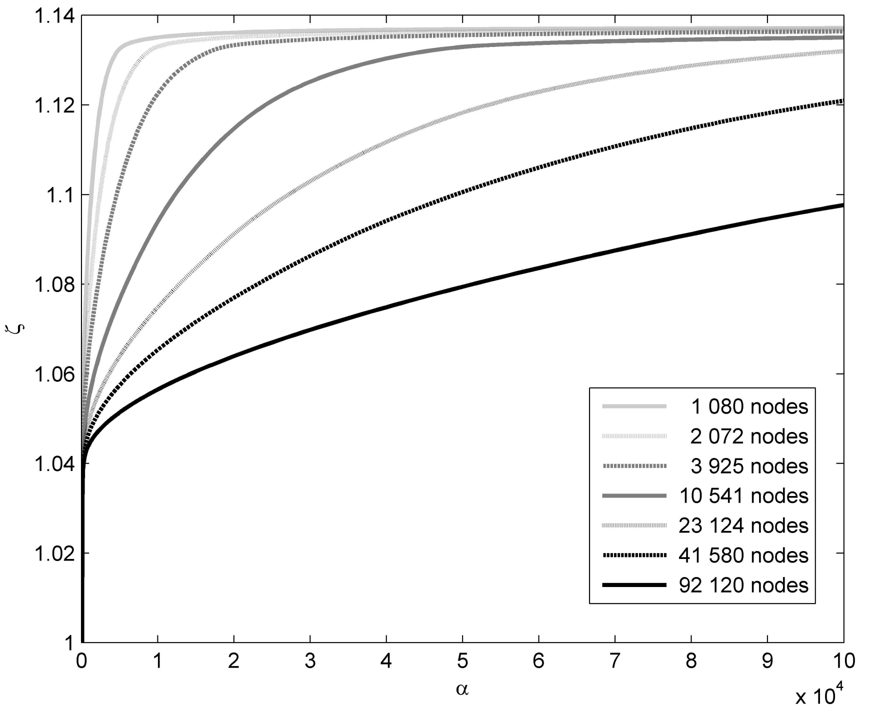

Figure 6: Loading paths up to $\alpha = 100000$ (zoom).

paper [10], it is shown that limit analysis is much simpler for such models and a truncation method is suggested there for models with unbounded yield surfaces.

Acknowledgements

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