Higgs mass predicted from the Standard Model with asymptotically safe gravity

Laura Laulumaa

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Supervisor: Kimmo Kainulainen

Abstract

The aim of this thesis is to predict the Higgs boson mass from the Standard model of particle physics with gravity as an asymptotically safe theory. To reach this goal, the running of four standard model couplings in addition to the Higgs self-coupling is derived at one-loop level in the $\overline{\text{MS}}$-scheme. Standard model β-functions are supplemented by asymptotically safe gravity corrections at very large energies, and differential equation group of β-functions is solved numerically. At low energies a partial finite renormalisation of the Standard Model is performed to derive the connection between the self-coupling in the $\overline{\text{MS}}$-scheme and the physical, measurable parameters.

Relating the value for the Higgs self-coupling given by the running of couplings to the physical one leads to a prediction $\hat{\lambda} \approx 0.131$ for the Higgs self-coupling which corresponds to a mass $\hat{m}_h \approx 126\,\text{GeV}$. In 2012 at CERN the Higgs mass was measured to be $\hat{m}_h \approx 125\,\text{GeV}$ corresponding to the value of $\hat{\lambda} \approx 0.13$ for the self-coupling. The prediction got in this thesis is very close to the measured one when one takes into account that calculations were done only at one-loop level.

Tiivistelmä

Tässä Pro Gradu -tutkielmassa tavoitteena on ennustaa Higgsin bosonin massa ottaen lähtökohdaksi hiukkasfysiikan standardimalli, johon on kytketty gravitaatio ns. asymptoottisesti turvallisena teoriana. Ennusteen laskemiseksi selvitetään Higgsin bosonin itseiskytkennän ja neljän muun standardimallin kytkinvakion juokseminen, eli kytkinvakioiden käyttäytyminen energiaskaalan funktiona, johtavassa kertaluvussa $\overline{\text{MS}}$-skeemassa. Standardimallista saatuihin β-funktioihin lisätään asymptoottisesti turvallisen gravitaation antamat korjaukset suurilla energiaskaaloilla, jonka jälkeen β-funktioiden muodostama differentiaaliyhtälöryhmä ratkaistaan numeerisesti. Standardimallin osittainen äärellinen remormalisaatio matalilla energioilla tarvitaan, kun halutaan johtaa relaatio Higgsin itseiskytkennän $\overline{\text{MS}}$-skeemassa saaman arvon ja fysikaalisten parametrien välille.

Liittämällä kytkinvakioiden juoksemisesta saatu Higgsin itseiskytkennän arvo fysikaalisiin parametreihin, saadaan Higgsin itseiskytkennälle ennuste $\hat{\lambda} \approx 0{,}131$, jota vastaava Higgsin massa on $\hat{m}_h \approx 126\,\text{GeV}$. Laskettu tulos osuu hyvin lähelle vuonna 2012 CERN:ssä mitattua arvoa $\hat{m}_h \approx 125\,\text{GeV}$, jota vastaa itseiskytkentä $\hat{\lambda} \approx 0{,}13$, kun otetaan huomioon, että tässä työssä laskut tehtiin vain johtavassa kertaluvussa.

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Contents

1 Introduction

The Standard Model of particle physics (SM) is a quantum field theory (QFT) about elementary particles and three of fundamental forces of nature (electromagnetic, weak and strong interactions). In fall 2012 the SM was completed when the last missing particle, Higgs boson, was discovered at the LHC in CERN with mass around 125 GeV [1, 2].

Despite being so beautiful and self-consistent theory, it is widely believed that the SM can not be 'the final theory'. The latest research has shown that the SM can not explain several observed phenomena in the universe. The most popular problems are the so called dark matter (DM) and dark energy (DE) problems. It has been shown that DM and DE, yet unknown constituents, account almost all of the energy content of the universe while the ordinary matter, formed by the SM particles, account only for a few per cent of the total [3]. Another problem with the SM is related to the size of the Higgs mass, the so called hierarchy problem. When one takes into account quantum corrections to Higgs self-energy, one would expect Higgs to have a very large mass of the size of the fundamental cut off. The measured one is at roughly the same scale as W- and Z-boson, at the electroweak scale. The question is, what makes the Higgs mass to be so small. Furthermore, the discovery of the Higgs boson at the LHC exposed the problem of non-stability of electroweak vacuum. It turns out that for the measured value of Higgs boson mass, assuming that there exists only the SM particles, at large scales Higgs self-coupling turns negative, which leads to unstable electroweak vacuum [4].

These problems have induced numerous extensions of the SM but a truly convincing model in any account is still missing. One possibility for solving the vacuum stability is presented in [5]. Coupling the gravity minimally to the SM as an asymptotically safe theory, the SM may stay perturbative at all scales. Asymptotic safety is a generalisation of the notion of renormalisability. As an asymptotically safe theory gravity will induce corrections to SM results and keep Higgs self-coupling positive at large scales without adding new particles to model. In this thesis my aim is to go through ideas in [5] and derive a prediction for the Higgs mass.

I start with an overview of the notion of the running couplings and discuss in particular the running of Higgs self-coupling. To derive the running of a coupling, a renormalised Lagrangian for interactions is needed. In Section 2 I go through the renormalisation of the Yang-Mills Lagrangian. After that I present the renormalisation of the Higgs Lagrangian and the Yukawa interaction term of the top quark. There is also a quick reminder about the Lagrangian and properties of a general Yang-Mills theory. The β-functions of the five SM couplings at one-loop level are calculated in Section 3. If the reader is already familiar with these one-loop calculations, it is recommended to skip Section 3 and go straight to Section 4 in which I consider the notion of asymptotic safety. I briefly go through the definition of an asymptotically safe theory, discuss the possibility that gravity is asymptotically safe and lastly present the corrections to β-functions given by the asymptotically safe gravity. Differential equation group of β-functions for the five SM couplings is solved numerically in Section 5, and in Section 6 those results are used to get a prediction for the Higgs mass. The last section, Section 7, is for discussion and conclusions.

1.1 Some useful results and notations

In this section I list some useful results relevant for loop calculations. Most of these results one can find for example in [6] and they are easy to verify directly.

The following identities for γ-matrices in a d-dimensional spacetime are often needed:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \tag{1a}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu \tag{1b}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{1c}$$

$$\gamma^\mu\gamma^\alpha\gamma^\nu = \gamma^\mu g^{\alpha\nu} - \gamma^\alpha g^{\mu\nu} + \gamma^\nu g^{\mu\alpha} - \mathrm{i}\epsilon^{\mu\alpha\nu\rho}\gamma^5\gamma_\rho \tag{1d}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = dg^{\mu\nu} \tag{1e}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = d(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}) \ . \tag{1f}$$

In symmetric integrals with respect to k one can use the following replacement:

$$k_\mu k_\nu \to \frac{1}{d}k^2 g_{\mu\nu} \ , \tag{2}$$

where d is again the spacetime dimension, i.e. the dimension of the phase space element in the integral.

Projection operators of left- and right-handed components are always needed when one is considering weak interactions. Operators P_L and P_R are defined as

$$P_\mathrm{L} = \frac{\mathbb{1}-\gamma^5}{2} \quad \text{and} \quad P_\mathrm{R} = \frac{\mathbb{1}+\gamma^5}{2}$$

and they obey

$$P_\mathrm{L} + P_\mathrm{R} = \mathbb{1} \tag{3a}$$

$$P_\mathrm{L}^2 = P_\mathrm{L} \ , \ P_\mathrm{R}^2 = P_\mathrm{R} \tag{3b}$$

$$P_{\mathrm{L}}P_{\mathrm{R}} = P_{\mathrm{R}}P_{\mathrm{L}} = 0 \tag{3c}$$

$$\gamma_\mu P_{\mathrm{L}} = P_{\mathrm{R}}\gamma_\mu \quad \forall\mu \in \{0,1,2,3\}\ . \tag{3d}$$

Calculations in Sections 3 and 6 can be done using relations presented in this section and anti-commutation rules for Dirac γ-matrices.

Notations in Feynman diagrams In Sections 3 and 6 there are several Feynman diagrams needed. Here I list the notations I have used when drawing those diagrams.

A fermion is always drawn with a directed solid line and a gluon with a curly line. A wavy line is for a photon, Z- and W-boson and is specified with labels γ, Z and W if needed. Thet Higgs boson, Goldstone bosons and ghosts are all drawn with a dashed line. For ghosts the dashed line is directed. Goldstone bosons are labelled with χ and ghosts with c. If there is no label, or label h, a dashed line marks the Higgs boson.

When there is a loop in a diagram, one diagram includes all possible combinations of that kind of particles. For example in Figure 11 diagram (3) includes both Z- and W-boson loops.

2 Counter term Lagrangian

In the SM Lagrangian the mass of a particle is given by some combination of the SM parameters, involving coupling constants and the vacuum expectation value (vev for short) v of the Higgs field. For example, the Higgs mass in the SM is defined to be $m_h = \sqrt{2\lambda v^2}$, where λ is the Higgs self-coupling. Thus if one somehow gets information about the behaviour of coupling constants of the model, one could get information about the behaviour of the masses.

Running of coupling and β-function Despite their name, coupling constants in a Lagrangian are not constant but change as a function of energy scale. One describes this dependence on the scale with the notion of running of the coupling. Formally the running is determined via the so called β-function, defined as [7]

$$\beta_g = \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} \ . \tag{4}$$

Hence, β-function is the derivative of coupling with respect to the scale multiplied with the scale. The running of the coupling is found by solving the differential equation (4) given the precise form of this β-function. If a theory contains more than one coupling constant, their running may depend on each other. In that case the β-functions form a system of differential equations:

$$\mu \frac{\mathrm{d}g_i}{\mathrm{d}\mu} = \beta_i(\{g_1\}) \ .$$

The SM and Higgs mass The leading idea in this thesis is to derive the running for the Higgs self-coupling at one-loop level. Here the term one-loop level stands for the first non-trivial order of perturbation theory used to compute the β-function. In practice, n-point functions are formed as a sum of all those diagrams that contain one loop at the most. Examining the behaviour of the self-coupling, one can rule out values of self-coupling that lead to unwanted behaviour at large energy scales. What is the role of other SM couplings here? In principle, all particles that interact with the Higgs field may affect the running of Higgs self-coupling. In the SM Higgs interacts with all particles of non-zero mass: with W- and Z-bosons via U(1) and SU(2) gauge couplings and with fermions via Yukawa couplings. Thus, besides of the running of Higgs self-coupling, also the running of U(1), SU(2) and Yukawa couplings has to be derived. Furthermore, although gluons do not interact with Higgs, the running of SU(3) gauge coupling is needed for the running of Yukawa couplings for quarks.

In the SM Lagrangian there is a different Yukawa coupling for each massive fermion and the strength of the coupling depends on the mass of the fermion. Since all other fermions are very light and much lighter than the top quark, I may neglect their contribution to the Higgs β-function with a very good accuracy.

For one-loop level calculations the SM Lagrangian must be renormalised. In this section I go through the renormalisation procedure for those parts of the SM Lagrangian which give the counter term Lagrangian for each of the couplings whose running I am considering. The gauge sector of the SM is formed by three gauge symmetries, namely by U(1), SU(2) and SU(3). The two last are special cases of a general Yang-Mills theory. Thus it is worth considering first a general SU(N) Yang-Mills theory, and then use its properties in the cases N = 2,3. U(1) gauge theory differs from the other two by being an Abelian theory and that U(1) hypercharge field couples differently with different fermions. However, the running of U(1) coupling can be derived from the general SU(N) results with small modifications.

Next I go through renormalisation of certain parts of the SM Lagrangian. The task is to find out the counter term Lagrangian of each part. I start with a general Yang-Mills theory. After that I will consider the Higgs Lagrangian and top Yukawa interaction term.

2.1 Yang–Mills Lagrangian

The Yang–Mills Lagrangian describes a theory with a fermion ψ and a non-Abelian gauge field A_μ and their interaction. The general form of the Lagrangian is [6]

$$\mathcal{L}_{\mathrm{YM}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \,, \tag{5}$$

where the covariant derivative is $D_\mu = \partial_\mu - igt^a A^a_\mu$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$ is the field strength tensor of the Yang–Mills gauge field. Matrices t^a are the generators of the symmetry group, i.e. the group under which the Lagrangian is symmetric. There are $N^2 - 1$ different generators for an N-dimensional group. The fermion is in a representation r of the symmetry group. The Faddeev-Popov gauge fixing induces to the Yang–Mills Lagrangian the ghost field c for which the covariant derivative is $D^{ac}_\mu = \partial_\mu \delta^{ac} + gf^{abc} A^b_\mu$. The third term in the Lagrangian gives the kinetic term of the ghost field and interaction between ghost and gauge field whereas the fourth gives an additional gauge depending kinetic term for the gauge field.

It is worth recalling some of the properties of the generators t^a.[1] The

[1] For further reading, see [6].

commutation relation of the generators, the Lie algebra, is usually written as

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c \ ,$$

where the structure constants f^{abc} are fully antisymmetric in all indices. Each generator is traceless. A product of two generators obeys

$$\mathrm{Tr}\left[t^a t^b\right] = (t^a t^b)_{ii} = C(r)\delta^{ab} \ , \tag{6}$$

were $C(r)$ is a constant depending on a representation r for the generators. It can also be shown that a generator squared is proportional to the unit matrix

$$t^a t^a = \mathbb{1}_{d(r)} C_2(r) \ . \tag{7}$$

The unit matrix $\mathbb{1}$ has dimension $d(r)$, where $d(r)$ is the dimension of the (irreducible) representation r, and $C_2(r)$ is a quadratic Casimir operator whose value depends on the representation.

For the group SU(N) one of the most common representation is the N-dimensional complex vector, called fundamental representation, for which

$$C(N) = \frac{1}{2} \quad \text{and} \quad C_2(N) = \frac{N^2-1}{2N} \ .$$

Fermions are usually, as well as in the SM, put into fundamental representation. Another common representation is the one to which generators of the algebra belong, called adjoint representation. The gauge field of the symmetry group is usually in this representation and

$$C(G) = C_2(G) = N \ .$$

Renormalisation of Yang–Mills theory Start from the Yang–Mills Lagrangian (5). First interpret all fields and parameters in the Lagrangian as bare quantities. Then rescale the bare fields to the renormalised ones by

$$\begin{cases} A^a_{0\,\mu} & \to Z_A^{1/2} A^a_\mu \\ c^a_0 & \to Z_c^{1/2} c^a \\ \psi_0 & \to Z_\psi^{1/2} \psi \end{cases}$$

after which the Lagrangian becomes

$$\mathcal{L}_{\mathrm{YM}} = \bar{\psi}(\mathrm{i} Z_\psi \partial\!\!\!/ + g_0 Z_\psi Z_A^{1/2} t^a A\!\!\!/^a - Z_\psi m_0)\psi \\ - \frac{1}{4} Z_A (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{Z_A}{2\xi_0}(\partial^\mu A^a_\mu)^2 + \dots \ ,$$

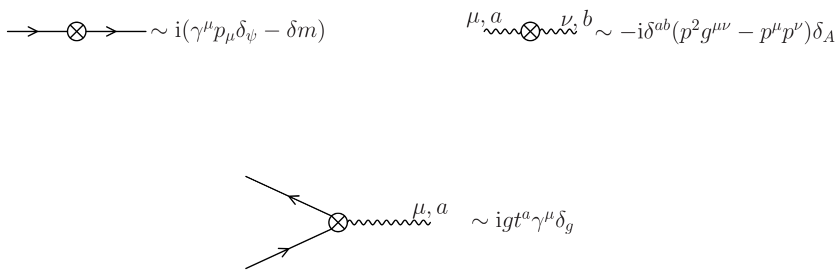


Figure 1: Counter term Feynman rules for fermion, gauge field and their interaction.

where three dots stand for three- and four-field interaction terms of the gauge field and ghost terms. Redefine mass and coupling by $Z_\psi m_0 = m + \delta m$, $g_0 Z_\psi Z_A^{1/2} = g Z_g$ and define counter terms $\delta_i = Z_i - 1$ for $i = \psi, A, g$. Furthermore one may redefine $\xi_0 Z_A^{-1} = \xi$ since ξ is just an arbitrary constant to choose the gauge. After these redefinitions the Lagrangian is

$$\mathcal{L}_{\mathrm{YM}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \mathrm{i}\bar{\psi}(\delta_\psi \not{\partial} - \delta m)\psi + g\delta_g t^a \bar{\psi}\not{A}^a \psi - \frac{1}{4}\delta_A(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \dots \,.$$

Similarly as before, three dots are for three and four self-interaction terms of the gauge field and ghost terms. The three last terms shown are counter terms for the fermion propagator, fermion-gauge-interaction and gauge field propagator. Feynman rules for these terms can be calculated as usually (see Appendix A) and they are written down in Figure 1.

2.2 Higgs Lagrangian

Next I will renormalise the Lagrangian of a real Higgs field. For the Higgs field the Lagrangian is similar to the Lagrangian of the ϕ^4-theory and coupling is the Higgs self-coupling λ. Essential terms in the bare Lagrangian are

$$\mathcal{L}_H \ni \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2} m^2_{h,0}\phi_0^2 - \frac{1}{4}\lambda_0 \phi_0^4 \,.$$

Surely the Higgs Lagrangian includes similar terms for Goldstone bosons. However, those terms are ignored since I was looking for counter terms only for the physical Higgs field.

Figure 2: Counter term Feynman rules for Higgs propagator and four-point interaction.

Rescale the Higgs field by $\phi_0 \to Z_\phi^{1/2}\phi$ similarly as the fields were rescaled in the case of a Yang-Mills theory. Redefining coupling by $\lambda_0 Z_\phi^2 = \lambda Z_\lambda$ and mass by $Z_\phi m_{h,0} = m_h + \delta m_h$ and defining counter terms by $\delta_\phi = Z_\phi - 1$ and $\lambda Z_\lambda = \lambda + \delta_\lambda$ yields

$$\mathcal{L}_H \ni \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_h^2\phi^2 - \frac{1}{4}\lambda\phi^4 + \frac{1}{2}\delta_\eta(\partial_\mu\phi)^2 - \frac{1}{2}\delta m_h \phi^2 - \frac{1}{4}\delta_\lambda\phi^4 \ .$$

The last three terms are the Higgs propagator, mass and four-point counter terms. Feynman rules for these terms are in Figure 2. Notice here the difference in the definition of the Higgs self-coupling counter term δ_λ to for example the coupling constant counter term in the case of the Yang–Mills theory. Counter term for the self-coupling λ is $\delta_\lambda = \lambda(Z_\lambda - 1)$ i.e. there is an additional λ on the right hand side. Due to this difference there is no λ in the Feynman rule for the Higgs four-point counter term.

2.3 Top Yukawa counter term

For the top Yukawa coupling there is an interaction term between the top quark and the Higgs boson. As a difference to what is done before is the change of handedness of the quark field in interaction with Higgs; the left- and right-handed components of the quark field have to be rescaled separately, and they will have different counter terms. With bare fields and couplings, the top Yukawa interaction term is

$$\mathcal{L}_{\mathrm{Y}} \ni \frac{y_{t,0}}{\sqrt{2}}\bar{t}_{0,\mathrm{R}} t_{0,\mathrm{L}} \phi_0 \ ,$$

where t_{L} (t_{R}) is the left-handed (right-handed) component of the top field and ϕ_0 is the neutral component of the Higgs doublet. Handling left- and right-handed components of the top quark as two different fields, rescaling of fields $\phi_0 \to Z_\phi^{1/2}\phi$ and $t_{0,\mathrm{L/R}} \to Z_{t,\mathrm{L/R}}^{1/2} t_{\mathrm{L/R}}$ with a redefinition of top Yukawa

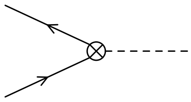


Figure 3: Feynman rule for Top Yukawa counter term.

coupling $y_{t,0}(Z_{t,\mathrm{L}}Z_{t,\mathrm{R}}Z_\phi)^{1/2} \equiv y_t Z_{y_t}$ yields

$$\mathcal{L}_\mathrm{Y} \ni (1+\delta_{y_t})\frac{y_t}{\sqrt{2}}\bar{t}_\mathrm{R} t_\mathrm{L}\phi \ ,$$

where the top Yukawa counter term is defined as $\delta_{y_t} = Z_{y_t} - 1$. Feynman rule for the top Yukawa counter term one can easily see from the equation above, and it is written down in Figure 3.

3 β-functions from the SM

As I discussed in the previous section, the running of the SM couplings is needed for the prediction of the Higgs mass. In this section I will derive β-functions of five SM coupling constants g_1, g_2, g_3, y_t and λ corresponding to the U(1), SU(2), SU(3), top Yukawa and Higgs self-interaction couplings respectively.

I start with deriving an expression of the β-function in the general case at one-loop level. I will then apply this expression to the special cases of SU(N), top Yukawa and Higgs self-interaction. After I have derived the β-function for a general SU(N) coupling, I get the β-functions for the SU(2) and SU(3) cases just by setting N = 2,3. The β-function for the U(1) coupling needs to be computed separately because all fermions couple differently to it.

For one-loop level calculations I have to choose a renormalisation scheme and fix the gauge. I choose to do calculations in the modified minimal subtraction renormalisation scheme, $\overline{\text{MS}}$ for short, in which counter terms are defined to 'eat' not only divergences of integrals but also finite terms proportional to $-\gamma_{\text{E}} + \log(4\pi)$, and I define $\frac{2}{\epsilon} - \gamma_{\text{E}} + \log(4\pi) \equiv \frac{2}{\epsilon_{\overline{\text{MS}}}}$. In this scheme calculations, especially those of three point functions, are easier since all masses can be ignored, as they only affect the finite parts of integrals. I also choose to do calculations in the Feynman gauge ($\xi = 1$). In this gauge there are more diagrams than in the Landau gauge ($\xi = 0$) since also ghosts contribute, but the diagrams with gauge boson self-couplings are easier to handle.

3.1 General form of β-function

Take a theory with a set of bare coupling constants $g_{j,0}$. In a d-dimensional spacetime (here it is defined $d \equiv 4 - \epsilon$) the bare couplings may not be dimensionless. Denote by α_j the mass dimension of a coupling $g_{j,0}$ in the d-dimensional spacetime. That is, $[g_{j,0}] = M^{\alpha_j}$ such that the product $g_{j,0}\mu^{-\alpha_j}$ is dimensionless. An interaction term with a coupling g_j in the bare Lagrangian is of the form $\mu^{-\alpha_j} g_{j,0}\psi^n_{i,0}$, where the fields ψ_i may be either scalar, fermion or gauge fields. Rescaling the fields ψ_i, as it was done in the previous section, yields

$$\mu^{-\alpha_j} g_{j,0} Z^{n/2}_{\psi_i} \psi^n_i \ .$$

Now redefining the coupling by $g_j Z_{g_j} = \mu^{-\alpha_j} g_0 Z^{n/2}_i$ gives

$$g_j = \mu^{-\alpha_j} g_{j,0} Z^{n/2}_{\psi_i} Z^{-1}_{g_j} \equiv \mu^{-\alpha} g_{j,0} Z_j \ .$$

The coupling g_j is now the renormalised coupling.

Recall that the β-function was defined to be the derivative of a coupling with respect to scale. Thus for coupling g_j

$$\beta_{g_j} = \mu \frac{\mathrm{d}g_j}{\mathrm{d}\mu} = -\alpha_j g_j + \mu^{-\alpha_j} g_{j,0} \mu \frac{\mathrm{d}Z_j}{\mathrm{d}\mu} \ .$$

How does Z_j depend on the scale? The answer is that Z_j does not depend explicitly on the scale but only implicitly via couplings. In general Z_j may depend on all couplings in the theory, and the dependence in the bare expansion is $Z_j = Z_j(\{g_{i,0}\mu^{-\alpha_i}\})$. Using the chain rule, one then finds

$$\beta_{g_j} \approx -\alpha_j g_j - g_j \sum_l \alpha_l g_l \frac{\partial}{\partial g_l} Z_j \ . \tag{8}$$

This result is an approximation at one-loop level because I replaced $g_{l,0}\mu^{\alpha_l}$ by g_l in the sum on the right hand side. Recall that counter terms were defined to be $\delta_i = Z_i - 1$. Thus a β-function can always be expressed with derivatives of counter terms with respect to coupling constants of the theory. Indeed, this is how I derive the β-functions for the SM couplings. After I have calculated the mass dimensions of SU(N), top Yukawa and Higgs self-interaction couplings, Equation (8) gives immediately the β-functions for them.

Remembering the fact that every term in the Lagrangian must have a dimension M^d in the d-dimensional spacetime, the mass dimensions of the SM couplings, consistent with $d = 4 - \epsilon$, are

$$\begin{cases} [\lambda] &= M^{\epsilon} \\ [y_f] &= M^{\epsilon/2} \\ [g_i] &= M^{\epsilon/2}, \quad \forall i = 1{,}2{,}3 \ . \end{cases} \tag{9}$$

Since I now have a general expression for a β-function and know the mass dimensions of the SM couplings, I can start deriving β-functions for them. For each coupling I find out an expression for the function Z with counter terms and then determine those counter terms in the $\overline{\mathrm{MS}}$ renormalisation scheme. I start with the SU(N) gauge coupling.

3.2 SU(N) gauge theory

To calculate the β-function for the SU(N) coupling constant g I have to decide which term in the Lagrangian I use. Due to the different combination of fields, Z-functions are not the same for different terms in the Lagrangian.

This leads to different combinations of counter terms. I choose to use the interaction term between a gauge boson and fermion[2].

After rescaling of fields this term is

$$g_0 Z_\psi Z_A^{1/2} t^a \bar{\psi} \not{A}^a \psi \ .$$

Function Z defined in the previous section is now

$$\begin{aligned} Z &= Z_\psi Z_A^{1/2} Z_g^{-1} = (1+\delta_\psi)(1+\delta_A)^{1/2}(1+\delta_g)^{-1} \\ &\approx 1 + \delta_\psi + \frac{1}{2}\delta_A - \delta_g \ . \end{aligned} \tag{10}$$

Furthermore, there is only one coupling with the mass dimension $[g] = M^{\epsilon/2}$. Using the general form (8), the β-function for coupling g is

$$\beta_g = -\frac{\epsilon}{2} g \left(1 + g \frac{\partial}{\partial g} \left(\delta_\psi + \frac{1}{2}\delta_A - \delta_g \right) \right) \ . \tag{11}$$

Thus to get the running of coupling g I have to calculate three counter terms: fermion and gauge boson wave function counter terms δ_ψ and δ_A and counter term for the coupling g itself. The wave function counter terms one obtains by renormalising the fermion and the gauge boson propagators at one-loop level and the counter term for g by renormalising the three-point function between the fermion and the gauge boson. Feynman rules for these counter terms I already derived in Section 2.1, see Figure 1.

3.2.1 Counter terms

In the general Yang–Mills theory there is only one gauge field that couples to fermions. The situation needed in this thesis is however more complicated, because in the SM there are three different gauge fields, several different fermions and one scalar. To handle this I divide the problem in different parts. I start with just one gauge field coupled to fermions and later on discuss how adding a new gauge field or scalar affects these results.

Let us start with fermion propagator. Since there is only one gauge field that interacts with the fermion, there is only one one-loop diagram, shown in Figure 4. Using the labels written in the figure this diagram is (for Feynman

[2]The β-function is the same despite of which term in the Lagrangian and which set of counter terms one uses. I chose the vertex between fermions and gauge field just for that counter terms needed in that case are much easier compared to for example the gauge boson three-vertex.

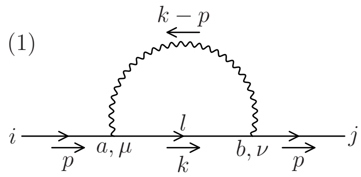


Figure 4: One-loop correction to fermion propagator given by a gauge field.

rules see Appendix A)

$$-\mathrm{i}\Sigma_\psi^{(1)} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}(-\mathrm{i}gt^b_{jl}\gamma^\nu)\frac{\mathrm{i}\not{k}}{k^2+\mathrm{i}\epsilon_\mathrm{F}}(-\mathrm{i}gt^a_{li}\gamma^\mu)\frac{-\mathrm{i}g_{\mu\nu}\delta^{ab}}{(p-k)^2+\mathrm{i}\epsilon_\mathrm{F}} \, .$$

In a d-dimensional spacetime

$$\begin{aligned}-\mathrm{i}\Sigma_\psi^{(1)} &= -g^2 t^a_{jl}t^a_{li}\mu^{4-d}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^\nu\not{k}\gamma_\nu}{(k^2+\mathrm{i}\epsilon_\mathrm{F})((p-k)^2+\mathrm{i}\epsilon_\mathrm{F})}\\ &= -g^2C_2(r)\delta_{ij}\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-(d-2)\not{k}}{(k^2+\mathrm{i}\epsilon_\mathrm{F})((p-k)^2+\mathrm{i}\epsilon_\mathrm{F})} \, ,\end{aligned}$$

where in the last step I used identity (1b) for γ-matrices and (7) for the product of two generators. Notice here that the indices i, j and l are fermion indices. Thus r in $C_2(r)$ refers to the representation of group SU(N) in which fermions are. Now Passarino–Veltman reduction integrals (see Appendix B) yield

$$\begin{aligned}-\mathrm{i}\Sigma_\psi^{(1)} &= g^2C_2(r)\delta_{ij}(d-2)\gamma^\mu B_\mu(p,0,0)\\ &= g^2C_2(r)\delta_{ij}(2-\epsilon)\gamma^\mu\frac{p_\mu}{p^2}\frac{1}{2}q^2B_0(p,0,0)\\ &= \mathrm{i}\delta_{ij}\not{p}\left(\frac{2g^2C_2(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}+\text{f.t.}\right) \, , \end{aligned} \tag{12}$$

where 'f.t.' refers to 'finite terms', i.e. terms that are either constants or at least linear in ϵ .

Recall that the Feynman rule for the fermion wave function counter term is $\mathrm{i}\not{p}\delta_{ij}\delta_\psi$, and in the $\overline{\mathrm{MS}}$-scheme a counter term is defined to be such that it cancels the $1/\epsilon_{\overline{\mathrm{MS}}}$-divergence. Thus one needs to define δ_ψ to be exactly the opposite of the factor multiplying $\mathrm{i}\delta_{ij}\not{p}$:

$$\delta_\psi = -\frac{2g^2C_2(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} \, . \tag{13}$$

Consider next one-loop corrections to the vertex between a fermion and a gauge field. There are two one-loop diagrams. One involves a virtual

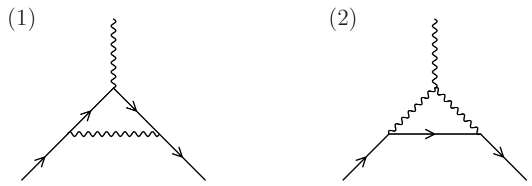


Figure 5: One-loop corrections to fermion-gauge-vertex in the case that there is only one gauge field.

gauge boson between two outgoing fermions and the other virtual gauge bosons between fermions and outgoing gauge boson, see Figure 5. A simple calculation shows that these diagrams give contributions

$$-\mathrm{i}\Gamma_g^{(1)} = \mathrm{i}g t^a_{ij}\gamma^\mu \left(\frac{2g^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}(C_2(r) - \frac{1}{2}C_2(G)) + \text{f.t.} \right)$$

and

$$-\mathrm{i}\Gamma_g^{(2)} = \mathrm{i}g t^a_{ij}\gamma^\mu \left(\frac{3g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) .$$

Comparing the two previous equations to the counter term Feynman rule for the fermion-gauge vertex in Figure 1, the counter term for the coupling g is

$$\delta_g = -\frac{2g^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}(C_2(r) + C_2(G)) . \tag{14}$$

Here again r refers to the fermion representation and G to the adjoint representation for the gauge field.

I have now calculated one-loop corrections to the fermion propagator and vertex between fermions and gauge boson to get counter terms δ_ψ and δ_g. The one still missing is δ_A. To get that I have to calculate one-loop corrections to the gauge boson propagator. Recall that in a Yang–Mills theory there are also ghosts to take into account. Figure 6 shows all one-loop diagrams which are formed by adding a fermion, a gauge boson or a ghost loop to the tree-level propagator. The fermion loop is

$$-\mathrm{i}\Pi_A^{ab\mu\nu,(1)} = -\mathrm{i}(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta^{ab} \left(\frac{8}{3}\frac{g^2 C(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} N_{\mathrm{F}} + \text{f.t.} \right) , \tag{15}$$

where N_{F} is the number of different fermions coupled to the gauge field. Notice here that I have assumed that for all fermions the coupling to gauge

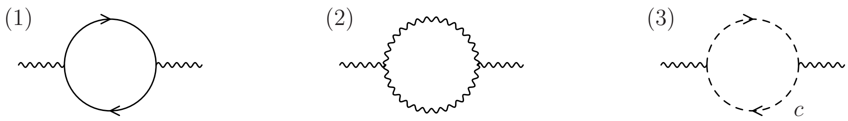


Figure 6: One-loop corrections to gauge boson propagator given by fermions, gauge field itself and ghosts.

bosons is the same. The gauge boson itself and ghosts turn out to give contributions

$$-\mathrm{i}\Pi_A^{ab\mu\nu,(2)} = \left(-\frac{11}{3}p^\mu p^\nu + \frac{19}{6}p^2 g^{\mu\nu}\right)\delta^{ab}\left(\frac{\mathrm{i}g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.}\right)$$

and

$$-\mathrm{i}\Pi_A^{ab\mu\nu,(3)} = \left(\frac{1}{3}p^\mu p^\nu + \frac{1}{6}p^2 g^{\mu\nu}\right)\delta^{ab}\left(\frac{\mathrm{i}g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.}\right).$$

One can see that gauge boson and ghost diagrams themselves are not gauge invariant since they are not proportional to the gauge invariant factor $(p^2 g^{\mu\nu} - p^\mu p^\nu)$. However, adding these two diagrams up gives

$$-\mathrm{i}\Pi_A^{ab\mu\nu,(2+3)} = -\mathrm{i}(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\left(-\frac{10}{3}\frac{g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.}\right), \tag{16}$$

so that the sum is gauge invariant as it should be. From equations (15) and (16) one obtains

$$\delta_A = \frac{10}{3}\frac{g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} - \frac{8}{3}\frac{g^2 C(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}N_{\mathrm{F}}\ . \tag{17}$$

Now I have derived the three counter terms needed for the β-function for the coupling g.

3.2.2 β-function for the Yang-Mills theory

The β-function for the Yang-Mills theory can now be calculated from Equation (11). Substituting counter terms (13), (14) and (17) into (11) yields

$$\begin{aligned}\beta_g &= -\frac{\epsilon}{2}g\Big(1 + g\Big(-\frac{4g}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}C_2(r) + \frac{4g}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}(C_2(r) + C_2(G)) \\ &\quad - \frac{8}{3}\frac{gC(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}N_{\mathrm{F}} + \frac{10}{3}\frac{g}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}C_2(G)\Big)\Big) \\ &= \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C_2(G) + \frac{4}{3}C(r)N_{\mathrm{F}}\right) + \mathcal{O}(\epsilon)\end{aligned} \tag{18}$$

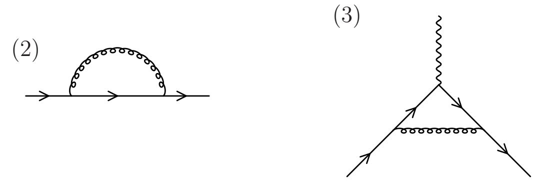


Figure 7: One-loop corrections given by the new gauge field.

where in the last step I used that $\frac{\epsilon}{\epsilon_{\overline{\rm MS}}} = 1 + \mathcal{O}(\epsilon)$. Equation (18) holds for the pure Yang–Mills theory. In the next section it is discussed how adding a new gauge or scalar field to theory affects the Yang–Mills β-function.

3.3 SU(N) theory with additional scalar and gauge fields

So far I have assumed that there is only one gauge boson and not any scalars. As I mentioned before, in the case of the SM I have to know how different gauge fields affect each other, and there is one scalar field too. Next I will discuss what kind of diagrams there are if there is a new gauge or scalar field, and what kind of contributions these diagrams give to counter terms.

Adding a new gauge field Consider the case in which a new gauge field is added to the theory I was considering before. This new field does interact with fermions but not with the original gauge field. Since fermions interact with the new gauge field, there is a new diagram for the fermion propagator similar to that in Figure 4 involving the new gauge field. Considering fermion-gauge-vertex, in diagram (1) in Figure 5 one may change the gauge field between two fermions to the new one. At one-loop level the new field does not change the gauge field propagator because gauge fields are not coupled to each other. Thus corrections by the new gauge field are one diagram to the fermion propagator and to the fermion-gauge-vertex shown in Figure 7.

The correction to the fermion propagator is obviously the same as (12) since the calculation did not depend on which gauge field one was considering. Thus

$$-\mathrm{i}\Sigma_{\psi}^{(2)} = \mathrm{i}\delta_{ij}\not{q}\left(\frac{2g'^2 C_2(r')}{16\pi^2\epsilon_{\overline{\rm MS}}} + \text{f.t.}\right) , \tag{19}$$

where g' is a coupling between fermion and the new gauge field and r' is the representation of fermion in the new gauge group. To calculate the correction to the fermion-gauge-vertex, note that a fermion carries an index related to both of the two gauge fields. Fermion index related to an interaction with one gauge field does not change in an interaction with the other. Keeping this in mind the contribution of the diagram (3) in Figure 7 is

$$-\mathrm{i}\Gamma_g^{(3)} = \mathrm{i}g t^a_{ji}\gamma^\mu \left(\frac{2g'^2 C_2(r')}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.} \right) . \tag{20}$$

One can see here that (19) gives exactly the same contribution to the fermion wave function counter term δ_ψ as (20) gives to coupling constant counter term δ_g. In the formula of β_g, Equation (11), these counter terms appear with opposite signs, which means that corrections from (19) and (20) cancel each other. Hence, adding a new gauge field to the theory does not change the β-function of coupling g if two gauge fields do not interact with each other.

Adding a scalar field Adding a scalar field to the theory is a little more complicated compared to adding a gauge field. Difficulties arise from the coupling between a fermion and a scalar, i.e. the Yukawa coupling. Also corrections given by the scalar depend not only on the representation in which the scalar is, but also on whether the scalar couples to the gauge field or not.

There are two cases of interest in the SM. In the first one the original gauge field corresponds to a gluon which does not interact with the Higgs field. In this case there are only two one-loop diagrams similarly as in the case of an additional gauge field: fermion propagator diagram and vertex correction in which the scalar is added between two fermions. See Figure 8.

Using Feynman rules from the Appendix A, the fermion two-point diagram in Figure 8 is

$$-\mathrm{i}\Sigma_\psi^{(3)} = \mathrm{i}\delta_{ji}\not{q} \left(\frac{y_f^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.} \right) . \tag{21}$$

The vertex correction is

$$-\mathrm{i}\Gamma_g^{(4)} = \mathrm{i}g t^a_{ji}\gamma^\mu \left(\frac{y_f^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.} \right) \tag{22}$$

from which one again can see that fermion propagator and vertex contribution cancel each other as in the case of the additional gauge field. Hence, there is no scalar contribution to the β-function of the coupling g if the scalar does not interact with the gauge field.

In the second case, the original gauge field corresponds to SU(2) field in the SM, and the new scalar, the Higgs field, interacts with it. In addition to

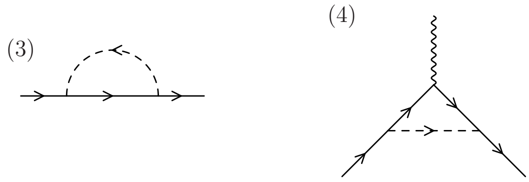


Figure 8: One-loop corrections to fermion propagator and fermion-gauge-vertex given by the new scalar field.

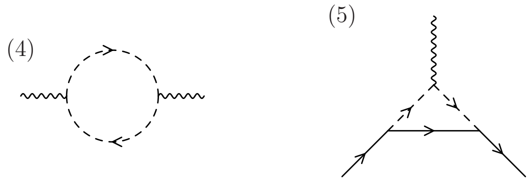


Figure 9: One-loop corrections to gauge field propagator and fermion-gauge-vertex given by the new scalar field if the scalar is coupled to the original gauge field.

corrections in Figure 8, there is a correction to the gauge field propagator and a correction to the fermion-gauge-vertex for which in the loop there are two scalar fields and one fermion. The two new diagrams are shown in Figure 9.

To calculate these diagrams, suppose that the scalar is in the fundamental representation. Here the chirality of the fermion has to be taken into account.

The correction to the fermion propagator, diagram (3) in Figure 8, remains the same. The gauge field diagram with a scalar loop, number (4) in Figure 9, is

$$-\mathrm{i}\Pi_A^{ab\mu\nu,(4)} = -\mathrm{i}(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{2}{3}\frac{g^2 C(r')}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}N_{\mathrm{S}} + \mathrm{f.t.}\right) , \tag{23}$$

where N_{S} is the number of scalars coupled to the gauge field and r' is the representation of the gauge group in which scalars are[3].

The two vertex corrections in Figure 8 and 9 are the most difficult ones. Let us first consider the one in Figure 8, where in the loop there are two

[3]Here I chose the scalar to be in the fundamental representation but it turns out that the result is the same if one chooses the adjoint representation. I do not prove this, but I keep the representation general since the result is.

fermion and one scalar propagator. The outgoing gauge field is now the SU(2) field, which couples only to the left-handed fermion. Interaction with the scalar changes the handedness of the fermion. Thus the two outgoing fermions can not be left-handed. Indeed, if they were, in the loop fermions should be right-handed, but they do not interact with the SU(2) field. Hence, outgoing fermions are right-handed, and the two fermions in the loop are left-handed. The coupling between fermions and gauge field is proportional to the group generator t^a_{ji}, where j,i are fermion indices. Since right-handed fermions do not carry the fermion index, the two vertices between the scalar and fermion fields forces the fermion indices to be the same, and there is a sum over fermion indices. Thus the diagram (4) in Figure 8 is proportional to the trace of the group generator, which is zero. As a conclusion, this diagram gives no contribution.

The only correction left to the fermion-gauge-vertex is the diagram (5) in Figure 9 with two scalar and one fermion propagators in the loop and with the SU(2) field as an external gauge field. By similar reasoning as before, if the outgoing fermions are right-handed, this diagram gives no contribution since it is proportional to the trace of the group generator. For left-handed outgoing fermions diagram is

$$-\mathrm{i}\Gamma^{(5)}_g = \mathrm{i}g t^a_{ji}\gamma^\mu P_L \left(\frac{y_f^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) . \tag{24}$$

This is the only correction to the fermion-gauge-vertex in the case where the original gauge field and the additional scalar are coupled. Comparing the correction to the counter term δ_g given by (24) to the correction to counter term δ_ψ given by (21), one can see that again these corrections are exactly the same. Hence, the only remaining contribution by the scalar field is to the gauge field propagator from Equation (23).

3.3.1 β-function for SU(N) theory with additional scalar field

The previous calculations showed that if one adds a gauge or scalar field to the general Yang–Mills theory, the only change to counter terms that affects the β-function is a correction to the gauge field propagator given by the scalar field. The corrected counter term is

$$\delta_A = \frac{10}{3}\frac{g^2 C_2(G)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} - \frac{8}{3}\frac{g^2 C(r)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} N_\mathrm{F} + \frac{2}{3}\frac{g^2 C(r')}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} N_\mathrm{S} \tag{25}$$

from Equations (17) and (23). With this change, the β-function is

$$\beta_g = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C_2(G) + \frac{4}{3}C(r)N_\mathrm{F} + \frac{1}{3}C(r')N_\mathrm{S} \right) + \mathcal{O}(\epsilon), \tag{26}$$

where N_{F} (N_{S}) is the number of fermions (scalars) coupled to the gauge field, G is for the adjoint representation of the symmetry group and r and r' respectively the representations of the symmetry group in which fermions and scalars are. In general one chooses both fermions and scalars to be in the fundamental representation, which yields (see Section 2.1)

$$\beta_g = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}N + \frac{2}{3}N_{\mathrm{F}} + \frac{1}{6}N_{\mathrm{S}}\right) , \tag{27}$$

where I dropped the terms proportional to ϵ, for the SU(N) gauge coupling g.

3.4 SU(2) and SU(3) gauge theories

Now I am ready to write down β-functions for the two non-Abelian gauge theories of SM using the general form of the SU(N) β-function (27).

In SU(2) isospin symmetry $N = 2$, $N_{\mathrm{F}} = 6$ (for 3 different lepton and 3 different quark families) and $N_{\mathrm{S}} = 1$ (for Higgs boson). Substituting these into Equation (27) the β-function of the SU(2) coupling g_2 is

$$\beta_{g_2}^{SM} = -\frac{19 g_2^3}{96\pi^2} . \tag{28}$$

Correspondingly in SU(3) colour symmetry $N = 3$, $N_{\mathrm{F}} = 6$ (for different flavours of quarks) and $N_{\mathrm{S}} = 0$ yielding

$$\beta_{g_3}^{SM} = -\frac{7 g_3^3}{16\pi^2} . \tag{29}$$

These were two of the three gauge theories in the SM. Next I turn to the case of the U(1) theory which needs some extra discussion.

3.5 U(1) gauge theory

The β-function of the U(1) gauge coupling cannot be obtained from the previous results because each fermion and scalar couples differently to the U(1) field. Indeed, unlike in the SU(N) theory where the coupling between the gauge field and any fermion was $\mathrm{i}g t^a_{ji}\gamma^\mu$, in the case of the U(1) theory the coupling depends not only on the fermion type but also handedness of fermion. The coupling is $\mathrm{i}g_1\frac{Y_{L,f}}{2}$ for a left-handed and $\mathrm{i}g_1\frac{Y_{R,f}}{2}$ for a right-handed fermion, where $Y_{L,f}$ and $Y_{R,f}$ are hypercharges of the fermion (see Table 1) defined by the so called Gell-Mann–Nishijima formula [8]

$$Y = 2(Q - T^3) ,$$

where Q is the electric charge and T^3 is the third component of weak aisospin of the particle. Making a replacement $gt^a_{ji} \to g_1 \frac{Y_{L,f}}{2}\delta_{ij}, g_1 \frac{Y_{R,f}}{2}\delta_{ij}$, the one-loop correction to the fermion propagator similar to the diagram in Figure 4 is

$$-\,\mathrm{i}\Sigma_\psi = \mathrm{i}\delta_{ij}\not{p}\left(\frac{2g^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}\frac{1}{2}\left(\left(\frac{Y_{L,f}}{2}\right)^2 + \left(\frac{Y_{R,f}}{2}\right)^2\right) + \text{f.t.}\right) . \tag{30}$$

The extra half on the right hand side came from the fact that left-handed and right-handed components of one fermion are considered separately. Comparing Equation (30) to (12), one can see that the only difference is that $C_2(r)$ in (12) is replaced with sum

$$\frac{1}{2}\left(\left(\frac{Y_{L,f}}{2}\right)^2 + \left(\frac{Y_{R,f}}{2}\right)^2\right) .$$

The same happens with all other fermion diagrams. Thus in the Yang–Mills β-function (26) one can make a replacement

$$\frac{4}{3}C(r)N_{\mathrm{F}} \to \frac{4}{3}\sum_f \frac{1}{2}\left(\left(\frac{Y_{L,f}}{2}\right)^2 + \left(\frac{Y_{R,f}}{2}\right)^2\right) = \frac{1}{6}\sum_f \left(Y^2_{L,f} + Y^2_{R,f}\right) .$$

Note that int taking the sum over all fermions, one has to take into account different families as well as different colours for quarks.

Similarly for scalars, he coupling $\mathrm{i}gt^a_{ij}$ is replaced by $\mathrm{i}g_1\frac{Y_\phi}{2}$, where Y_ϕ is the hypercharge of a scalar, yielding

$$\frac{1}{3}C(r)N_{\mathrm{S}} \to \frac{1}{3}\sum_\phi \left(\frac{Y_\phi}{2}\right)^2 = \frac{1}{12}\sum_\phi Y^2_\phi .$$

The term proportional to $C_2(G)$ in (26) is due to the ghost diagram and diagrams with the gauge field self-interaction. Thus for the U(1) theory one has to drop this term.

With these changes to (26), the β-function of the U(1) theory is

$$\beta^{SM}_{g_1} = \frac{g_1^3}{16\pi^2}\left(\frac{1}{6}\sum_f \left(Y^2_{L,f} + Y^2_{R,f}\right) + \frac{1}{12}\sum_\phi Y^2_\phi\right) .$$

Substituting hypercharges from Table 1 one finds

$$\beta^{SM}_{g_1} = \frac{41 g_1^3}{96\pi^2} . \tag{31}$$

particle	charge Q	isospin T^3	hypercharge Y
l_L	-1	$-1/2$	-1
l_R	-1	0	-2
ν_L	0	$1/2$	-1
u_L	$2/3$	$1/2$	$1/3$
u_R	$2/3$	0	$4/3$
d_L	$-1/3$	$-1/2$	$1/3$
d_R	$-1/3$	0	$-2/3$
ϕ^0	0	$-1/2$	1
ϕ^+	$+1$	$1/2$	1

Table 1: Charges, isospins and hypercharges of the SM particles. Here $l_{L,R}$ denotes left- and right-handed leptons, $u_{L,R}$ upper quarks, $d_{L,R}$ lower quarks, ν_L neutrinos, and $\phi^{0,+}$ neutral and charged scalar fields.

This was the last one of the gauge couplings in the SM. Notice here that for every gauge coupling in the SM the running depends only on the coupling itself. This fact was not obvious since fermions interact with all of them and thus at one-loop level could have induced dependence on other gauge couplings. But as I showed in Section 3.3, the correction by other gauge fields to the fermion wave function counter term cancels out the correction to counter term for the coupling itself, and thus there is no contribution to the β-function.

3.6 Higgs self-coupling

The next β-function to calculate is the one for Higgs self-coupling λ. In this section I start to use physical Higgs and gauge fields. Feynman rules for those are found in Appendix A.2.

For the β-function of the self-coupling λ I use the Higgs four-point interaction term. With the renormalised Higgs field that term is

$$\frac{1}{4}\lambda_0 Z_\phi^2 \phi^4 \ .$$

Recall that I defined the Higgs self-coupling counter term by $\lambda Z_\lambda \equiv \lambda + \delta_\lambda$, which gives $Z_\lambda = 1 + \frac{1}{\lambda}\delta_\lambda$. Thus the Z-function in the case of the self-coupling

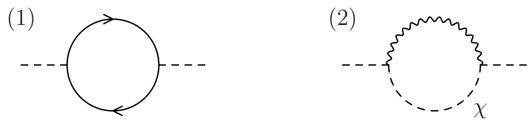


Figure 10: Diagrams contributing to Higgs wave function renormalisation at one loop.

λ is

$$Z = Z_\phi^2 Z_\lambda^{-1} \approx 1 + 2\delta_\phi - \frac{1}{\lambda}\delta_\lambda$$
$$= 1 + \frac{1}{\lambda}(2\lambda\delta_\phi - \delta_\lambda) \ . \tag{32}$$

Inserting this into the expression (8) for a general β-function gives

$$\beta_\lambda = -\epsilon\lambda\Big(1 + \Big(\lambda\frac{\partial}{\partial\lambda} + \frac{1}{2}\Big(g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2} + y_t\frac{\partial}{\partial y_t}\Big)\Big)\frac{1}{\lambda}(2\lambda\delta_\phi - \delta_\lambda)\Big) \ . \tag{33}$$

Here I have used Equation (9) for mass dimensions of the SM couplings. Furthermore, Higgs propagator and four-point function cannot include gluons at one-loop level because in all diagrams particle propagating in a loop must be coupled to Higgs. Thus the counter terms δ_ϕ and δ_λ cannot depend on the strong coupling constant g_3, and I am allowed to drop the derivative with respective to g_3 in (33).

3.6.1 Higgs self-coupling and wave function counter terms

To get the counter terms δ_ϕ and δ_λ one has to calculate the Higgs propagator and four-point function at one-loop level. I start with the wave function renormalisation counter term δ_ϕ. There are quite many diagrams for the Higgs propagator at one-loop level, but fortunately only two of them give a contribution to δ_ϕ, see Figure 10. Diagrams contributing to δ_ϕ must be proportional to Higgs momentum squared p^2, and thus there has to be a momentum variable in the numerator of the loop-integral. There are two possibilities to get this: either there are fermions in the loop, or the vertex Feynman rule is proportional to Higgs momentum.

The first case with fermions in the loop, shown by the first diagram in Figure 10, gives a contribution

$$-\mathrm{i}\Pi_\phi^{(1)} = \mathrm{i}p^2\left(\frac{6y_f^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right) \ . \tag{34}$$

The other relevant correction is given by a gauge boson and a Goldstone boson in the loop because the three-vertex between Higgs, gauge boson and Goldstone boson is the only vertex proportional to Higgs momentum, see Feynman rules in Appendix A.2. There are two possibilities: either both the gauge boson and Goldstone are neutral, or both of them are charged. These two cases are included in the second diagram in Figure 10 and they give

$$\begin{aligned} -\mathrm{i}\Pi_\phi^{(2)} &= \mathrm{i}p^2 \left(-\frac{g_2^2 + g_1^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} - \frac{2g_2^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) \\ &= \mathrm{i}p^2 \left(-\frac{3g_2^2 + g_1^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) . \end{aligned} \tag{35}$$

Extracting divergent parts from Equations (34) and (35), the Higgs wave function counter term is found to be

$$\delta_\phi = \frac{1}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} \left(3g_2^2 + g_1^2 - 6y_f^2 \right) . \tag{36}$$

Next in turn there is the Higgs self-coupling counter term δ_λ. In Section 2.2 I got the four-point counter term Feynman rule to be $-\frac{\mathrm{i}}{4}\delta_\lambda$. Since there are identical particles, this diagram has to be calculated as the others. The symmetry factor of the diagram is $S = \frac{1}{4!}$ so

$$-\mathrm{i}\Gamma_\phi^{\delta_\lambda} = \frac{1}{S} \left(-\frac{\mathrm{i}}{4}\delta_\lambda \right) = -6\mathrm{i}\delta_\lambda . \tag{37}$$

The six diagrams for the Higgs four-point function at one-loop are shown in Figure 11. There are in fact more diagrams, but the rest are finite and thus do not affect the counter terms in the $\overline{\mathrm{MS}}$-scheme. Note here that all but the box diagrams, diagrams (4) and (5), must be permuted, i.e. take into account s-, t- and u-channels. Furthermore, since diagrams include many identical particles, symmetry factors are highly non-trivial.

The first correction is simply given by Higgs itself and is

$$-\mathrm{i}\Gamma_\lambda^{(1)} = -6\mathrm{i} \left(-\frac{18\lambda^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) . \tag{38}$$

The next two diagrams are given by neutral and charged Goldstone and gauge bosons. Their contributions are

$$-\mathrm{i}\Gamma_\lambda^{(2)} = -6\mathrm{i} \left(-\frac{6\lambda^2}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.} \right) \tag{39}$$

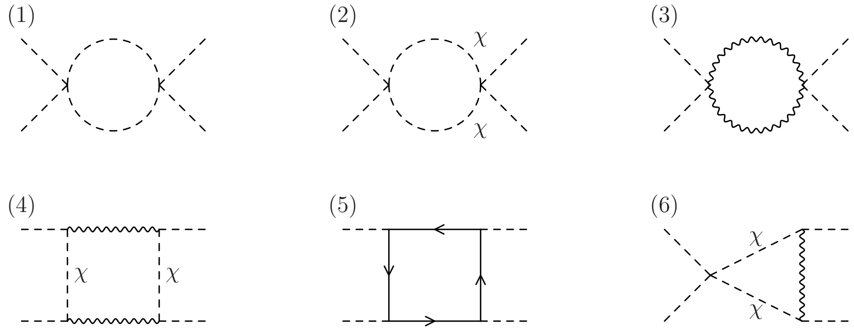


Figure 11: Higgs four-point self interaction at one-loop level. All but the box diagrams, diagrams (4) and (5), one must permute, i.e. take s-, t- and u-channels.

and

$$-\mathrm{i}\Gamma_\lambda^{(3)} = -6\mathrm{i}\left(-\frac{1}{4}\frac{(g_2^2+g_1^2)^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} - \frac{1}{2}\frac{g_2^4}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right) . \tag{40}$$

Box diagrams are finite unless there is at least a factor k^4 of the loop-momentum in the numerator since the denominator is of the order $(k^2)^4 = k^8$. As I discussed before, only the fermion propagator and the gauge-Goldstone-Higgs-vertex give k to the numerator. Hence, the only box diagrams which are not finite are the fermion box and a box with two gauge fields and Goldstones in turn, i.e. diagrams (4) and (5) in Figure 11. Of course, in the diagram with gauge fields and Goldstone bosons there are both neutral and charged particles included. These boxes give

$$-\mathrm{i}\Gamma_\lambda^{(4)} = -6\mathrm{i}\left(-\frac{1}{8}\frac{(g_2^2+g_1^2)^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} - \frac{1}{4}\frac{g_2^4}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right) \tag{41}$$

and

$$-\mathrm{i}\Gamma_\lambda^{(5)} = -6\mathrm{i}\left(\frac{6y_t^4}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right) . \tag{42}$$

The last diagram, number (6) in Figure 11, is a triangle diagram with two Goldstones and one gauge field. It is also permuted in three channels for both charged and neutral fields and gives

$$-\mathrm{i}\Gamma_\lambda^{(6)} = -6\mathrm{i}\left(\frac{\lambda(g_2^2+g_1^2)}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \frac{2\lambda g_2^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right) . \tag{43}$$

From Equations (38), (39), (40), (41), (42) and (43) the counter term δ_λ for self-coupling can be read remembering the factor $-6i$ in Equation (37) and that δ_λ must cancel all divergences in one-loop diagrams. Thus

$$\delta_\lambda = \frac{1}{16\pi^2 \epsilon_{\overline{\mathrm{MS}}}} \left(24\lambda^2 - 6y_f^4 + \frac{3}{8}\left(2g_2^4 + \left(g_2^2 + g_1^2\right)^2\right) - \lambda\left(3g_2^2 + g_1^2\right)\right) . \quad (44)$$

When I derived the δ_ϕ and δ_λ counter terms I just calculated diagrams for any fermions with a Yukawa coupling y_f. However, in the beginning of Section 2 I explained that I am allowed to forget all Yukawa couplings but that of top quark. Thus whenever there is a coupling y_f, it may be replaced with the top Yukawa coupling y_t.

3.6.2 β-function for Higgs self-coupling

Above I derived expressions for the Higgs wave function and self-coupling counter terms. Substituting into (33) these counter terms from (36) and (44), the β-function of the Higgs self-coupling is found to be

$$\beta_\lambda^{SM} = \frac{1}{16\pi^2}\Big(24\lambda^2 - 6y_t^4 + \frac{3}{8}\left(2g_2^4 + \left(g_2^2 + g_1^2\right)^2\right) + 12\lambda y_t^2 - 3\lambda\left(3g_2^2 + g_1^2\right)\Big) . \quad (45)$$

One can see here that to get running of the Higgs self-coupling, running of the all other SM couplings is needed. I already have calculated the cases of the SM gauge couplings. Thus the only one left is the running of the top Yukawa coupling.

3.7 Top Yukawa coupling

As in the case of Higgs self-coupling, the calculations in this section are done using physical gauge and Higgs fields. To calculate the running of the top Yukawa coupling I use the top mass term which in the bare Lagrangian is

$$\frac{y_{t,0} v_0}{\sqrt{2}} \bar{t}_{\mathrm{R},0} t_{\mathrm{L},0} \, .$$

Here the left- and right-handed top fields can be rescaled as before, $t_{\mathrm{L},0} \to Z_{\mathrm{L}}^{1/2} t_{\mathrm{L}}$ and $t_{\mathrm{R},0} \to Z_{\mathrm{R}}^{1/2} t_{\mathrm{R}}$. The Higgs vev is renormalised as the Higgs field, i.e. $v_0 \to Z_\phi v$. This yields

$$\frac{y_{t,0} v_0}{\sqrt{2}} \bar{t}_{\mathrm{R},0} t_{\mathrm{L},0} \to \frac{y_{t,0} v}{\sqrt{2}} \left(Z_{t,\mathrm{L}} Z_{t,\mathrm{R}} Z_\phi\right)^{1/2} \bar{t}_{\mathrm{R}} t_{\mathrm{L}} \, . \quad (46)$$

There is a novelty considering the Higgs vev. In Section 6 I will show that the Higgs vacuum expectation value is a scale dependent quantity as well as mass or coupling. Denoting by $\hat{v}$ the scale independent, physical Higgs vev which is known to have a value $\hat{v} \approx 246\,\text{GeV}$, one-loop correction to the scale dependent vev is

$$v = \hat{v}(1 + \frac{1}{2}\delta_v) \ , \tag{47}$$

where, ignoring finite parts, I will find

$$\delta_v = \frac{1}{16\pi^2 \epsilon_{\overline{\text{MS}}}}(3g_2^2 + g_1^2) \ . \tag{48}$$

Equation (46) with (47) now defines the Z-function in the case of top Yukawa to be

$$\begin{aligned} Z &= (1 + \frac{1}{2}\delta_v)(Z_{t,\text{L}} Z_{t,\text{R}} Z_\phi)^{1/2} Z_{y_t}^{-1} \\ &\approx 1 + \frac{1}{2}(\delta_v + \delta_{t,\text{L}} + \delta_{t,\text{R}} + \delta_\phi) - \delta_{y_t} \ . \end{aligned} \tag{49}$$

Here the counter terms $\delta_{t,\text{L/R}}$ are the wave function counter terms for left- and right-handed parts of the top propagator, δ_ϕ is the Higgs wave function counter term, already derived in the previous section, and δ_{y_t} is the counter term for the interaction between the top quark and Higgs. Equation (8) with (49) yields

$$\begin{aligned} \beta_{y_t} = &-\frac{\epsilon}{2} y_t - \epsilon y_t \left(\lambda \frac{\partial}{\partial \lambda} + \frac{1}{2} \left(g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2} + g_3 \frac{\partial}{\partial g_3} + y_t \frac{\partial}{\partial y_t} \right) \right) \\ &\times \left(\frac{1}{2}(\delta_v + \delta_{t,\text{L}} + \delta_{t,\text{R}} + \delta_\phi) - \delta_{y_t} \right) \ . \end{aligned} \tag{50}$$

In the next section I will derive the remaining three counter terms, namely $\delta_{t,\text{L/R}}$ and δ_{y_t}. These counter terms are found by calculating the top propagator and Yukawa vertex at one-loop level.

3.7.1 Top Yukawa and top wave function counter terms

Calculate the counter term δ_t at first. Possible one-loop diagrams of the top propagator are the one with a Higgs loop, the one with a photon, Z-boson or W-boson loop and the third with a gluon loop, see Figure 12. The diagram with the Higgs boson is the easiest one and gives a contribution

$$-\text{i}\Sigma_t^{(1)} = \text{i}\not{p} \left(\frac{y_t^2}{2} \frac{1}{16\pi^2 \epsilon_{\overline{\text{MS}}}} + \text{f.t.} \right) (P_\text{L} + P_\text{R}) \ . \tag{51}$$

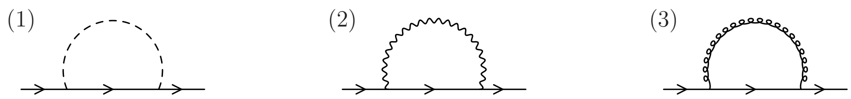


Figure 12: Diagrams forming top quark propagator at one-loop level. The diagram (2) stands for photon, Z and W loops.

The correction is the same for both left- and right-handed components. The photon diagram does not care about handedness as the Higgs diagram, but for the W- and Z-boson loops the correction is different for different components. The sum of Z, W and photon loops, collectively depicted by diagram (2) in Figure 12, is given by

$$-\mathrm{i}\Sigma_t^{(2)} = \mathrm{i}\not{p}\left(\frac{3}{2}\frac{g_2^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \frac{1}{18}\frac{g_1^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right)P_{\mathrm{L}} \tag{52}$$

$$+\mathrm{i}\not{p}\left(\frac{8}{9}\frac{g_1^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right)P_{\mathrm{R}}\ . \tag{53}$$

For the gluon loop the calculation is very similar to the one with an SU(N) gauge boson in Section 3.2. Diagram (3) in Figure 12 is given by

$$-\mathrm{i}\Sigma_t^{(3)} = \mathrm{i}\not{p}\left(\frac{8g_3^2}{3}\frac{1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right)(P_{\mathrm{L}} + P_{\mathrm{R}})\ . \tag{54}$$

Defining the left- and right-handed top wave function counter terms such that they cancel the $\frac{1}{\epsilon_{\overline{\mathrm{MS}}}}$-divergences in Equations (51), (53) and (54) yields

$$\delta_{t,\mathrm{L}} = \frac{-1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}\left(\frac{1}{2}y_t^2 + \frac{8}{3}g_3^2 + \frac{3}{2}g_2^2 + \frac{1}{18}g_1^2\right) \tag{55}$$

and

$$\delta_{t,\mathrm{R}} = \frac{-1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}\left(\frac{1}{2}y_t^2 + \frac{8}{3}g_3^2 + \frac{8}{9}g_1^2\right)\ . \tag{56}$$

The top Yukawa counter term is defined from one-loop corrections to the top Yukawa interaction. For the vertex between the top quark and Higgs there are three non-finite diagrams, see Figure 13. Diagrams are formed similarly as for the propagator. The first of the diagrams in Figure 13 is obtained by adding a Higgs boson between the external top quarks. Another diagram with additional Higgs between external Higgs and fermions, similar to (4) in Figure 9, is finite. Thus the Higgs contribution is

$$-\mathrm{i}\Gamma_{y_t}^{(1)} = -\frac{\mathrm{i}y_t}{\sqrt{2}}\left(-\frac{y_t^2}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \text{f.t.}\right)\ . \tag{57}$$

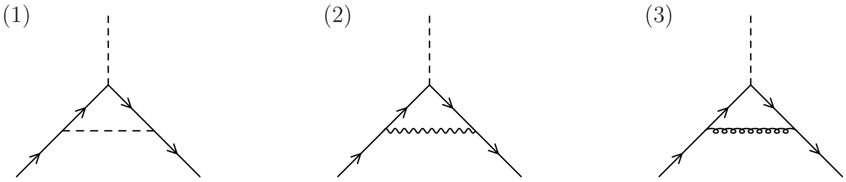


Figure 13: Non-finite top Yukawa vertex diagrams at one-loop level. In diagram (2) there are only photon or Z-boson as an additional gauge field.

Adding a photon, Z- or W-boson to the top Yukawa vertex is very similar as adding Higgs boson. There are two kinds of diagrams: one with a boson between top quarks and one with a boson between top quarks and Higgs. The latter of them is possible only with an additional Z- or W-boson since photons do not interact with Higgs, but for Z and W it is finite. The former one, diagram (2) in Figure 13, cannot include a W-boson, since then thea fermion in the loop would be a bottom quark for whose interaction with Higgs was neglected. Thus the diagram (2) is the sum of diagrams with an additional photon or Z-boson and is

$$-\mathrm{i}\Gamma_{y_t}^{(2)} = -\frac{\mathrm{i}y_t}{\sqrt{2}}\left(\frac{8g_1^2}{9}\frac{1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.}\right) . \tag{58}$$

When one is adding a gluon to the top Yukawa vertex, the only possibility, since Higgs and gluon do not interact with each other, is to add it between the top quarks, see diagram (3) in Figure 13. This diagram gives

$$-\mathrm{i}\Gamma_{y_t}^{(3)} = -\frac{\mathrm{i}y_t}{\sqrt{2}}\left(\frac{32g_3^2}{3}\frac{1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}} + \mathrm{f.t.}\right) . \tag{59}$$

The final top Yukawa counter term

$$\delta_{y_t} = \frac{1}{16\pi^2\epsilon_{\overline{\mathrm{MS}}}}\left(y_t^2 - \frac{32}{3}g_3^2 - \frac{8}{9}g_1^2\right) \tag{60}$$

is obtained collecting divergences from Equations (57), (58) and (59).

3.7.2 Top Yukawa β-function

Inserting the Higgs wave function counter term (36) and vev the counter term (48) in addition to the top counter terms (55), (56) and (60) in the expression (50), the β-function of the top Yukawa coupling is

$$\beta_{y_t}^{SM} = \frac{1}{16\pi^2}\left(\frac{9}{2}y_t^3 - 8g_3^2 y_t - \frac{9}{4}g_3^2 y_t - \frac{17}{12}g_1^2 y_t\right) . \tag{61}$$

Here one can see that apart from top Yukawa itself, the β-function of top Yukawa depends on the SM gauge couplings g_1, g_2 and g_3. The fact that β_{y_t} does not depend on the Higgs self-coupling is not obvious, since the correction to the top Yukawa vertex with an additional Higgs between the top quarks and Higgs is proportional to λ. However, this diagram is finite. Hence in the $\overline{\text{MS}}$-scheme λ does not affect running of the top Yukawa coupling.

With the top Yukawa β-function I conclude this section. I have now derived at one-loop level the β-functions for the SM gauge couplings, Higgs self-interaction and top Yukawa assuming that no other particles are included in the SM. In [5] these SM results were corrected by taking into account gravitation as an asymptotically safe theory. In the next section asymptotic safety and corrections to the SM β-functions are considered.

4 Asymptotic safety

So far I have considered pure SM, which is known to be a renormalisable and consistent theory. The problem occurs when one would like to describe gravity with a quantum theory. Quantising the gravity is not a problem as such, but the fact that quantum gravity one obtains is not a renormalisable theory. In Einstein's theory of gravity the coupling constant is Newton's gravitational constant G_{N} which has a mass dimension -2, that is $[G_{\mathrm{N}}] = M^{-2}$. For such a theory every Green's function is divergent at sufficiently high order, and an infinite number of redefinitions are needed to cancel those divergences. [6, 8]

Despite all the effort that has been put into the subject, no solution for non-renormalisability of quantum gravity has been found yet. Several frameworks have been suggested, such as loop quantum gravity or supergravity, but none of them has yet solved the problem [8,9]. A slightly different type of solution involves a notion called asymptotic safety (AS), proposed by Steven Weinberg in late 70's [10].

4.1 What it is to be asympotically safe?

According to Weinberg [10]: "A theory is said to be asymptotically safe if the 'essential' coupling parameters approach a fixed point as the momentum scale of their renormalisation point goes to infinity." What does this mean?

Consider a QFT with some set of coupling constants that determine the theory. Define the essential coupling constants to be those which cannot be eliminated by rescaling of the fields[4]. The set of essential couplings is denoted by $\{g_i(\mu)\}$. Despite that g_i are called constants, they may depend on the energy scale μ and evolve when the scale changes. Similarly as it was done in the beginning of Section 3.1, one denotes by α_i the mass dimension of the coupling. One can form a set of dimensionless couplings $\{\bar{g}_i(\mu)\}$ such that $g_i = \mu^{\alpha_i}\bar{g}_i(\mu)$ and use the set $\{\bar{g}_i(\mu)\}$ to span a space called theory space.

In Section 2 the β-function of a coupling $\bar{g}_i$ was defined to be

$$\beta_i(\bar{g}(\mu)) = \mu\frac{\partial}{\partial\mu}\bar{g}_i(\mu) \ .$$

Notice that β_i does depend explicitly only on couplings $\bar{g}_i$. The scale dependence is implicit via couplings. This is justified by fact that β cannot have an explicit scale dependence since β is a dimensionless quantity and the only dimensional parameter is the scale μ [10]. With a given set of initial conditions the functions β_i determine an unique trajectory in the theory space.

[4] For example, the field renormalisation constant Z_ϕ is not an essential constant.

Since a QFT is fully determined by the evolution of its coupling constants, each trajectory in the theory space, and thus each set of initial conditions and β_i's, corresponds to one theory.

Assume that there is a point $g*$ in the theory space for which $g_i(\mu) \to g_i*$ when $\mu \to \infty$ and

$$\beta_i(g*) = 0 \quad \forall i \ .$$

The point $g*$ is said to be a ultraviolet (UV) fixed point of theory. If $g* = 0$, the point is called Gaussian, otherwise non-Gaussian, or interacting, fixed point [6, 10–12].

Recall that a set of initial conditions of β-functions determined one trajectory in the theory space. The set of all trajectories which hit the fixed point g^* form the UV critical surface. A theory is said to be asymptotically safe if its coupling constants lie on a finite dimensional UV critical surface of some fixed point [10–12]. The finiteness of the dimension of the UV critical surface is crucial since the dimension defines the number of undetermined parameters. If the dimension is finite, there is only a finite number of parameters to be fixed, which corresponds to a generalised condition of renormalisability.

What is special about AS theories is that even though they are not renormalisable they have a well defined UV limit. A special case of AS theories are all theories which lie on a critical surface of a fixed point $g^* = 0$. These are eventually renormalisable and asymptotically free. There are cases too, where an AS theory is renormalisable even if $g^* \neq 0$ [10].

4.2 Quantum theory of gravity and asymptotic safety

The next question is whether the quantum gravity is asymptotically safe. It has been proven that in a 2-dimensional spacetime the quantum gravity indeed is an AS theory. Let us see how this happens. In two dimensional spacetime the theory of pure gravity with Lagrangian $-\frac{1}{16\pi G_N}\sqrt{g}R$ has a dimensionless coupling constant G_{N}. In $2+\epsilon$ dimensions the mass dimension of G_{N} is $[G_{\mathrm{N}}] = M^{-\epsilon}$. At the limit $\epsilon \to 0$ the Laurent expansion of the bare coupling is [10]

$$G_{\mathrm{N},0}\mu^{\epsilon} = G_{\mathrm{N}}(\mu) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} b_{\nu}(G_{\mathrm{N}}(\mu)) \ ,$$

where b_ν are coefficients of the poles of order ν at two dimensions. This leads to a β-function [10]

$$\beta_{G_{\mathrm{N}}} = \epsilon G_{\mathrm{N}} + b_1(G_{\mathrm{N}}) - G_{\mathrm{N}} b_1'(G_{\mathrm{N}}) \ . \tag{62}$$

Expanding b_1 in G_{N} yields

$$b_1(G_{\mathrm{N}}) = bG_{\mathrm{N}}^2 + \mathcal{O}(G_{\mathrm{N}}^3) \ ,$$

and substituting this back to (62) gives

$$\beta_{G_{\mathrm{N}}} = \epsilon G_{\mathrm{N}} - bG_{\mathrm{N}}^2 + \mathcal{O}(G_{\mathrm{N}}^3) \ .$$

Thus one can see that if the coefficient b is positive, there is a fixed point at $G_{\mathrm{N}}^* = \epsilon/b + \mathcal{O}(\epsilon^2)$. The precise values of b depends of the number of fermion, gauge and scalar fields in different models. Examples of b computed for various models are found in [10]. The conclusion is that gravitation in two-dimensional spacetime may indeed be considered as an asymptotically safe theory.

Continuation of the previous result to four dimensions has turned out to be not so easy. Hard work with functional renormalisation group methods has given some evidence of existence of a non-Gaussian fixed point in four dimensions, but there is no actual proof that quantum gravity in four dimensions is an AS theory [13–18]. Furthermore, there is research done about AS with different models and those models coupled to gravity [19–21].

4.3 Gravitational contribution to beta-functions

In this section I discuss gravitational corrections to the SM β-function for each SM coupling. Deriving these results would be beyond the scope of this thesis. A lot of new methods would be needed to do that, so I just collect the results published elsewhere. In [5] a general form of the gravitational correction to the SM coupling g_j is given to be

$$\beta_{g_j}^{grav} = \frac{a_j}{8\pi}\frac{\mu^2}{M_{\mathrm{P}}^2(\mu)} g_j \ , \tag{63}$$

where a_j is a constant which depends on which coupling one is considering and μ is the energy scale. In the denominator $M_{\mathrm{P}}^2(\mu)$ is a scale dependent Planck mass defined as $M_{\mathrm{P}}^2(\mu) = M_{\mathrm{P}}^2 + \xi\mu^2$, where $\xi \approx 0.024$ is a constant.

According to [22] the gravitational correction at one-loop level for a general Yang-Mills theory minimally coupled to gravity is negative and $a_g \approx -1$. This result has been criticised because of its possible gauge dependence [23, 24]. However, work done in [25, 26] supports the results in [22].

In [5] the gravitational correction for the Higgs self-coupling is positive and $a_\lambda \approx 3$ based on [27–29] and for top Yukawa $a_{y_t} \approx -0.5$ based on [30].

5 The running of the SM couplings

I start this section by collecting the β-functions of the five SM couplings. The SM parts are derived in Section 3 and gravitational corrections are from the previous section.

$$\begin{aligned}
\beta_{g_1} &= \frac{41}{96\pi^2} g_1^3 - \frac{1}{8\pi}\frac{\mu^2}{M_P^2(\mu)} g_1 \\
\beta_{g_2} &= -\frac{19}{96\pi^2} g_2^3 - \frac{1}{8\pi}\frac{\mu^2}{M_P^2(\mu)} g_2 \\
\beta_{g_3} &= -\frac{7}{16\pi^2} g_3^3 - \frac{1}{8\pi}\frac{\mu^2}{M_P^2(\mu)} g_3 \\
\beta_{y_t} &= \frac{1}{16\pi^2}\left(\frac{9}{2} y_t^3 - 8 y_t g_3^2 - \frac{9}{4} y_t g_2^2 - \frac{17}{12} y_t g_1^2\right) - \frac{1}{16\pi}\frac{\mu^2}{M_P^2(\mu)} y_t \\
\beta_\lambda &= \frac{1}{16\pi^2}\Big(24\lambda^2 - 6y_t^4 + \frac{3}{8}\left(2g_2^4 + \left(g_2^2 + g_1^2\right)^2\right) + 12\lambda y_t^2 \\
&\quad - 3\lambda\left(3g_2^2 + g_1^2\right)\Big) + \frac{3}{8\pi}\frac{\mu^2}{M_P^2(\mu)}\lambda
\end{aligned}$$

For a cross-check of the SM parts, see for example [4, 5, 31–33]. The task of this section is to solve the set of differential equations represented by the β-functions. The idea is to solve the β-functions for a range of initial values of λ using the measured values as initial conditions for all other couplings.

It is worth mentioning that since the SM calculations are done in the $\overline{\text{MS}}$-scheme, I have to have my initial conditions in that scheme too. Furthermore, the initial value for λ that is obtained as a result is in the $\overline{\text{MS}}$-scheme at a certain scale. Certainly it is important to specify the scale since, this is the whole point of β-functions, the value of the coupling in the $\overline{\text{MS}}$-scheme depends on the energy scale.

5.1 Initial conditions

To solve differential equations numerically I need to give initial values for the four SM couplings g_1, g_2, g_3 and y_t. Note that since their β-functions were calculated at one-loop level, also the initial values of couplings have to be one-loop results. The numerical values of the two electroweak gauge couplings and top Yukawa coupling in the $\overline{\text{MS}}$-scheme calculated at one-loop at the top quark pole mass $\hat{m}_t = 173.34\,\text{GeV}$ are [34]

$$\begin{cases} g_1(\hat{m}_t) &= 0.35940 \\ g_2(\hat{m}_t) &= 0.64754 \\ y_t(\hat{m}_t) &= 0.95113 \ . \end{cases} \tag{64}$$

The third gauge coupling g_3 can be expressed in terms of the strong fine structure constant α_{S}:

$$\alpha_S \equiv \frac{g_3^2}{4\pi} \quad \Rightarrow \quad g_3 = \sqrt{4\pi\alpha_S} \ . \tag{65}$$

The strong fine structure constant in the $\overline{\text{MS}}$-scheme at the Z-boson mass scale at one-loop level is [36]

$$\alpha_{\text{S}}(\hat{m}_Z) = 0.1172 \ , \tag{66}$$

where $\hat{m}_Z = 91.1876\,\text{GeV}$. This yields a value

$$g_3 = 1.2136 \tag{67}$$

for the strong coupling constant. However, this is not the desired initial condition yet since the scale at which it is defined is different to the top pole mass scale at which the other three couplings were given. The strong coupling constant can be defined at the top pole mass by running it from the m_Z-scale using its β-function. At small scales, much less than the Planck scale, the gravitational corrections can be neglected, and thus the β-function of g_3 can be solved analytically at one-loop level. The solution is

$$g_3(\mu) = \left(\frac{g_3^2(\hat{m}_Z)}{1 + \frac{7}{16\pi^2} g_3^2(\hat{m}_Z) \log\left(\frac{\mu^2}{\hat{m}_Z^2}\right)} \right)^{1/2} \ .$$

Substituting here $\mu = \hat{m}_t = 173.34\,\text{GeV}$, the strong coupling is

$$g_3(\hat{m}_t) = 1.1888 \ . \tag{68}$$

I will use this value with (64) as the initial conditions for the β-functions.

5.2 Solution for β-functions

Above I derived the initial values at the top quark pole mass scale for the three SM gauge couplings and the top Yukawa coupling at one-loop level.

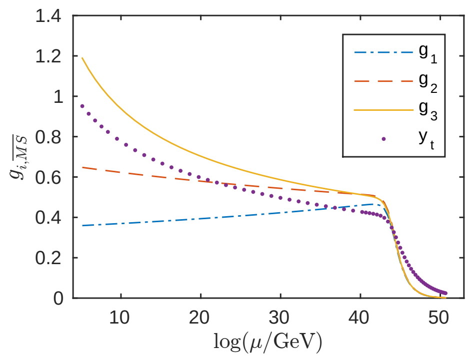


Figure 14: The running of the SM gauge and top Yukawa couplings with $\xi = 0.024$.

With these initial conditions the running of the gauge and top Yukawa couplings can be solved.

I start by solving the running of the gauge coupling constants, since the β-function of each gauge coupling depends only on the coupling itself and thus it can be solved independently of the other couplings at one-loop level. After that, running of the top Yukawa coupling can be calculated, since its β-function depends on the gauge couplings but not on the Higgs self-coupling. Solutions of these β-functions are shown in Figure 14. Very similar results were found in [22]. The drop at $\log(\mu) \approx 45$, i.e. at the Planck mass scale, is due to gravitational corrections. From Equation (63) one can see that gravitational corrections are negligible until $\mu \approx M_{\mathrm{P}}$ because of the large denominator $M_{\mathrm{P}}^2 + \xi\mu^2$. At the Planck mass scale gravitational corrections became significant and being negative corrections they force the running towards zero. This happens also for the hypercharge coupling g_1 which in the SM blows up at very large scales. As a conclusion, the gauge and top Yukawa couplings are asymptotically free and the three gauge couplings are unified at the Planck scale if gravitation is taken in as an asymptotically safe theory.

Solution for the Higgs self-coupling is still missing however. For λ I do not have an initial condition from measurements but the aim is to find it numerically from the condition that it stays finite to arbitrarily high scales. To this end I solve the differential equation group for $\lambda \in \{0, 0.05, 0.1, \ldots, 1.2\}$ and find the correct λ by iterating.

From Figure 15a one can see that running of the Higgs self-coupling λ is very unstable as the initial value changes, and that for the set of chosen initial values the running either drops below zero or blows up at high scales. However, iterating the initial value between 0.15 and 0.20, between the last

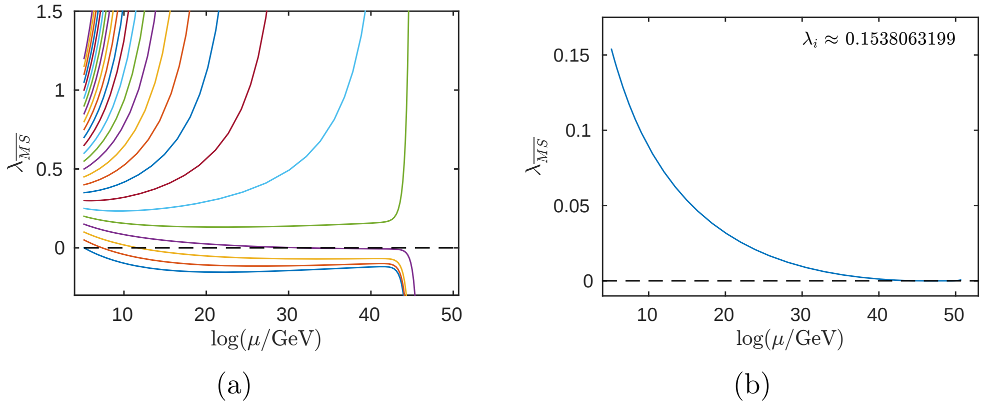


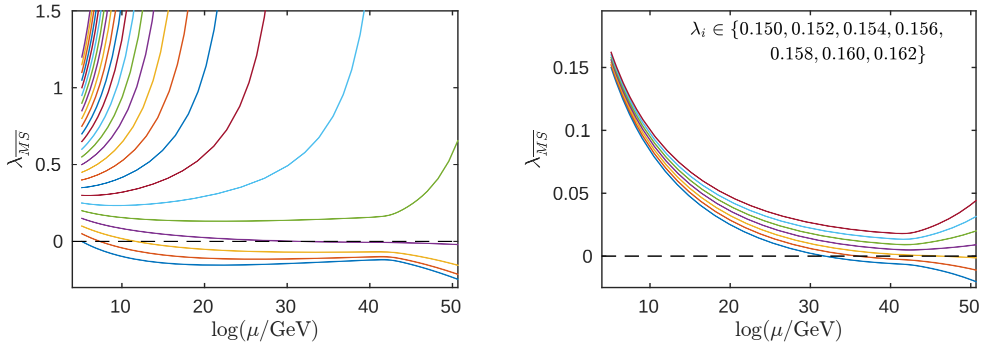
Figure 15: The running of the Higgs self-coupling with $\xi = 0.024$. In (a) the initial values of the Higgs self-coupling are $\lambda \in \{0, 0.05, 0.1, \ldots, 1.2\}$ and in (b) $\lambda \approx 0.1538$.

line that has negative values (purple line) and the first that blows up (green line), for a single initial value $\lambda \approx 0.1538$ the Higgs self-coupling remains finite and positive up to Planck scale, see Figure 15b. This behaviour shows that the Higgs self-coupling has an unstable UV fixed point. The conclusion is that the Higgs self-coupling in the $\overline{\text{MS}}$-scheme calculated at one-loop level at the top pole mass is $\lambda \approx 0.1538$.

5.2.1 Changing constant ξ

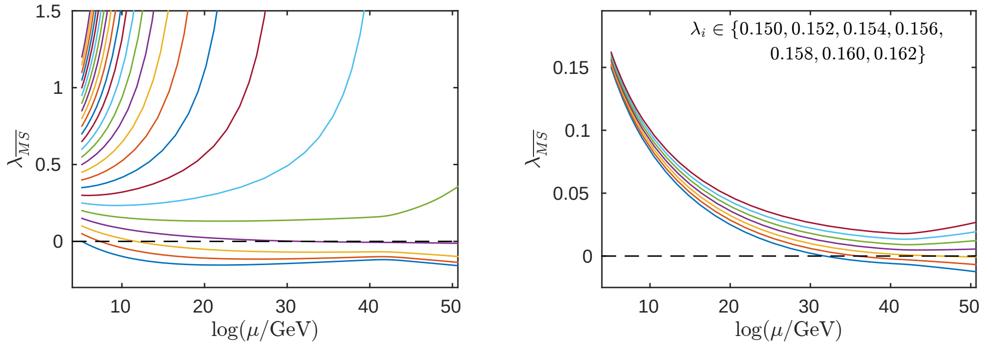
In [5] the constant ξ is chosen to be $\xi \approx 0.024$ based on the results in [28, 29, 35]. It is interesting to see how strongly this choice affects the running of the SM couplings. To see this, I solve the running of the couplings for three other values of ξ in addition to the value $\xi = 0.024$: cases $\xi = 0.5$,1 and 5. What happens when one increases ξ is that the ratio $^{\mu^2}/_{M_P^2(\mu)}$ is getting smaller at all scales. Thus increasing ξ means switching the gravitational corrections on only at larger scales. This can be seen in Equation (63).

In Figures 16, 17 and 18 there are shown the running of the Higgs self-coupling λ with $\xi = 0.5$,1,5. One can see that bigger values of ξ smooth the behaviour of λ at large scales for initial values around 0.155. Since running is smoother for bigger ξ, there actually is a range of acceptable initial values, not just a single point as for $\xi = 0.024$. This means that the method I have used to predict the Higg self-coupling is more inaccurate for bigger values of ξ since the prediction for λ is not just a single point but a range of initial values. Furthermore, the coupling λ is not asymptotically free for all of those initial values. However, the first initial value for which λ remains positive



(a) (b)

Figure 16: The running of the Higgs self-coupling with $\xi = 0.5$.

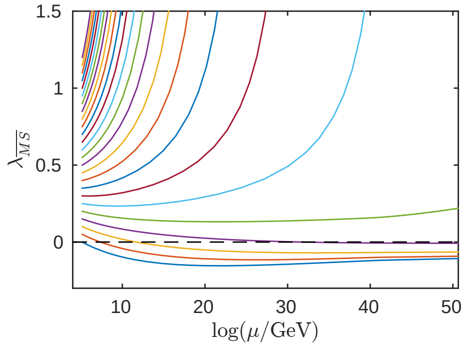


(a) (b)

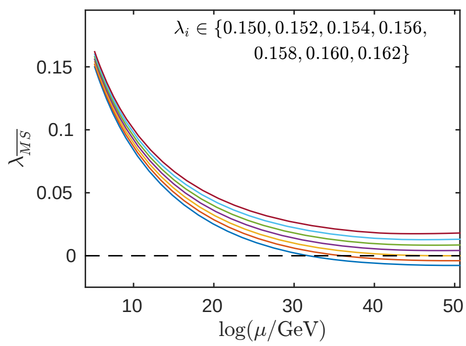
Figure 17: The running of the Higgs self-coupling with $\xi = 1$.

at all scales up to the Planck mass is roughly the same as in the case of $\xi = 0.024$.

In Figure 19 there are running of the three gauge couplings and top Yukawa coupling with $\xi = 0.5$,1,5. One can see that for $\xi = 0.5$,1 the gravitational corrections are still able to little affect the running of the gauge and top Yukawa couplings. However, none of them is asymptotically free already at the Planck scale as in the case of $\xi = 0.024$. For values $\xi = 0.5$,1 the trend of the running of the U(1) hyperchage coupling turns decreasing from increasing at the Planck scale, so also g_1 remains finite at all scales. For the value $\xi = 5$ the gravitational corrections barely affect the running of the gauge or top Yukawa couplings. Gravitational corrections cannot change the behaviour of the running of g_1 but it eventually blows up at very large scales.

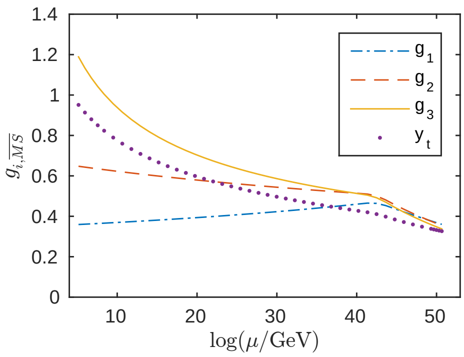


(a)

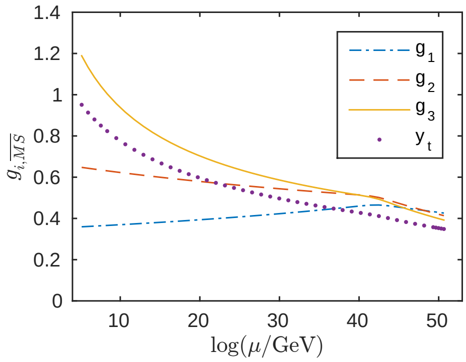


(b)

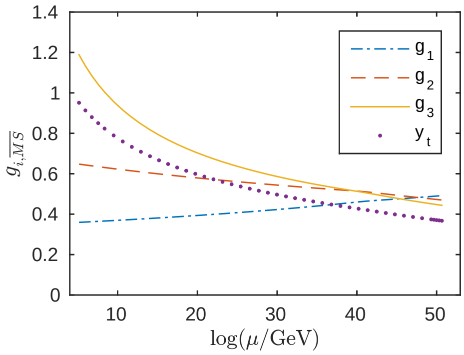
Figure 18: The running of the Higgs self-coupling with $\xi = 5$.



(a) $\xi = 0.5$



(b) $\xi = 1$



(c) $\xi = 5$

Figure 19: The running of the gauge and top Yukawa couplings with $\xi = 0.5, 1{,}5$.

6 Higgs mass

In the previous section numerical solutions of β-functions gave to Higgs self-coupling the value $\lambda \approx 0.1538$ at the top pole mass scale. The next thing is to calculate the Higgs mass corresponding to that value. If one used blindly the tree-level relation between mass and coupling, $m_h = \sqrt{2\lambda v^2}$, it would give $m_h \approx 136\,\mathrm{GeV}$. This is clearly much bigger than the physical mass $m_h \approx 125\,\mathrm{GeV}$.

The reason is that I chose to use the $\overline{\mathrm{MS}}$ renormalisation scheme to define the coupling. Thus $\lambda \approx 0.1538$ is the value given in that scheme at the top pole mass scale. That is why I got too big a mass using just tree-level relation for mass and coupling. The goal in this section is to find the approximative one-loop relation between the physical Higgs self-coupling and the one in the $\overline{\mathrm{MS}}$-scheme.

6.1 Relation between self-coupling in $\overline{\mathrm{MS}}$- and physical schemes

Here I first go through the idea of relating the self-coupling in a physical scheme to the one defined in the $\overline{\mathrm{MS}}$-scheme. The argument is based on the fact that the difference between the coupling in two renormalisation schemes is always equal to the difference between the counter terms of that coupling in those schemes. This fact is quite easy to derive using the definition of the renormalised coupling in terms of the bare coupling. Thus the hard thing is not to relate the $\overline{\mathrm{MS}}$-coupling to a physical coupling but to find out the counter term in the physical scheme. The counter terms in $\overline{\mathrm{MS}}$ I derived already in Section 3.6.

I start with the Higgs potential renormalisation which I can use to get the Higgs self-coupling and the mass counter terms at $p^2 = 0$. Going through mass renormalisation in the physical scheme I can obtain the self-coupling λ at $p^2 = 0$ as a function of physical mass. Since the physical mass and physical coupling are easily related, I actually have λ in $p^2 = 0$ -scheme as a function of physical coupling. Here a problem arises however. The relation includes the Higgs vev defined as the minimum of Higgs potential in the renormalisation scheme I have used. How is this vev related to the physical vev $\hat{v} = 246\,\mathrm{GeV}$? It turns out that the Higgs vev has to be corrected at one-loop level as well as masses and couplings. The relevant one-loop corrections of the vev can be derived from muon decay. Given an equation for λ in the $p^2 = 0$ -scheme and another for the one-loop corrected Higgs vev, and yet another for the physical self-coupling, I can easily get the difference between the physical and $p^2 = 0$ couplings from the difference between the counter terms in those schemes.

Define the physical self-coupling $\hat{\lambda}$ such that

$$\hat{m}_h^2 \equiv 2\hat{\lambda}\hat{v}^2 , \tag{69}$$

where $\hat{m}_h$ is the physical pole mass and the physical Higgs vev is defined by

$$\frac{\hat{G}_{\rm F}}{\sqrt{2}} \equiv \frac{1}{2\hat{v}^2} , \tag{70}$$

were $\hat{G}_{\rm F}$ is the Fermi constant. In Section 5 I obtained the Higgs self-coupling in the $\overline{\rm MS}$-scheme. The task is now to relate that result to the physical coupling $\hat{\lambda}$ and thus by (69) get a relation between physical mass and $\lambda_{\overline{\rm MS}}$.

The relation between the bare and the renormalised couplings, in an arbitrary scheme R, is given by

$$\lambda_0 \equiv \lambda_R + \delta\lambda_R . \tag{71}$$

This relation I can use as a link between couplings in different renormalisation schemes. The physical coupling is related to the one in $\overline{\rm MS}$ by

$$\begin{aligned} \lambda_{\overline{\rm MS}} + \delta\lambda_{\overline{\rm MS}} &= \lambda_0 = \hat{\lambda} + \delta\hat{\lambda} \\ \Rightarrow \hat{\lambda} &= \lambda_{\overline{\rm MS}} + \delta\lambda_{\overline{\rm MS}} - \delta\hat{\lambda} . \end{aligned} \tag{72}$$

The previous equation shows that in switching from one renormalisation scheme to another, the difference between couplings arises from the difference in their counter terms. The counter term in $\overline{\rm MS}$ may be calculated from Equation (71) and using Equations (36) and (44). In Section 2.2 I defined

$$\begin{aligned} \lambda_0 Z_\phi^2 &= \lambda_{\overline{\rm MS}} + \delta_\lambda^{\overline{\rm MS}} \\ \Rightarrow \lambda_0 &\approx \lambda_{\overline{\rm MS}} + \delta_\lambda^{\overline{\rm MS}} - 2\lambda_{\overline{\rm MS}}\delta_\phi^{\overline{\rm MS}} \end{aligned}$$

so Equation (71) yields at one-loop level

$$\delta\lambda_{\overline{\rm MS}} \approx \delta_\lambda^{\overline{\rm MS}} - 2\lambda_{\overline{\rm MS}}\delta_\phi^{\overline{\rm MS}} . \tag{73}$$

The problem is to find out what is $\delta\hat{\lambda}$, which is a nontrivial task. It requires determining the Higgs wave function and self-coupling counter terms in the $p^2 = 0$ renormalisation scheme and the one-loop renormalisation of the muon decay. I start with the derivation of the $p^2 = 0$ counter terms.

Higgs potential renormalisation The Higgs potential, written with the bare field and coupling, is

$$V(\Phi_0) = \mu_0^2 \Phi_0^\dagger \Phi_0 + \lambda_0 (\Phi_0^\dagger \Phi_0)^2 \ .$$

Renormalising the field by writing $\Phi_0 = Z_\Phi^{1/2} \Phi$ and redefining couplings $\mu_0^2 Z_\Phi = \mu^2 + \delta\mu^2$ and $\lambda_0 Z_\Phi^2 = \lambda + \delta_\lambda$ yields

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 + \delta\mu^2 \Phi^\dagger \Phi + \lambda \delta_\lambda (\Phi^\dagger \Phi)^2 \ .$$

The last two terms form the counter term part of the potential. Expanding the field around a minimum v, one defines the field Φ as

$$\Phi = \begin{pmatrix} \chi^+ \\ \frac{1}{\sqrt{2}}(v + \phi + \mathrm{i}\chi_0) \end{pmatrix} ,$$

where χ^+ and χ_0 are charged and neutral Goldstone bosons and ϕ is the Higgs boson. The potential for the Higgs field ϕ now becomes

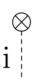
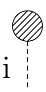
$$\begin{aligned} V = {} & (\mu^2 v + \lambda v^3 + \delta\mu^2 v + \delta_\lambda v^3)\phi + \frac{1}{2}(\mu^2 + 3\lambda v^2 + \delta\mu^2 + 3\delta_\lambda v^3)\phi^2 \\ & + (\lambda v + \delta_\lambda v)\phi^3 + \frac{1}{4}(\lambda + \delta_\lambda)\phi^4 + \dots \ , \end{aligned}$$

where I have ignored all terms not proportional to the Higgs field ϕ.

I chose to renormalise the theory with the following renormalisation conditions:

$$\left. \frac{\mathrm{d}V}{\mathrm{d}\phi} \right|_{\Phi = \binom{0}{v}} = 0 \quad \text{and} \quad \left. \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2} \right|_{\Phi = \binom{0}{v}} = m^2_{h, p^2=0} \ . \tag{74}$$

The first of these renormalisation conditions sets v to be the minimum of the potential and the second defines the Higgs mass at that minimum. The physical significance of the first of the renormalisation conditions (74) is to keep the vev at place and kill the tadpole of the Higgs field. The first derivative of the potential is the one-point function at $p^2 = 0$ [6, 8]:

$$\left. \frac{\mathrm{d}V}{\mathrm{d}\phi} \right|_{\Phi = \binom{0}{v}} = -\Gamma^1(p^2 = 0) = \mathrm{i}$$  $$+\, \mathrm{i}$$ 

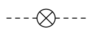
$$= v(\mu^2 + \lambda v^2 + \delta\mu^2 + \delta_\lambda v^2 + D) \ ,$$

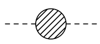
where I have denoted the sum of all one-loop Higgs tadpole diagrams at $p^2 = 0$ by $-\mathrm{i}vD$. Using the first of renormalisation conditions (74)

$$\mu^2 + \lambda v^2 + \delta\mu^2 + \delta_\lambda v^2 + D = 0 \ . \tag{75}$$

The counter term $\delta\mu^2$ is now fixed in terms of D and δ_λ. The second derivative of the potential is related to the two-point function such that

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right|_{\Phi=\binom{0}{v}} = -\Gamma^2(p^2=0) = \mathrm{i}\ \text{[diagram]} + \mathrm{i}\ \text{[diagram]}$$





$$= \mu^2 + 3\lambda v^2 + \delta\mu^2 + 3\delta_\lambda v^2 + \Pi_{\phi,\text{1-loop}}(0)\ .$$

Inserting the counter term $\delta\mu^2$ from Equation (75) one finds

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right|_{\Phi=\binom{0}{v}} = 2\lambda v^2 + 2\delta_\lambda v^2 - D + \Pi_{\phi,\text{1-loop}}(0)$$
$$= m^2_{h,p^2=0}\ , \qquad (76)$$

where I used the second renormalisation condition in (74). I have not yet fixed the coupling constant counter term δ_λ. It can be chosen such that in the previous equation the counter term cancels the one- and two-point functions, that is

$$2\delta_\lambda v^2 - D + \Pi_{\phi,\text{1-loop}}(0) = 0$$
$$\Rightarrow \delta_\lambda = \frac{D - \Pi_{\phi,\text{1-loop}}(0)}{2v^2}\ . \qquad (77)$$

Using this definition, Equation (76) now gives the familiar relation between Higgs mass and self-coupling

$$m^2_{h,p^2=0} = 2\lambda v^2\ . \qquad (78)$$

In the potential V the terms proportional to ϕ^2 form the mass and mass counter terms of Higgs. Thus the Higgs mass counter term at $p^2 = 0$ is

$$\begin{aligned}\delta m^2_{h,p^2=0} &= \delta\mu^2 + 3\delta_\lambda \\ &= -D - \delta_\lambda v^2 - \lambda v^2 - \mu^2 + 3\delta_\lambda v^2 \\ &= -D + 2\delta_\lambda v^2 = -\Pi_{\phi,\text{1-loop}}(0)\ ,\end{aligned} \qquad (79)$$

where I used the two counter terms $\delta\mu$ and δ_λ from Equations (75) and (77) and the fact that $-\mu^2 \equiv 2\lambda v^2$.

In Equation (77) I have written a wave function counter term, but this is not yet the physical counter term I need. The mass I have defined in (78) is not physical either, but it can be related to the physical mass quite easily.

Relation to physical mass The full inverse propagator for Higgs calculated at $p^2 = 0$ is [8]

$$\mathrm{i}\Delta^{-1}(p^2) = p^2 - 2\lambda v^2 - \Pi_{\phi,\text{1-loop}}(p^2) + p^2\delta_\phi^{p^2=0} - \delta m^2_{h,p^2=0} \;, \tag{80}$$

where $\Pi_{\phi,\text{1-loop}}$ is the sum of the Higgs two-point diagrams at one-loop level. Expanding the two-point function $\Pi_{\phi,\text{1-loop}}$ around $p^2 = 0$ gives

$$\Pi_{\phi,\text{1-loop}}(p^2) = \Pi_{\phi,\text{1-loop}}(0) + \Pi'_{\phi,\text{1-loop}}(0)p^2 + \tilde{\Pi}_{\phi,\text{1-loop}}(p^2) \;,$$

where the function $\tilde{\Pi}_{\phi,\text{1-loop}}(p^2)$ and its derivative vanish at $p^2 = 0$ [5]. Inserting this result into Equation (80), the inverse propagator becomes

$$\begin{aligned}\mathrm{i}\Delta^{-1}(p^2) = p^2 - 2\lambda v^2 - \Pi_{\phi,\text{1-loop}}(0) - \Pi'_{\phi,\text{1-loop}}(0)p^2 + p^2\delta_\phi^{p^2=0}\\ - \delta m^2_{h,p^2=0} + \tilde{\Pi}_{\phi,\text{1-loop}}(p^2) \;.\end{aligned}$$

Defining the counter terms such that all divergences are cancelled leads to conditions

$$\delta m^2_{h,p^2=0} = -\Pi_{\phi,\text{1-loop}}(0) \tag{81}$$

and

$$\delta_\phi^{p^2=0} = \Pi'_{\phi,\text{1-loop}}(0) \;. \tag{82}$$

I am then left with the inverse propagator

$$\mathrm{i}\Delta^{-1}(p^2) = p^2 - 2\lambda v^2 + \tilde{\Pi}_{\phi,\text{1-loop}}(p^2)$$

which sets the $p^2 = 0$ mass to be $m_{h,p^2=0} = 2\lambda v^2$. These results are consistent with my renormalisation conditions (74) and (75). Indeed, see Equations (78) and (79), which were derived using renormalisation conditions (74). The only new information here was that the wave function counter term at $p^2 = 0$ is the derivative of the two-point function.

Physical mass is defined to be the mass at which the inverse propagator has a pole[6]. Thus at the physical pole at $p^2 = \hat{m}_h^2$ the inverse propagator is

$$\begin{aligned}\mathrm{i}\Delta^{-1}(p^2) = p^2 - 2\lambda v^2 - \Pi_{\phi,\text{1-loop}}(\hat{m}_h^2) - \Pi'_{\phi,\text{1-loop}}(\hat{m}_h^2)(p^2 - \hat{m}_h^2)\\ + (p^2 - \hat{m}_h^2)\delta_\phi^{p^2=0} + \hat{m}_h^2\hat{\delta}_\phi - \delta m^2_{h,p^2=0} + \tilde{\Pi}_{\phi,\text{1-loop}}(p^2) \;,\end{aligned} \tag{83}$$

[5] In practise, $\tilde{\Pi}_{\phi,\text{1-loop}}$ is higher order terms of Taylor expansion for $\Pi_{\phi,\text{1-loop}}$.

[6] and that's why it is also called pole mass

where I have expanded $\Pi_{\phi,\text{1-loop}}$ around $p^2 = \hat{m}_h^2$ so function $\tilde{\Pi}_{\phi,\text{1-loop}}$ satisfies $\tilde{\Pi}_{\phi,\text{1-looop}}(\hat{m}_h^2) = \tilde{\Pi}'_{\phi,\text{1-looop}}(\hat{m}_h^2) = 0$. Notice here that mass counter term is the one calculated at $p^2 = 0$ because it is fixed by the renormalisation conditions. On the other hand the inverse propagator is

$$\mathrm{i}\Delta^{-1}(p^2) \equiv p^2 - \hat{m}_h^2 + \tilde{\Pi}_{\phi,\text{1-loop}}(p^2)$$

which yields a wave function counter term

$$\hat{\delta}_\phi = \Pi'_{\phi,\text{1-loop}}(\hat{m}_h^2) \tag{84}$$

and physical mass

$$\hat{m}_h^2 = 2\lambda_{p^2=0}v^2 + \Pi_{\phi,\text{1-loop}}(\hat{m}_h^2) - \hat{m}_h^2\hat{\delta}_\phi + \delta m^2_{h,p^2=0}\ .$$

Substituting here counter terms from Equations (81) and (84)

$$\begin{aligned} 2\lambda_{p^2=0}v^2 &= \hat{m}_h^2 - \Big(\Pi_{\phi,\text{1-loop}}(\hat{m}_h^2) - \Pi_{\phi,\text{1-loop}}(0) - \hat{m}_h^2\Pi'_{\phi,\text{1-loop}}(\hat{m}_h^2)\Big) \\ &\equiv \hat{m}_h^2(1 - \delta_{\hat{m}_h})\ , \end{aligned} \tag{85}$$

where

$$\delta_{\hat{m}_h} = \frac{\Pi_{\phi,\text{1-loop}}(\hat{m}_h^2) - \Pi_{\phi,\text{1-loop}}(0) - \hat{m}_h^2\Pi'_{\phi,\text{1-loop}}(\hat{m}_h^2)}{\hat{m}_h^2}\ . \tag{86}$$

Now I have derived a relation between the self-coupling at $p^2 = 0$ and physical mass. But the question remains what is v in Equation (85)? The renormalisation conditions were chosen such that v is always the minimum of the Higgs potential. How is it related to number 246 GeV get from Equation (70)? At tree-level they are the same, but at one-loop there is a correction. That correction can be calculated renormalising muon decay at one-loop level.

Muon decay At tree level muon decays via a W-boson to an electron, electron antineutrino and muon neutrino, see Figure 20. Effectively this process is described by the Fermi four-point interaction Lagrangian

$$\Gamma_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} = \frac{G_\mathrm{F}}{\sqrt{2}}\left(\bar{u}_e\gamma^\alpha(\mathbb{1} - \gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1} - \gamma^5)u_\mu\right)\ , \tag{87}$$

where the measured effective coupling for the muon decay, Fermi constant, is $\hat{G}_\mathrm{F} \approx 1.1663787 \cdot 10^{-5}$ GeV^{-2} [36]. A tree level calculation from the SM gives

$$\Gamma^\text{tree}_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} = \frac{g_2^2}{8m_W^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1} - \gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1} - \gamma^5)u_\mu\right)\ . \tag{88}$$

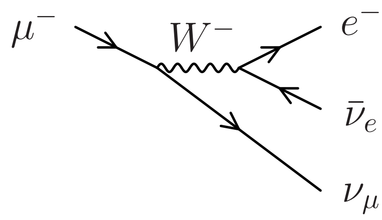


Figure 20: Muon decay at tree level.

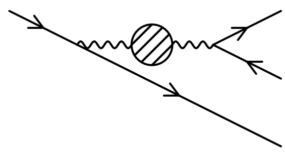


Figure 21: W propagator corrections at one loop to muon decay.

Comparing Equations (87) and (88) and using $m_W = \frac{gv}{2}$, one has a relation

$$\frac{\hat{G}_{\mathrm{F}}}{\sqrt{2}} = \frac{1}{2v^2} \tag{89}$$

which yields a familiar value for the Higgs vev $v \approx 246\,\mathrm{GeV}$. This holds at tree level. Now I need to redo the calculation at one-loop level. Several different classes of diagrams contribute to muon decay.

First, there are those diagrams with a loop in the W-propagator, called oblique corrections to muon decay, see Figure 21. This diagram gives [37]

$$\begin{aligned}\Gamma^{\text{oblique}}_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} &= \frac{g_2^2}{8m_W^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right)\left(\mathrm{i}\Pi_W(p^2)\frac{-\mathrm{i}}{p^2-m_W^2}\right)\Bigg|_{p^2=0} \\ &\quad\times\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right) \\ &\approx -\frac{1}{2v^2}\frac{\Pi_W(0)}{m_W^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right)\ .\end{aligned}$$

Note here that I calculated the W self-energy Π_W at $p^2 = 0$. This is not an exact result, since in the muon rest frame the muon momentum squared is m_μ^2. However, since the muon mass is much smaller than the W mass, I may approximate $m_\mu^2 \approx 0$ and thus take $p^2 = 0$. Since $\Pi_W(0) = \Pi_{W,\text{1-loop}}(0) +$

δm_W^2, the full oblique correction is

$$\Gamma^{\text{oblique}}_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} = -\frac{1}{2v^2}\frac{\Pi_{W,\text{1-loop}}(0)+\delta m_W^2}{m_W^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right) \times \left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right) \ . \tag{90}$$

Here $\Pi_{W,\text{1-loop}}(0)$ is the sum of the one-loop corrections to the $g_{\mu\nu}$-part of the W propagator calculated at $p^2=0$. The mass counter term δm_W^2 can be determined as follows. The W mass term in the bare Lagrangian is

$$\frac{g_{2,0}^2 v_0^2}{4}W_0^2 \ .$$

Renormalising the W field and Higgs vev it becomes

$$\left(\frac{g_{2,0}}{g_2}\right)^2 Z_\phi Z_W \frac{g_2^2 v^2}{4}W^2 \equiv (m_W^2+\delta m_W^2)W^2 \ . \tag{91}$$

I can determine the ratio of the bare and renormalised couplings using some interaction term including W. The idea is to use a term for which the counter terms are easy to calculate. From the three-vertex between leptons and W one finds

$$g_{2,0}l_0\nu_{l,0}W_0 \to g_{2,0}(Z_l Z_{\nu_l} Z_W)^{1/2} l\nu_l W \equiv g_2 Z_{g_2} l\nu_l W$$
$$\Rightarrow \frac{g_{2,0}}{g_2} = Z_{g_2}(Z_l Z_{\nu_l} Z_W)^{-1/2} \ .$$

Now remember that the measured value for g_2, that I use, is given in the $\overline{\text{MS}}$-scheme. Therefore, I have to renormalise the vertex between a W and leptons in that scheme too. Thus I find

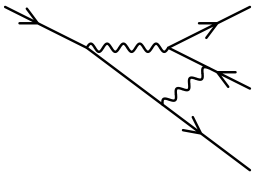
$$\frac{g_{2,0}}{g_2} = Z_{g_2}^{\overline{\text{MS}}}(Z_l^{\overline{\text{MS}}} Z_{\nu_l}^{\overline{\text{MS}}} Z_W^{\overline{\text{MS}}})^{-1/2} \ .$$

Substituting this to the left hand side of (91) yields

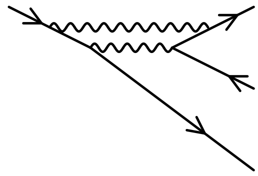
$$(m_W^2+\delta m_W^2) = \frac{g_2^2 v^2}{4}(Z_{g_2}^{\overline{\text{MS}}})^2 (Z_l^{\overline{\text{MS}}} Z_{\nu_l}^{\overline{\text{MS}}} Z_W^{\overline{\text{MS}}})^{-1}\hat{Z}_\phi Z_W^{\overline{\text{MS}}}$$
$$\approx m_W^2(1+2\delta_{g_2}^{\overline{\text{MS}}} - \delta_l^{\overline{\text{MS}}} - \delta_{\nu_l}^{\overline{\text{MS}}} + \hat{\delta}_\phi)$$

so the W mass counter term is fixed to

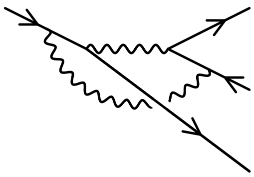
$$\delta m_W^2 = m_W^2(2\delta_{g_2}^{\overline{\text{MS}}} - \delta_l^{\overline{\text{MS}}} - \delta_{\nu_l}^{\overline{\text{MS}}} + \hat{\delta}_\phi) \ . \tag{92}$$



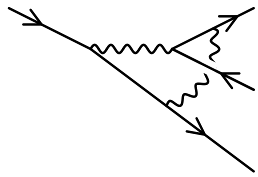
(a) Box with Z between neutrinos.



(b) Box with Z or γ between muon and electron.



(c) Box with Z between muon and electron neutrino.



(d) Box with Z between muon neutrino and electron.

Figure 22: Muon decay box diagrams at one-loop.

Note here that the W wave function counter term cancelled in the expression (92). Also note that the Higgs wave function counter term is calculated in the physical scheme. With this mass counter term the oblique correction to the muon decay is fully determined.

Another class of one-loop corrections contributing to the muon decay are so-called box diagrams, in which there is an additional boson between fermions, see Figure 22. I denote them by Γ_μ^{box}. Although this is much less evident than in the case of oblique corrections, it turns out that also box diagrams reduce to the form of the tree-level Lagrangian with only some additional factors in front.

The last class of muon decay diagrams at one-loop level are vertex corrections for the interaction between fermions and W shown in Figure 23. Similarly to the oblique corrections and the box diagrams, these diagrams are also proportional to tree-level amplitude with just an additional multiplicative factor from the three-point function:

$$\begin{aligned}\Gamma^{\text{vertex}}_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} = &\frac{g_2^2}{8m_w^2}\left(\bar{u}_e\Gamma_e(0)\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right)\\ &+\frac{g_2^2}{8m_w^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\Gamma_\mu(0)\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right)\ ,\end{aligned}$$

(a) Muon vertex correction. (b) Electron vertex correction.

Figure 23: Vertex corrections to muon decay.

where $\Gamma_l(0)$ is a three-point function between the W and lepton and its neutrino. Furthermore in the $\overline{\text{MS}}$ scheme $\Gamma_l(0) = \Gamma_{l,\text{1-loop}}(0) + \delta_{g_2}^{\overline{\text{MS}}}$, so the sum of vertex corrections is

$$\begin{aligned}\Gamma^{\text{vertex}}_{\mu^- \to e^- \bar{\nu}_e \nu_\mu} = \frac{1}{2v^2} &\left(\Gamma_{e,\text{1-loop}}(0) + \Gamma_{\mu,\text{1-loop}}(0) + 2\delta_{g_2}^{\overline{\text{MS}}}\right) \left(\bar{u}_e \gamma^\alpha (\mathbb{1} - \gamma^5) v_{\bar{\nu}_e}\right) \\ &\times \left(\bar{u}_{\nu_\mu} \gamma_\alpha (\mathbb{1} - \gamma^5) u_\mu\right) \ . \end{aligned} \tag{93}$$

All in all, comparing Equation (87) to (90), (93) and using Equation (92) for the W mass counter term, the relation between Fermi constant $\hat{G}_\text{F}$ and Higgs vev at one-loop level becomes

$$\begin{aligned}\frac{\hat{G}_\text{F}}{\sqrt{2}} &= \frac{1}{2v^2}\Big(1 - \frac{\Pi_{W,\text{1-loop}}(0)}{m_W^2} - 2\delta_{g_2}^{\overline{\text{MS}}} + \delta_l^{\overline{\text{MS}}} + \delta_{\nu_l}^{\overline{\text{MS}}} - \hat{\delta}_\phi \\ &\quad + \Gamma^{\text{box}}_\mu + \Gamma_{e,\text{1-loop}}(0) + \Gamma_{\mu,\text{1-loop}}(0) + 2\delta_{g_2}^{\overline{\text{MS}}}\Big) \\ &\equiv \frac{1}{2v^2}(1 + \delta_v) \ , \end{aligned} \tag{94}$$

where

$$\delta_v = -\frac{\Pi_{W,\text{1-loop}}(0)}{m_W^2} + \delta_l^{\overline{\text{MS}}} + \delta_{\nu_l}^{\overline{\text{MS}}} - \hat{\delta}_\phi + \Gamma^{\text{box}}_\mu + \Gamma_{e,\text{1-loop}}(0) + \Gamma_{\mu,\text{1-loop}}(0) \ . \tag{95}$$

Equation (94) implies that in one-loop level calculations the Higgs vev v has to be corrected with respect to the tree-level expression.

Final result The reason to renormalise the muon decay was to find out how the Higgs vev v changes at one-loop. Solving v from Equation (94) and inserting it in Equation (85) now leads to

$$\lambda_{p^2=0}\frac{\sqrt{2}}{\hat{G}_{\rm F}}(1+\delta_v) = \hat{m}_h^2(1-\delta_{\hat{m}_h})$$
$$\Rightarrow \lambda_{p^2=0} = \frac{\hat{G}_{\rm F}}{\sqrt{2}}\hat{m}_h^2(1-\delta_{\hat{m}_h}-\delta_v)\ .$$

This equation is the one I wanted, namely the equation between Higgs self-coupling defined at $p^2=0$ and the physical scheme. Indeed, using definitions of the physical Fermi constant (69) and physical mass (70), the previous equation becomes

$$\lambda_{p^2=0} = \hat{\lambda}(1-\delta_{\hat{m}_h}-\delta_v)\ . \tag{96}$$

On the other hand, using equation (71), the difference between λ in these two renormalisation schemes is $\lambda_{p^2=0} = \hat{\lambda}+\delta\hat{\lambda}-\delta\lambda_{p^2=0}$. Comparing this to Equation (96) one gets

$$\delta\hat{\lambda} = \delta\lambda_{p^2=0} - \hat{\lambda}(\delta_{\hat{m}_h}+\delta_v)\ . \tag{97}$$

This counter term finally is the one I need to derive a relation between the physical Higgs self-coupling and the numerical value in the $\overline{\rm MS}$-scheme.

Inserting counter term of physical coupling (97) in Equation (72)

$$\begin{aligned}\hat{\lambda} &= \lambda_{\overline{\rm MS}} + \delta\lambda_{\overline{\rm MS}} - \delta\lambda_{p^2=0} + \hat{\lambda}(\delta_{\hat{m}_h}+\delta_v)\\ &= \lambda_{\overline{\rm MS}} + \delta_\lambda^{\overline{\rm MS}} - 2\lambda_{\overline{\rm MS}}\delta_\phi^{\overline{\rm MS}} - \delta_\lambda^{p^2=0} + 2\lambda_{p^2=0}\delta_\phi^{p^2=0} + \hat{\lambda}(\delta_{\hat{m}_h}+\delta_v)\ .\end{aligned}$$

On the right hand side I am allowed to replace $\lambda_{p^2=0}$ and $\hat{\lambda}$ with $\lambda_{\overline{\rm MS}}$ since they are multiplying counter terms, already the first order corrections. Thus

$$\hat{\lambda} = \lambda_{\overline{\rm MS}} + \delta_\lambda^{\overline{\rm MS}} - \delta_\lambda^{p^2=0} + \lambda_{\overline{\rm MS}}\left(2\delta_\phi^{p^2=0} - 2\delta_\phi^{\overline{\rm MS}} + \delta_{\hat{m}_h} + \delta_v\right)\ . \tag{98}$$

Substituting here counter terms (77), (82), (86) and (95) with (84), the physical self-coupling as a function of self-coupling in the $\overline{\rm MS}$-scheme becomes

$$\begin{aligned}\hat{\lambda} = \lambda_{\overline{\rm MS}} + \delta_\lambda^{\overline{\rm MS}} - \frac{D - \Pi_{\phi,\text{1-loop}}(0)}{2v^2} + \lambda_{\overline{\rm MS}}\Big(2\Pi'_{\phi,\text{1-loop}}(0) - 2\delta_\phi^{\overline{\rm MS}}\\ + \frac{\Pi_{\phi,\text{1-loop}}(m^2_{h,\overline{\rm MS}}) - \Pi_{\phi,\text{1-loop}}(0) - m^2_{h,\overline{\rm MS}}\Pi'_{\phi,\text{1-loop}}(m^2_{h,\overline{\rm MS}})}{m^2_{h,\overline{\rm MS}}} - \frac{\Pi_{W,\text{1-loop}}(0)}{m_W^2}\\ + \delta_l^{\overline{\rm MS}} + \delta_{\nu_l}^{\overline{\rm MS}} - \Pi'_{\phi,\text{1-loop}}(m^2_{h,\overline{\rm MS}}) + \Gamma_\mu^{\rm box} + \Gamma_{e,\text{1-loop}}(0) + \Gamma_{\mu,\text{1-loop}}(0)\Big)\ .\end{aligned} \tag{99}$$

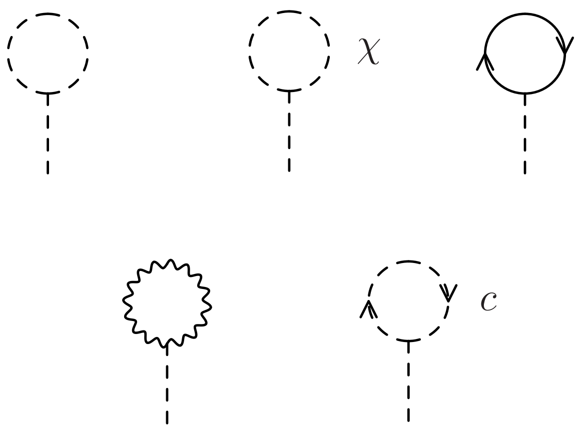


Figure 24: Higgs tadpole at one-loop level. Corrections are by Higgs boson itself, Goldstone bosons, fermions (top quark), Z- and W-bosons and ghosts.

On the right hand side of (99) I have replaced the physical mass $\hat{m}_h^2$ with the one in the $\overline{\text{MS}}$-scheme. As I mentioned before, I am allowed to replace these quantities with the $\overline{\text{MS}}$ ones in the first order corrections. Before I get the numerical value for the physical self-coupling, or mass, some one-loop calculations are still left to do to determine the Higgs tadpole and the two-point function, the lepton and neutrino two-point functions, the muon decay box diagrams and the W-electron-electron neutrino and W-muon-muon neutrino three-point functions. This is done in the next section.

6.2 Some loop calculations once more

Higgs one- and two-point functions Calculate first a one-loop tadpole of the Higgs boson. There are five different diagrams at one-loop level, shown in Figure 24. Unlike with my calculations to obtain the β-functions, here also the finite parts need to be evaluated. Due to that, it is much more reasonable to write down the results using Passarino–Veltman integrals instead of writing explicit integral expressions of diagrams. Recall that I denoted the sum of tadpole diagrams with $-\mathrm{i}vD$. The sum of diagrams in Figure 24 is then found to be

$$\begin{aligned}
-\mathrm{i}vD &= 3\lambda v A_0(m_h) + \lambda v(A_0(m_Z) + 2A_0(m_W)) - \frac{3}{\sqrt{2}} y_t m_t d A_0(m_t) \\
&\quad + \frac{v}{4}(g_1^2 + g_2^2)(dA_0(m_Z) - A_0(m_Z)) + \frac{v}{2} g_2^2 (dA_0(m_W) - A_0(m_W)) \,.
\end{aligned}$$

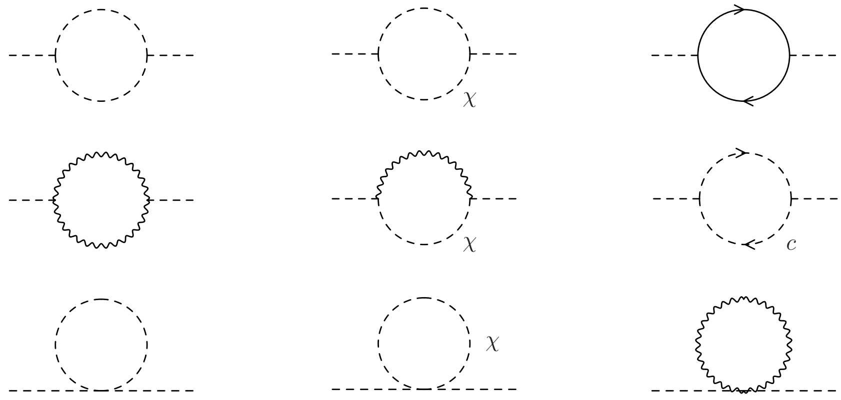


Figure 25: Diagrams contributing to Higgs two-point function at one-loop level.

In the previous equation $d = 4 - \epsilon$ is the dimension of spacetime in the dimensional regularisation. The function A_0 is defined in Appendix B. Using the relation between the fermion mass and Yukawa coupling, $m_f = \frac{y_f v}{\sqrt{2}}$, I get

$$D = 3\mathrm{i}\lambda A_0(m_h) + \mathrm{i}\lambda(A_0(m_Z) + 2A_0(m_W)) - \frac{3\mathrm{i}}{2} y_t^2 d A_0(m_t) + \frac{\mathrm{i}}{4}(g_1^2 + g_2^2)(d-1)A_0(m_Z) + \frac{\mathrm{i}}{2} g_2^2 (d-1) A_0(m_W)\ . \quad (100)$$

In Section 3.6.1 I calculated the two diagrams for the Higgs propagator at one-loop level. There are many more diagrams that I need to calculate to get the finite corrections. In fact, to get the full two-point function there are nine diagrams at one loop level, shown in Figure 25. The combined result from these diagrams to the Higgs self-energy is

$$\begin{aligned}\Pi_{h,\text{1-loop}}(p^2) = D &+ 9\mathrm{i}\lambda m_h^2 B_0(p, m_h^2, m_h^2) \\ &+ \frac{1}{2}\left(2\lambda m_h^2 + (g_1^2 + g_2^2)\left((d-1)m_Z^2 - p^2\right)\right)\mathrm{i}B_0(p, m_Z, m_Z) \\ &+ \left(2\lambda m_h^2 + g_2^2\left((d-1)m_W^2 - p^2\right)\right)\mathrm{i}B_0(p, m_W, m_W) \\ &- \frac{3}{2} d y_t^2 \left(2m_t^2 - \frac{1}{2}p^2\right)\mathrm{i}B_0(p, m_t, m_t)\ . \quad (101)\end{aligned}$$

For the wave function counter terms the derivative of the two-point function was also needed including finite parts. I do not calculate it analytically, but only numerically.

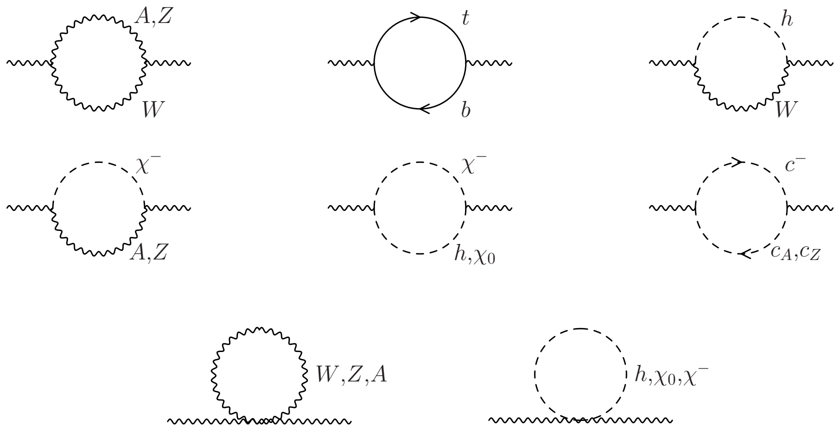


Figure 26: W two-point function at one-loop level. The outgoing boson is always W-boson. To clarify which diagrams are possible, I have written all labels explicitly.

W **two-point function** For the W two-point function there are 16 diagrams drawn in Figure 26. Since I have taken all fermions but top quark massless, only top and bottom quarks contribute in the fermion loop. The full W propagator at one loop level is

$$\Pi^{\mu\nu}_{W,\text{1-loop}}(p) = (g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})\Pi_{W,\text{1-loop}}(p^2) \ .$$

At $p^2 = 0$ the terms proportional $p^\mu p^\nu$ vanish. Thus the coefficient of $g^{\mu\nu}$ directly gives the function $\Pi_{W,\text{1-loop}}$. The sum of one-loop diagrams shown in Figure 26 yields

$$\begin{aligned}\Pi_{W,\text{1-loop}}(0) = &\left(-\frac{6(d-1)}{d} + \frac{2}{d}\sin^2\theta_W + (d-1) + \frac{1}{2}\right) g_2^2 \mathrm{i}A_0(m_W) \\ &+ \left(-\frac{1}{d} + \frac{2}{d}\cos^2\theta_W + (d-1)\cos^2\theta_W + \frac{1}{4}\right) g_2^2 \mathrm{i}A_0(m_Z) \\ &- \left(\frac{1}{d} - \frac{1}{4}\right) g_2^2 \mathrm{i}A_0(m_h) - \left(\frac{1}{d} - 1\right) g_2^2 m_W^2 \mathrm{i}B_0(0,m_h,m_W) \\ &+ \left(-\frac{6(d-1)}{d} - \frac{1}{d} + \frac{2}{d}\cos^2\theta_W\right) g_2^2 m_W^2 \mathrm{i}B_0(0,m_Z,m_W) \\ &+ g_1^2 \sin^2\theta_W m_W^2 \mathrm{i}B_0(0,m_Z,m_W) + g_2^2 \sin^2\theta_W m_W^2 \mathrm{i}B_0(0,m_W,0) \\ &- \frac{3}{2}(d-2) m_W^2 y_t^2 \mathrm{i}B_0(0,m_t,0) \ . \end{aligned} \quad (102)$$

Lepton and neutrino two-point functions Start with a neutrino two-point function. There are two diagrams, namely the W- and Z-boson loops. Summing these diagrams the neutrino two-point function at one-loop level gives

$$\begin{aligned}-\mathrm{i}\Sigma_{\nu_l,\text{1-loop}}(p^2) = \mathrm{i}\not{p}P_{\mathrm{L}}\Big(&- \mathrm{i}\frac{g_2^2}{2}\frac{d-2}{2p^2}\left(A_0(m_W) + (p^2 - m_W^2)B_0(p,0,m_W)\right) \\ &- \mathrm{i}\frac{g_2^2}{\cos^2\theta_W}\frac{d-2}{8p^2}\left(A_0(m_Z) + (p^2 - m_Z^2)B_0(p,0,m_Z)\right)\Big) \ .\end{aligned} \tag{103}$$

The previous result is the full neutrino two-point function at one-loop level containing both divergent and finite parts. To calculate the W mass counter term, the neutrino and lepton wave function counter terms were needed in the $\overline{\text{MS}}$-scheme. Thus I am interested only in the finite parts of these counter terms. Using the results shown in Appendix B, the divergent parts of A_0 and B_0 in the $\overline{\text{MS}}$-scheme are

$$\text{div}\left(A_0(m)\right) = \frac{2\mathrm{i}m^2}{16\pi^2\epsilon_{\overline{\text{MS}}}}$$

and

$$\text{div}\left(B_0(p, m_1, m_2)\right) = \frac{2\mathrm{i}}{16\pi^2\epsilon_{\overline{\text{MS}}}} \ .$$

Using these results it is easy to extract the neutrino wave function counter term in the $\overline{\text{MS}}$-scheme from Equation (103):

$$\delta_{\nu_l}^{\overline{\text{MS}}} = -\frac{1}{2}(3g_2^2 + g_1^2)\frac{1}{16\pi^2\epsilon_{\overline{\text{MS}}}} \ . \tag{104}$$

For lepton two-point function there is a photon loop in addition to the W- and Z-boson loops. The W- and Z-boson diagrams I can calculate similarly as for neutrino, ignoring lepton mass. In the photon diagram, since photon is massless, I have to keep the lepton mass non-zero[7]. The left-handed component of the two-point function is

$$\begin{aligned}-\mathrm{i}\Sigma_{l,\text{1-loop}}(p^2) = \mathrm{i}\not{p}P_{\mathrm{L}}\Big(&- \mathrm{i}\frac{g_2^2}{2}\frac{d-2}{2p^2}\left(A_0(m_W) + (p^2 - m_W^2)B_0(p,0,m_W)\right) \\ &- \mathrm{i}\frac{g_2^2}{\cos^2\theta_W}\frac{d-2}{8p^2}(1 - 2\sin^2\theta_W)^2\big(A_0(m_Z) \\ &+ (p^2 - m_Z^2)B_0(p,0,m_Z)\big) \\ &- \mathrm{i}g_2^2\sin^2\theta_W\frac{d-2}{2p^2}\left(A_0(m_l) + (p^2 - m_l^2)\right)B_0(p,0,m_l)\Big)\end{aligned}$$

[7] Another way to calculate this diagram is to give to photon small mass m_p to regulate the integral. However, the divergent part of this diagram would be the same.

from which the wave function counter term in the $\overline{\text{MS}}$-scheme is

$$\delta_l^{\overline{\text{MS}}} = -\frac{1}{2}(3g_2^2 + g_1^2)\frac{1}{16\pi^2\epsilon_{\overline{\text{MS}}}} \ . \tag{105}$$

Thus the neutrino and lepton counter terms are the same in the $\overline{\text{MS}}$-scheme, as they should. Note also that these counter terms do not depend on which lepton I am considering. This verifies that indeed in the $\overline{\text{MS}}$-scheme the g_2 counter terms in the muon decay are the same in the both ends of W-boson.

Muon decay box diagrams The task is to compute the muon decay box diagrams shown in Figure 22. Due to four propagators in the loop, these corrections are all finite. They are straightforward to calculate, when one uses the fact (1d) for product of three gamma matrices and expression for the C_0-function in terms of the B_0-functions given in Appendix B. However, the box with a photon between muon and electron (diagram 22b in Figure 22) needs some extra discussion. In diagrams with an additional Z-boson the large masses of Z and W ensure that neglecting muon momentum or muon mass is a valid approximation. A box with an additional photon however, is infrared sensitive at the limit where the muon mass is set to zero since the photon is massless. I have shown however that one finds the leading log result (in the $\log\left(\frac{m_\mu}{m_Z}\right)$) by setting the muon momentum to zero and keeping the muon mass finite. The electron mass can be neglected as before. Thus the sum of box diagrams is

$$\begin{aligned}\Gamma^{\text{box}}_{\mu^-\to e^-\bar{\nu}_e\nu_\mu} = &\frac{\mathrm{i}g_2^2}{8m_W^2}\left(\bar{u}_e\gamma^\alpha(\mathbb{1}-\gamma^5)v_{\bar{\nu}_e}\right)\left(\bar{u}_{\nu_\mu}\gamma_\alpha(\mathbb{1}-\gamma^5)u_\mu\right)\\ &\times\Big(\frac{g_2^2}{cos^2\theta_W}\frac{-1+4\sin^2\theta_W}{d(m_W^2-m_Z^2)}\big(B_0(0{,}0{,}m_W)-B_0(0{,}0{,}m_Z)\big)\\ &+g_2\sin^2\theta_W\frac{4}{m_W^2-m_\mu^2}\frac{1}{d}\big(B_0(0{,}0{,}m_W)-B_0(0{,}0{,}m_\mu)\big)\Big) \ .\end{aligned}$$

In the previous equation the first line on the right hand side is the tree-level result, so the box corrections are

$$\begin{aligned}\Gamma^{\text{box}}_\mu = &\frac{\mathrm{i}g_2^2}{cos^2\theta_W}\frac{1}{m_W^2-m_Z^2}\frac{-1+4\sin^2\theta_W}{d}\big(B_0(0{,}0{,}m_W)-B_0(0{,}0{,}m_Z)\big)\\ &+\mathrm{i}g_2\sin^2\theta_W\frac{4}{m_W^2-m_\mu^2}\frac{1}{d}\big(B_0(0{,}0{,}m_W)-B_0(0{,}0{,}m_\mu)\big) \ .\end{aligned} \tag{106}$$

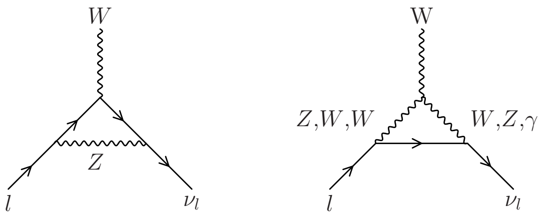


Figure 27: One-loop corrections to three-point function between W, lepton and its neutrino.

Three-point function between leptons and W The last set of one-loop diagrams I need are the three-point function between W, muon and muon neutrino and the same with electron and electron neutrino. These two I can calculate considering W interacting just with a lepton and its neutrino. There are two kinds of diagrams, see Figure 27. In the first diagram there is an additional Z-boson between the lepton and neutrino. There is no such a diagram with an additional W-boson because of charge conservation. In the second diagram the external W decays to two gauge fields which then interact with a charged lepton and neutrino. Those two gauge fields can be Z and W or W and γ. The order of W and Z matters, because Z interacts with charged leptons differently with than neutrinos. Sum of these four diagrams is

$$\begin{aligned}-\mathrm{i}\Gamma_{W,l,\nu_l} = \frac{\mathrm{i}g_2^2}{\sqrt{2}}\gamma^\mu P_\mathrm{L}\Big(&- \frac{\mathrm{i}g_2^2}{\cos^2\theta_W}\frac{(d-2)^2}{2d}(-\frac{1}{2}+\sin^2\theta_W)B_0(0,0,m_Z) \\ &+ \mathrm{i}g_2^2\frac{4(d-1)}{d}(-1+\sin^2\theta_W)B_0(0,m_W,m_Z) \\ &- \mathrm{i}g_2^2\sin^2\theta_W\frac{4(d-1)}{d}B_0(0,0,m_W)\Big) \; .\end{aligned}$$

Here one can see that the three-point functions do not depend on the lepton type in the limit of massless leptons. Thus electron and muon vertex corrections are the same and

$$\begin{aligned}\Gamma_{e/\mu,\text{1-loop}}(0) = &-\frac{\mathrm{i}g_2^2}{\cos^2\theta_W}\frac{(d-2)^2}{2d}(-\frac{1}{2}+\sin^2\theta_W)B_0(0,0,m_Z) \\ &+ \mathrm{i}g_2^2\frac{4(d-1)}{d}(-1+\sin^2\theta_W)B_0(0,m_W,m_Z) \\ &- \mathrm{i}g_2^2\sin^2\theta_W\frac{4(d-1)}{d}B_0(0,0,m_W) \; . \qquad (107)\end{aligned}$$

This was the last function that was needed in the Equation (99).

6.3 Numerical results

Above I calculated the two- three- and four-point functions that were needed in Equation (99). The last thing to do before I can calculate $\hat{\lambda}$ is to check that there are no divergences left on the right hand side of Equation (99). The difference between a coupling in two different renormalisation schemes is always finite. Hence, if there are divergences left in the Equation (99), or in (98), something has gone wrong. First, comparing divergences of δ_λ in $p^2 = 0$ and $\overline{\text{MS}}$ schemes

$$\text{div}\left(\delta_\lambda^{\overline{\text{MS}}} - \frac{D - \Pi_{\phi,\text{1-loop}}(0)}{2v^2}\right) = -\frac{\lambda_{\overline{\text{MS}}}}{16\pi^2\epsilon_{\overline{\text{MS}}}}\left(3g_2^2 + g_1^2\right) .$$

Second, the divergences between the Higgs wave function counter terms are cancelled. Indeed,

$$\text{div}\left(\Pi'_{\phi,\text{1-loop}}(0) - \delta_\phi^{\overline{\text{MS}}}\right) = 0 .$$

Thus to make the right hand side of (99) or (98) finite, the mass correction $\delta_{\hat{m}}$ and vev correction δ_v together have to give a divergence proportional to $(3g_2^2 + g_1^2)$. The mass correction itself is finite however. Subtracting $\Pi_{\phi,\text{1-loop}}(0)$ from $\Pi_{\phi,\text{1-loop}}(m^2_{h,\overline{\text{MS}}})$ one is left with terms proportional to p^2. Divergence of those terms at $p^2 = m^2_{h,\overline{\text{MS}}}$ is just the same as the divergence of $\Pi'_{\phi,\text{1-loop}}(m^2_{h,\overline{\text{MS}}})$ multiplied with $m^2_{h,\overline{\text{MS}}}$, and thus $\delta_{\hat{m}}$ is finite. This is consistent with the fact that the difference between masses in two renormalisation schemes cannot be divergent. Due to the finiteness of $\delta_{\hat{m}}$, the missing divergence has to be given by δ_v. Indeed, recalling that box-corrections are finite and W-lepton tree-vertices the same, the divergence of δ_v is

$$\Delta_v \equiv \text{div}\left(-\frac{\Pi_{W,\text{1-loop}}(0)}{m_W^2} + \delta_l^{\overline{\text{MS}}} + \delta_{\nu_l}^{\overline{\text{MS}}} - \Pi'_{\phi,\text{1-loop}}(\hat{m}_h^2) + 2\Gamma_{l,\text{1-loop}}(0)\right) .$$

After somewhat lengthy calculation one finds that

$$\Delta_v = (3g_2^2 + g_1^2)\frac{1}{16\pi^2\epsilon_{\overline{\text{MS}}}} ,$$

which indeed proves that difference between $\hat{\lambda}$ and $\lambda_{\overline{\text{MS}}}$ is finite. Hence, when I solve Equation (99), I can deal with only finite parts of functions.

Numerical values of coupling constants in the $\overline{\text{MS}}$-scheme were given in Section 5. The values for masses are found by using tree-level relations and those $\overline{\text{MS}}$-values for couplings, and the Weinberg angle θ_W is defined via the tree-level relation [6]

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}} .$$

The scale is set to be $\mu = \hat{m}_t$. Certainly (99) can not depend on scale since equations with physical quantities are always scale independent and finite. The scale has to be set to the top pole mass simply because the values of the coupling constants were given at that scale. If the scale was changed, then also the values of the coupling constants would change.

Inserting one-loop results from the previous section with numerical values of couplings and masses in Equation (99), the physical values of the Higgs self-coupling and mass are

$$\begin{cases} \hat{\lambda} & \approx 0.131 \\ \hat{m}_h & \approx 126\,\mathrm{GeV}\ . \end{cases} \tag{108}$$

7 Discussion

In this thesis I have predicted the Higgs boson mass from the SM with gravitation as an asymptotically safe theory. The prediction is found by requiring that all couplings remain finite to very high energies using the running of the five standard model couplings. I started with renormalisation of the Yang-Mills Lagrangian, Higgs Lagrangian and top Yukawa interaction term in Section 2. In Section 3 I derived one-loop β-functions needed for the differential equations that give the running of couplings for the three gauge couplings of the SM, top Yukawa coupling and Higgs self-coupling from the SM. Corrections to the SM results given by asymptotically safe gravity were discussed in Section 4. Numerical solution of the group of β-functions in Section 5 gave the value $\lambda \approx 0.1538$ for the Higgs self-coupling in the $\overline{\text{MS}}$-scheme at the top quark pole mass scale. In Section 6 that value was related to the physical parameters. It turned out that to get the relation between the $\overline{\text{MS}}$ and physical self-couplings requires among other things one-loop renormalisation of the muon decay. The one-loop level relation leads to the values

$$\begin{cases} \hat{\lambda} & \approx 0.131 \\ \hat{m}_h & \approx 126\,\text{GeV} \end{cases}$$

for the physical Higgs self-coupling and mass.

The measured value for Higgs mass was $m_h \approx 125\,\text{GeV}$ which corresponds to the self-coupling $\lambda \approx 0.13$. Remembering the fact that calculations in this thesis were done at first non-trivial order, this result is very good. This result actually is approximately the same than given in [5], although their calculations were done at two-loop level. This may be because in [5] the value of the top pole mass is smaller than the one I used. Even small changes in the top Yukawa coupling strongly affects the running of the Higgs self-coupling. Furthermore, uncertainties in the value of strong fine structure constant cause uncertainty to the Higgs self-coupling via top Yukawa since of the three SM gauge couplings the strong coupling has the largest contribution to the running of the top Yukawa coupling. For example for the top Yukawa and Higgs self-coupling β-functions two-loop corrections are significant [33]. To get a more accurate prediction, one thus should do calculations at two-loop level. Two-loop results for the SM β-functions are well-known. At higher orders there are only dozens of diagrams more to be calculate, but the idea remains the same. Careful error analysis with relation to uncertainties in the strong fine structure constant and top mass would also lead to more accurate predictions. Obviously, uncertainties in the U(1) and SU(2) couplings as well as neglecting almost all fermion masses cause some error to final results, but their role is much less significant.

When considering final results, also gravitational corrections are to be discussed. First of all, the absence of the proof for asymptotic safety of gravity in four dimensions is a thing to be taken into account. In four dimensions there are only 'strong hints' for an existence of A non-trivial fixed point. The second thing is the form of gravitational corrections. As I mentioned in Section 4.3 there has been discussion about a possible gauge dependence of results I have used. These two issues would need more careful research.

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A Feynman rules

In this appendix I briefly go through the method I have used for deriving Feynman rules. First I'll describe the general idea and after that I derive one rule step by step in the case of an SU(N) theory as an example. In the end of this appendix there is a complete list of Feynman rules that are needed in this thesis.

Assume that there is a term $\phi_1\phi_2^2$, where ϕ_1 and ϕ_2 are some quantum fields, in the Lagrangian. I would like to derive the Feynman rule for the vertex between these two fields. The idea is to derive the T-matrix element between an incoming state of one ϕ_1 and two ϕ_2 fields and an outgoing vacuum state[8]. In practice, I replace the fields ϕ_1 and ϕ_2 with their field operators and then calculate the matrix element of that operator between the incoming state $|\phi_1(p_1)\phi_2(p_2)\phi_2(p_3)\rangle$ with those fields and outgoing vacuum state $\langle\Omega|$. In my example the corresponding operator is

$$\phi_1\phi_2^2 \to \hat{\phi}_1(x)\hat{\phi}_2(x)\hat{\phi}_2(x)$$

and the matrix element to be calculated is

$$\int \mathrm{d}^4x \; \langle\Omega|\, \hat{\phi}_1(x)\hat{\phi}_2(x)\hat{\phi}_2(x)\, |\phi_1(p_1)\phi_2(p_2)\phi_2(p_3)\rangle \,.$$

Extracting the factor $(2\pi)^4\delta^4(p_1+p_2+p_3)$ for four-momentum conservation and (if there is) spinors and polarization vectors for external legs, the rest is the Feynman rule for $\phi_1\phi_2^2$-vertex. When there are identical particles taking part in the interaction, I divide the result with a symmetry factor of the vertex to get a rule without them and to save myself from many difficulties with higher order diagrams.

Field operators Just as a reminder, I list here the field operators for real and complex scalar, gauge and fermion fields.

- Real scalar field:

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_\mathbf{k}\mathrm{e}^{-\mathrm{i}k\cdot x} + a_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}k\cdot x} \right)$$

[8] I could as well take one ϕ_1 field to form the incoming and two ϕ_2 fields to form the outgoing state or all fields in the outgoing state. Just for simplicity I keep all fields in the incoming state to get all momenta flowing in.

- Complex scalar field:

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_\mathbf{k}\mathrm{e}^{-\mathrm{i}k\cdot x} + b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}k\cdot x}\right)$$
$$\hat{\phi}^*(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_\mathbf{k}^\dagger\mathrm{e}^{\mathrm{i}k\cdot x} + b_\mathbf{k} \mathrm{e}^{-\mathrm{i}k\cdot x}\right)$$

- Gauge field:

$$\hat{A}_\mu^a(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_{\mu\mathbf{k}}^a \epsilon_\mu^a(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + a_{\mu\mathbf{k}}^{a\,\dagger}\epsilon_\mu^{a\,*}(k) \mathrm{e}^{\mathrm{i}k\cdot x}\right)$$

- Fermion field:

$$\hat{\psi}_i(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_{i\mathbf{k}} u_i(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + b_{i\mathbf{k}}^\dagger v_i(k) \mathrm{e}^{\mathrm{i}k\cdot x}\right)$$
$$\hat{\bar{\psi}}_i(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \left(a_{i\mathbf{k}}^\dagger \bar{u}_i(k)\mathrm{e}^{\mathrm{i}k\cdot x} + b_{i\mathbf{k}} \bar{v}_i(k) \mathrm{e}^{-\mathrm{i}k\cdot x}\right)$$

As a shorthand notation one often writes

$$\widetilde{d^3\mathbf{k}} \doteq \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \ .$$

One-particle states are normalised such that

$$|\mathbf{k}\rangle = \sqrt{2E_\mathbf{k}} a_\mathbf{k}^\dagger |\Omega\rangle \ . \tag{109}$$

With this normalisation, the inner product of two one-particle states is Lorentz invariant [6].

A.1 Feynman rules for Yang–Mills theory

In this section I derive step by step the Feynman rule for the vertex between fermions and a gauge field starting from the Yang-Mills Lagrangian. After that I just list all other Feynman rules in SU(N) theory needed in this thesis.

The Lagrangian describing a general SU(N) Yang–Mills theory is (see Section 2.1)

$$\mathcal{L} = \bar{\psi}_j(\mathrm{i}\not{D}_{ji} - m_\psi\delta_{ij})\psi_i - \frac{1}{4}F_{\mu\nu}^a F^{a\,\mu\nu} - \frac{1}{2}(\partial^\mu A_\mu^a)^2 + \bar{c}^a(-\partial^\mu D_\mu^{ab})c^b \ ,$$

where the covariant derivative and field strength tensor are

$$
\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\
D^{ab}_\mu &= \delta^{ab}\partial_\mu + g f^{abc} A^c_\mu .
\end{aligned}
$$

The fermion part, apart from fermion's mass term, is $\bar{\psi}_j \mathrm{i}(\not{D})_{ji}\psi_i = \mathrm{i}\bar{\psi}_i \not{\partial}\psi_i + g t^a_{ji}\gamma^\mu \bar{\psi}_j A^a_\mu \psi_i$. Here the first term is the kinetic term of fermion. The second term is the one giving the interaction between a fermion and an SU(N) gauge field. Let us find the Feynman rule for that vertex. The T-matrix element is

$$
\int \mathrm{d}^4x \, \langle\Omega| \, g t^b_{lm} \gamma^\alpha \hat{\bar{\psi}}_l \hat{A}^b_\alpha \hat{\psi}_m \, |\mathbf{k},\mu,a;\mathbf{p}_1,i;\mathbf{p}_2,j\rangle \ ,
$$

where in the initial state the momentum $\mathbf{k}$ is for the gauge field, $\mathbf{p}_1$ for the fermion and $\mathbf{p}_2$ for the antifermion. Inserting field operators and writing the initial state with creation operators using (109) yields

$$
\begin{aligned}
\int & \mathrm{d}^4x \, \widetilde{d^3\mathbf{q}_1}\widetilde{d^3\mathbf{q}_2}\widetilde{d^3\mathbf{q}_3} \, g t^b_{lm} \, \langle\Omega| \left(a^\dagger_{l\mathbf{q}_1} \bar{u}_l(q_1) \mathrm{e}^{\mathrm{i}q_1\cdot x} + b_{l\mathbf{q}_1} \bar{v}_l(q_1) \mathrm{e}^{-\mathrm{i}q_1\cdot x} \right) \gamma^\alpha \\
& \times \left(a^b_{\alpha\mathbf{q}_2} \epsilon^b_\alpha(q_2) \mathrm{e}^{-\mathrm{i}q_2\cdot x} + a^{b\,\dagger}_{\alpha\mathbf{q}_2} \epsilon^{b\,*}_\alpha(q_2) \mathrm{e}^{\mathrm{i}q_2\cdot x} \right) \\
& \times \left(a_{m\mathbf{q}_3} u_m(q_3) \mathrm{e}^{-\mathrm{i}q_3\cdot x} + b^\dagger_{m\mathbf{q}_3} v_m(q_3) \mathrm{e}^{\mathrm{i}q_3\cdot x} \right) \\
& \times \sqrt{2E_\mathbf{k}} a^{a\,\dagger}_{\mu\mathbf{k}} \sqrt{2E_{\mathbf{p}_1}} a^\dagger_{i\mathbf{p}_1} \sqrt{2E_{\mathbf{p}_2}} b^\dagger_{j\mathbf{p}_2} \, |\Omega\rangle \ .
\end{aligned}
$$

Using the fact that a creation operator destroys an outgoing and an annihilation operator an incoming vacuum and that the vacuum expectation value of the product of annihilation and creation operators is

$$
\langle\Omega| \, a_{\mu\mathbf{k}} a^\dagger_{\nu\mathbf{q}} \, |\Omega\rangle = (2\pi)^3 \delta_{\mu\nu} \delta^{(3)}(\mathbf{k}-\mathbf{q}) \ ,
$$

leads to the next expression for T-matrix element:

$$
\begin{aligned}
& \int \mathrm{d}^4x \, \frac{\mathrm{d}^3\mathbf{q}_1}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{q}_2}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{q}_3}{(2\pi)^3} \, g t^b_{lm} \mathrm{e}^{-\mathrm{i}(q_1+q_2+q_3)\cdot x} (2\pi)^3 \delta_{lj} \delta^{(3)}(\mathbf{q}_1-\mathbf{p}_2) \bar{v}_l(\mathbf{q}_1) \gamma^\alpha \\
& \quad \times (2\pi)^3 \delta_{\alpha\mu} \delta^{ab} \delta^{(3)}(\mathbf{q}_2-\mathbf{k}) \epsilon^b_\alpha(q_2) (2\pi)^3 \delta_{mi} \delta^{(3)}(\mathbf{q}_3-\mathbf{p}_1) u_m(\mathbf{q}_3) \\
=& \int \mathrm{d}^4x \, g t^a_{ji} \mathrm{e}^{-\mathrm{i}(p_2+k+p_1)\cdot x} \bar{v}_j(\mathbf{p}_2) \gamma^\mu u_i(\mathbf{p}_1) \epsilon^a_\mu(k) \\
=& (2\pi)^4 \delta^{(4)}(k+p_1+p_2) g t^a_{ji} \bar{v}_j(\mathbf{p}_2) \gamma^\mu u_i(\mathbf{p}_1) \epsilon^a_\mu(k) \, .
\end{aligned}
$$

In the last step I used the following expression of the delta-function:

$$
\int \mathrm{d}^4x \, \mathrm{e}^{-\mathrm{i}k\cdot x} = (2\pi)^4 \delta^{(4)}(k) \ ,
$$

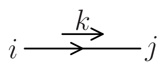
By extracting factors $(2\pi)^4\delta^{(4)}(k+p_1+p_2)$ for four-momentum conservation, spinors of fermions and polarization vector of gauge field, one is left with the factor $gt^a_{ji}\gamma^\mu$. Furthermore, one has to multiply the result with the imaginary unit i because there is an additional i in the action $S = \exp(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L})$ in the Green's function. Finally, the Feynman rule for the vertex between fermions and a gauge field is $\mathrm{i}gt^a_{ji}\gamma^\mu$. Here the index j labels the outgoing and index i the incoming fermion.

Notice here that Feynman rule was just the constants of the interaction term in the Lagrangian multiplied with complex unit i. If there was a derivative of the field in the interaction term, there would be an additional factor $(\pm \mathrm{i}p_\mu)$. This can be seen from the fact that taking a derivative of field operator, the derivative acts on the exponential function dropping $(\pm \mathrm{i}p_\mu)$ in front. Keeping this in mind the Feynman rules can be seen straight from the Lagrangian without any trouble of calculating vacuum expectation values.

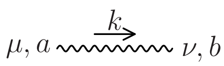
A.1.1 List of SU(N) Feynman rules

The rest of the SU(N) Feynman rules can be derived similarly as I derived the rule for vertex between fermions and gauge field. Here I list all those rules.

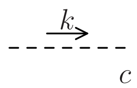
- Propagators



$\sim \frac{\mathrm{i}(k^\mu\gamma_\mu+m_f)}{k^2-m_f^2+\mathrm{i}\epsilon_\mathrm{F}}\delta_{ij}$

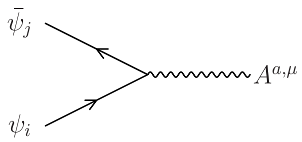


$\sim \frac{-\mathrm{i}g^{\mu\nu}}{k^2+\mathrm{i}\epsilon_\mathrm{F}}\delta^{ab}$

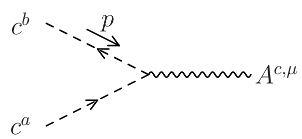


$\sim \frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon_\mathrm{F}}$

- Three-point vertices between gauge field and fermions or ghosts:

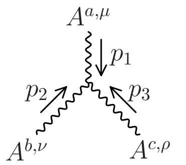


$\sim \mathrm{i}gt^a_{ji}\gamma^\mu$



$\sim \mathrm{i}f^{abc}p^\mu$

- Three-point vertex of gauge fields

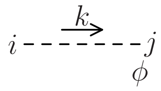


$\sim g f^{abc} V^{\mu\nu\rho}(p_1, p_2, p_3)$

$$V_{\mu\nu\rho}(p_1,p_2,p_3) = (p_3 - p_1)_\nu g_{\mu\rho} + (p_1 - p_2)_\rho g_{\nu\mu} + (p_2 - p_3)_\mu g_{\rho\nu}$$

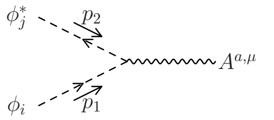
Feynman rules for an additional scalar, for which the interaction with a gauge field is $|(D_\mu)_{ji}\phi_i|^2$ and with a fermion is $y\bar{\psi}_{L,i}\phi_j\psi_R +$ h.c., are

- Propagator

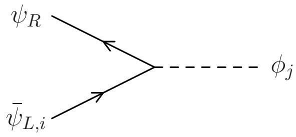


$\sim \frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon_\mathrm{F}}\delta_{ij}$

- Three-point vertices between scalar and gauge field or fermions



$\sim -\mathrm{i}g t^a_{ji}(p_2^\mu - p_1^\mu)$

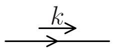


$\sim \mathrm{i}y_f\delta_{ij}P_\mathrm{L}$

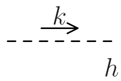
A.2 SM Feynman rules

In this Appendix I list all SM Feynman rules that are needed in this thesis. These rules do not include symmetry factors if there are identical particles interacting, but one has to compute them when calculating a diagram. For a cross-check, see for example [38] (notice differences due to symmetry factors).

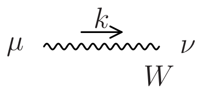
- Propagators



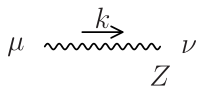
$\sim \frac{\mathrm{i}(k^\mu \gamma_\mu + m_f)}{k^2 - m_f^2 + \mathrm{i}\epsilon_\mathrm{F}}$



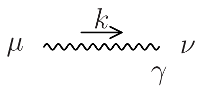
$\sim \frac{\mathrm{i}}{k^2 - m_h^2 + \mathrm{i}\epsilon_\mathrm{F}}$



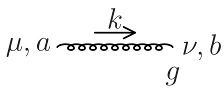
$\sim \frac{-\mathrm{i}g^{\mu\nu}}{k^2 - m_W^2 + \mathrm{i}\epsilon_\mathrm{F}}$



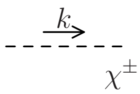
$\sim \frac{-\mathrm{i}g^{\mu\nu}}{k^2 - m_Z^2 + \mathrm{i}\epsilon_\mathrm{F}}$



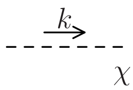
$\sim \frac{-\mathrm{i}g^{\mu\nu}}{k^2 + \mathrm{i}\epsilon_\mathrm{F}}$



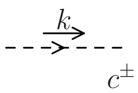
$\sim \frac{-\mathrm{i}g^{\mu\nu}}{k^2 + \mathrm{i}\epsilon_\mathrm{F}}\delta^{ab}$



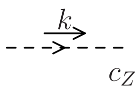
$\sim \frac{\mathrm{i}}{k^2 - m_W^2 + \mathrm{i}\epsilon_\mathrm{F}}$



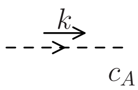
$\sim \frac{\mathrm{i}}{k^2 - m_Z^2 + \mathrm{i}\epsilon_\mathrm{F}}$



$\sim \frac{\mathrm{i}}{k^2 - m_W^2 + \mathrm{i}\epsilon_\mathrm{F}}$

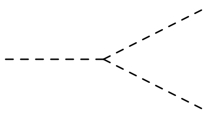


$\sim \frac{\mathrm{i}}{k^2 - m_Z^2 + \mathrm{i}\epsilon_\mathrm{F}}$

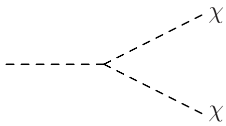


$\sim \frac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon_\mathrm{F}}$

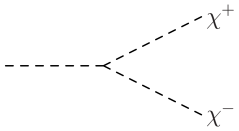
- Three-point vertices with Higgs boson



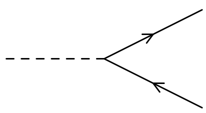
$\sim -\mathrm{i}\lambda v$



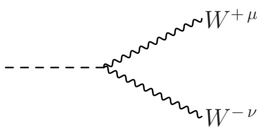
$\sim -\mathrm{i}\lambda v$



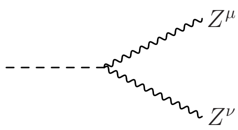
$\sim -2\mathrm{i}\lambda v$



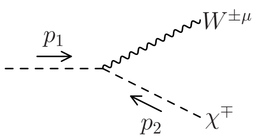
$\sim -\frac{\mathrm{i}y_f}{\sqrt{2}}$



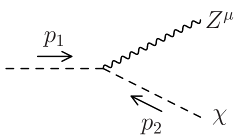
$\sim \frac{\mathrm{i}g_2^2 v}{2} g^{\mu\nu}$



$\sim \frac{\mathrm{i}g_2^2 v}{4\cos^2\theta_W} g^{\mu\nu}$

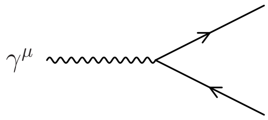


$\sim \pm\frac{\mathrm{i}g_2}{2}(p_1^\mu - p_2^\mu)$

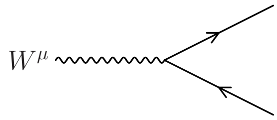


$\sim \frac{g_2}{2\cos\theta_W}(p_1^\mu - p_2^\mu)$

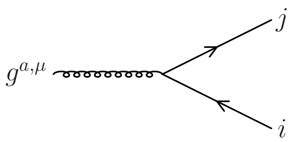
- Three-point vertices between fermions and gauge bosons



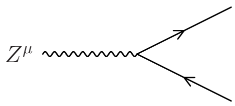
$\sim \mathrm{i}eQ_f\gamma^\mu$



$\sim \frac{\mathrm{i}g_2}{\sqrt{2}}\gamma^\mu P_\mathrm{L}$



$\sim \mathrm{i}g_3 t^a_{ji}\gamma^\mu$

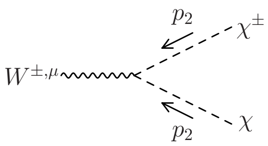


$\sim \frac{\mathrm{i}g_2}{\cos\theta_W}\gamma^\mu(C_{\mathrm{L,f}}P_\mathrm{L} + C_{\mathrm{R,f}}P_\mathrm{R})$

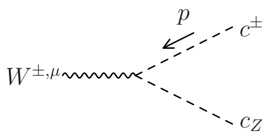
$$\begin{cases} C_{\mathrm{L,f}} &= T^3_f - Q_f \sin^2\theta_W \\ C_{\mathrm{R,f}} &= -Q_f \sin^2\theta_W \end{cases}$$

	T^3_f	Q_f
l_L	$-1/2$	-1
l_R	0	-1
ν_L	$1/2$	0
u_L	$1/2$	$2/3$
d_L	$-1/2$	$-1/3$
u_R	0	$2/3$
d_R	0	$-1/3$

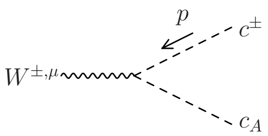
- Three-point vertices with W-boson



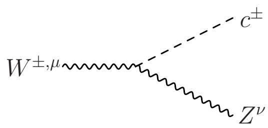
$\sim \frac{g_2}{2}(p_1^\mu - p_2^\mu)$



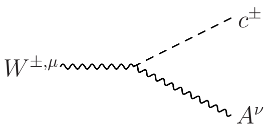
$\sim \pm \mathrm{i} g_2 \cos\theta_W$



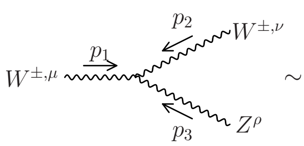
$\sim \mp \mathrm{i} g_2 \sin\theta_W$



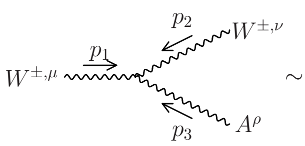
$\sim -\mathrm{i} g_1 m_W g^{\mu\nu}$



$\sim \mathrm{i} e m_W g^{\mu\nu}$

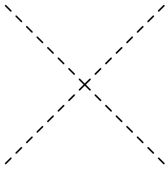


$\sim \mathrm{i} g_2 \cos\theta_W V^{\mu\nu\rho}(p_1, p_2, p_3)$

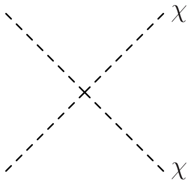


$\sim \mathrm{i} g_2 \sin\theta_W V^{\mu\nu\rho}(p_1, p_2, p_3)$

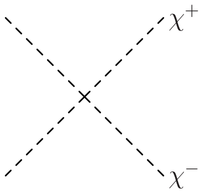
- Four-point vertices



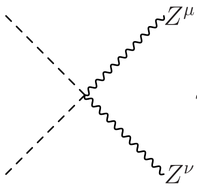
$\sim -\mathrm{i}\frac{\lambda}{4}$



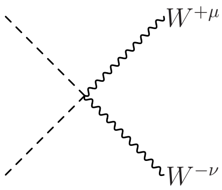
$\sim -\mathrm{i}\frac{\lambda}{2}$



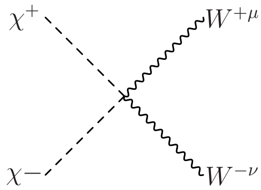
$\sim -\mathrm{i}\lambda$



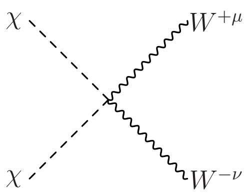
$\sim \mathrm{i}\frac{g_2^2}{8\cos^2\theta_W}g^{\mu\nu}$



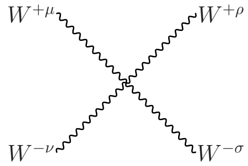
$\sim \mathrm{i}\frac{g_2^2}{4}g^{\mu\nu}$



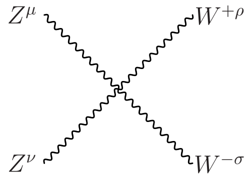
$\sim \mathrm{i}\frac{g_2^2}{2}g^{\mu\nu}$



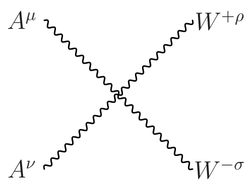
$\sim \mathrm{i}\frac{g_2^2}{4}g^{\mu\nu}$



$$\sim \mathrm{i}g_2^2(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$



$$\sim -\mathrm{i}\tfrac{g_2^2}{2}\cos^2\theta_W(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$$



$$\sim -\mathrm{i}\tfrac{g_2^2}{2}\sin^2\theta_W(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$$

B Passarino–Veltman reduction integrals

In this section I list a set of integral forms known as Passarino–Veltman reduction integrals [37, 39, 40].

When calculating Feynman diagrams, one quite soon after few integrals finds out that every time in dimensional regularization it is about the same tricks one after another, namely going to the d-dimensions, doing the Wick rotation, using the Feynman parametrization and using a certain representation of the beta function and the properties of the gamma function. So why not to do all this once for some basic integrals and then write all diagrams as a sum of those integrals? The results of Section 6.2 also show the usefulness of these functions when one writes down loop-calculations. The sum of several diagrams can be written in a very elegant form using the Passarino–Veltman integrals. From those expressions also the divergent parts of functions can be seen quite easily.

Here I write down those Passarino–Veltman integrals that are used in this thesis. The scalar one-point function is defined as

$$\begin{aligned} A_0(m) &= \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon_{\mathrm{F}}} \\ &= \frac{\mathrm{i}m^2}{16\pi^2} \left[\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log(4\pi) + 1 - \log(\frac{m^2}{\mu^2}) + \mathcal{O}(\epsilon) \right] , \end{aligned}$$

where we assumed that $m \neq 0$, we have $\epsilon = 4 - d$ and γ_{E} is Euler–Mascheroni constant ($\gamma_E \approx 0.577$). Another scalar function is the two-point function

$$\begin{aligned} B_0(p,m_1,m_2) &= \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2 + \mathrm{i}\epsilon_{\mathrm{F}})((k-p)^2 - m_2^2 + \mathrm{i}\epsilon_{\mathrm{F}})} \\ &= \frac{\mathrm{i}}{16\pi^2} \Big[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) - \log\Big(\frac{p^2 + i\epsilon_{\mathrm{F}}}{\mu^2} \Big) + 2 \\ &\quad - \sum_{\pm} \Big((1 - x_\pm) \log(1 - x_\pm) + x_\pm \log(-x_\pm) \Big) \Big] , \end{aligned}$$

where

$$x_\pm = \frac{p^2 + m_1^2 - m_2^2}{2p^2} \pm \frac{\sqrt{(p^2 + m_1^2 - m_2^2)^2 - 4p^2 m_1^2 + 4\mathrm{i}\epsilon_{\mathrm{F}} p^2}}{2p^2} .$$

Notice here that the divergence of $A_0(m)$ is the same as that of $mB_0(p, m_1, m_2)$. Furthermore, it is a small exercise to prove that B_0 is symmetric under an exchange of masses, i.e.

$$B_0(p,m_1, m_2) = B_0(p,m_2,m_1) .$$

It turns out that all diagrams in my thesis can be written as a sum of these two scalar functions. A clever reader notices that A_0 function is obtained when a diagram has only one internal propagator, and diagrams with two internal propagators are proportional to B_0. Of course, if the Feynman rules for vertices in a diagram give an integration momentum in the numerator, then some of the propagators in the denominator may be cancelled and thus diagrams with more than one or two internal propagators may be proportional to A_0 and B_0. This just proves the usefulness of these functions.

There are two a bit more complicated functions, vector and tensor functions, which are also needed. Luckily, they both reduce to a sum of functions A_0 and B_0. The two-point vector function is

$$\begin{aligned} B_\mu(p,m_1,m_2) &= \mu^{4-d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k_\mu}{(k^2-m_1^2+\mathrm{i}\epsilon_{\mathrm{F}})((k-p)^2-m_2^2+\mathrm{i}\epsilon_{\mathrm{F}})} \\ &= \frac{p_\mu}{p^2}B_1(p,m_1,m_2)\,, \end{aligned}$$

where

$$B_1(p,m_1,m_2) = \frac{1}{2}\left(A_0(m_2) - A_0(m_1) + \left(p^2+m_1-m_2\right)B_0(p,m_1,m_2)\right)$$

and the tensor function is

$$\begin{aligned} B_{\mu\nu}(p,m_1,m_2) &= \mu^{4-d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k_\mu k_\nu}{(k^2-m_1^2+\mathrm{i}\epsilon_{\mathrm{F}})((k-p)^2-m_2^2+\mathrm{i}\epsilon_{\mathrm{F}})} \\ &= p_\mu p_\nu B_{21}(p,m_1,m_2) + g_{\mu\nu}B_{22}(p,m_1,m_2)\,, \end{aligned}$$

where

$$\begin{aligned} B_{21}(p,m_1,m_2) = \frac{-1}{p^2(d-1)}\Big[&\big(1-\frac{d}{2}-\frac{d}{4p^2}(p^2+m_1^2-m_2^2)\big)A_0(m_2) \\ &+\frac{d}{4p^2}(p^2+m_1^2-m_2^2)A_0(m_1) \\ &+\big(m_1^2-\frac{d}{4p^2}(p^2+m_1^2-m_2^2)^2\big)B_0(p,m_1,m_2)\Big] \end{aligned}$$

and

$$\begin{aligned} B_{22}(p,m_1,m_2) = \frac{1}{d-1}\Big[&\big(\frac{1}{2}-\frac{1}{4p^2}(p^2+m_1^2-m_2^2)\big)A_0(m_2) \\ &+\frac{1}{4p^2}(p^2+m_1^2-m_2^2)A_0(m_1) \\ &+\big(m_1^2-\frac{1}{4p^2}(p^2+m_1^2-m_2^2)^2\big)B_0(p,m_1,m_2)\Big]\,. \end{aligned}$$

The third scalar integral is the one with three propagators in denominator, called C_0-integral. In this thesis it is needed only in a special case with all external momenta zero and two non-zero masses where it can be reduced to a sum of two B_0-functions

$$\begin{aligned} C_0(m_1,m_2) &\equiv \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+\mathrm{i}\epsilon_{\mathrm{F}})(k^2-m_1^2+\mathrm{i}\epsilon_{\mathrm{F}})(k^2-m_2^2+\mathrm{i}\epsilon_{\mathrm{F}})} \\ &= \frac{1}{m_2^2-m_1^2}\left(B_0(0,0,m_2)-B_0(0,0,m_1)\right) \ . \end{aligned}$$

This function is finite in four dimensions since divergences are the same in the two B_0-functions.