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Review Article

Neutrinoless Double $\beta^+$/EC Decays

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The relation of neutrino masses to neutrino oscillations and the nuclear double beta decay is highlighted. In particular, the neutrinoless $\beta^+\beta^+$, $\beta^+\text{EC}$, and resonant ECEC decays are investigated using microscopic nuclear models. Transitions to the ground state and excited $0^+$ states are analyzed. Systematics of the related nuclear matrix elements are studied and the present status of the resonant ECEC decays is reviewed.

1. Introduction

The modern neutrino oscillation experiments have brought the study of neutrino properties to the era of precision measurements. At the same time the fundamental character (Majorana or Dirac) of the neutrino is still unknown, as is also its absolute mass scale. To gain information on these two unknowns the atomic nuclei can be engaged as the mediators of the Majorana-neutrino triggered neutrinoless double beta ($0\nu2\beta$) decays. The key issue here is how to cope with the involved nuclear-structure issues of the decays, crystallized in the form of the nuclear matrix elements (NMEs) [1–3]. To be able to exploit the potential data extracted from the $0\nu2\beta$-decay experiments one needs to evaluate the NMEs in a reliable enough way. It has become customary to employ the neutrino-emitting correspondent of $0\nu2\beta$ decay, the two-neutrino double beta ($2\nu2\beta$) decay, to confine the nuclear-model degrees of freedom in the NME calculations.

The $2\nu2\beta$ decay is a second-order process in the standard model of electroweak interactions and the associated half-lives have been measured for several nuclei [4].

The neutrinoless double $\beta^-$ ($0\nu\beta^-\beta^-$) decays have been studied intensively over the years [2, 3] due to their favorable decay $Q$ values. The positron-emitting modes of decays, $\beta^-\beta^+$, $\beta^-\text{EC}$, and EEC, are much less studied. From here on we will denote all these decay modes as $0\nu\beta^-/EC$ decays. The general, nuclear model independent frameworks of theory for these decays have been investigated in [7] for the $0\nu\beta^-/EC$-decay channels $\beta^-\beta^+$ and $\beta^-\text{EC}$. The formalism for the resonant neutrinoless double electron capture ($R0\nu\text{ECEC}$) was first developed in [8] and later discussed and extended to its radiative variant ($0\nu\gamma\text{ECEC}$) in [9]. Due to the resonant nature of the $R0\nu\text{ECEC}$ decay its studies have called for precise measurements of the mass differences of the atoms involved in the decays. The resonant mode of $0\nu\text{ECEC}$ decays is studied intensively for its potential enhanced sensitivity to discover the Majorana mass of the neutrino and that is why much experimental effort is being invested in observing this mode of decay.

2. Neutrino Masses and Oscillations

In the calculations of transition rates of the $0\nu\beta^+/EC$ decays, the neutrino-physics part and nuclear-physics part factorize. We will start by considering the neutrino-physics part. The weak-interaction Lagrangian of leptons is diagonal in the neutrino fields $\nu_e, \nu_\mu$, and $\nu_\tau$, called flavor eigenfields. The charged-current interaction part of the Lagrangian of the Standard Model of electroweak interactions, which is relevant to the considerations of this presentation, is given by

\[
\mathcal{L}_{CC} = -\frac{g}{2\sqrt{2}} \sum \bar{\nu}_\ell Y_\mu (1 - Y_5) \ell W^\mu + \text{h.c.} = -\frac{g}{\sqrt{2}} \sum \bar{\nu}_\ell Y_\mu e_\ell^\dagger W^\mu + \text{h.c.,}
\] (1)
where $\ell$ refers to the three lepton flavors, $\ell = e, \mu, \tau$, $W^\nu$ is a vector field corresponding to the charged weak boson $W^\nu$, $\nu_{\ell L}$ and $\ell_R$ are the left-handed chiral components of the neutrino and charged lepton fields, and $g$ is the gauge coupling constant. In all phenomena studied so far neutrinos appear as ultrarelativistic particles, but it is known that, albeit being extremely light compared with other fermions, neutrinos do have mass, evidenced by observations of many neutrino-flavor-oscillation phenomena (see, e.g., [10–18]). In neutrino oscillations transitions between neutrino flavors take place, indicating that neutrinos mix with each other. This mixing arises through the mechanism that gives neutrinos their mass. The mass part of the neutrino Lagrangian is hence not diagonalized by the flavor fields $\nu_i$ but by fields $\nu_i$ ($i = 1, 2, 3$) that have definite masses $m_i$, known as the mass eigenfields. The left-handed flavor eigenfields appearing in the interaction Lagrangian (1) are superpositions of the left-handed components of the mass eigenfields:

$$\nu_{\ell L} = \sum_{\ell = e, \mu, \tau} U_{\ell \alpha} \nu_{\ell L}.$$

where $U$ is a unitary $3 \times 3$ matrix, called the neutrino mixing matrix or Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [19, 20].

The mass of the left-handed neutrinos can arise from mass terms of the form

$$\frac{1}{2} M_{\ell e}^2 (\nu_{\ell e}^\dagger \nu_{\ell e}) + \text{h.c.},$$

the so-called Majorana mass terms. They can arise in the Standard Model of particle physics through nonrenormalizable interactions between neutrinos and neutral Higgs bosons: $- (Y_{\ell e} / \Lambda) (\nu_{\ell e}^\dagger \nu_{\ell e}) (H^0)^2 \nu_{\ell L} + \text{h.c.}$, where $Y_{\ell e}$ is the Yukawa coupling constant, $\Lambda$ is the energy scale of some new physics not present in the Standard Model, and $H^0$ is a neutral Higgs field. In the Standard Model, the vacuum expectation value of the neutral Higgs field is nonzero, $(H^0)^2 = \nu / 2 \neq 0$, giving rise to the following Majorana mass term for the left-handed neutrinos:

$$- \frac{\nu^2 Y_{\ell e}^2 (\nu_{\ell e}^\dagger \nu_{\ell e})^2}{4 \Lambda} \nu_{\ell L} + \text{h.c.},$$

that is, $M_{\ell e}^2 = \nu^2 Y_{\ell e}^2 / 2 \Lambda$.

One assigns leptons an additive quantum number called the lepton number $L$, such that $L = +1$ for particles and $L = −1$ for antiparticles. The lepton number is conserved in the standard electroweak interactions, like in the charged-current interactions described by the Lagrangian (1), but the Majorana mass term (3) breaks it by two units; that is, Majorana mass terms are sources or sinks of the lepton number. No empirical evidence of nonconservation of the lepton number exists so far.

If one assumes that there exist, in addition to the left-handed neutrino fields $\nu_{\ell L}$, right-handed neutrino fields $\nu_{R\ell}$, the neutrino mass Lagrangian may contain also the Dirac mass terms $- M_{\ell e} (\nu_{\ell e}^\dagger \nu_{R e}) + \text{h.c.}$ and another type of Majorana mass terms $- (1/2) M_{\ell e}^2 (\nu_{\ell e}^\dagger \nu_{\ell e}) + \text{h.c.}$. Unless the Majorana mass terms vanish, the fields $\nu_i$ that diagonalize the full mass Lagrangian are two-component Majorana fields obeying the condition (“Majorana condition”)

$$\nu_i = \nu_i^C.$$  

There are in this case altogether six mass states. It is generally assumed that $M_{\ell e}^2 \gg M_{\ell e}^2 \gg M_{\ell e}^2$ (the so-called seesaw model [21–25]), implying that three of these six states are light, corresponding to the three ordinary neutrinos appearing in (2), while the other three are very heavy and decouple from the low-energy physics. Even if the mixing between light and heavy sectors is neglected, the relation (2) is still applicable.

A lot of empirical information on the neutrino mixing, that is, the elements of the matrix $U$, and the neutrino masses has been obtained via solar, atmospheric, reactor, and accelerator neutrino oscillation experiments. The mixing matrix $U$ can be presented in terms of six measurable parameters, three rotation angles and three phases, as follows [26]:

$$U = \left( \begin{array}{ccc} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{i \delta} \\ - s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta} & s_{12} s_{23} c_{13} \\ s_{12} s_{23} c_{13} & s_{12} s_{23} s_{13} e^{i \delta} & c_{23} c_{13} \end{array} \right),$$

where $P = \text{diag}(1, e^{i \alpha}, e^{i \beta})$, $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$, and $\delta$ is called the Dirac phase and $\alpha$ and $\beta$ the Majorana phases. The probability for the oscillatory transition from the neutrino flavor $\nu_\alpha$ to the flavor $\nu_\beta$ as a function of the distance of flight $L$ and neutrino energy $E$ is given by (see, e.g., [27])

$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha \beta} - 4 \sum_{i > j} \text{Re} \left[ U_{\alpha i}^* U_{\beta j}^* U_{\beta i} U_{\alpha j} \right] \sin^2 \frac{\Delta m_{ij}^2 L}{4E},$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$. As can be seen from this formula, the neutrino oscillations do not bring us any information about the absolute neutrino mass scale, only about the squared mass differences $\Delta m_{ij}^2$. One can also easily show that neutrino oscillations are not sensitive to the Majorana phases $\alpha$ and $\beta$. 
A global fit to oscillation data yields the following values for the parameters [5]:

\[
\begin{align*}
\Delta m_{21}^2 &= 7.54^{+0.26}_{-0.22} \times 10^{-5} \text{eV}^2, \\
\Delta m_{31}^2 &= \Delta m_{32}^2 = 2.43^{+0.06}_{-0.10} \times 10^{-3} \text{eV}^2, \\
\sin^2 \theta_{12} &= 0.307^{+0.18}_{-0.16}, \\
\sin^2 \theta_{23} &= 0.386^{+0.24}_{-0.21}, \\
\sin^2 \theta_{13} &= 0.0241^{+0.0025}_{-0.0025}. 
\end{align*}
\] (8)

Here the normal mass hierarchy \( m_3 > m_1, m_2 \) is assumed; the values are slightly varied for the inverse hierarchy \( m_3 < m_1, m_2 \) (see [5]).

The main goals of the forthcoming neutrino oscillation experiments are to measure the value of the CP phase \( \delta \) and to determine the neutrino mass hierarchy, whether it is normal or inverted. The other important open questions of neutrino physics include determining the absolute mass scale of neutrinos and finding out whether neutrinos are Dirac particles or Majorana particles. These latter two questions could be at least partially solved by neutrinoless double beta decay and other lepton number violating processes. Information about the absolute neutrino mass can be also obtained by determining the effective electron neutrino mass \( m_\beta = \sqrt{\sum_i |U_{ei}|^2 m_i^2} \) in beta decay experiments, as well as from the cosmological precision measurements of the sum of neutrino masses \( \sum_i m_i \). The current experimental upper limits for \( m_\beta \) are 2.3 eV [28] and 2.1 eV [29], and for the sum of neutrino masses \( \sum_i m_i \) the recent Planck satellite data [6] imply the upper limit 0.66 eV.

3. Neutrino Masses and Double Beta Decay

In the standard picture the neutrinoless double beta decays \( (A,Z) \to (A,Z + 2) + 2e^- \) and \( (A,Z) \to (A,Z - 2) + 2e^+ \) are mediated by light neutrinos. These processes are of great importance from the particle-physics point of view, as they would indicate the violation of lepton number, which in turn would imply that light neutrinos are Majorana particles. This would be valuable information for understanding the origin of fermion masses.

We are considering in this work particularly the positron-emission mode \( (A,Z) \to (A,Z - 2) + 2e^+ \) (see Figure 1). In the electroweak model the leptonic part of this process is described by a second-order perturbation given by

\[
\left( \frac{G_F}{\sqrt{2}} \right)^2 e^+ \gamma \nu_e' (1 + \gamma_5) \nu_e \bar{\nu}_\mu' (1 - \gamma_5) e^-, \tag{9}
\]
where $e^+$, $e^-$, $\nu_e$, and $\bar{\nu}_e$ are the field operators of the electron, positron, electron neutrino, and electron antineutrino, respectively. The strength of the interaction is governed by the Fermi coupling constant $G_F/\sqrt{2} = g^2/(8\pi m_W^2)$, where $g$ is the fundamental gauge coupling of the electroweak theory and $m_W$ is the mass of $W^\pm$. The propagator describing the internal neutrino is given by

$$\langle 0 | \bar{\nu}_e(x) \nu_e(y) | 0 \rangle = \sum_i (U^*_{ei})^2 \langle 0 | \bar{\nu}_i(x) \nu_i(y) | 0 \rangle$$

$$= -i \sum_i (U^*_{ei})^2 \int \frac{d^4q}{(2\pi)^4} \frac{g + m_i}{q^2 - m_i^2} \exp(-iq \cdot (x - y)), \quad (10)$$

where the condition (5) is used.

The amplitude of the process $(A, Z) \rightarrow (A, Z - 2) + 2e^+$ is proportional to

$$\sum_i G_F^2 |U_{ei}|^2 \gamma_{\mu L} \frac{g + m_i}{q^2 - m_i^2} \gamma_\mu \gamma_L = \sum_i G_F^2 |U_{ei}|^2 \frac{m_i}{q^2 - m_i^2} \gamma_\mu \gamma_L \gamma_\mu, \quad (11)$$

where $q$ is the momentum of the exchanged neutrino and $\gamma_{\mu L(R)}$ are the chirality projection matrices $\gamma_{\mu L(R)} = (1 - i\gamma^5/2)/2$. Note that the $q$ part of the neutrino propagator does not contribute due to chirality mismatch. Typically $q = 100$ MeV, in accordance with a typical nuclear distance of 1 fm. Given that neutrinos are expected to be in the sub-eV mass scale, one can safely approximate the denominator of the neutrino propagator by $q^2$, leading to

$$G_F^2 \left( \sum_i |U_{ei}|^2 m_i \right) \frac{1}{q^2} \gamma_\mu \gamma_L \gamma_\mu. \quad (12)$$

The essential part of the amplitude from neutrino-physics point of view is the quantity:

$$\langle m_\nu \rangle = \sum_i U^*_{ei} m_i, \quad (13)$$

whose absolute value is called the effective neutrino mass; that is,

$$m_{\text{eff}} = |\langle m_\nu \rangle|. \quad (14)$$

Although this quantity depends on a great number of observables, it is associated with just one single parameter of the fundamental Lagrangian, the Majorana mass term of the left-handed electron neutrino $M_{\nu L}$ (see (3)).

The modes $\beta^0 EC$ and ECEC (Figure 2) are described by the same operator (9) as the $\beta^+ \beta^-$ mode, which is easily understandable since the antiparticle creation operator is always associated with the particle annihilation operator in the fermion fields. Hence all these processes probe the same effective neutrino mass. The decay rates of the processes are proportional to $|\langle m_\nu \rangle|^2$.

Using the standard parametrization (6) of the mixing matrix $U$, one can cast $\langle m_\nu \rangle$ in the following form:

$$\langle m_\nu \rangle = c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 e^{-i\phi_{13}} m_2 + s_{13}^2 e^{-i\phi_{13}} m_3, \quad (15)$$

where $\phi_{12} = \alpha$ and $\phi_{13} = \beta - \delta$. Depending on the phases $\phi_{12}$ and $\phi_{13}$, the contributions of the three neutrino mass states will add up constructively or destructively. In the case the CP symmetry is conserved, the phase factors assume the values $+1$ or $-1$, depending on the intrinsic CP quantum numbers of the mass states, which in turn depend on the detailed structure of the mass matrix. There are four possible sign combinations which lead to different values for $\langle m_\nu \rangle$. Any values of the phases different from $\pm 1$ would mean violation of the CP symmetry.

The amplitude of the electron-electron decay mode is proportional to the complex conjugate of $\langle m_\nu \rangle$. As the decay widths are proportional to $|\langle m_\nu \rangle|^2$, the modes $\beta^+ \beta^+$ and $\beta^- \beta^-$, as well as of the modes $\beta^0 EC$ and ECEC, probe neutrino physics through the same quantity. Hence the CP is not manifestly broken in neutrinoless double beta decay, although the Majorana phases $\phi_{12}$ and $\phi_{13}$ appear in $\langle m_\nu \rangle$. One can understand this also as a consequence of the fact that in the limit $q^2 \gg m_i^2$ the amplitudes depend on just one parameter of the mass Lagrangian, the element $M_{\nu L}^{ij}$, allowing for no measurable phases. To be sensitive to the Majorana CP phases, one should be able to distinguish between the mass states $\gamma_i$.

Apart from the CP phases $\phi_{12}$ and $\phi_{13}$, which are not observables of neutrino oscillations (the possible CP violation in oscillation phenomena is due to the Dirac phase $\delta$), there are two unknowns in the expression of the effective mass, namely, the absolute neutrino mass scale, say the mass $m_0$ of the lightest neutrino, and the mass hierarchy, that is, whether $m_3 > m_1, m_2$ (normal hierarchy) or $m_3 < m_1, m_2$ (inverted hierarchy). All three neutrino masses can be
expressed in terms of the absolute mass $m_0$: in the case of the normal hierarchy

$$m_1 = m_0,$$
$$m_2 = \sqrt{m_0^2 + \Delta m_{21}^2},$$
$$m_3 = \sqrt{m_0^2 + \Delta m_{31}^2},$$
and in the case of inverted hierarchy

$$m_1 = \sqrt{m_0^2 + |\Delta m_{21}^2|},$$
$$m_2 = \sqrt{m_0^2 + \Delta m_{31}^2 + |\Delta m_{31}^2|},$$
$$m_3 = m_0.$$

The squared mass difference $\Delta m_{21}^2$ and the absolute value of the mass difference $|\Delta m_{31}^2|$ are known from neutrino oscillation experiments. The neutrino hierarchy will be determined in the forthcoming neutrino oscillation experiments. This information would be crucial for interpretation of the results of the double beta decay experiments. In the case of inverted hierarchy, $|\langle m_1 \rangle|$ has lower limit of the order of $10^{-2}$ eV, as can be inferred from Figure 3, where the effective mass, $m_{\text{eff}} = |\langle m_1 \rangle|$, is presented as a function of the mass $m_0$ of the lightest neutrino for all possible values of the Majorana phases $\phi_{12}$ and $\phi_{13}$. If no signal of double beta decay is obtained above this limit, it would mean that either the hierarchy has to be the normal one or the neutrino is not a Majorana particle. An observation of double beta transition with $|\langle m_1 \rangle| < 10^{-2}$ eV would mean that the mass hierarchy is normal and the neutrino is a Majorana particle. On the other hand, nonobservation of the transition would not mean that the process does not exist, since in the case of the normal mass hierarchy the effective mass and hence the decay width can be arbitrarily small.

### 4. Double Beta Decays on the $\beta^+ / \text{Electron-Capture Side}$

In this section a rather detailed account of the basic theoretical ingredients of the half-life calculations is given. In this way the reader can have a unified picture of the formalisms used for various types of double beta transitions.

#### 4.1. Half-Lives and Nuclear Matrix Elements

In this work it is assumed that the $0^+\beta^+ / \text{EC}$ decays proceed exclusively via the exchange of massive Majorana neutrinos, as discussed in Section 3. The inverse half-lives for the neutrinoless $\beta^+\beta^+$ and $\beta^+\text{EC}$ decays can be cast in the form

$$[T_{\alpha}^{\text{EC}}(0^+)]^{-1} = C_{\alpha}(0^+) |M^{(0\alpha)}|^2 (m_{\text{eff}} \text{[eV]})^2,$$

where $m_{\text{eff}}$ is the effective neutrino mass (14) that should be given in (18) in units of eV. The decays described by (18) proceed via the available phase space for the final state leptons and the phase-space integrals $G_{0\nu}^{\beta^+\beta^+}(0^+)$ and $G_{0\nu}^{\text{EC}}(0^+)$ are defined in [7]. The involved nuclear matrix element (NME) can be written (see, e.g., [45–47]) in terms of the Gamow-Teller (GT), Fermi (F), and tensor (T) matrix elements in the form

$$M^{(0\alpha)} = \left( \frac{g_A}{1.25} \right)^2 \left[ M^{(0\alpha)}_{\text{GT}} - \left( \frac{g_V}{g_A} \right)^2 M^{(0\alpha)}_{\text{EC}} + M^{(0\alpha)}_{\text{T}} \right],$$

where $g_A = 1.25$ corresponds to the bare-nucleon value of the axial-vector coupling constant and $g_V = 1.00$ is the vector coupling constant. The tensor matrix element is neglected in the present calculations since its contribution is very small [48, 49]. The above defined NME is convenient since the phase-space factor to be used with it is always the one defined for $g_A = 1.25$ independent of the value of $g_A$ used in (19).

In the case of the neutrinoless double electron capture, $0^+\text{EC}$, there are no leptons available in the final state to carry away the decay energy. In this case one has to engage some additional mechanism to rid the initial atom of the excess energy of decay. There are two proposed mechanisms to cope with this situation: the radiative $0^+\text{EC}$ decay [9] and the resonant decay, $R\nu\text{EC}$ [8]. The resonance condition—close degeneracy of the initial and final (excited) atomic states—can enhance the decay rate by a factor as large as $10^6$. The $R\nu\text{EC}$ process is of the form

$$e^- + e^- + (A, Z) \rightarrow (A, Z - 2)^+ \rightarrow (A, Z - 2) + \gamma + 2X,$$

where the capture of two atomic electrons leaves the final atom in an excited state, in most cases having the final nucleus in an excited state. The excited state of the nucleus decays by one or more gamma rays and the atomic vacancies is filled by outer electrons with emission of X-rays.

Fulfillment of the resonance condition depends on the so-called degeneracy parameter $Q - E$, where $E$ is the excitation energy of the final atomic state and $Q$ is the difference between the initial and final atomic masses. Possible candidates for such resonant decays are many and a representative list will be displayed in Section 7. The final nuclear states with spin-parity $0^+$ are the most favorable ones and the only ones discussed as examples in this review. The inverse half-life for transitions to $0^+$ states can be written as

$$[T_{\alpha}^{\text{EC}}(0^+)]^{-1} = g_{\nu\alpha}^{\text{EC}}(0^+) |M^{(0\alpha)}|^2 m_{\text{eff}}^2 \Gamma (Q - E)^2 + \Gamma^2 / 4,$$

where the daughter state $(A, Z - 2)^+$ is a virtual state with energy

$$E = E^* + E_{\text{TH}} + E_{\text{TH'}} + E_{\text{HH}},$$

including the possible nuclear excitation energy and the binding energies of the two captured electrons. The last term accounts for the Coulomb repulsion between the two holes.
The quantity $\Gamma$ denotes the combined nuclear and atomic radiative widths where the atomic widths dominate and are a few tens of electron volts [50]. The factor $g_{\text{EC}}$ can be called the atomic factor and it contains the information about the density distributions of the involved atomic orbitals at the nucleus. It can be written as

$$g_{\text{EC}} = \left( \frac{G_F \cos \theta_C}{\sqrt{2}} \right)^4 \frac{g_A^4}{4\pi^2 \ln R^2_A} m^6 \mathcal{N}_{n=0} \mathcal{N}_{n'=0},$$

(23)

where $\mathcal{N}_{n=0}$ is the normalization of the relativistic Dirac wave function for the atomic orbital specified by the quantum numbers $(n,\kappa)$ [7] in the presence of a uniformly charged spherical nucleus.

The Gamow-Teller and Fermi NMEs appearing in (19) can be written explicitly in the form

$$M_K^{(0)} = \sum_{j'j} \sum_{k'k} (-1)^{j+j'} \left[ m_K^2 (m', p' ; j'; k_1, k_2) \right] \times \left( 0_j \left| \left[ c_{j'}^2 \bar{c}_{j'} \right] \right| J_{k_1} \right) \langle f_{k_1} | k \rangle \langle J_{k_1} | f_{k_2} \rangle \left( 0_{j'} \right),$$

(24)

where $K = F, GT$ and $k_1$ and $k_2$ label the different nuclear-model solutions for a given multipole $J^n$, the set $k_1$ stemming from the calculation based on the final nucleus and the set $k_2$ stemming from the calculation based on the initial nucleus. Here the one-body transition densities are $\langle 0_j | \left[ c_{j'}^2 \bar{c}_{j'} \right] | J_{k_1} \rangle$ and $\langle J_{k_1} | f_{k_2} \rangle$, and they are given separately for the different types of $0^+$ final states in Section 4.3.

The two-particle matrix element of (24) can be written as

$$m_K (m', p' ; j'; k_1, k_2) = \sum_{\lambda} \left( \begin{array}{ccc} \lambda & \lambda & \lambda \\ 1 & 2 & S \\ J_{k_1} & J_{k_2} & J' \end{array} \right) \times \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{array} \right) \times \sum_{n,n'L} M_A (n_1 n L; n_1 n' L; n_1 n') \times \sum_{n_2 n' L} M_A (n_2 L n L; n_1 p p' L; n_2 p p') \times \int d^3 r \phi_{n_2} (r) h_K \left( r_{12}, \frac{1}{2} (E_{k_1} + E_{k_2}) \right) \phi_{n_1} (r),$$

(25)

where $\tilde{j} = \sqrt{j^2 + 1}$ and $r_{12} = |r_1 - r_2|$ is the relative distance between the two decaying protons. The following auxiliary quantities have been defined

$$F_F = 1, \quad F_{GT} = 6 (-1)^{j+1} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & S \\ 1 & 1 & 1 \end{array} \right\}.$$  

(26)

The quantities $M_A$ are the Moshinsky brackets that mediate the transformation from the laboratory coordinates $r_1$ and $r_2$ to the center-of-mass coordinate $R = (1/\sqrt{2}) (r_1 + r_2)$ and the relative coordinate $r = (1/\sqrt{2}) (r_1 - r_2)$. In this way the short-range correlations of the two decaying protons are easily incorporated in the theory. The wave functions $\phi_{n}(r)$ are taken to be the eigenfunctions of the isotropic harmonic oscillator.

The neutrino potential $h_K (r_{12}, E) = F, GT$, in the integral of (26) is defined as

$$h_K (r_{12}, E) = \frac{2}{\pi} R_A \int dq \frac{q \tilde{h}_K (q^2)}{q + E - (E_i + E_f) / 2} j_0 (q r_{12}),$$

(27)

where $j_0$ is the spherical Bessel function and the integration is performed over the exchanged momentum $q$. Here $E_i = M_{\pi^-}^2$ is the ground-state mass energy of the initial nucleus and $E_f$ the (ground-state or excited-state) mass energy of the final nucleus. In practice the lowest pnQRPA energies of the two sets $k_1$ and $k_2$ are normalized such that the energy difference of these energies and the mass energy of the initial nucleus match the corresponding experimental energy difference. The term $h_K (q^2)$ in (27) includes the contributions arising from the short-range correlations, nucleon form factors, and higher-order terms of the nucleonic weak current [51]. For all the $0\nu\beta\beta$/EC transitions of this work the NMEs have been computed by the use of both the Jastrow short-range correlations [52] and the UCOM correlations [53, 54]. Both short-range correlators have been recently used in many $0\nu\beta\beta$ calculations [48, 49, 55–58] and in some $0\nu\beta\beta$/EC calculations [36, 59–61].

### 4.2. Nuclear Models and Model Parameters

In this work the wave functions of the nucleon states involved in the double beta decay transitions are calculated by the use of the quasiparticle random-phase approximation (QRPA) in realistically large single-particle model spaces. The $J^n$ states of the intermediate nucleus of the $\beta\beta$ decays are generated by the usual proton-neutron QRPA (pnQRPA) [2, 62] in the form

$$| J^n K^+ M \rangle = Q^n (J^n K^+ M) \langle \text{QRPA}$$

$$= \sum_{\alpha \beta} \left\{ X_{\alpha \beta} \left( \begin{array}{c} a_\alpha^+ a_\beta^+ \end{array} \right) \right\}_{JM} - Y_{\alpha \beta} \left( \begin{array}{c} a_\alpha^+ a_\beta^+ \end{array} \right) \langle \text{QRPA}$$

(28)

where $X$ and $Y$ are the forward- and backward-going amplitudes of the pnQRPA, obtained by solving the pnQRPA
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Figure 4: Schematic representation of the possible $0\beta^\pm$/EC decay modes of $^{106}$Cd. The atomic resonance at 2766.26 keV includes both the nuclear excitation energy and the energy related to the two electron holes in the atomic K shell as given in (22).

Figure 5: Computed partial $0\beta^\pm$/EC decay half-lives of the various decay transitions from $^{106}$Cd. The value $m_{\text{eff}} = 0.3$ eV is adopted for the effective neutrino mass and the UCOM short-range correlations have been assumed. The half-lives are given in units of years.

The excited states $I_i^\pm$ in the final even–even nuclei are described by the phonons of the charge-conserving QRPA (ccQRPA), expressed as

$$
\left| I_i^\pm, M \right> = Q^I \left( I_i^\pm, M \right)_{\text{QRPA}} = \sum_{ab} \left( Z_{ab}^{I_i^\pm} \left\{ a^+_a d^-_b \right\}_{JM} - W_{ab}^{I_i^\pm} \left\{ a^+_a d^-_b \right\}_{JM} \right) |\text{QRPA}>,
$$

where the symmetrized amplitudes $Z$ and $W$ are obtained from the usual ccQRPA amplitudes $X$ and $Y$ [62] through the transformation

$$
Z_{ab}^{I_i^\pm} = \begin{cases} 
X_{ab}^{I_i^\pm}, & \text{if } a = b \\
\frac{1}{2} X_{ab}^{I_i^\pm}, & \text{if } a < b \\
\frac{1}{2} X_{ba}^{I_i^\pm}, & \text{if } a > b 
\end{cases}
$$

and similarly for $W$ in terms of $Y$.

Now one can take a $I_i^\pm = 2_1^+$ phonon of (29) and build an ideal two-phonon $I^\pm$ state of the form

$$
\left| I_{2-\text{ph}}^\pm \right> = \frac{1}{\sqrt{2}} \left( Q^I (2_1^+) Q^I (2_1^+) \right)_I \left| \text{QRPA} \right>,
$$

An ideal two-phonon state consists of partner states $I^\pm = 0^+, 2^+, 4^+$ that are degenerate in energy and exactly at an energy twice the excitation energy of the $2_1^+$ state. In practice this degeneracy is always lifted by the residual interaction between the one- and two-phonon states [63]. The one- and two-phonon states in the final even–even nucleus are connected to the $J^\pi$ states of the intermediate nucleus by transition amplitudes obtained from a higher-QRPA framework called the multiple-commutator model (MCM), first introduced in [64] and further extended in [65].

The calculations were done in sufficiently large single-particle spaces and the single-particle energies were generated by the use of a spherical Coulomb-corrected Woods-Saxon (WS) potential with a standard parametrization [66],...
Figure 7: Computed partial $0\beta^+$/EC decay half-lives of the various decay transitions from $^{124}$Xe. The value $m_{\text{eff}} = 0.3$ eV is adopted for the effective neutrino mass and the UCOM short-range correlations have been assumed. The half-lives are given in units of years.

Figure 8: Values of the computed nuclear matrix elements for $0\beta^-$/EC decay transitions to the ground state ($0^+_{\text{gs}}$), the first excited $0^+$ state ($0^+_{\text{1}}$) and the resonant $0^+$ state ($0^+_{\text{res}}$) for all the nuclei discussed in this paper. The quasiparticles are treated in the BCS formalism and the pairing matrix elements are scaled by a common factor, separately for protons and neutrons. In practice these factors are fitted such that the lowest quasiparticle energies obtained from the BCS match the experimental pairing gaps for protons and neutrons, respectively [62].

As explained in detail in [45] the particle-hole and particle-particle parts of the proton-neutron two-body interaction are separately scaled by the particle-hole parameter $g_{\text{ph}}$ and particle-particle parameter $g_{\text{pp}}$. The value of the particle-hole parameter was fixed by the available systematics [62] on the location of the Gamow-Teller giant resonance (GTGR) state. The value of the $g_{\text{pp}}$ parameter regulates the $\beta^-$-decay amplitude of the first $1^+$ state in the intermediate nucleus [68] and hence also the decay
The transition density corresponding to the final ground state reads

$$\left( \frac{0^+_{\text{gs}}}{\left| c_{n1}^+ c_{p1}^- \right|} \right| I_{k}^+ )
= \tilde{J}^{-1} \frac{1}{L+I+1} \left[ \frac{-\bar{u}_p \bar{v}_n X_{p1}^{T \pi} + \bar{v}_p \bar{u}_n Y_{p1}^{T \pi}}{\bar{u}_p \bar{v}_{n1} \bar{X}_{p1}^{T \pi} + \bar{v}_p \bar{u}_{n1} Y_{p1}^{T \pi}} \right],$$

(33)

where $\nu$ ($\bar{V}$) and $u$ ($\bar{U}$) correspond to the BCS occupation and unoccupation amplitudes of the initial (final) even-even nucleus. The amplitudes $X$ and $Y$ ($\bar{X}$ and $\bar{Y}$) come from the pnQRPA calculation starting from the initial (final) nucleus of the $0\nu\beta^-$/EC decay.

For the excited states the multiple-commutator model (MCM) [64, 65] is used. It is designed to connect excited states of an even-even reference nucleus to states of the neighboring odd-odd nucleus. The states of the odd-odd nucleus are given by the pnQRPA and the excited states of the even-even nucleus are generated by the ccQRPA [71].

The ccQRPA phonon (29) defines a state in the final nucleus

$$\left( \frac{0^+_{\text{gs}}}{\left| c_{n1}^+ c_{p1}^- \right|} \right| I_{k}^+ )$$

instead of the expression (33) for the ground-state transition. Again $\nu$ ($\bar{V}$) and $u$ ($\bar{U}$) correspond to the BCS occupation and unoccupation amplitudes of the initial (final) even-even nucleus. The amplitudes $X$ and $Y$ ($\bar{X}$ and $\bar{Y}$) come from the pnQRPA calculation starting from the initial (final) nucleus of the $0\nu\beta^-$/EC decay. The ampli-

FIGURE 10: Negative of the ratio between the Gamow-Teller and Fermi NMEs for decays to the ground state ($0^+_0$), the first excited $0^+$ state ($0^+_1$) and the resonant $0^+$ state ($0^+_\text{res}$) for all the nuclei discussed in this paper.
In the case of the two-phonon excitation the transition density to be inserted in (24) attains the form
\[
\left( I_{2-\text{ph}}^+ \right| \left[ \psi_{\alpha}^{+} \right] \right| J_{\alpha}^+ )
\]
\[
= \frac{40}{\sqrt{2}} \hat{I} \hat{J} (\hat{I} \hat{J})_{-1}^{+} \hat{j}_\nu \hat{j}_\alpha^{+}
\times \sum_{p_i n_i} \left[ \overline{\psi}_p \overline{\varphi}_{n_i} X_{n_i}^{n_1} Z_{p_i}^{j_1} \bar{Z}_{n_i}^{j_1} \overline{W}_{p_i}^{j_2} \overline{W}_{n_i}^{j_2} \right]
\times \left\{ \begin{array}{c} j_\nu \ j_1 \ 2 \\ J_\nu \ J_1 \ 2 \\ L \ j \ I \end{array} \right\} ,
\]
where, as usual, the barred quantities denote amplitudes obtained for the final nucleus of double beta decay. In the present work we use only \( I_{2-\text{ph}}^+ = 0^+ \).

5. Typical Examples

In the present paper the neutrinoless \( \beta^+ \beta^+ \) and \( \beta^+ \) EC transitions in various nuclei are discussed. Considered are the transitions to the ground state, \( 0^+_{gs} \), and to the first \( 0^+ \) state, \( 0^+_1 \).

The \( 0\nu\beta^+ / EC \) decays to only the \( 0^+ \) states are considered since large suppression of the mass mode for the decays to \( 2^+ \) states is expected [72]. Furthermore, the \( 0\nu\beta^+ / EC \) transitions to the possible resonant states are considered. In the present work the analysis of the \( 0\nu\beta^+ / EC \) half-lives is performed by assuming a \( 0^+ \) assignment for the resonant states. This assignment leads to a very likely enhancement in
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The decay rate. Since this assignment is in many cases only tentative or even unlikely, the calculated half-lives should be taken as optimistic estimates or as lower limits for the half-life.

All the discussed decay transitions are displayed in Figure 4, where the decay of $^{106}$Cd serves as paradigm. Both $0\nu\beta^+\beta^-$ and $0\nu\beta^+\gamma$ EC transitions to the ground state are possible whereas only the $0\nu\beta^+\beta^-$ EC mode is possible for the decay to the $0^+_1$ state for phase-space reasons since the $0\nu\beta^+\beta^-$ decay has a negative $Q$ value for this transition. The resonant $0\nu\beta^+\gamma$EC transition is also shown with the total energy (including the electron-hole contributions, see (22)) of the resonant atomic state.

The various $0\nu\beta^+\gamma$/EC decay modes can now be treated by applying the formalisms outlined in Section 4. In particular, the BCS is used to create the quasiparticles in the chosen single-particle valence space and the pnQRPA is used to produce the intermediate $f^n$ states involved in the NME (24). The excited states in the final nucleus are produced by the use of the ccQRPA and the final states are connected to the intermediate $f^n$ states by the MCM prescriptions. After adjusting the parameters of the model Hamiltonian the rates related to the various decay transitions can be evaluated. The results are shown in our paradigm case in Figure 5 for the range $g_A = 1.00 - 1.25$ of the axial-vector coupling constant and for the value $m_{\text{eff}} = 0.3$ eV of the effective neutrino mass (14).

As seen from Figure 5, the fastest decay mode is $0\nu\beta^+\gamma$ EC to the ground state of $^{106}$Pd with a half-life of $(1.5-1.7) \times 10^{26}$ years. This could be in the range of detection sensitivity of the next generation of double beta experiments. The resonance transition proceeds by the capture of two K electrons and emission of two K X-rays, has a half-life of $(2.3-6.3) \times 10^{26}$ years, and is thus very hard to be detected in the foreseeable future.

In Figure 6 the half-lives of decay transitions in $^{96}$Ru are shown for the range $g_A = 1.00 - 1.25$ and for the value $m_{\text{eff}} = 0.3$ eV. Again the fastest transition is $0\nu\beta^+\gamma$ EC to the ground state of $^{96}$Mo with a half-life of $(5.5-6.3) \times 10^{27}$ years, slightly slower than in the case of $^{106}$Cd decay. Interestingly enough there are decays to two excited $0^+$ states at energies 1148.13 keV and 1330 keV. The latter state is assumed to be a two-phonon state discussed in this paper, the former one being a one-ccQRPA-phonon state. In this case the resonant decay proceeds with the capture of two $L_1$ electrons and emission of two $L_1$ X-rays. The computed half-life for the resonant decay is $(4.9-22) \times 10^{27}$ years which is impossible to be detected in the foreseeable future.

The last example of this section pertains to the $0\nu\beta^+\gamma$/EC decays in $^{124}$Xe, shown in Figure 7. The ranges $g_A = 1.00 - 1.25$ and $m_{\text{eff}} = 0.3$ eV were adopted in the calculations. The $0\nu\beta^+\gamma$ EC decay to the ground state of $^{124}$Te is the fastest with a half-life of $(1.2-4.2) \times 10^{27}$ years, being in the range of

![Figure 12](image-url)
the corresponding decay transition in $^{106}$Cd. The decay to the resonance state at 2854.87 keV proceeds with the capture of two K electrons and emission of two K X-rays. The computed half-life is $(1.9 \pm 5.6) \times 10^{30}$ years and is thus the fastest of the three discussed example cases, though hard to be detected in the near future.

The computed half-lives can be expressed by the use of the auxiliary quantities $C_{\beta \beta'}$ and $C_{\beta \beta'EC}$ in the following form:

$$T_{1/2}^{\beta \beta'} = C_{\beta \beta'} (m_{\text{eff}} [\text{eV}])^{-2},$$

$$T_{1/2}^{\beta \beta'EC} = C_{\beta \beta'EC} (m_{\text{eff}} [\text{eV}])^{-2},$$

(36)

where the effective neutrino mass should be inserted in units of eV. In Table 1 the auxiliary factors of the above equations are given for the nuclei and transitions under discussion. The UCOM short-range correlations have been used combining the results for the possible different basis sets used in the nuclear structure calculations and for the range $g_A = 1.00$–1.25 of the axial-vector coupling constant.

From Table 1 it can be evidenced that generally the fastest transitions are the $0\nu\beta\beta$ EC transitions to the ground state and transitions to the excited $0^+$ state(s) are quite much suppressed relative to the ground-state transitions.

6. Systematic Features of the Nuclear Matrix Elements

There are not too many nuclei that have reasonable $Q$ values and decay by $0\nu\beta\beta/EC$ decays, and only part of these can have a reasonable chance of decaying via the resonant neutrinoless double EC channel. It is nevertheless instructive to have a fresh view at the systematic features of the involved NMEs.

6.1. The $0\nu\beta\beta/EC$ NMEs. A systematics of the computed NMEs of the $0\nu\beta\beta/EC$ decays is shown in Figure 8. The values of NMEs for decays to the ground state ($0^+_g$), first excited $0^+$ state ($0^+_1$) and the resonant $0^+$ state ($0^+_{\text{res}}$) are shown. As mentioned before the assignment of $J^\pi = 0^+$ to the resonant states has to be taken in some cases, like for the $^{106}$Cd decay, with a grain of salt. From the figure one notices that the ground-state NMEs are rather large ($5.0$ or more), except for $^{92}$Mo. This means that matrix-element-wise the $0\nu\beta\beta/EC$ decays are not suppressed relative to the $0\nu\beta\beta$ decays. This can be further evidenced in Figure 9 where these NMEs are shown together with those of $0\nu\beta\beta$ decays for nuclei with $70 \leq A \leq 100$ (a) and for nuclei with $104 \leq A \leq 136$ (b).

In Figure 8 it is seen that the NMEs corresponding to the resonant $0^+$ states are larger than the NMEs corresponding to the decays to $0^+_1$ states. This stems from the fact that the resonant states are treated as one-ccQRPA-phonon states.
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from Figure 9 it is seen that the 0νβ+EC NMEs show local maxima in 96Ru, 106Cd, 124Xe, 136Ba, and 138Ce for the ground-state transitions and even a global maximum: 138Ba. On the other hand, 92Mo shows a global minimum. For the decays to the 0+ states the 0νβ+EC NMEs are small relative to the 0νβ−β− NMEs, except for 96Ru which has a relatively large NME.

6.2. Fermi and Gamow-Teller Parts of the 0νβ+EC NMEs. One can also scrutinize the decomposition of the 0νβ+EC NMEs to their Fermi and Gamow-Teller constituents. This decomposition is shown in Figure 10 where the negative of the ratio of these two constituents has been plotted for decays to the different final states, 0gs, 01+, and 03+. In Figure II the same has been done in a global context by including also the ratios for the 0νβ−β− emitters. In this figure one notices that the ratios have a rather universal value of roughly 2.0, except in the case of 78Kr that has a ratio of about 6.0. The ratios for the 01+ transitions show pronounced peaks for the 0νβ−β− emitters 78Kr and 96Ru and for the 0νβ−β− emitters 110Pd and 116Cd, whereas pronounced minima occur for 106Cd and 124Xe. Most of these ratios for the 01+ are slightly above 2.0.

All in all, more variation in the Gamow-Teller/Fermi ratio is seen for the 01+ states than for the ground states. From Figure 10 it is clear that the ratios for the resonant states are much higher than for the 01+ or 03+ states in the corresponding nuclei.

6.3. Decompositions of the 0νβ−EC NMEs. The 0νβ−EC NMEs can be decomposed into contributions of different intermediate multipole states as done in [39] for 98Ru. Let us use here the decay of 124Xe as an example. The decomposition of the 0νβ−EC NMEs \( M_{\beta^{-}}(\nu) \) (19) can be made in two ways, either through the different multipole states \( J^\pi \) of the intermediate nucleus (in this case the states of 124Xe) or through different couplings \( J^\prime \) of the two decaying nucleons [57, 73]. For the Gamow-Teller NME these decompositions can be schematically written as

\[
M_{\beta^{-}}(\nu) = \sum_{J^\pi} \sum_{J^\prime} M_{\beta^{-}}^{GT}(J^\pi, J^\prime),
\]

where \( M_{\beta^{-}}^{GT}(J^\pi, J^\prime) \) is given explicitly in (24). The decompositions (37) are shown for the Gamow-Teller NMEs of the decays of 124Xe in Figures 12, 13, and 14. All the figures refer to calculations using the Jastrow short-range correlations and the value \( g_A = 1.25 \) for the axial-vector coupling constant.

From the decomposition figures one can make the following general observations. For the ground-state NME the decomposition in terms of \( J^\pi \) is the typical one of the pnQRPA calculations [45, 57, 59] and the decomposition in terms of \( J^\prime \) is typical of the shell-model [73] and pnQRPA [36, 57, 59] calculations. Here typical for the \( J^\pi \) decomposition are the strong contributions of the high-multipole components \( 2^-, 3^-, 4^-, \) and \( 5^- \). In this case the 1+ contribution is modest contrary to that of the 98Ru
Table 2: Comparison of the ground-state NMEs for the QRPA, IBM-2 [31], and PHFB [32] models using the Jastrow short-range correlations. The results for $^{136}$Ce are taken from [30] with $g_A = 1.261$.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>QRPA ($g_A = 1.25$)</th>
<th>IBM-2 ($g_A = 1.269$)</th>
<th>PHFB ($g_A = 1.254$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_{GT}^{(0)}$</td>
<td>$M_t^{(0)}$</td>
<td>$M_t^{(0)'}$</td>
</tr>
<tr>
<td>$^{78}$Kr</td>
<td>3.271</td>
<td>-0.331</td>
<td>3.482</td>
</tr>
<tr>
<td>$^{90}$Ru</td>
<td>2.589</td>
<td>-0.988</td>
<td>3.222</td>
</tr>
<tr>
<td>$^{106}$Cd</td>
<td>4.920</td>
<td>-1.586</td>
<td>5.935</td>
</tr>
<tr>
<td>$^{124}$Xe</td>
<td>3.491</td>
<td>-1.889</td>
<td>4.700</td>
</tr>
<tr>
<td>$^{130}$Ba</td>
<td>5.412</td>
<td>-2.528</td>
<td>7.031</td>
</tr>
<tr>
<td>$^{136}$Ce</td>
<td>4.282</td>
<td>-1.961</td>
<td>5.537</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the $0^+_1$ NMEs for the QRPA and IBM-2 [31] models using the Jastrow short-range correlations.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>QRPA ($g_A = 1.25$)</th>
<th>IBM-2 ($g_A = 1.269$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_{GT}^{(0)}$</td>
<td>$M_t^{(0)}$</td>
</tr>
<tr>
<td>$^{78}$Kr</td>
<td>0.039</td>
<td>-0.008</td>
</tr>
<tr>
<td>$^{90}$Ru</td>
<td>2.004</td>
<td>-0.396</td>
</tr>
<tr>
<td>$^{106}$Cd</td>
<td>0.317</td>
<td>-0.537</td>
</tr>
<tr>
<td>$^{124}$Xe</td>
<td>-0.005</td>
<td>-0.050</td>
</tr>
</tbody>
</table>

6.4. Comparison of NMEs Produced by Different Models. In Table 2 we present the results of recent calculations for the NMEs of the discussed nuclei. The QRPA results are the ones of this work, the IBM-2 results are taken from [31], and the projected Hartree-Fock-Bogoliubov (PHFB) results are taken from [32]. The IBA-2 model is based on a phenomenological Hamiltonian with connections to the underlying shell model via a mapping procedure. The PHFB is a mean-field model with phenomenological Hamiltonians. Both IBM-2 and PHFB give results that are relevant for nuclear structure calculations. Since IBM-2 and PHFB quote their results using the Jastrow short-range correlations, the QRPA calculations have been done by using these correlations. All the quoted calculations in Table 2 use practically the same value of the axial-vector coupling constant $g_A$.

From Table 2 one observes that the NMEs computed by the use of the QRPA and IBM-2 are rather similar whereas the PHFB NMEs deviate from them notably. These trends are similar to the ones for the $0\nu\beta^+\beta^-$ decaying nuclei as discussed extensively in [74].

In Table 3 the NMEs corresponding to the $0\nu\beta^+\beta^-$ decays to the first excited $0^+$ state, $0^+_1$, are shown for the QRPA and the IBM-2. The PHFB model cannot access these NMEs since it is by definition a mean-field model describing only ground-state transitions. What is striking in Table 3 are the very different NMEs and their trends predicted by the two models. The QRPA produces small NMEs for $^{78}$Kr, $^{106}$Cd, and $^{124}$Xe and rather large NME for $^{90}$Ru. For IBM-2 the opposite happens. This tension between the two calculations is more drastic than in the case of the $0\nu\beta^+\beta^-$ transitions, as analyzed in [74].

6.5. Experimental Limits for the Half-Lives. Up to now only limits of half-lives have been extracted for the various $0\nu\beta^+\beta^-$ processes. Measurements have been done, for example, for $^{74}$Se [75], for $^{96}$Ru [76, 77], for $^{106}$Cd [78], for $^{112}$Sn [79], for $^{136}$Ce and $^{136}$Ca [80], and for $^{64}$Zn and $^{180}$W [81]. The obtained lower limits are of the order of $10^{30}$ years for $^{96}$Ru [77] and $^{106}$Cd [78] and $10^{13}$ years for $^{136}$Ce [80]. These limits are still very far from the theoretical estimates as implied by Table 1.

7. Present Status of the Resonant Processes

Table 4 lists the known cases of $0\nu\beta^+\beta^-$ transitions in various nuclei where Q-value measurements have been conducted recently. These Q values have been meas-
Table 4: R0vECEC decay transitions with the final-state spin-parity indicated in the second column and the degeneracy parameters $Q - E$ in the third column. Also the involved atomic orbitals have been given in the fourth column. The second last column lists the currently available half-life estimates with the references to the third column. Also the involved atomic orbitals have been given in the fourth column. These second last column lists the currently available half-life estimates with the references to the third column.

<table>
<thead>
<tr>
<th>Transition</th>
<th>$J^e$</th>
<th>$Q - E$ [keV]</th>
<th>Orbitals</th>
<th>$C^{ECEC}$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{74}$Se $\rightarrow$ $^{74}$Ge</td>
<td>$2^+$</td>
<td>2.23</td>
<td>$L_2L_1$</td>
<td>(0.2–100) $\times 10^{13}$</td>
<td>[33]</td>
</tr>
<tr>
<td>$^{96}$Ru $\rightarrow$ $^{96}$Mo</td>
<td>$2^+$</td>
<td>8.92 (13)</td>
<td>$L_1L_3$</td>
<td></td>
<td>[34]</td>
</tr>
<tr>
<td>$^{102}$Pd $\rightarrow$ $^{102}$Ru</td>
<td>$0^+$</td>
<td>-3.90 (13)</td>
<td>$L_1L_1$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{106}$Cd $\rightarrow$ $^{106}$Pd</td>
<td>$0^+$</td>
<td>75.26 (36)</td>
<td>$KL_1$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{112}$Sn $\rightarrow$ $^{112}$Cd</td>
<td>$0^+$</td>
<td>-0.33 (41)</td>
<td>$KK$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{124}$Xe $\rightarrow$ $^{124}$Te</td>
<td>$0^+$</td>
<td>1.86 (15)</td>
<td>$KK$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{130}$Ba $\rightarrow$ $^{130}$Xe</td>
<td>$0^+$</td>
<td>10.18 (30)</td>
<td>$KK$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{136}$Ce $\rightarrow$ $^{136}$Ba</td>
<td>$0^+$</td>
<td>-11.67</td>
<td>$KK$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{144}$Sm $\rightarrow$ $^{144}$Nd</td>
<td>$2^+$</td>
<td>171.89 (87)</td>
<td>$KL_1$</td>
<td></td>
<td>[35]</td>
</tr>
<tr>
<td>$^{152}$Gd $\rightarrow$ $^{152}$Sm</td>
<td>$0^+$</td>
<td>0.91 (18)</td>
<td>$KL_1$</td>
<td></td>
<td>[40, 41]</td>
</tr>
<tr>
<td>$^{156}$Dy $\rightarrow$ $^{156}$Gd</td>
<td>$1^-$</td>
<td>0.75 (10)</td>
<td>$L_2L_3$</td>
<td></td>
<td>[42]</td>
</tr>
<tr>
<td>$^{162}$Er $\rightarrow$ $^{162}$Dy</td>
<td>$2^+$</td>
<td>0.04 (10)</td>
<td>$MN_3$</td>
<td></td>
<td>[42]</td>
</tr>
<tr>
<td>$^{164}$Er $\rightarrow$ $^{164}$Dy</td>
<td>$0^+$</td>
<td>2.69 (30)</td>
<td>$KL_1$</td>
<td></td>
<td>[34]</td>
</tr>
<tr>
<td>$^{168}$Yb $\rightarrow$ $^{168}$Er</td>
<td>$2^+$</td>
<td>6.81 (13)</td>
<td>$L_1L_1$</td>
<td></td>
<td>[41, 43]</td>
</tr>
<tr>
<td>$^{180}$W $\rightarrow$ $^{180}$Hf</td>
<td>$0^{+}$</td>
<td>11.24 (27)</td>
<td>$KK$</td>
<td></td>
<td>[41, 44]</td>
</tr>
</tbody>
</table>

where the effective neutrino mass should be given in units of eV. In all the listed cases where $C^{ECEC}$ has been computed the decay rates are suppressed by the rather sizable magnitude of the degeneracy parameter. Decays to $0^+$ states are favored over the decays to $2^+$ or $1^+, 2^+, 3^+$, and so forth states due to the involved nuclear wave functions and/or higher-order transitions. Also captures from atomic orbitals with orbital angular momentum $I > 0$ are suppressed [33].

There are some favorable values of degeneracy parameters listed in Table 4, like $^{106}$Cd $\rightarrow$ $^{106}$Pd$(2, 3)^-$ and $^{156}$Dy $\rightarrow$ $^{156}$Gd$(0^+, 1^-, 2^+)$ but the associated nuclear matrix elements are still waiting for their evaluation. Strong suppression of the NMEs related to final states with $J > 0$ is, however, expected. In case of the $^{156}$Dy decay the deformation also plays an important role. At the moment the most favorable case with a half-life estimate is the case $^{152}$Gd $\rightarrow$ $^{152}$Sm$(0^{+}, 0^{+})$ which describes a decay transition to the ground state.

8. Summary and Conclusions

Neutrino masses and their influence on neutrino oscillations and on the nuclear double beta decay have been addressed. The various positron-emitting and/or electron-capture modes of the neutrinoless double beta decays have been investigated for the associated nuclear matrix elements and decay half-lives. A QRPA-based theory framework with G-matrix-based two-body interactions and realistically large single-particle bases has been used in the calculations. The computed values of the nuclear matrix elements have been analyzed and contrasted globally with the double beta minus nuclear matrix elements. Special attention has been paid to the resonant neutrinoless double electron capture process to survey its potential for Majorana-mass detection in dedicated experiments. Generally, the resonance condition is poorly satisfied and the emerging half-lives are extremely hard to measure. Few exceptions occur but the associated nuclear matrix elements are not known. Further theoretical efforts in these cases are stringently called for.

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