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Aleksandr Poretskii

Electromagnetic Wave Propagation in Non-Homogeneous Waveguides

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Editors
Timo Männikkö
Department of Mathematical Information Technology, University of Jyväskylä
Pekka Olsbo, Ville Korkiakangas
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ABSTRACT

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We investigate an electromagnetic waveguide, having several cylindrical ends. The waveguide is assumed to be empty and to have a perfectly conductive boundary. We study the electromagnetic field, excited in the waveguide in the presence of charges and currents. The field can be described as a solution of the stationary Maxwell system with conductive boundary conditions and "intrinsic" radiation conditions at infinity. We prove the problem to be well-posed. Electromagnetic waves propagation in the waveguide can be described by means of a scattering matrix. We introduce such a matrix for all values of the spectral parameter k in the waveguide continuous spectrum and study its properties. Moreover, we propose and justify a method for approximating the scattering matrix for all k in the continuous spectrum, including thresholds; the presence of waveguide eigenvalues does not influence the method statement.

The results of the thesis extend the area of electromagnetic waveguide theory and have numerous applications. Particularly, the asymptotic and numerical methods, developed in the thesis, can be used for design and analysis of complex waveguides with resonators, SHF splitters, etc. To prove the results, we extend the over-determined Maxwell system to an elliptic problem and study the latter in detail. The information on the Maxwell system comes from that, obtained for the elliptic problem.

Keywords: The stationary Maxwell system, waveguides, elliptic extension, intrinsic radiation conditions, radiation principle, scattering matrix, method for approximating the scattering matrix, minimizer of a quadratic functional, exponential convergence rate, thresholds, stable basis, extended scattering matrix, limits of the scattering matrix at thresholds.

Author Aleksandr Poretskii
Department of Mathematical Information Technology
University of Jyväskylä, Finland
E-mail: `poras1990@list.ru`

Supervisors Prof. Dr. Pekka Neittaanmäki
Department of Mathematical Information Technology
University of Jyväskylä, Finland

Prof. Dr. Boris A. Plamenevskii
Department of Higher Mathematics and Mathematical Physics
Saint-Petersburg State University, Russia

Reviewers Prof. Dr. Lev M. Baskin
Department of Higher Mathematics
The Bonch-Bruevich Saint - Petersburg State University of Telecommunications, Russia

Prof. Dr. Dirk Pauly
Fakultät für Mathematik
Universität Duisburg-Essen, Campus Essen, Germany

Opponent Prof. Dr. Dmitri Sokolovski
Department of Physical Chemistry
UPV/EHU, Ikerbasque, Basque Country

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1 INTRODUCTION

1.1 Preliminaries

As a model for a non-homogeneous electromagnetic waveguide, we consider a boun-dary value problem for the stationary Maxwell system

$$\begin{aligned} i\,\mathrm{rot}\,u^2(x) - ku^1(x) &= f^1(x), \qquad -i\,\mathrm{div}\,u^2(x) = h^1(x), \\ -i\,\mathrm{rot}\,u^1(x) - ku^2(x) &= f^2(x), \qquad i\,\mathrm{div}\,u^1(x) = h^2(x), \qquad x \in G, \end{aligned} \tag{1.1.1}$$

with the conductive boundary conditions

$$\nu(x) \times u^1(x) = 0, \ \langle u^2(x), \nu(x)\rangle = 0, \ x \in \partial G, \tag{1.1.2}$$

in a three-dimensional domain G having several cylindrical ends. Here u^1 and u^2 are electric and magnetic vectors while ν stands for the unit outward normal to the boundary ∂G.

The theory of electromagnetic waveguides basically deals with two problems: the problem of scattering (diffraction) of an electromagnetic wave on the waveguide inhomogeneities and the problem of electromagnetic field excitation by given charges and currents. Among numerous mathematical works devoted to the problems, we outline two directions: in the papers of the first direction (see [20], [8],[9] and references therein), there are considered cylindrical waveguides with filling medium that does not depend on the cylinder axial variable; some local perturbations (in a bounded domain) of the waveguide shape and filling medium are allowed as well. Another direction is related to the Wiener-Hopf technique and the mode matching method. One considers the case when the Maxwell system can be reduced to the Helmholtz equation and a waveguide consists of finitely many model domains. Surveys of the methods are given in monographs [35], [24], [27].

An actual problem is to extend the class of electromagnetic waveguides admitting a mathematically accurate investigation and to develop mathematical scattering theory of such waveguides. In particular, the problem is to define a scattering matrix in the scattering theory framework, to develop asymptotic methods for the matrix investigation, as well as to elaborate and justify a method for approximating the matrix.

In the present thesis, we abandon the limitations related to cylindrical shape of waveguide and admit waveguides having arbitrary finite number of cylindrical outlets to infinity. In a bounded domain the waveguide may have an arbitrary shape with a smooth boundary. For such waveguides, we propose and justify a radiation principle, introduce a scattering matrix that depends on the spectral parameter and is defined at the waveguide continuous spectrum. For any spectral parameter, the matrix is unitary and have a finite size that changes at thresholds and remains constant between two neighbouring thresholds. Moreover, in the thesis we propose and justify a method for approximating the scattering matrix.

Results of the thesis extend the area of electromagnetic waveguide theory and have numerous applications. A waveguide excitation is described by the radiation principle and the wave propagation is described by the scattering matrix. A method for approximating the scattering matrix can be applied for computer modelling of a real-life wave propagation process. Particularly, the asymptotic and numerical methods, developed in the thesis, can be used to analyze quality of complex waveguides with resonators and SHF splitters, etc.

We use neither the methods nor the results of the works mentioned in the second paragraph. Our investigation begins with extension of the overdetermined Maxwell system (1.1.1)- (1.1.2) to an elliptic system. To this end, we use the orthogonal extension method suggested by I.S. Gudovich, S.G. Krein, and I.M. Kulikov (see [15] and references therein) and come to the system

$$\begin{aligned} i\,\mathrm{rot}\,u^2(x) + i\nabla a^2(x) - ku^1(x) = f^1(x), &\quad -i\,\mathrm{div}\,u^2(x) - ka^1(x) = h^1(x), \\ -i\,\mathrm{rot}\,u^1(x) - i\nabla a^1(x) - ku^2(x) = f^2(x), &\quad i\,\mathrm{div}\,u^1(x) - ka^2(x) = h^2(x), \end{aligned} \tag{1.1.3}$$

$x \in G$, with the boundary conditions

$$-\langle u^1(x), \tau_2(x)\rangle = 0, \langle u^1(x), \tau_1(x)\rangle = 0, \langle u^2(x), \nu(x)\rangle = 0, a^2(x) = 0, \tag{1.1.4}$$

$x \in \partial G$. Here u^1, u^2 are, as previously mentioned, three-dimensional vector functions, while a^1 and a^2 stand for scalar functions. Vectors $\tau_1(x)$ and $\tau_2(x)$ are unit tangent vectors to ∂G and $\nu(x)$ is the unit vector of outward normal, (τ_1, τ_2, ν) is a right-hand triple of vectors. Such an extension was used, in particular, in papers by Birman and Solomyak [7] for investigating spectrum of the Maxwell system in domains with non-smooth boundary. In the case of waveguides, such an "elliptization" of the Maxwell system is employed for the first time, possibly, because a theory of waveguides for elliptic systems was developed lately with a sufficient generality. We refer the reader to the works of Picard et al. [28], [29], Matioukevitch et al. [22] for other examples of the Maxwell system elliptization.

Elliptic boundary-value problems (for systems of equations) in waveguides, having several cylindrical outlets to infinity, were studied in Nazarov and Plamenevskii [25] (see also an extensive bibliography therein). In particular, the intrinsic radiation conditions were described, the solvability of the boundary value problem with those radiation conditions was established, the unitary scattering matrix was introduced. Essentially, the matrix was defined on the waveguide continuous spectrum and for every value of spectral parameter the matrix has a finite size equal to the multiplicity of waveguide continuous spectrum.

The theory of elliptic problems in waveguides was developed in further papers by Nazarov, Plamenevskii and their coauthors. We outline a series of papers devoted to a method for approximate computation of the scattering matrix. For the first time the method was proposed in Grikurov et al. [14] for approximating the scattering matrix of the Helmholtz equation in diffraction gratings. Then it was developed by Plamenevskii et al. [30], [31] for the case of waveguides. In [30] the authors considered the two-dimensional waveguides and the Helmholtz operator, while the waveguides of arbitrary dimensions and the self-adjoint elliptic systems of any order were discussed in [31]. The approach suggested in [31] turns out to be new for the Helmholtz operator as well and more simple than that in [30]. Let us mention that the method for computing the scattering matrix of the Helmholtz operator in n-dimensional waveguides was described with detailed proof by Baskin et al. [5].

We apply the methods of the elliptic theory described in the two preceding paragraphs to the elliptic problem (1.1.3)- (1.1.4). Then we clarify specific properties of the problem coming from the Maxwell system. In particular, we investigate in detail the operator pencils generated by the elliptic problem (1.1.3)- (1.1.4). Finally, we derive the information on the Maxwell system (1.1.1)- (1.1.2) from that obtained about the elliptic problem (1.1.3)- (1.1.4).

The method for approximating the scattering matrix was discussed in [31] under the condition that the spectral parameter does not coincide with thresholds. When the spectral parameter crosses a threshold, the scattering matrix changes its size and the method becomes inapplicable. We propose and justify in Chapter 6 a modification of the method for computing the waveguide scattering matrix of the Helmholtz equation in a neighborhood of a threshold. The problem is simpler than that for the Maxwell system from a technical point of view, but it represents a good illustration how to deal with "threshold effects" in general. We construct a wave basis being "stable" in a neighbourhood of a threshold and introduce an "augmented scattering matrix", corresponding to the basis that keeps its size and depends smoothly on the spectral parameter of the neighbourhood. We compute the augmented matrix using a modification of the method [31] and express the ordinary scattering matrix in terms of the augmented one.

Note that the "stable arguments" are not uncommon in asymptotic studies of various "threshold" situations. In this connection we refer to the work of Costabel et al. [10] and the work of Maz'ya et al. [23], where the asymptotics of solutions to elliptic boundary value problems were investigated near singularities of the boundary. In the work of Kamotskii et al. [18], the asymptotics of the scattering matrix for a two dimensional diffraction grating was justified, in essence, with the help of a stable basis in the space of waves.

The method for approximating the waveguide scattering matrix, described in the thesis, has been successfully implemented in real computations. A series of papers [2] - [5], [17] is devoted to investigating electron resonant tunneling in quantum waveguides of variable cross-section. The basic characteristic of resonant tunneling is a "transmission coefficient" which can be expressed in terms of the waveguide scattering matrix. In [2] - [5] the method was applied for computing the waveguide scattering matrix of the Helmholtz equation as well as Pauli equation in various geometric situations. In [17] the scattering matrix of the Helmholtz equation is computed in the case when the spectral parameter is close to a threshold. To this end the authors apply both the "non-threshold" method (described in Chapter 3 of the thesis) and the "threshold" method (described in Chapter 5 of the thesis) and compare the obtained results.

1.2 Review of results

We consider a domain G in a three-dimensional space $\mathbb{R}^3$, coinciding outside a large ball with a union of finitely many non-overlapping semi-cylinders $\Pi_+^1 \cup \cdots \cup \Pi_+^{\mathcal{T}}$, where $\Pi_+^q = \{(y^q, t^q) : y^q \in \Omega^q, t^q > 0\}$, and cross-section Ω^q is a bounded domain in $\mathbb{R}^2$.

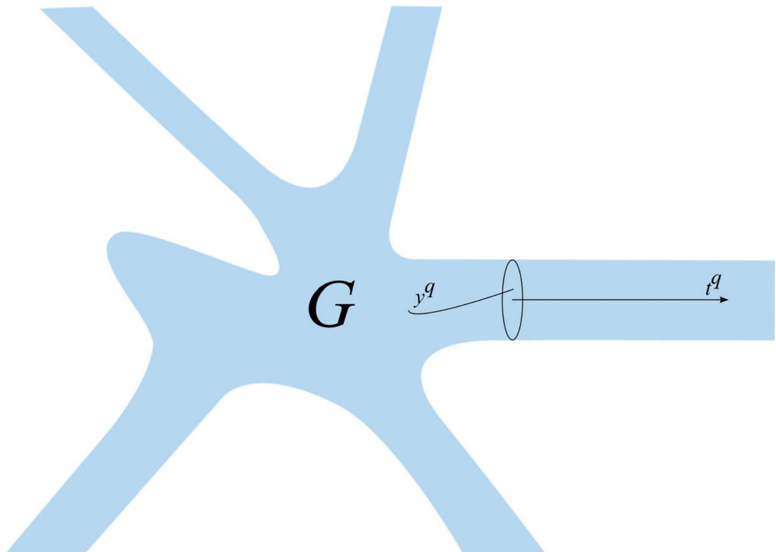


FIGURE 1 An example of a waveguide, having several cylindrical ends.

Furthermore, in the introduction, for the sake of statement simplicity we suppose $\mathcal{T} = 1$ and denote the single cylindrical outlet by $\Pi_+ = \{y \in \Omega, t > 0\}$. The boundary ∂G of the domain G is assumed to be smooth. The Maxwell system of equations

$$
\begin{aligned}
i \operatorname{rot} u^2(x) - k u^1(x) &= f^1(x), & -i \operatorname{div} u^2(x) &= h^1(x), & & \\
-i \operatorname{rot} u^1(x) - k u^2(x) &= f^2(x), & i \operatorname{div} u^1(x) &= h^2(x), & x &\in G,
\end{aligned} \tag{1.2.1}
$$

with the boundary conditions

$$
\nu(x) \times u^1(x) = 0, \ \langle u^2(x), \nu(x) \rangle = 0, \ x \in \partial G, \tag{1.2.2}
$$

describes an electromagnetic field, excited in an empty waveguide G with perfectly conductive boundary in the presence of charges and currents. Here u^1, u^2 are $\mathbb{C}^3$-valued functions, which stand for the vectors of electric and magnetic field, $\langle \cdot, \cdot \rangle$ is an inner product in $\mathbb{C}^3$, $\cdot \times \cdot$ is a vector product in $\mathbb{R}^3$, and ν is the outward normal to ∂G. The system (1.2.1) is overdetermined (eight equations and only six indeterminate functions). The compatibility conditions

$$
\begin{aligned}
\operatorname{div} f^1(x) - ikh^2(x) &= 0, \ x \in G, \\
\operatorname{div} f^2(x) + ikh^1(x) &= 0, \ x \in G, \\
\langle f^2(x), \nu(x) \rangle &= 0, \ x \in \partial G
\end{aligned} \tag{1.2.3}
$$

are necessary for the solvability of problem (1.2.1), (1.2.2).

In the Chapter 2, we introduce an elliptic boundary value problem

$$
\mathcal{A}(D,k)\mathcal{U}(x) = \mathcal{F}(x), \ x \in G, \quad \mathcal{B}(x)\mathcal{U}(x) = \mathcal{G}(x), \ x \in \partial G, \tag{1.2.4}
$$

obtained from (1.2.1)- (1.2.2) by means of "orthogonal extension" method [15]. Here

$$
\mathcal{U} = (u^1, a^1, u^2, a^2), \quad \mathcal{F} = (f^1, h^1, f^2, h^2), \quad \mathcal{G} = (g^1, g^2, g^3, g^4),
$$

u^j, f^j are three-dimensional and a^j, h^j are scalar functions in G, $j = 1, 2$, and g^l is a scalar function in ∂G, $l = 1, \dots, 4$. The differential operator $\mathcal{A}(D,k)$ and the boundary operator $\mathcal{B}$ are given by

$$
\mathcal{A}(D,k)\mathcal{U} = \begin{pmatrix} i \operatorname{rot} u^2 + i \nabla a^2 - k u^1 \\ -i \operatorname{div} u^2 - k a^1 \\ -i \operatorname{rot} u^1 - i \nabla a^1 - k u^2 \\ i \operatorname{div} u^1 - k a^2 \end{pmatrix}, \quad x \in G, \tag{1.2.5}
$$

$$
\mathcal{B}\mathcal{U} = (-\langle u^1, \tau_2 \rangle, \langle u^1, \tau_1 \rangle, \langle u^2, \nu \rangle, a^2), \quad x \in \partial G,
$$

where τ_1, τ_2, ν is a right-hand triple of orthonormal vectors in $\mathbb{R}^3$: ν is an outward normal to ∂G, and τ_1, τ_2 are tangent vectors. For $\mathcal{U}, \mathcal{V} \in C_c^\infty(\overline{G}; \mathbb{C}^8)$ and the boundary operator $\mathcal{Q}$: $\mathcal{Q}\mathcal{U} = -i(\langle u^2, \tau_1 \rangle, \langle u^2, \tau_2 \rangle, a^1, -\langle u^1, \nu \rangle), x \in \partial G$, the Green formula

$$
(\mathcal{A}(D,k)\mathcal{U}, \mathcal{V})_G + (\mathcal{B}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\partial G} = (\mathcal{U}, \mathcal{A}(D,k)\mathcal{V})_G + (\mathcal{Q}\mathcal{U}, \mathcal{B}\mathcal{V})_{\partial G}, \tag{1.2.6}
$$

holds, where $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_{\partial G}$ are inner products in $L_2(G;\mathbb{C}^8)$ and $L_2(\partial G;\mathbb{C}^4)$ respectively. (In what follows we skip number of components in the notations of vector function linear spaces.)

Solutions of the homogeneous problem (1.2.4) in the cylinder $\Pi = \Omega \times \mathbb{R}$ play the role of waves propagating in the cylindrical outlet Π_+ of the waveguide G. We will seek solutions of the form

$$\Pi \ni (y,t) \mapsto \exp(i\lambda t)P(y), \quad t \in \mathbb{R}, y \in \Omega.$$

Here λ and P are an eigenvalue and a corresponding eigenvector of the operator pencil

$$\mathfrak{A}(D_y,\lambda;k)P(y) = \exp(-i\lambda t)\mathcal{A}(D,k)(\exp(i\lambda t)P(y)), \quad y \in \Omega, t \in \mathbb{R}, \qquad (1.2.7)$$

with $\mathcal{A}(D,k)$ of the form (1.2.5). In Section 2.2, the spectrum of the operator pencil is investigated in detail. We describe its eigenvalues, corresponding eigenvectors and generalized eigenvectors.

In the domain $\mathcal{D}(\mathfrak{A})$ of the pencil (1.2.7), we include vector functions $P = (\varphi,\alpha,\psi,\beta)$ with components $\varphi,\psi \in C^1(\Omega;\mathbb{C}^3)$ and $\alpha,\beta \in C^1(\Omega;\mathbb{C})$, satisfying on $\partial\Omega$ the boundary conditions $\varphi_3 = 0$, $\varphi_1\nu_2 - \varphi_2\nu_1 = 0$, $\psi_1\nu_1 + \psi_2\nu_2 = 0$, $\beta = 0$, where (ν_1,ν_2) is an outward normal to $\partial\Omega$. A value λ is said to be an eigenvalue of the pencil $\mathfrak{A}(\cdot,k)$ if there exists a non-zero vector function $P \in \mathcal{D}(\mathfrak{A})$, such that

$$\mathfrak{A}(\lambda,k)P(y) = 0,\ y \in \Omega.$$

The vector function P is said to be an eigenvector. Its components are smooth functions in $\overline{\Omega}$. A restriction of the pencil $\mathfrak{A}(\cdot,k)$ to $\mathcal{D}(\mathfrak{M}) = \{P = (\varphi,0,\psi,0) \in \mathcal{D}(\mathfrak{A})\}$ is called a Maxwell pencil and denoted by $\mathfrak{M}(\cdot,k)$. In other words, $\mathfrak{M}(\cdot,k)$ is an operator pencil of the form (1.2.7), defined for problem (1.2.1)- (1.2.2). A value λ is said to be an eigenvalue of the pencil $\mathfrak{M}(\cdot,k)$, if there exists a non-zero vector function $P \in \mathcal{D}(\mathfrak{M})$, satisfying the equality $\mathfrak{A}(\lambda,k)P = 0$.

Given $k \in \mathbb{R}$, the eigenvalues of the pencil $\mathfrak{A}(\cdot,k)$ ($\mathfrak{M}(\cdot,k)$) lie on the real and imaginary axes, symmetrically about the origin (numbers $+\lambda$ and $-\lambda$ are eigenvalues at once and the dimensions of the spaces $\ker\mathfrak{A}(\lambda,k)$ and $\ker\mathfrak{A}(-\lambda,k)$ ($\ker\mathfrak{M}(\lambda,k)$ and $\ker\mathfrak{M}(-\lambda,k)$) are equal). The real axis contains finitely many eigenvalues. If for some $k \in \mathbb{R}\setminus\{0\}$ we have an eigenvalue $\lambda = 0$ for the pencil $\mathfrak{A}$ ($\mathfrak{M}$), then for any corresponding eigenvector $P \in \mathcal{D}(\mathfrak{A})$ ($P \in \mathcal{D}(\mathfrak{M})$) there exists a generalized eigenvector in $\mathcal{D}(\mathfrak{A})$ ($\mathcal{D}(\mathfrak{M})$) (an exact definition of generalized eigenvectors see in Section 2.2.2). The corresponding value of k is called a threshold of problem (1.2.4). There are no generalized eigenvectors for $\lambda \neq 0$. Thresholds are located symmetrically relative to zero and can accumulate only at infinity.

For a given eigenvalue λ of the pencil $\mathfrak{M}(\cdot,k)$ we fix a basis set of eigenvectors $\{P_{\mathfrak{M},j}\}$ in the space $\ker\mathfrak{M}(\lambda,k)$. Furthermore, the set is augmented by vectors $\{P_{\nabla,l}\}$ up to a basis of the space $\ker\mathfrak{A}(\lambda,k)$. The basis eigenvectors $\{P_{\mathfrak{M},j}\}$, $\{P_{\nabla,l}\}$ and corresponding generalized eigenvectors (if they arise) are chosen in a specific way.

In the second part of Chapter 2 (Sections 2.3-2.5), we investigate problem (1.2.4) in G, describe its continuous spectrum, introduce a unitary scattering matrix and justify a “radiation principle” (a well-posed statement of the problem with intrinsic radiation conditions). To this end, we apply the scheme for investigating elliptic boundary value problems in domains with cylindrical ends proposed in Nazarov and Plamenevskii [25].

If for a number k there exists a solution $\mathcal{U}$ to the homogeneous problem (1.2.4) that satisfies $\mathcal{U}(x) = O(|x|)$ for $|x| \to \infty$ and does not belong to $L_2(G)$, then the k is said to be a point in the continuous spectrum and the $\mathcal{U}$ is said to be a corresponding continuous spectrum eigenfunction (CSE). We denote the linear hull of CSEs by $E(k)$. A number k is said to be an eigenvalue of problem (1.2.4), if there exists a solution in $L_2(G)$. Eigenvalues can not accumulate at a finite distance. For the sake of simplicity, we assume in the introduction that k is distinct from zero and eigenvalues of problem (1.2.4) (in the thesis we deal with general case).

We first assume, that k is fixed, $k > 0$, and does not coincide within a threshold. The asymptotics of CSEs is described in terms of incoming and outgoing waves. For every real eigenvalue λ of the operator pencil $\mathfrak{A}(\cdot, k)$ and every eigenvector P in the set $\{P_{\mathfrak{M},j}\}$, $\{P_{\nabla,l}\}$, corresponding to λ, we introduce a function u given at $\Pi_+ \cap G$ by the equality

$$u(y, t; k) = \exp(i\lambda t)P(y; k), \quad y \in \Omega,\ t > T \tag{1.2.8}$$

for a sufficiently large T and extended smoothly to the rest of G. The obtained functions satisfy the homogeneous problem (1.2.4) for a large $|x|$ and are called waves. If λ is negative (positive), then the corresponding wave u of the form (1.2.8) is said to be incoming (outgoing) and is denoted by u^+ (u^-). Since the spectra of the pencils $\mathfrak{A}(\cdot, k)$ and $\mathfrak{M}(\cdot, k)$ are symmetric about the origin, the number of waves in the sets $\{u^+\}$ and $\{u^-\}$ are equal. We enumerate both sets with the index $j = 1, \dots, \Upsilon$. The linear hull of the functions $u_1^+, \dots, u_\Upsilon^+, u_1^-, \dots, u_\Upsilon^-$ is called the space of waves and is denoted by $\mathcal{W}(k)$.

According to the elliptic theory [25], in the space $E(k)$ there exists the basis of CSEs $Y_1^+, \dots, Y_\Upsilon^+$, subjected to the relations

$$Y_j^+(\cdot, k) = u_j^+(\cdot, k) + \sum_{l=1}^{\Upsilon} S_{jl}(k) u_l^-(\cdot, k) + O(\exp(-\delta|x|)), \quad j = 1, \dots, \Upsilon \tag{1.2.9}$$

for a large $|x|$ and $\delta < \delta_0(k)$ (where $\delta_0(k) = \min|\operatorname{Im}\lambda|$ over imaginary eigenvalues λ of the pencil $\mathfrak{A}(\cdot, k)$). The dimension $\Upsilon(k)$ of $E(k)$ is called the multiplicity of continuous spectrum at k. The matrix $S(k)$ of the size $\Upsilon(k) \times \Upsilon(k)$ with elements $S_{jl}(k)$ is unitary and is called the scattering matrix.

Let us describe a radiation principle of the elliptic problem. To this end, we introduce a weighted Sobolev space $H^l_\beta(G)$, $l \geq 0$ with the norm

$$\|u; H^l_\beta(G)\| := \|\rho_\beta u; H^l(G)\| = \left(\sum_{|\alpha|=0}^{l} \int_G |D^\alpha(\rho_\beta u)|^2\, dx \right)^{1/2},$$

where ρ_β is a smooth function on $\overline{G}$, coinciding on $G \cap \Pi_+$ with the map $(y,t) \mapsto \exp(\beta t)$. Let us denote by $H_\beta^{l+1/2}(\partial G)$ the space of traces on ∂G of functions in $H_\beta^{l+1}(G)$ (as a rule, we will use the same notations for the spaces of vector functions with components in $H_\beta^l(G)$ and $H_\beta^{l+1/2}(\partial G)$). The operator $\{\mathcal{A}(D,k), \mathcal{B}\}$ of the boundary value problem (1.2.4) implements a continuous mapping

$$\mathcal{L}_\beta : H_\beta^{l+1}(G) \to H_\beta^l(G) \times H_\beta^{l+1/2}(\partial G) =: \mathcal{H}_\beta^l(G) \tag{1.2.10}$$

for any $\beta \in \mathbb{R}$ and $l = 0, 1, \ldots$. The operator (1.2.10) is a Fredholm operator if and only if the line $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = \beta\}$ is free from the spectrum of the pencil $\mathfrak{A}(\cdot, k)$, the number k being fixed. Recall that an operator is called a Fredholm operator if its range is closed and the kernel and cokernel are finite dimensional. Let us chose $0 < \delta < \delta_0$, where δ_0 is chosen in (1.2.9).

Proposition 1.2.1. *Let $\{\mathcal{F}, \mathcal{G}\}$ belong to the space $\mathcal{H}_\delta^l(G)$. Then:*
1. Problem (1.2.4) *has a unique solution $\mathcal{U}$, subject to the radiation conditions*

$$\mathcal{V} = \mathcal{U} - c_1 u_1^- - \cdots - c_\Upsilon u_\Upsilon^- \in H_\delta^{l+1}(G). \tag{1.2.11}$$

2. The coefficients c_j in the asymptotics (1.2.11) *are given by*

$$c_j = i(\mathcal{F}, Y_j^-)_G + i(\mathcal{G}, \mathcal{Q}Y_j^-)_{\partial G},$$

where $Y_j^- = \sum_{l=1}^{\Upsilon} S_{jl}^{-1} Y_l^+$ with functions Y_l^+ from (1.2.9)*, and $\mathcal{Q}$ is the operator from* (1.2.6).
3. The inequality

$$\|\mathcal{V}; H_\delta^{l+1}(G)\| + |c_1| + \cdots + |c_\Upsilon| \leq \text{const}\|\{\mathcal{F}, \mathcal{G}\}; \mathcal{H}_\delta^l(G)\|$$

holds.

Chapter 3 is devoted to formulation and justification of a method for approximating the scattering matrix $S(k)$. For elliptic boundary value problems with a semi-bounded operator a method for approximating the scattering matrix was essentially justified in papers by Plamenevskii, Sarafanov et al. (see [31], [5]). An operator of problem (1.2.4) is not semi-bounded and a statement, as well as justification of the method needs an essential modification. Such a modification is described in this Chapter. We assume here k being distinct from the thresholds. A method for computing the scattering matrix in a neighbourhood of a threshold is discussed in Chapter 5.

Let $\Pi_+^R = \{(y,t) \in \Pi : t > R\}$, $G^R = G \setminus \Pi_+^R$ for large R (see FIG. 2). Then $\partial G^R \setminus \partial G = \Gamma^R = \{(y,t) \in \Pi : t = R\}$. Introduce a boundary value problem

$$\begin{aligned} \mathcal{A}(D,k)\mathcal{U}(x) &= \mathcal{F}(x), \quad x \in G^R, \\ \mathcal{B}(x)\mathcal{U}(x) &= \mathcal{G}(x), \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{B}(x) + i\mathcal{Q}(x))\mathcal{U}(x) &= \mathcal{H}(x), \quad x \in \Gamma^R. \end{aligned} \tag{1.2.12}$$

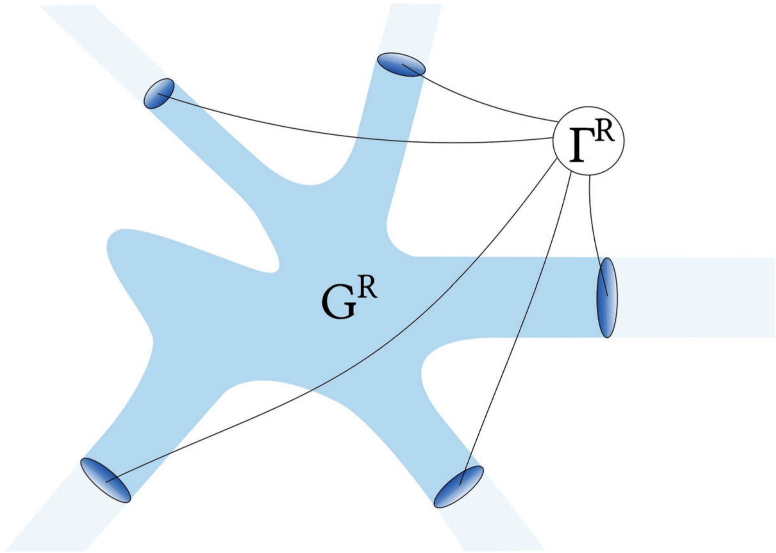


FIGURE 2 A waveguide with truncated cylindrical ends.

The boundary ∂G^R contains an edge $\partial\Gamma^R$. Let $r(x) = \operatorname{dist}(x, \partial\Gamma^R)$. We define spaces $V^0_{1/2}(G^R)$ and $V^1_{1/2}(G^R)$ as the completions of $C^\infty_c(\overline{G^R} \setminus \partial\Gamma^R)$, with respect to norms

$$\|u; V^0_{1/2}(G^R)\|^2 = \int_{G^R} r|u|^2 dx, \qquad \|u; V^1_{1/2}(G^R)\|^2 = \int_{G^R} (r|\nabla u|^2 + r^{-1}|u|^2) dx.$$

Let $V^{1/2}_{1/2}(\Gamma^R)$ and $V^{1/2}_{1/2}(\partial G^R \setminus \Gamma^R)$ denote the spaces of traces on Γ^R and $\partial G^R \setminus \Gamma^R$ of functions in $V^1_{1/2}(G^R)$.

Theorem 1.2.2. *For any right-hand-side*

$$\{\mathcal{F}, \mathcal{G}, \mathcal{H}\} \in V^0_{1/2}(G^R) \times V^{1/2}_{1/2}(\partial G^R \setminus \Gamma^R) \times V^{1/2}_{1/2}(\Gamma^R),$$

problem (1.2.12) *has a unique solution* $\mathcal{U} \in V^1_{1/2}(G^R)$. *If* $\mathcal{F} = 0$, $\mathcal{G} = 0$, *then the solution* $\mathcal{U}$ *satisfies the equality* $\|\mathcal{U}; L_2(\Gamma^R)\| = \|\mathcal{H}; L_2(\Gamma^R)\|$.

Let $X^R_l(\cdot, k; a)$ stand for a solution to the problem (1.2.12) with right-hand-side

$$\mathcal{F} = 0, \quad \mathcal{G} = 0, \quad \mathcal{H} = \Big(\mathcal{B} + i\mathcal{Q}\Big)\Big(u^+_l(\cdot, k) + \sum_{j=1}^{\Upsilon} a_j u^-_j(\cdot, k)\Big),$$

where $a = (a_1, \dots, a_\Upsilon)$ is an arbitrary vector in $\mathbb{C}^\Upsilon$. Introduce a functional

$$a \mapsto J^R_l(a; k) = \Big\| \mathcal{Q}\Big(X^R_l(\cdot, k; a) - u^+_l(\cdot, k) - \sum_{j=1}^{\Upsilon} a_j u^-_j(\cdot, k)\Big); L_2(\Gamma^R)\Big\|^2. \quad (1.2.13)$$

Let $Z^{\pm}_{j,R}$ be a solution to the problem (1.2.12) with right-hand-side $\mathcal{F} = 0, \mathcal{G} = 0, \mathcal{H} = (\mathcal{B} + i\mathcal{Q})u^{\pm}_l$. We have $X^R_l = Z^+_{l,R} + \sum a_j Z^-_{j,R}$, and the functional takes the form

$$J^R_l(a;k) = \langle aE^R, a\rangle + 2\mathrm{Re}\,\langle F^R_l, a\rangle + G^R_l. \tag{1.2.14}$$

Here E^R, F^R are square matrices of $\Upsilon \times \Upsilon$, with elements

$$E^R_{ij} = (\mathcal{Q}(Z^-_{i,R} - u^-_i), \mathcal{Q}(Z^-_{j,R} - u^-_j))_{\Gamma^R}, F^R_{ij} = (\mathcal{Q}(Z^+_{i,R} - u^+_i), \mathcal{Q}(Z^-_{j,R} - u^-_j))_{\Gamma^R},$$

F^R_l is a row vector with number l of the matrix F^R, and $G^R_l = \|\mathcal{Q}(Z^+_{l,R} - u^+_l)\|^2_{\Gamma^R}$. As an approximation to the row $(S_{l1}, \dots, S_{l\Upsilon})$ of the scattering matrix $S(k)$, we take a minimizer $a^0(R,k) = (a^0_1(R,k), \dots, a^0_\Upsilon(R,k))$ of the functional (1.2.13). Formula (1.2.14) implies the minimizer $a^0(R,k)$ to solve the equation

$$a^0(R,k)E^R + F^R_l = 0.$$

Hence, a matrix S^R, composed of such minimizers, satisfies $S^R E^R + F^R = 0$ and serves as an approximation to the scattering matrix S.

Theorem 1.2.3. *Let an interval* $[k_1, k_2]$ *of the continuous spectrum of problem* (1.2.4) *contain no thresholds. Then for all* $k \in [k_1, k_2]$ *and* $R > R_0$*, where* R_0 *is a sufficiently large number, there exists a unique minimizer* $a^0(R,k)$ *of the functional* (1.2.13) *and*

$$|a^0_j(R,k) - S_{lj}(k)| \le Ce^{-\delta R}, \; j = 1, \dots, \Upsilon,$$

where $0 < \delta < \min_{[k_1,k_2]} \delta_0(k)$*,* $\delta_0(k)$ *is the same as in* (1.2.9)*, and the constant* $C = C(\delta)$ *is independent of* k *and* R*.*

Chapter 4 is devoted to coming back from the elliptic problem (1.2.4) to the original nonaugmented one (1.2.1)- (1.2.2). We start with a description of the continuous spectrum of the Maxwell system. The space $E(k)$ of continuous spectrum eigenfunctions of the problem (1.2.4) can be represented as a direct sum of subspaces (the representation follows directly from the equations (1.1.3))

$$E(k) = E^{\mathfrak{M}}(k) \dotplus E^{\nabla}(k).$$

Here the space $E^{\mathfrak{M}}(k)$ consists of CSEs of the form $(u^1, 0, u^2, 0)$. A solution of the form $\mathcal{U} = (u^1, 0, u^2, 0)$ to the problem $\mathcal{A}(D,k)\mathcal{U} = \mathcal{F}$, $\mathcal{B}\mathcal{U} = 0$ is said to be a Maxwell-type. Then components u^1, u^2 of the Maxwell-type solution satisfy the original problem (1.2.1)- (1.2.2). So, the space $\mathcal{E}(k)$ of CSEs of problem (1.2.1)- (1.2.2) can be identified with the space $E^{\mathfrak{M}}(k)$ of Maxwell-type CSEs of the problem (1.2.4). To introduce a scattering matrix for the problem (1.2.1)- (1.2.2), we have to prove that in the space $E^{\mathfrak{M}}(k)$ there exists a basis, subject to relations of the form (1.2.9).

Let us introduce more detailed notations. The waves $u^{\pm}_l$, corresponding to (Maxwell) eigenvectors $\{P_{\mathfrak{M},j}\}$, are of the form $u^{\pm}_l = (u^1, 0, u^2, 0)$. Such waves are called Maxwell-type waves and are denoted by $e^{\pm}_l$, $l = 1, \dots, v$. The waves

$u_l^{\pm}$ corresponding to (non-Maxwell) eigenvectors $\{P_{\nabla,j}\}$, are called gradient-type and are denoted by $\gamma_l^{\pm}$, $l = 1, \dots, \Upsilon - v$. We prove that the scattering matrix $S(k)$ is block-diagonal $S(k) = \operatorname{diag}(s(k), \sigma(k))$. In other words, the Maxwell-type (gradient-type) incoming waves e_j^+ (γ_j^+) are scattered only on the Maxwell-type (gradient-type) outgoing waves e_l^- (γ_l^-). Let us denote by $\mathcal{E}_j^+$ the continuous spectrum eigenfunction, with the asymptotics (1.2.9) that contains the wave e_j^+, $j = 1, \dots, v$. The functions $\mathcal{E}_j^+$, $j = 1, \dots, v$, form a basis in the space $E^{\mathfrak{M}}(k)$. After introducing new notations we come to the following theorem

Theorem 1.2.4. *In the space* $\mathcal{E}(k)$ *of problem* (1.2.1)- (1.2.2) *CSEs, there exists a basis* $\hat{\mathcal{E}}_1^+, \dots, \hat{\mathcal{E}}_v^+$*, subject to the relations*

$$\hat{\mathcal{E}}_j^+(\cdot, k) = \hat{e}_j^+(\cdot, k) + \sum_{l=1}^{v} s_{jl}(k)\hat{e}_l^-(\cdot, k) + O(\exp(-\delta|x|)), \quad j = 1, \dots, v. \tag{1.2.15}$$

Here the functions $\hat{\mathcal{E}}_j^+$ and $\hat{e}_j^{\pm}$ are obtained from $\mathcal{E}_j^+$ and $e_j^{\pm}$ by crossing of (zero) components a^1, a^2. The matrix s is unitary and by definition, is a scattering matrix of the problem (1.2.1)- (1.2.2).

Furthermore, from the elliptic problem radiation principle (Proposition 1.2.1), we derive a radiation principle for problem (1.2.1)- (1.2.2). Let $\mathcal{U} = (u^1, a^1, u^2, a^2)$ be a solution to the problem (1.2.4) with the radiation conditions (1.2.11) and a right-hand-side $\{\mathcal{F}, 0\}$, subject to compatibility conditions (1.2.3). Then the component a^1 (a^2) of the solution solves the homogeneous Neumann (Dirichlet) problem for the Helmholtz equation with intrinsic radiation conditions. By virtue of uniqueness theorem, such a solution must be zero. Hence, the solution $\mathcal{U}$ is Maxwell-type, that is, is of the form $\mathcal{U} = (u^1, 0, u^2, 0)$. The obtained assertion is still not completely satisfactory for our purpose (the return to the Maxwell system (1.2.1)- (1.2.2)), since the statement of Proposition 1.2.1 contains other reminders of the elliptic problem: the radiation conditions include the gradient-type waves and the coefficients in (1.2.11) are calculated by means of elliptic CSEs. However if compatibility conditions (1.2.3) hold, then the gradient-type waves appear in (1.2.11) with zero coefficients and the radiation conditions include the Maxwell-type waves e_j^- only. Since the scattering matrix $S(k)$ is block diagonal, the coefficients of the Maxwell-type waves e_j^- are calculated by means of the Maxwell-type CSEs $\mathcal{E}_j^- = \sum_{l=1}^{v} s_{jl}^{-1}\mathcal{E}_l^+$. After introducing new notations we get

Theorem 1.2.5. *Let* δ *satisfy* $0 < \delta < \delta_0$ *and* $\mathcal{F} = (f^1, h^1, f^2, h^2)$ *in* $H_\delta^l(G; \mathbb{C}^8)$ *be subjected to compatibility conditions* (1.2.3). *Then:*
1. Problem (1.2.1)- (1.2.2) *has a unique solution* $U = (u^1, u^2)$*, subject to the radiation conditions*

$$V = U - c_1\hat{e}_1^- - \cdots - c_v\hat{e}_v^- \in H_\delta^{l+1}(G; \mathbb{C}^6). \tag{1.2.16}$$

2. The coefficients c_j *in asymptotics* (1.2.16) *are given by* $c_j = i\,(f, \hat{\mathcal{E}}_j^-)_G$*, where* $f = (f^1, f^2)$ *and* $\hat{\mathcal{E}}_j^- = \sum_{l=1}^{v} s_{jl}^{-1}\hat{\mathcal{E}}_l^+$ *with* $\hat{\mathcal{E}}_l^+$ *from* (1.2.15).

3. The inequality

$$\|V; H^{l+1}_{\delta}(G;\mathbb{C}^6)\| + |c_1| + \cdots + |c_v| \leq \text{const}\|\mathcal{F}; H^{l}_{\delta}(G;\mathbb{C}^8)\|$$

holds.

Finally, in Chapter 5 we modify arguments of Chapter 3 and propose a method for computing the scattering matrix in a neighbourhood of a threshold (such a modification was developed in [33]). Instead of the Maxwell system, we consider the Helmholtz equation with Dirichlet boundary conditions. The problem is simpler than that for the Maxwell system from a technical point of view, but it represents a good illustration how to deal with "threshold effects" in general. We have already justified analogous results for the Maxwell system, and a paper on the topic is being prepared for the press.

The waveguide is described by the Dirichlet problem for the operator $-\Delta - \mu$, where μ is a spectral parameter and Δ is the Laplace operator. The continuous spectrum of the problem coincides with the semiaxis $\{\mu \in \mathbb{R} : \tau_1 \leq \mu\}$, the τ_1 being a positive number. For every point $\mu \in [\tau_1, +\infty)$ there exist $\varkappa(\mu)$ solutions, $\varkappa(\mu) < \infty$, to the homogeneous problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= 0, \qquad x \in G, \\ u(x) &= 0, \qquad x \in \partial G, \end{aligned} \tag{1.2.17}$$

satisfying $|u(x)| \leq \text{Const}(1 + |x|)$ in G and linearly independent modulo $L_2(G)$. Such solutions are called the continuous spectrum eigenfunctions and the number $\varkappa(\mu)$ is called the multiplicity of the continuous spectrum. The threshold values (thresholds) form a sequence $\tau_1 < \tau_2, \ldots, \tau_n \to +\infty$. The multiplicity $\varkappa(\mu)$ is constant on every interval $[\mu', \mu'']$ of the continuous spectrum containing no threshold. The function $\mu \mapsto \varkappa(\mu)$ has discontinuity at every threshold being continuous from the right. This is an increasing function, so $\varkappa(\mu) \to +\infty$ as $\mu \to +\infty$. It is known [25] that, for every $\mu \in [\tau_1, +\infty]$ in the space of continuous spectrum eigenfunctions, there exists a basis $Y_1(\cdot, \mu), \ldots, Y_{\varkappa(\mu)}(\cdot, \mu)$ modulo $L_2(G)$, such that

$$Y_j(x, \mu) = u_j^+(x, \mu) + \sum_{k=1}^{\varkappa(\mu)} S_{jk}(\mu) u_k^-(x, \mu) + O(e^{-\varepsilon|x|}) \tag{1.2.18}$$

for $|x| \to \infty$ and $j = 1, \ldots, \varkappa(\mu)$. Here ε is a sufficiently small number, $u_j^+(\cdot, \mu)$ is an "incoming" wave, and $u_j^-(\cdot, \mu)$ is an "outgoing" one (precise definitions see in Section 5.1). The matrix $S(\mu) = \|S_{jk}(\mu)\|$ is unitary; it is called the scattering matrix. Plamenevskii et al. [30], [31] discusses a method for approximating the matrix $S(\mu)$, under the condition that μ varies on an interval $[\mu', \mu'']$ of the continuous spectrum containing no thresholds. In a neighbourhood of a threshold the method turns out to be inapplicable and needs an essential modification. Such a modification is described in this Chapter.

Let $\tau' < \tau < \tau''$ be three succeeding thresholds. On the interval (τ, τ''), one can choose a basis of incoming waves $w_1^+(\cdot, \mu), \ldots, w_\varkappa^+(\cdot, \mu)$ and outgoing waves $w_1^-(\cdot,\mu),\dots,w_\varkappa^-(\cdot,\mu)$ with analytic functions $(\tau,\tau'')\ni\mu\mapsto w_j^\pm(\cdot,\mu)$, which admit the analytic continuation to (τ',τ''), where $\varkappa=\varkappa(\mu'')$ (recall that $\varkappa(\mu)=\text{const}$ for $\mu\in[\tau,\tau'')$). Such a basis is called stable at the threshold τ. For $\mu\in(\tau',\tau)$, some incoming waves and the same number of outgoing waves turn out to be exponentially growing as $x\to\infty$. On the interval (τ,τ''), in the space of continuous spectrum eigenfunctions, there exists a basis $\mathcal{Y}_1(\cdot,\mu),\dots,\mathcal{Y}_\varkappa(\cdot,\mu)$ satisfying the conditions

$$\mathcal{Y}_j(x,\mu)=w_j^+(x,\mu)-\sum_{k=1}^{M}\mathcal{S}_{jk}(\mu)w_k^-(x,\mu)+O(e^{-\varepsilon|x|}). \tag{1.2.19}$$

The functions $\mu\mapsto\mathcal{Y}_j(\cdot,\mu)$ and $\mu\mapsto\mathcal{S}_{jk}(\mu)$ are analytic and admit the analytic continuation to (τ',τ''). In contrast to $S(\mu)$, the new matrix $\mathcal{S}(\mu)=\|\mathcal{S}_{jk}(\mu)\|$ keeps its size on the interval, the matrix is unitary for all $\mu\in(\tau',\tau'')$. The entries of $S(\mu)$ can be expressed in terms only related to the matrix $\mathcal{S}(\mu)$. In particular, this enables us to prove the existence of finite limits $\lim S(\mu)$ as $\mu\to\tau\pm0$, to calculate the limits, and in essence to reduce the approximation of the matrix $S(\mu)$ with $\mu\in[\mu',\mu'']$ to that of the augmented matrix $\mathcal{S}(\mu)$. As an approximation to a row of $\mathcal{S}(\mu)$, we take a minimizer of a quadratic functional $\mathcal{J}^R(\cdot,\mu)$. To construct such a functional we use a boundary value problem in the bounded domain G^R obtained from G by cutting off the cylindrical ends at a distance R. We set

$$\begin{gathered}\Pi_+^{r,R}=\{(y^r,t^r)\in\Pi^r:t^r>R\},\quad G^R=G\setminus\cup_{r=1}^{\mathcal{T}}\Pi_+^{r,R},\\ \partial G^R\setminus\partial G=\Gamma^R=\cup_r\Gamma^{r,R},\quad \Gamma^{r,R}=\{(y^r,t^r)\in\Pi^r:t^r=R\}\end{gathered}$$

for a large R and introduce the boundary value problem

$$\begin{aligned}-\Delta\mathcal{X}_j^R-\mu\mathcal{X}_j^R&=0,\quad x\in G^R;\\ \mathcal{X}_j^R&=0\quad x\in\partial G^R\setminus\Gamma^R;\\ (-\partial_n+i\zeta)\mathcal{X}_j^R&=(-\partial_n+i\zeta)(w_j^++\textstyle\sum_{k=1}^M a_kw_k^-),\ x\in\Gamma^R,\end{aligned}$$

where $w_j^\pm$ is a stable basis in the space of waves, $\zeta\in\mathbb{R}\setminus\{0\}$ is an arbitrary fixed number, and a_k are complex numbers. As an approximation to the row $(\mathcal{S}_{j1}(\mu),\dots,\mathcal{S}_{jM}(\mu))$, we take the minimizer $a^0(R,\mu)=(a_1^0(R,\mu),\dots,a_M^0(R,\mu))$ of the functional

$$\mathcal{J}_j^R(a_1,\dots,a_M)=\|\mathcal{X}_j^R(\cdot,\mu)-w_j^+(\cdot,\mu)-\sum_{k=1}^M a_kw_k^-(\cdot,\mu);L_2(\Gamma^R)\|^2,$$

where $\mathcal{X}_j^R(\cdot,\mu)$ is a solution to the boundary value problem. If $\tau\in[\mu',\mu'']\subset(\tau',\tau'')$, then the inequality

$$\|a(R,\mu)-\mathcal{S}_j(\mu)\|\leqslant C(\Lambda)e^{-\Lambda R}$$

holds for all $\mu\in[\mu',\mu'']$ and $R\geqslant R_0$ with positive constants Λ and $C(\Lambda)$ independent of μ and R.

2 ELLIPTIC EXTENSION OF THE MAXWELL SYSTEM. THE SCATTERING MATRIX AND THE RADIATION PRINCIPLE

In this Chapter we introduce an extended (elliptic) Maxwell system and specify the results established by Nazarov et al. [25] for the general elliptic problems, self-adjoint with respect to a Green formula.

The extended Maxwell system and the boundary value problem for the system in the domain G are introduced in Section 2.1. In Section 2.2 we consider the problem in a cylinder $\Omega \times \mathbb{R}$ and associate with the problem an operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ on functions in Ω, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary. Here we investigate in detail spectral properties of the pencil and make up Jordan chains of eigenvectors and generalized eigenvectors, which satisfy special orthogonality and normalization conditions. Moreover, in Sections 2.3 and 2.4 we introduce the space of waves, describe the continuous spectrum eigenfunctions, and define the unitary scattering matrix. Finally, we present a well-posed problem with intrinsic radiation conditions in Section 2.5.

2.1 Extended Maxwell system

The system (1.2.1) is overdetermined. The compatibility conditions needed for the solvability of problem (1.2.1), (1.2.2) are described by Proposition 2.1.1. Let us recall the formulas

$$(\operatorname{rot} u, v)_G = -(u, \nu \times v)_{\partial G} + (u, \operatorname{rot} v)_G, \tag{2.1.1}$$

$$(\nabla \alpha, u)_G = (\alpha, \langle u, \nu \rangle)_{\partial G} - (\alpha, \operatorname{div} u)_G, \tag{2.1.2}$$

where α is a scalar function, u and v are three dimensional vector functions, while $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_{\partial G}$ in (2.1.1) and (2.1.2) on the left denote the inner products on $L_2(G, \mathbb{C}^3)$ and $L_2(\partial G, \mathbb{C}^3)$, and on the right hand-side of (2.1.2) is the inner product on $L_2(G, \mathbb{C})$ and $L_2(\partial G, \mathbb{C})$. As a rule, we further denote by $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_{\partial G}$

the inner products on $L_2(G,\mathbb{C}^p)$ and $L_2(\partial G,\mathbb{C}^p)$ for various p; the context excludes misunderstanding.

Proposition 2.1.1. *Assume that* (1.2.1) *and* (1.2.2) *hold for some sufficiently smooth functions* u^j, f^j, h^j, $j = 1,2$. *Then*

$$\operatorname{div} f^1(x) - ikh^2(x) = 0, \ x \in G, \tag{2.1.3}$$

$$\operatorname{div} f^2(x) + ikh^1(x) = 0, \ x \in G, \tag{2.1.4}$$

$$\langle f^2(x), \nu(x)\rangle = 0, \ x \in \partial G. \tag{2.1.5}$$

Proof. We apply div to the first equality (1.2.1) and obtain $\operatorname{div} f^1 + k \operatorname{div} u^1 = 0$. Since $i \operatorname{div} u^1 = h^2$, we arrive at (2.1.3). In a similar way, we can establish (2.1.4). For (2.1.5), it suffices to verify $(f^2, \nabla\psi)_G = -(\operatorname{div} f^2, \psi)_G$ for all $\psi \in C_c^\infty(\overline{G})$ (see (2.1.2)). We have

$$(f^2, \nabla\psi)_G = -i(\operatorname{rot} u^1, \nabla\psi)_G - k(u^2, \nabla\psi)_G. \tag{2.1.6}$$

From (1.2.2) and (2.1.2), it follows that

$$(u^2, \nabla\psi)_G = -(\operatorname{div} u^2, \psi)_G.$$

Moreover, (2.1.1) and boundary condition (1.2.2) lead to

$$(\operatorname{rot} u^1, \nabla\psi)_G = (u^1, \operatorname{rot} \nabla\psi)_G + (\nabla\psi, \nu \times u^1)_{\partial G} = 0.$$

Therefore, (2.1.6) implies

$$(f^2, \nabla\psi)_G = k(\operatorname{div} u^2, \psi)_G = -(\operatorname{div} f^2, \psi)_G. \ \square$$

We now pass on to the "orthogonal extension" of system (1.2.1) (cf [15]). Namely, in the domain G we introduce the boundary value problem for the "augmented Maxwell system":

$$\begin{array}{rcl} i \operatorname{rot} u^2(x) + i\,\nabla a^2(x) - ku^1(x) &=& f^1(x), \\ -i \operatorname{div} u^2(x) - ka^1(x) &=& h^1(x), \\ -i \operatorname{rot} u^1(x) - i\,\nabla a^1(x) - ku^2(x) &=& f^2(x), \\ i \operatorname{div} u^1(x) - ka^2(x) &=& h^2(x) \end{array} \tag{2.1.7}$$

with boundary conditions

$$\begin{array}{lll} -\langle u^1(x), \tau_2(x)\rangle = g^1(x), & \langle u^1(x), \tau_1(x)\rangle = g^2(x), & \\ \langle u^2(x), \nu(x)\rangle = g^3(x), & a^2(x) = g^4(x), & x \in \partial G; \end{array} \tag{2.1.8}$$

where u^1 and u^2 are three dimensional vector-valued functions, a^1, a^2 are for scalar functions in G. Moreover, $\tau_1(x), \tau_2(x)$ are tangent vectors and $\nu(x)$ is outward normal to ∂G, the vectors $\tau_1(x), \tau_2(x), \nu(x)$ form a right-hand triple. The next assertion can be verified by immediate calculation.

Proposition 2.1.2. *Problem* (2.1.7), (2.1.8) *is elliptic.*

Rewrite (2.1.7), (2.1.8) in the form

$$\begin{aligned} \mathcal{A}(D,k)\mathcal{U}(x) &= \mathcal{F}(x),\ x \in G, \\ \mathcal{B}\mathcal{U}(x) &= \mathcal{G}(x),\ x \in \partial G, \end{aligned} \tag{2.1.9}$$

where $D = (D_1, D_2, D_3)$, $D_j = -i\partial/\partial x_j, \mathcal{U} = (u^1, a^1, u^2, a^2)$. The next proposition follows from (2.1.1) and (2.1.2).

Proposition 2.1.3. *The Green formula holds for*

$$(\mathcal{A}(D,k)\mathcal{U},\mathcal{V})_G + (\mathcal{B}\mathcal{U},\mathcal{Q}\mathcal{V})_{\partial G} = (\mathcal{U},\mathcal{A}(D,k)\mathcal{V})_G + (\mathcal{Q}\mathcal{U},\mathcal{B}\mathcal{V})_{\partial G} \tag{2.1.10}$$

with $\mathcal{U} = (u^1, a^1, u^2, a^2)$, $\mathcal{V} = (v^1, b^1, v^2, b^2)$, *where* a^j, b^j *are in* $C_c^\infty(\bar{G};\mathbb{C})$, u^j, v^j *are in* $C_c^\infty(\bar{G};\mathbb{C}^3)$, *and*

$$\begin{aligned} &\mathcal{B}\mathcal{U} = (-\langle u^1, \tau_2\rangle, \langle u^1, \tau_1\rangle, \langle u^2, \nu\rangle, a^2), \\ &\mathcal{Q}\mathcal{V} = -i(\langle v^2, \tau_1\rangle, \langle v^2, \tau_2\rangle, b^1, -\langle v^1, \nu\rangle). \end{aligned} \tag{2.1.11}$$

The operator of problem (2.1.9) *is self-adjoint, with respect to Green formula* (2.1.10).

Since k remains fixed, we will frequently drop it from future notations.

2.2 Elliptic and Maxwell operator pencils

Let us consider the operator $\{\mathcal{A}(D), \mathcal{B}\}$ of problem (2.1.9) in the cylinder $\Omega \times \mathbb{R} = \{x = (x_1, x_2, x_3) : (x_1, x_2) \in \Omega, x_3 \in \mathbb{R}\}$, where Ω is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Assume that $\Phi = (\varphi, \alpha, \psi, \beta)$ with components φ, ψ from $C^\infty(\bar{\Omega};\mathbb{C}^3)$ and α, β in $C^\infty(\bar{\Omega};\mathbb{C})$ satisfies the boundary condition

$$\mathcal{B}\Phi = (\nu \times \varphi, \langle \psi, \nu\rangle, \beta) = 0 \tag{2.2.1}$$

on $\partial\Omega$, where ν is the outward normal to $\partial\Omega$. We consider that ν is a three dimensional vector of the form $(\nu_1, \nu_2, 0)$. Denoting by u_τ and u_ν the tangent and normal components of u on $\partial\Omega$, we rewrite (2.2.1) in the form

$$\varphi_\tau = 0,\ \psi_\nu = 0,\ \beta|\partial\Omega = 0.$$

For the vectors Φ with above properties, we introduce the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ by

$$\mathfrak{A}(\lambda)\Phi(x_1, x_2) = \exp(-i\lambda x_3)\mathcal{A}(D)(\exp(i\lambda x_3)\Phi(x_1, x_2)). \tag{2.2.2}$$

For the usual operators ∇, rot, div, and Δ in $\Omega \times \mathbb{R}$, let us define in Ω the operators $\nabla(\lambda)$, $\mathrm{rot}(\lambda)$, $\mathrm{div}(\lambda)$, $\Delta(\lambda)$ by

$$\begin{aligned} \nabla(\lambda)\alpha(x_1, x_2) &= \exp(-i\lambda x_3)\nabla(\exp(i\lambda x_3)\alpha(x_1, x_2)), \\ \mathrm{rot}(\lambda)\varphi(x_1, x_2) &= \exp(-i\lambda x_3)\mathrm{rot}(\exp(i\lambda x_3)\varphi(x_1, x_2)), \end{aligned}$$

etc. The formulas for the usual operators can immediately be extended to the operations with parameter. For instance, from the equality rot rot $= \nabla \operatorname{div} - \Delta$ it follows that $\operatorname{rot}(\lambda)\operatorname{rot}(\lambda) = \nabla(\lambda)\operatorname{div}(\lambda) - \Delta(\lambda)$. For φ, ψ in $C^\infty(\bar{\Omega};\mathbb{C}^3)$ and α in $C^\infty(\bar{\Omega};\mathbb{C})$, we have

$$\begin{aligned} (\nabla(\lambda)\alpha, \varphi)_\Omega &= (\alpha, \langle\varphi, \nu\rangle)_{\partial\Omega} - (\alpha, \operatorname{div}(\bar{\lambda})\varphi)_\Omega, && (2.2.3)\\ (\operatorname{rot}(\lambda)\varphi, \psi)_\Omega &= (\varphi, \psi \times \nu)_{\partial\Omega} + (\varphi, \operatorname{rot}(\bar{\lambda})\psi)_\Omega, && (2.2.4) \end{aligned}$$

ν being the outward normal to $\partial\Omega$.

Denote by $H^l(\Omega;\mathbb{C}^8)$, $l = 0, 1, \dots$ the space of vectors with eight components in the Sobolev space $H^l(\Omega;\mathbb{C})$ of complex functions in Ω. The elements $\Phi \in H^l(\Omega;\mathbb{C}^8)$ will be written as $\Phi = (\varphi, \alpha, \psi, \beta)$, where $\varphi, \psi \in H^l(\Omega;\mathbb{C}^3)$ and $\alpha, \beta \in H^l(\Omega;\mathbb{C})$. For $l = 1, 2, \dots$ we set

$$\mathcal{D}H^l(\Omega) = \{\Phi \in H^l(\Omega;\mathbb{C}^8) : \varphi_\tau = 0, \psi_\nu = 0, \beta|\partial\Omega = 0\}. \qquad (2.2.5)$$

Let us consider the operator $\mathfrak{A}(\lambda)$ given by (2.2.2) on the domain $\mathcal{D}H^l(\Omega)$. According to the general theory of elliptic operator pencils (see [1]), for $\lambda \in \mathbb{C}$ except some isolated points the mapping $\mathfrak{A}(\lambda) : \mathcal{D}H^l(\Omega) \to H^{l-1}(\Omega;\mathbb{C}^8)$ is an isomorphism. The mentioned isolated values are the eigenvalues of the pencil $\lambda \mapsto \mathfrak{A}(\lambda)$ of finite algebraic multiplicity. The components of eigenvectors and generalized eigenvectors are smooth functions in $\overline{\Omega}$. For $\mathcal{U} = (\varphi, \alpha, \psi, \beta) \in \mathcal{D}H^l(\Omega)$, from (2.1.7) and (2.2.2) it follows that

$$\mathfrak{A}(\lambda) : \begin{pmatrix} \varphi \\ \alpha \\ \psi \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} i\operatorname{rot}(\lambda)\,\psi + i\,\nabla(\lambda)\beta - k\varphi \\ -i\operatorname{div}(\lambda)\,\psi - k\alpha \\ -i\operatorname{rot}(\lambda)\,\varphi - i\,\nabla(\lambda)\alpha - k\psi \\ i\operatorname{div}(\lambda)\,\varphi - k\beta \end{pmatrix}. \qquad (2.2.6)$$

The pencil $\mathfrak{A}$ is called elliptic and its restriction to the set $\{\mathcal{U} \in \mathcal{D}H^l(\Omega) : \mathcal{U} = (\varphi, 0, \psi, 0)\}$ is called Maxwell and denoted by $\mathfrak{M}$. In other words, $\mathfrak{M}$ is the operator pencil defined for the problem (1.2.1), (1.2.2). The pencils $\mathfrak{A}$ and $\mathfrak{M}$ depend on the two parameters k and λ. In what follows k remains fixed while λ plays the role of a spectral parameter. A number λ_0 is an eigenvalue of the pencil $\mathfrak{M}(\cdot, k)$ if there exists a smooth nonzero vector $\Phi = (\varphi, 0, \psi, 0)$, that is subject to the boundary conditions $\varphi_\tau = 0, \psi_\nu = 0$ on $\partial\Omega$ and satisfies the equation $\mathfrak{M}(\lambda_0, k)\Phi = 0$ (and therefore the equation $\mathfrak{A}(\lambda_0, k)\Phi = 0$).

2.2.1 Eigenvalues and eigenvectors of the pencils $\mathfrak{A}$ and $\mathfrak{M}$

In this Section, when describing the eigenvalues and eigenvectors of the elliptic pencil $\mathfrak{A}$, we modify the approach known for the Maxwell pencil $\mathfrak{M}$ (for example see [37]). In the domain Ω we consider the system

$$\begin{aligned} i\operatorname{rot}(\lambda)\,\psi + i\,\nabla(\lambda)\beta - k\varphi = 0,\\ -i\operatorname{div}(\lambda)\,\psi - k\alpha = 0,\\ -i\operatorname{rot}(\lambda)\,\varphi - i\,\nabla(\lambda)\alpha - k\psi = 0,\\ i\operatorname{div}(\lambda)\,\varphi - k\beta = 0. \end{aligned} \qquad (2.2.7)$$

Apply $\mathrm{rot}(\lambda)$ to the rot-equations in (2.2.7). Since $\mathrm{rot}(\lambda)\mathrm{rot}(\lambda) = \nabla(\lambda)\mathrm{div}(\lambda) - \Delta(\lambda)$, we obtain

$$i\,\nabla(\lambda)\mathrm{div}(\lambda)\psi - i\,\Delta(\lambda)\psi = k\,\mathrm{rot}(\lambda)\varphi, \tag{2.2.8}$$

$$-i\,\nabla(\lambda)\mathrm{div}(\lambda)\varphi + i\,\Delta(\lambda)\varphi = k\,\mathrm{rot}(\lambda)\psi. \tag{2.2.9}$$

Taking account of the second and the third equations (2.2.7), rewrite (2.2.8) in the form of

$$\Delta(\lambda)\psi + k^2\psi = 0. \tag{2.2.10}$$

From (2.2.9), the first and the fourth equations in (2.2.7) it follows that

$$\Delta(\lambda)\varphi + k^2\varphi = 0. \tag{2.2.11}$$

Let us turn to α and β. Applying $\mathrm{div}(\lambda)$ to the first equation in (2.2.7), we obtain

$$i\,\mathrm{div}(\lambda)\nabla(\lambda)\beta - k\,\mathrm{div}(\lambda)\varphi = 0.$$

By virtue of the fourth equation in (2.2.7), the last equation can be rewritten as

$$\Delta(\lambda)\beta + k^2\beta = 0. \tag{2.2.12}$$

In a similar way, we derive

$$\Delta(\lambda)\alpha + k^2\alpha = 0. \tag{2.2.13}$$

Denote by φ_j and ψ_j the components of φ and ψ respectively, $j = 1, 2, 3$, and write down the rot-equations (2.2.7) through their projections onto the coordinate axes. From the resulting equations we choose two pairs in order to form two linear algebraic systems. In one of the system

$$\begin{aligned} k\psi_1 + \lambda\varphi_2 &= -i\partial_2\varphi_3 - i\partial_1\alpha, \\ \lambda\psi_1 + k\varphi_2 &= -i\partial_1\psi_3 + i\partial_2\beta, \end{aligned} \tag{2.2.14}$$

where $\partial_q = \partial/\partial x_q$, the role of unknowns is played by ψ_1 and φ_2. In the other system

$$\begin{aligned} \lambda\psi_2 - k\varphi_1 &= -i\partial_2\psi_3 - i\partial_1\beta, \\ k\psi_2 - \lambda\varphi_1 &= i\partial_1\varphi_3 - i\partial_2\alpha, \end{aligned} \tag{2.2.15}$$

we take ψ_2 and φ_1 as unknowns. Assuming that $k^2 - \lambda^2 \neq 0$ and solving both of these systems, we obtain on Ω the relations

$$\begin{aligned} \varphi_1 &= (k^2 - \lambda^2)^{-1}[i\lambda\partial_1\varphi_3 + ik\partial_2\psi_3 - i\lambda\partial_2\alpha + ik\partial_1\beta], \\ \varphi_2 &= (k^2 - \lambda^2)^{-1}[i\lambda\partial_2\varphi_3 - ik\partial_1\psi_3 + i\lambda\partial_1\alpha + ik\partial_2\beta], \\ \psi_1 &= (k^2 - \lambda^2)^{-1}[-ik\partial_2\varphi_3 + i\lambda\partial_1\psi_3 - ik\partial_1\alpha - i\lambda\partial_2\beta], \\ \psi_2 &= (k^2 - \lambda^2)^{-1}[ik\partial_1\varphi_3 + i\lambda\partial_2\psi_3 - ik\partial_2\alpha + i\lambda\partial_1\beta]. \end{aligned} \tag{2.2.16}$$

Thus, if $(\varphi, \alpha, \psi, \beta)$ satisfies (2.2.7), then every vector-valued function φ, ψ, and every scalar function α, β are solutions to the Helmholtz equation in Ω, see (2.2.10), (2.2.11) and (2.2.12), (2.2.13). Moreover, φ_1, φ_2, ψ_1, and ψ_2 are expressed by (2.2.16) through φ_3, ψ_3, α, β.

The immediate task is to derive from (2.2.1) (see also (2.2.5)) some boundary conditions on $\partial\Omega$ for the last four functions. Conditions (2.2.1) are equivalent to the system of the following four equalities on $\partial\Omega$:

$$\varphi_3 = 0, \qquad \beta = 0, \tag{2.2.17}$$

$$\nu_1\varphi_2 - \nu_2\varphi_1 = 0, \qquad \nu_1\psi_1 + \nu_2\psi_2 = 0. \tag{2.2.18}$$

Taking into account (2.2.16), we rewrite (2.2.18) in the form

$$ik[(\nu_1\partial_2 - \nu_2\partial_1)\varphi_3 + (\nu_1\partial_1 + \nu_2\partial_2)\alpha] - i\lambda[(\nu_1\partial_1 + \nu_2\partial_2)\psi_3 - (\nu_1\partial_2 - \nu_2\partial_1)\beta] = 0,$$
$$-i\lambda[(\nu_1\partial_2 - \nu_2\partial_1)\varphi_3 + (\nu_1\partial_1 + \nu_2\partial_2)\alpha] + ik[(\nu_1\partial_1 + \nu_2\partial_2)\psi_3 - (\nu_1\partial_2 - \nu_2\partial_1)\beta] = 0.$$

By assumption, $k^2 - \lambda^2 \neq 0$. Therefore the two last equalities imply that

$$\begin{aligned} (\nu_1\partial_2 - \nu_2\partial_1)\varphi_3 + (\nu_1\partial_1 + \nu_2\partial_2)\alpha &= 0, \\ (\nu_1\partial_1 + \nu_2\partial_2)\psi_3 - (\nu_1\partial_2 - \nu_2\partial_1)\beta &= 0. \end{aligned}$$

Besides, in view of (2.2.17)

$$(\nu_1\partial_2 - \nu_2\partial_1)\varphi_3 = 0, \qquad (\nu_1\partial_2 - \nu_2\partial_1)\beta = 0.$$

Therefore, on $\partial\Omega$

$$\partial_\nu\psi_3 = (\nu_1\partial_1 + \nu_2\partial_2)\psi_3 = 0, \qquad \partial_\nu\alpha = (\nu_1\partial_1 + \nu_2\partial_2)\alpha = 0.$$

We have proved the following

Proposition 2.2.1. *Let λ be an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$ and $(\varphi, \alpha, \psi, \beta)$ a corresponding eigenvector, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\psi = (\psi_1, \psi_2, \psi_3)$. Moreover, assume that $k^2 - \lambda^2 \neq 0$. Then*

$$\Delta(\lambda)\alpha + k^2\alpha = 0 \text{ in } \Omega, \qquad \partial_\nu\alpha = 0 \text{ on } \partial\Omega, \tag{2.2.19}$$
$$\Delta(\lambda)\beta + k^2\beta = 0 \text{ in } \Omega, \qquad \beta = 0 \text{ on } \partial\Omega, \tag{2.2.20}$$
$$\Delta(\lambda)\varphi_3 + k^2\varphi_3 = 0 \text{ in } \Omega, \qquad \varphi_3 = 0 \text{ on } \partial\Omega, \tag{2.2.21}$$
$$\Delta(\lambda)\psi_3 + k^2\psi_3 = 0 \text{ in } \Omega, \qquad \partial_\nu\psi_3 = 0 \text{ on } \partial\Omega, \tag{2.2.22}$$

while φ_j, ψ_j for $j = 1, 2$ are defined by (2.2.16). *Conversely, any nonzero vector $(\varphi, \alpha, \psi, \beta)$ whose component satisfy* (2.2.19) – (2.2.22) *and* (2.2.16) *is an eigenvector of the pencil $\mathfrak{A}(\cdot, k)$ corresponding to λ.*

If λ (such that $k^2 - \lambda^2 \neq 0$) is an eigenvalue for one of the pencils $\mathfrak{A}(\cdot, k)$ and $\mathfrak{M}(\cdot, k)$, then it is an eigenvalue for the other pencil as well. A number λ turns out to be an eigenvalue of these pencils if and only if λ is an eigenvalue for at least one of problems (2.2.19) and (2.2.20). *The equalities*

$$\varkappa_{\mathfrak{A}}(\lambda, k) = 2\varkappa_{\mathfrak{M}}(\lambda, k) = 2\varkappa_{\mathcal{D}}(\lambda, k) + 2\varkappa_{\mathcal{N}}(\lambda, k),$$

hold, where $\varkappa_{\mathfrak{A}}(\lambda, k)$ and $\varkappa_{\mathfrak{M}}(\lambda, k)$ are the geometric multiplicities of λ for the pencils $\mathfrak{A}(\cdot, k)$ and $\mathfrak{M}(\cdot, k)$, while $\varkappa_{\mathcal{N}}(\lambda, k)$ and $\varkappa_{\mathcal{D}}(\lambda, k)$ are those for problems (2.2.19) *and* (2.2.20).

We are now coming to describing the eigenvalues and eigenvectors under the condition $k^2 - \lambda^2 = 0$. Formulas (2.2.10), (2.2.11), (2.2.12), and (2.2.13) take the form

$$\Delta\psi = 0, \qquad \Delta\varphi = 0, \qquad \Delta\beta = 0, \qquad \Delta\alpha = 0, \tag{2.2.23}$$

where $\Delta := \Delta(0) = \partial_1^2 + \partial_2^2$. The second and third equalities in (2.2.23), together with (2.2.17), give us

$$\varphi_3 = 0, \qquad \beta = 0 \qquad \text{on } \Omega. \tag{2.2.24}$$

In what follows, we consider the cases of $\lambda^2 = k^2 \neq 0$ and $\lambda^2 = k^2 = 0$ separately.

First we consider $\lambda^2 = k^2 \neq 0$. From (2.2.14) and (2.2.24), we derive $\partial_1(\psi_3 - (\lambda/k)\alpha) = 0$, while (2.2.15) implies $\partial_2(\psi_3 - (\lambda/k)\alpha) = 0$. That is

$$\psi_3 - (\lambda/k)\alpha = C_0 = \text{const}.$$

Let us show, that $C_0 = 0$. For the function $u(x) = \exp{(i\lambda x_3)}\varphi(x_1, x_2)$, we write the Stokes formula

$$\int_\Omega (\operatorname{rot} u)_n \, dx_1 dx_2 = \int_{\partial\Omega} u_\tau \, ds, \tag{2.2.25}$$

where $(\operatorname{rot} u)_n$ is the projection of $\operatorname{rot} u$ onto the vector $n = (0, 0, 1)$, directed along the axis of the cylinder $\Omega \times \mathbb{R}$. From (2.2.25), it follows that

$$\int_\Omega (\operatorname{rot}(\lambda)\, \varphi)_n \, dx_1 dx_2 = \int_{\partial\Omega} \varphi_\tau \, ds = 0 \tag{2.2.26}$$

because $\varphi_\tau = 0$. We project the second rot-equation (2.2.7) onto the axis x_3:

$$(\operatorname{rot}(\lambda)\, \varphi)_n - ik(\psi_3 - (\lambda/k)\alpha) = 0,$$

integrate this equality over Ω, take account of (2.2.26), and arrive at $C_0 = 0$. Thus,

$$\psi_3 - (\lambda/k)\alpha = 0 \qquad \text{on } \Omega. \tag{2.2.27}$$

Let us obtain the boundary conditions for α. We use the second equation of (2.2.14) and the first equation of (2.2.15), which in view of (2.2.24) take the form

$$k\psi_1 + \lambda\varphi_2 = -i\partial_1\alpha, \qquad k\psi_2 - \lambda\varphi_1 = -i\partial_2\alpha.$$

This and (2.2.18) lead to

$$i\partial_\nu\alpha = -k(\nu_1\psi_1 + \nu_2\psi_2) - \lambda(\nu_1\varphi_2 - \nu_2\varphi_1) = 0 \qquad \text{on } \partial\Omega. \tag{2.2.28}$$

We now turn to φ_j, ψ_j, $j = 1, 2$. Consider the third components of the rot-equations and the div-equations (2.2.7)

$$\begin{aligned} &i(\partial_1\psi_2 - \partial_2\psi_1) + i(i\lambda)\beta - k\varphi_3 = 0, && i(\partial_1\varphi_2 - \partial_2\varphi_1) + i(i\lambda)\alpha + k\psi_3 = 0, \\ &-i(\partial_1\psi_1 + \partial_2\psi_2 + i\lambda\psi_3) - k\alpha = 0, && i(\partial_1\varphi_1 + \partial_2\varphi_2 + i\lambda\varphi_3) - k\beta = 0, \end{aligned}$$

which, in accordance with (2.2.24) and (2.2.27), take the form

$$\begin{aligned} &\partial_1\psi_2 - \partial_2\psi_1 = 0, && \partial_1\varphi_2 - \partial_2\varphi_1 = 0, \\ &\partial_1\psi_1 + \partial_2\psi_2 = 0, && \partial_1\varphi_1 + \partial_2\varphi_2 = 0. \end{aligned} \tag{2.2.29}$$

We assume that Ω is a 1-connected domain. According to the first and fourth equalities of (2.2.29), there exist (one-valued) potentials H and G in Ω, such that

$$\varphi_1 = \partial_2 H, \qquad \varphi_2 = -\partial_1 H$$
$$\psi_j = \partial_j G, \qquad j = 1, 2. \tag{2.2.30}$$

By virtue of the other two equalities in (2.2.29), we have

$$\Delta H = 0, \qquad \Delta G = 0 \qquad \text{on } \Omega. \tag{2.2.31}$$

Substituting the expressions (2.2.30) into (2.2.18), we obtain the boundary conditions for H and G:

$$\partial_\nu H = 0, \qquad \partial_\nu G = 0 \qquad \text{on } \partial\Omega. \tag{2.2.32}$$

Therefore $H = \text{const}$ and $G = \text{const}$ in Ω and in view of (2.2.30)

$$\varphi_1 = \varphi_2 = 0, \qquad \psi_1 = \psi_2 = 0 \qquad \text{on } \Omega. \tag{2.2.33}$$

Now we consider the case $\lambda^2 = k^2 = 0$. From the second equation of (2.2.14) we obtain $\partial_1 \alpha = 0$, while the first equation of (2.2.15) implies $\partial_2 \alpha = 0$, that is $\alpha = \text{const}$. Moreover, according to the first equation of (2.2.14) we get $\partial_1 \psi_3 = 0$, and from the second equation of (2.2.15) we get $\partial_2 \psi_3 = 0$, thus $\psi_3 = \text{const}$. As before, we have (2.2.33).

Summarizing the above discussion on the case $k^2 - \lambda^2 = 0$, we obtain the following

Proposition 2.2.2. *Let Ω be a 1-connected domain. Then the following assertions are valid:*

1. If $\lambda^2 = k^2 \neq 0$, then λ is an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$. The corresponding eigenspace is one-dimensional and spanned by the vector $\Phi = (\varphi, \alpha, \psi, \beta)$ with components

$$\varphi = 0, \qquad \alpha = \text{const} \neq 0, \qquad \psi_1 = \psi_2 = 0, \qquad \psi_3 = (\lambda/k)\alpha, \qquad \beta = 0.$$

The vector Φ does not belong to the domain of the pencil $\mathfrak{M}(\cdot, k)$, and the number λ is not an eigenvalue of $\mathfrak{M}(\cdot, k)$.

2. For $k = 0$ the number $\lambda = 0$ is an eigenvalue of $\mathfrak{A}(\cdot, k)$. The corresponding eigenspace is spanned by the vectors $\widehat{\Phi} = (\hat{\varphi}, \hat{\alpha}, \hat{\psi}, \hat{\beta})$ and $\widetilde{\Phi} = (\tilde{\varphi}, \tilde{\alpha}, \tilde{\psi}, \tilde{\beta})$, where

$$\hat{\varphi} = 0, \qquad \hat{\alpha} = \text{const} \neq 0, \qquad \hat{\psi} = 0, \qquad \hat{\beta} = 0,$$

$$\tilde{\varphi} = 0, \qquad \tilde{\alpha} = 0, \qquad \tilde{\psi}_1 = \tilde{\psi}_2 = 0, \qquad \tilde{\psi}_3 = \text{const} \neq 0, \qquad \tilde{\beta} = 0.$$

The vector $\widehat{\Phi}$ does not belong to the domain of $\mathfrak{M}(\cdot, k)$, while $\widetilde{\Phi}$ is an eigenvector of $\mathfrak{M}(\cdot, k)$. Thus to the eigenvalue $\lambda = 0$ of $\mathfrak{M}(\cdot, 0)$ there corresponds the eigenspace spanned by the vector $\widetilde{\Phi}$.

2.2.2 Generalized eigenvectors

We first recall some definitions in the general theory of operator pencils (e.g. see [13]). Let $\lambda \mapsto \mathfrak{A}(\lambda)$ be an elliptic operator pencil. An ordered collection of vectors $\Phi^0, \dots, \Phi^{\kappa-1}$ is called a Jordan chain corresponding to an eigenvalue λ_0, if

$$\sum_{q=0}^{\nu} \frac{1}{q!} \partial_\lambda^q \mathfrak{A}(\lambda_0) \Phi^{\nu-q} = 0, \qquad \nu = 0, \dots, \kappa - 1.$$

It is clear that Φ^0 is an eigenvector. The elements $\Phi^1, \dots, \Phi^{\kappa-1}$ are called generalized eigenvectors and κ is the length of the chain. The rank of an eigenvector Φ^0 (rank Φ^0) is the maximal length of all Jordan chains, with the same eigenvector Φ^0. Any eigenvalue λ_0 is isolated, and $\dim\ker \mathfrak{A}(\lambda_0) < \infty$, and the ranks of all eigenvectors are finite.

Let $J = \dim\ker \mathfrak{A}(\lambda_0)$ and let $\Phi^{0,1}, \dots, \Phi^{0,J}$ be a system of vectors, such that rank $\Phi^{0,1}$ is maximal among the ranks of all eigenvectors corresponding to λ_0, and rank $\Phi^{0,j}$, where $j = 2, \dots, J$, is maximal among the ranks of all eigenvectors in a direct complement in $\ker\mathfrak{A}(\lambda_0)$ to the linear span $\mathcal{L}(\Phi^{0,1}, \dots, \Phi^{0,j-1})$. The numbers $\kappa_j = \operatorname{rank}\Phi^{0,j}$ are called the partial multiplicities of the eigenvalue λ_0, and the sum $\kappa_1 + \cdots + \kappa_J$ is called the (total) multiplicity of λ_0. If for each $j = 1, \dots, J$ the vectors $\Phi^{0,j}, \Phi^{1,j}, \dots, \Phi^{\kappa_j-1,j}$ form a Jordan chain, then the system of vectors $\{\Phi^{0,j}, \Phi^{1,j}, \dots, \Phi^{\kappa_j-1,j} : j = 1, \dots, J\}$ is called a canonical system of Jordan chains corresponding to the eigenvalue λ_0.

Generalized eigenvectors of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, k)$ exist only for some isolated values the parameter $k \in \mathbb{R}$. Such values are called the thresholds. In the present thesis, we use some information about the Jordan chains to define the scattering matrix and to prove the basic theorems for the thresholds as well. Next we will describe the generalized eigenvectors of the elliptic pencil $\lambda \mapsto \mathfrak{A}(\lambda, k)$ given by (2.2.6). Let $\Phi^0 = (\varphi^0, \alpha^0, \psi^0, \beta^0)$ be an eigenvector corresponding to an eigenvalue λ_0 of the pencil $\mathfrak{A}(\cdot, k)$. A generalized eigenvector Φ^1 is a solution to the equation

$$\mathfrak{A}(\lambda_0, k)\Phi^1 + \partial_\lambda \mathfrak{A}(\lambda_0, k)\Phi^0 = 0. \tag{2.2.34}$$

If such a solution exists, then it is determined up to adding any eigenvector corresponding to λ_0. We set $\Phi^1 = (\varphi^1, \alpha^1, \psi^1, \beta^1)$ and rewrite the equation (2.2.34) more elaborately

$$\begin{aligned} i \operatorname{rot}(\lambda_0)\, \psi^1 + i\, \nabla(\lambda_0)\beta^1 - k\varphi^1 + i\, \partial_\lambda \operatorname{rot}(\lambda_0)\, \psi^0 + i\, \partial_\lambda \nabla(\lambda_0)\beta^0 &= 0, \\ -i \operatorname{div}(\lambda_0)\, \psi^1 - k\alpha^1 - i\, \partial_\lambda \operatorname{div}(\lambda_0)\, \psi^0 &= 0, \\ -i \operatorname{rot}(\lambda_0)\, \varphi^1 - i\, \nabla(\lambda_0)\alpha^1 - k\psi^1 - i\, \partial_\lambda \operatorname{rot}(\lambda_0)\, \varphi^0 - i\, \partial_\lambda \nabla(\lambda_0)\alpha^0 &= 0, \\ i \operatorname{div}(\lambda_0)\, \varphi^1 - k\beta^1 + i\, \partial_\lambda \operatorname{div}(\lambda_0)\, \varphi^0 &= 0. \end{aligned} \tag{2.2.35}$$

Notice that

$$\begin{aligned} &\partial_\lambda \operatorname{rot}(\lambda_0)\, \varphi = (-i\varphi_2, i\varphi_1, 0), \\ &\partial_\lambda \operatorname{div}(\lambda_0)\, \varphi = i\varphi_3, \\ &\partial_\lambda \nabla(\lambda_0)\alpha = (0, 0, i\alpha), \end{aligned} \tag{2.2.36}$$

where φ is a vector function with components $(\varphi_1, \varphi_2, \varphi_3)$ and α is a scalar function. As in the preceding Section, we apply $\mathrm{rot}(\lambda_0)$ to the rot-equations in (2.2.35). Taking into account (2.2.36), (2.2.7), and the second and third equations in (2.2.35), we obtain

$$\Delta(\lambda_0)\psi^1 + k^2\psi^1 - 2\lambda_0\psi^0 = 0. \tag{2.2.37}$$

In a similar way we derive

$$\Delta(\lambda_0)\varphi^1 + k^2\varphi^1 - 2\lambda_0\varphi^0 = 0. \tag{2.2.38}$$

Let us turn to α^1 and β^1. Applying $\mathrm{div}(\lambda_0)$ to the first equation in (2.2.35), we have

$$i\,\mathrm{div}(\lambda_0)\nabla(\lambda_0)\beta^1 - k\,\mathrm{div}(\lambda_0)\varphi^1 + i\,\mathrm{div}(\lambda_0)\partial_\lambda\nabla(\lambda_0)\beta^0 = 0.$$

In view of the fourth equation in (2.2.35) and the third equation in (2.2.36), the last relation can be rewritten in the form

$$\Delta(\lambda_0)\beta^1 + k^2\beta^1 - 2\lambda_0\beta^0 = 0. \tag{2.2.39}$$

In a like manner we obtain

$$\Delta(\lambda_0)\alpha^1 + k^2\alpha^1 - 2\lambda_0\alpha^0 = 0. \tag{2.2.40}$$

Let us write the rot-equations through their projections onto the coordinate axes. From the resulting six equations we choose two pairs in order to form two linear algebraic systems. In one of the system

$$\begin{aligned} k\psi_1 + \lambda_0\varphi_2^1 &= -i\partial_2\varphi_3^1 - i\partial_1\alpha^1 - \varphi_2^0, \\ \lambda_0\psi_1^1 + k\varphi_2^1 &= -i\partial_1\psi_3^1 + i\partial_2\beta^1 - \psi_1^0, \end{aligned} \tag{2.2.41}$$

the role of unknowns is played by ψ_1^1 and φ_2^1. In the other system

$$\begin{aligned} \lambda_0\psi_2^1 - k\varphi_1^1 &= -i\partial_2\psi_3^1 - i\partial_1\beta^1 - \psi_2^0, \\ k\psi_2^1 - \lambda_0\varphi_1^1 &= i\partial_1\varphi_3^1 - i\partial_2\alpha^1 + \varphi_1^0, \end{aligned} \tag{2.2.42}$$

we take ψ_2^1 and φ_1^1 as unknowns. Assuming that $k^2 - \lambda_0^2 \neq 0$ and solving these systems, we obtain on Ω the relations

$$\begin{aligned} \varphi_1^1 &= (k^2-\lambda_0^2)^{-1}[i\lambda_0(\partial_1\varphi_3^1 - i\varphi_0^1) + ik(\partial_2\psi_3^1 - i\psi_2^0) - i\lambda_0\partial_2\alpha^1 + ik\partial_1\beta^1], \\ \varphi_2^1 &= (k^2-\lambda_0^2)^{-1}[i\lambda_0(\partial_2\varphi_3^1 - i\varphi_2^0) - ik(\partial_1\psi_3^1 - i\psi_1^0) + i\lambda_0\partial_1\alpha^1 + ik\partial_2\beta^1], \\ \psi_1^1 &= (k^2-\lambda_0^2)^{-1}[-ik(\partial_2\varphi_3^1 - i\varphi_2^0) + i\lambda_0(\partial_1\psi_3^1 - i\psi_1^0) - ik\partial_1\alpha^1 - i\lambda_0\partial_2\beta^1], \\ \psi_2^1 &= (k^2-\lambda_0^2)^{-1}[ik(\partial_1\varphi_3^1 - i\varphi_1^0) + i\lambda_0(\partial_2\psi_3^1 - i\psi_2^0) - ik\partial_2\alpha^1 + i\lambda_0\partial_1\beta^1]. \end{aligned} \tag{2.2.43}$$

Thus the components of Φ^1 are expressed by (2.2.43) through φ_3^1, ψ_3^1, α^1, β^1. A generalized eigenvector satisfies the same boundary conditions (2.2.1) as eigenvector. From this we derive boundary conditions for φ_3^1, ψ_3^1, α^1, β^1. Conditions (2.2.1) are equivalent to the following system on $\partial\Omega$:

$$\varphi_3^1 = 0, \qquad \beta^1 = 0, \tag{2.2.44}$$

$$\nu_1\varphi_2^1 - \nu_2\varphi_1^1 = 0, \qquad \nu_1\psi_1^1 + \nu_2\psi_2^1 = 0. \tag{2.2.45}$$

Taking account of (2.2.43), we write (2.2.45) in the form

$$
\begin{aligned}
&k[i(\nu_1\partial_2-\nu_2\partial_1)\varphi_3+(\nu_1\varphi_2^0-\nu_2\varphi_1^0)+i(\nu_1\partial_1+\nu_2\partial_2)\alpha^1]-\\
&-\lambda_0[i(\nu_1\partial_1+\nu_2\partial_2)\psi_3^1+(\nu_1\psi_1^0+\nu_2\psi_2^0)-i(\nu_1\partial_2-\nu_2\partial_1)\beta^1]=0,\\
&-\lambda_0[i(\nu_1\partial_2-\nu_2\partial_1)\varphi_3^1+(\nu_1\varphi_2^0-\nu_2\varphi_1^0)+i(\nu_1\partial_1+\nu_2\partial_2)\alpha^1]+\\
&+k[i(\nu_1\partial_1+\nu_2\partial_2)\psi_3^1+(\nu_1\psi_1^0+\nu_2\psi_2^0)-i(\nu_1\partial_2-\nu_2\partial_1)\beta^1]=0.
\end{aligned}
$$

The terms, containing components of the eigenvector, vanish due to the boundary conditions. Recall that the case $k^2-\lambda_0^2\neq 0$ is assumed. Therefore the last two equalities lead to

$$
\begin{aligned}
(\nu_1\partial_2-\nu_2\partial_1)\varphi_3^1+(\nu_1\partial_1+\nu_2\partial_2)\alpha^1 &= 0,\\
(\nu_1\partial_1+\nu_2\partial_2)\psi_3^1-(\nu_1\partial_2-\nu_2\partial_1)\beta^1 &= 0.
\end{aligned}
$$

Moreover, in view of (2.2.44)

$$(\nu_1\partial_2-\nu_2\partial_1)\varphi_3^1=0, \qquad (\nu_1\partial_2-\nu_2\partial_1)\beta^1=0.$$

Hence on $\partial\Omega$

$$\partial_\nu\psi_3^1=(\nu_1\partial_1+\nu_2\partial_2)\psi_3^1=0, \qquad \partial_\nu\alpha^1=(\nu_1\partial_1+\nu_2\partial_2)\alpha^1=0. \tag{2.2.46}$$

We conclude that each of the pairs $(\varphi_3^0,\varphi_3^1)$, (β^0,β^1) consists of an eigenvector and a generalized eigenvector corresponding to the eigenvalue λ_0 of the Dirichlet problem for the Helmholtz equation (see (2.2.38), (2.2.44) and (2.2.39), (2.2.44)). Each pair (ψ_3^0,ψ_3^1), (α^0,α^1) consists of an eigenvector and a generalized vector corresponding to the eigenvalue λ_0 of the Neumann problem for the Helmholtz equation (see (2.2.37), (2.2.46) and (2.2.40), (2.2.46)). (If λ_0 is an eigenvalue of none of the problems, then the corresponding "eigenvector" vanishes and there is no generalized eigenvector.) Let us clarify conditions that provide generalized eigenvectors for the Dirichlet and Neumann problems. For instance, consider the problem

$$\Delta(\lambda)X+k^2X=0 \text{ in } \Omega, \qquad X=0 \text{ on } \partial\Omega. \tag{2.2.47}$$

Let λ_0 and X^0 be an eigenvalue and an eigenvector of problem (2.2.47). The solvability condition of the problem

$$\Delta(\lambda_0)X^1+k^2X^1-2\lambda_0X^0=0 \text{ in } \Omega, \qquad X^1=0 \text{ on } \partial\Omega$$

is $2\lambda_0(X^0,X^0)_\Omega=0$, so the Jordan chain X^0,X^1 exists only for $\lambda_0=0$ (if the number is an eigenvalue). The same is valid for the Neumann problem.

Thus the pencil $\lambda\mapsto\mathfrak{A}(\lambda,k)$ given by (2.2.6) can have a generalized eigenvector only in the case that 0 is an eigenvalue. Assume that the condition is fulfilled and there exists a Jordan chain Φ^0,Φ^1 corresponding to $\lambda_0=0$, where $\Phi^0=(\varphi^0,\alpha^0,\psi^0,\beta^0)$ is an eigenvector. Then (2.2.16) and (2.2.43) imply that a generalized vector Φ^1 is of the form

$$\Phi^1 = \frac{1}{k^2}\begin{pmatrix} i\,k\partial_2\psi_3^1 + i\,k\partial_1\beta^1 + i\partial_1\varphi_3^0 - i\partial_2\alpha^0 \\ -ik\partial_1\psi_3^1 + i\,k\partial_2\beta^1 + i\partial_2\varphi_3^0 + i\partial_1\alpha^0 \\ k^2\,\varphi_3^1 \\ k^2\,\alpha^1 \\ -ik\partial_2\varphi_3^1 - i\,k\partial_1\alpha^1 + i\partial_1\psi_3^0 - i\partial_2\beta^0 \\ ik\partial_1\varphi_3^1 - i\,k\partial_2\alpha^1 + i\partial_2\psi_3^0 + i\partial_1\beta^0 \\ k^2\,\psi_3^1 \\ k^2\,\beta^1 \end{pmatrix}. \tag{2.2.48}$$

In order to make certain, that a generalized vector exists it suffices to verify by immediate calculation that a vector Φ^1 of the form (2.2.48) turns out to be a generalized eigenvector. Such a vector is determined up to adding an arbitrary eigenvector of the pencil $\mathfrak{A}(\cdot,k)$ corresponding to $\lambda_0 = 0$.

A chain Φ^0, Φ^1 is continued by a vector Φ^2 if there exists a solution Φ^2 to the equation

$$\mathfrak{A}(\lambda_0,k)\Phi^2 + \partial_\lambda\mathfrak{A}(\lambda_0,k)\Phi^1 + \frac{1}{2}\partial_\lambda^2\mathfrak{A}(\lambda_0,k)\Phi^0 = 0.$$

Since $\partial_\lambda^2\mathfrak{A}(\lambda,k) \equiv 0$, this equation takes the form

$$\mathfrak{A}(\lambda_0,k)\Phi^2 + \partial_\lambda\mathfrak{A}(\lambda_0,k)\Phi^1 = 0.$$

Repeating with slight modifications the argument done for the first eigenvector, we obtain that the components $\varphi_3^2, \alpha^2, \psi_3^2, \beta^2$ of Φ^2 must satisfy the boundary value problems

$$\Delta(\lambda_0)\varphi_3^2 + k^2\varphi_3^2 - 2\lambda_0\varphi_3^1 - \varphi_3^0 = 0 \text{ in } \Omega, \quad \varphi_3^2 = 0 \text{ on } \partial\Omega, \tag{2.2.49}$$

$$\Delta(\lambda_0)\alpha^2 + k^2\alpha^2 - 2\lambda_0\alpha^1 - \alpha^0 = 0 \text{ in } \Omega, \quad \partial_\nu\alpha^2 = 0 \text{ on } \partial\Omega, \tag{2.2.50}$$

$$\Delta(\lambda_0)\psi_3^2 + k^2\psi_3^2 - 2\lambda_0\psi_3^1 - \psi_3^0 = 0 \text{ in } \Omega, \quad \partial_\nu\psi_3^2 = 0 \text{ on } \partial\Omega, \tag{2.2.51}$$

$$\Delta(\lambda_0)\beta^2 + k^2\beta^2 - 2\lambda_0\beta^1 - \beta^0 = 0 \text{ in } \Omega, \quad \beta^2 = 0 \text{ on } \partial\Omega. \tag{2.2.52}$$

Since $\lambda_0 = 0$, the third term of each equation above vanishes on Ω. The last (fourth) term of these equations is an eigenfunction of the operator $\Delta(\lambda) + k^2$, which corresponds to the eigenvalue λ_0 and is subject to the Dirichlet boundary condition (φ_3^0 and β^0), or the Neumann boundary condition (ψ_3^0 and α^0). Therefore none of the problems (2.2.49) – (2.2.52) is solvable, so neither chain Φ^0, Φ^1 admits continuation and any partial multiplicity κ of $\lambda_0 = 0$ equals 2. Thus we have proved the following assertion:

Proposition 2.2.3. *Let λ_0 be an eigenvalue of the pencil $\mathfrak{A}(\cdot,k)$ with the corresponding eigenvector $(\varphi^0, \alpha^0, \psi^0, \beta^0)$, $\varphi^0 = (\varphi_1^0, \varphi_2^0, \varphi_3^0)$ and $\psi^0 = (\psi_1^0, \psi_2^0, \psi_3^0)$. Assume, that $k^2 - \lambda_0^2 \neq 0$. Then in the case $\lambda_0 \neq 0$, there is no generalized eigenvector corresponding to λ_0. If $\lambda_0 = 0$, then for any eigenvector there exists a generalized eigenvector Φ^1*

satisfying (2.2.48) *and*

$$\begin{aligned}
\Delta(0)\varphi_3^1 + k^2\varphi_3^1 &= 0 \textit{ in } \Omega, \qquad \varphi_3^1 = 0 \textit{ on } \partial\Omega, && (2.2.53)\\
\Delta(0)\alpha^1 + k^2\alpha^1 &= 0 \textit{ in } \Omega, \qquad \partial_\nu\alpha^1 = 0 \textit{ on } \partial\Omega, && (2.2.54)\\
\Delta(0)\psi_3^1 + k^2\psi_3^1 &= 0 \textit{ in } \Omega, \qquad \partial_\nu\psi_3^1 = 0 \textit{ on } \partial\Omega, && (2.2.55)\\
\Delta(0)\beta^1 + k^2\beta^1 &= 0 \textit{ in } \Omega, \qquad \beta^1 = 0 \textit{ on } \partial\Omega, && (2.2.56)
\end{aligned}$$

with $\Delta(0) = \partial_1^2 + \partial_2^2$. *Any partial multiplicity of* $\lambda_0 = 0$ *is equal to 2.*

The elements φ_3^1, α^1, ψ_3^1, and β^1 are generalized eigenvectors related to the eigenvectors φ_3^0, α^0, ψ_3^0, and β^0 respectively. The generalized eigenvectors can be chosen up to adding the eigenvectors. In particular, we could choose all elements φ_3^1, α^1, ψ_3^1, and β^1 (or some of them) to be zero (see (2.2.53) - (2.2.56)).

Let us see whether generalized eigenvectors exist under the condition $k^2 - \lambda_0^2 = 0$. The formulas (2.2.37), (2.2.38), (2.2.39), and (2.2.40) take the form of

$$\Delta\psi^1 - 2\lambda_0\psi^0 = 0, \quad \Delta\varphi^1 - 2\lambda_0\varphi^0 = 0, \quad \Delta\beta^1 - 2\lambda_0\beta^0 = 0, \quad \Delta\alpha^1 - 2\lambda_0\alpha^0 = 0, \tag{2.2.57}$$

where $\Delta := \Delta(0) = \partial_1^2 + \partial_2^2$. The second and the third equalities in (2.2.57), together with (2.2.24) and (2.2.44), mean that

$$\varphi_3^1 = 0, \qquad \beta^1 = 0 \qquad \text{on } \Omega. \tag{2.2.58}$$

Next, we consider the cases $\lambda_0^2 = k^2 \neq 0$ and $\lambda_0^2 = k^2 = 0$ separately.

First, we consider the case of $\lambda_0^2 = k^2 \neq 0$. From (2.2.41), (2.2.58), and (2.2.33), we obtain $\partial_1(\psi_3^1 - (\lambda_0/k)\alpha^1) = 0$, while (2.2.42) leads to $\partial_2(\psi_3^1 - (\lambda_0/k)\alpha^1) = 0$. That is

$$\psi_3^1 - (\lambda_0/k)\alpha^1 = C_0 = \text{const}.$$

Repeating the arguments in (2.2.26), we have

$$\int_\Omega (\operatorname{rot}(\lambda_0)\,\varphi^1)_n\,dx_1dx_2 = 0. \tag{2.2.59}$$

We project the second rot-equation (2.2.35) onto the axis x_3, make use of

$$(-i\partial_\lambda \operatorname{rot}(\lambda_0)\varphi^0)_n = 0,$$

and obtain

$$(\operatorname{rot}(\lambda_0)\,\varphi^1)_n - ik(\psi_3^1 - (\lambda_0/k)\alpha^1) = 0.$$

We now integrate the equality over Ω, taking into account (2.2.59), and conclude that $C_0 = 0$. Thus,

$$\psi_3^1 - (\lambda_0/k)\alpha^1 = 0 \qquad \text{on } \Omega. \tag{2.2.60}$$

Let us consider the third components of the rot-equations and the div-equations in (2.2.35)

$$\begin{aligned}
i(\partial_1\psi_2^1 - \partial_2\psi_1^1) + i(i\lambda_0)\beta^1 - k\varphi_3 - \beta^0 &= 0,\\
i(\partial_1\varphi_2^1 - \partial_2\varphi_1^1) + i(i\lambda_0)\alpha^1 + k\psi_3^1 - \alpha^0 &= 0,\\
-i(\partial_1\psi_1^1 + \partial_2\psi_2^1 + i\lambda_0\psi_3^1) - k\alpha^1 + \psi_3^0 &= 0,\\
i(\partial_1\varphi_1^1 + \partial_2\varphi_2^1 + i\lambda_0\varphi_3^1) - k\beta^1 - \varphi_3^0 &= 0,
\end{aligned}$$

which according to (2.2.24), (2.2.58), and (2.2.60) take the form

$$\begin{aligned} \partial_1\psi_2^1 - \partial_2\psi_1^1 &= 0, & \partial_1\varphi_2^1 - \partial_2\varphi_1^1 + i\alpha^0 &= 0, \\ \partial_1\psi_1^1 + \partial_2\psi_2^1 + i\psi_3^0 &= 0, & \partial_1\varphi_1^1 + \partial_2\varphi_2^1 &= 0. \end{aligned} \tag{2.2.61}$$

As before, we assume the domain Ω to be one-connected. According to the first and the forth equations in (2.2.61), there exist (one-valued) potentials H^1 and G^1 in Ω, such that

$$\begin{aligned} &\varphi_1 = \partial_2 H^1, \qquad \varphi_2 = -\partial_1 H^1 \\ &\psi_j = \partial_j G^1, \qquad j = 1, 2. \end{aligned} \tag{2.2.62}$$

In view of the rest two equalities (2.2.61), we have

$$\Delta H^1 = i\alpha^0, \qquad \Delta G^1 = -i\psi_3^0 \qquad \text{on } \Omega. \tag{2.2.63}$$

Substitute (2.2.62) into (2.2.45), which gives us the boundary conditions for the potentials H^1 and G^1:

$$\partial_\nu H^1 = 0, \qquad \partial_\nu G^1 = 0 \qquad \text{on } \partial\Omega. \tag{2.2.64}$$

The solvability of problems (2.2.63) and(2.2.64) necessitates the equalities

$$(\psi_3^0, 1) = 0, \qquad (\alpha^0, 1) = 0,$$

which means that $\psi_3^0 = 0, \alpha^0 = 0$ since $\psi_3^0 = \text{const}$ and $\alpha^0 = \text{const}$. Therefore, assuming the existence of a generalized eigenvector, we obtain a contradiction because an eigenvector cannot be zero.

Now we consider the case of $\lambda_0^2 = k^2 = 0$. As before, we have (2.2.61),(2.2.63), and (2.2.64). Hence the assumption of a generalized eigenvector existence again leads to contradiction. We arrive at the following proposition:

Proposition 2.2.4. *Let Ω be a one-connected domain and λ_0 an eigenvalue of the pencil* (2.2.6), *with $\lambda_0^2 = k^2$. Then there is no generalized eigenvector corresponding to λ_0.*

2.2.3 Canonical systems of Jordan chains

Let λ_0 be an eigenvalue of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, k)$. Next, we construct canonical systems of Jordan chains in the cases $k^2 = \lambda_0$ and $k^2 \neq \lambda_0^2$ separately. 1^o. *First, we consider the case* $k^2 - \lambda_0^2 \neq 0$.

Let $J_{\mathcal{D}} = J_{\mathcal{D}}(\lambda_0)$ ($J_{\mathcal{N}} = J_{\mathcal{N}}(\lambda_0)$) denote the dimension of the kernel of the Dirichlet problem (Neumann problem) for the operator $\Delta(\lambda_0) + k^2$ in Ω. Let

$$A^1, \dots, A^{J_{\mathcal{N}}};\ B^1, \dots, B^{J_{\mathcal{D}}};\ \Phi^1, \dots, \Phi^{J_{\mathcal{D}}};\ \Psi^1, \dots, \Psi^{J_{\mathcal{N}}} \tag{2.2.65}$$

be the basis for $\ker \mathfrak{A}(\lambda_0, k)$. We write every vector in (2.2.65) as $(\varphi^j, \alpha^j, \psi^j, \beta^j)$, where $\varphi^j = (\varphi_1^j, \varphi_2^j, \varphi_3^j)$ and $\psi^j = (\psi_1^j, \psi_2^j, \psi_3^j)$, and define the vector components for the vectors of the four collections $\{A^j\}$, $\{B^j\}$, $\{\Phi^j\}$, and $\{\Psi^j\}$.

For A^j, we choose α^j, such that $\{\alpha^1, \dots, \alpha^{J_\mathcal{N}}\}$ form a basis for the kernel of the Neumann problem for the operator $\Delta(\lambda_0) + k^2$. We set $\varphi_3^j = \psi_3^j = \beta^j = 0$ and define φ_1^j, φ_2^j, ψ_1^j, ψ_2^j according to (2.2.16), and substituting φ_3^j to φ_3, etc. Then A^j satisfies

$$A^j = (k^2 - \lambda_0^2)^{-1}(-i\lambda_0\partial_2\alpha^j,\, i\lambda_0\partial_1\alpha^j,\, 0,\, (k^2 - \lambda_0^2)\alpha^j,\, -ik\partial_1\alpha^j,\, -ik\partial_2\alpha^j,\, 0,\, 0). \tag{2.2.66}$$

By Proposition 2.2.1, the vectors A^j are linearly independent in $\ker \mathfrak{A}(\lambda_0, k)$.

The components $\beta^1, \dots, \beta^{J_\mathcal{D}}$ of the vectors $B^1, \dots, B^{J_\mathcal{D}}$ form a basis of the kernel of the Dirichlet problem for the operator $\Delta(\lambda_0) + k^2$. Besides, $\varphi_3^j = \psi_3^j = \alpha^j = 0$, and define φ_1^j, φ_2^j, ψ_1^j, and ψ_2^j according to (2.2.16), with the new $\varphi_3 = \varphi_3^j$, etc. Then

$$B^j = (k^2 - \lambda_0^2)^{-1}(ik\partial_1\beta^j,\, ik\partial_2\beta^j,\, 0,\, 0,\, -i\lambda_0\partial_2\beta^j,\, i\lambda_0\partial_1\beta^j,\, 0,\, (k^2 - \lambda_0^2)\beta^j). \tag{2.2.67}$$

By Proposition 2.2.1, the vectors B^j are linearly independent in $\ker \mathfrak{A}(\lambda_0, k)$.

We now define the vectors Φ^j. Choose φ_3^j, such that $\varphi_3^1, \dots, \varphi_3^{J_\mathcal{D}}$ form a basis of the kernel of the Dirichlet problem, set $\psi_3^j = \alpha^j = \beta^j = 0$, and define φ_1^j, φ_2^j, ψ_1^j, and ψ_2^j according to (2.2.16) where $\varphi_3 = \varphi_3^j$, etc., are adjusted again. We have

$$\Phi^j = (k^2 - \lambda_0^2)^{-1}(i\lambda_0\partial_1\varphi_3^j,\, i\lambda_0\partial_2\varphi_3^j,\, (k^2 - \lambda_0^2)\varphi_3^j,\, 0,\, -ik\partial_2\varphi_3^j,\, ik\partial_1\varphi_3^j,\, 0,\, 0). \tag{2.2.68}$$

Finally, we describe the vectors Ψ^j. We choose ψ_3^j, such that $\psi_3^1, \dots, \psi_3^{J_\mathcal{N}}$ form a basis for the kernel of the Neumann problem, set $\varphi_3^j = \alpha^j = \beta^j = 0$ and then define the rest components according to (2.2.16). As a result, we obtain

$$\Psi^j = (k^2 - \lambda_0^2)^{-1}(ik\partial_2\psi_3^j,\, -ik\partial_1\psi_3^j,\, 0,\, 0,\, i\lambda_0\partial_1\psi_3^j,\, i\lambda_0\partial_2\psi_3^j,\, (k^2 - \lambda_0^2)\psi_3^j,\, 0). \tag{2.2.69}$$

Proposition 2.2.1 and the definition of the vectors (2.2.65) lead to

Proposition 2.2.5. *Let λ_0 be an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$, with $\lambda_0 \neq 0$ and $k^2 - \lambda_0^2 \neq 0$. Then the vectors* (2.2.65) *form a basis for* $\ker \mathfrak{A}(\lambda_0, k)$. *There are no generalized eigenvectors. The vectors $\Phi^1, \dots, \Phi^{J_\mathcal{D}}, \Psi^1, \dots, \Psi^{J_\mathcal{N}}$ form a basis for* $\ker \mathfrak{M}(\lambda_0, k)$, *where $\mathfrak{M}$ is the Maxwell pencil.*

Assume now, that $\lambda_0 = 0$ is an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$. The vectors (2.2.65), defined by the equalities (2.2.66)-(2.2.69) for $\lambda_0 = 0$, form a basis for $\ker \mathfrak{A}(0, k)$. We denote these vectors by $A^{0,j}, \dots, \Psi^{0,j}$ (instead of $A^j, \dots, \Psi^j$). According to Proposition 2.2.3, to construct a canonical system of Jordan chains, it suffices to indicate a generalized eigenvector $A^{1,j}, \dots, \Psi^{1,j}$ for every vector $A^{0,j}$, ..., $\Psi^{0,j}$. Each generalized eigenvector will be written in the form of (2.2.48), with $\varphi_3^{1,j}$, $\alpha^{1,j}$, $\psi_3^{1,j}$, $\beta^{1,j}$ instead of φ_3^1, α^1, ψ_3^1, β^1, and with $\varphi_3^{0,j}, \alpha^{0,j}, \psi_3^{0,j}, \beta^{0,j}$ instead of $\varphi_3^0, \alpha^0, \psi_3^0, \beta^0$. Let us define new notations for the four collections of the vectors $\{A^{1,j}\}$, $\{B^{1,j}\}$, $\{\Phi^{1,j}\}$, $\{\Psi^{1,j}\}$. We write the eigenvectors $A^{0,j}, \dots, \Psi^{0,j}$, into the form of $(\varphi^{0,j}, \alpha^{0,j}, \psi^{0,j}, \beta^{0,j})$; it is always clear from the context which vector is meant.

We will start with the vectors of the collection $A^{1,j}$. Recall that the component $\alpha^{0,j}$ of $A^{0,j}$ is an eigenvector of the Neumann problem for the operator $\Delta(0)+k^2$ in the domain Ω. As for $\alpha^{1,j}$, we take a generalized eigenvector adjoined to $\alpha^{0,j}$. Set $\varphi_3^{1,j}=\psi_3^{1,j}=\beta^{1,j}=0$ and define $A^{1,j}$ according to (2.2.48). The generalized eigenvectors $B^{1,j}$, $\Phi^{1,j}$, $\Psi^{1,j}$ are defined with the evident modifications.

As a result, we obtain the four collections of Jordan chains, each consisting of two elements:

$$A^{0,j}, A^{1,j}; \quad B^{0,j}, B^{1,j}; \quad \Phi^{0,j}, \Phi^{1,j}; \quad \Psi^{0,j}, \Psi^{1,j} \tag{2.2.70}$$

with

$$\begin{aligned} A^{0,j} &= k^{-2}(0,\,0,\,0,\,k^2\alpha^{0,j},\,-ik\partial_1\alpha^{0,j},\,-ik\partial_2\alpha^{0,j},\,0,\,0),\\ A^{1,j} &= k^{-2}(-i\partial_2\alpha^{0,j},\,i\partial_1\alpha^{0,j},\,0,\,k^2\alpha^{1,j},\,-ik\partial_1\alpha^{1,j},\,-ik\partial_2\alpha^{1,j},\,0,\,0), \end{aligned} \tag{2.2.71}$$

where $j=1,\dots,J_{\mathcal{N}}(0)$;

$$\begin{aligned} B^{0,j} &= k^{-2}(ik\partial_1\beta^{0,j},\,ik\partial_2\beta^{0,j},\,0,\,0,\,0,\,0,\,0,\,k^2\beta^{0,j}),\\ B^{1,j} &= k^{-2}(ik\partial_1\beta^{1,j},\,ik\partial_2\beta^{1,j},\,0,\,0,\,-i\partial_2\beta^{0,j},\,i\partial_1\beta^{0,j},\,0,\,k^2\beta^{1,j}), \end{aligned} \tag{2.2.72}$$

where $j=1,\dots,J_{\mathcal{D}}(0)$;

$$\begin{aligned} \Phi^{0,j} &= k^{-2}(0,\,0,\,k^2\varphi_3^{0,j},\,0,\,-ik\partial_2\varphi_3^{0,j},\,ik\partial_1\varphi_3^{0,j},\,0,\,0),\\ \Phi^{1,j} &= k^{-2}(i\partial_1\varphi_3^{0,j},\,i\partial_2\varphi_3^{0,j},\,k^2\varphi_3^{1,j},\,0,\,-ik\partial_2\varphi_3^{1,j},\,ik\partial_1\varphi_3^{1,j},\,0,\,0), \end{aligned} \tag{2.2.73}$$

where $j=1,\dots,J_{\mathcal{D}}(0)$;

$$\begin{aligned} \Psi^{0,j} &= k^{-2}(ik\partial_2\psi_3^{0,j},\,-ik\partial_1\psi_3^{0,j},\,0,\,0,\,0,\,0,\,k^2\psi_3^{0,j},\,0),\\ \Psi^{1,j} &= k^{-2}(ik\partial_2\psi_3^{1,j},\,-ik\partial_1\psi_3^{1,j},\,0,\,0,i\partial_1\psi_3^{0,j},\,i\partial_2\psi_3^{0,j},\,k^2\psi_3^{1,j},\,0), \end{aligned} \tag{2.2.74}$$

where $j=1,\dots,J_{\mathcal{N}}(0)$.

Proposition 2.2.6. *Assume that $k^2\neq 0$ and $\lambda_0=0$ is an eigenvalue of the pencil $\lambda\mapsto\mathfrak{A}(\lambda,k)$. Then the chains* (2.2.70)*, defined by* (2.2.71)–(2.2.74) *form a canonical system of Jordan chains of $\mathfrak{A}$, corresponding to the eigenvalue $\lambda_0=0$. The chains* (2.2.73) *and* (2.2.74) *belong to the domain of the Maxwell pencil $\mathfrak{M}(0,k)$, while the chains* (2.2.71) *and* (2.2.72) *do not.*

2.2.4 Special choice of Jordan chains

We use freedom in choosing Jordan chains to form a canonical system of chains subject to some orthogonality and normalization conditions. For the statement of problem with radiation conditions and for the definition of the scattering matrix it suffices to consider only the real eigenvalues of the pencil $\mathfrak{A}(\cdot,k)$. Therefore in this Section we discuss Jordan chains with real eigenvalues λ_0.

2.2.4.1 The case of $\lambda_0 \neq 0$

Let λ_0 be an eigenvalue of $\mathfrak{A}(\cdot, k)$, such that $\lambda_0 \in \mathbb{R} \setminus 0$ and $k^2 - \lambda_0^2 \neq 0$; then $\lambda_0^2 < k^2$. Denote by $\alpha^1, \dots, \alpha^{J_\mathcal{N}}$ ($\beta^1, \dots, \beta^{J_\mathcal{D}}$) an orthogonal basis for the kernel of the Neumann (Dirichlet) problem for the operator $\Delta(\lambda_0) + k^2$. We assume, that

$$\|\alpha^\zeta; L_2(\Omega)\| = \|\beta^\tau; L_2(\Omega)\| = |2k\lambda_0|^{-1/2}. \tag{2.2.75}$$

We also assume, that the bases $\psi_3^1, \dots, \psi_3^{J_\mathcal{N}}$ and $\varphi_3^1, \dots, \varphi_3^{J_\mathcal{D}}$ in (2.2.69) and (2.2.68) coincide, respectively, with the previously defined orthogonal bases $\alpha^1, \dots, \alpha^{J_\mathcal{N}}$ and $\beta^1, \dots, \beta^{J_\mathcal{D}}$. Let us define

$$\begin{aligned}\Psi^\zeta &= (k^2 - \lambda_0^2)^{-1/2}(ik\partial_2\alpha^\zeta,\ -ik\partial_1\alpha^\zeta,\ 0,\ 0,\ i\lambda_0\partial_1\alpha^\zeta,\ i\lambda_0\partial_2\alpha^\zeta,\ (k^2 - \lambda_0^2)\alpha^\zeta,\ 0),\\ A^\zeta &= (0,\ 0,\ 0,\ k\alpha^\zeta,\ -i\partial_1\alpha^\zeta,\ -i\partial_2\alpha^\zeta,\ \lambda_0\alpha^\zeta,\ 0),\end{aligned} \tag{2.2.76}$$

and set

$$\begin{aligned}\Phi^\tau &= (k^2 - \lambda_0^2)^{-1/2}(i\lambda_0\partial_1\beta^\tau,\ i\lambda_0\partial_2\beta^\tau,\ (k^2 - \lambda_0^2)\beta^\tau,\ 0,\ -ik\partial_2\beta^\tau,\ ik\partial_1\beta^\tau,\ 0,\ 0),\\ B^\tau &= (i\partial_1\beta^\tau,\ i\partial_2\beta^\tau,\ -\lambda_0\beta^\tau,\ 0,\ 0,\ 0,\ 0,\ k\beta^\tau).\end{aligned} \tag{2.2.77}$$

Define $\mathfrak{A}' := \partial_\lambda \mathfrak{A}(\lambda, k)$. From (2.2.6) and (2.2.36), it follows that

$$\mathfrak{A}'\Phi = (\psi_2, -\psi_1, -\beta, \psi_3, -\varphi_2, \varphi_1, \alpha_1, -\varphi_3) \tag{2.2.78}$$

for any λ and for every vector $\Phi = (\varphi, \alpha, \psi, \beta)$, where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$. In particular, the matrix $\mathfrak{A}'$ is symmetric and independent of λ and k.

Proposition 2.2.7. *Let λ_0 be an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$, such that $\lambda_0 \in \mathbb{R} \setminus 0$ and $k^2 - \lambda_0^2 \neq 0$. Then the eigenvectors* (2.2.76) *and* (2.2.77) *satisfy the orthogonality and normalization conditions*

$$\begin{aligned}(\mathfrak{A}'A^\tau, A^\zeta) = (\mathfrak{A}'\Psi^\tau, \Psi^\zeta) &= \operatorname{sgn}(k\lambda_0)\delta_{\tau,\zeta}, \quad \tau, \zeta = 1, \dots J_\mathcal{N},\\ (\mathfrak{A}'B^\tau, B^\zeta) = (\mathfrak{A}'\Phi^\tau, \Phi^\zeta) &= \operatorname{sgn}(k\lambda_0)\delta_{\tau,\zeta}, \quad \tau, \zeta = 1, \dots J_\mathcal{D}.\end{aligned} \tag{2.2.79}$$

Moreover, for all τ and ζ

$$(\mathfrak{A}'P^\tau, R^\zeta) = 0, \tag{2.2.80}$$

where one may change P and R for any of the letters A, B, Φ, Ψ to obtain a pair of two distinct letters. In other words, the left-hand-side of (2.2.80)*, obtained in such a manner, must differ from those in* (2.2.79).

Proof. To explain the definitions (2.2.76) and (2.2.77), we return to (2.2.66)–(2.2.69). To avoid misunderstanding, the vectors A^j, B^j, Φ^j and Ψ^j in these formulas we will temporarily denote by $\widetilde{A}^j$, $\widetilde{B}^j$, $\widetilde{\Phi}^j$ and $\widetilde{\Psi}^j$. The corresponding bases $\{\alpha^j\}$, $\{\beta^j\}$, $\{\varphi_3^j\}$ and $\{\psi_3^j\}$ we assume to be orthogonal and, for the time being, introduce no other requirements. We first take up the eigenvectors

$$\begin{aligned}\widetilde{A}^\tau &= (k^2 - \lambda_0^2)^{-1}(-i\lambda_0\partial_2\alpha^\tau,\ +i\lambda_0\partial_1\alpha^\tau,\ 0,\ (k^2 - \lambda_0^2)\alpha^\tau,\ -ik\partial_1\alpha^\tau,\ -ik\partial_2\alpha^\tau,\ 0,\ 0),\\ \widetilde{\Psi}^\tau &= (k^2 - \lambda_0^2)^{-1}(ik\partial_2\psi_3^\tau,\ -ik\partial_1\psi_3^\tau,\ 0,\ 0,\ i\lambda_0\partial_1\psi_3^\tau,\ i\lambda_0\partial_2\psi_3^\tau,\ (k^2 - \lambda_0^2)\psi_3^\tau,\ 0).\end{aligned} \tag{2.2.81}$$

A straightforward calculation (using (2.2.78) and integration by parts) leads to

$$\begin{aligned}(\mathfrak{A}'\widetilde{A}^\tau, \widetilde{A}^\zeta) &= 2k\lambda_0(\alpha^\tau, \alpha^\zeta)/(k^2-\lambda_0^2),\\ (\mathfrak{A}'\widetilde{\Psi}^\tau, \widetilde{\Psi}^\zeta) &= 2k\lambda_0(\psi_3^\tau, \psi_3^\zeta)/(k^2-\lambda_0^2),\\ (\mathfrak{A}'\widetilde{A}^\tau, \widetilde{\Psi}^\zeta) &= -2\lambda_0^2(\alpha^\tau, \psi_3^\zeta)/(k^2-\lambda_0^2),\end{aligned} \tag{2.2.82}$$

where $\tau, \zeta = 1, \dots, J_{\mathcal{N}}$. Moreover,

$$(\mathfrak{A}'\widetilde{A}^\tau, \widetilde{B}^\zeta) = (\mathfrak{A}'\widetilde{A}^\tau, \widetilde{\Phi}^\zeta) = (\mathfrak{A}'\widetilde{B}^\zeta, \widetilde{\Psi}^\tau) = (\mathfrak{A}'\widetilde{\Phi}^\zeta, \widetilde{\Psi}^\tau) = 0 \tag{2.2.83}$$

for all ζ and τ. We now assume, that the bases $\alpha^1, \dots, \alpha^{J_{\mathcal{N}}}$ and $\psi_3^1, \dots, \psi_3^{J_{\mathcal{N}}}$ coincide and satisfy the normalization condition (2.2.75). Now define $\breve{A}^\tau = (k^2-\lambda_0^2)^{1/2}\widetilde{A}^\tau$ and $\breve{\Psi}^\zeta = (k^2-\lambda_0^2)^{1/2}\widetilde{\Psi}^\zeta$. Then (2.2.82) takes the form

$$\begin{aligned}(\mathfrak{A}'\breve{A}^\tau, \breve{A}^\zeta) &= \mathrm{sgn}(k\lambda_0)\delta_{\tau,\zeta},\\ (\mathfrak{A}'\breve{\Psi}^\tau, \breve{\Psi}^\zeta) &= \mathrm{sgn}(k\lambda_0)\delta_{\tau,\zeta},\\ (\mathfrak{A}'\breve{A}^\tau, \breve{\Psi}^\zeta) &= -|\lambda_0/k|\,\delta_{\tau,\zeta}.\end{aligned} \tag{2.2.84}$$

The formulas for $\tau = \zeta$ in the third line of (2.2.84) do not correspond to (2.2.80) and have to be recast. We set $\widehat{A}^\tau = \breve{A}^\tau + (\lambda_0/k)\breve{\Psi}^\tau$ and obtain

$$(\mathfrak{A}'\widehat{A}^\tau, \breve{\Psi}^\tau) = (\mathfrak{A}'\breve{A}^\tau, \breve{\Psi}^\tau) + \frac{\lambda_0}{k}(\mathfrak{A}'\breve{\Psi}^\tau, \breve{\Psi}^\tau) = -\left|\frac{\lambda_0}{k}\right| + \left|\frac{\lambda_0}{k}\right| = 0. \tag{2.2.85}$$

From (2.2.81) and the equalities $\alpha^\tau = \psi_3^\tau$, it follows that

$$\widehat{A}^\tau = \frac{(k^2-\lambda_0^2)^{1/2}}{k}(0,\,0,\,0,\,k\alpha^\tau,\,-i\partial_1\alpha^\tau,\,-i\partial_2\alpha^\tau,\,\lambda_0\alpha^\tau,\,0). \tag{2.2.86}$$

In view of (2.2.84),

$$\begin{aligned}(\mathfrak{A}'\widehat{A}^\tau, \widehat{A}^\tau) &= (\mathfrak{A}'(\breve{A}^\tau + (\lambda_0/k)\breve{\Psi}^\tau), \breve{A}^\tau + (\lambda_0/k)\breve{\Psi}^\tau) =\\ = (\mathfrak{A}'\breve{A}^\tau, \breve{A}^\tau) + 2\frac{\lambda_0}{k}(\mathfrak{A}'\breve{A}^\tau, \breve{\Psi}^\tau) &+ \frac{\lambda_0^2}{k^2}(\mathfrak{A}'\breve{\Psi}^\tau, \breve{\Psi}^\tau)) = \frac{k^2-\lambda_0^2}{k^2}\mathrm{sgn}(k\lambda_0).\end{aligned}$$

Let us define A^τ as

$$A^\tau := k(k^2-\lambda_0^2)^{-1/2}\widehat{A}^\tau = (0,\,0,\,0,\,k\alpha^\tau,\,-i\partial_1\alpha^\tau,\,-i\partial_2\alpha^\tau,\,\lambda_0\alpha^\tau,\,0)$$

and set $\Psi^\tau := \breve{\Psi}^\tau$; such A^τ and Ψ^τ coincide with those in the formulation of Proposition 2.2.7. From the above argument it follows, that

$$(\mathfrak{A}'A^\tau, A^\tau) = \mathrm{sgn}(k\lambda_0), \quad (\mathfrak{A}'A^\tau, \Psi^\tau) = 0.$$

Thus we have obtained (2.2.79) and, moreover, (2.2.83) with A^τ and Ψ^ζ instead of $\widetilde{A}^\tau$ and $\widetilde{\Psi}^\zeta$. The further transformation (going from $\widetilde{B}$, $\widetilde{\Phi}$ to B, Φ) does not involve A^τ and Ψ^ζ and does not "spoil" the formulas (2.2.83). We now consider the eigenvectors

$$\widetilde{B}^\tau = (k^2-\lambda_0^2)^{-1}(ik\partial_1\beta^\tau,\, ik\partial_2\beta^\tau,\, 0,\, 0,\, -i\lambda_0\partial_2\beta^\tau,\, i\lambda_0\partial_1\beta^\tau,\, 0,\, (k^2-\lambda_0^2)\beta^\tau),$$
$$\widetilde{\Phi}^\zeta = (k^2-\lambda_0^2)^{-1}(i\lambda_0\partial_1\varphi_3^\zeta,\, i\lambda_0\partial_2\varphi_3^\zeta,\, (k^2-\lambda_0^2)\varphi_3^\zeta,\, 0,\, -ik\partial_2\varphi_3^\zeta,\, ik\partial_1\varphi_3^\zeta,\, 0,\, 0), \tag{2.2.87}$$

where $\tau,\zeta = 1,\dots,J_{\mathcal{D}}$ (see (2.2.67 and (2.2.68)). We have

$$(\mathfrak{A}'\widetilde{B}^\tau,\widetilde{B}^\zeta) = 2k\lambda_0(\beta^\tau,\beta^\zeta)/(k^2-\lambda_0^2),$$
$$(\mathfrak{A}'\widetilde{\Phi}^\tau,\widetilde{\Phi}^\zeta) = 2k\lambda_0(\varphi_3^\tau,\varphi_3^\zeta)/(k^2-\lambda_0^2), \tag{2.2.88}$$
$$(\mathfrak{A}'\widetilde{B}^\tau,\widetilde{\Phi}^\zeta) = 2\lambda_0^2(\beta^\tau,\varphi_3^\zeta)/(k^2-\lambda_0^2).$$

Let us assume that the bases $\{\varphi_3^\tau\}$ and $\{\beta^\tau\}$ coincide and satisfy (2.2.75). Define the vectors $\check{\Phi}^\tau = (k^2-\lambda_0^2)^{1/2}\widetilde{\Phi}^\tau$ and $\check{B}^\zeta = (k^2-\lambda_0^2)^{1/2}\widetilde{B}^\zeta$. Then (2.2.88) takes the form of

$$(\mathfrak{A}'\check{B}^\tau,\check{B}^\zeta) = \operatorname{sgn}(k\lambda_0)\delta_{\tau,\zeta},$$
$$(\mathfrak{A}'\check{\Phi}^\tau,\check{\Phi}^\zeta) = \operatorname{sgn}(k\lambda_0)\delta_{\tau,\zeta}, \tag{2.2.89}$$
$$(\mathfrak{A}'\check{B}^\tau,\check{\Phi}^\zeta) = |\lambda_0/k|\delta_{\tau,\zeta}.$$

We set $\widehat{B}^\tau := \check{B}^\tau - (\lambda_0/k)\check{\Phi}^\tau$. Then $(\mathfrak{A}'\widehat{B}^\tau,\check{\Phi}^\tau) = 0$ and

$$\widehat{B}^\tau = (k^2-\lambda_0^2)^{1/2}/k(i\partial_1\beta^\tau,\, i\partial_2\beta^\tau,\, -\lambda_0\beta^\tau,\, 0,\, 0,\, 0,\, 0,\, k\beta^\tau).$$

From (2.2.89), we get

$$(\mathfrak{A}'\widehat{B}^\tau,\widehat{B}^\tau) = \frac{(k^2-\lambda_0^2)}{k^2}\operatorname{sgn}(k\lambda_0).$$

Define $B^\tau := k(k^2-\lambda_0^2)^{-1/2}\widehat{B}^\tau$ and $\Phi^\tau := \check{\Phi}^\tau$. Taking into account $\varphi_3^\tau = \beta^\tau$ in (2.2.87), we obtain

$$B^\tau = (i\partial_1\beta^\tau,\, i\partial_2\beta^\tau,\, -\lambda_0\beta^\tau,\, 0,\, 0,\, 0,\, 0,\, k\beta^\tau),$$
$$\Phi^\tau = (k^2-\lambda_0^2)^{-1/2}(i\lambda_0\partial_1\beta^\tau,\, i\lambda_0\partial_2\beta^\tau,\, (k^2-\lambda_0^2)\beta^\tau,\, 0,\, -ik\partial_2\beta^\tau,\, ik\partial_1\beta^\tau,\, 0,\, 0),$$

with $(\mathfrak{A}'B^\tau,B^\tau) = \operatorname{sgn}(k\lambda_0)$ and $(\mathfrak{A}'B^\tau,\Phi^\tau) = 0$. □

Proposition 2.2.8. *Let Ω be a one-connected domain and $\lambda^2 = k^2 \neq 0$. Then λ is an eigenvalue of the pencil $\mathfrak{A}(\cdot,k)$. To this eigenvalue, there corresponds a one dimensional eigenspace, spanned by the vector $A = k(\varphi,\alpha,\psi,\beta)$, where $\varphi = 0$, $\alpha = \text{const} \neq 0$, $\psi_1 = \psi_2 = 0$, $\psi_3 = (\lambda/k)\alpha$, and $\beta = 0$. The vector A, normalized by $2|\lambda k||\alpha|^2\,|\Omega| = 1$, $|\Omega|$ being the measure of Ω, satisfies*

$$(\mathfrak{A}'A,A) = \operatorname{sgn}(\lambda k). \tag{2.2.90}$$

Proof. It suffices to apply Proposition 2.2.2 and take into account (2.2.78). □

Remark 2.2.9. *The formula* (2.2.76) *holds for the vector A in the case of $\lambda^2 = k^2 \neq 0$ as well, however the vectors B, Φ, and Ψ do not exist in the case.*

2.2.4.2 The case of $\lambda_0 = 0$

We now assume, that $\lambda_0 = 0$ is an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$ and $k^2 - \lambda_0^2 \neq 0$. Let $\alpha^{0,1}, \dots, \alpha^{0,J_\mathcal{N}}$ ($\beta^{0,1}, \dots, \beta^{0,J_\mathcal{D}}$) be an orthogonal basis for the kernel of the Neumann (Dirichlet) problem for the operator $\Delta + k^2$, where $\Delta := \Delta(0) = \partial_1^2 + \partial_2^2$. The vectors of the bases are normalized by the conditions

$$\|\alpha^{0,\zeta}; L_2(\Omega)\| = \|\beta^{0,\tau}; L_2(\Omega)\| = |k|^{-1/2}. \tag{2.2.91}$$

Let the bases $\psi_3^{0,1}, \dots, \psi_3^{0,J_\mathcal{N}}$ and $\varphi_3^{0,1}, \dots, \varphi_3^{0,J_\mathcal{D}}$ coincide respectively with the bases $\alpha^{0,1}, \dots, \alpha^{0,J_\mathcal{N}}$ and $\beta^{0,1}, \dots, \beta^{0,J_\mathcal{D}}$. Moreover, assume that

$$\alpha^{1,\zeta} = \psi_3^{1,\zeta} = \beta^{1,\tau} = \varphi_3^{1,\tau} = 0, \quad \tau = 1, \dots J_D, \; \zeta = 1, \dots J_N. \tag{2.2.92}$$

We set

$$\begin{aligned} A^{0,j} &= (0,\, 0,\, 0,\, k\alpha^{0,j},\, -i\partial_1\alpha^{0,j},\, -i\partial_2\alpha^{0,j},\, 0,\, 0), \\ A^{1,j} &= (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, \alpha^{0,j},\, 0), \end{aligned} \tag{2.2.93}$$

where $j = 1, \dots, J_\mathcal{N}(0)$;

$$\begin{aligned} B^{0,j} &= (i\partial_1\beta^{0,j},\, i\partial_2\beta^{0,j},\, 0,\, 0,\, 0,\, 0,\, 0,\, k\beta^{0,j}), \\ B^{1,j} &= (0,\, 0,\, -\beta^{0,j},\, 0,\, 0,\, 0,\, 0,\, 0), \end{aligned} \tag{2.2.94}$$

where $j = 1, \dots, J_\mathcal{D}(0)$;

$$\begin{aligned} \Phi^{0,j} &= (0,\, 0,\, k\varphi_3^{0,j},\, 0,\, -i\partial_2\varphi_3^{0,j},\, i\partial_1\varphi_3^{0,j},\, 0,\, 0), \\ \Phi^{1,j} &= k^{-1}(i\partial_1\varphi_3^{0,j},\, i\partial_2\varphi_3^{0,j},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0), \end{aligned} \tag{2.2.95}$$

where $j = 1, \dots, J_\mathcal{D}(0)$;

$$\begin{aligned} \Psi^{0,j} &= (i\partial_2\psi_3^{0,j},\, -i\partial_1\psi_3^{0,j},\, 0,\, 0,\, 0,\, 0,\, k\psi_3^{0,j},\, 0), \\ \Psi^{1,j} &= k^{-1}(0,\, 0,\, 0,\, 0, i\partial_1\psi_3^{0,j},\, i\partial_2\psi_3^{0,j},\, 0,\, 0), \end{aligned} \tag{2.2.96}$$

where $j = 1, \dots, J_\mathcal{N}(0)$.

Proposition 2.2.10. *Let $\lambda_0 = 0$ be an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$. Then the Jordan chains* (2.2.93)-(2.2.96) *satisfy the orthogonality and normalization conditions*

$$\begin{aligned} (\mathfrak{A}' A^{i,\tau}, A^{j,\zeta}) = (\mathfrak{A}'\Psi^{i,\tau}, \Psi^{j,\zeta}) &= \operatorname{sgn}(k)\delta_{\tau,\zeta}\delta_{i,1-j}, \quad \tau, \zeta = 1, \dots J_\mathcal{N}, \\ (\mathfrak{A}' B^{i,\tau}, B^{j,\zeta}) = (\mathfrak{A}'\Phi^{i,\tau}, \Phi^{j,\zeta}) &= \operatorname{sgn}(k)\delta_{\tau,\zeta}\delta_{i,1-j}, \quad \tau, \zeta = 1, \dots J_\mathcal{D}, \end{aligned} \tag{2.2.97}$$

for any $i, j = 0, 1$. Moreover, for all τ, ζ and $i, j = 0, 1$ there holds the equality

$$(\mathfrak{A}' P^{i,\tau}, R^{j,\zeta}) = 0, \tag{2.2.98}$$

where one can change P and R for any of A, B, Φ, and Ψ to obtain a pair consisting of two distinct letters.

Proof. We temporarily denote the vectors $A^{i,\zeta}$, $B^{i,\tau}$, $\Phi^{i,\tau}$, and $\Psi^{i,\zeta}$ in (2.2.71)-(2.2.74) by $\widetilde{A}^{i,\zeta}$, $\widetilde{B}^{i,\tau}$, $\widetilde{\Phi}^{i,\tau}$, and $\widetilde{\Psi}^{i,\zeta}$, and assume the corresponding bases $\{\alpha^{0,\zeta}\}$, $\{\beta^{0,\tau}\}$, $\{\varphi_3^{0,\tau}\}$, and $\{\psi_3^{0,\zeta}\}$ to be orthogonal. The generalized eigenvectors $\{\alpha^{1,\zeta}\}$, $\{\beta^{1,\tau}\}$, $\{\varphi_3^{1,\tau}\}$, and $\{\psi_3^{1,\zeta}\}$ are for the time being arbitrary. Now we have

$$(\mathfrak{A}'\widetilde{A}^{i,\tau},\widetilde{B}^{j,\zeta}) = (\mathfrak{A}'\widetilde{A}^{i,\tau},\widetilde{\Phi}^{j,\zeta}) = (\mathfrak{A}'\widetilde{B}^{j,\zeta},\widetilde{\Psi}^{i,\tau}) = (\mathfrak{A}'\widetilde{\Phi}^{j,\zeta},\widetilde{\Psi}^{i,\tau}) = 0 \qquad (2.2.99)$$

for all τ,ζ with $i,j = 0,1$. Therefore, we can discuss the orthogonality (2.2.98) and normalization (2.2.97) conditions independently for the collections $\{\widetilde{A}^{i,\tau}\},\{\widetilde{\Psi}^{i,\tau}\}$ and $\{\widetilde{B}^{j,\zeta}\},\{\widetilde{\Phi}^{j,\zeta}\}$. Let us begin with A and Ψ. A straightforward calculation leads to

$$\begin{aligned}
&(\mathfrak{A}'\widetilde{\Psi}^{0,\tau},\widetilde{\Psi}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\widetilde{\Psi}^{1,\tau},\widetilde{\Psi}^{0,\zeta}) = (\mathfrak{A}'\widetilde{\Psi}^{0,\tau},\widetilde{\Psi}^{1,\zeta}) = (1/k)(\psi^{0,\tau},\psi^{0,\zeta}), \quad \tau,\zeta = 1,\dots J_N, \qquad (2.2.100)\\
&(\mathfrak{A}'\widetilde{\Psi}^{1,\tau},\widetilde{\Psi}^{1,\zeta}) = (1/k)[(\psi^{0,\tau},\psi^{1,\zeta}) + (\psi^{1,\tau},\psi^{0,\zeta})],
\end{aligned}$$

$$\begin{aligned}
&(\mathfrak{A}'\widetilde{A}^{0,\tau},\widetilde{\Psi}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\widetilde{A}^{1,\tau},\widetilde{\Psi}^{0,\zeta}) = (\mathfrak{A}'\widetilde{A}^{0,\tau},\widetilde{\Psi}^{1,\zeta}) = 0, \quad \tau,\zeta = 1,\dots J_N, \qquad (2.2.101)\\
&(\mathfrak{A}'\widetilde{A}^{1,\tau},\widetilde{\Psi}^{1,\zeta}) = (-1/k^2)(\alpha^{0,\tau},\psi^{0,\zeta}),
\end{aligned}$$

$$\begin{aligned}
&(\mathfrak{A}'\widetilde{A}^{0,\tau},\widetilde{A}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\widetilde{A}^{1,\tau},\widetilde{A}^{0,\zeta}) = (\mathfrak{A}'\widetilde{A}^{0,\tau},\widetilde{A}^{1,\zeta}) = (1/k)(\alpha^{0,\tau},\alpha^{0,\zeta}), \quad \tau,\zeta = 1,\dots J_N, \qquad (2.2.102)\\
&(\mathfrak{A}'\widetilde{A}^{1,\tau},\widetilde{A}^{1,\zeta}) = (1/k)[(\alpha^{0,\tau},\alpha^{1,\zeta}) + (\alpha^{1,\tau},\alpha^{0,\zeta})].
\end{aligned}$$

We now require that the bases $\alpha^{0,1},\dots,\alpha^{0,J_N}$ and $\psi^{0,1},\dots,\psi^{0,J_N}$ coincide and are normalized by (2.2.91). We choose the generalized eigenvectors to be zero: $\alpha^{1,\zeta} = \psi^{1,\zeta} = 0$ for $\zeta = 1,\dots J_N$ (see the paragraph after the formulation of Proposition 2.2.3). As in the case of $\lambda_0 \neq 0$, define the vectors $\breve{A}^{\tau} = k\widetilde{A}^{\tau}$ and $\breve{\Psi}^{\zeta} = k\widetilde{\Psi}^{\zeta}$. Then (2.2.100) and (2.2.101) take the form

$$\begin{aligned}
&(\mathfrak{A}'\breve{\Psi}^{0,\tau},\breve{\Psi}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\breve{\Psi}^{1,\tau},\breve{\Psi}^{0,\zeta}) = (\mathfrak{A}'\breve{\Psi}^{0,\tau},\breve{\Psi}^{1,\zeta}) = \operatorname{sgn}(k)\delta_{\tau,\zeta}, \quad \tau,\zeta = 1,\dots J_N, \qquad (2.2.103)\\
&(\mathfrak{A}'\breve{\Psi}^{1,\tau},\breve{\Psi}^{1,\zeta}) = 0,
\end{aligned}$$

$$\begin{aligned}
&(\mathfrak{A}'\breve{A}^{0,\tau},\breve{\Psi}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\breve{A}^{1,\tau},\breve{\Psi}^{0,\zeta}) = (\mathfrak{A}'\breve{A}^{0,\tau},\breve{\Psi}^{1,\zeta}) = 0, \quad \tau,\zeta = 1,\dots J_N. \qquad (2.2.104)\\
&(\mathfrak{A}'\breve{A}^{1,\tau},\breve{\Psi}^{1,\zeta}) = (-1/|k|)\delta_{\tau,\zeta},
\end{aligned}$$

The third equality in (2.2.104) for $\tau = \zeta$ contradicts the orthogonality condition (2.2.98) and has to be modified. Set $\widehat{A}^{0,\tau} = \breve{A}^{0,\tau}$ and $\widehat{A}^{1,\tau} = \breve{A}^{1,\tau} + 1/k\breve{\Psi}^{0,\tau}$ and obtain

$$\begin{aligned}
&(\mathfrak{A}'\widehat{A}^{0,\tau},\breve{\Psi}^{0,\zeta}) = 0,\\
&(\mathfrak{A}'\widehat{A}^{1,\tau},\breve{\Psi}^{0,\zeta}) = (\mathfrak{A}'\widehat{A}^{0,\tau},\breve{\Psi}^{1,\zeta}) = 0, \quad \tau,\zeta = 1,\dots J_N, \qquad (2.2.105)\\
&(\mathfrak{A}'\widehat{A}^{1,\tau},\breve{\Psi}^{1,\zeta}) = 0.
\end{aligned}$$

The vectors $\widehat{A}^{0,\tau}, \widehat{A}^{1,\tau}$ satisfy the conditions (2.2.97):

$(\mathfrak{A}'\widehat{A}^{0,\tau}, \widehat{A}^{0,\zeta}) = 0,$
$(\mathfrak{A}'\widehat{A}^{1,\tau}, \widehat{A}^{0,\zeta}) = (\mathfrak{A}'\widehat{A}^{0,\tau}, \widehat{A}^{1,\zeta}) = \operatorname{sgn}(k)\delta_{\tau,\zeta}, \quad \tau, \zeta = 1, \dots J_N,$
$(\mathfrak{A}'\widehat{A}^{1,\tau}, \widehat{A}^{1,\zeta}) = 0.$

For $\Psi^{j,\zeta} = \check{\Psi}^{j,\zeta}$ and $A^{i,\tau} = \widehat{A}^{i,\tau}$ the relations (2.2.93),(2.2.96) and (2.2.97),(2.2.98) hold. The derivation of (2.2.94),(2.2.95) for the chains B and Φ do not involve the chains A and Ψ. $\square$

2.3 Waves

In the Sections 2.3-2.5 we specify for the problem (2.1.9) the results, established by Nazarov et al. [25] for the general elliptic problems self-adjoint with respect to a Green formula. To this end, we use the knowledge of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, k)$, obtained in Section 2.2. We introduce the space of waves, describe the continuous spectrum eigenfunctions, define the unitary scattering matrix, and present a well-posed problem for the augmented Maxwell system with intrinsic radiation conditions.

First we briefly summarize Section 2.2. Let us consider the problem (2.1.9) taking as domain G the cylinder

$$\Pi = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega, x_3 \in \mathbb{R}\},$$

where Ω is a one-connected domain with smooth boundary $\partial\Omega$. Assume the parameter k to be real. Let $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda, k)$ be the operator pencil (2.2.6) with domain $\mathcal{D}H^l(\Omega)$ (see (2.2.5)). From Section 2.2, it follows that for every $k \in \mathbb{R}$ there exist real eigenvalues of the pencil $\mathfrak{A}(\cdot, k)$. They are symmetric about the origin; the collections of the partial multiplicities of symmetric eigenvalues coincide. The nonzero eigenvalues have no generalized eigenvectors. The number 0 is an eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$ if and only if k^2 is an eigenvalue for at least one of the boundary value problems

$$\begin{aligned} (\partial_1^2 + \partial_2^2)u(x) + k^2 u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega; \\ (\partial_1^2 + \partial_2^2)u(x) + k^2 u(x) = 0, \quad x \in \Omega, \quad \partial_\nu u(x) = 0, \quad x \in \partial\Omega. \end{aligned} \qquad (2.3.1)$$

If in addition $k \neq 0$, then every partial multiplicity of the eigenvalue 0 is equal to 2. If $k = 0$, then to the eigenvalue 0 there correspond two linearly independent eigenvectors of the pencil $\mathfrak{A}(\cdot, 0)$, while the generalized eigenvectors do not exist. Thus for each k, the sum of the total multiplicities of all real eigenvalues of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, k)$ turns out to be even.

A value k is called a threshold of the pencil $\mathfrak{A}$, if k^2 is an eigenvalue for at least one of the problems in (2.3.1). (The first threshold on the semiaxis $0 < k < \infty$ is $\sqrt{\mu_2}$ because $\mu_2 < \lambda_1$, where μ_2 (λ_1) is the second (first) eigenvalue of the Neumann (Dirichlet) problem, see [11] and references there.) Denote by $\Sigma_{\mathfrak{A}}(k)$ and

$\Sigma_{\mathfrak{M}}(k)$ the sum of the total multiplicities of all real eigenvalues of the pencils $\mathfrak{A}(\cdot, k)$ and $\mathfrak{M}(\cdot, k)$, respectively. Both functions $\Sigma_{\mathfrak{A}}$ and $\Sigma_{\mathfrak{M}}$ are even. On the semiaxis $0 < k < \infty$, each of these functions is constant on any open interval whose ends coincide with neighboring thresholds and step-wise increases at every threshold being continuous from the right. There hold the equalities

$$\Sigma_{\mathfrak{A}}(k) = 2 \text{ for } k \in (-\sqrt{\mu_2}, \sqrt{\mu_2}), \tag{2.3.2}$$

$$\Sigma_{\mathfrak{M}}(k) = 0 \text{ for } k \in (-\sqrt{\mu_2}, \sqrt{\mu_2}) \setminus 0, \qquad \Sigma_{\mathfrak{M}}(0) = 1. \tag{2.3.3}$$

We first suppose, that a (fixed) number k is not a threshold. Then all eigenvalues of the pencil $\mathfrak{A}(\cdot, k)$ are nonzero, the generalized eigenvectors do not exist, and the chosen bases of eigenvectors are described by Proposition 2.2.7. Denote by P any vector of such a basis corresponding to an eigenvalue λ_0 and introduce the vector function

$$\Pi \ni x \mapsto \mathcal{P}(x) := \exp(i\lambda_0 x_3) P(x_1, x_2). \tag{2.3.4}$$

The function $\mathcal{P}$ satisfies the problem (2.1.9) in the cylinder Π. Let $\mathcal{R}$ be a function of the same type with any basis vector R in Proposition 2.2.7, that corresponds possibly to another eigenvalue λ_1. Let also χ be a smooth cut-off function, $\chi \geq 0$, $\chi(t) = 0$ for $t \leq T - 1$ and $\chi(t) = 1$ for $t \geq T$. We introduce the bilinear form

$$q_N(\chi\mathcal{P}, \chi\mathcal{R}) := (\mathcal{A}(D)\chi\mathcal{P}, \chi\mathcal{R})_{\Pi(N)} - (\chi\mathcal{P}, \mathcal{A}(D)\chi\mathcal{R})_{\Pi(N)}, \tag{2.3.5}$$

where $\Pi(N) = \{x = (x_1, x_2, x_3) \in \Pi : x_3 < N\}$, while $T < N$. We have

$$\mathcal{A}(D)(\chi\mathcal{P}) = \chi\mathcal{A}(D)\mathcal{P} + [\mathcal{A}(D), \chi]\mathcal{P} = [\mathcal{A}(D), \chi]\mathcal{P},$$

where $[\mathcal{A}(D), \chi] = \mathcal{A}(D)\chi - \chi\mathcal{A}(D)$. Therefore the first inner product on the right remains constant when N is increasing; owing to a similar reason, the second product remains constant as well. Hence we can define the form

$$q(\chi\mathcal{P}, \chi\mathcal{R}) := (\mathcal{A}(D)\chi\mathcal{P}, \chi\mathcal{R})_{\Pi} - (\chi\mathcal{P}, \mathcal{A}(D)\chi\mathcal{R})_{\Pi}. \tag{2.3.6}$$

Proposition 2.3.1. *Assume, that k is not a threshold and let λ_0 and λ_1 be eigenvalues of the pencil $\mathfrak{A}(\cdot, k)$ (the equality $\lambda_0 = \lambda_1$ is not ruled out). We also assume that*

$$\mathcal{P}(x) = \exp(i\lambda_0 x_3) P(x_1, x_2), \quad \mathcal{R}(x) = \exp(i\lambda_1 x_3) R(x_1, x_2),$$

where P is a vector in the list (2.2.76), (2.2.77) *corresponding to λ_0, and R is a vector in such a list corresponding to λ_1. Then*

$$q(\chi\mathcal{P}, \chi\mathcal{P}) = -i \operatorname{sgn}(k\lambda_0); \quad q(\chi\mathcal{R}, \chi\mathcal{R}) = -i \operatorname{sgn}(k\lambda_1); \tag{2.3.7}$$

$$q(\chi\mathcal{P}, \chi\mathcal{R}) = 0 \quad \textit{for} \quad \mathcal{P} \neq \mathcal{R}. \tag{2.3.8}$$

Proof. We write down the operator $\mathcal{A}$ in the form

$$\mathcal{A}(D_1, D_2, D_3) = \mathcal{A}(D_1, D_2, 0) + \mathcal{A}' D_3, \tag{2.3.9}$$

where $\mathcal{A}'$ is a numerical matrix and integrate by parts the first term on the right in (2.3.5). Since the operator of problem (2.1.9) is formally self-adjoint (Proposition 2.1.3), we obtain

$$q_N(\chi\mathcal{P},\chi\mathcal{R}) = -i\exp\left(i(\lambda_0-\lambda_1)N\right)(\mathcal{A}'P,R)_\Omega.$$

As before, we have $q_N(\chi\mathcal{P},\chi\mathcal{R}) = q(\chi\mathcal{P},\chi\mathcal{R})$. On the other hand, from (2.2.2) it follows that $\mathfrak{A}' = \mathcal{A}'$, so

$$q(\chi\mathcal{P},\chi\mathcal{R}) = -i\exp\left(i(\lambda_0-\lambda_1)N\right)(\mathfrak{A}'P,R)_\Omega. \tag{2.3.10}$$

If the eigenvectors P and R of the pencil $\mathfrak{A}(\cdot,k)$ correspond to the same eigenvalue, then (2.3.10) and Proposition 2.2.7 provide the equalities (2.3.7). In the case of $\lambda_0 \neq \lambda_1$, the right-hand-side in (2.3.10) is independent of N only if $(\mathfrak{A}'P,R)_\Omega = 0$, whereas the left-hand-side is always independent of N. Therefore, the equality $(\mathfrak{A}'P,R)_\Omega = 0$ holds for all eigenvectors P and R corresponding to distinct eigenvalues of $\mathfrak{A}(\cdot,k)$, so by virtue of (2.3.10), the equality (2.3.8) holds as well. □

We now suppose that k is a threshold and $k \neq 0$. The case in which $k = 0$ is special and we will discuss it separately in Section 4.5. Then all nonzero eigenvalues of $\mathfrak{A}(\cdot,k)$ are simple and the corresponding vector functions $\mathcal{P}$ are constructed as in the non-threshold case (see (2.3.4)). The Jordan chains for $\lambda_0 = 0$ are described by Proposition 2.2.10. Let $\{P^0,P^1\}$ be such a chain. Introduce the vector functions

$$\Pi \ni (x_1,x_2,x_3) \mapsto \mathcal{P}^{\pm}(x_1,x_2,x_3) := \frac{(ix_3 \pm 1)P^0(x_1,x_2) + P^1(x_1,x_2)}{\sqrt{2}}. \tag{2.3.11}$$

Proposition 2.3.2. *The functions $\mathcal{P}^{\pm}$ satisfy the problem* (2.1.9) *in the cylinder* Π.

Proof. Taking account of (2.3.9), we have

$$\mathcal{A}(D_1,D_2,D_3)\mathcal{P}^{\pm}(x) = \frac{\mathcal{A}(D_1,D_2,0)P^1(x_1,x_2) + \mathcal{A}'P^0(x_1,x_2)}{\sqrt{2}}. \tag{2.3.12}$$

From (2.2.2) it follows that $\mathfrak{A}(0,k) = \mathcal{A}(D_1,D_2,0)$ and $\mathfrak{A}' = \mathcal{A}'$. The right-hand-side (2.3.12) vanishes by virtue of (2.2.34). □

For the nonzero eigenvalues of the pencil $\mathfrak{A}(\cdot,k)$, we define functions $\mathcal{P}$ of the form (2.3.4), and for the eigenvalue 0 introduce the functions (2.3.11). For functions $\mathcal{P}$ and $\mathcal{R}$ of such type, corresponding to (possibly, distinct) eigenvalues of $\mathfrak{A}(\cdot,k)$, we introduce the form q as before by the formula (2.3.6).

Proposition 2.3.3. *Let k be a threshold, $k \neq 0$, and let λ_0 and λ_1 be eigenvalues of the pencil $\mathfrak{A}(\cdot,k)$ (the equality $\lambda_0 = \lambda_1$ is not ruled out). Assume that $\mathcal{P}$ and $\mathcal{R}$ are defined for λ_0 and λ_1 relatively in the manner above. Then the following equalities hold*

$$\begin{aligned} q(\chi\mathcal{P},\chi\mathcal{P}) &= -i\,\mathrm{sgn}\,(k\lambda_0) \quad \textit{for} \quad \lambda_0 \neq 0; \\ q(\chi\mathcal{P},\chi\mathcal{P}) &= -i\,\mathrm{sgn}\,(\pm k) \quad \textit{for} \quad \lambda_0 = 0, \quad \mathcal{P} = \mathcal{P}^{\pm}, \end{aligned} \tag{2.3.13}$$

and those obtained by replacing $\mathcal{P}$ and λ_0 by $\mathcal{R}$ and λ_1. Moreover,

$$q(\chi\mathcal{P},\chi\mathcal{R}) = 0 \quad \textit{for} \quad \mathcal{P} \neq \mathcal{R}. \tag{2.3.14}$$

Proof. We repeat the argument in the proof of Proposition 2.3.1. Let us write down the operator $\mathcal{A}$ in the form (2.3.9) and integrate by parts the first term in (2.3.5). Since the operator of problem (2.1.9) is formally self-adjoint, we have

$$q(\chi\mathcal{P},\chi\mathcal{R}) = -i(\mathcal{A}'\mathcal{P},\mathcal{R})_\Omega|_{x_3=N} = -i\exp\left(i(\lambda_0-\lambda_1)N\right)(a_2N^2+a_1N+a_0),$$

where a_2, a_1, a_0 are the constants determined by the functions $\mathcal{P}$ and $\mathcal{R}$. If $\lambda_1 \neq \lambda_0$, then the right-hand-side is independent of N only in the case that it vanishes. Thus $q(\chi\mathcal{P},\chi\mathcal{R}) = 0$ for $\lambda_1 \neq \lambda_0$.

Assume now, that the eigenvalues coincide. If $\lambda_0 = \lambda_1 \neq 0$, we repeat the proof given in the non-threshold situation (Proposition 2.3.1). It remains to consider the case of $\lambda_0 = \lambda_1 = 0$. Let $\mathcal{P},\mathcal{R}$ be given by (2.3.11), where $\mathcal{P} = \mathcal{P}^\sigma, \mathcal{R} = \mathcal{R}^\tau$, and σ, τ take values $+$ and $-$. Then

$$q(\chi\mathcal{P}^\sigma,\chi\mathcal{R}^\tau) = -i/2\Big(N^2(\mathfrak{A}'P^0,R^0)_\Omega + (\mathfrak{A}'(\sigma P^0+P^1),\tau R^0+R^1)_\Omega + \\ +iN\big((\mathfrak{A}'P^0,\tau R^0+R^1)_\Omega - (\mathfrak{A}'(\sigma P^0+P^1),R^0)_\Omega\big)\Big) \qquad (2.3.15)$$

The right-hand-side is independent of N only if

$$(\mathfrak{A}'P^0,R^0)_\Omega = 0, \quad (\mathfrak{A}'P^0,R^1)_\Omega - (\mathfrak{A}'P^1,R^0)_\Omega = 0. \qquad (2.3.16)$$

According to Proposition 2.2.10, if the Jordan chains $\{P^0,P^1\}$ and $\{R^0,R^1\}$ are distinct and/or $\sigma \neq \tau$, then the right-hand-side of (2.3.15) vanishes. If the chains coincide and $\sigma = \tau$ (that is $\mathcal{R} = \mathcal{P} = \mathcal{P}^\sigma$), the second equality of (2.3.13) holds. □

Let us return to the domain G with N outlets to infinity (defined at the beginning of Section 2.1). We assume that ρ_β is a smooth positive function on $\overline{G}$, given on $\Pi^q_+ \cap G$ by $\rho_\beta(y^q,t^q) = \exp(\beta t^q)$, where $q = 1,\dots,N$, and β is a real number. For $l = 0,1,\dots$ we introduce the space $H^l_\beta(G)$ of functions on G with the norm

$$\|u;H^l_\beta(G)\| := \|\rho_\beta u;H^l(G)\| := \Big(\sum_{|\alpha|=0}^{l}\int_G |D^\alpha(\rho_\beta u)|^2\,dx\Big)^{1/2}. \qquad (2.3.17)$$

Denote by $H^{l-1/2}_\beta(\partial G)$ for $l = 1,2,\dots$ the space of traces on ∂G of functions in $H^l_\beta(G)$. We will denote the spaces of vector functions with components in $H^l_\beta(G)$ and $H^{l-1/2}_\beta(\partial G)$ in the same manner as their scalar analogues. The operator $\{\mathcal{A},\mathcal{B}\}$ of problem (2.1.9) implements a continuous mapping

$$\mathcal{L}_\beta = \{\mathcal{A},\mathcal{B}\} : H^l_\beta(G) \to H^{l-1}_\beta(G) \times H^{l-1/2}_\beta(\partial G) \qquad (2.3.18)$$

for $l = 1,2,\dots$ and any $\beta \in \mathbb{R}$. The operator (2.3.18) is a Fredholm operator if and only if the line $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta\}$ is free from the spectrum of every pencil $\lambda \mapsto \mathfrak{A}^q(\lambda,k)$ defined for the problem (2.1.9) in the cylinder Π^q, $q = 1,\dots,N$, the number k being fixed. Recall that an operator is called a Fredholm operator if its range is closed and the kernel and cokernel are finite dimensional.

Denote by 2Y the sum of total multiplicities of all real eigenvalues of the pencils $\mathfrak{A}^q(\cdot,k)$, and by δ a small positive number such that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \leq \delta\}$ contains no eigenvalues of the pencils $\mathfrak{A}^q(\cdot,k)$ except the real ones.

Proposition 2.3.4. *There hold the equalities*

$$\dim \operatorname{coker} \mathcal{L}_{-\delta} = \dim \ker \mathcal{L}_{\delta} =: d,$$
$$\dim \operatorname{coker} \mathcal{L}_{\delta} = \dim \ker \mathcal{L}_{-\delta} = d + \mathrm{Y}.$$

Let us explain these formulas. Since the operator $\mathcal{L}_\delta$ is Fredholm (as well as $\mathcal{L}_{-\delta}$), the number d is finite. A basis $z_1, \dots, z_d$ in the space $\ker \mathcal{L}_\delta$ consists of solutions to the homogeneous problem (2.1.9) exponentially vanishing at infinity. One can obtain a basis in $\ker \mathcal{L}_{-\delta}$ by adding to the collection $z_1, \dots, z_d$ some solutions $\zeta_1, \dots, \zeta_{\mathrm{Y}}$ of the homogeneous problem (2.1.9) in $H^l_{-\delta}(G)$ linearly independent modulo $H^l_\delta(G)$ (a linear combination $c_1\zeta_1 + \dots + c_{\mathrm{Y}}\zeta_{\mathrm{Y}}$ with constant coefficients c_j belongs to $H^l_\delta(G)$ if and only if $c_1 = \dots = c_{\mathrm{Y}} = 0$).

Let k be a threshold of none of the pencils $\mathfrak{A}^q(\cdot,k)$. We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $t > T$ and $\chi(t) = 0$ for $t < T-1$, where T is sufficiently large. For every real eigenvalue of $\mathfrak{A}^q(\cdot,k)$ and all solutions of the form (2.3.4) to the homogeneous problem (2.1.9) in the cylinder Π^q, we introduce the functions $\Pi^q \ni (y^q,t^q) \mapsto \chi(t^q)\mathcal{P}(y^q,t^q)$. Extend these functions by zero to the domain G. If k is a threshold of the pencil $\mathfrak{A}^q(\cdot,k)$, then we in addition introduce the functions $\Pi^q \ni (y^q,t^q) \mapsto \chi(t^q)\mathcal{P}(y^q,t^q)$ that correspond to the eigenvalue $\lambda_0 = 0$ and all solutions of the form (2.3.11). We also extend such functions by zero to G. As a result, we obtain the 2Y functions $v_1, \dots, v_{2\mathrm{Y}}$ on G. Denote by $\mathfrak{N}$ the space spanned by $v_1, \dots, v_{2\mathrm{Y}}$ and introduce the quotient space $\mathcal{W} = (\mathfrak{N} \dotplus H^l_\delta(G))/H^l_\delta(G)$ whose dimension $\dim \mathcal{W}$ equals 2Y. The elements of $\mathcal{W}$ are called waves.

By arguments similar to those in the first part of Section 2.3, including Proposition 2.3.1 and its proof, one can verify that for $u, v \in \mathfrak{N} \dotplus H^l_\delta(G)$ the bilinear form

$$q_G(u,v) := (\mathcal{A}u,v)_G + (\mathcal{B}u,\mathcal{Q}v)_{\partial G} - (u,\mathcal{A}v)_G - (\mathcal{Q}u,\mathcal{B}v)_{\partial G} \tag{2.3.19}$$

takes finite values. If at least one of the elements u, v belongs to $H^l_\delta(G)$, then $q_G(u,v) = 0$ and hence the form q_G is defined on $\mathcal{W} \times \mathcal{W}$. For any waves $U, V \in \mathcal{W}$ there holds the equality $q_G(U,V) = -\overline{q_G(V,U)}$, so for every wave $U \in \mathcal{W}$ the number $q_G(U,U)$ is imaginary. A wave U is called by outgoing (incoming), if $iq_G(U,U)$ is a positive (negative) number.

Proposition 2.3.5. 1^0. *In the space $\mathcal{W}$, there exists a basis $U_1^+, \dots, U_{\mathrm{Y}}^+, U_1^-, \dots, U_{\mathrm{Y}}^-$ subject to the orthogonality and normalization conditions*

$$q_G(U_j^\pm, U_k^\mp) = 0 \quad iq_G(U_j^\pm, U_k^\pm) = \mp\delta_{jk}, \quad j,k = 1, \dots, \mathrm{Y}. \tag{2.3.20}$$

Thus the waves $U_1^-, \dots, U_{\mathrm{Y}}^-$ are outgoing and the waves $U_1^+, \dots, U_{\mathrm{Y}}^+$ are incoming.

2^0. *If a basis $W_1, \dots, W_{2\mathrm{Y}}$ consists of incoming and outgoing orthogonal waves, then it contains* Y *ougoing and* Y *incoming waves.*

The proof of Proposition 2.3.1 shows that the equalities (2.3.7) remain valid if we change q for q_G and as $\mathcal{P}$ and $\mathcal{Q}$ take functions from the collection $v_1, \ldots, v_{2\Upsilon}$ defined before (2.3.19). Then (possibly, after renumbering) the cosets $V_1, \ldots, V_{2\Upsilon}$ of $v_1, \ldots, v_{2\Upsilon}$ mod $H^l_\delta(G)$ form a basis in $\mathcal{W}$ satisfying (2.3.20).

2.4 Eigenfunctions of continuous spectrum. Scattering matrix

By definition, $k \in \mathbb{R}$ belongs to the continuous spectrum of the operator $\mathcal{L}_\delta = \mathcal{L}_\delta(k)$ if there exists a function in $\ker \mathcal{L}_{-\delta}(k)$ that does not belong to $H^1_\delta(G)$. Such functions are called the continuous spectrum eigenfunctions of the operator $\mathcal{L}_\delta(k)$ (or problem (2.1.9)). To define the scattering matrix related to the (homogeneous) problem (2.1.9), we need a special basis for the space of continuous spectrum eigenfunction.

Proposition 2.4.1. *Let $U^+_1, \ldots, U^+_\Upsilon, U^-_1, \ldots, U^-_\Upsilon$ be the basis of $\mathcal{W}$, mentioned in Proposition* 2.3.5 (1^0). *Then there exist bases $Y^+_1, \ldots, Y^+_\Upsilon$ and $Y^-_1, \ldots, Y^-_\Upsilon$ in* $\ker \mathcal{L}_{-\delta}(k)$ *modulo $H^1_\delta(G)$, such that*

$$\dot{Y}^-_j = U^-_j + \sum_{l=1}^{\Upsilon} T_{jl} U^+_l, \tag{2.4.1}$$

$$\dot{Y}^+_j = U^+_j + \sum_{l=1}^{\Upsilon} S_{jl} U^-_l, \tag{2.4.2}$$

where $j = 1, \ldots, \Upsilon$ and $\dot{x}$ is the coset representative of $x \in \mathfrak{N} \dotplus H^l_\delta(G)$. The $\Upsilon \times \Upsilon$-matrix $S(k) = \|S_{jl}(k)\|$ that consists of the coefficients in (2.4.2) *is unitary , $S^* = S^{-1}$. Moreover, $S^{-1} = T = \|T_{jl}\|$, where T_{jl} are the coefficients in* (2.4.1).

The matrix $S(k) = \|S_{jl}(k)\|$ is independent of the mentioned coset representatives and is called the scattering matrix (recall that $U^-_1, \ldots, U^-_\Upsilon$ are outgoing waves and $U^+_1, \ldots, U^+_\Upsilon$ are incoming ones).

From Proposition 2.4.1, formula (2.3.2) and the paragraph in Section 2.3 preceding this formula, it follows that the continuous spectrum of problem (2.1.9) covers the whole real axis. The dimension of the space of continuous spectrum eigenfunction (modulo $H^1_\delta(G)$) at a point k is called the multiplicity of continuous spectrum at k. The multiplicity is equal to

$$\Sigma^1_{\mathfrak{A}}(k)/2 + \cdots + \Sigma^N_{\mathfrak{A}}(k)/2,$$

where N is the number of the outlets of G to infinity and $\Sigma^j_{\mathfrak{A}}(k)$ is the sum of the total multiplicities of the real eigenvalues of the pencil $\mathfrak{A}^j(\cdot, k)$.

2.5 Radiation principle

Let $U_1^+,\dots,U_\Upsilon^+,U_1^-,\dots,U_\Upsilon^-$ be the basis in Proposition 2.3.5 (1^0). We choose any $u_j^- \in U_j^-$, $j=1,\dots,\Upsilon$, and denote by $\mathfrak{S}$ the linear hull of the functions $u_1^-,\dots,u_\Upsilon^-$. On the space $\mathfrak{S} \dotplus H_\delta^l(G)$, we consider the restriction $\mathbb{L}$ of $\mathcal{L}_{-\delta}(k)$, which is a continuous mapping

$$\mathbb{L} : \mathfrak{S} \dotplus H_\delta^l(G) \to H_\delta^{l-1}(G) \times H_\delta^{l-1/2}(\partial G).$$

Proposition 2.5.1. *Let $z_1,\dots,z_d$ be a basis in* $\ker \mathcal{L}_\delta(k)$ *and $\{\mathcal{F},\mathcal{G}\} \in H_\delta^{l-1}(G) \times H_\delta^{l-1/2}(\partial G)$ with*

$$(\mathcal{F},z_j)_G + (\mathcal{G},\mathcal{Q}z_j)_{\partial G} = 0, \quad j=1,\dots,d,$$

where $\mathcal{Q}$ is the operator in the Green formula (2.1.10). *Then*

1^0. *There exists a solution $\mathcal{U} \in \mathfrak{S} \dotplus H_\delta^l(G)$ to $\mathbb{L}\mathcal{U} = \{\mathcal{F},\mathcal{G}\}$ determined up to an arbitrary term in* $\ker \mathcal{L}_\delta(k)$.

2^0. *The inclusion*

$$\mathcal{V} := \mathcal{U} - c_1 u_1^- - \dots - c_\Upsilon u_\Upsilon^- \in H_\delta^l(G), \tag{2.5.1}$$

holds with

$$c_j = i(\mathcal{F},Y_j^-)_G + i(\mathcal{G},\mathcal{Q}Y_j^-)_{\partial G}, \quad j=1,\dots,\Upsilon, \tag{2.5.2}$$

where $\zeta_1,\dots,\zeta_\Upsilon$ are elements in $\ker \mathcal{L}_{-\delta}(k)$ *satisfying* (2.4.1).

3^0. *For such a solution $\mathcal{U}$ there holds the inequality*

$$\begin{aligned} &\|\mathcal{V};H_\delta^l(G)\| + |c_1| + \dots + |c_\Upsilon| \\ &\le \mathrm{const}(\|\mathcal{F};H_\delta^{l-1}(G)\| + \|\mathcal{G};H_\delta^{l-1/2}\partial G\| + \|\rho_\delta\mathcal{V};L_2(G)\|). \end{aligned} \tag{2.5.3}$$

A solution $\mathcal{U}_0$ that satisfies the additional conditions $(\mathcal{U}_0,z_j)_G = 0$, $j=1,\dots,d$, is unique and subject to the estimate (2.5.3) *with the right-hand-side replaced by* $\mathrm{const}(\|\mathcal{F};H_\delta^{l-1}(G)\| + \|\mathcal{G};H_\delta^{l-1/2}\partial G\|)$.

The inclusion (2.5.1) involves only representatives of outgoing waves. Such an inclusion is called an intrinsic radiation condition. Then Proposition 2.5.1 is entitled as the “radiation principle”.

Remark 2.5.2. *Proposition* 2.5.1 *remains valid if the incoming and outgoing waves exchange their roles. A radiation condition involving only incoming waves will also be named intrinsic. If required, we will explain what kind of intrinsic radiation is meant, "incoming" or "outgoing".*

3 METHOD FOR COMPUTING THE SCATTERING MATRIX BETWEEN THRESHOLDS

In this Chapter, we propose and justify a method for approximating the scattering matrix under the condition, that the spectral parameter k belongs to an interval of the continuous spectrum containing no thresholds. A minimizer of a quadratic functional $J_l^R(\cdot)$ serves as an approximation to a row S_l of the scattering matrix S. To construct such a functional, we solve an auxiliary boundary value problem in the bounded domain G^R, which is obtained by cutting off the cylindrical ends of the domain G from a distance R,(see FIG. 2). The auxiliary problem is proved to have a unique solution. As R approaches infinity, the minimizer approaches S_l at an exponential rate uniformly with respect to the spectral parameter.

In Section 3.1, we outline the description of the method. In Section 3.2 we prove that the auxiliary problem has a unique solution in a function class. Section 3.3 is devoted to justification of the method, where we prove the existence of the functional minimizer and establish a convergence estimate.

3.1 Description of the method

For approximating the scattering matrix S of elliptic problem (2.1.9) we will use the scheme given in [31]. We set

$$\Pi_+^{q,R} = \{(y^q, t^q) \in \Pi^q : t^q > R\}, \quad G^R = G \setminus \cup_{q=1}^{\mathcal{T}} \Pi_+^{q,R} \tag{3.1.1}$$

for a sufficiently large R; then $\partial G^R \setminus \partial G = \Gamma^R = \cup_{q=1}^{\mathcal{T}} \Gamma^{q,R}$, where $\Gamma^{q,R} = \{(y^q, t^q) \in \Pi^q : t^q = R\}$. We are going to calculate the line $(S_{l1}, \dots, S_{l\Upsilon})$, $l = 1, \dots, \Upsilon$, of the matrix S. As an approximation to this line, we take a minimizer $a^0 = (a_1^0, \dots, a_\Upsilon^0) \in \mathbb{C}^\Upsilon$ of the quadratic functional

$$J_l^R(a_1, \dots, a_\Upsilon) = \| \mathcal{Q}(X_l^R(\cdot; a) - u_l^+ - \sum_{j=1}^{\Upsilon} a_j u_j^-); L_2(\Gamma^R) \|^2. \tag{3.1.2}$$

Here $X_l^R(\cdot;a)$ is a solution to the problem

$$\begin{aligned} \mathcal{A}(D,k)\mathcal{U}(x) &= \mathcal{F}(x), \quad x \in G^R, \\ \mathcal{B}(x)\mathcal{U}(x) &= \mathcal{G}(x), \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{B}(x) + i\mathcal{Q}(x))\mathcal{U}(x) &= \mathcal{H}(x), \quad x \in \Gamma^R, \end{aligned} \tag{3.1.3}$$

with right-hand-side

$$\mathcal{F} = 0, \quad \mathcal{G} = 0, \quad \mathcal{H} = \Big(\mathcal{B} + i\mathcal{Q}\Big)\Big(u_l^+(\cdot,k) + \sum_{j=1}^{\Upsilon} a_j u_j^-(\cdot,k)\Big). \tag{3.1.4}$$

Let $Z_{j,R}^{\pm}$ be a solution to the problem (3.1.3) with $\mathcal{F} = 0$, $\mathcal{G} = 0$, and $\mathcal{H} = \big(\mathcal{B} + i\mathcal{Q}\big)u_j^{\pm}(\cdot,k)$. We have $X_l^R = Z_{l,R}^+ + \sum a_j Z_{j,R}^-$ and can write the functional in the form

$$J_l^R(a;k) = \langle aE^R, a\rangle + 2\mathrm{Re}\,\langle F_l^R, a\rangle + G_l^R, \tag{3.1.5}$$

where E^R and F^R are $\Upsilon \times \Upsilon$-matrices with entries

$$\begin{aligned} E_{ij}^R &= (\mathcal{Q}(Z_{i,R}^- - u_i^-), \mathcal{Q}(Z_{j,R}^- - u_j^-))_{\Gamma^R}, \\ F_{ij}^R &= (\mathcal{Q}(Z_{i,R}^+ - u_i^+), \mathcal{Q}(Z_{j,R}^- - u_j^-))_{\Gamma^R}. \end{aligned}$$

F_l^R is the row with number l of the matrix F^R, and $G_l^R = \|\mathcal{Q}(Z_{l,R}^+ - u_l^+)\|_{\Gamma^R}^2$. A minimizer $a^0(R)$ of functional (3.1.5) satisfies

$$a^0(R)E^R + F_l^R = 0. \tag{3.1.6}$$

The following theorem is a basic result of Chapter 3.

Theorem 3.1.1. *Let an interval $[k_1, k_2]$ in the continuous spectrum of problem* (2.1.9) *be free from thresholds. Then for all $k \in [k_1, k_2]$ and $R > R_0$, R_0 being a sufficiently large number, there exists a unique minimizer $a^0(R,k) = (a_1^0(R,k), \dots, a_\Upsilon^0(R,k))$ of the functional $J_l^R(\cdot,k)$ in* (3.1.2). *The inequalities*

$$|a_j^0(R,k) - S_{lj}(k)| \le Ce^{-\delta R},\ j = 1, \dots, \Upsilon, \tag{3.1.7}$$

hold with the same δ as in (2.4.1) *and a constant C independent of k and R.*

3.2 Solvability of auxiliary problem in G^R

The boundary ∂G^R of G^R contains the edges $\partial\Gamma^{q,R}$. We set $\Gamma^R = \cup_{q=1}^{\mathcal{T}} \partial\Gamma^{q,R}$ and denote by r a smooth positive function on $\overline{G^R} \setminus \partial\Gamma^R$ coinciding, near $\partial\Gamma^R$, with the distance $\operatorname{dist}(x, \partial\Gamma^R)$. Let $V_\beta^l(G^R)$, $l = 0,1$, be the closure of $C_c^\infty(\overline{G^R} \setminus \partial\Gamma^R)$ in the norm

$$\|u; V_\beta^l(G^R)\| = \Big(\sum_{|\alpha| \le l} \int_{G^R} r(x)^{2(\beta - l + |\alpha|)} |D_x^\alpha u(x)|^2 dx\Big)^{1/2}$$

amd let $V^{1/2}_\beta(\Gamma^R)$ and $V^{1/2}_\beta(\partial G^R \setminus \Gamma^R)$ be the space of traces of the functions in $V^1_\beta(G^R)$ on Γ^R and $\partial G^R \setminus \Gamma^R$ respectively.

Theorem 3.2.1. *For a sufficiently small $\varepsilon > 0$, $\beta \in (1/2-\varepsilon, 1/2+\varepsilon)$, and any*

$$\{\mathcal{F}, \mathcal{G}, \mathcal{H}\} \in V^0_\beta(G^R) \times V^{1/2}_\beta(\partial G^R \setminus \Gamma^R) \times V^{1/2}_\beta(\Gamma^R),$$

there exists a unique solution $\mathcal{U} \in V^1_\beta(G^R)$ to problem (3.1.3).

To prove Theorem 3.2.1, we make use of a scheme in the theory of elliptic boundary value problems in domains with piecewise smooth boundary (e.g. see [25], Chapter 8, and references therein).

Let $\mathbb{K} = \{(x_1, x_2) : x_1, x_2 > 0\}$, $\mathcal{O} = (0,0)$, and $\partial\mathbb{K}_j = \{(x_1, x_2) \in \partial\mathbb{K} : x_j > 0\}$, where $j = 1,2$. In the wedge $\mathbb{D} = \mathbb{K} \times \mathbb{R}$ with edge $M = \mathcal{O} \times \mathbb{R}$, we consider the problem

$$\begin{aligned} \mathcal{A}^0(D_1, D_2, D_3)\mathcal{U}(x) &= \mathcal{F}(x), \quad x \in \mathbb{D} = \mathbb{K} \times \mathbb{R}, \\ \mathcal{B}_1\mathcal{U}(x) &= \mathcal{G}(x), \quad x \in \partial\mathbb{D}_1 = \partial\mathbb{K}_1 \times \mathbb{R}, \\ (\mathcal{B}_2 + i\mathcal{Q}_2)\mathcal{U}(x) &= \mathcal{H}(x), \quad x \in \partial\mathbb{D}_2 = \partial\mathbb{K}_2 \times \mathbb{R}; \end{aligned} \tag{3.2.1}$$

here $\mathcal{A}^0(D) = \mathcal{A}(D;0)$ denotes the principal part of the differential operator $\mathcal{A}(D;k)$, while $\mathcal{B}_j = \mathcal{B}(x)|(x \in \partial\mathbb{D}_j)$ and $\mathcal{Q}_j = \mathcal{Q}(x)|(x \in \partial\mathbb{D}_j)$ are boundary operators with constant coefficients

$$\begin{aligned} \mathcal{B}_1\mathcal{U} &= (-u^1_3, u^1_1, -u^2_2, a^2), \quad \mathcal{Q}_1\mathcal{U} = -i(u^2_1, u^2_3, a^1, u^1_2), \\ \mathcal{B}_2\mathcal{U} &= (-u^1_2, u^1_3, -u^2_1, a^2), \quad \mathcal{Q}_2\mathcal{U} = -i(u^2_3, u^2_2, a^1, u^1_1). \end{aligned} \tag{3.2.2}$$

Problem (3.2.1) reduces to the problem

$$\begin{aligned} \mathcal{A}^0(D_\eta, \omega)U(\eta, \xi) &= F(\eta, \xi), \quad \eta \in \mathbb{K}, \\ \mathcal{B}_1 U(\eta, \xi) &= G(\eta, \xi), \quad \eta \in \partial\mathbb{K}_1, \\ (\mathcal{B}_2 + i\mathcal{Q}_2)U(\eta, \xi) &= H(\eta, \xi), \quad \eta \in \partial\mathbb{K}_2, \end{aligned} \tag{3.2.3}$$

in the angle $\mathbb{K}$ by means of the Fourier transform $\mathscr{F} = \mathscr{F}_{x_3 \to \xi}$ and the change of variables

$$\eta = (\eta_1, \eta_2) = |\xi|(x_1, x_2), \quad D_\eta = (D_{\eta_1}, D_{\eta_2}), \quad \omega = \xi/|\xi|, \tag{3.2.4}$$

$$U(\eta, \xi) = (\mathscr{F}\mathcal{U})(\eta/|\xi|, \xi), \quad F(\eta, \xi) = |\xi|^{-1}(\mathscr{F}\mathcal{F})(\eta/|\xi|, \xi), \tag{3.2.5}$$

$$H(\eta, \xi) = (\mathscr{F}\mathcal{H})(\eta/|\xi|, \xi), \quad G(\eta, \xi) = (\mathscr{F}\mathcal{G})(\eta/|\xi|, \xi). \tag{3.2.6}$$

According to Theorem 8.3.1 in [25], the operator of problem (3.1.3)

$$\mathcal{L}_\beta = \{\mathcal{A}(D;k), \mathcal{B}, \mathcal{B} + i\mathcal{Q}\} : V^1_\beta(G^R) \to V^0_\beta(G^R) \times V^{1/2}_\beta(\partial G^R \setminus \Gamma^R) \times V^{1/2}_\beta(\Gamma^R) \tag{3.2.7}$$

is Fredholm if and only if the operator of problem (3.2.3)

$$\mathcal{L}_\beta(\omega) = \{\mathcal{A}^0(D_\eta, \omega), \mathcal{B}_1, \mathcal{B}_2 + i\mathcal{Q}_2\} : E^1_\beta(\mathbb{K}) \to E^0_\beta(\mathbb{K}) \times E^{1/2}_\beta(\partial\mathbb{K}_1) \times E^{1/2}_\beta(\partial\mathbb{K}_2) \tag{3.2.8}$$

implements an isomorphism for $\omega = \pm 1$. Here $E^l_\beta(\mathbb{K})$, $l \geq 0$, is the completion of the set $C^\infty_c(\overline{\mathbb{K}} \setminus \mathcal{O})$ in the norm

$$\|u; E^l_\beta(\mathbb{K})\|^2 = \sum_{k_1+k_2 \leq l} \int_{\mathbb{K}} |\eta|^{2\beta}(1 + |\eta|^{2(k_1+k_2-l)})|D^{k_1}_{\eta_1} D^{k_2}_{\eta_2} u(\eta_1, \eta_2)|^2 d\eta_1 d\eta_2 \tag{3.2.9}$$

and $E^{1/2}_\beta(\partial\mathbb{K}_j)$ is the space of traces on $\partial\mathbb{K}_j$ of the functions in $E^1_\beta(\mathbb{K})$.

Let us find the values of β for which operator (3.2.8) implements an isomorphism. We introduce polar coordinates (r, φ) in $\mathbb{K}$, $r \in (0, +\infty)$, $\varphi \in (0, \pi/2)$, and define, on the space $C^1[0, \pi, 2]$, the operator pencil $\mathfrak{L}(\lambda) = \{\mathfrak{C}(D_\varphi, \lambda), \mathcal{B}_1, \mathcal{B}_2 + i\mathcal{Q}_2\}$, where

$$\mathfrak{C}(D_\varphi, \lambda)\Phi(\varphi) = r^{1-i\lambda}\mathcal{A}^0(D_1, D_2, 0)(r^{i\lambda}\Phi(\varphi)), \tag{3.2.10}$$

$\mathcal{B}_j$ and $\mathcal{Q}_j$ are given by (3.2.2), and

$$\mathfrak{C}(D_\varphi, \lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\varphi & \cos\varphi \\ 0 & 0 & 0 & 0 & -\cos\varphi & -\sin\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\varphi & -\cos\varphi & 0 & 0 \\ 0 & 0 & -\cos\varphi & \sin\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\varphi & -\cos\varphi & 0 & 0 & 0 & 0 \\ \cos\varphi & \sin\varphi & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} i\partial_\varphi & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & i\partial_\varphi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\partial_\varphi & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & i\partial_\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\partial_\varphi & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & i\partial_\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\partial_\varphi & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & i\partial_\varphi \end{pmatrix}. \tag{3.2.11}$$

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of the pencil $\mathfrak{L}$, if there exists a nonzero function $\Phi \in \mathbb{C}^1[0, \pi/2]$, satisfying

$$\mathfrak{C}(D_\varphi, \lambda)\Phi(\varphi) = 0, \quad \varphi \in (0, \pi/2), \tag{3.2.12}$$

$$\mathcal{B}_1\Phi(0) = 0, \quad (\mathcal{B}_2 + i\mathcal{Q}_2)\Phi(\pi/2) = 0. \tag{3.2.13}$$

Proposition 3.2.2. *There exist constant β_* and β^* such that $\beta_* < 1/2 < \beta^*$ and the strip $\{\lambda \in \mathbb{C} : \beta_* < \operatorname{Im}\lambda < \beta^*\}$ is free of the eigenvalues of the pencil $\mathfrak{L}$.*

Proof. We first show that the line $\operatorname{Im}\lambda = 1/2$ contains no eigenvalues of $\mathfrak{L}$. We will use the Green formula

$$\begin{aligned} 0 = (\mathfrak{C}(D_\varphi, \lambda)\Phi, \Psi)_{(0,\pi/2)} + \langle \mathcal{B}_1\Phi(0), \mathcal{Q}_1\Psi(0)\rangle + \langle \mathcal{B}_2\Phi(\pi/2), \mathcal{Q}_2\Psi(\pi/2)\rangle - \\ -(\Phi, \mathfrak{C}^*(D_\varphi, \bar{\lambda})\Psi)_{(0,\pi/2)} - \langle \mathcal{Q}_1\Phi(0), \mathcal{B}_1\Psi(0)\rangle - \langle \mathcal{Q}_2\Phi(\pi/2), \mathcal{B}_2\Psi(\pi/2)\rangle, \end{aligned} \tag{3.2.14}$$

where

$$\mathfrak{C}^*(D_\varphi,\lambda)\Phi(\varphi)=r^{2-i\lambda}\mathcal{A}^0(D_{x_1},D_{x_2},0)(r^{i\lambda-1}\Phi(\varphi))=\mathfrak{C}(D_\varphi,\lambda+i)\Phi(\varphi). \tag{3.2.15}$$

Let us assume, that there is an eigenvalue λ of the pencil $\mathfrak{L}$ with $\operatorname{Im}\lambda=1/2$ and let Φ be a corresponding eigenvector. Then $\bar\lambda+i=\lambda$ and, in view of (3.2.15) and (3.2.12),

$$\mathfrak{C}^*(D_\varphi,\bar\lambda)\Phi(\varphi)=\mathfrak{C}(D_\varphi,\bar\lambda+1)\Phi(\varphi)=\mathfrak{C}(D_\varphi,\lambda)\Phi(\varphi)=0. \tag{3.2.16}$$

According to (3.2.12), (3.2.13) and (3.2.16), equality (3.2.14) for $\Psi=\Phi$ reduces to

$$0=\langle\mathcal{B}_2\Phi(\pi/2),\mathcal{Q}_2\Phi(\pi/2)\rangle-\langle\mathcal{Q}_2\Phi(\pi/2),\mathcal{B}_2\Phi(\pi/2)\rangle.$$

From this, taking into account the second condition of (3.2.13), we obtain

$$0=\langle\mathcal{Q}_2\Phi(\pi/2),\mathcal{Q}_2\Phi(\pi/2)\rangle=\langle\mathcal{B}_2\Phi(\pi/2),\mathcal{B}_2\Phi(\pi/2)\rangle. \tag{3.2.17}$$

Together with (3.2.2), these equalities mean that

$$\Phi(\pi/2)=0. \tag{3.2.18}$$

Equation (3.2.12) consists of the four independent ordinary differential equation systems

$$\Phi'_{2j}(\varphi)=(-1)^j i\lambda\,\Phi_{2j-1}(\varphi),\quad \Phi'_{2j-1}(\varphi)=(-1)^{j+1}i\lambda\,\Phi_{2j}(\varphi), \tag{3.2.19}$$

where $j=1,2,3,4$ and Φ_k is the k-th component of Φ. From (3.2.18) and (3.2.19) it follows that

$$\Phi''_{2j}(\varphi)=(-1)^j i\lambda\Phi'_{2j-1}(\varphi)=\lambda^2\,\Phi_{2j}(\varphi),\quad \varphi\in(0,\pi/2), \tag{3.2.20}$$

$$\Phi'_{2j}(\pi/2)=(-1)^j i\lambda\,\Phi_{2j-1}(\pi/2)=0. \tag{3.2.21}$$

In a similar way, we obtain

$$\Phi''_k(\varphi)-\lambda^2\Phi_k(\varphi)=0,\quad \varphi\in(0,\pi/2), \tag{3.2.22}$$

$$\Phi_k(\pi/2)=0,\quad \Phi'_k(\pi/2)=0. \tag{3.2.23}$$

for $k=1,\dots,8$. The general solution to the equation (3.2.22) is of the form $\Phi_k(\varphi)=c_0\mathrm{ch}(\lambda\varphi)+c_1\mathrm{sh}(\lambda\varphi)$. In view of conditions, (3.2.23) the constants c_0 and c_1 are equal to zero.

Thus, the vector function Φ is identically 0 and the line $\operatorname{Im}\lambda=1/2$ contains no eigenvalues of the pencil $\mathfrak{L}$. From the ellipticity of problem (3.1.3) it follows that for any $N>0$ in the strip $\{\lambda\in\mathbb{C}:|\operatorname{Im}\lambda|<N\}$, there are at most finitely many eigenvalues of the pencil $\mathfrak{L}$. Let λ_* and λ^* be the eigenvalues nearest to the line $\operatorname{Im}\lambda=1/2$ satisfying $\operatorname{Im}\lambda_*<1/2<\operatorname{Im}\lambda^*$; we set $\beta_*=\operatorname{Im}\lambda_*$ and $\beta^*=\operatorname{Im}\lambda^*$. □

Proposition 3.2.3. *The operator* (3.2.8) *implements an isomorphism for* $\omega = \pm 1$ *and all* β*, such that* $|\beta - 1/2| < \varepsilon = \min(\beta^* - 1/2, 1/2 - \beta_*)$.

Proof. Here we only consider the case $\omega = +1$. The case $\omega = -1$ can be handled in a similar fashion as the case $\omega = +1$. By Proposition 3.2.2 and Theorem 8.2.3 [25], the operator $\mathcal{L}_\beta(\omega)$ is Fredholm for $\beta \in (\beta_*, \beta^*)$. It remains to verify, that the kernel and cokernel of $\mathcal{L}_\beta(\omega)$ are trivial. We begin with proving the triviality of the kernel and will assume that $\beta = 1/2$. The known inequality (see Section 4.3 from Nazarov et al. [25])

$$\|\eta \mapsto |\eta|^{\beta-1/2} v(\eta); L_2(\partial\mathbb{K})\| \le c\, \|v; V_\beta^{1/2}(\partial\mathbb{K})\| \tag{3.2.24}$$

provides the continuity of the embedding

$$E_{1/2}^{1/2}(\partial\mathbb{K}) \subset V_{1/2}^{1/2}(\partial\mathbb{K}) \subset L_2(\partial\mathbb{K}). \tag{3.2.25}$$

Here $V_\beta^l(\mathbb{K})$ stands for the space, which is defined like $V_\beta^l(G^R)$ with the change of G^R, x, and $r(x)$ for $\mathbb{K}$, η, and $|\eta|$, respectively. $V_\beta^{1/2}(\partial\mathbb{K})$ is the space of traces on $\partial\mathbb{K}$ of the functions in $V_\beta^1(\mathbb{K})$. The maps

$$\begin{aligned} \mathcal{A}^0(D_\eta, \omega) :& E_{1/2}^1(\mathbb{K}) \to E_{1/2}^0(\mathbb{K}) = V_{1/2}^0(\mathbb{K}), \\ & E_{1/2}^1(\mathbb{K}) \subset E_{-1/2}^0(\mathbb{K}) = V_{-1/2}^0(\mathbb{K}), \\ \mathcal{B}_j, \mathcal{Q}_j :& E_{1/2}^1(\mathbb{K}) \to E_{1/2}^{1/2}(\partial\mathbb{K}_j) \subset L_2(\partial\mathbb{K}_j) \end{aligned} \tag{3.2.26}$$

are continuous due to the continuity of operator (3.2.8) and embedding (3.2.25). Therefore, the Green formula

$$\begin{aligned} &(\mathcal{A}^0(D_\eta, \omega)U, V)_{\mathbb{K}} + (\mathcal{B}_1 U, \mathcal{Q}_1 V)_{\partial\mathbb{K}_1} + (\mathcal{B}_2 U, \mathcal{Q}_2 V)_{\partial\mathbb{K}_2} = \\ &= (U, \mathcal{A}^0(D_\eta, \omega)V)_{\mathbb{K}} + (\mathcal{Q}_1 U, \mathcal{B}_1 V)_{\partial\mathbb{K}_1} + (\mathcal{Q}_2 U, \mathcal{B}_2 V)_{\partial\mathbb{K}_2} \end{aligned} \tag{3.2.27}$$

makes sense for U and V in $E_{1/2}^1(\mathbb{K})$.

Now we assume, that $V = U \in \ker \mathcal{L}_{1/2}(\omega)$, where $\mathcal{L}_{1/2}(\omega)$ is operator (3.2.8) for $\beta = 1/2$. Since U belongs to $\in E_{1/2}^1(\mathbb{K})$ and satisfies homogeneous problem (3.2.3), from (3.2.27) it follows, that

$$0 = (\mathcal{B}_2 U, \mathcal{Q}_2 U)_{\partial\mathbb{K}_2} - (\mathcal{Q}_2 U, \mathcal{B}_2 U)_{\partial\mathbb{K}_2} = -2i(\mathcal{Q}_2 U, \mathcal{Q}_2 U)_{\partial\mathbb{K}_2} = 2i(\mathcal{B}_2 U, \mathcal{B}_2 U)_{\partial\mathbb{K}_2}. \tag{3.2.28}$$

Hence $\mathcal{Q}_2 U = \mathcal{B}_2 U = 0$ on $\partial\mathbb{K}_2$; taking into account (3.2.2), we obtain

$$U(\eta) = 0, \quad \eta \in \partial\mathbb{K}_2. \tag{3.2.29}$$

Being subject to the condition (3.2.29), a solution U to homogeneous problem (3.2.3) is trivial. To show this, we will verify that the components of such a vector U satisfy the homogeneous Helmholtz equation in $\mathbb{K}$ and the homogeneous Dirichlet and Neumann conditions on $\partial\mathbb{K}_2$. Then, according to the unique continuation theorem (e.g., see [6]), the function U is identically 0 on $\mathbb{K}$.

We first note, that the function U is smooth on $\overline{\mathbb{K}} \setminus \{\mathcal{O}\}$ because U is a solution to homogeneous elliptic problem (3.2.3). Applying $\mathcal{A}^0(D_\eta, \omega)$ to the equality

$$\mathcal{A}^0(D_\eta, \omega)U(\eta) = 0,\ \eta \in \mathbb{K}^\rho, \tag{3.2.30}$$

we obtain

$$-(\Delta_2 - 1)U(\eta) = 0,\ \eta \in \mathbb{K}^\rho. \tag{3.2.31}$$

Here $\Delta_2 = \partial_1^2 + \partial_2^2$, $\partial_j = \partial/\partial\eta_j$, and $j = 1, 2$. Let $V = \mathcal{A}^0(D_\eta, \omega)W$, where $W = (W_1, \dots, W_8)$ is an arbitrary vector-valued function in $C_c^\infty(\mathbb{K} \cup \partial\mathbb{K}_2)$. Then the Green formula (3.2.27) and (3.2.29) implies, that

$$(\mathcal{A}^0(D_\eta, \omega)U, \mathcal{A}^0(D_\eta, \omega)W)_{\mathbb{K}} = (U, \mathcal{A}^0(D_\eta, \omega)\mathcal{A}^0(D_\eta, \omega)W)_{\mathbb{K}} = \tag{3.2.32}$$

$$= -(U, (\Delta_2 - 1)W)_{\mathbb{K}}. \tag{3.2.33}$$

From (3.2.30), (3.2.31), and (3.2.32), we get

$$-((\Delta_2 - 1)U, W)_{\mathbb{K}} = 0 = -(U, (\Delta_2 - 1)W)_{\mathbb{K}}. \tag{3.2.34}$$

Now we set $W_j = 0$ for $j = 2, \dots, 8$ in (3.2.34). In view of the Green formula for the Laplacian, the obtained equality $(\Delta_2 U_1, W_1)_{\mathbb{K}} = (U_1, \Delta_2 W_1)_{\mathbb{K}}$ takes the form of $(\partial_\nu U_1, W_1)_{\partial\mathbb{K}} = (U_1, \partial_\nu W_1)_{\partial\mathbb{K}}$. This and (3.2.29) lead to $(\partial_\nu U_1, W_1)_{\partial\mathbb{K}} = 0$. Since W_1 is an arbitrary function in $C_c^\infty(\mathbb{K} \cup \partial\mathbb{K}_2)$, we have

$$\partial_\nu U_1(x) = 0,\ x \in \partial\mathbb{K}_2. \tag{3.2.35}$$

Thus, U_1 satisfies homogeneous elliptic equation (3.2.31) and Cauchy data (3.2.29) and (3.2.35). Therefore, by the unique continuation theorem, $U_1 \equiv 0$ in $\mathbb{K}$. The relations $U_j \equiv 0$ in $\mathbb{K}$ for $j = 2, \dots, 8$ can be verified like that for $j = 1$. Hence the kernel of operator (3.2.8) is trivial for $\beta = 1/2$. Since the strip $\{\lambda : \beta_* < \operatorname{Im}\lambda < \beta^*\}$ is free of the spectrum of the pencil $\mathfrak{L}$, the kernel of operator (3.2.8) is trivial for all $\beta \in (\beta_*, \beta^*)$.

Cokernel of the operator $\mathcal{L}_\beta(\omega)$ coincides with the kernel of the adjoint operator

$$\mathcal{L}_\beta(\omega)^* : E_\beta^0(\mathbb{K})^* \times E_\beta^{1/2}(\partial\mathbb{K}_1)^* \times E_\beta^{1/2}(\partial\mathbb{K}_2)^* \to E_\beta^1(\mathbb{K})^*. \tag{3.2.36}$$

Arguing in the same way as in the proof of Proposition 4.3.8 [25] and using Proposition 8.2.6 [25], we obtain

$$\{u, v, w\} \in E_{1-\beta}^1(\mathbb{K}) \times E_{1-\beta}^{1/2}(\partial\mathbb{K}_1) \times E_{1-\beta}^{1/2}(\partial\mathbb{K}_2) \tag{3.2.37}$$

for any element $\{u, v, w\}$ in the kernel of operator (3.2.36). The function u satisfies the homogeneous problem (3.2.3) with $\mathcal{B}_2 + i\mathcal{Q}_2$ changed for $\mathcal{B}_2 - i\mathcal{Q}_2$ in the third equation,

$$v(x) = \mathcal{Q}_1 u(x),\ x \in \partial\mathbb{K}_1, \quad \text{and} \quad w(x) = \mathcal{Q}_2 u(x),\ x \in \partial\mathbb{K}_2. \tag{3.2.38}$$

As in the first part of the proof, with β and $\mathcal{B}_2 + i\mathcal{Q}_2$ replaced by $1 - \beta$ and $\mathcal{B}_2 - i\mathcal{Q}_2$ respectively, we find $v \equiv 0$. Thus, the kernel and cokernel of operator (3.2.8) are trivial for $\beta \in (\beta_*, \beta^*) \cap (1 - \beta^*, 1 - \beta_*)$. □

Let us return to problem (3.1.3) in the domain G^R. In view of Proposition 3.2.3, operator (3.2.7) is Fredholm (see Theorem 8.3.1 [25]). When proving the triviality of its kernel and cokernel, we will use the following lemma.

Lemma 3.2.4. *Let $\beta \le 1/2$. Then:*
1. The Green formula

$$(\mathcal{A}(D,k)\mathcal{U},\mathcal{V})_{G^R} + (\mathcal{B}\mathcal{U},\mathcal{Q}\mathcal{V})_{\partial G^R\setminus\Gamma^R} - (\mathcal{U},\mathcal{A}(D,k)\mathcal{V})_{G^R} - (\mathcal{Q}\mathcal{U},\mathcal{B}\mathcal{V})_{\partial G^R\setminus\Gamma^R} = \\ = -(\mathcal{B}\mathcal{U},\mathcal{Q}\mathcal{V})_{\Gamma^R} + (\mathcal{Q}\mathcal{U},\mathcal{B}\mathcal{V})_{\Gamma^R} \quad (3.2.39)$$

holds for any $\mathcal{U}$ and $\mathcal{V}$ in $V^1_\beta(G^R)$.
2. If $\mathcal{U}_0$ belongs to $V^1_\beta(G^R)$ and satisfies (3.1.3) *for $\mathcal{F}=0$ and $\mathcal{G}=0$, then*

$$\|\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^8)\|^2 = \|\mathcal{B}\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^4)\|^2 + \|\mathcal{Q}\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^4)\|^2 = \\ = \|(\mathcal{B}+i\mathcal{Q})\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^4)\|^2. \quad (3.2.40)$$

3. If $\mathcal{U}_0$ belongs to $V^1_\beta(G^R)$ and satisfies (3.1.3) *for $\mathcal{H}=0$, then*

$$\|\mathcal{B}\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^4)\|^2 = \|\mathcal{Q}\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^4)\|^2 = \|\mathcal{U}_0; L_2(\Gamma^R;\mathbb{C}^8)\|^2/2. \quad (3.2.41)$$

Proof. For $\beta < \gamma$ the space $V^1_\beta(G^R)$ is continuously embedded into $V^1_\gamma(G^R)$, therefore, it suffices to verify the lemma for $\beta = 1/2$. We first prove the continuity of the embedding

$$V^{1/2}_{1/2}(\partial G^R) \subset L_2(\partial G^R). \quad (3.2.42)$$

In view of (3.2.24), for $v \in C^\infty_c(\overline{\partial\mathbb{D}}\setminus M)$ and $u \in C^\infty_c(\overline{\mathbb{D}}\setminus M)$, such that $u|_{\partial\mathbb{D}} = v$, the inequality

$$\|v(\cdot,x_3); L_2(\partial\mathbb{K})\|^2 \le c^2\|v(\cdot,x_3); V^{1/2}_{1/2}(\partial\mathbb{K})\|^2 \le c^2\|u(\cdot,x_3); V^1_{1/2}(\mathbb{K})\|^2 = \\ = c^2\int_{\mathbb{K}} dx_1dx_2\Big\{|x_1^2+x_2^2|^{-1/2}|u(x_1,x_2,x_3)|^2 + |x_1^2+x_2^2|^{1/2}\sum_{j=1,2}|D_ju(x_1,x_2,x_3)|^2\Big\} \\ \le c^2\int_{\mathbb{K}} dx_1dx_2\Big\{|x_1^2+x_2^2|^{-1/2}|u(x_1,x_2,x_3)|^2 + |x_1^2+x_2^2|^{1/2}\sum_{j=1,2,3}|D_ju(x_1,x_2,x_3)|^2\Big\} \quad (3.2.43)$$

holds with constant c independent of x_3 and u. Integrating over $x_3 \in \mathbb{R}$, we have

$$\|v; L_2(\partial\mathbb{D})\|^2 \le c^2\|u; V^1_{1/2}(\mathbb{D})\|^2. \quad (3.2.44)$$

We extend, by continuity, this inequality to all $u \in V^1_{1/2}(\mathbb{D})$ and obtain

$$\|v; L_2(\partial\mathbb{D})\|^2 \le c^2\|v; V^{1/2}_{1/2}(\partial\mathbb{D})\|^2 \quad (3.2.45)$$

because the norm on the right is equal to $\inf\|u; V^1_{1/2}(\mathbb{D})\|$ for $u : u|_{\partial\mathbb{D}} = v$. Using a partition of unity, we arrive at (3.2.42).

The continuity of operator (3.2.7) and embedding (3.2.42) imply that of the maps

$$\begin{aligned} \mathcal{A}(D,k) :& V^1_{1/2}(G^R) \to V^0_{1/2}(G^R), \\ & V^1_{1/2}(G^R) \subset V^0_{-1/2}(G^R), \\ \mathcal{B}, \mathcal{Q} :& V^1_{1/2}(G^R) \to L_2(\partial G^R), \end{aligned} \tag{3.2.46}$$

which allows to extend the Green formula

$$(\mathcal{A}(D,k)\mathcal{U}, \mathcal{V})_{G^R} + (\mathcal{B}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\partial G^R} - (\mathcal{U}, \mathcal{A}(D,k)\mathcal{V})_{G^R} - (\mathcal{Q}\mathcal{U}, \mathcal{B}\mathcal{V})_{\partial G^R} = 0 \tag{3.2.47}$$

to the functions $\mathcal{U}$ and $\mathcal{V}$ in $V^1_{1/2}(G^R)$.

Let us prove item 2. According to the identity

$$\begin{aligned} ((\mathcal{B}+i\mathcal{Q})\mathcal{U}, (\mathcal{B}+i\mathcal{Q})\mathcal{V})_{\Gamma^R} =& (\mathcal{B}\mathcal{U}, \mathcal{B}\mathcal{V})_{\Gamma^R} + (\mathcal{Q}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\Gamma^R} - \\ & - i(\mathcal{B}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\Gamma^R} + i(\mathcal{Q}\mathcal{U}, \mathcal{B}\mathcal{V})_{\Gamma^R}, \end{aligned} \tag{3.2.48}$$

the right-hand-side of (3.2.39) can be written in the form

$$-i\left\{((\mathcal{B}+i\mathcal{Q})\mathcal{U}, (\mathcal{B}+i\mathcal{Q})\mathcal{V})_{\Gamma^R} - (\mathcal{B}\mathcal{U}, \mathcal{B}\mathcal{V})_{\Gamma^R} - (\mathcal{Q}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\Gamma^R}\right\}. \tag{3.2.49}$$

The left-hand-side of (3.2.39) vanishes for $\mathcal{V} = \mathcal{U} = \mathcal{U}_0$ and

$$\|(\mathcal{B}+i\mathcal{Q})\mathcal{U}_0; L_2(\Gamma^R)\|^2 - \|\mathcal{B}\mathcal{U}_0; L_2(\Gamma^R)\|^2 - \|\mathcal{Q}\mathcal{U}_0; L_2(\Gamma^R)\|^2 = 0. \tag{3.2.50}$$

It remains to take account of the equality

$$\|\mathcal{U}_0; L_2(\Gamma^R; \mathbb{C}^8)\|^2 = \|\mathcal{B}\mathcal{U}_0; L_2(\Gamma^R; \mathbb{C}^4)\|^2 + \|\mathcal{Q}\mathcal{U}_0; L_2(\Gamma^R; \mathbb{C}^4)\|^2, \tag{3.2.51}$$

which follows from (2.1.11).

Item 3 immediately follows from the third equation in (3.1.3) and (3.2.51). □

Now we are ready to complete the proof of Theorem 3.2.1. We use the same scheme as in the proof of Proposition 3.2.3 and outline some basic steps. By Theorem 8.3.1 [25], the operator (3.2.7) is Fredholm. Let us show that the kernel and cokernel of this operator are trivial. We first assume that $\beta = 1/2$ and $\mathcal{U} \in V^1_{1/2}(G^R)$ is a solution to homogeneous problem (3.1.3). From item 2 of Lemma 3.2.4, it follows that

$$\mathcal{U}(x) = 0, \quad x \in \Gamma^R. \tag{3.2.52}$$

The function $\mathcal{U}$ is smooth on $\overline{G^R} \setminus \partial\Gamma^R$. Applying $\mathcal{A}(D,-k)$ to the equality

$$\mathcal{A}(D,k)\mathcal{U}(x) = 0, \quad x \in G^R, \tag{3.2.53}$$

we obtain

$$-(\Delta + k^2)\mathcal{U}(x) = 0, \quad x \in G^R. \tag{3.2.54}$$

Let us verify that the components $\mathcal{U}_j$, $j = 1, \dots, 8$, of $\mathcal{U}$ satisfy the homogeneous Neumann condition on Γ^R. By virtue of (3.2.53), (3.2.54), and (3.2.39),

$$(\Delta\mathcal{U}, \mathcal{V})_{G^R} = (\mathcal{U}, \Delta\mathcal{V})_{G^R} \tag{3.2.55}$$

for an arbitrary $\mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_8) \in C_c^\infty(G^R \cup \Gamma^R)$. We set $\mathcal{V}_j = 0$ for $j = 2, \dots, 8$; then (3.2.55) and (3.2.52) lead to

$$\partial_\nu \mathcal{U}_1(x) = 0, \quad x \in \Gamma^R. \tag{3.2.56}$$

By the unique continuation theorem, $\mathcal{U}_1 = 0$ in G^R. The equalities $\mathcal{U}_j = 0$ in G^R for $j = 2, \dots, 8$ are obtained like that for $\mathcal{U}_1$. Thus, the kernel of operator (3.2.7) is trivial for $\beta = 1/2$. According to the Theorem 8.3.2 [25], the kernel of operator (3.2.7) is trivial for all $\beta \in (1/2 - \varepsilon, 1/2 + \varepsilon)$. The triviality of the cokernel follows from Theorem 8.3.3 [25] in a similar way.

3.3 Justification of the method

To complete the proof of Theorem 3.1.1, it remains to verify that the matrix E^R in (3.1.5) is non-singular and to establish estimate (3.1.7). All inequalities of this Section are uniform with respect to $k \in [k_1, k_2]$. Furthermore, we consider functions belonging to the space $L_2(\Gamma^R; \mathbb{C}^8)$, the values of operators $\mathcal{B}$ and $\mathcal{Q}$ on such functions are in the space $L_2(\Gamma^R; \mathbb{C}^4)$. To simplify formulae as a rule we do not indicate the spaces in notations of inner products and norms. We set

$$\varphi_l(x, a) = u_l^+(x) + \sum_{j=1}^{\Upsilon} a_j u_j^-(x), \tag{3.3.1}$$

where $a = (a_1, \dots, a_\Upsilon)$ is an arbitrary vector in $\mathbb{C}^\Upsilon$ and $x \in G^R$. From item 3 of Lemma 3.2.4, the functional (3.1.2) takes the form of

$$J_l^R(a) = 1/2 \|\mathrm{X}_l^R(\cdot; a) - \varphi_l(\cdot, a); L_2(\Gamma^R; \mathbb{C}^8)\|^2 \tag{3.3.2}$$

and the entries of the matrices E^R and F^R can be written as

$$E_{ij}^R = 1/2((Z_{i,R}^- - u_i^-), (Z_{j,R}^- - u_j^-)), \tag{3.3.3}$$

$$F_{ij}^R = 1/2((Z_{i,R}^+ - u_i^+), (Z_{j,R}^- - u_j^-)), \tag{3.3.4}$$

where $(\cdot, \cdot)$ denotes the inner product in $L_2(\Gamma^R; \mathbb{C}^8)$.

Lemma 3.3.1. *Let $u_j^\pm$, $j = 1, \dots, \Upsilon$, be incoming and outgoing waves subject to*

$$iq_G(u_j^\pm, u_l^\mp) = 0, \quad iq_G(u_j^\pm, u_l^\pm) = \mp\delta_{jl}, \quad j, l = 1, \dots, \Upsilon. \tag{3.3.5}$$

Then for large R and $j, l = 1, \dots, \Upsilon$

$$(u_j^\pm, u_l^\pm) - ((\mathcal{B} + i\mathcal{Q})u_j^\pm, (\mathcal{B} + i\mathcal{Q})u_l^\pm) = \mp\delta_{jl}, \tag{3.3.6}$$

$$(u_j^\pm, u_l^\mp) - ((\mathcal{B} + i\mathcal{Q})u_j^\pm, (\mathcal{B} + i\mathcal{Q})u_l^\mp) = 0. \tag{3.3.7}$$

Proof. We rearrange the right-hand-side of Green formula (3.2.39) like in the proof of Lemma 3.2.4. For $\mathcal{U}, \mathcal{V} \in \mathcal{W}(k)$, the left-hand-side of (3.2.39) coincides with definition (2.3.5) of the form $q_G(\cdot,\cdot)$. We have

$$iq_G(\mathcal{U},\mathcal{V}) = (\mathcal{U},\mathcal{V}) - ((\mathcal{B}+i\mathcal{Q})\mathcal{U}, (\mathcal{B}+i\mathcal{Q})\mathcal{V}). \tag{3.3.8}$$

It remains to take into account (3.3.5). □

Proposition 3.3.2. *The matrix E^R with entries* (3.3.3) *is non-singular for all $R \geq R_0$, where R_0 is a sufficiently large number.*

Proof. Assume to the contrary, that for any R_0 there exist a number $R > R_0$ and a vector $c = (c_1, \dots, c_\Upsilon)$, such that $|c| = 1$ and $E^R c = 0$. Then, in view of (3.3.3), the functions $\mathcal{U} = \sum_j c_j u_j^-$ and $\mathcal{V} = \sum_j c_j Z_{j,R}^-$ satisfy

$$\mathcal{U} = \mathcal{V} \text{ on } \Gamma^R. \tag{3.3.9}$$

From item 2 of Lemma 3.2.4, it follows that

$$\|\mathcal{V}\|^2 - \|(\mathcal{B}+i\mathcal{Q})\mathcal{V}\|^2 = 0. \tag{3.3.10}$$

Taking into account (3.3.9) and Lemma 3.3.1, we obtain

$$0 = \|\mathcal{U}\|^2 - \|(\mathcal{B}+i\mathcal{Q})\mathcal{U}\|^2 = \sum_j |c_j|^2 = 1, \tag{3.3.11}$$

which is a contradiction, q.e.d. □

Embedding theorems and the definition (2.3.17) of the space $H^1_\beta(G)$ implies the following Lemma.

Lemma 3.3.3. *For every function $\mathcal{U} \in H^1_\beta(G)$ the inequality*

$$\|\mathcal{U}; L_2(\Gamma^R)\| \leq Ce^{-\beta R}\|\mathcal{U}; H^1_\beta(G)\|,$$

holds, where the constant C is independent of R, β and $\mathcal{U}$.

Proof. The described result is local, so we firstly prove an analogous inequality in a cylinder domain and then, using a partition of unity, we turn to the domain with cylindrical ends. Let $\Pi^q = \Omega^q \times \mathbb{R} = \{(y,t) : y \in \Omega^q, t \in \mathbb{R}\}$, and $\Gamma^{q,R} = \{(y,t) : y \in \Omega^q,\, t = R\}$. Then by an embedding theorem, for every $u \in H^1(\Pi^q)$ the inequality

$$\|u; L_2(\Gamma^{q,R})\| \leq C\|u; H^1(\Pi^q)\| \tag{3.3.12}$$

holds with the constant $C = C(\Omega^q)$, independent of R and u. Substituting the function $u(y,t) = \exp(\beta t)v(y,t)$ into the inequality, we obtain

$$e^{\beta R}\|v; L_2(\Gamma^{q,R})\| \leq C\|v; H^1_\beta(\Pi^q)\|, \tag{3.3.13}$$

for every $v \in H^1_\beta(\Pi^q)$, where $H^1_\beta(\Pi^q)$ is a closure of the lineal $C^\infty_c(\Pi^q)$ by the norm

$$\|v; H^1_\beta(\Pi^q)\| = \|(y,t) \mapsto \exp(\beta t) v(y,t); H^1(\Pi^q)\|.$$

Let T be a sufficiently large number and $\chi_q : G \to [0,1]$ be a smooth cutting function supported in $G \cap \Pi^{q,T}_+$, such that $\chi_q(x) = 1$ for $x \in G \cap \Pi^{q,T+1}_+$, where $\Pi^{q,T}_+ = \{(y^q, t^q) \in \Pi^q : t^q > T\}$. For $\mathcal{U} \in H^1_\beta(G)$, we introduce a function $\mathcal{U}_q$ in Π^q

$$\mathcal{U}_q(x) = \chi_q(x)\mathcal{U}(x),\ x \in \Pi^{q,T}_+, \qquad \mathcal{U}_q(x) = 0,\ x \in \Pi^q \setminus \Pi^{q,T}_+.$$

The function $\mathcal{U}_q$ belongs to the space $H^1_\beta(\Pi^q)$ and $\|\mathcal{U}_q; H^1_\beta(\Pi^q)\| = \|\chi_q\mathcal{U}; H^1_\beta(G)\|$. Using inequality (3.3.13) for $\mathcal{U}_q$ and $R > T+1$, we have

$$\|\mathcal{U}; L_2(\Gamma^{q,R})\| \le C(\Omega^q) e^{-\beta R} \|\chi_q \mathcal{U}; H^1_\beta(G)\|.$$

Summing up the inequalities over $q = 1, \dots, \mathcal{T}$, we come to

$$\|\mathcal{U}; L_2(\Gamma^R)\| \le C e^{-\beta R} \|\mathcal{U}; H^1_\beta(G)\|, \tag{3.3.14}$$

where $C = \max C(\Omega^q)$. □

Proposition 3.3.4. *Let $a^0(R) = (a^0_1(R), \dots, a^0_\Upsilon(R))$ be a minimizer of the functional $J^R_l(a)$ in* (3.1.2)*. Then*

$$J^R_l(a^0(R)) = O(e^{-2\delta R}) \text{ as } R \to \infty, \tag{3.3.15}$$

where δ is the number in (2.4.1)*. For all $R \ge R_0$ the componenents of $a^0(R)$ are uniformly bounded,*

$$|a^0_j(R)| \le \text{const} < \infty, \quad j = 1, \dots, \Upsilon. \tag{3.3.16}$$

Proof. We set

$$\mathbb{N}(R) := \|Y^+_l - \varphi_l(S_l); L_2(\Gamma^R; \mathbb{C}^8)\|, \tag{3.3.17}$$

where $\varphi_l(S_l)$ is the function $x \mapsto \varphi_l(x, S_l)$ in (3.3.1) for $a = S_l = (S_{l1}, \dots, S_{l\Upsilon})$, and S_{lj} are the entries of the scattering matrix S of problem (2.1.9). According to Lemma 3.3.3 the equation (2.4.1) implies $\mathbb{N}(R) \le Ce^{-\delta R}$, where the constant C is independent of R. Let $X^R_l(S_l)$ be a solution to problem (3.1.3) with right-hand-side (3.1.4) for $a = S_l$. We have

$$\|(\mathcal{B} + i\mathcal{Q})(X^R_l(S_l) - Y^+_l)\|^2 = \|(\mathcal{B} + i\mathcal{Q})(\varphi_l(S_l) - Y^+_l)\|^2 \le \text{const}\, \mathbb{N}(R)^2.$$

Here const stands for an universal constant. From item 2 of Lemma 3.2.4, we obtain

$$\|X^R_l(S_l) - Y^+_l\|^2 = \|(\mathcal{B} + i\mathcal{Q})(X^R_l(S_l) - Y^+_l)\|^2 \le \text{const}\mathbb{N}(R)^2.$$

From the previous estimate, (3.3.17), and (3.3.2), we get

$$J_l^R(S_l) = 1/2\|X_l^R(S_l) - \varphi_l(S_l)\|^2 \leq \text{const}\mathbb{N}(R)^2.$$

To obtain (3.3.15), it remains to take into account the inequality $J_l^R(a^0(R)) \leq J_l^R(S_l)$.

To justify an uniform boundedness of the minimizer $a^0(R)$ we show that the value $||a^0(R)| - 1|$ decays at exponential rate as $R \to \infty$. By virtue Lemma 3.3.1

$$|a^0|^2 - 1 = \sum_{j=1}^{\Upsilon} |a_j^0|^2 - 1 = \|\varphi_l(a^0)\|^2 - \|(\mathcal{B} + i\mathcal{Q})\varphi_l(a^0)\|^2. \tag{3.3.18}$$

Let us figure the right-hand-side of (3.3.18) in the form of

$$\left(\|\varphi_l(a^0)\| - \|(\mathcal{B} + i\mathcal{Q})\varphi_l(a^0)\|\right)\left(\|\varphi_l(a^0)\| + \|(\mathcal{B} + i\mathcal{Q})\varphi_l(a^0)\|\right) \tag{3.3.19}$$

Taking into account the definition of $X_l^R(a^0)$ and item 2 of Lemma 3.2.4, we have

$$\|(\mathcal{B} + i\mathcal{Q})\varphi_l(a^0)\|^2 = \|(\mathcal{B} + i\mathcal{Q})X_l^R(a^0)\|^2 = \|X_l^R(a^0)\|^2. \tag{3.3.20}$$

Making use of (3.3.20) and (3.3.2), we estimate the first factor of (3.3.19)

$$\begin{aligned}\Big|\|\varphi_l(a^0)\| - \|(\mathcal{B} + i\mathcal{Q})\varphi_l(a^0)\|\Big| &= \Big|\|\varphi_l(a^0)\| - \|X_l^R(a^0)\|\Big| \leq \\ &\leq \|\varphi_l(a^0) - X_l^R(a^0)\| = (2J_l^R(a^0))^{1/2}.\end{aligned}$$

Let us majorize the second factor of (3.3.19) by const$\|\varphi_l(a^0)\|$. Furthermore,

$$\|\varphi_l(a^0)\| = \|u_l^+ + \sum a_j u_j^-\| \leq \text{const}(\max\|u_j^\pm\|)(1 + |a^0|). \tag{3.3.21}$$

Taking into account the estimate for $J_l^R(a^0)$, we obtain

$$\Big||a^0(R)| - 1\Big| \leq \text{const}\mathbb{N}_1\mathbb{N}(R),$$

where the constant $\mathbb{N}_1 = \max\|u_j^\pm\| = \max(k/\lambda_j)^{1/2}$ is independent of R and δ and $\mathbb{N}(R) \leq C(\delta)e^{-\delta R}$. □

Proof. (of Theorem 3.1.1). Let $\varphi_l(a^0)$ and $\varphi_l(S_l)$ be defined as before by equalities (3.3.1), and $X_l^R(a^0)$ and $X_l^R(S_l)$ be the corresponding solutions to the problem (3.1.3). We will use the same arguments as in the proof of the previous proposition. From Lemma 3.3.1, we have

$$|a^0 - S_l|^2 = \sum_{j=1}^{\Upsilon} |a_j^0(R) - S_{lj}|^2 = \|\varphi_l(a^0) - \varphi_l(S_l)\|^2 - \|(\mathcal{B} + i\mathcal{Q})(\varphi_l(a^0) - \varphi_l(S_l))\|^2, \tag{3.3.22}$$

and from Lemma 3.2.4, we obtain

$$\|(\mathcal{B} + i\mathcal{Q})(\varphi_l(a^0) - \varphi_l(S_l))\|^2 = \|X_l^R(a^0) - X_l^R(S_l)\|^2. \tag{3.3.23}$$

The right-hand-side of (3.3.22) we figure as

$$\Big(\|\varphi_l(a^0)-\varphi_l(S_l)\|-\|(\mathcal{B}+i\mathcal{Q})(\varphi_l(a^0)-\varphi_l(S_l))\|\Big)\times$$
$$\times\Big(|\|\varphi_l(a^0)-\varphi_l(S_l)\|+\|(\mathcal{B}+i\mathcal{Q})(\varphi_l(a^0)-\varphi_l(S_l))\|\Big).$$

The first factor is majorized by

$$\|(\varphi_l(a^0)-X_l^R(a^0))-(\varphi_l(S_l)-X_l^R(S_l))\|\le(2J_l^R(a^0))^{1/2}+(2J_l^R(S_l))^{1/2}\le$$
$$\le \mathrm{const}\mathbb{N}(R),$$

and the second factor by $\mathrm{const}\|\varphi_l(a^0)-\varphi_l(S_l)\|$, while

$$\|\varphi_l(a^0)-\varphi_l(S_l)\|\le\mathrm{const}\mathbb{N}_1|a^0-S_l|.$$

Then

$$|a^0-S_l|^2\le\mathrm{const}\mathbb{N}_1\mathbb{N}(R)|a^0-S_l|, \tag{3.3.24}$$

where the constant $\mathbb{N}_1=\max(k/\lambda_j)^{1/2}$ is independent of R and δ and $\mathbb{N}(R)\le C(\delta)e^{-\delta R}$. □

4 THE MAXWELL SYSTEM IN A WAVEGUIDE

Here we derive corollaries for the (non-augmented) Maxwell system from the results of Chapter 2 on the elliptic system. We describe the continuous spectrum eigenfunctions; define the scattering matrix and establish that it is unitary (Section 4.3); state the problem with intrinsic radiation conditions and prove its well-posedness (Section 4.4). Sections 4.1 and 4.2 are devoted to preparations for going from the elliptic system to the Maxwell system. The problems for the augmented and non-augmented Maxwell systems are considered for $k = 0$ (statics) in Section 4.5.

4.1 On radiation properties of the elliptic problem with right-hand-side subject to compatibility conditions

We introduce notations more detailed than those in 2.3-2.5. First assume that the domain G has only one outlet to infinity. We fix a real $k \neq 0$ and let λ_0 be a real eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$ such that $k^2 - \lambda_0^2 \neq 0$.

Consider the case $k\lambda_0 > 0$. The vectors A^ζ, Ψ^ζ ($\zeta = 1, \dots J_\mathcal{N}(\lambda_0)$) and B^τ, Φ^τ ($\tau = 1, \dots, J_\mathcal{D}(\lambda_0)$) given by (2.2.76) and (2.2.77) define a basis for the kernel $\ker \mathfrak{A}(\lambda_0, k)$. These vectors satisfy (2.2.79) and (2.2.80). We denote by E_j^- the waves, each being determined by one of the vectors Φ^τ or Ψ^ζ, $j = 1, \dots, J_\mathcal{D}(\lambda_0) + J_\mathcal{N}(\lambda_0)$ (see the definition of waves before (2.3.19)); the waves E_j^- are enumerated in arbitrary order. Similarly, let Γ_j^- be the waves corresponding to vectors of the form A^ζ or B^τ.

For every eigenvalue λ_q of the pencil $\mathfrak{A}(\cdot, k)$, such that $k\lambda_q > 0$, we construct similar collections $\{E_j^-\}$ and $\{\Gamma_j^-\}$, numbering each collection (individually) with one index. Thus we have composed the sets

$$\{E_j^-\}_{j=1}^J, \quad \{\Gamma_j^-\}_{j=1}^J, \quad \text{where} \quad J = \sum_q (J_\mathcal{D}(\lambda_q) + J_\mathcal{N}(\lambda_q)), \tag{4.1.1}$$

where the summation is taken over all real eigenvalues of the pencil $\mathfrak{A}(\cdot, k)$, subject to $k\lambda_q > 0$ and $k^2 - \lambda_q^2 \neq 0$.

The waves E_j^- correspond to the eigenvectors of $\mathfrak{A}(\cdot,k)$ in the domain of the pencil $\mathfrak{M}(\cdot,k)$, whereas the waves Γ_j^- are generated by the eigenvectors that do not belong to the domain of $\mathfrak{M}(\cdot,k)$; all of these waves, E_j^- and Γ_j^-, are outgoing.

When composing the collections (4.1.1), we did not take into account the wave related to the eigenvalue $\lambda = k$ of the pencil $\mathfrak{A}(\cdot,k)$ (see Remark 2.2.9). The eigenvector A does not belong to the domain of $\mathfrak{M}(\cdot,k)$, so we include the wave in the collection $\{\Gamma_j^-\}$, which now has one element more than $\{E_j^-\}$.

Recall, that the real spectrum of the pencil $\mathfrak{A}(\cdot,k)$ is symmetric about the coordinate origin. If the wave E_j^- is generated by an eigenvector of the form Φ (Ψ) corresponding to an eigenvalue λ_0, then E_j^+ will stand for the wave generated by the vector Φ (Ψ), corresponding to the eigenvalue $-\lambda_0$. According to the same rule, we define the waves in $\{E_j^+\}$ and $\{\Gamma_j^+\}$; all of these waves, E_j^+ and Γ_j^+, are incoming.

If k is a thershold, then $\lambda = 0$ turns out to be an eigenvalue of the pencil $\mathfrak{A}(\cdot,k)$. To define the waves corresponding to that eigenvalue, we make use of the solutions (2.3.11) of the homogeneous problem in the cylinder Π. If such a solution is generated by a Jordan chain in the domain of the Maxwell pencil $\mathfrak{M}(\cdot,k)$, then we include the related incoming (outgoing) wave in the collection E^+ (E^-). If the chain is not Maxwell, we send the wave to Γ^+ (Γ^-).

Let us finally turn to the domain G that has a total N outlets to infinity and assume that a (fixed) k coincides with none of the threshold values of the pencils $\mathfrak{A}^1(\cdot,k),\dots,\mathfrak{A}^N(\cdot,k)$. We combine the collections of waves defined for all outlets into the four collections

$$E^\pm := \{E_j^\pm\}_{j=1}^{v}, \quad \Gamma^\pm := \{\Gamma_j^\pm\}_{j=1}^{v+N}, \tag{4.1.2}$$

numbering each collection (individually) by one index and keeping the above correspondence between E_j^+ and E_j^- as well as between Γ_j^+ and Γ_j^-. Taken together, the collections (4.1.2) form a basis in the wave space $\mathcal{W}$ satisfying

$$iq_G(E_j^+,E_j^+) = -1,\ iq_G(E_j^-,E_j^-) = 1 \text{ for } j = 1,\dots,v, \tag{4.1.3}$$
$$iq_G(\Gamma_j^+,\Gamma_j^+) = -1,\ iq_G(\Gamma_j^-,\Gamma_j^-) = 1 \text{ for } j = 1,\dots,v+N, \tag{4.1.4}$$
$$q_G(P,R) = 0; \tag{4.1.5}$$

in (4.1.5), the symbols P, R each can take any of the values $E_j^\pm$, $\Gamma_j^\pm$ to form a pair distinct from those in (4.1.3) and (4.1.4).

Let us consider the system (2.1.7) with homogeneous boundary conditions (2.1.8), where g^1, g^2, and g^3 are equal to zero. If the right-hand-side (f^1,h^1,f^2,h^2) of (2.1.7) is in $H_\delta^{l-1}(G)$ for $l \geq 2$, then the compatibility conditions (2.1.3)-(2.1.5) are immediately understandable. We explain how to interpret these conditions in the case $l = 1$. Introduce the space

$$H_\delta(\operatorname{div},G) = \{f \in H_\delta^0(G) : \operatorname{div} f \in H_\delta^0(G)\}$$

of functions $G \ni x \mapsto f(x) \in \mathbb{C}^3$ with the norm

$$\|f; H_\delta(\mathrm{div}, G)\| = (\|f; H^0_\delta(G)\|^2 + \|\mathrm{div}\, f; H^0_\delta(G)\|^2)^{1/2}.$$

We denote by $W_{\delta,0}(\mathrm{div}, G)$ the closed subspace in $H_\delta(\mathrm{div}, G)$ whose elements satisfy

$$(f, \nabla\psi)_G = (\mathrm{div}\, f, \psi)_G \tag{4.1.6}$$

for all $\psi \in C^\infty_c(\overline{G})$. With $f \in H^{l-1}_\delta(G)$ for $l \geq 2$, the relation (4.1.6) is equivalent to

$$\langle f(x), \nu(x)\rangle = 0 \quad \text{for} \quad x \in G; \tag{4.1.7}$$

for $l = 1$, the equality (4.1.7) has to be replaced by (4.1.6). Thus for $l = 1$ the compatibility conditions mean, that

$$\begin{gathered} f^1 \in H_\delta(\mathrm{div}, G), \quad f^2 \in W_{\delta,0}(\mathrm{div}, G), \quad h^1, h^2 \in H^0_\delta(G), \\ \mathrm{div}\, f^1(x) - ikh^2(x) = 0, \quad \mathrm{div}\, f^2(x) - ikh^1(x) = 0, \quad x \in G. \end{gathered} \tag{4.1.8}$$

Proposition 4.1.1. *Assume that k is a real number, distinct from zero, and a vector $\mathcal{F} = (f^1, h^1, f^2, h^2)$ in $H^{l-1}_\delta(G)$, which satisfies the compatibility conditions* (2.1.3)-(2.1.5) *for $l \geq 2$ or* (4.1.8) *for $l = 1$. Let also $\mathcal{U} = (u^1, a^1, u^2, a^2)$ be a solution to $\mathbb{L}\mathcal{U} = \{\mathcal{F}, 0\}$ satisfying the intrinsic radiation conditions* (2.5.1). *Then the component a^1 is a solution to the boundary value problem*

$$(\Delta + k^2)a^1(x) = 0, \quad x \in G, \quad \partial_\nu a^1(x) = 0, \quad x \in \partial G, \tag{4.1.9}$$

being subject to the intrinsic radiation conditions defined for the problem (4.1.9), *whereas a^2 is a solution to the problem*

$$(\Delta + k^2)a^2(x) = 0, \quad x \in G, \quad a^2(x) = 0, \quad x \in \partial G, \tag{4.1.10}$$

that is subject to the intrinsic radiation conditions defined for the problem (4.1.10). *Therefore, a^1 (a^2) is distinct from zero if and only if it turns out to be an eigenfunction of the problem* (4.1.9) *(problem* (4.1.10)*).*

Proof. We first show that a^2 is a solution to problem (4.1.10). The components of $\mathcal{U}$ satisfy

$$\begin{aligned} i\,\mathrm{rot}\, u^2(x) + i\,\nabla a^2(x) - ku^1(x) &= f^1(x), \\ -i\,\mathrm{div}\, u^2(x) - ka^1(x) &= h^1(x), \\ -i\,\mathrm{rot}\, u^1(x) - i\,\nabla a^1(x) - ku^2(x) &= f^2(x), \\ i\,\mathrm{div}\, u^1(x) - ka^2(x) &= h^2(x) \end{aligned} \tag{4.1.11}$$

with boundary conditions

$$\nu(x) \times u^1(x) = 0, \; \langle u^2(x), \nu(x)\rangle = 0, \; a^2(x) = 0, \; x \in \partial G. \tag{4.1.12}$$

Multiply the first equation in (4.1.11) by $\nabla\zeta$ and the forth equation by $ik\zeta$, where $\zeta \in C^\infty_c(G)$. Now we have

$$\begin{aligned} i(\mathrm{rot}\, u^2, \nabla\zeta)_G + i(\nabla a^2, \nabla\zeta)_G - k(u^1, \nabla\zeta)_G &= (f^1, \nabla\,\zeta)_G, \\ -k(\mathrm{div}\, u^1, \zeta)G - ik^2(a^2, \zeta)_G &= ik(h^2, \zeta)_G. \end{aligned}$$

Let us integrate by part all terms in the first line and add the result to the second line:

$$-i(\Delta a^2 + k^2 a^2, \zeta)_G = (-\operatorname{div} f^1 + ikh^2, \zeta)_G.$$

Since the right-hand-side vanishes due to the compatibility conditions, we obtain the problem (4.1.10) for a^2.

We shall take up the component a^1. Let $\eta \in C_c^\infty(\overline{G})$ with $\partial_\nu \eta | \partial G = 0$. Let us multiply the third equation in (4.1.11) by $\nabla \eta$ and the second equation by $ik\eta$. We have

$$\begin{aligned} -i(\operatorname{rot} u^1, \nabla\eta)_G - i(\nabla a^1, \nabla \eta)_G - k(u^2, \nabla\eta)_G &= (f^2, \nabla\eta)_G, \\ k(\operatorname{div} u^2, \eta)_G - ik^2(a^1, \eta)_G &= ik(h^1, \eta)_G. \end{aligned} \tag{4.1.13}$$

Taking into account the boundary conditions (4.1.12) and the corresponding compatibility condition, we integrate by parts all terms in the first line, then

$$i(a^1, \Delta\eta)_G + k(\operatorname{div} u^2, \eta)_G = -(\operatorname{div} f^2, \eta)_G.$$

Now we subtract this equality from the second equation of (4.1.13) and obtain

$$-(a^1, \Delta\eta + k^2\eta)_G = (\operatorname{div} f^2 + ikh^1, \eta)_G.$$

According to the compatibility condition, it follows that

$$(a^1, \Delta\eta + k^2\eta)_G = 0 \tag{4.1.14}$$

for all $\eta \in C_c^\infty(\overline{G})$ such that

$$\partial_\nu \eta(x) = 0 \quad \text{for} \quad x \in G. \tag{4.1.15}$$

The equality (4.1.14) remains valid for all $\eta \in H_\delta^2(G)$, subject to the condition (4.1.15). We introduce the operator

$$A_{\mathcal{N}}(\delta) : \eta \mapsto \Delta\eta + k^2\eta, \quad A_{\mathcal{N}}(\delta) : \mathcal{D}(A_{\mathcal{N}}(\delta)) \to H_\delta^0(G)$$

with domain

$$\mathcal{D}(A_{\mathcal{N}}(\delta)) = \{\eta \in H_\delta^2(G) : \partial_\nu\eta = 0 \quad \text{on} \quad \partial G\}.$$

The equalities (4.1.14) and (4.1.15) mean that $a^1 \in \operatorname{coker} A_{\mathcal{N}}(\delta)$. It follows (e.g. see [21] (Ch.2, 5.2, 5.3) and [25], (Theorem 5.1.4)), that a^1 belongs to $H_{-\delta}^l(G)$ with any $l = 1, 2, \ldots$ and satisfies (4.1.9).

Let us consider the radiation conditions for a^1 and a^2. Given the new notation, (2.5.1) takes the form of

$$\mathcal{U} - \sum_{j=1}^{\upsilon} c_j e_j^- - \sum_{j=1}^{\upsilon+N} d_j \gamma_j^- \in H_\delta^l(G). \tag{4.1.16}$$

Here e_j^- and γ_j^- are representatives of the waves E_j^- and Γ_j^-, whereas c_j and d_j are constant coefficients. Specifically,

$$\mathcal{U} = (u^1, a^1, u^2, a^2) = (u_1^1, u_2^1, u_3^1, a^1, u_1^2, u_2^2, u_3^2, a^2).$$

We first consider the nonthreshold case. Each function e_j^- is generated by one of the vectors Ψ^ζ in (2.2.76) or Φ^τ in (2.2.77) (see the definition of waves before (2.3.19)). The forth and eighth components of such functions are equal to zero, so the functions e_j^- contribute neither in the asymptotics of a^1 nor in that of a^2. It is also clear that the asymptotics of a^1 is independent of γ_j^- generated by vectors of the form B^τ in (2.2.77), whereas the asymptotics of a^2 is independent of γ_j^- generated by vectors of the form A^ζ in (2.2.76). Therefore, from (4.1.16) it follows that

$$a^1 - \sum_A d_j \gamma_{j,4}^- \in H_\delta^l(G), \quad a^2 - \sum_B d_j \gamma_{j,8}^- \in H_\delta^l(G), \tag{4.1.17}$$

where the sums $\sum_A$ and $\sum_B$ contain all the γ_j^- corresponding to vectors of the form A^ζ and B^τ, while $\gamma_{j,m}^-$ stands for the m-th component of the vector function γ_j^-.

We show that the relations (4.1.17) are the intrinsic radiation conditions for the solutions a^1 and a^2 of problems (4.1.9) and (4.1.10). Let us consider the first of relations (4.1.17) (containing a^1). Each functions $\gamma_{j,4}^-$ in the sum $\sum_A$ is generated by a vector of the form A^ζ with

$$(\mathfrak{A}'(\lambda_0)A^\zeta, A^\zeta) = \operatorname{sgn}(k\lambda_0),$$

where λ_0 is an eigenvalue of the pencil $\mathfrak{A}$, corresponding to A^ζ (Proposition 2.2.7) (it is not excluded that $\lambda_0 = k$, see Remark 2.2.9). Since (4.1.16) contains only outgoing waves, the expressions $iq(\gamma_j^-, \gamma_j^-)$ for all terms in $\sum_A$ are numbers of the same sign. In problem (4.1.9), the function $\gamma_{j,4}^-$ is a representative of incoming or outgoing wave according to the sign of $iq_{\mathcal{N}}(\gamma_{j,4}^-, \gamma_{j,4}^-)$, where

$$q_{\mathcal{N}}(u,v) = ((\Delta + k^2)u, v)_G - (\partial_\nu u, v)_{\partial G} - (u, (\Delta + k^2)v)_G + (u, \partial_\nu v)_{\partial G}.$$

Let $\lambda \mapsto \mathfrak{A}_{\mathcal{N}}(\lambda)$ be the operator pencil corresponding to the problem (4.1.9) in the same domain Ω where has been given the pencil $\mathfrak{A}$. The pencil $\mathfrak{A}_{\mathcal{N}}$ is defined by the equality

$$\mathfrak{A}_{\mathcal{N}}(\lambda)u = (\partial_1^2 + \partial_2^2 - \lambda^2 + k^2)u$$

on the set of smooth functions in Ω satisfying $\partial_\nu u = 0$. Since α^ζ and A^ζ are connected by (2.2.76), we have

$$-k(\mathfrak{A}_{\mathcal{N}}'(\lambda_0)\alpha^\zeta, \alpha^\zeta) = (\mathfrak{A}'(\lambda_0)A^\zeta, A^\zeta). \tag{4.1.18}$$

Considering (2.3.10), we derive from (4.1.18), that

$$-ikq_{\mathcal{N}}(\gamma_{j,4}^-, \gamma_{j,4}^-) = iq(\gamma_j^-, \gamma_j^-). \tag{4.1.19}$$

Therefore all the expressions $iq_{\mathcal{N}}(\gamma_{j,4}^-, \gamma_{j,4}^-)$ are numbers of the same sign. This means, that the first of the relations (4.1.17) turns out to be an intrinsic radiation condition for the solution a^1 to problem (4.1.9). For the second relation (4.1.17), the required relation can be proved in a similar way.

We now assume, that k coincides within a threshold. Let us consider for instance the first of the relations (4.1.17) (containing a^1). Each function γ_j^- in the sum $\sum_A$ corresponds to a real eigenvalue of the pencil $\mathfrak{A}(\cdot, k)$. If such an eigenvalue λ_0 differs from zero, then one can again obtain (4.1.19) by repeating word for word the argument given for the non-threshold case. If $\lambda_0 = 0$, then γ_j^- is a function of the form $\chi A_\zeta^\pm$, where $A_\zeta^\pm$ has been defined by (2.3.11) for the Jordan chain $\{A^{0,\zeta}, A^{1,\zeta}\}$,

$$A_\zeta^\pm(x) := \frac{(ix_3 \pm 1)A^{0,\zeta}(x_1, x_2) + A^{1,\zeta}(x_1, x_2)}{\sqrt{2}}.$$

As before, let $\lambda \mapsto \mathfrak{A}_\mathcal{N}(\lambda)$ be the operator pencil of problem (4.1.9). Incoming and outgoing waves $\alpha_\zeta^\pm$ corresponding to problem (4.1.9) are of the form

$$\alpha_\zeta^\pm(x) := \frac{(ix_3 \pm 1)\alpha^{0,\zeta}(x_1, x_2) + \alpha^{1,\zeta}(x_1, x_2)}{\sqrt{2}},$$

where $\{\alpha^{0,\zeta}, \alpha^{1,\zeta}\}$ is the Jordan chain for the pencil $\mathfrak{A}_\mathcal{N}$ connected with $\{A^{0,\zeta}, A^{1,\zeta}\}$ by (2.2.71). From Proposition 2.2.10, we get

$$\begin{aligned}
&(\mathfrak{A}' A^{0,\tau}, A^{0,\zeta}) = 0,\\
&(\mathfrak{A}' A^{1,\tau}, A^{0,\zeta}) = (\mathfrak{A}' A^{0,\tau}, A^{1,\zeta}) = k(\alpha^{0,\tau}, \alpha^{0,\zeta}),\\
&(\mathfrak{A}' A^{1,\tau}, A^{1,\zeta}) = k[(\alpha^{0,\tau}, \alpha^{1,\zeta}) + (\alpha^{1,\tau}, \alpha^{0,\zeta})], \quad \tau, \zeta = 1, \dots J_N. \qquad (4.1.20)
\end{aligned}$$

It follows that

$$-ikq_\mathcal{N}(\chi\alpha_\zeta^\pm, \chi\alpha_\zeta^\pm) = iq(\chi A_\zeta^\pm, \chi A_\zeta^\pm). \qquad (4.1.21)$$

To verify this equality, it suffices to fulfil for the Helmholtz operator the calculations similar to the proof of Proposition 2.3.3 and to make use of (4.1.20). Thus intrinsic radiation conditions (4.1.16) imply the intrinsic radiation conditions for a^1 in the sense of problem (4.1.9). □

Corollary 4.1.2. *Let the hypotheses of Proposition* 4.1.1 *be fulfilled. Then*

$$\mathcal{U} - \sum_{j=1}^{v} c_j e_j^- \in H_\delta^l(G), \qquad (4.1.22)$$

so the coefficients d_j, $j = 1, \dots, \gamma + N$, *vanish in* (4.1.16).

Proof. Since a^1 satisfies the homogeneous problem (4.1.9), we have either $a^1 \equiv 0$ (if k^2 is not an eigenvalue of the problem) or $a^1 \in H_\delta^l(G)$ (if a^1 is an eigenfunction). In both cases, all coefficients d_j vanish from the sum $\sum_A$ (see (4.1.17)). The coefficients d_j in the sum $\sum_B$ vanish for the same reason. Therefore (4.1.16) takes the form (4.1.22). □

The representatives e_j^- of the waves E_j^- are generated by eigenvectors of the pencil $\mathfrak{M}(\cdot, k)$, so one can choose them to be "Maxwell" whose components $e_{j,4}^-$

and $e^-_{j,8}$ vanish. However the inclusion (4.1.22) is still not completely satisfactory for our purpose (the return to the Maxwell system), because the coefficients c_j are defined in the terms of the elliptic problem. Moreover, if k turns out to be an eigenvalue of the elliptic problem, then, for the existence of a solution $\mathcal{U}$, the right-hand-side must be orthogonal to the eigenvectors of the problem. Finally, in this case the elliptic problem reminds of itself also by the fact that a solution $\mathcal{U}$ is determined up to adding its arbitrary eigenvector corresponding to the eigenvalue k.

4.2 Decomposing an eigenfunction of the elliptic problem into the solenoidal and gradient terms

Proposition 4.2.1. *Let $\mathcal{U} = (u^1, a^1, u^2, a^2)$ be an eigenvector of problem* (2.1.9) *that corresponds to an eigenvalue $k \neq 0$ and belongs to the space $H^l_\delta(G)$ with $l = 1$ (and, consequently, with any $l = 2, 3, \dots$). Then $\mathcal{U} = V + W$, where*

$$V = ((i/k)\nabla a^2, a^1, -(i/k)\nabla a^1, a^2),$$
$$W = ((i/k)\mathrm{rot}\, u^2, 0, -(i/k)\mathrm{rot}\, u^1, 0),$$

so at least one of the vectors V and W is nonzero. If $V(W)$ is a nonzero vector, then it is an eigenvector of the problem (2.1.9). *The vectors V and W are orthogonal in $L_2(G)$.*

Proof. The vector $\mathcal{U}$ satisfies the homogeneous system in the domain G

$$\begin{aligned} i\,\mathrm{rot}\, u^2(x) + i\,\nabla a^2(x) - ku^1(x) &= 0, \\ -i\,\mathrm{div}\, u^2(x) - ka^1(x) &= 0, \\ -i\,\mathrm{rot}\, u^1(x) - i\,\nabla a^1(x) - ku^2(x) &= 0, \\ i\,\mathrm{div}\, u^1(x) - ka^2(x) &= 0 \end{aligned} \tag{4.2.1}$$

with boundary conditions

$$u^1_\tau(x) = 0,\ u^2_\nu(x) = 0,\ a^2(x) = 0,\ x \in \partial G. \tag{4.2.2}$$

The representations $\mathcal{U} = V + W$ immediately follows from (4.2.1). As in the proof of Proposition 4.1.1, we find out that a^1 satisfies the problem

$$(\Delta + k^2)a^1(x) = 0, \quad x \in G, \quad \partial_\nu a^1(x) = 0, \quad x \in \partial G. \tag{4.2.3}$$

This time $a^1 \in H^l_\delta(G)$, so either a^1 is an eigenfunction of this problem or $a^1 = 0$. In a similar way, we obtain that a^2 vanishes or turns out to be an eigenfunction of the problem

$$(\Delta + k^2)a^2(x) = 0, \quad x \in G, \quad a^2(x) = 0, \quad x \in \partial G. \tag{4.2.4}$$

Let us show that V and W satisfy the boundary conditions (4.2.2). We have to verify the equalities on ∂G for V

$$(\nabla a^2)_\tau = 0, \quad (\nabla a^1)_\nu = 0, \quad a^2 = 0, \tag{4.2.5}$$

and also the equalities for W on ∂G

$$(\text{rot}\, u^2)_\tau = 0, \quad (\text{rot}\, u^1)_\nu = 0. \tag{4.2.6}$$

From $a^2(x) = 0$ on ∂G (see (4.2.2)), it follows, that $(\nabla a^2)_\tau = 0$. This, the condition $u^1_\tau = 0$, and the first equation of the system (4.2.1) imply that $(\text{rot}\, u^2)_\tau = 0$. The equalities $u^2_\nu = 0$ in (4.2.2), $(\nabla a^1)_\nu = 0$ in (4.2.3), and the third equation in (4.2.1) lead to $(\text{rot}\, u^1)_\nu = 0$. Thus all conditions (4.2.5) and (4.2.6) are fulfilled.

We now show, that each of the vectors V and W is a solution to the system (4.2.1). As before, denoting the operator of the system by $\mathcal{A}(D)$, we have

$$\mathcal{A}(D)V = (0, -(1/k)\Delta a^1 - ka^1, 0, -(1/k)\Delta a^2 - ka^2).$$

Hence $\mathcal{A}(D)V = 0$is obtained (4.2.3) and (4.2.4). Therefore if $V \neq 0$, then V is an eigenvector of problem (2.1.9).

We apply rot to the rot-equations of the system (4.2.2) and obtain

$$i\, \text{rot}\,\text{rot}\, u^2 - k\, \text{rot}\, u^1 = 0, \quad -i\, \text{rot}\,\text{rot}\, u^1 - k\, \text{rot}\, u^2 = 0. \tag{4.2.7}$$

Since

$$\mathcal{A}(D)W = ((1/k)\text{rot}\,\text{rot}\, u^1 - i\, \text{rot}\, u^2, 0\,, (1/k)\, \text{rot}\,\text{rot}\, u^2 + i\, \text{rot}\, u^1, 0),$$

we arrive at $\mathcal{A}(D)W = 0$. If the vector W is nonzero, then it is an eigenvector of problem (2.1.9). The orthogonality of V and W can be verified by integration by parts. □

Corollary 4.2.2. *1. A number k is an eigenvalue of the elliptic problem* (4.2.1), (4.2.2) *if and only if at least one of the three conditions holds: (i) k is an eigenvalue of the problem* (1.2.1), (1.2.2) *for the Maxwell system; (ii) k^2 is an eigenvalue of the Neumann problem* (4.2.3) *for the Helmholtz equation; (iii) k^2 is an eigenvalue of the Dirichlet problem* (4.2.4) *for the Helmholtz equation.*

2. An eigenvalue k of elliptic problem (4.2.1), (4.2.2) *turns out to be an eigenvalue of problem* (1.2.1), (1.2.2) *as well only when at least one of the eigenvectors of the elliptic problem corresponding to k has a solenoidal projection distinct from zero.*

The notations in Proposition 4.2.3 are the same as those in Proposition 4.2.1.

Proposition 4.2.3. *Let k^2 be an eigenvalue of at least one of the problems* (4.2.3) *and* (4.2.4) *and let $V = ((i/k)\nabla a^2, a^1, -(i/k)\nabla a^1, a^2)$ be an eigenvector of problem* (4.2.1), (4.2.2), *corresponding to the eigenvalue k. If a vector $F = (f^1, h^1, f^2, h^2)$ satisfies the compatibility conditions* (2.1.3) - (2.1.5), *then $(F, V)_G = 0$.*

To verify this assertion it suffices to substitute in $(F, V)_G$ the expressions of the vector projections and then integrate by parts.

4.3 The continuous spectrum eigenfunctions of the Maxwell system. The scattering matrix

Let δ be a small positive number, such that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \leq \delta\}$ contains no eigenvalues of the pencils $\mathfrak{M}^q(\cdot, k)$, $q = 1, \dots, N$, except the real eigenvalues. The functions in $H^1_{-\delta}(G)$ satisfying the homogeneous problems (1.2.1), (1.2.2) and not belonging to $H^1_\delta(G)$ are called eigenfunctions continuous spectrum of the Maxwell system. By $\mathcal{M}_\delta(k)$ we denote the operator of the system (1.2.1) defined on the functions in $H^1_\delta(G)$ satisfying the boundary conditions (1.2.2).

We set $\mu_{\mathfrak{M}} = \min\{\mu^1_2, \dots, \mu^N_2\}$, where μ^j_2 is the second eigenvalue of the Neumann problem (2.3.1) in the domain Ω^j which is the cross-section of the cylindrical outlet Π^j_+). For $\mu_{\mathfrak{M}} \leq k^2$, there exist continuous spectrum eigenfunctions of the Maxwell system. In the Section, we show that the wave E^+_j in the elliptic system is scattered only into the waves E^-_q, $q = 1, \dots, v$. Therefore the elliptic system has a solenoidal continuous spectrum eigenfunction whose asymptotics is a linear combination of representatives of the waves $E^+_j, E^-_1, \dots, E^-_v$ up to a term in $H^l_\delta(G)$. Such functions form a basis in the space of the continuous spectrum eigenfunctions of the Maxwell system. Moreover the gradient wave Γ^+_l can be scattered only into the waves Γ^-_m, $m = 1, \dots, v + N$. Therefore the scattering matrix of elliptic system is block diagonal with two blocks ("solenoidal" and "gradient") on the principal diagonal. Since this matrix is unitary, each of the mentioned blocks turns out to be unitary.

If $0 < k^2 < \mu_{\mathfrak{M}}$, then the Maxwell waves do not exist, so there are no the continuous spectrum eigenfunctions of the Maxwell system. In Section 4.5, we show that for $k = 0$, there is no a Maxwell wave transporting energy. The scattering matrix for the (homogeneous) problem (1.2.1), (1.2.2) is defined only on the set $\{k \in \mathbb{R} : k^2 \geq \mu_{\mathfrak{M}}\}$, which we call the continuous spectrum of the aforementioned problem. The multiplicity of the continuous spectrum at a point k is equal to the sum

$$\Sigma^1_{\mathfrak{M}}(k)/2 + \cdots + \Sigma^N_{\mathfrak{M}}(k)/2.$$

The next Proposition 4.3.1 describes a basis in the space of continuous spectrum eigenfunctions of the Maxwell system.

Proposition 4.3.1. *Let e^+_j and e^-_j be representatives of the waves E^+_j and E^-_j respectively. Then there exists a solution $\mathcal{E}^+_j$ to the homogeneous problem* (1.2.1), (1.2.2), *such that*

$$\mathcal{E}^+_j(\cdot, k) - e^+_j(\cdot, k) - \sum_{q=1}^{v} s_{jq}(k) e^-_q(\cdot, k) \in H^1_\delta(G) \qquad (4.3.1)$$

for $j = 1, \dots, v$. If k is not an eigenvalue of problem (1.2.1), (1.2.2) *for the Maxwell system (that is the homogeneous problem with given k has no nonzero solutions in $H^1_\delta(G)$), then the solution $\mathcal{E}^+_j$ is unique. The functions $\mathcal{E}^+_1(\cdot, k), \dots, \mathcal{E}^+_v(\cdot, k)$ form a basis in the space of continuous spectrum eigenfunctions of the Maxwell system corresponding to the number k. In the case that k is an eigenvalue for the problem* (1.2.1), (1.2.2), *the solution*

$\mathcal{E}_j^+(\cdot, k)$ *is determined up to a term in* ker $\mathcal{M}_\delta(k)$, *whereas* $\mathcal{E}_j^+(\cdot, k), \dots, \mathcal{E}_v^+(\cdot, k)$ *form a basis modulo* ker $\mathcal{M}_\delta(k)$.

Proof. We restrict ourselves to verifying the proposition under the condition that k is not an eigenvalue for the problem (1.2.1), (1.2.2). Assume, that $\chi \in C^\infty(\mathbb{R})$, $\chi \geq 0$, $\chi(t) = 0$ for $t \leq T - 1$ and $\chi(t) = 1$ for $t \geq T$. Moreover let e_j^+ coincide with the mapping $(y^q, t^q) \mapsto \chi(t^q)\mathcal{P}(y^q, t^q)$ in one of the cylindrical outlets of the waveguide and be extended by zero to G. Here $\mathcal{P}$ is a solution of the homogeneous problem for the Maxwell system in the cylinder, generated by a vector Φ^τ or Ψ^ζ in (2.2.77) and (2.2.76)(or by Jordan chains $\{\Phi^{0,\tau}, \Phi^{1,\tau}\}$, $\{\Psi^{0,\zeta}, \Psi^{1,\zeta}\}$ in (2.2.95), (2.2.96)) (see the definition of waves before (2.3.19)). We have

$$\mathcal{A}(D)e_j^+(x) = [\mathcal{A}(D), \chi]\mathcal{P}(x), \;\; x \in G, \qquad \mathcal{B}e_j^+(x) = 0, \;\; x \in \partial G. \tag{4.3.2}$$

The fourth and the eighth components of the vector function e_j^+ vanish, so in accordance with Proposition 2.1.1, the right-hand-side of (4.3.2) satisfies the compatibility conditions (2.1.3) -(2.1.5).

Let us consider the problem

$$\mathcal{A}(D)\mathcal{U}(x) = [\mathcal{A}(D), \chi]\mathcal{P}(x), \;\; x \in G, \;\; \mathcal{B}\mathcal{U}(x) = 0, \;\; x \in \partial G. \tag{4.3.3}$$

We first discuss the case that not only k differs from the eigenvalues of the Maxwell system, but also k^2 is not an eigenvalue for the problems (4.1.9) and (4.1.10). From Proposition 2.5.1 and 4.1.1, there exists a unique solution $\mathcal{U}$ of problem (4.3.3) with intrinsic radiation conditions, whereas the fourth and the eighth components of $\mathcal{U}$ are equal to zero. We set $\mathcal{E}_j^+ := e_j^+ - \mathcal{U}$ and check that $\mathcal{E}_j^+$ has the required properties. From (4.3.2) and (4.3.3), it follows that $\mathcal{E}_j^+$ is a solution to the homogeneous problem (1.2.1), (1.2.2). The properties of $\mathcal{U}$ just indicated lead to

$$\mathcal{U} = \sum_{q=1}^{v} c_q e_q^- + v,$$

where $v \in H_\delta^1(G)$. Therefore the equality $\mathcal{E}_j^+ = e_j^+ - \mathcal{U}$ implies (4.3.1) with constants s_{jq}.

We now drop the additional assumption on k^2, that is we assume, that k^2 is an eigenvalue at least one of the two problems (4.1.9) and (4.1.10). Then k is an eigenvalue of the elliptic problem (4.2.1), (4.2.2) with "gradient" eigenvectors of the form

$$V = ((i/k)\nabla a^2, a^1, -(i/k)\nabla a^1, a^2) \tag{4.3.4}$$

(see Proposition 4.2.1). From (4.3.2) it follows, that

$$([\mathcal{A}(D), \chi]\mathcal{P}), V)_G = (\mathcal{A}(D)e_j^+, V)_G = (e_j^+, (\mathcal{A}(D)V)_G = 0$$

(the second equality can be obtained by integration by parts in view of the inclusion $V \in H_\delta^1(G)$). Hence there exists a solution $\mathcal{U}$ to problem (4.3.3) with intrinsic radiation conditions. This time such a solution has been determined up to a term of the form (4.3.4) in $\ker \mathcal{L}_\delta(k)$. As before, we set $\mathcal{E}_j^+ := e_j^+ - \mathcal{U}$; for any choice of $\mathcal{U}$, the function $\mathcal{E}_j^+$ satisfies the homogeneous elliptic problem (4.2.1), (4.2.2). We make use of the arbitrariness in the definition of $\mathcal{U}$ in order to provide the orthogonality of $\mathcal{E}_j^+$ to $\ker \mathcal{L}_\delta(k)$. Let us show that then $\mathcal{E}_j^+$ is a unique solution to the homogeneous problem (1.2.1), (1.2.2), which satisfies the inclusion (4.3.1). We write down $\mathcal{E}_j^+ = (u^1, a^1, u^2, a^2)$ and obtain, as in the proof of Proposition 4.2.1, that $\mathcal{E}_j^+ = V + W$, where V and W are defined by the same formulas as in Proposition 4.2.1. Each of the vectors V and W is a solution to (4.2.1) with boundary conditions (4.2.5) and (4.2.6). This, in particular, implies that $V \in \ker \mathcal{L}_\delta(k)$, so $(\mathcal{E}_j^+, V)_G = 0$ and moreover $(V, W)_G = 0$. Therefore $(V, V)_G = (\mathcal{E}_j^+, V)_G - (W, V)_G = 0$ and finally $\mathcal{E}_j^+ = W$.

It remains to verify that $\mathcal{E}_j^+$, $j = 1, \dots, v$, make up a basis in the space of continuous spectrum eigenfunctions. Let $\mathcal{E}$ be any continuous spectrum eigenfunction of the Maxwell system corresponding to k. Then

$$\mathcal{E} - \sum_{j=1}^{v} \alpha_j e_j^+ - \sum_{j=1}^{v} \beta_j e_j^- \in H_\delta^1(G),$$

where α_j and β_j are certain constants. Therefore $\mathcal{U} := \mathcal{E} - \sum_{j=1}^{v} \alpha_j \mathcal{E}_j^+$ satisfies

$$\mathcal{U} - \sum_{j=1}^{v} d_j e_j^- \in H_\delta^1(G)$$

with constant coefficients d_j. This means that $\mathcal{U}$ is a solution to the homogeneous elliptic problem (4.2.1), (4.2.2), which is subject to intrinsic radiation conditions and orthogonal to $\ker \mathcal{L}_\delta(k)$ (in the case that the lineal is nonzero, it consists of the gradient eigenvectors V, see Proposition 4.2.1). Therefore $\mathcal{U} = 0$ and $\mathcal{E} = \sum_{j=1}^{v} \alpha_j \mathcal{E}_j^+$. □

Proposition 4.3.2. *Let γ_j^+ and γ_j^- be representatives of the gradient waves Γ_j^+ and Γ_j^- respectively. Then there exists a solution $\mathcal{G}_j^+$ to the elliptic problem* (4.2.1), (4.2.2)*, such that*

$$\mathcal{G}_j^+ - \gamma_j^+ - \sum_{p=1}^{v+N} \sigma_{jp} \gamma_p^- \in H_\delta^1(G) \tag{4.3.5}$$

for $j = 1, \dots, v + N$. If k^2 is an eigenvalue neither of the Neumann problem (4.1.9) nor of the Dirichlet problem (4.1.10), then the solution $\mathcal{G}_j^+$ is unique; the functions $\mathcal{G}_1^+, \dots, \mathcal{G}_{v+N}^+$ make up the basis in the space of gradient continuous spectrum eigenfunctions of the elliptic problem corresponding to k. In the case, that k^2 is an eigenvalue of at least one of the problems (4.1.9) and (4.1.10), then $\ker \mathcal{L}_\delta(k)$ *contains a nonzero subspace $\mathcal{V}$ of gradient vectors and the solution $\mathcal{G}_j^+$ is defined up to arbitrary term in $\mathcal{V}$, whereas $\mathcal{G}_1^+, \dots, \mathcal{G}_{v+N}^+$ constitute a basis modulo $\mathcal{V}$.*

Proof: The proof is similar to the proof of Proposition 4.3.1; let us outline the proof. Assume, that the wave Γ_j^+ corresponds to $\lambda_0 \neq 0$, generated, for example, by a vector of the form

$$B = (i\partial_1\beta^\tau, i\partial_2\beta^\tau, -\lambda_0\beta^\tau, 0, 0, 0, 0, k\beta^\tau)$$

in (2.2.77). When proving Proposition 4.3.1, we chose for the solenoid wave E_j^+ a "special" representative e_j^+ with fouth and eighth components being zero. Dealing now with the gradient wave Γ_j^+, we take a "special " gradient representative γ_j^+ which coincides in one of the cylindrical waveguide outlets with the mapping

$$(y,t) \mapsto (i\nabla(\chi(t)\exp(i\lambda_0 t)\beta^\tau(y)), 0, 0, 0, 0, k\chi(t)\exp(i\lambda_0 t)\beta^\tau(y)),$$

and vanishes in the rest part of G . Applying the operator of problem (2.1.9) to γ_j^+, we obtain

$$\mathcal{A}(D,k)\gamma_j^+ = (0,\ldots,0, i\,\mathrm{div}\, i\nabla(\chi\exp(i\lambda_0 t)\beta^\tau) - k^2\chi\exp(i\lambda_0 t)\beta^\tau), \quad \mathcal{B}\gamma_j^+ = 0. \tag{4.3.6}$$

Let us denote the right-hand-side of (4.3.6) by F, and consider

$$\mathcal{A}(D,k)\mathcal{U}(x) = F(x),\ x \in G;\ \mathcal{B}\mathcal{U}(x) = 0,\ x \in \partial G. \tag{4.3.7}$$

For simplicity, we assume that k is not an eigenvalue for the problem, that is, $\ker \mathcal{L}_\delta(k)$=0. According to Proposition 2.5.1, there exists a unique solution $\mathcal{U}$ to the problem (4.3.7) with intrinsic radiation conditions

$$\mathcal{U} - \sum c_j e_j^- - \sum d_p \gamma_p^- \in H_\delta^1(G),$$

where $c_j = (F, \mathcal{E}_j^-)_G$ and $\mathcal{E}_j^-$ are continuous spectrum eigenfunctions satisfying (4.3.8) ($\mathcal{E}_j^-$ are defined like $\mathcal{E}_j^+$ but the roles of the incoming and outgoing waves are interchanged). The fourth and the eighth components of every function $\mathcal{E}_j^-$ vanish, whereas all components of F, except the eighth component, are equal to zero. Therefore $c_j = 0$ for all j and we can set $\mathcal{G}_j^+ := \gamma_j^+ - \mathcal{U}$.

If k is a threshold, then for a gradient wave corresponding to the eigenvalue $\lambda_0 = 0$, we choose a "special" representative which coincides with the mapping

$$(y,t) \mapsto \frac{(i\nabla(\chi(t)(it+1)\beta^{0,\tau}(y)), 0, 0, 0, 0, k\chi(t)(it+1)\beta^{0,\tau}(y))}{\sqrt{2}},$$

in one of the cylindrical outlets and vanishes in the rest part of G. □

The next assertion is justified by Propositions 2.4.1, 4.3.1, and 4.3.2.

Proposition 4.3.3. *The scattering matrix $S(k)$ of the elliptic problem (for the augmented Maxwell system) is block diagonal, $S(k) = (s(k), \sigma(k))$. In the basis* (4.1.2) *the expansions* (4.3.1) *and* (4.3.5) *hold, so the $v \times v$-block $s(k) = \|s_{jq}\|$ consists of the coefficients* *in* (4.3.1)*, while the* $(v+N)\times(v+N)$*-block* $\sigma(k) = \|\sigma_{jq}(k)\|$ *consists of those in* (4.3.5)*. Each of the blocks is unitary. Moreover*

$$\mathcal{E}_j^-(\cdot,k) - e_j^-(\cdot,k) - \sum_{q=1}^{v} t_{jq}(k)e_q^+(\cdot,k) \in H_\delta^1(G) \qquad (4.3.8)$$

with $t = s^{-1}$*, where* $t = \|t_{jq}\|$*.*

The block $s(k)$ is called the scattering matrix of the homogeneous problem (1.2.1), (1.2.2). In essence, Proposition 4.3.3 coincides with Theorem 1.2.4.

Remark 4.3.4. *Applying Theorem 3.1.1, we compute an approximation* S^R *to the scattering matrix* S *of the elliptic system. The matrix is block-diagonal* $S = \operatorname{diag}(s,\sigma)$*. Hence, as an approach to the matrix* s *serves the corresponding block of* S^R*.*

4.4 The radiation principle for the Maxwell system

A correct boundary value problem with intrinsic radiation conditions is called the radiation principle. We are now in a position to derive the radiation principle for the Maxwell system from that of the elliptic problem (Proposition 2.5.1). Using the results of Sections 4.1 - 4.3, we first adapt for the purpose the statement of the radiation principle of the elliptic problem.

Let $E_1^+,\dots,E_v^+$ be incoming waves and $E_1^-,\dots,E_v^-$ outgoing waves in (4.1.2). For the waves $E_j^\pm$ we choose representatives $e_j^\pm$ which coincide with the function $(y,t)\mapsto\chi(t)\mathcal{P}(y,t)$ in one of the cylindrical outlets and vanish elsewhere in G. Here $\mathcal{P}(y,t)$ is the solution of the homogeneous problem (2.1.9) in cylinder given by (2.3.4) (by (2.3.11))(see 2.3). Such $e_j^\pm$ has the fourth and the eighth components equal to zero in G, and satisfy the boundary conditions $\mathcal{B}e_j^\pm = 0$ on ∂G. As before, let δ be a small positive number, such that the strip $\{\lambda\in\mathbb{C} : |\operatorname{Im}\lambda| < \delta\}$ contains no eigenvalues of the pencils $\lambda\mapsto\mathfrak{A}^q(\lambda,k)$, $q=1,\dots,N$, except the real ones. If the space $\ker\mathcal{L}_\delta(k)$ is nonzero, we represent every element as the sum of orthogonal solenoidal and gradient eigenvectors of problem (4.2.1), (4.2.2); such a possibility is provided by Proposition 4.2.1. Let $\ker^\perp\mathcal{L}_\delta(k)$ denote the linear hull of the solenoid eigenvectors in $\ker\mathcal{L}_\delta(k)$.

Theorem 4.4.1. *Let* $\mathcal{Z}_1,\dots,\mathcal{Z}_m$ *be a basis in* $\ker^\perp\mathcal{L}_\delta(k)$ *and* $\mathcal{F} = (f^1,h^1,f^2,h^2)$ *a vector in* $H_\delta^{l-1}(G)$ *subject to the compatibility conditions* (2.1.3) - (2.1.5) *and the orthogonality conditions* $(\mathcal{F},\mathcal{Z}_j) = 0$ *for* $j=1,\dots,m$*. Then:*

1. If $k^2 \ge \mu_{\mathfrak{M}}$ *(that is,* k *belongs to the continuous spectrum of the Maxwell system), then there exists a solution to the problem*

$$\mathcal{A}(D,k)\mathcal{U}(x) = \mathcal{F}(x),\ x\in G,\ \mathcal{B}\mathcal{U}(x) = 0,\ x\in\partial G, \qquad (4.4.1)$$

with radiation conditions

$$\mathcal{V} := \mathcal{U} - c_1e_1^- - \dots - c_ve_v^- \in H_\delta^l(G)$$

and with zero fourth and eighth components. Here $c_j = (\mathcal{F}, \mathcal{E}_j^-)$ *with* $\mathcal{E}_j^-$ *satisfying the homogeneous problem* (4.4.1) *and the inclusion* (4.3.8). *Such a solution* $\mathcal{U}$ *is determined up to an arbitrary term in* $\ker^{\perp} \mathcal{L}_\delta(k)$ *and*

$$\|\mathcal{V}; H_\delta^l(G)\| + |c_1| + \cdots + |c_v| \leq \mathrm{const}(\|\mathcal{F}; H_\delta^{l-1}(G)\| + \|\rho_\delta \mathcal{V}; L_2(G)\|). \quad (4.4.2)$$

A solution $\mathcal{U}_0$*, satisfying the additional conditions* $(\mathcal{U}_0, \mathcal{Z}_j)_G = 0$*, is unique and the estimate* (4.4.2) *holds with right-hand-side replaced by* $\mathrm{const}\|\mathcal{F}; H_\delta^{l-1}(G)\|$.

2. *If* $0 < k^2 < \mu_{\mathfrak{M}}$*, then there exists a solution to problem (4.4.1) satisfying* $\mathcal{U} \in H_\delta^l(G)$ *and having the fourth and the eighth components equal to zero. Such a solution* $\mathcal{U}$ *is determined up to an arbitrary term in* $\ker^{\perp} \mathcal{L}_\delta(k)$ *and*

$$\|\mathcal{U}; H_\delta^l(G)\| \leq \mathrm{const}(\|\mathcal{F}; H_\delta^{l-1}(G)\| + \|\rho_\delta \mathcal{U}; L_2(G)\|). \quad (4.4.3)$$

A solution $\mathcal{U}_0$*, satisfying the additional conditions* $(\mathcal{U}_0, \mathcal{Z}_j)_G = 0$*, is unique and the estimate* (4.4.3) *holds with right-hand-side replaced by* $\mathrm{const}\|\mathcal{F}; H_\delta^{l-1}(G)\|$.

Proof. 1. Since $\mathcal{F}$ satisfies the compatibility conditions, the equalities $(\mathcal{F}, V)_G = 0$ hold for all gradient vectors $V \in \ker \mathcal{L}_\delta(k)$ (Proposition 4.2.3). Thus the vector $\mathcal{F}$ is orthogonal to $\ker \mathcal{L}_\delta(k)$. According to Proposition 2.5.1, there exists a solution $\mathcal{U}$ to problem (4.4.1) subject to the intrinsic radiation conditions, which in view of Corollary 4.1.2 take the form of

$$\mathcal{U} - c_1 e_1^- - \cdots - c_v e_v^- \in H_\delta^l(G).$$

Again using Proposition 2.5.1, we obtain $c_j = (\mathcal{F}, \mathcal{E}_j^-)_G$. Such a solution $\mathcal{U}$ is determined up to an arbitrary term in $\ker \mathcal{L}_\delta(k)$. Let us choose a special solution $\mathcal{U}'$ and write down the general solution $\mathcal{U} = \mathcal{U}' + V + W$ with gradient V and solenoidal W vectors in $\ker \mathcal{L}_\delta(k)$. Proposition 4.1.1 implies that $\mathcal{U}' = (u^1, a^1, u^2, a^2)$, a^1 and a^2 being eigenfunctions of problems (4.1.9) and (4.1.10) respectively. Hence there exists a unique gradient vector V', such that $\mathcal{U}' + V' + W = (v^1, 0, v^2, 0)$ for all $W \in \ker^{\perp} \mathcal{L}_\delta(k)$. Finally choose a vector W' so that the solution $\mathcal{U}_0 := \mathcal{U}' + V' + W'$ satisfies $(\mathcal{U}_0, \mathcal{Z}_j)_G = 0$ with $j = 1, \ldots, m$. Then $\mathcal{U}_0$ is uniquely determined and the estimate holds given in the formulation of the theorem.

2. If $0 < k^2 < \mu_{\mathfrak{M}}$, then the real axis is free from the eigenvalues of the pencils $\mathfrak{M}^j(\cdot, k)$. Therefore the functions $\mathcal{E}_j^-$, e_j^- are absent as well as the corresponding radiation conditions. It remains to repeat with evident modifications the argument in the first part of the proof. □

In order to get a radiation principle directly for problem (1.2.1), (1.2.2), we have to somewhat change the notation in Theorem 4.4.1. Namely, the domain of the operator of problem (4.4.1) consists of vectors with eight components, whereas the fourth and the eighth components are equal to zero. Crossing out these zero components, we replace the 8-vectors by the "Maxwell" 6-vectors and obtain problem (1.2.1), (1.2.2) instead of (4.4.1). When going from (4.4.1) to (1.2.1), (1.2.2), we change the notation $\mathcal{U}, e_j^{\pm}, \mathcal{E}_j^{\pm}, \mathcal{L}$ for $U, \hat{e}_j^{\pm}, \hat{\mathcal{E}}_j^{\pm}, \mathcal{M}$, respectively. Moreover, in the notation of the spaces of vector functions we explicitly indicate the number of components, e. g., $H_\delta^l(G, \mathbb{C}^6)$. Then Theorem 4.4.1 in essence coincides with Theorem 1.2.5.

4.5 Statics: $k = 0$

In this Section we show that for $k = 0$ the Maxwell operator has N linearly independent waves participating in a well-posed boundary value problem, N being the number of cylindrical ends of the domain G. The waves do not transfer electromagnetic energy. The operator of the boundary value problem describes a closed system without energy dissipation. For $k = 0$ there are no nontrivial solutions to the homogeneous problem exponentially decaying at infinity. Therefore for any right-hand-side subject to the compatibility conditions the boundary value problem is uniquely solvable. In the absence of magnetic monopolies, the right-hand-side $\mathcal{F} = (f^1, h^1, f^2, h^2)$ of (1.2.1) satisfies $f^2 = 0$ and $h^1 = 0$. Then the statement of the well-posed boundary problem contains no aforementioned waves.

In what follows we assume that the domain G and all the cross-sections Ω^q are one-connected while the boundary ∂G is connected.

4.5.1 Waves

Let $\lambda \mapsto \mathfrak{A}(\lambda, 0)(= \mathfrak{A}(\lambda, 0))$ be the operator pencil of the augmented (elliptic) Maxwell system for $k = 0$ in a one-connected domain Ω. The number $\lambda_0 = 0$ is a unique real eigenvalue of the pencil. The eigenspace is spanned by the vectors

$$\Phi_\pm = (2\,|\Omega|)^{-1/2}(0, 0, 0, 1, 0, 0, \pm 1, 0) \tag{4.5.1}$$

with orthogonality and normalization conditions

$$(\mathfrak{A}'\Phi_\pm, \Phi_\mp) = 0, \qquad (\mathfrak{A}'\Phi_\pm, \Phi_\pm) = \pm 1. \tag{4.5.2}$$

There are no generalized eigenvectors (see Propositions 2.2.2 and 2.2.4). In G, the dimension of the wave space is equal to $2N$, N being the number of cylindrical ends of the domain G. In the wave space we choose a basis $V_1, \dots, V_{2N}$ with representatives $v_1, \dots, v_{2N}$. The supports of v_q, v_{q+N} belong to $G \cap \Pi^q_+$; on the supports, the functions are given by

$$v_q(y^q, t^q) = \chi(t^q)\Phi^q_+, \qquad v_{q+N}(y^q, t^q) = \chi(t^q)\Phi^q_-, \qquad q = 1, .., N,$$

where χ is a cut-off function, $0 \le \chi \le 1$, $\chi(t) = 0$ for $t \le T - 1$, and $\chi(t) = 1$ for $t \ge T$; the vectors $\Phi^q_\pm$ are defined by (4.5.1) on Ω^q. According to the argument after Proposition 2.3.5, the basis $V_1, \dots, V_{2N}$ in the wave space $\mathcal{W}$ satisfies (2.3.20).

Note that the forth and the seventh components of the outgoing $V_1, \dots, V_N$ and incoming $V_{N+1}, \dots, V_{2N}$ waves are nonzero. However there exist N linearly independent Maxwell waves $W_j = (V_j - V_{j+N})$, $j = 1, \dots, N$. The waves do not transfer energy, that is, $q_G(W_j, W_k) = 0$, $j, k = 1, ..., N$.

4.5.2 Continuous spectrum eigenfunctions. Scattering matrix

Recall, that the elements in $\ker\mathcal{L}_{-\delta}$ not belonging to $H^1_\delta(G)$ are called the continuous spectrum eigenfunctions (CSE) of the operator $\mathcal{L}_\delta = \mathcal{L}_\delta(0)$. The number of CSE linearly independent modulo $H^1_\delta(G)$ is half of the wave space dimension (see Proposition) 2.4.1) and is equal to N. In this Section we describe CSE in more detail and calculate the scattering matrix.

Proposition 4.5.1. *Let $\mathcal{U} = (u^1, a^1, u^2, a^2) \in \ker\mathcal{L}_{-\delta}(0)$, where as usual u^1, u^2 are three dimensional vector functions and a^1, a^2 are scalar functions. Then*

$$u^1 = 0, \quad a^1 = \text{const}, \quad u^2 = \nabla\omega, \quad a^2 = 0, \tag{4.5.3}$$

while ω belongs to $H^l_{-\delta}(G)$, $l = 2, 3, \dots$, and

$$\Delta\omega(x) = 0, \quad x \in G; \qquad \partial_\nu\omega(x) = 0, \quad x \in \partial G. \tag{4.5.4}$$

Proof. The waves $V_1, \dots, V_{2N}$ form a basis for the space of waves, so

$$\mathcal{U} - c_1 v_1 - \dots - c_{2N} v_{2N} \in H^l_\delta(G) \tag{4.5.5}$$

with certain constant $c_1, \dots, c_{2N}$. It follows, that

$$a^1 - \chi_1(c_1 + c_N)(2|\Omega^1|)^{-1/2} - \dots - \chi_N(c_N + c_{2N})(2|\Omega|^N)^{-1/2} \in H^l_\delta(G), \tag{4.5.6}$$

$$a^2 \in H^l_\delta(G), \tag{4.5.7}$$

where χ_j is a cut-off function with support in $G \cap \Pi^j_+$, equal to 1 near infinity. The function $\mathcal{U}$ satisfies the homogeneous problem (2.1.7),(2.1.8). Applying div to the rot-equations in (2.1.7) and taking account of (2.1.8), we obtain the problems for a^1 and a^2:

$$\Delta a^1(x) = 0, \quad x \in G, \qquad \partial_\nu a^1(x) = 0, \quad x \in \partial G, \tag{4.5.8}$$

$$\Delta a^2(x) = 0, \quad x \in G, \qquad a^2(x) = 0, \quad x \in \partial G. \tag{4.5.9}$$

We integrate $(\Delta a^1, a^1)_G$ and $(\Delta a^2, a^2)_G$ by parts, making use of (4.5.6), (4.5.7) and (4.5.8), (4.5.9). As a result, we obtain

$$0 = (\Delta a^j, a^j)_G = -(\nabla a^j, \nabla a^j)_G, \qquad j = 1, 2,$$

so $a^1 = \text{const}$ and $a^2 = 0$ in G. Now from (2.1.7) it follows, that

$$\operatorname{rot} u^1 = 0, \quad \operatorname{div} u^1 = 0; \qquad \operatorname{rot} u^2 = 0, \quad \operatorname{div} u^2 = 0.$$

Since the domain G is one-connected, we have $u^1 = \nabla\tau$ and $u^2 = \nabla\omega$, where ω and τ are harmonic functions in G. In view of (2.1.8), we obtain the problems

$$\Delta\,\omega(x) = 0, \quad x \in G, \qquad \partial_\nu\omega(x) = 0, \quad x \in \partial G, \tag{4.5.10}$$

$$\Delta\,\tau(x) = 0, \quad x \in G, \qquad \tau(x) = \text{const}, \quad x \in \partial G. \tag{4.5.11}$$

The set ∂G is one-connected, so "const" in (4.5.11) stands for the same constant everywhere on ∂G. The change τ for $\tau - \text{const}$ has no effect on u^1, and one can assume that const $= 0$ in (4.5.11). Therefore $\tau = 0$ and $u^1 = 0$ in G. □

Proposition 4.5.2. *There holds the equality* $\ker\mathcal{L}_\delta(0) = 0$.

Proof. Let $\mathcal{U} \in H^1_\delta(G)$ and $\mathcal{U} = (u^1, a^1, u^2, a^2)$. Since $\ker\mathcal{L}_\delta(0) \subset \ker\mathcal{L}_{-\delta}(0)$, the conclusion of Proposition 4.5.1 holds for u^j and a^j. This time $a^1 \in H^1_\delta(G)$, so $a^1 = 0$. Being subject to $\nabla\omega \in H^1_\delta(G)$, a solution ω of (4.5.4) admits asymptotics of the form $\omega(y,t) = c_j + O(\exp(-\delta t))$ with $c_j = \text{const}$ on every set $G \cap \Pi^j_+$ as $t \to +\infty$. Now we can integrate $(\Delta\omega, \omega)_G$ by parts and obtain $\|\nabla\omega; L_2(G)\| = 0$. Thus $u^2 = \nabla\omega = 0$. □

Proposition 4.5.3. *Let $\zeta_1, \dots, \zeta_N$ be the basis for the space of CSE, subject to* (2.4.1). *The vectors of the basis are of the form $\zeta_j = (\vec{0}, a^1_j, \nabla\omega_j, 0)$, where $\vec{0}$ stands for the three-functions component vector $(0,0,0)$. Then:*

1. *The vector functions $\eta_j = (\vec{0}, a^1_j, -\nabla\omega_j, 0)$ satisfy* (2.4.2).
2. *The scattering matrix s in Proposition* 2.4.1 *is self-adjoint with entries*

$$s_{jk} = \begin{cases} 2/d_j - 1, & k = j, \\ 2/\sqrt{d_j d_k}, & k \neq j, \end{cases} \tag{4.5.12}$$

where

$$d_j = \frac{\sum_{l=1}^N |\Omega_l|}{|\Omega_j|}. \tag{4.5.13}$$

3. *There holds the equality*

$$a^1_j = \frac{\sqrt{2\,|\Omega_j|}}{\sum_{l=1}^N |\Omega_l|}. \tag{4.5.14}$$

Proof. The formulas (2.4.1) for the eigenfunctions $\zeta_j = (\vec{0}, a^1_j, \nabla\omega_j, 0)$ are equivalent to (2.4.2) for the eigenfunctions $\eta_j = (\vec{0}, a^1_j, -\nabla\omega_j, 0)$ with matrix $s = t = s^*$. Therefore the unitary matrix s turns out to be self-adjoint (see 2.4). This and the equalities $g^1_j = \text{const}$ lead to (4.5.12), while a^1_j can be obtained from (4.5.12) and (2.4.1). □

Remark 4.5.4. *We set $Y := (\vec{0}, 1, \vec{0}, 0)$. It is clear, that $Y \in \ker\mathcal{L}_\delta(0)$, so Y turns out to be a CSE for $\mathcal{L}_\delta(G)$. The forth component of each ζ_j is constant (and equal to a^1_j). Therefore the fourth and the eighth components of any $Z_j := \zeta_j - a^1_j Y$ vanish, and Z_j satisfies the homogeneous problem* (1.2.1), (1.2.2). *Among the functions $Z_1, \dots, Z_N$ there are $N-1$ linearly independent functions, because the linear hull of $Y, Z_1, \dots, Z_N$ coincides with* $\ker\mathcal{L}_\delta(0)$. *Thus in the space* $\ker\mathcal{L}_\delta(0)$ *there exists a basis consisting of $N-1$ solenoidal functions and one gradient function Y.*

4.5.3 Radiation principle for the augmented Maxwell system

We are now in a position to specify Proposition 2.5.1. Let $V_1, \dots, V_{2N}$ be the basis as in Proposition 2.3.5 (1^0). We choose the wave representatives $v_1, \dots, v_{2N}$ in the same way as in 4.5.2. Denote by $\mathfrak{S}$ the linear hull of $v_1, \dots, v_N$. On the space $\mathfrak{S} \dotplus H^l_\delta(G)$ we consider the restriction $\mathbb{L}$ of $\mathcal{L}_{-\delta}(0)$, which is a continuous mapping

$$\mathbb{L} : \mathfrak{S} \dotplus H^l_\delta(G) \to H^{l-1}_\delta(G) \times H^{l-1/2}_\delta(\partial G).$$

Theorem 4.5.5. *Let* $\{\mathcal{F},\mathcal{G}\} \in H_\delta^{l-1}(G) \times H_\delta^{l-1/2}(\partial G)$ *l and let the compatibility conditions* (2.1.3)-(2.1.5) *be fulfilled. Then:*

1. *There exists a unique solution* $\mathcal{U} = (u^1, a^1, u^2, a^2) \in \mathfrak{S} \dotplus H_\delta^l(G)$ *to the equation*

$$\mathbb{L}\mathcal{U} = \{\mathcal{F},\mathcal{G}\}.$$

2. *There holds the equalities*

$$a^1 = \text{const}, \quad a^2 = 0. \tag{4.5.15}$$

Moreover, if $\mathcal{G} = 0$ *then*

$$a^1 = \frac{i(h^1,1)_G}{\sum_{l=1}^N |\Omega_l|} \tag{4.5.16}$$

3. *The vector* (u^1, u^2) *satisfies* (1.2.1), (1.2.2) *(for the non-augmented Maxwell system) and*

$$w^2 := u^2 - a^1 \sum_{l=1}^N \chi_l(0,0,1) \in H_\delta^l(G), \qquad w^1 := u^1 \in H_\delta^l(G).$$

4. *For the solution* (u^1, u^2) *of* (1.2.1), (1.2.2) *there is valid the estimate*

$$\|w^1; H_\delta^l(G)\| + \|w^2; H_\delta^l(G)\| + |a_1| \leq \text{const}(\|\mathcal{F}; H_\delta^{l-1}(G)\| + \|\mathcal{G}; H_\delta^{l-1/2}\partial G\|). \tag{4.5.17}$$

Proof. The existence and uniqueness of solutions follow from Proposition 2.5.1 and the triviality of the kernel $\ker \mathcal{L}_\delta(0)$. Let us verify (4.5.15). Applying div to the rot-equations in (2.1.7), and taking into account the compatibility conditions (2.1.3), (2.1.4), we arrive at $\Delta a^1 = \Delta a^2 = 0$ in G. By virtue of (1.2.2) and (2.1.5) we have $\partial_\nu a^1 = a^2 = 0$ on ∂G. The inclusions (4.5.6) and (4.5.7) follow from Proposition 2.5.1 (2^0). As in Proposition 4.5.1, we can now derive $a^1 = \text{const}$ and $a^2 = 0$. It remains to calculate the constant a^1. To this end, it suffices to find its asymptotics in one of the cylindrical outlets. According to (2.5.1) and (2.5.2),

$$a^1 = (i(\mathcal{F},\zeta_j)_G + i(\mathcal{G},\mathcal{Q}\zeta_j)_{\partial G}) / \sqrt{2\,|\Omega_j|}. \tag{4.5.18}$$

The expressions with $\nabla\omega_j^1$ vanish after integrating by parts. For $\mathcal{G} = 0$ we have

$$a^1 = \frac{i(h^1, a_j^1)_G}{\sqrt{2\,|\Omega_j|}} = \frac{i(h^1,1)_G}{\sum_{l=1}^N |\Omega_l|}. \tag{4.5.19}$$

Since $a^1 = \text{const}$ and $a^2 = 0$, the vector (u^1, u^2) satisfies (1.2.1), (1.2.2). The asymptotcs for u^1, u^2 follows from (2.5.1), while (4.5.17) can be obtained from (2.5.3). □

4.5.4 Radiation principle for the Maxwell operator

Denote by $\mathcal{L}_{\mathcal{M},-\delta}(0)$ the operator of boundary value problem (1.2.1), (1.2.2) for $k = 0$ with domain $H^l_{-\delta}(G)$. Let $\mathfrak{S}_{\mathcal{M}}$ be the space spanned by the function $\sum_{l=1}^{N} \chi_l(0,0,0,0,0,1)$. We consider the restriction $\mathbb{L}_{\mathcal{M}}$ of $\mathcal{L}_{\mathcal{M},-\delta}(0)$ to $\mathfrak{S}_{\mathcal{M}} \dot{+} H^l_\delta(G)$; the mapping

$$\mathbb{L}_{\mathcal{M}} : \mathfrak{S}_{\mathcal{M}} \dot{+} H^l_\delta(G) \to H^{l-1}_\delta(G) \times H^{l-1/2}_\delta(\partial G).$$

is continuous. The next assertion immediately follows from Theorem 4.5.5.

Theorem 4.5.6. *Let* $\mathcal{F} \in H^{l-1}_\delta(G)$ *and let the compatibility conditions* (2.1.3)-(2.1.5) *be fulfilled. Then there exists a unique solution* $(u^1, u^2) \in \mathfrak{S}_{\mathcal{M}} \dot{+} H^l_\delta(G)$ *of the equation*

$$\mathbb{L}_{\mathcal{M}}(u^1, u^2) = \{\mathcal{F}, 0\}.$$

The vector (u^1, u^2) *satisfies*

$$w^2 := u^2 - a^1 \sum_{l=1}^{N} \chi_l(0,0,1) \in H^l_\delta(G), \qquad w^1 := u^1 \in H^l_\delta(G),$$

where $a^1 = i(h^1, 1)_G / \sum_{l=1}^{N} |\Omega_l|$. *The solution* (u^1, u^2) *admits the estimate*

$$\|w^1; H^l_\delta(G)\| + \|w^2; H^l_\delta(G)\| + |a^1| \leq \mathrm{const}\|\mathcal{F}; H^{l-1}_\delta(G)\|.\square$$

Formula (4.5.16) shows that for the right-hand-side of (1.2.1) with $f^2 = 0$ and $h^1 = 0$, the constant a^1 equals zero. In other words, in the absence of magnetic monopolies, for any $k \in (-k_0, k_0)$, there exists a unique solution to the problem (1.2.1), (1.2.2), which decays exponentially at infinity.

5 METHOD FOR COMPUTING THE SCATTERING MATRIX IN A NEIGHBOURHOOD OF A THRESHOLD

In this Chapter, we consider a waveguide described by the Dirichlet problem for the Helmholtz equation

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= 0, \qquad x \in G, \\ u(x) &= 0, \qquad x \in \partial G. \end{aligned} \tag{5.0.20}$$

The waveguide scattering matrix $S(\mu)$ changes its size when spectral parameter μ crosses a threshold. To calculate $S(\mu)$ in a neighborhood of a threshold, we introduce an "augmented" scattering matrix $\mathcal{S}(\mu)$ that keeps its size near the threshold, where the matrix $\mathcal{S}(\mu)$ is analytic in μ. A minimizer of a quadratic functional $\mathcal{J}^R(\cdot, \mu)$ serves as an approximation to a row of the matrix $\mathcal{S}(\mu)$. To construct such a functional, we solve an auxiliary boundary value problem in the bounded domain obtained by cutting off, at a distance R, the waveguide outlets to infinity. As $R \to \infty$, the minimizer $a(R, \mu)$ at exponential rate tends to the corresponding row of $\mathcal{S}(\mu)$ uniformly, with respect to μ in a neighborhood of the threshold. Finally, we express the elements of the “ordinary”, scattering matrix $S(\mu)$ through those of the augmented matrix $\mathcal{S}(\mu)$.

Section 5.1 is devoted to constructing a stable basis of waves in a neighborhood of a threshold for the waveguide in the domain G. The continuous spectrum eigenfunctions and the scattering matrices $S(\mu)$ and $\mathcal{S}(\mu)$ are introduced in Section 5.2; here we also prove the analyticity of the matrices on the corresponding intervals of the continuous spectrum. We describe the connections between the matrices $S(\mu)$ and $\mathcal{S}(\mu)$ and calculate the one-sided limits of $S(\mu)$ at a threshold in Section 5.3. The last Section 5.4 contains the statement and justification of the method for approximating of the scattering matrices.

5.1 Augmented space of waves

5.1.1 The waves in a cylinder

In the cylinder $\Pi = \{(y,t) : y = (y_1,\ldots,y_n) \in \Omega, t \in \mathbb{R}\}$, we consider the problem

$$\begin{aligned} (-\Delta - \mu)u(y,t) &= 0, \ \ (y,t) \in \Pi, \\ u(y,t) &= 0 \ \ \ (y,t) \in \partial\Pi, \end{aligned} \tag{5.1.1}$$

where

$$\Delta = \Delta_y + \partial_t^2, \quad \Delta_y = \sum_{j=1}^{n} \partial_j^2, \quad \partial_j = \partial/\partial y_j.$$

Let us connect with problem (5.1.1) an operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda,\mu)$, by setting

$$\mathfrak{A}(\lambda,\mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \quad y \in \Omega; \quad v|_{\partial\Omega} = 0. \tag{5.1.2}$$

We also consider the problem

$$\begin{aligned} (-\Delta_y - \mu)v(y) &= 0, \quad y \in \Omega, \\ v(y) &= 0, \quad y \in \partial\Omega. \end{aligned} \tag{5.1.3}$$

The eigenvalues of problem (5.1.3) are called the thresholds of problem (5.1.1). The thresholds form a positive sequence $\tau_1 < \tau_2 < \ldots$, which strictly increases to infinity. Let us introduce the nondecreasing sequence $\{\mu_k\}_{k=1}^{\infty}$ of the eigenvalues of problem (5.1.3), counted according to their multiplicity (generally, the numbering of τ_l and that of μ_k are different; every μ_k coincides with one of the thresholds τ_l). We assume, that the corresponding eigenvectors φ_k are orthogonal and normalized by the condition

$$\int_{\Omega} \varphi_k(y)\overline{\varphi_k(y)}\, dy = 1. \tag{5.1.4}$$

The spectrum of the operator pencil (for every fixed $\mu \in \mathbb{R}$) consists of isolated eigenvalues on the axes of complex plane. For any μ the eigenvalues $\lambda_k^{\pm}$ of the pencil $\lambda \mapsto \mathfrak{A}(\lambda,\mu)$ are defined by $\lambda_k^{\pm} = \pm(\mu - \mu_k)^{1/2}$. If $\lambda_k^{\pm} \neq 0$, to the eigenvalues $\lambda_k^{\pm}$ there corresponds the same eigenvector φ_k, which is also an eigenvector of problem (5.1.3), corresponding to the eigenvalue μ_k. There is no generalized eigenvector in this case. If $\mu_{k-1} < \mu < \mu_k$, then $\lambda_k^{\pm}, \lambda_{k+1}^{\pm}, \ldots$ are imaginary and $\lambda_1^{\pm}, \ldots, \lambda_{k-1}^{\pm}$ are real. In the case of $\mu = \mu_k$, to the eigenvalue $0 = \lambda_k^{+} = \lambda_k^{-}$, there corresponds a Jordan chain φ^0, φ_k^1, where φ_k^0 is an eigenvector and φ_k^1 is a generalized eigenvector.

We fix a real $\mu \neq \mu_k$, $k = 1, 2, \ldots$, that is, the μ is not a threshold, and introduce the linear complex space spanned by the functions

$$(y,t) \mapsto \exp(i\lambda_k^{\pm} t)\varphi_k(y) \tag{5.1.5}$$

with real $\lambda_k^{\pm} = \pm(\mu - \mu_k)^{1/2}$; the functions (5.1.5) satisfy (5.1.1). We denote the space by $W(\mu)$ and will call it the space of waves. Its dimension is equal to the doubled number of μ_k (counted according to their multiplicities), such that $\mu_k < \mu$. The functions

$$u_k^{\pm}(y,t;\mu) = (2|\lambda_k^{\mp}|)^{-1/2} \exp(i\lambda_k^{\mp} t)\varphi_k(y) \tag{5.1.6}$$

form a basis in $W(\mu)$. We call $u_k^+(\cdot,\mu)$ a wave incoming from $+\infty$, and $u_k^-(\cdot,\mu)$ a wave outgoing to $+\infty$.

Assume now, that $\mu = \tau$ is a threshold and, consequently, μ is an eigenvalue of (5.1.3) with multiplicity $\varkappa \geq 1$. Then $\varkappa$ numbers μ_l satisfy $\mu_l = \tau$. For each l the functions $\exp(i\lambda_l^+ t)\varphi_l(y)$ and $\exp(i\lambda_l^- t)\varphi_l(y)$ coincide. Therefore the number of linearly independent functions of the form (5.1.5) for $\mu = \tau$ is $\varkappa$ less than the number of such functions for μ satisfying $\tau < \mu < \tau + \beta$ with small $\beta > 0$. However for a more general notion of the waves, the dimension of the space $W(\mu)$ is continuous from the right at the threshold. In such a case, the definition of incoming and outgoing waves is based on energy reasons as in the Sommerfeld and Mandelstamm principles.

For the definition we introduce the form

$$\begin{aligned} q_N(u,v) := {} & ((-\Delta - \mu)u, v)_{\Pi(N)} + (u, -\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ & -(u,(-\Delta-\mu)v)_{\Pi(N)} - (-\partial_\nu u, v)_{\partial\Pi(N)\cap\partial\Pi}, \end{aligned} \tag{5.1.7}$$

where $\Pi(N) = \{(y,t) \in \Pi : t < N\}$, the number $\mu \in \mathbb{R}$ is for the time being not a threshold, $u = \chi f$ and $v = \chi g$, while f and g are any of the functions (5.1.6), corresponding to real $\lambda_k^{\pm}(\mu)$ (possibly with distinct indices); χ stands for a smooth cut-off function, $\chi(t) = 0$ for $t < T - 1$ and $\chi(t) = 1$ for $t > T$ with $T < N$. Integrating by parts, we see that

$$iq_N(\chi u_k^{\pm}, \chi u_l^{\mp}) = 0 \quad \text{for all } k, l, \tag{5.1.8}$$

$$iq_N(\chi u_k^{\pm}, \chi u_l^{\pm}) = \mp\delta_{kl}, \tag{5.1.9}$$

so the result is independent of N and χ; in what follows we drop N but keep χ. We name the wave u_k^+ (u_k^-) incoming (outgoing) for $-(+)$ on the right in (5.1.9) and obtain the definition of incoming (outgoing) waves equivalent to the old definition.

We are going to consruct a basis in the (augmented) space of waves "stable at a threshold". Let $\mu \in \mathbb{R}$ be a regular value of the spectral paremeter of problem (5.1.3) and μ_m the eigenvalue with the greatest number satisfying $\mu_m < \mu$. We also assume that $\mu_l < \mu_{l+1} = \cdots = \mu_m$. Then the numbers $\tau' := \mu_l$, $\tau := \mu_{l+1} = \cdots = \mu_m$, and $\tau'' := \mu_{m+1}$ turn out to be three successive thresholds $\tau' < \tau < \tau''$ of problem (5.1.1) in the cylinder Π. (We discuss the general situation; the cases $l + 1 = m$, $m = 1$ (and so on) can be considered with evident simplifications.)

We set

$$w_k^{\pm}(y,t;\mu) = 2^{-1/2}\left(\frac{e^{it\sqrt{\mu-\mu_k}} + e^{-it\sqrt{\mu-\mu_k}}}{2} \mp \frac{e^{it\sqrt{\mu-\mu_k}} - e^{-it\sqrt{\mu-\mu_k}}}{2\sqrt{\mu-\mu_k}}\right)\varphi_k(y), \tag{5.1.10}$$

$$w_p^{\pm}(y,t;\mu) = u_p^{\pm}(y,t;\mu), \tag{5.1.11}$$

where $k = l+1,\dots,m$, $p = 1,\dots,l$, and $u_p^{\pm}$ are defined in (5.1.6).

Proposition 5.1.1. *The functions $\mu \mapsto w_k^{\pm}(y,t;\mu)$, $k = l+1,\dots,m$, admit the analytic continuation to the whole complex plane. These analytic functions smoothly depend on the parameters $y \in \bar{\Omega}$ and $t \in \mathbb{R}$ (i.e., any derivatives in y and t are analytic functions as well).*

The functions $\mu \mapsto w_p^{\pm}(y,t;\mu)$ are analytic on the complex plane with cut along the ray $\{\mu \in \mathbb{R} : -\infty < \mu \le \mu_p\}$, $p = 1,\dots,l$; they smoothly depend on y and t.

All the functions $w_k^{\pm}$, $k = 1,\dots,m$, are solutions to problem (5.1.1). *For every μ in $(\tau' < \mu < +\infty)$ the functions* (5.1.10), (5.1.11) *satisfy the orthogonality and normalization conditions*

$$iq(\chi w_r^{\pm}(\cdot\,;\mu), \chi w_s^{\mp}(\cdot\,;\mu)) = 0 \quad \text{for all } r,s = 1,\dots,m, \tag{5.1.12}$$

$$iq(\chi w_r^{\pm}(\cdot\,;\mu), \chi w_s^{\pm}(\cdot\,;\mu)) = \mp\delta_{rs}. \tag{5.1.13}$$

Proof. The first and second expression inside the parentheses in (5.1.10) can be decomposed in the series

$$\sum_{l\ge 0}\frac{(\mu_k-\mu)^l t^{2l}}{(2l)!} \quad \text{and} \quad it\sum_{l\ge 0}\frac{(\mu_k-\mu)^l t^{2l}}{(2l+1)!}, \tag{5.1.14}$$

which are absolutely and uniformly convergent on any compact $K \subset \{(\mu,t) : \mu \in \mathbb{C}, t \in \mathbb{R}\}$. This implies the analyticity properties of $w_k^{\pm}(y,t;\mu)$ for $k = l+1,\dots,m$. The corresponding assertions about $w_p^{\pm}(y,t;\mu)$ with $p = 1,\dots,l$ are evident.

It remains to verify the orthogonality and normalization conditions. We first assume that $\mu > \tau$ and consider, for instance, (5.1.13). If r and s are distinct, then the equalities in (5.1.13) follow from the orthogonality of φ_r and φ_s (as well as (5.1.8)and (5.1.9)). In the case $r = s \le l$, (5.1.9) contains the needed formula. Finally assume that $r = s > l$ and substitute the expressions (5.1.10) into $q(\chi w_r^{\pm}, \chi w_s^{\pm})$. Setting $\lambda := \sqrt{\mu-\tau}$, we obtain

$$\begin{aligned} iq(\chi w_s^{\pm}, \chi w_s^{\pm}) &= \lambda^{-2}((\lambda\pm 1)(\lambda\mp 1)iq^{+-} + (\lambda\mp 1)(\lambda\pm 1)iq^{-+} \\ &+ (\lambda\mp 1)^2 iq^{++} + (\lambda\pm 1)^2 iq^{--}), \end{aligned} \tag{5.1.15}$$

where, for example, $q^{+-} = 2^{-3}q(\chi e^{it\lambda}\varphi_s, \chi e^{-it\lambda}\varphi_s)$, and so on. Taking account of (5.1.6), (5.1.8), and (5.1.9), we arrive at (5.1.13).

We now consider the function

$$\begin{aligned} \mathbb{C} \ni \mu \mapsto q_N(u,v;\mu) := &((-\Delta-\mu)u,v)_{\Pi(N)} + (u, -\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ &-(u,(-\Delta-\bar{\mu})v)_{\Pi(N)} - (-\partial_\nu u, v)_{\partial\Pi(N)\cap\partial\Pi}, \end{aligned} \tag{5.1.16}$$

where $\Pi(N)$, N, and χ are the same as in (5.1.7), $u = \chi w_r^{\pm}(\cdot\,;\mu)$, and $v = \chi w_s^{\mp}(\cdot\,;\bar{\mu}))$. Since u and $\bar{v}$ are analytic, the function $\mu \mapsto q_N(u,v;\mu)$ is analytic as well. Therefore, the equalities (5.1.13) (with $r = s > l$) are valid for all $\mu \in \mathbb{C}$. □

From (5.1.10) it follows that $w_k^{\pm}(y,t;\tau) = 2^{-1/2}(1 \mp it)\varphi_k(y)$, $k = l+1,\dots,m$, and, in the case $\mu < \tau$, the amplitudes of the waves exponentially grow as $t \to \infty$. The space spanned by the waves (5.1.10) and (5.1.11) is called the augmented space of waves for $\tau' < \mu < \tau$ and denoted by $W_a(\mu)$. Let $W(\mu)$ denote the linear hull of the functions (5.1.10) and (5.1.11) for $\tau \le \mu < \tau''$ and the linear hull of the functions (5.1.11) for $\tau' < \mu < \tau$. The lineal $W(\mu)$ is called the space of waves. An element $w \in W_a(\mu)$ (or $W(\mu)$) is called a wave incoming from $+\infty$ (outgoing to $+\infty$), if $iq(\chi w, \chi w) < 0$ ($iq(\chi w, \chi w) > 0$).

The collection of waves $\{w^{\pm}\}_{k=1}^{m}$ defined by (5.1.10) and (5.1.11) is called a basis of waves stable in a neighborhood of the threshold τ. A basis of waves of the form (5.1.6) is by definition stable on (μ', μ'') if the interval $[\mu', \mu'']$ contains no thresholds.

5.1.2 Waves in the domain G

Let G be a domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1 \cup \dots \cup \Pi_+^{\mathcal{T}}$ of finitely many nonoverlapping semicylinders

$$\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r,\ t^r > 0\},$$

where (y^r, t^r) are local coordinates in Π_+^r and Ω^r is a bounded domain in $\mathbb{R}^n$.

We introduce the problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) = 0,\ \ x \in G,\\ u(x) = 0,\ \ x \in \partial G. \end{aligned} \tag{5.1.17}$$

With every Π_+^r we associate a problem of the form (5.1.1) in the cylinder $\Pi^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r \in \mathbb{R}\}$. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function, $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t > 1$. Each wave in Π^r we multiply by the function $t \mapsto \chi(t^r - t_0^r)$ with a certain $t_0^r > 0$ and then extend it by zero to the domain G. All functions (for all Π^r), obtained in such a way, we call waves in G. A number τ is called a threshold for problem (5.1.17) if the τ is a threshold at least for one of problems of the form (5.1.1) in Π^r, $r = 1,\dots,\mathcal{T}$. Let $\tau' < \tau < \tau''$ be three successive thresholds for problem (5.1.17); then the intervals (τ', τ) and (τ, τ'') are free from the thresholds.

For $\mu \in (\tau', \tau)$ we introduce the augmented space $\mathcal{W}_a(\mu, G)$ of waves in G as the union of the waves in G corresponding to the waves in $W_a(\mu)$ for Π^r, $r = 1,\dots,\mathcal{T}$; if a space $W_a(\mu)$ is not introduced on the interval $\tau' < \mu < \tau$ for a certain Π^r (which means that the τ is not a threshold for problem (5.1.1) in such a cylinder), then, from this cylinder, we include into the space $\mathcal{W}_a(\mu, G)$ the waves generated by the elements of the corresponding $W(\mu)$. By definition, for $\mu \in (\tau', \tau'')$ the space $\mathcal{W}(\mu, G)$ of waves in G is the union of the waves in G that correspond to the waves in $W(\mu)$ for all Π^r.

The bases $\{u_j^{\pm}(\cdot,\mu)\}$ and $\{w_j^{\pm}(\cdot,\mu)\}$ of waves in $\mathcal{W}(\mu,G)$ and $\mathcal{W}_a(\mu,G)$ comprise the waves obtained in G from the basis waves in Π^r, $r=1,\dots,\mathcal{T}$. The basis waves in the spaces $\mathcal{W}(\mu,G)$ and $\mathcal{W}_a(\mu,G)$ are subject to orthogonality and normalization conditions like (5.1.8) and (5.1.9) or (5.1.12) and (5.1.13) with the form q in a cylinder replaced by the form q_G in G:

$$\begin{aligned} q_G(u,v) := ((-\Delta-\mu)u,v)_G + (u,-\partial_\nu v)_{\partial G} \\ -(u,(-\Delta-\mu)v)_G - (-\partial_\nu u,v)_{\partial G}. \end{aligned} \tag{5.1.18}$$

An element w in $\mathcal{W}_a(\mu,G)$ (or in $\mathcal{W}(\mu,G)$) is called a wave incoming from ∞ (outgoing to ∞), if $iq_G(\chi w,\chi w)<0$ ($iq_G(\chi w,\chi w)>0$).

A basis of waves in G is called stable near a value ν of the spectral parameter if the basis consists of bases in the cylinders $\Pi^1,\dots,\Pi^{\mathcal{T}}$ stable near ν.

5.2 Continuous spectrum eigenfunctions. Scattering matrices

Let $\tau'<\tau<\tau''$ be three successive thresholds for problem (5.1.17). For simplicity, we assume that these three numbers are thresholds for a problem of the form (5.1.1) only in one of the cylinders $\Pi^1,\dots,\Pi^{\mathcal{T}}$, for instance in $\Pi^1=\Omega^1\times\mathbb{R}$. Moreover, we suppose that $\tau'=\mu_l$, $\tau=\mu_{l+1}=\dots=\mu_m$, and $\tau''=\mu_{m+1}$, where μ_k are eigenvalues of problem (5.1.3) in Ω^1. Thus for $\Pi=\Pi^1$ we deal with the situation considered in 5.1.1.

5.2.1 Intrinsic and expanded radiation principles

We consider the boundary value problem

$$\begin{aligned} -\Delta u(x)-\mu u(x)=f(x),\ \ x\in G, \\ u(x)=g(x),\ \ x\in\partial G, \end{aligned} \tag{5.2.1}$$

and recall two correct statements of the problem with radiation conditions at infinity: the intrinsic and expanded radiation principles. In the first principle, the intrinsic radiation conditions contain only outgoing waves in the space $\mathcal{W}(\mu,G)$. The second (expanded) principle includes the outgoing waves in the augmented space $\mathcal{W}_a(\mu,G)$. For the general elliptic problems self-adjoint with respect to the Green formula, the first statement was discussed by Nazarov et al. [25] and the second one was considered by Nazarov et al. [26] and Kamotskii et al. [19] (for various geometric situations). We will apply the intrinsic principle with spectral parameter outside a neighborhood of the thresholds. In vicinity of a threshold, we make use of the expanded principle employing the stable basis of waves in $\mathcal{W}_a(\mu,G)$ constructed in Section 5.1.

We first define needed function spaces. For integer $l\ge 0$ we denote by

$H^l(G)$ the Sobolev space with the norm

$$\|v; H^l(G)\| = \left(\sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^\alpha v(x)|^2 \, dx \right)^{1/2},$$

and let $H^{l-1/2}(\partial G)$ with $l \geq 1$ stand for the space of traces on ∂G of the functions in $H^l(G)$. Assume that ρ_γ is a smooth positive on $\overline{G}$ function given on Π_+^r by the equality $\rho_\gamma(y^r, t^r) = \exp(\gamma t^r)$ with $\gamma \in \mathbb{R}$. We also introduce the spaces $H_\gamma^l(G)$ and $H_\gamma^{l-1/2}(\partial G)$ with norms $\|u; H_\gamma^l(G)\| = \|\rho_\gamma u; H^l(G)\|$ and $\|v; H_\gamma^{l-1/2}(\partial G)\| = \|\rho_\gamma v; H^{l-1/2}(\partial G)\|$. The operator of problem (5.2.1) implements the continuous mapping

$$\mathcal{A}_\gamma(\mu) : H_\gamma^2(G) \to H_\gamma^0(G) \times H_\gamma^{3/2}(\partial G). \tag{5.2.2}$$

As is known, the operator (5.2.2) is Fredholm if and only if the line $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \gamma\}$ is free of the eigenvalues of the pencils $\lambda \mapsto \mathfrak{A}^r(\lambda, \mu)$, $r = 1, \dots, \mathcal{T}$, where $\mathfrak{A}^r$ is a pencil of the form (5.1.2) for the problem (5.1.1) in the cylinder Π^r. (An operator is called Fredholm, if its range is closed and the kernel and cokernel are finite dimensional.)

We now proceed to the intrinsic radiation principle. Assume that μ does not coincide within a threshold, $\mu \in (\tau', \tau'')$, and $\mu \neq \tau$. Let δ denote a small positive number, such that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq \delta\}$ contains only real eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$; we denote the number of such eigenvalues (counted with their multiplicities) by $2M = 2M(\mu)$. There exist collections of elements $\{Y_1^+(\cdot, \mu), \dots, Y_M^+(\cdot, \mu)\}$ and $\{Y_1^-(\cdot, \mu), \dots, Y_M^-(\cdot, \mu)\}$ in the kernel $\ker\mathcal{A}_{-\delta}(\mu)$ of $\mathcal{A}_{-\delta}(\mu)$, such that

$$\left(Y_j^+(\cdot, \mu) - u_j^+(\cdot, \mu) - \sum_{k=1}^{M} S_{jk}(\mu) u_k^-(\cdot, \mu) \right) \in H_\delta^2(G), \tag{5.2.3}$$

$$\left(Y_j^-(\cdot, \mu) - u_j^-(\cdot, \mu) - \sum_{k=1}^{M} T_{jk}(\mu) u_k^+(\cdot, \mu) \right) \in H_\delta^2(G), \tag{5.2.4}$$

where $S(\mu) = \|S_{jk}(\mu)\|$ is a unitary scattering matrix and $S(\mu)^{-1} = T(\mu) = \|T_{jk}(\mu)\|$. Every collection $\{Y_1^+(\cdot, \mu), \dots, Y_M^+(\cdot, \mu)\}$ and $\{Y_1^-(\cdot, \mu), \dots, Y_M^-(\cdot, \mu)\}$ is a basis modulo $\ker\mathcal{A}_\delta(\mu)$ in $\ker\mathcal{A}_{-\delta}(\mu)$. This means, that any $v \in \ker\mathcal{A}_{-\delta}(\mu)$ is a linear combination of the functions $Y_1^+(\cdot, \mu), \dots, Y_M^+(\cdot, \mu)$ up to a term in $\ker\mathcal{A}_\delta(\mu)$. The same is true also for $Y_1^-(\cdot, \mu), \dots, Y_M^-(\cdot, \mu)$. If μ is not an eigenvalue of operator (5.2.2), that is, $\ker\mathcal{A}_\delta(\mu) = 0$, every collection $\{Y_j^+\}$ and $\{Y_j^-\}$ is a basis for the kernel $\ker\mathcal{A}_{-\delta}(\mu)$ in the usual sense.

The elements $Y(\cdot, \mu)$ in $\ker\mathcal{A}_{-\delta}(\mu) \setminus \ker\mathcal{A}_\delta(\mu)$ are called the continuous spectrum eigenfunctions of problem (5.1.17) corresponding to μ.

Denote by $\mathfrak{N}$ the linear hull $\mathfrak{L}(u_1^-, \dots, u_M^-)$. We define the norm of $u = \sum c_j u_j^- + v \in \mathfrak{N} \dotplus H_\delta^2(G)$ with $c_j \in \mathbb{C}$ and $v \in H_\delta^2(G)$ by

$$\|u\| = \sum |c_j| + \|v; H_\delta^2(G)\|.$$

Let $\mathbf{A}(\mu)$ be the restriction of the operator $\mathcal{A}_{-\delta}(\mu)$ to the space $\mathfrak{N} \dotplus H^2_\delta(G)$. The map

$$\mathbf{A}(\mu) : \mathfrak{N} \dotplus H^2_\delta(G) \to H^0_\delta(G) \times H^{3/2}_\delta(\partial G) =: \mathcal{H}_\delta(G) \tag{5.2.5}$$

is continuous. The next theorem provides the statement of problem (5.2.1) with intrinsic radiation conditions at infinity (the numbers μ and δ are supposed to satisfy the requirements given just before (5.2.3)).

Theorem 5.2.1. *Let $z_1, \dots, z_d$ be a basis in the space* $\ker\mathcal{A}_\delta(\mu)$, *$\{f, g\} \in \mathcal{H}_\delta(G)$ and $(f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G} = 0$, $j = 1, \dots, d$. Then:*

1. There exists a solution $u \in \mathfrak{N} \dotplus H^2_\delta(G)$ of the equation $\mathbf{A}(\mu)u = \{f, g\}$ determined up to an arbitrary term in $\ker\mathcal{A}_\delta(\mu)$.

2. The inclusion

$$v \equiv u - c_1 u_1^- - \cdots - c_M u_M^- \in H^2_\delta(G) \tag{5.2.6}$$

holds with $c_j = i(f, Y_j^-)_G + i(g, -\partial_\nu Y_j^-)_{\partial G}$.

3. The inequality

$$\|v; H^2_\delta(G)\| + |c_1| + \cdots + |c_M| \leq \text{const}\left(\|\{f, g\}; \mathcal{H}_\delta(G)\| + \|\rho_\delta v; L_2(G)\|\right). \tag{5.2.7}$$

holds with v and $c_1, \dots, c_M$ in (5.2.6). *A solution u_0 that is subject to the additional conditions $(u_0, z_j)_G = 0$ for $j = 1, \dots, d$ is unique and satisfies* (5.2.7) *with right-hand-side changed for* $\text{const}\|\{f, g\}; \mathcal{H}_\delta(G)\|$.

4. If $\{f, g\} \in \mathcal{H}_\delta(G) \cap \mathcal{H}_{\delta'}(G)$ and the strip $\{\lambda \in \mathbb{C} : \min\{\delta, \delta'\} \leq \operatorname{Im}\lambda \leq \max\{\delta, \delta'\}\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, then the solutions $u \in \mathfrak{N} \dotplus H^2_\delta(G)$ and $u' \in \mathfrak{N} \dotplus H^2_{\delta'}(G)$ coincide, while the choice between δ and δ' in essence effects only the constant in (5.2.7).

Remark 5.2.2. In Theorem 5.2.1, one can take the numbers δ and "const" in (5.2.7) invariant for all μ in $[\mu', \mu''] \subset (\tau, \tau'')$ (in $[\mu', \mu''] \subset (\tau', \tau)$). If μ'' approaches τ'' (τ), the δ must tend to zero: an admissible interval for δ has to be narrowed because the imaginary eigenvalues of the pencils move closer to the real axis; the constant in (5.2.7) increases. From the proof of Theorem 5.2.1 in [25] one can see that the constant also increases when μ' approaches τ (or τ').

We now turn to the expanded radiation principle in a neihgborhood of τ. To this end, for problem (5.1.17) we construct a basis of waves stable at the threshold τ. We make up such a basis from the waves generated by the functions (5.1.10), (5.1.11) and from the waves corresponding to the real eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 2, \dots, \mathcal{T}$. According to our assumption (at the beginning of Section 5.2), the interval $[\tau', \tau'']$ contains no threshold for problems of the form (5.1.1) in the cylinders $\Pi^2, \dots, \Pi^{\mathcal{T}}$. Therefore the number of real eigenvalues for every of the pencils $\mathbb{R} \ni \lambda \to \mathfrak{A}^r(\lambda, \mu)$, $r = 2, \dots, \mathcal{T}$, remains invariant for $\mu \in [\tau', \tau'']$. Thus when passing from the cylinder Π^1 to the domain G, the dimension of wave space increases by the same number for all $\mu \in (\tau', \tau'')$. We set $2L = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau', \tau)$ and $2M = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$; then $M - L = m - l$, where m and l are the same as in (5.1.10), (5.1.11), and $\dim \mathcal{W}_a(\mu, G) = 2M$ for $\mu \in (\tau', \tau)$.

We choose the number γ for the operators $\mathcal{A}_{\pm\gamma}(\mu)$ to be proper for all μ in a neighborhood of the threshold $\tau = \mu_m$. Let us explain such a choice. We have $\lambda_k^{\pm}(\mu) = \pm(\mu - \mu_k)^{1/2}$, $\mu_{l+1} = \ldots, \mu_m$, so $\lambda_k^{\pm}(\tau) = 0$ with $k = l+1, \ldots, m$. The interval of the imaginary axis with ends $-i(\mu_{m+1} - \mu_m)^{1/2}$, $i(\mu_{m+1} - \mu_m)^{1/2}$ punctured at the coordinate origin is free of the spectra of the pencils $\mathfrak{A}^q(\cdot, \mu_m)$, $q = 1, \ldots, \mathcal{T}$. If μ a little moves along $\mathbb{R}$, the eigenvalues of the pencils $\mathfrak{A}^q(\cdot, \mu)$ slightly shift along the coordinate axes. Therefore, for a small $\alpha > 0$ there exists $\beta > 0$, such that for $\mu \in (\mu_m - \beta, \mu_m + \beta)$ the intervals $iI_{\pm\alpha} := \pm i(\alpha, (\mu_{m+1} - \mu_m)^{1/2} - \alpha)$ are free of the spectra of the pencils $\mathfrak{A}^q(\cdot, \mu)$. So the lines $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \pm\gamma\}$ with $\gamma \in I_\alpha$ do not intersect the spectra of $\mathfrak{A}^q(\cdot, \mu)$, while the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \leq \gamma\}$ contains only the real eigenvalues of the pencils and the numbers $\lambda_k^{\pm}(\mu) = \pm(\mu - \mu_k)^{1/2} = \pm(\mu - \mu_m)^{1/2}$ in (5.1.10), $k = l+1, \ldots, m$.

Let $\mu \in (\tau - \beta, \tau + \beta)$, $\gamma \in I_\alpha$, and let $\{w_k^{\pm}(\cdot, \mu)\}$ be the stable basis of waves in G described in 5.1.1 and 5.1.2. In the kernel $\ker\mathcal{A}_{-\gamma}(\mu)$ of $\mathcal{A}_{-\gamma}(\mu)$, there exist collections of elements $\{\mathcal{Y}_1^+(\cdot, \mu), \ldots, \mathcal{Y}_M^+(\cdot, \mu)\}$ and $\{\mathcal{Y}_1^-(\cdot, \mu), \ldots, \mathcal{Y}_M^-(\cdot, \mu)\}$, such that

$$\left(\mathcal{Y}_j^+(\cdot, \mu) - w_j^+(\cdot, \mu) - \sum_{k=1}^{M} \mathcal{S}_{jk}(\mu) w_k^-(\cdot, \mu)\right) \in H_\gamma^2(G), \tag{5.2.8}$$

$$\left(\mathcal{Y}_j^-(\cdot, \mu) - w_j^-(\cdot, \mu) - \sum_{k=1}^{M} \mathcal{T}_{jk}(\mu) w_k^+(\cdot, \mu)\right) \in H_\gamma^2(G), \tag{5.2.9}$$

where $\mathcal{S}(\mu) = \|\mathcal{S}_{jk}(\mu)\|$ is the unitary matrix and $\mathcal{S}(\mu)^{-1} = \mathcal{T}(\mu) = \|\mathcal{T}_{jk}(\mu)\|$. Every collection $\{\mathcal{Y}_1^+(\cdot, \mu), \ldots, \mathcal{Y}_M^+(\cdot, \mu)\}$ and $\{\mathcal{Y}_1^-(\cdot, \mu), \ldots, \mathcal{Y}_M^-(\cdot, \mu)\}$ is a basis (modulo $\ker\mathcal{A}_\gamma(\mu)$) in $\ker\mathcal{A}_{-\gamma}(\mu)$.

The elements $\mathcal{Y}(\cdot, \mu)$ in $\ker\mathcal{A}_{-\gamma}(\mu) \setminus \ker\mathcal{A}_\gamma(\mu)$ are called the continuous spectrum eigenfunctions of problem (5.1.17) corresponding to the number μ. The matrix $\mathcal{S}(\mu)$ (with $\mu \in (\tau - \beta, \tau)$) is called the augmented scattering matrix.

Let $\mathfrak{K}$ denote the linear hull $\mathfrak{L}(w_1^-, \ldots, w_M^-)$. We define a norm of $w = \sum c_j w_j^- + v \in \mathfrak{K} \dotplus H_\gamma^2(G)$, where $c_j \in \mathbb{C}$ and $v \in H_\gamma^2(G)$, by the equality

$$\|w\| = \sum |c_j| + \|v; H_\gamma^2(G)\|.$$

Let $\mathbf{A}(\mu)$ be the restriction of $\mathcal{A}_{-\gamma}(\mu)$ to the space $\mathfrak{K} \dotplus H_\gamma^2(G)$; then the mapping

$$\mathbf{A}(\mu) : \mathfrak{K} \dotplus H_\gamma^2(G) \to H_\gamma^0(G) \times H_\gamma^{3/2}(\partial G) =: \mathcal{H}_\gamma(G). \tag{5.2.10}$$

is continuous.

Theorem 5.2.3. *Let $\mu \in (\tau - \beta, \tau + \beta)$, $\gamma \in I_\alpha$, and let $\{w_k^{\pm}(\cdot, \mu)\}$ be the aforementioned stable basis of waves in G. Assume $z_1, \ldots, z_d$ to be a basis in the space* $\ker\mathcal{A}_\gamma(\mu)$, *$\{f, g\} \in \mathcal{H}_\gamma(G)$ and $(f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G} = 0$, $j = 1, \ldots, d$. Then:*

1). *There exists a solution $w \in \mathfrak{K} \dotplus H_\gamma^2(G)$ to the equation $\mathbf{A}(\mu)w = \{f, g\}$ determined up to an arbitrary term in the lineal $\mathcal{L}(z_1, \ldots, z_d)$.*

2). *The inclusion*

$$v \equiv w - c_1 w_1^- - \cdots - c_M w_M^- \in H^2_\gamma(G), \tag{5.2.11}$$

holds with $c_j = i(f, \mathcal{Y}_j^-)_G + i(g, -\partial_\nu \mathcal{Y}_j^-)_{\partial G}$.

3). *Such a solution* w *satisfies the inequality*

$$\|v; H^2_\gamma(G)\| + |c_1| + \cdots + |c_M| \leq \text{const}\,(\|\{f,g\}; \mathcal{H}_\gamma(G)\| + \|\rho_\gamma v; L_2(G)\|)\,. \tag{5.2.12}$$

A solution w_0*, subject to the conditions* $(w_0, z_j)_G = 0$ *for* $j = 1, \dots, d$*, is unique and the estimate* (5.2.12) *holds with the right-hand-side changed for* const$\|\{f,g\}; \mathcal{H}_\gamma(G)\|$.

4). *If* $\{f,g\} \in \mathcal{H}_\gamma(G) \cap \mathcal{H}_{\gamma'}(G)$ *and the strip* $\{\lambda \in \mathbb{C} : \min\{\gamma,\gamma'\} \leq \text{Im}\lambda \leq \max\{\gamma,\gamma'\}\}$ *contains no eigenvalues of the pencils* $\mathfrak{A}^r(\cdot,\mu)$*,* $r = 1, \dots, \mathcal{T}$*, the solutions* $w(\cdot,\mu) \in \mathfrak{K} \dotplus H^2_\gamma(G)$ *and* $w'(\cdot,\mu) \in \mathfrak{K} \dotplus H^2_{\gamma'}(G)$ *of the equation* $\mathbf{A}(\mu)w = \{f,g\}$ *coincide, while the choice between* γ *and* γ'*, in essence, effects only the constant in* (5.2.12).

We would like to extend relations of the form (5.2.8) and (5.2.9) to the interval (τ', τ'') with analytic functions $\mu \mapsto \mathcal{Y}_j^\pm(\mu)$. Unlike the situation in Remark 5.2.2, it is not possible, generally speaking, to extend (5.2.8) and (5.2.9) to any interval $[\mu', \mu''] \subset (\tau', \tau'')$ with the same index γ. However, to that purpose, one can use a finite collection of indices for various parts of $[\mu', \mu'']$. The following Lemma explains how to compile such a collection.

Lemma 5.2.4. *For any interval* $[\mu', \mu''] \subset (\tau', \tau'')$ *there exists a finite covering* $\{U_p\}_{p=0}^N$ *consisting of open intervals and a collection of indices* $\{\gamma^p\}_{p=0}^N$ *subject to the following conditions (with a certain nonnegative number N):*

1) $\mu' \in U_0$*,* $\mu'' \in U_N$*;* $U_0 \cap U_p = \varnothing$*,* $p = 2, \dots, N$*;* $U_N \cap U_p = \varnothing$*,* $p = 0, \dots, N-2$*; moreover,* U_p *only overlaps* U_{p-1} *and* U_{p+1}*,* $1 \leq p \leq N-1$.

2) *The line* $\{\lambda \in \mathbb{C} : \text{Im}\lambda = \gamma^p\}$ *is free from the spectra of the pencils* $\mathfrak{A}^r(\cdot,\mu)$*,* $r = 1, \dots, \mathcal{T}$*, for all* $\mu \in U_p \cap [\mu', \mu'']$ *and* $p = 0, \dots, N$.

3) *The strip* $\{\lambda \in \mathbb{C} : \gamma^p \leq \text{Im}\lambda \leq \gamma^{p+1}\}$ *is free from the spectra of the pencils* $\mathfrak{A}^r(\cdot,\mu)$*,* $r = 1, \dots, \mathcal{T}$ *for all* $\mu \in U_p \cap U_{p+1}$ *and* $p = 0, \dots, N-1$.

4) *The inequality* $|\text{Im}(\mu - \tau)^{1/2}| < \gamma^p$ *holds for* $\mu \in U_p \cap [\mu', \mu'']$ *(recall that* $\pm(\mu - \tau)^{1/2}$ *are eigenvalues of* $\mathfrak{A}^1(\cdot,\mu)$*,* $\tau = \mu_{l+1} = \cdots = \mu_m$*); there are no other eigenvalues of the pencils* $\mathfrak{A}^r(\cdot,\mu)$*,* $r = 1, \dots, \mathcal{T}$*, in the strip* $\{\lambda \in \mathbb{C} : |\text{Im}\lambda| \leq \gamma^p\}$ *except the real ones,* $p = 0, \dots, N$.

Proof. Let us consider an interval $[\mu', \mu'']$ and let $\tau \in (\mu', \mu'')$. Just before formulas (5.2.8) and (5.2.9), we have defined the interval $(\tau - \beta, \tau + \beta)$ that can be taken as an element of the desired covering. It was earlier shown that as an index γ for such an element one can choose any number in $I_\alpha = (\alpha, (\mu_{m+1} - \mu_m)^{1/2} - \alpha)$ with small positive α; the number β depends on α.

Let us take some $\nu \in (\tau, \tau + \beta)$. The eigenvalue $\lambda_m(\mu) = (\mu - \mu_m)^{1/2}$ of the pencil $\mathfrak{A}^1(\cdot,\mu)$ is real for $\mu > \nu$, the eigenvalue $\lambda_{m+1}(\mu) = i(\mu_{m+1} - \mu)^{1/2}$ of the pencil tends to zero when μ increases from ν to $\tau'' = \mu_{m+1}$, and the interval $\{z \in \mathbb{C} : z = it, 0 < t < (\mu_{m+1} - \mu'')^{1/2}\}$ of the imaginary axis remains free from the spectra of the pencils $\mathfrak{A}^r(\cdot,\mu)$, $\mu' \leq \mu \leq \mu''$, $r = 1, \dots, \mathcal{T}$. Therefore

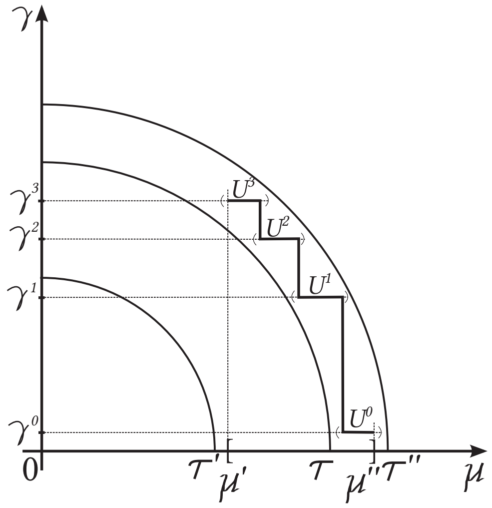


FIGURE 3 A collection of γ^p and corresponding U_p from Lemma 5.2.4. Here the curves depict the functions $\gamma = \operatorname{Im}\lambda_k(\mu) = \operatorname{Im}(\mu - \mu_k)^{1/2}$ with $k = l, m, m+1$.

the interval $(\nu, \tilde{\nu})$ with $\mu'' < \tilde{\nu} < \tau''$ can serve as an element of the covering and any number $\gamma \in (0, (\mu_{m+1} - \mu'')^{1/2})$ can be an index for the element. Finally we choose the elements U_p to the left of the threshold τ so that the graphs of the functions $U_p \ni \mu \mapsto \gamma^p = \text{const}$ are located between the graphs of the functions $(\tau', \tau) \ni \mu \mapsto \operatorname{Im}\lambda_k(\mu) = (\mu_k - \mu)^{1/2}$, $k = m, m+1$, and the indices form a decreasing sequence $\gamma^0 > \gamma^1 > \dots$. □

5.2.2 Analyticity of scattering matrices with respect to spectral parameter

Let us consider the bases $\{\mathcal{Y}_j^+\}$ and $\{\mathcal{Y}_j^-\}$ for the space of continuous spectrum eigenfunctions (CSE) defined near the threshold τ (see (5.2.8) and (5.2.9)). We first show that the functions $\mu \mapsto \mathcal{Y}_j^\pm(\cdot, \mu)$ admit the analytic extension to the interval (τ', τ''). In what follows by the analyticity of a function on an interval, we mean the possibility of analytic continuation of the function in a complex neighborhood of every point in the interval. Then we prove the analyticity of the scattering matrix $\mu \mapsto \mathcal{S}(\mu)$ on (τ', τ''). The analyticity does not exclude the existence of eigenvalues of problem (5.1.17) embedded into the continuous spectrum. However, the analyticity eliminates the arbitrariness in the choice of CSE. Moreover, we establish the analyticity of the elements $\mu \mapsto Y_j^\pm(\cdot, \mu)$ in (5.2.3) and (5.2.4) as well as the analyticity of the corresponding scattering matrix $\mu \mapsto S(\mu)$ on (τ', τ) and (τ, τ'').

In a neighborhood of any point of the interval (τ', τ''), one can define an operator $\mathcal{A}_\gamma(\mu)$, which is needed for relations like (5.2.8) and (5.2.9). The index γ has been provided by Lemma 5.2.4: the same number γ^p can serve for all $\mu \in U_p$. Therefore, for $\mu \in U_p$ there exist the families $\{\mathcal{Y}_j^\pm(\cdot, \mu)\} \subset \ker \mathcal{A}_{-\gamma^p}(\mu)$ satisfying relations like (5.2.8) and (5.2.9) with unitary matrix $\mathcal{S}(\mu)$, so Theorem 5.2.3 holds with $\mu \in U_p$. Thus, it suffices to prove the analyticity of the "local families" $\{\mathcal{Y}_j^{\pm}(\cdot,\mu)\}$ and that of the matrix $\mathcal{S}(\mu)$ on U_p and to verify the compatibility of such families on the intersections of neighborhoods.

We first obtain a representation of the operator $\mathbf{A}(\mu)^{-1}$, where $\mathbf{A}(\mu)$ is operator (5.2.5) or (5.2.10), in a neighborhood of an eigenvalue of problem (5.1.17). To this end we recall some facts in the theory of holomorphic operator-valued functions (e. g., see [12]). Let $\mathcal{D}$ be a domain in a complex plane, B_1 and B_2 Banach spaces, and $\mathbb{A}$ a holomorphic operator-valued function $\mathcal{D} \ni \mu \mapsto \mathbb{A}(\mu) : B_1 \to B_2$. The spectrum of the function $\mathbb{A}(\cdot)$ is the set of points $\mu \in \mathcal{D}$, such that $\mathbb{A}(\mu)$ is a noninvertible operator. A number μ_0 is called an eigenvalue of $\mathbb{A}$ if there exists a nonzero vector $\varphi_0 \in B_1$, such that $\mathbb{A}(\mu_0)\varphi_0 = 0$. Then φ_0 is called an eigenvector. Let μ_0 and φ_0 be an eigenvalue and an eigenvector. Elements $\varphi_1, \dots, \varphi_{m-1}$ are called generalized eigenvectors, if

$$\sum_{q=0}^{n} \frac{1}{q!}(\partial_\mu^q \mathbb{A})(\mu_0)\varphi_{n-q} = 0,$$

where $n = 1, \dots, m$. A holomorphic function $\mathbb{A}$ is said to be Fredholm, if the operator $\mathbb{A}(\mu) : B_1 \to B_2$ is Fredholm for all $\mu \in \mathcal{D}$ and is invertible at least for one μ. The spectrum of a Fredholm function $\mathbb{A}$ consists of isolated eigenvalues of finite algebraic multiplicity. The holomorphic function $\mathbb{A}^*$ adjoint to $\mathbb{A}$ is defined on the set $\{\mu : \bar{\mu} \in \mathcal{D}\}$ by the equality $\mathbb{A}^*(\mu) = (\mathbb{A}(\bar{\mu}))^* : B_1^* \to B_2^*$. If one of the functions $\mathbb{A}$ and $\mathbb{A}^*$ is Fredholm, then the other one is also Fredholm. A number μ_0 is an eigenvalue of $\mathbb{A}$ if and only if $\bar{\mu}_0$ is an eigenvalue of $\mathbb{A}^*$; the algebraic and geometric multipicities of $\bar{\mu}_0$ coincide with those of μ_0.

Let us consider the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$ in (5.2.5) or (5.2.10) on an interval $[\mu', \mu'']$ that belongs to one of the intervals (τ', τ) or (τ, τ''). Taking account of Remark 5.2.2, we choose the same index δ in (5.2.5) and in Theorem 5.2.1 for all $\mu \in [\mu', \mu'']$. When considering the function $\mu \mapsto \mathbf{A}(\mu)$ in (5.2.10) on an interval $[\mu', \mu''] \subset (\tau', \tau'')$, we suppose the interval to be so small that Lemma 5.2.4 enables us to take the same γ in (5.2.10) and in Theorem 5.2.3 for all $\mu \in [\mu', \mu'']$. According to Proposition 5.1.1, the waves in the definitions of operators (5.2.5) and (5.2.10) are holomorphic in a complex neighborhood of the corresponding interval $[\mu', \mu'']$. Therefore, the functions $\mu \mapsto \mathbf{A}(\mu)$ in Theorems 5.2.1 and 5.2.3 are holomorphic in the same neighborhood.

Proposition 5.2.5. 1). *Let $\mu \mapsto \mathbf{A}(\mu)$ be the function in Theorem* 5.2.3, *μ_0 an eigenvalue of operator* (5.2.2), *and $(z_1, \dots, z_d)$ a basis for* $\ker \mathcal{A}_\gamma(\mu_0)$. *Then in a punctured neighborhood of μ_0 there holds the representation*

$$\mathbf{A}^{-1}(\mu)\{f,g\} = (\mu - \mu_0)^{-1}\mathbf{P}\{f,g\} + \mathbf{R}(\mu)\{f,g\}, \tag{5.2.13}$$

where $\{f,g\} \in \mathcal{H}_\gamma(G)$,

$$\mathbf{P}\{f,g\} = -\sum_{j=1}^{d} \left((f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G}\right) z_j, \tag{5.2.14}$$

and the function $\mathbf{R}(\mu) : \mathcal{H}_\gamma(G) \to \mathfrak{K} \dotplus H^2_\gamma(G)$ *is holomorphic in a neighborhood of* μ_0.

2). *Let* $\mu \mapsto \mathbf{A}(\mu)$ *be the operator-valued function in Theorem* 5.2.1, μ_0 *an eigenvalue of operator* (5.2.2) *in* (τ', τ) *or* (τ, τ''), *and* $(z_1, \dots, z_d)$ *a basis for* $\ker\mathcal{A}_\delta(\mu_0)$. *Then in a punctured neighborhood of* μ_0 *there holds representation* (5.2.13), *where* $\mathbf{P}\{f, g\}$ *is defined by* (5.2.14) *and the function* $\mathbf{R}(\mu) : \mathcal{H}_\delta(G) \to \mathfrak{N} \dotplus H^2_\delta(G)$ *is holomorphic in a neighborhood of* μ_0.

Proof. 1). By Theorem 5.2.3, 1), the operator $\mathbf{A}(\mu)$ is Fredholm at any $\mu \in [\mu'_m, \mu''_m]$. We can consider that $\mathbf{A}(\mu)$ is Fredholm in a neighborhood U (the Fredholm property is stable with respect to perturbations that are small in the operator norm). Moreover, the operator $\mathbf{A}(\mu)$ is invertible for all $\mu \in [\mu'_m, \mu''_m]$ except the eigenvalues of operator (3.2), which are real and isolated. Hence the function $\mu \mapsto \mathbf{A}(\mu)$ is Fredholm in a neighborhood of μ_0 in the complex plane. From Theorem 5.2.3, 4), it follows that the eigenspaces of operators (3.10) and (3.2) coincide, that is, $\ker\mathbf{A}(\mu_0) = \ker\mathcal{A}_\gamma(\mu_0) \subset H^2_\gamma(G)$. It is easy to verify that the operator-valued function $\mathbf{A}$ has no generalized eigenvectors at μ_0. Then the Keldysh theorem on the resolvent of holomorphic operator-valued fuction (see [12]) provides the equality

$$\mathbf{A}^{-1}(\mu)\{f, g\} = (\mu - \mu_0)^{-1}\mathbf{T}\{f, g\} + \mathbf{R}(\mu)\{f, g\}, \qquad (5.2.15)$$

where $\mathbf{T}\{f, g\} = \sum_{j=1}^d \langle\{f, g\}, \{\psi_j, \chi_j\}\rangle z_j$, the duaity $\langle\cdot, \cdot\rangle$ on the pair $\mathcal{H}_\gamma(G)$, $\mathcal{H}_\gamma(G)^*$ is defined by $\langle\{f, g\}, \{\psi, \chi\}\rangle = (f, \psi)_G + (g, \chi)_{\partial G}$, and $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_{\partial G}$ are the extensions of the inner products on $L_2(G)$ and $L_2(\partial G)$ to the pairs $H^0_\gamma(G)$, $H^0_\gamma(G)^*$ and $H^{3/2}_\gamma(\partial G)$, $H^{3/2}_\gamma(\partial G)^*$. The elements $\{\psi_j, \chi_j\} \in \ker\mathbf{A}(\mu_0)^* \subset W(G; \gamma)^*$ are subject to the orthogonality and normalization conditions

$$\langle(\partial_\mu\mathbf{A})(\mu_0)z_j, \{\psi_k, \chi_k\}\rangle = \delta_{jk}, \quad j, k = 1, \dots, d. \qquad (5.2.16)$$

Furthermore, $(\partial_\mu\mathbf{A})(\mu_0)z_j = \{-z_j, 0\} \in W(G; \gamma)$. The elements $\{\psi_k, \chi_k\}$ can be interpreted in terms of the Green formula and, in view of (5.2.16), can be rewritten in the form $\{\psi_k, \chi_k\} = \{-z_k, \partial_\nu z_k\}$ (e.g., see [25]). Now $\mathbf{T}\{f, g\}$ coincides with $\mathbf{P}\{f, g\}$ in (5.2.14) and (5.2.15) takes the form of (5.2.13).

2). One can repeat with evident modifications the argument in 1). □

We are now ready to discuss the analyticity of bases in the space of the continuous spectrum eigenfunctions. For instance, we proceed to the basis $\{\mathcal{Y}^+_j\}$ in (5.2.8). From the definition of the wave w^+_j in G (see 5.1.2), it follows that the function $G \ni x \mapsto w^+_j(x, \mu)$ is supported by one of the cylindrical ends of G,

$$-\Delta w^+_j(x, \mu) - \mu w^+_j(x, \mu) = f_j(x, \mu), \quad x \in G,$$
$$w^+_j(x, \mu) = 0, \quad x \in \partial G,$$

and the support of the function $x \mapsto f_j(x, \mu)$ is compact. Let us consider the equation

$$\mathbf{A}(\mu)w(\cdot, \mu) = \{f_j(\cdot, \mu), 0\} \qquad (5.2.17)$$

on an interval $[\mu', \mu''] \subset (\tau', \tau'')$. We first assume that the interval $[\mu', \mu'']$ is free of the eigenvalues of the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$. In view of Theorem 5.2.3, for all $\mu \in [\mu', \mu'']$ there exists a unique solution $w = v + c_1 w_1^- + \cdots + c_M w_M^-$ to equation (5.2.17),

$$w(\cdot, \mu) = \{c_1(\mu), \ldots, c_M(\mu), v(\cdot, \mu)\} \in \mathfrak{K} \dotplus H^2_\gamma(G). \qquad (5.2.18)$$

Since the functions $\mu \mapsto \mathbf{A}(\mu)^{-1}$ and $\mu \mapsto f_j(\cdot, \mu)$ are holomorphic in a complex neighborhood of the interval $[\mu', \mu'']$, the components of the vector-valued function $\mu \mapsto w(\cdot, \mu)$ are holomorphic as well. Therefore, the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\cdot, \mu)$ in the same neighborhood follows from the equality

$$\mathcal{Y}_j^+ = w_j^+ - w. \qquad (5.2.19)$$

Assume now that the interval $[\mu', \mu'']$ contains an eigenvalue μ_0 of the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$. We find the residue $\mathbf{P}\{f, g\}$ in (5.2.13) for $\{f, g\} = \{f_j, 0\}$ in the right-hand-side of (5.2.17). For $z \in \ker \mathcal{A}_\gamma(\mu_0)$, we have

$$(f, z)_G + (g, -\partial_\nu z)_{\partial G} = (f_j, z)_G = (-\Delta w_j^+ - \mu w_j^+, z)_G = (w_j^+, -\Delta z - \mu z)_G = 0.$$

Hence $\mathbf{P}\{f_j, 0\} = 0$ and, by virtue of (5.2.13),

$$w(\cdot, \mu) = \mathbf{A}(\mu)^{-1}\{f_j, 0\} = \mathbf{R}(\mu)\{f_j, 0\},$$

which means that the function $\mu \mapsto w(\cdot, \mu)$ is analytic in a neighborhood of μ_0. This implies the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\cdot, \mu)$.

The analyticity of the functions $\mu \mapsto \mathcal{Y}_j^-(\cdot, \mu)$ can be proved in the same way. When verifying the analyticity of functions of the form $\mu \mapsto Y_j^+(\cdot, \mu)$ and $\mu \mapsto Y_j^-(\cdot, \mu)$ in (5.2.3) and (5.2.4) in a complex neighborhood of the interval $[\mu', \mu''] \subset (\tau', \tau)$ or $[\mu', \mu''] \subset (\tau, \tau'')$, one has to make only evident modification of the above argument.

Lemma 5.2.4 and Theorem 5.2.3, 4) enable us to extend formulas (5.2.8) and (5.2.9) to the whole interval (τ', τ'') for the analytic families $\mu \mapsto \mathcal{Y}_j^\pm(\cdot, \mu)$; however, one index γ has to be replaced by a collection of indices. Nontheless, in a neighdorhood of any given point $\mu \in (\tau', \tau'')$, one can do with one index γ. Remark 5.2.2 and Theorem 5.2.1, 4) allow to extend (5.2.3) and (5.2.4) to the intervals (τ', τ) and (τ, τ'') for the analytic families $\mu \mapsto Y_j^\pm(\cdot, \mu)$.

Theorem 5.2.6. *Let τ' and τ'' be thresholds of problem* (5.1.17) *such that $\tau' < \tau''$ and the interval (τ', τ'') contains the only threshold τ. We also suppose that the three thresholds relate to the same cylindrical end. Then:*

1). *On the intervals (τ', τ) and (τ, τ''), there exist analytic bases $\{\mu \mapsto Y_j^\pm(\cdot, \mu)\}$ in the spaces of continuous spectrum eigenfunctions of problem* (5.1.17) *satisfying* (5.2.3) *and* (5.2.4) *with the scattering matrix $\mu \mapsto S(\mu)$ analytic on the mentioned intervals.*

2). *On the interval (τ', τ'') there exist analytic bases $\{\mu \mapsto \mathcal{Y}_j^\pm(\cdot, \mu)\}$ in the spaces of continuous spectrum eigenfunctions of problem* (5.1.17) *satisfying* (5.2.8) *and* (5.2.9) *with the scattering matrix $\mu \mapsto \mathcal{S}(\mu)$ analytic on (τ', τ'').*

Proof. From the argument in 5.2.2, it suffices to verify the analyticity of the scattering matrices. For example, let us consider the matrix $\mu \mapsto \mathcal{S}(\mu)$. Equality (5.2.19), the representation $w = v + c_1 w_1^- + \cdots + c_M w_M^-$, and inclusion (5.2.18) lead to

$$\mathcal{Y}_j^+(\cdot,\mu) = w_j^+(\cdot,\mu) - \sum_{k=1}^{M} c_k(\mu) w_k^-(\cdot,\mu) \in H_\gamma^2(G).$$

Therefore, $\mathcal{S}_{jk}(\mu) = -c_k(\mu)$, $k = 1, \ldots, M$. It remains to take into account that the functions $\mu \mapsto c_k(\mu)$ are analytic on (τ', τ''). □

For the basis $\{\mathcal{Y}_j^+(\cdot,\mu)\}_j^M$ (see Theorem 5.2.6, 2)), we introduce the columns $\mathcal{Y}_{(\mathbf{1})}^+ = (\mathcal{Y}_1^+, \ldots, \mathcal{Y}_L^+)^t$ and $\mathcal{Y}_{(\mathbf{2})}^+ = (\mathcal{Y}_{L+1}^+, \ldots, \mathcal{Y}_M^+)^t$ and write down the scattering matrix in the form

$$\mathcal{S}(\mu) = \begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix},$$

where $\mathcal{S}_{(11)}(\mu)$ is a block of size $L \times L$ and $\mathcal{S}_{(22)}(\mu)$ is a block of size $(M-L) \times (M-L)$, while $\mu \in (\tau', \tau'')$. We also set

$$D = ((\mu - \tau)^{1/2} + 1)/((\mu - \tau)^{1/2} - 1)$$

with $(\mu - \tau)^{1/2} = i(\tau - \mu)^{1/2}$ for $\mu \le \tau$ and $(\tau - \mu)^{1/2} \ge 0$. The next assertion will be of use in Section 5.3.

Lemma 5.2.7. *Assume that $\mu \in (\tau', \tau]$ and $\mathcal{S}(\mu)$ is the scattering matrix in Theorem* 5.2.6, 2)*. Then*

$$\ker(D + \mathcal{S}_{(22)}(\mu)) \subset \ker \mathcal{S}_{(12)}(\mu), \tag{5.2.20}$$

$$\operatorname{Im}(D + \mathcal{S}_{(22)}(\mu)) \supset \operatorname{Im} \mathcal{S}_{(21)}(\mu). \tag{5.2.21}$$

Therefore the operator $\mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}$ is defined on $\operatorname{Im}(D + \mathcal{S}_{(22)}(\mu))$.

Proof. Let us consider (5.2.20). We assume that $h \in \ker(D + \mathcal{S}_{(22)}(\mu))$ and $(0, h)^t \in \mathbb{C}^M$. Then

$$\begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} S_{(12)}(\mu) h \\ -Dh \end{pmatrix}.$$

Since the matrix $\mathcal{S}(\mu)$ is unitary and $|D| = 1$, we have $\|h\|^2 = \|\mathcal{S}_{(12)}(\mu)h\|^2 + \|h\|^2$, so $\mathcal{S}_{(12)}(\mu)h = 0$ and (5.2.20) is valid. Inclusion (5.2.21) is equivalent to

$$\ker(D + \mathcal{S}_{(22)}(\mu))^* \subset \ker \mathcal{S}_{(21)}(\mu)^*. \tag{5.2.22}$$

Moreover,

$$\mathcal{S}(\mu)^* = \begin{pmatrix} \mathcal{S}_{(11)}(\mu)^* & \mathcal{S}_{(21)}(\mu)^* \\ \mathcal{S}_{(12)}(\mu)^* & \mathcal{S}_{(22)}(\mu)^* \end{pmatrix}$$

and the matrix $\mathcal{S}(\mu)^*$ is unitary, therefore (5.2.22) may be proven by the same argument as (5.2.20). □

5.3 Other properties of the scattering matrices

Here we clarify the connection between the matrices $\mathcal{S}(\mu)$ and $S(\mu)$ on the interval $\tau' < \mu < \tau$, prove the existence of the one-side finite limits $\lim S(\mu)$ as $\mu \to \tau \pm 0$, and describe the transformation of the scattering matrix under changes of basis in the space of waves $\mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$. We keep the assumptions introduced at the very beginning of Section 5.2.

5.3.1 The connection between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau' < \mu < \tau$

Let us recall the description of the stable basis chosen for definition of $\mathcal{S}(\mu)$. In the semicylinder Π_+^1, we introduce the functions

$$\Pi_+^1 \ni (y,t) \mapsto e_k^{\pm}(y,t;\mu) := \chi(t) \exp\left(\pm it\sqrt{\mu - \mu_k}\right)\varphi_k(y), \tag{5.3.1}$$

where $k = l+1, \dots, m$ (the notation is the same as in (5.1.10); as before, $\mu_{l+1} = \dots = \mu_m = \tau$). We extend the functions by zero to the whole domain G and set

$$w_{L+j}^{\pm}(\cdot\,;\mu) = 2^{-1/2}\left(\frac{e_{l+j}^{+}(\cdot\,;\mu) + e_{l+j}^{-}(\cdot\,;\mu)}{2} \mp \frac{e_{l+j}^{+}(\cdot\,;\mu) - e_{l+j}^{-}(\cdot\,;\mu)}{2\sqrt{\mu - \mu_{l+j}}}\right) \tag{5.3.2}$$

for $j = 1, \dots, m-l = M-L$ (the equality $m-l = M-L$ was explained just after Remark 5.2.2). All the rest waves with supports in Π_+^1 that was obtained from the functions (5.1.11) and the waves of the same type with supports in $\Pi_+^2, \dots, \Pi_+^{\mathcal{T}}$, we number by one index $j = 1, \dots, L$ and denote by $w_1^{\pm}(\cdot\,;\mu), \dots, w_L^{\pm}(\cdot\,;\mu)$. The obtained collection $\{w_1^{\pm}, \dots, w_M^{\pm}\}$ is a basis of waves in G stable in a neighborhood of the threshold τ. Finally, we introduce the columns $\mathbf{w}_{(1)}^{\pm} = (w_1^{\pm}, \dots, w_L^{\pm})^t$, $\mathbf{w}_{(2)}^{\pm} = (w_{L+1}^{\pm}, \dots, w_M^{\pm})^t$, and $(\mathbf{w}_{(1)}^{\pm}, \mathbf{w}_{(2)}^{\pm}) = (w_1^{\pm}, \dots, w_M^{\pm})^t$, where t stands for matrix transposing. The components of the vector $\mathbf{w}_{(1)}^{\pm}$ are bounded, while the components of $\mathbf{w}_{(2)}^{\pm}$ exponentially grow at infinity in Π_+^1. Let $\mathbf{e}_{(1)}^{\pm} = (e_1^{\pm}, \dots, e_L^{\pm})^t$ and $\mathbf{e}_{(2)}^{\pm} = (e_{L+1}^{\pm}, \dots, e_M^{\pm})^t$, then

$$\mathbf{w}_{(2)}^{\pm} = D^{\mp}\mathbf{e}_{(2)}^{+} + D^{\pm}\mathbf{e}_{(2)}^{-} \tag{5.3.3}$$

with

$$D^{\pm} = ((\mu-\tau)^{1/2} \pm 1)/2\sqrt{2}(\mu-\tau)^{1/2}.$$

The following assertion is, in essence, from Nazarov et al. [26].

Proposition 5.3.1. *Let $\mu \in (\tau', \tau)$ and let $S(\mu)$ and $\mathcal{S}(\mu)$ be the scattering matrices in Theorem* 5.2.6. *Then*

$$S(\mu) = \mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu), \tag{5.3.4}$$

with

$$D = D^{+}/D^{-} = ((\mu-\tau)^{1/2} + 1)/((\mu-\tau)^{1/2} - 1).$$

Proof. We verify (5.3.4). Rewrite (5.2.8) in the form

$$\begin{aligned}&\mathcal{Y}^+_{(\mathbf{1})} - \mathbf{w}^+_{(\mathbf{1})} - \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(12)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G),\\ &\mathcal{Y}^+_{(\mathbf{2})} - \mathbf{w}^+_{(\mathbf{2})} - \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(22)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G).\end{aligned} \tag{5.3.5}$$

Recall, that $\gamma > 0$ has been chosen according to Lemma 5.2.4, so the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \gamma\}$ contains the eigenvalues $\pm(\mu - \tau)^{1/2}$ of the pencil $\mathfrak{A}^1(\cdot, \mu)$. We take $\delta > 0$, such that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \delta\}$ contains only the real eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$. Then $\delta < \gamma$ and $H^2_\gamma(G) \subset H^2_\delta(G)$. Instead of $\mathbf{w}^\pm_{(\mathbf{2})}$, we substitute into (5.3.5) their expressions in (5.3.3). For the aforementioned δ the vector-valued function $\mathbf{e}^+_{(\mathbf{2})}$ belongs to $H^2_\delta(G)$. As a result we obtain

$$\mathcal{Y}^+_{(\mathbf{1})} = \mathbf{w}^+_{(\mathbf{1})} + \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} + \mathcal{S}_{(12)}D^-\mathbf{e}^-_{(\mathbf{2})} + \Re_{(\mathbf{1})}, \tag{5.3.6}$$

$$\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D + \mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \Re_{(\mathbf{2})}, \tag{5.3.7}$$

where $\Re_{(\mathbf{1})}, \Re_{(\mathbf{2})} \in H^2_\delta(G)$. Introduce the orthogonal projector

$$\mathcal{P} : \mathbb{C}^{M-L} \to \mathrm{Im}(D + \mathcal{S}_{(22)}(\mu)).$$

Taking account of (5.2.21) and (5.3.7), we arrive at

$$\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D + \mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \mathcal{P}\Re_{(\mathbf{2})}. \tag{5.3.8}$$

We apply the operator $\mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}$ to both sides of (5.3.8) and subtract the resulting equality from (5.3.6). We have

$$Z = \mathbf{w}^+_{(\mathbf{1})} + (\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu))\mathbf{w}^-_{(\mathbf{1})} + R, \tag{5.3.9}$$

where

$$Z = \mathcal{Y}^+_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})}, \tag{5.3.10}$$

$$R = \Re_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\Re_{(\mathbf{2})}. \tag{5.3.11}$$

The components of the vectors $\mathcal{Y}^+_{(\mathbf{1})}$ and $\mathcal{Y}^+_{(\mathbf{2})}$ satisfy problem (5.1.17). In view of (5.3.10), the same is true for the components of the vector Z. Moreover from $\Re_{(\mathbf{1})}, \Re_{(\mathbf{2})} \in H^2_\delta(G)$, and (5.3.11) it follows that $R \in H^2_\delta(G)$. Hence the formula (5.3.9) describes the scattering of the vector $\mathbf{w}^+_{(\mathbf{1})}$ of incoming waves in the basis $\mathbf{w}^+_{(\mathbf{1})}, \mathbf{w}^-_{(\mathbf{1})}$ as well as (5.2.3), so we obtain (5.3.4).□

5.3.2 The connection between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau < \mu < \tau''$

We consider two bases for the wave space $\mathcal{W}(\mu, G)$ for $\tau < \mu < \tau''$. One of the bases consists of the waves in G corresponding to functions of the form $u^\pm_q(\cdot, \mu)$ in (5.1.6), while the other one consists of the waves generated by the functions $w^\pm_q(\cdot, \mu)$ (see (5.1.10), (5.1.11)). As before, the scattering matrices defined in these bases are denoted by $S(\mu)$ and $\mathcal{S}(\mu)$ (see Theorem 5.2.6); this time, that is, for $\mu \in (\tau, \tau'')$, the matrices are of the same size $M \times M$.

The scattering matrices are independent of the choice of the cut-off function χ in the definition of the space $\mathcal{W}(\mu, G)$. Identifying "equivalent" waves, one can omit such a cut-off function from consideration. To this end we introduce the quotient space

$$\dot{\mathcal{W}}(\mu, G) := (\mathcal{W}(\mu, G) \dot{+} H^2_\gamma(G))/H^2_\gamma(G).$$

Let $\dot{v}$ stand for the class in $\dot{\mathcal{W}}(\mu, G)$, with representative $v \in \mathcal{W}(\mu, G)$. In what follows, waves of the form $\chi u^\pm_q(\cdot, \mu)$ and $\chi w^\pm_q(\cdot, \mu)$ in G are denoted by $u^\pm_q(\cdot, \mu)$ and $w^\pm_q(\cdot, \mu)$. The collections $\{\dot{u}^\pm_q(\cdot, \mu)\}^M_{j=1}$ and $\{\dot{w}^\pm_k(\cdot, \mu)\}^M_{k=1}$ are bases for the space $\dot{\mathcal{W}}(\mu, G)$, so $\dim \dot{\mathcal{W}}(\mu, G) = 2M$. The form $q_G(u, v)$ in (5.1.18) is independent of the choice of representatives in $\dot{u}$ and $\dot{v}$; hence it is defined on $\dot{\mathcal{W}}(\mu, G) \times \dot{\mathcal{W}}(\mu, G)$. From (5.1.8) and (5.1.9) it follows that

$$iq_G(\dot{u}^\pm_k(\cdot\,;\mu), \dot{u}^\mp_l(\cdot\,;\mu)) = 0 \quad \text{for all } k, l = 1, \dots, M, \tag{5.3.12}$$
$$iq_G(\dot{u}^\pm_k(\cdot\,;\mu), \dot{u}^\pm_l(\cdot\,;\mu)) = \mp\delta_{kl}, \tag{5.3.13}$$

while (5.1.12) and (5.1.13) lead to

$$iq_G(\dot{w}^\pm_r(\cdot\,;\mu), \dot{w}^\mp_s(\cdot\,;\mu)) = 0 \quad \text{for all } r, s = 1, \dots, M, \tag{5.3.14}$$
$$iq_G(\dot{w}^\pm_r(\cdot\,;\mu), \dot{w}^\pm_s(\cdot\,;\mu)) = \mp\delta_{rs}. \tag{5.3.15}$$

Thus $\dot{\mathcal{W}}(\mu, G)$ turns out to be a $2M$-dimensional complex space with indefinite inner product $< \dot{u}, \dot{v} >:= -iq_G(\dot{u}, \dot{v})$. The projection

$$\pi : \mathcal{W}(\mu, G) \dot{+} H^2_\gamma(G) \to \dot{\mathcal{W}}(\mu, G) \tag{5.3.16}$$

maps the space of continuous spectrum eigenfunctions onto a subspace in $\dot{\mathcal{W}}(\mu, G)$ of dimension M; denote the subspace by $\mathcal{E}(\mu)$.

Let $V_1, \dots, V_{2M}$ be a basis for $\dot{\mathcal{W}}(\mu, G)$, subject to the orthogonality and normalization conditions

$$< V_j, V_l >= \delta_{jl}, \; < V_{j+M}, V_{l+M} >= -\delta_{jl} \text{ for } j, l = 1, \dots, M. \tag{5.3.17}$$

The elements $V_1, \dots, V_M$ are called incoming waves while the elements $V_{M+1}, \dots,$ V_{2M} are called outgoing waves. Assume that $X_1, \dots, X_M$ is a basis of $\mathcal{E}(\mu)$ that defines, in the basis of waves $V_1, \dots, V_{2M}$, the scattering matrix $\mathfrak{S}(\mu)$ of size $M \times M$ (compare with (5.2.3)). We represent the vectors X_j as coordinate rows and form the $M \times 2M$-matrix $X = (X_1, \dots, X_M)^t$ (which is a column of the letters $X_1, \dots, X_M$). Finally, let I stand for the unit matrix of size $M \times M$. Then a relation of the form (5.2.3) leads to

$$X = (I, \mathfrak{S}(\mu))V, \tag{5.3.18}$$

where V is the $2M \times 2M$-matrix $(V_1, \dots, V_{2M})^t$ consisting of the coordinate rows of the vectors V_j and $(I, \mathfrak{S}(\mu))$ is a matrix of size $M \times 2M$.

Assume that $\tilde{V}_1, \dots, \tilde{V}_{2M}$ is another basis of waves subject to conditions of the form (5.3.17), $\tilde{X}_1, \dots, \tilde{X}_M$ is a basis of $\mathcal{E}(\mu)$, and $\tilde{\mathfrak{S}}(\mu)$ is the corresponding scattering matrix, such that

$$\tilde{X} = (I, \tilde{\mathfrak{S}}(\mu))\tilde{V}. \tag{5.3.19}$$

Assume, that $\tilde{V} = \mathfrak{T}V$ and write down the $2M \times 2M$-matrix $\mathfrak{T}$ as $\mathfrak{T} = (\mathfrak{T}_{(k\,l)})_{k,l=1}^2$ with blocks $\mathfrak{T}_{(k\,l)}$ of size $M \times M$.

Lemma 5.3.2. *The matrices* $\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}$ *and* $\mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}$ *are invertible and*

$$\mathfrak{S}(\mu) = (\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)})^{-1}(\mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}). \tag{5.3.20}$$

Proof. For the bases $X_1, \dots, X_M$ and $\tilde{X}_1, \dots, \tilde{X}_M$ there exists a nonsingular $M \times M$-matrix B, such that $\tilde{X} = BX$. Therefore, from (5.3.19), we have

$$BX = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}V.$$

Taking into account (5.3.18), we obtain $B(I, \mathfrak{S}(\mu))V = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}V$, so

$$B(I, \mathfrak{S}(\mu)) = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}.$$

Let us write this equality in the form of

$$(B, B\mathfrak{S}(\mu)) = (\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}, \mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}).$$

Now the assertions of Lemma are evident. □

We intend to make use of (5.3.20) taking as $\tilde{V}$ the image, under canonical projection (5.3.16), of the stable basis of $\mathcal{W}(\mu, G)$ in (5.2.8) and as V the image of the wave basis in (5.2.3). As $\tilde{\mathfrak{S}}(\mu)$ and $\mathfrak{S}(\mu)$, we choose $\mathcal{S}(\mu)$ and $S(\mu)$ respectively. We proceed to computing the matrix $\mathfrak{T}$ in the equality $\tilde{V} = \mathfrak{T}V$. In doing so, instead of $\tilde{V}$ and V we can consider their just mentioned preimages in $\mathcal{W}(\mu, G)$. We set

$$u_j := u_j^+, \qquad u_{j+M} := u_j^-, \qquad j = 1, \dots, M, \tag{5.3.21}$$

where $u_j^\pm$ are the waves in $\mathcal{W}(\mu, G)$, generated by the functions of the form (5.1.6). We also introduce

$$\begin{aligned} w_j &:= w_j^+ = u_j^+, \qquad w_{j+M} := w_j^- = u_j^-, \qquad j = 1, \dots, L, \\ w_p &:= w_p^+, \qquad w_{p+M} := w_p^-, \qquad p = L+1, \dots, M, \end{aligned} \tag{5.3.22}$$

where $w_p^\pm$ are the waves in $\mathcal{W}(\mu, G)$, generated by the functions (5.1.10). For the matrix $\mathfrak{T}$, the equality $w = \mathfrak{T}u$ holds with the columns $w = (w_1, \dots, w_{2M})^t$ and $u = (u_1, \dots, u_{2M})^t$. For convenience, we will here denote functions (5.1.10) in the same way as the waves $w_p^\pm$; let us write down functions in the form

$$w_p^\pm(\mu) = 2^{-1/2}((e^{it\lambda} + e^{-it\lambda})/2) \mp (e^{it\lambda} - e^{-it\lambda})/2\lambda)\varphi_p,$$

where $\lambda = \sqrt{\mu - \tau}$ and τ is a threshold; we also write the functions (5.1.6) in the form

$$u_p^{\pm}(\mu) = (2\lambda)^{-1/2} e^{\mp it\lambda} \varphi_p.$$

Then we have

$$w_p^{\pm} = (1/2)(u_p^{+}(\lambda^{1/2} \pm \lambda^{-1/2}) + u_p^{-}(\lambda^{1/2} \mp \lambda^{-1/2})), \qquad p = L+1, \dots, M;$$

here by $w_p^{\pm}$ and $u_p^{\pm}$ one can mean the functions in the cylinder and the corresponding waves in the domain G alike. Together with (5.3.21) and (5.3.22), this leads to the following description of the blocks $\mathfrak{T}_{ij}$ of the matrix $\mathfrak{T}$.

Lemma 5.3.3. *Each of the matrices $\mathfrak{T}_{(ij)}$ consists of four blocks and is block-diagonal. The equalities*

$$\mathfrak{T}_{(11)}(\mu) = \mathfrak{T}_{(22)}(\mu) = \operatorname{diag}\{I_L,\, 2^{-1}(\lambda^{(1/2)} + \lambda^{(-1/2)}) I_{M-L}\}, \tag{5.3.23}$$

$$\mathfrak{T}_{(21)}(\mu) = \mathfrak{T}_{(12)}(\mu) = \operatorname{diag}\{O_L,\, 2^{-1}(\lambda^{1/2} - \lambda^{-1/2}) I_{M-L}\} \tag{5.3.24}$$

hold, where I_K is the unit matrix of size $K \times K$, O_L is the zero matrix of size $L \times L$, and $\lambda = \sqrt{\mu - \tau}$ with $\mu \in (\tau, \tau'')$.

We return to (5.3.20) with $\mathcal{S}$ and S instead of $\tilde{\mathfrak{S}}$ and $\mathfrak{S}$. Let us divide the matrix $\mathcal{S}$ into four blocks with $\mathcal{S}_{(11)}$ of size $L \times L$ and $\mathcal{S}_{(22)}$ of size $(M-L) \times (M-L)$. We also set $d^{\pm} = 2^{-1}(\lambda^{1/2} \pm \lambda^{-1/2})$. Then

$$\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)} = \begin{pmatrix} I_L & \mathcal{S}_{(12)} d^- \\ O & \mathcal{S}_{(22)} d^- + I_{M-L} d^+ \end{pmatrix}. \tag{5.3.25}$$

According to Lemma 5.3.2, the matrix $\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)}$ is invertible, so the matrix $\mathcal{S}_{(22)} d^- + I_{M-L} d^+$ is invertible as well, therefore

$$(\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)})^{-1} = \begin{pmatrix} I_L & -\mathcal{S}_{(12)} d^- (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} \\ O & (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} \end{pmatrix}. \tag{5.3.26}$$

In view of (5.3.20), we obtain

Proposition 5.3.4. *For $\mu \in (\tau, \tau'')$ the blocks $S_{(ij)}$ of the scattering matrix*

$$S(\mu) = (\mathfrak{T}_{(11)} + \mathcal{S}(\mu)\mathfrak{T}_{(21)})^{-1} (\mathfrak{T}_{(12)} + \mathcal{S}(\mu)\mathfrak{T}_{(22)})$$

admit the representations

$$S_{(11)} = \mathcal{S}_{(11)} - \mathcal{S}_{(12)} d^- (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} \mathcal{S}_{(21)}, \tag{5.3.27}$$

$$S_{(12)} = \mathcal{S}_{(12)} d^+ - \mathcal{S}_{(12)} d^- (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} (\mathcal{S}_{(22)} d^+ + I_{M-L} d^-), \tag{5.3.28}$$

$$S_{(21)} = (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} \mathcal{S}_{(21)}, \tag{5.3.29}$$

$$S_{(22)} = (\mathcal{S}_{(22)} d^- + I_{M-L} d^+)^{-1} (\mathcal{S}_{(22)} d^+ + I_{M-L} d^-). \tag{5.3.30}$$

5.3.3 The limits of $S(\mu)$ as $\mu \to \tau \pm 0$

To calculate the one-sided limits of $S(\mu)$, we make use of (5.3.4) as $\mu \to \tau - 0$ and apply (5.3.27) – (5.3.30) as $\mu \to \tau + 0$. The computation procedure depends on whether the number 1 is an eigenvalue of the matrix $\mathcal{S}_{22}(\tau)$.

5.3.3.1 The limits of $S(\mu)$ as $\mu \to \tau \pm 0$ provided 1 is not an eigenvalue of $\mathcal{S}_{(22)}(\tau)$

Recall, that the functions $\mu \mapsto \mathcal{S}_{(kl)}(\mu)$ are analytic in a neighborhood of $\mu = \tau$. Therefore from (5.3.4) it immediately follows, that

$$\lim_{\mu \to \tau - 0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau). \tag{5.3.31}$$

Let us proceed to compute $\lim S(\mu)$ as $\mu \to \tau + 0$. By virtue of (5.3.27) and (5.3.31),

$$\begin{aligned}\lim_{\mu \to \tau + 0} S_{(11)}(\mu) = &\lim_{\mu \to \tau + 0} (\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(\mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu))^{-1}\mathcal{S}_{(21)}(\mu)) \\ =&\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau) = \lim_{\mu \to \tau - 0} S(\mu).\end{aligned} \tag{5.3.32}$$

According to (5.3.30),

$$\begin{aligned}\lim_{\mu \to \tau + 0} S_{(22)}(\mu) = &\lim_{\mu \to \tau + 0} (\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)} d^+/d^- + 1) = \\ &= (\mathcal{S}_{(22)}(\tau) - 1)^{-1}(-\mathcal{S}_{(22)}(\tau) + 1) = -I_{M-L}.\end{aligned} \tag{5.3.33}$$

It follows from (5.3.29), that

$$S_{(21)}(\mu) = (\mathcal{S}_{(22)} + d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Since $d^-(\mu) = 2^{-1}((\mu - \tau)^{1/2} - 1)/(\mu - \tau)^{1/4}$, we arrive at

$$S_{(21)}(\mu) = O((\mu - \tau)^{1/4}) \to 0 \qquad \text{for} \qquad \mu \to \tau + 0. \tag{5.3.34}$$

Finally, consider $S_{(12)}(\mu)$. We rewrite (5.3.28) in the form

$$\begin{aligned}S_{(12)} &= \mathcal{S}_{(12)} d^+(1 - (\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)} + d^-/d^+)) = \\ &= \mathcal{S}_{(12)} d^+(\mathcal{S}_{(22)} + d^+/d^-)^{-1}(d^+/d^- - d^-/d^+).\end{aligned}$$

In view of

$$d^+(\mu)(d^+/d^- - d^-/d^+) = 2(\mu - \tau)^{1/4}/((\mu - \tau)^{1/2} - 1),$$

we obtain

$$S_{(12)}(\mu) = O((\mu - \tau)^{1/4}) \to 0 \qquad \text{for} \qquad \mu \to \tau + 0. \tag{5.3.35}$$

5.3.3.2 The limits of $S(\mu)$ as $\mu \to \tau \pm 0$ provided 1 is an eigenvaue of $\mathcal{S}_{(22)}(\tau)$

We set $\lambda = \sqrt{\mu - \tau}$ with $\mu = \tau + \lambda^2$ and consider the function $\lambda \mapsto \Phi(\lambda) : \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$,

$$\Phi(\lambda) := \mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu) = \mathcal{S}_{(22)}(\tau + \lambda^2) + (\lambda + 1)/(\lambda - 1). \quad (5.3.36)$$

The number $\lambda = 0$ is an eigenvalue of the function $\lambda \mapsto \Phi(\lambda)$, if and only if 1 is an eigenvalue of the matrix $\mathcal{S}_{(22)}(\tau)$; in such a case, $\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker \Phi(0)$. To calculate the limits of $S(\mu)$ as $\mu \to \tau \pm 0$, we need a knowledge of the resolvent $\lambda \mapsto \Phi(\lambda)^{-1}$ in a neighborhood of $\lambda = 0$. Propositions 5.3.5 and 5.3.6 provide the required information.

Proposition 5.3.5. *There holds the equality*

$$\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker(\mathcal{S}_{(22)}(\tau)^* - 1). \quad (5.3.37)$$

Proof. Assume that $h \in \ker(\mathcal{S}_{(22)}(\tau) - 1)$. Then, as shown in the proof of Lemma 5.2.7, the vector $(0, h)^t \in \mathbb{C}^M$ belongs to $\ker(\mathcal{S}(\tau) - 1)$ and $\mathcal{S}_{(12)}(\tau)h = 0$. The same argument with $\mathcal{S}(\tau)^*$ instead of $\mathcal{S}(\tau)$ shows that the inclusion $g \in \ker(\mathcal{S}_{(22)}(\tau)^* - 1)$ implies $(0, g)^t \in \ker(\mathcal{S}(\tau)^* - 1)$ and $\mathcal{S}_{(21)}(\tau)^* g = 0$. Since $\mathcal{S}(\tau)^* = \mathcal{S}(\tau)^{-1}$, we have

$$\ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1). \quad (5.3.38)$$

Let $h_1, \dots, h_\varkappa$ be a basis of $\ker(\mathcal{S}_{(22)}(\tau) - 1)$ and $g_1, \dots, g_\varkappa$ the basis of $\ker(\mathcal{S}_{(22)}(\tau)^* - 1)$. We set $\tilde{h}_j = (0, h_j)^t$ and $\tilde{g}_j = (0, g_j)^t$. Then (5.3.38) implies

$$\tilde{h}_j, \tilde{g}_j \in \ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1), \qquad j = 1, \dots, \varkappa.$$

Therefore, any vector of the collection $h_1, \dots, h_\varkappa$ is a linear combination of the vectors $g_1, \dots, g_\varkappa$ and vice versa. □

Proposition 5.3.6. *Let* Φ *be the matrix function in* (5.3.36) *and* $\dim \ker \Phi(0) = \varkappa > 0$. *Then, in a punctured neighborhood of* $\lambda = 0$, *the resolvent* $\lambda \mapsto \Phi(\lambda)^{-1}$ *admits the representation*

$$\Phi(\lambda)^{-1} = -(2\lambda)^{-1} \sum_{j=1}^{\varkappa} \{\cdot, h_j\} h_j + \Gamma(\lambda); \quad (5.3.39)$$

here $h_1, \dots, h_\varkappa$ *is an orthonormal basis of* $\ker(\mathcal{S}_{(22)}(\tau) - 1)$, $\{u, v\}$ *is the inner product on the space* $\mathbb{C}^{M-L}$, *and* $\lambda \mapsto \Gamma(\lambda) : \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$ *is a matrix function holomorphic in a neighborhood of* $\lambda = 0$.

Proof. It is known (e.g., see [12],[13]) that, under certain conditions, the resolvent $\mathfrak{A}(\lambda)^{-1}$ of a holomorphic operator function $\lambda \mapsto \mathfrak{A}(\lambda)$ in a punctured neighborhood of an isolated eigenvalue λ_0 admits the representation

$$\mathfrak{A}(\lambda)^{-1} = (\lambda - \lambda_0)^{-1} \sum_{j=1}^{\varkappa} (\cdot, \psi_j)\phi_j + \Gamma(\lambda), \quad (5.3.40)$$

where $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ are bases for the spaces $\ker\mathfrak{A}(\lambda_0)$ and $\ker\mathfrak{A}(\lambda_0)^*$ satisfying the orthogonality and normalization conditions

$$(\partial_\lambda \mathfrak{A}(\lambda_0)\phi_j, \psi_k) = \delta_{jk}, \qquad j, k = 1, \dots, \varkappa, \tag{5.3.41}$$

and Γ is an operator function holomorphic in a neighborhood of λ_0. Formula (5.3.40) is related to the case where the operator function $\lambda \mapsto \mathfrak{A}(\lambda)$ has no generalized eigenvectors at the point λ_0. To justify (5.3.39), we have to show that there are no generalized eigenvectors of the function $\lambda \mapsto \Phi(\lambda)$ at the point $\lambda = 0$ and to verify agreement between (5.3.39) and (5.3.40).

We first take up the generalized eigenvectors. Assume that $0 \neq h^0 \in \ker\Phi(0)$. The equation $\Phi(0)h^1 + (\partial_\lambda\Phi)(0)h^0 = 0$ for a generalized eigenvector h^1 is of the form

$$(\mathcal{S}_{(22)}(\tau) - 1)h^1 = 2h^0.$$

The orthogonality of h^0 to the lineal $\ker(\mathcal{S}_{(22)}(\tau)^* - 1) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$ is necessary for the solvability of this equation (see (5.3.37)). Since $0 \neq h^0 \in \ker\Phi(0) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$, the solvability condition is not fulfilled, so the generalized eigenvectors do not exist.

Let us compare (5.3.39) and (5.3.40). We have $(\partial_\lambda\Phi)(0) = -2I_{M-L}$. Moreover, in view of (5.3.37), the bases $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ in (5.3.40) can be chosen to satisfy $\phi_j = -\psi_j = h_j/\sqrt{2}$ and as $h_1, \dots, h_\varkappa$ there can be taken an orthonormal basis of $\ker(\mathcal{S}_{22}(\tau) - 1)$. Then

$$\{(\partial_\lambda\Phi)(0)\phi_j, \psi_k\} = \delta_{jk}, \qquad j, k = 1, \dots, \varkappa,$$

and the representation (5.3.40) takes the form of (5.3.39). □

Let us calculate $\lim S(\mu)$ as $\mu \to \tau - 0$. According to Lemma 5.2.7,

$$\operatorname{Im}(\mathcal{S}_{(22)}(\tau) - 1) \supset \operatorname{Im}\mathcal{S}_{(21)}(\tau).$$

Therefore, Proposition 5.3.5 leads to the equalities $\{\mathcal{S}_{(21)}(\tau)f, h_j\} = 0$ for any $f \in \mathbb{C}^L$ and $h_1, \dots, h_\varkappa$ in (5.3.39). Because the function $\mu \to \mathcal{S}_{(21)}(\mu)$ is analytic, we have $\mathcal{S}_{(21)}(\mu) = \mathcal{S}_{(21)}(\tau) + O(|\mu - \tau|)$; recall that $|\mu - \tau| = |\lambda|^2$. Applying (5.3.39), we obtain

$$(\mathcal{S}_{(22)}(\mu) + D(\mu))^{-1}\mathcal{S}_{(21)}(\mu) = \Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|). \tag{5.3.42}$$

Now from (5.3.4) it follows, that

$$\lim_{\mu \to \tau - 0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau); \tag{5.3.43}$$

Lemma 5.2.7 allows to treat the right-hand-side as the operator

$$\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau)$$

(see (5.3.31)). For $\mu \to \tau - 0$ there holds the estimate

$$S(\mu) - (\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau)) = O(|\mu - \tau|^{(1/2)}). \tag{5.3.44}$$

Let us proceed to computing the limits as $\mu \to \tau + 0$. Compute $\lim_{\mu\to\tau+0} S_{(11)}(\mu)$ in the same way as $\lim_{\mu\to\tau-0} S(\mu)$ and obtain

$$\lim_{\mu\to\tau+0} S_{(11)}(\mu) = \lim_{\mu\to\tau-0} S(\mu). \tag{5.3.45}$$

In view of (5.3.30),

$$\begin{aligned} S_{(22)}(\mu) &= \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1}\left(\mathcal{S}_{(22)}(\mu) + d^-/d^+\right) d^+/d^- \\ &= d^+/d^- + \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1}\left(d^-/d^+ - d^+/d^-\right) d^+/d^-. \end{aligned}$$

Applying resolvent representation (5.3.39), we write the last equality in the form of

$$S_{(22)}(\mu) = \frac{\lambda+1}{\lambda-1}\left(I + \frac{2}{\lambda^2-1}\sum_{j=1}^{\varkappa}(\cdot, h_j)h_j - \frac{4\lambda}{\lambda^2-1}\Gamma(\lambda)\right). \tag{5.3.46}$$

Hence

$$\lim_{\mu\to\tau+0} S_{(22)}(\mu) = 2\sum_{j=1}^{\varkappa}(\cdot, h_j)h_j - I = P - Q, \tag{5.3.47}$$

where $P = \sum_{j=1}^{\varkappa}(\cdot, h_j)h_j$ is the orthogonal projector $\mathbb{C}^{M-L}$ onto $\ker\left(\mathcal{S}_{(22)}(\tau) - 1\right)$ and $Q = I - P$. Moreover, for $\mu \to \tau + 0$, it follows from (5.3.46), that

$$S_{(22)}(\mu) - P + Q = O(|\mu-\tau|^{1/2}). \tag{5.3.48}$$

In accordance with (5.3.29),

$$S_{(21)}(\mu) = (\mathcal{S}_{22}(\mu) + I_{M-L}d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Taking account of (5.3.42) and of $d^- = (\lambda-1)/2\sqrt{\lambda}$, we obtain

$$S_{(21)}(\mu) = \left(\Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|)\right) 2\sqrt{\lambda}/(\lambda-1).$$

Consequently,

$$S_{(21)}(\mu) = O(|\mu-\tau|^{1/4}) \to 0 \qquad \text{for} \qquad \mu \to \tau + 0. \tag{5.3.49}$$

It remains to find the limit of $S_{(12)}(\mu)$. By virtue of (5.3.28),

$$S_{(12)}(\mu) = \mathcal{S}_{(12)}(\mu)d^+\left(I - (\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu) + d^-/d^+)\right).$$

Since

$$(\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu) + d^-/d^+) = I - \frac{4\lambda}{\lambda^2-1}(\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1},$$

we arrive at

$$S_{(12)}(\mu) = \frac{2\sqrt{\lambda}}{\lambda-1}\mathcal{S}_{(12)}(\mu)\left(-\frac{1}{2\lambda}\sum(\cdot, h_j)h_j + \Gamma(\lambda)\right).$$

Recall that $h_j \in \ker(\mathcal{S}_{22}(\tau) - 1) \subset \ker\mathcal{S}_{12}(\tau)$ (see (5.2.20)), $\mathcal{S}_{(12)}(\mu) = \mathcal{S}_{(12)}(\tau) + O(|\mu-\tau|)$, and $\mu - \tau = \lambda^2$. Therefore, as $\mu \to \tau + 0$ we have

$$S_{(12)}(\mu) = O(|\mu-\tau|^{1/4}) \to 0. \tag{5.3.50}$$

5.4 Method for computing the scattering matrix

We first recall the method for the scattering matrix $S(\mu)$ in Theorem 5.2.6, 1) with $\mu' \leqslant \mu \leqslant \mu''$, where $[\mu', \mu''] \subset (\tau', \tau)$ or $[\mu', \mu''] \subset (\tau, \tau'')$. The interval $[\mu', \mu'']$ may contain eigenvalues of the operator (5.2.5). The method was justified for the Laplace operator by Plamenevskii et al. [30] and generalized for elliptic systems by Plamenevskii et al. [31]. We set

$$\begin{gathered}\Pi_{+}^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^{\mathcal{T}} \Pi_{+}^{r,R}, \\ \partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}, \quad \Gamma^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r = R\}\end{gathered}$$

for large R and introduce the boundary value problem

$$\begin{aligned} -\Delta X_j^R(x,\mu) - \mu X_j^R(x,\mu) &= 0, \quad x \in G^R; \\ X_j^R(x,\mu) &= 0 \quad x \in \partial G^R \setminus \Gamma^R; \\ (-\partial_n + i\zeta) X_j^R(x,\mu) &= (-\partial_n + i\zeta)\left(u_j^+(x,\mu) + \textstyle\sum_{k=1}^M a_k u_k^-(x,\mu)\right), \ x \in \Gamma^R, \end{aligned} \tag{5.4.1}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ is an arbitrary fixed number, a_k are complex numbers, and $u_j^{\pm}$ are the waves in (5.1.6). As an approximation to the row $(S_{j1}, \ldots, S_{jM})$ there serves a minimizer $a^0(R,\mu) = (a_1^0(R,\mu), \ldots, a_M^0(R,\mu))$ of the functional

$$J_j^R(a_1, \ldots, a_M; \mu) = \| X_j^R(\cdot,\mu) - u_j^+(\cdot,\mu) - \sum_{k=1}^{M} a_k u_k^-(\cdot,\mu); L_2(\Gamma^R)\|^2, \tag{5.4.2}$$

where X_j^R is a solution to problem (5.4.1). To clarify the dependence of X_j^R on the parameters $a_1, \ldots, a_M$, we consider the problems

$$\begin{aligned} -\Delta v_j^{\pm} - \mu v_j^{\pm} &= 0, \quad x \in G^R, \\ v_j^{\pm} &= 0, \quad x \in \partial G^R \setminus \Gamma^R; \\ (-\partial_n + i\zeta) v_j^{\pm} &= (-\partial_n + i\zeta) u_j^{\pm}, \ x \in \Gamma^R, \quad j = 1, \ldots, M; \end{aligned} \tag{5.4.3}$$

we have $X_j^R = v_{j,R}^+ + \sum_k a_k v_{k,R}^-$. Let us introduce the $M \times M$-matrices with entries

$$\begin{aligned} \mathbf{E}_{jk}^R &= \left(v_j^- - u_j^-, v_k^- - u_k^-\right)_{\Gamma^R}, \\ \mathbf{F}_{jk}^R &= \left(v_j^+ - u_j^+, v_k^- - u_k^-\right)_{\Gamma^R}. \end{aligned}$$

We also set

$$\mathbf{G}_j^R = \left(v_j^+ - u_j^+, v_j^+ - u_j^+\right)_{\Gamma^R}.$$

Now functional (5.4.2) can be written in the form

$$J_j^R(a) = \langle a\mathbf{E}^R, a\rangle + 2\mathrm{Re}\,\langle \mathbf{F}_j^R, a\rangle + \mathbf{G}_j^R,$$

where $\mathbf{F}_j^R$ is the j-th row of the matrix $\mathbf{F}^R$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{C}^M$. The minimizer $a^0(R,\mu)$ satisfies $a^0(R,\mu)\mathbf{E}^R + \mathbf{F}_j^R = 0$; the matrix $\mathbf{E}^R$ is non-singular.

It was shown by Plamenevskii et al. [30] that the minimizer $a^0(R,\mu)$ tends to the row $(S_{j1},\dots,S_{jM})$ as $R\to+\infty$ at exponential rate. More precisely, the estimate

$$\sum_{k=1}^{M}|S_{jk}(\mu)-a_k^0(R,\mu)|\leqslant Ce^{-\delta R}$$

holds for any $R\geqslant R_0$, where δ is the number in (5.2.3), R_0 is a sufficiently large positive number, and the constant C is independent of R and $\mu\in[\mu',\mu'']$.

We now proceed to calculating the matrix $\mathcal{S}(\mu)$ in Theorem 5.2.6, 2) with $\mu\in[\mu',\mu'']\subset(\tau',\tau'')$. The interval $[\mu',\mu'']$ may contain the threshold τ as well as some eigenvalues of the operator (5.2.10). Introduce the boundary value problem

$$\begin{aligned}
-\Delta\mathcal{X}_j^R-\mu\mathcal{X}_j^R &= 0, \quad x\in G^R;\\
\mathcal{X}_j^R &= 0 \quad x\in\partial G^R\setminus\Gamma^R;\\
(-\partial_n+i\zeta)\mathcal{X}_j^R &= (-\partial_n+i\zeta)(w_j^+ +\textstyle\sum_{k=1}^{M}a_kw_k^-),\ x\in\Gamma^R,
\end{aligned}\tag{5.4.4}$$

where $w_j^\pm$ is a stable basis (5.1.10), (5.1.11) for the space of waves, $\zeta\in\mathbb{R}\setminus\{0\}$, and $a_k\in\mathbb{C}$. As an approximation to the row $(\mathcal{S}_{j1},\dots,\mathcal{S}_{jM})$, we suggest a minimizer $a^0(R)=(a_1^0(R),\dots,a_M^0(R))$ of the functional

$$\mathcal{J}_j^R(a_1,\dots,a_M)=\|\mathcal{X}_j^R-w_j^+-\sum_{k=1}^{M}a_kw_k^-;L_2(\Gamma^R)\|^2,\tag{5.4.5}$$

where $\mathcal{X}_j^R$ is a solution of problem (5.4.4). Let us consider the problems

$$\begin{aligned}
-\Delta z_j^\pm-\mu z_j^\pm &= 0, \quad x\in G^R;\\
z_j^\pm &= 0, \quad x\in\partial G^R\setminus\Gamma^R;\\
(-\partial_n+i\zeta)z_j^\pm &= (-\partial_n+i\zeta)w_j^\pm,\ x\in\Gamma^R; \quad j=1,\dots,M,
\end{aligned}$$

set

$$\begin{aligned}
\mathcal{E}_{jk}^R &= \left(z_j^- - w_j^-, z_k^- - w_k^-\right)_{\Gamma^R},\\
\mathcal{F}_{jk}^R &= \left(z_j^+ - w_j^+, z_k^- - w_k^-\right)_{\Gamma^R},\\
\mathcal{G}_j^R &= \left(z_j^+ - w_j^+, z_j^+ - w_j^+\right)_{\Gamma^R},
\end{aligned}\tag{5.4.6}$$

and rewrite functional (5.4.5) into the form of

$$\mathcal{J}_j^R(a)=\langle a\mathcal{E}^R,a\rangle+2\mathrm{Re}\,\langle\mathcal{F}_j^R,a\rangle+\mathcal{G}_j^R,$$

where $\mathcal{F}_j^R$ is the j-th row of the matrix $\mathcal{F}^R$. Thus the minimizer $a^0(R)$ is a solution to the system $a^0(R)\mathcal{E}^R+\mathcal{F}_j^R=0$.

The justification of the method is similar to the justification by Plamenevskii et al. [31]. The following Propositions 5.4.1 and 5.4.2 can be verified in the same way as the analogous assertions in [31].

Proposition 5.4.1. *The matrix $\mathcal{E}^R(\mu)$ with entries* (5.4.6) *is non-singular for all $\mu \in [\mu', \mu'']$ and $R \geqslant R_0$, where R_0 is sufficiently large number.*

Proposition 5.4.2. *Let u be a solution to the problem*

$$\begin{aligned} -\Delta u - \mu u &= 0, \quad x \in G^R, \\ u &= 0 \quad x \in \partial G^R \setminus \Gamma^R, \\ (-\partial_n + i\zeta)u &= h, \; x \in \Gamma^R, \end{aligned}$$

with $h \in L_2(\Gamma^R)$. Then

$$\|u; L_2(\Gamma^R)\| \leqslant \frac{1}{|\zeta|}\|h; L_2(\Gamma^R)\|. \tag{5.4.7}$$

Proposition 5.4.3. *Let $a^0(R,\mu) = (a_1^0(R,\mu), \dots, a_M^0(R,\mu))$ be a minimizer of the functional $\mathcal{J}_l^R$ in* (5.4.5). *Then*

$$\mathcal{J}_l^R \left(a^0(R,\mu)\right) \leqslant Ce^{-2\gamma R} \quad \text{for } R \to \infty, \tag{5.4.8}$$

where the constant C is independent of $R \geqslant R_0$, $\mu \in [\mu', \mu'']$, and $\gamma = \gamma(\mu)$ is the piecewise constant index described in Lemma 5.2.4. *For all $R \geqslant R_0$ and $\mu \in [\mu', \mu'']$ the components of the vector $a^0(R,\mu)$ are uniformly bounded,*

$$|a_j^0(R,\mu)| \leqslant \text{const} < \infty, \quad j = 1, \dots, M.$$

Proof. Relation (5.4.8) has been obtained in the same manner as in [31]. Let us verify the uniform boundedness of the minimizer $a^0(R,\mu)$. According to Lemma 5.2.4, for $[\mu', \mu'']$ there exists a finite covering I_p such that for each interval I_p one can choose a number $\gamma(\mu)$ in (5.2.8) (and consequently in (5.4.8)) independent of μ. Moreover, $\max_{\mu \in I_p} \mathrm{Re}\sqrt{\tau - \mu} < \gamma < \min_{\mu \in I_p} \mathrm{Re}\sqrt{\tau'' - \mu}$. We consider that μ runs through one of the covering intervals. Denote by Z_l^R the solution of problem (5.4.4) corresponding to $a^0(R,\mu) = (a_1^0(R,\mu), \dots, a_M^0(R,\mu))$. Setting $u = v = Z_l^R$ in the Green formula, we obtain

$$(-\partial_\nu Z_l^R, Z_l^R)_{\Gamma^R} - (Z_l^R, -\partial_\nu Z_l^R)_{\Gamma^R} = 0. \tag{5.4.9}$$

By virtue of (5.4.8),

$$\|Z_l^R - (w_l^+ + \sum_{j=1}^M a_j(R,\mu)w_j^-); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty, \tag{5.4.10}$$

uniformly with respect to μ. Since

$$(-\partial_\nu + i\zeta)Z_l^R|_{\Gamma^R} = (-\partial_\nu + i\zeta)(w_l^+ + \sum_{j=1}^M a_j^0(R)w_j^-)|_{\Gamma^R},$$

from (5.4.9) it follows, that

$$\| -\partial_\nu(Z_l^R - (w_l^+ + \sum_{j=1}^M a_j^0(R)w_j^-)); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty. \tag{5.4.11}$$

Recall, that for $\mu > \tau$, the waves $w_l^{\pm}$ are bounded functions; for $\mu < \tau$ the waves $w_l^{\pm}$ with $L < l \leqslant M$ defined by (5.3.2) grow at infinity as $O(e^{\sqrt{\tau-\mu}|x|})$ and as $O(|x|)$ for $\mu = \tau$. We use (5.4.10) and (5.4.11) to reduce (5.4.9) to the form

$$(-\partial_\nu \varphi_l, \varphi_l)_{\Gamma^R} - (\varphi_l, -\partial_\nu \varphi_l)_{\Gamma^R} = |a^0(R)|\, O(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)R}),$$

where $\varphi_l = w_l^+ + \sum a_j^0(R) w_j^-$; as before, $\sqrt{\tau-\mu} = i\sqrt{\mu-\tau}$ for $\mu > \tau$, ε being an arbitrary small positive number. In view of (5.1.12) and (5.1.13), the left-hand-side is equal to $-i(1 - \sum |a_j^0(R)|^2)$. Therefore,

$$|a^0(R)|^2 = 1 + o(|a^0(R)|),$$

which leads to $|a^0(R)| = 1 + o(1)$. Looking over all elements of the covering, we obtain the desired estimate everywhere on $[\mu', \mu'']$. □

Theorem 5.4.4. *For all $R \geqslant R_0$, where R_0 is a sufficiently large number, and for all $\mu \in [\mu', \mu''] \subset (\tau', \tau'')$ there exists a unique minimizer $a^0(R, \mu) = (a_1^0(R, \mu), \dots, a_M^0(R, \mu))$ of the functional $\mathcal{J}_l^R$ in* (5.4.2). *The estimates*

$$\sum_{k=1}^{M} |\mathcal{S}_{jk}(\mu) - a_k^0(R, \mu)| \leqslant C e^{-\Lambda R} \tag{5.4.12}$$

hold for all $R \geqslant R_0$, $\mu \in [\mu', \mu'']$, and $0 < \Lambda < \min_{\mu \in [\mu', \mu'']} \operatorname{Re}(\sqrt{\tau''-\mu} - \sqrt{\tau-\mu})$, where $\sqrt{\tau-\mu} = i\sqrt{\mu-\tau}$ for $\mu > \tau$ and the constant $C = C(\Lambda)$ is independent of R and μ.

Proof. As in the proof of the previous assertion, we assume that μ runs through an interval I_p of the covering of $[\mu', \mu'']$ in Lemma 5.2.4, so the number γ in (5.2.8), (5.4.10) and (5.4.11) is independent of μ, while $\max_{\mu \in I_p} \operatorname{Re} \sqrt{\tau-\mu} < \gamma < \min_{\mu \in I_p} \operatorname{Re} \sqrt{\tau''-\mu}$.

Let Y_l^R be a solution to problem (5.4.4), where a_j, $j = 1, \dots, M$, are taken to be the entries $\mathcal{S}_{lj}$ of the scattering matrix $\mathcal{S}$, and let Z_l^R and $(a_1^0(R, \mu), \dots, a_M^0(R, \mu))$ be the same as in Proposition 5.4.3. We substitute $u = v = U_l := Y_l - Z_l^R$ into the Green formula. Since U_l satisfies the first two equations in (5.4.4), we have

$$(-\partial_\nu U_l, U_l)_{\Gamma^R} - (U_l, -\partial_\nu U_l)_{\Gamma^R} = 0. \tag{5.4.13}$$

Setting

$$\varphi_l = w_l^+ + \sum_{j=1}^{M} a_j^0(R, \mu) w_j^-, \quad \psi_l = w_l^+ + \sum_{j=1}^{M} S_{lj}(\mu) w_j^-, \tag{5.4.14}$$

we write down U_l in the form

$$U_l = Y_l - Z_l^R = (Y_l - \psi_l) + (\psi_l - \varphi_l) + (\varphi_l - Z_l^R).$$

Note that $(Y_l - \psi_l)|_{\Gamma^R} = O(e^{-\gamma R})$ by virtue (5.2.8). Moreover, by Proposition 5.4.3, the components of the minimizer $a_j(R,\mu)$ are uniformly bounded. In view of (5.4.10)and (5.4.11), this leads from (5.4.13) to the relation

$$(-\partial_\nu(\psi_l - \varphi_l), (\psi_l - \varphi_l))_{\Gamma^R} - ((\psi_l - \varphi_l), -\partial_\nu(\psi_l - \varphi_l))_{\Gamma^R} = O(e^{-(\gamma - \sqrt{\tau-\mu} - \varepsilon)R}), \tag{5.4.15}$$

where ε is an arbitrary small positive number. Straightforward calculation shows that the left-hand-side is equal to $i\sum_{j=1}^{M} |a_j^0(R,\mu) - \mathcal{S}_{lj}(\mu)|^2$ (it suffices to use (5.4.13), (5.1.12), and (5.1.13)). Hence

$$\sum_{j=1}^{M} |a_j^0(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O(e^{-(\gamma - \sqrt{\tau-\mu} - \varepsilon)R})$$

and we arrive at (5.4.12) for $\mu \in I_p$ and $\Lambda \leqslant \min_{\mu \in I_p}(\gamma - \operatorname{Re}\sqrt{\tau - \mu} - \varepsilon)/2$.

We now prove that the inequality

$$\sum_{j=1}^{M} |a_j(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O(e^{-2(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1-2^{-N})R}) \tag{5.4.16}$$

holds for any positive integer N. For $N = 1$ the inequality has been obtained, so it suffices to derive from (5.4.16) the same estimate with $N + 1$ instead of N. We have

$$\psi_l - \varphi_l = \sum_{j=1}^{M} (\mathcal{S}_{lj}(\mu) - a_j(R,\mu)) w_j^- = O(e^{-[(\gamma - \sqrt{\tau-\mu})(1-2^{-N}) - \sqrt{\tau-\mu} - \varepsilon]R}).$$

So we can go from (5.4.13) to (5.4.15) with right-hand-side replaced by $O(e^z)$ with $z = -[(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N}) - \sqrt{\tau-\mu} - \varepsilon + \gamma]R = -2(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N-1})R$. Again calculating the left hand-side of (5.4.15), we obtain

$$\sum_{j=1}^{M} |a_j(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O(e^{-2(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1-2^{-N-1})R}).$$

Hence we have proved (5.4.12) for any positive integer N with $\mu \in I_p$ and $\Lambda \leqslant \min_{\mu \in I_p}(\gamma - \operatorname{Re}\sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N})$. Increasing N and decreasing ε, we can take Λ to be arbitrarily close to $\gamma - \operatorname{Re}\sqrt{\tau-\mu}$. Looking over all intervals I_p of the covering, we obtain the needed estimate everywhere on $[\mu', \mu'']$ with $\Lambda < \min_{\mu \in [\mu',\mu'']}(\gamma(\mu) - \operatorname{Re}\sqrt{\tau-\mu})$. Finally, for the difference between $\gamma(\mu)$ and $\operatorname{Re}\sqrt{\tau'' - \mu}$ to be so small as needed, it suffices to refine the covering of $[\mu', \mu'']$. □

In a neighborhood of the threshold τ, the matrix $\mathcal{S}(\mu)$ can be calculated by the method presented in this paper. Since the limits of $S(\mu)$ as $\mu \to \tau \pm 0$ are finite, the connection between $S(\mu)$ and $\mathcal{S}(\mu)$ allows to calculate $S(\mu)$ for μ in vicinity of τ.

5.5 Examples of the method application

The method for approximating the waveguide scattering matrix, described in Chapters 3 and 5, has been successfully implemented in real-life computational applications. A series of papers [2] - [5], [17] is devoted to the resonant tunneling in quantum waveguides. The authors consider the Dirichlet problem for the Helmholtz equation or for the Pauli equation in cylindrical waveguides, having two narrows of small diameter. The narrows play the role of effective potential barriers and the waveguide part between the narrows plays the role of a resonator. So, there arise conditions for resonant tunneling of electrons. The phenomenon consists in the fact that, for electron with energy E (E stands for the spectral parameter), the probability $T(E)$ of passing from one part of the waveguide to the other through the resonator has a sharp peak at a "resonant energy" E_{res}. The probability $T(E)$ is called a transmission coefficient.

In [2] - [5] the method, analogous to that in Chapter 3, was applied for computing the transmission coefficient $T(E)$. With a kind permission of the authors we reproduce here some plots computed by means of the method. Suppose E belonging to an interval between the first and the second threshold. Then the scattering matrix $S(E)$ is of size 2×2 and the transmission coefficient $T_1(E)$ is equal to $|S_{12}(E)|^2$. Its dependence on E is presented in the FIG. 4.

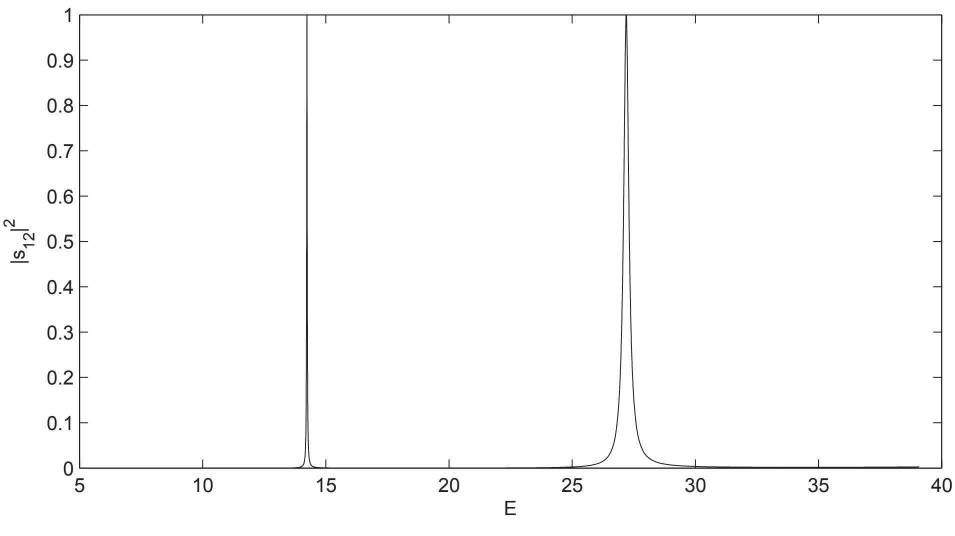


FIGURE 4 The transmission coefficient dependence on the spectral parameter E belonging to an interval between first and second thresholds. The plot is computed by the method, analogous to that in Chapter 3 [2], [4], [5].

If E is higher than the second threshold, then there arise several incoming and outgoing waves and we deal with multichannel scattering. Let $u_1^-, \dots, u_n^-$ be the outgoing waves propagating in the first cylindrical end and $u_{n+1}^-, \dots, u_{2n}^-$ be the outgoing waves propagating in the second cylindrical end, while $n = n(E)$. Then the total transmission coefficient of the incoming wave u_1^+ is equal to

$$T_1(E) = \sum_{j=n+1}^{2n} |S_{1j}(E)|^2. \tag{5.5.1}$$

Its dependence on E is presented in the FIG. 5.

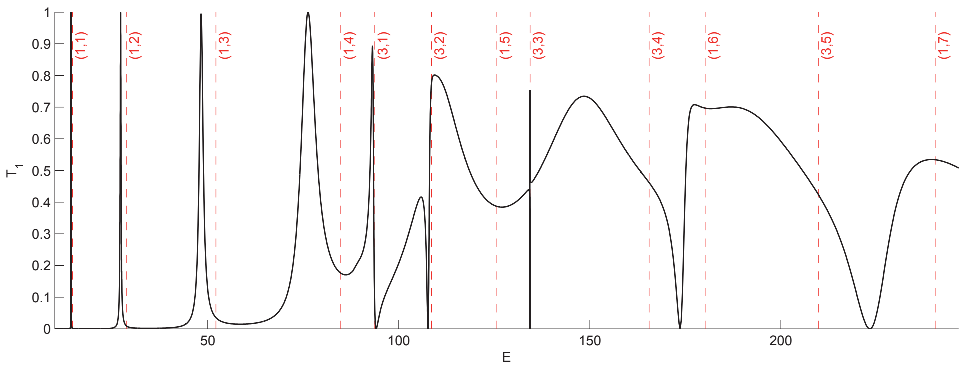
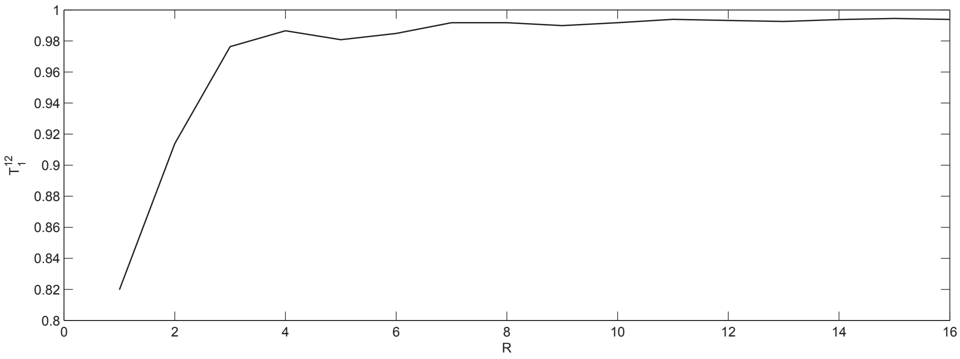
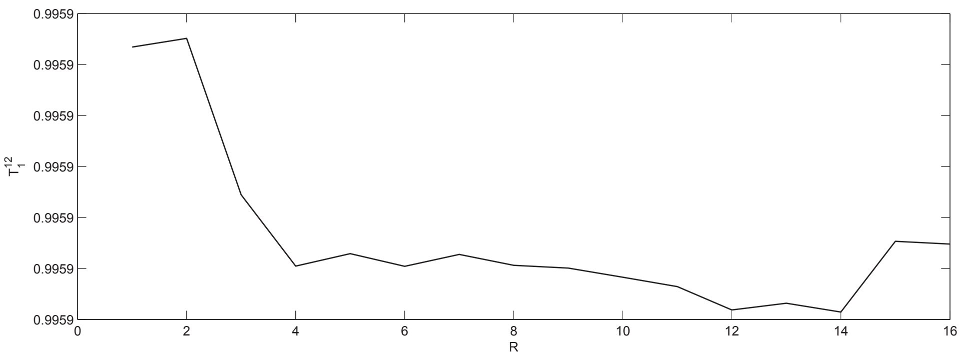


FIGURE 5 The total transmission coefficient dependence on the spectral parameter E in the case of multichannel scattering. The plot is computed by the method, analogous to that in Chapter 3 [4], [5].

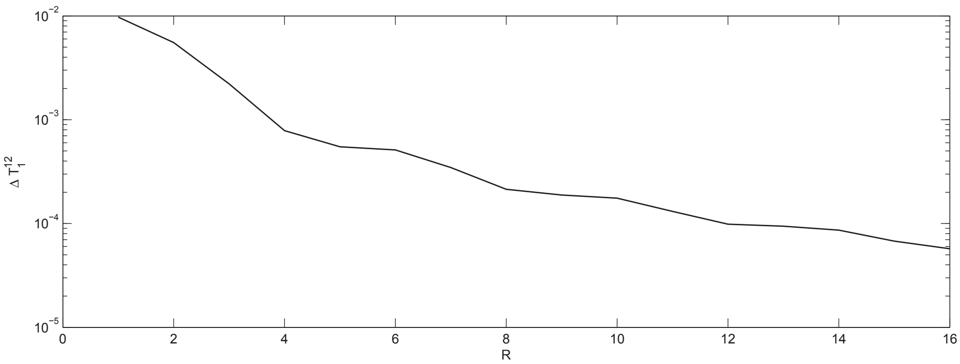
In [17] the scattering matrix is computed near thresholds. To this end the authors apply both the “non-threshold” method of Chapter 3 and the “threshold” method of Chapter 5 and compare the obtained results. It turns out that, when E is close to the threshold τ and $E > \tau$, the methods show good agreement in accuracy. When E is close to the threshold and $E < \tau$, the method of Chapter 5 converges much faster as R tends to infinity, than the method of Chapter 3 (see FIG. 6), as it was expected from theoretical prerequisites. But the advantage in accuracy of the “threshold” method becomes negligible for energies not too far from the threshold $\tau - E = 1 \simeq 0.03\tau$ (see FIG. 7). Far from thresholds the “non-threshold” method of Chapter 3 is preferrable, because the number of auxiliary boundary value problems to be solved is less than that, needed for the method of Chapter 5, and the accuracy is the same.



(1)

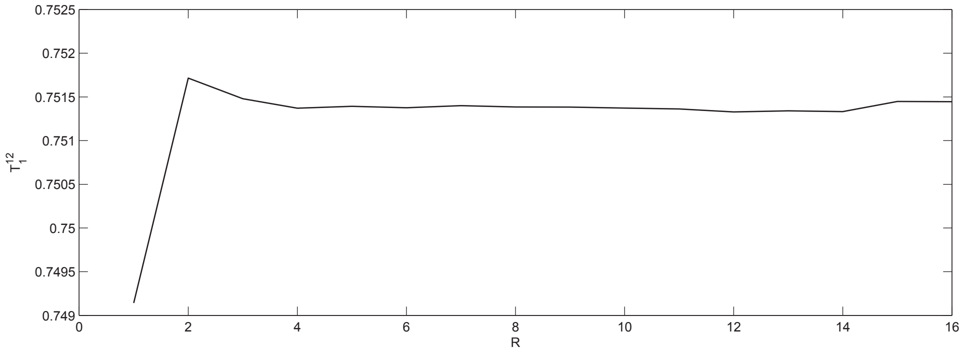


(2)

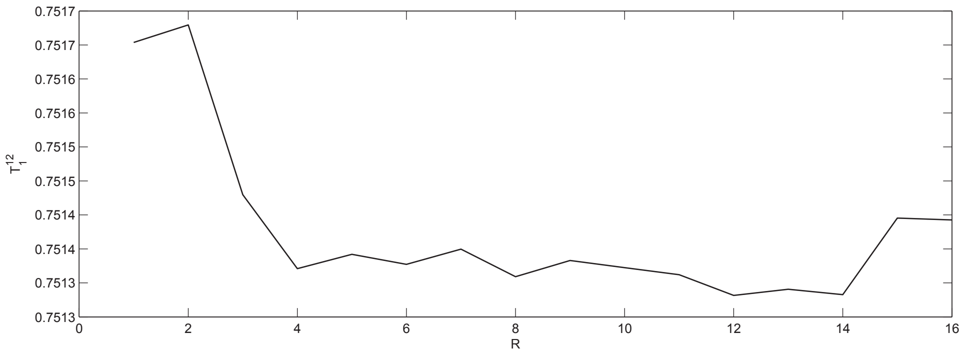


(3)

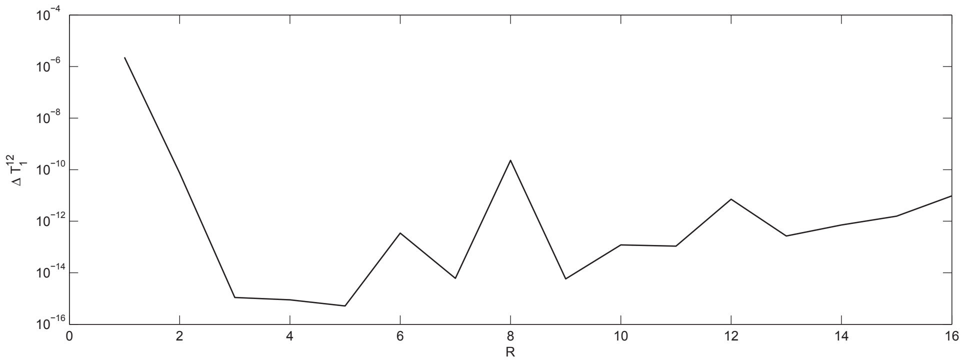
FIGURE 6 A comparison of convergence rate for the method of Chapter 3 and the method of Chapter 5. The computed transmission coefficient $T_{12}^1 := |S_{12}|^2$ (1) using the method of Chapter 3, (2) using the method of Chapter 5, (3) the computed difference for $\tau - E = 10^{-7}$ [17].



(1)



(2)



(3)

FIGURE 7 A comparison of convergence rate for the method of Chapter 3 and the method of Chapter 5. The computed transmission coefficient $T_{12}^1 := |S_{12}|^2$ (1) using the method of Chapter 3, (2) using the method of Chapter 5, (3) the computed difference for $\tau - E = 1$ [17].

6 CONCLUSIONS

In the thesis, we use an original method that allows us to extend significantly the class of electromagnetic waveguides, admitting mathematically exact investigation. Namely, we associate with the Maxwell system an elliptic boundary value problem. The problem is investigated by means of waveguide theory for elliptic systems. Finally, we derive the information about the Maxwell system from that, obtained for the elliptic one.

The elliptization of the Maxwell system provides all the advantages of an elliptic case, in particular, possibility of localization, a freedom in choosing waveguide geometry and filling medium. In the present thesis, we relax the restrictions of the geometry. We consider waveguides having finitely many cylindrical outlets to infinity. A waveguide is supposed to be empty, i.e. dielectric and magnetic permittivity are identity matrices, and the waveguide boundary is supposed to be perfectly conductive. The proposed method can be used to advance the developed theory, particularly, to study waveguides with non-homogeneous filling medium, with dissipation etc.

Main results of the work:

1. We propose and justify a well-posed problem statement with intrinsic radiation conditions, the "radiation principle". The intrinsic radiation conditions mean, that the principal part of solution asymptotics at infinity consists of outgoing waves only. A problem with the radiation conditions has a unique solution that depends continuously on right-hand-side.
2. We describe the waveguide continuous spectrum, introduce a scattering matrix, defined on the continuous spectrum, and study its properties. The matrix is unitary and has a finite size that is called the continuous spectrum multiplicity. The continuous spectrum multiplicity depends on the spectral parameter, the function is even, piecewise constant, and increasing on the positive semiaxis; the discontinuity points of the function are called the thresholds. The positive thresholds form a sequence increasing to infinity.
3. We propose and justify a method for approximating the scattering matrix under condition, that the spectral parameter belongs to an interval, containing no thresholds. A minimizer of a quadratic functional serves as an

approximation to the scattering matrix row. To construct the functional, we solve an auxiliary problem in a bounded domain, obtained from the waveguide by cutting off the cylindrical ends. The minimizer tends to the scattering matrix row at an exponential rate as the domain size tends to infinity.

We also investigate the threshold behaviour of the waveguide scattering matrix for the Helmholtz equation with Dirichlet boundary conditions and obtain the following results:

4. We prove that the scattering matrix depends analytically on the spectral parameter in any continuous spectrum interval, containing no thresholds. In a neighbourhood of a threshold, we introduce an augmented scattering matrix that keeps its size and depends analytically on the spectral parameter in the neighbourhood. Threshold behaviour of the (ordinary) scattering matrix is described in terms of the augmented one. Particularly, we show the scattering matrix to have finite left and right one-sided limits at thresholds.
5. We propose and justify a method for approximating the augmented scattering matrix in the neighbourhood of the threshold. The ordinary scattering matrix can be computed by way of computing the augmented scattering matrix.
6. Numerical simulations show that, in a neighbourhood of a threshold, the method of Chapter 5 converges much faster, than the method of Chapter 3. Far from the thresholds, the method of Chapter 3 is preferrable, since the methods have the same accuracy, but the method of Chapter 5 needs a larger amount of computational operations.

The results of 4 and 5 on the Helmholtz equation are generalized for the Maxwell system as well, and a paper on the topic is being prepared for the press.

YHTEENVETO (FINNISH SUMMARY)

Tutkimme sähkömagneettisia aallonjohtimia, joissa on useita sylinterimäisiä päitä. Aallonjohdin oletetaan tyhjäksi ja sen reuna täydellisen johtavaksi. Tutkimme varausten ja sähkövirran luomaa sähkömagneettista kenttää aallonjohtimen läheisyydessä. Sähkökenttä määräytyy Maxwellin yhtälöiden mukaan. Tarkasteltavassa systeemissä oletetaan johtavat reunaehdot ja säteilyehdot äärettömyydessä. Todistamme matemaattisen ongelman olevan hyvin asetettu. Sähkömagneettisen kentän värähtelyä aallonjohtimessa esittää erityinen sirontamatriisi. Tutkimuksessa esitellään sirontamatriisi kaikille spektraaliparametreille k aallonjohtimen jatkuvassa spektrissä ja tutkitaan sen ominaisuuksia. Lisäksi esitämme numeerisen menetelmän, jolla sirontamatriisia voidaan approksimoida. Menetelmä ei ole riippuvainen aallonjohtimen ominaisarvoista.

Väitöskirjan tulokset laajentavat sähkömagneettisen aallonjohtimen teoriaa. Väitöskirjassa esitellyt asymptoottinen analyysi ja numeeriset menetelmät mahdollistavat esimerkiksi monimutkaisten aallonjohdinten resonaattoreiden ja SHF jakajien tarkastelun. Matemaattinen tarkastelu perustuu ylimääräytyvän Maxwellin systeemin laajentamiseen elliptiseksi ongelmaksi.

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