

*Reports of the Department of Mathematical Information Technology*  
*Series B. Scientific Computing*  
**No. B. 11/2007**

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**Generalized Maxwell Equations**  
**in**  
**Exterior Domains V:**  
**Low Frequency Asymptotics**

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ISBN 978-951-39-2836-0  
ISSN 1456-436X

# Generalized Maxwell Equations in Exterior Domains V: Low Frequency Asymptotics\*

Dirk Pauly<sup>†</sup>

## Abstract

We continue the study of the operator of generalized Maxwell equations and completely discover the behavior of the solutions of the time-harmonic equations as the frequency tends to zero. Thereby we identify degenerate operators in terms of special ‘polynomially growing’ solutions of a corresponding static problem, which must be added to the ‘usual’ Neumann series in order to describe the low frequency asymptotic adequately.

**Key words:** Maxwell’s equations, low frequency asymptotics, exterior boundary value problems, variable coefficients, electro-magneto static, Hodge-Helmholtz decompositions, radiating solutions, asymptotic expansions, spherical harmonics, Hankel functions

**AMS MSC-classifications:** 35Q60, 78A25, 78A30

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\*This research was supported by the *Deutsche Forschungsgemeinschaft* via the project ‘We 2394: Untersuchungen der Spektralschar verallgemeinerter Maxwell-Operatoren in unbeschränkten Gebieten’ and by the *Department of Mathematical Information Technology* of the University of Jyväskylä, where the author spent a sabbatical of three month during spring 2007 as a postdoc fellow.

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## 1 Introduction and main results

We continue (and finish) our studies started in [25] and [26] on the low frequency behaviour of the solution operator

$$\mathcal{L}_\omega : \begin{array}{l} L_s^{2,q,q+1}(\Omega) \longrightarrow L_t^{2,q,q+1}(\Omega) \\ (F, G) \longmapsto (E, H) \end{array} , \quad \forall s, -t > 1/2$$

of the generalized<sup>1</sup> time-harmonic Maxwell equation

$$\operatorname{div} H + i\omega\varepsilon E = F \quad , \quad \operatorname{rot} E + i\omega\mu H = G$$

shortly written as

$$(M + i\omega\Lambda)(E, H) = (F, G) \tag{1.1}$$

with homogeneous (Dirichlet) electric boundary condition

$$\Gamma_t E = 0$$

(modeling a perfect conductor) in an exterior domain  $\Omega \subset \mathbb{R}^N$  of dimension  $N \in \mathbb{N}$  (i.e. a connected open set with compact complement), where  $\Gamma_t := \iota^*$  denotes the tangential trace and  $\iota : \partial\Omega \hookrightarrow \bar{\Omega}$  the natural embedding of the boundary. We note that the range of  $\mathcal{L}_\omega$  is even contained in  $\mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)$ . Here  $(E, H), (F, G)$  are pairs of alternating differential forms of rank  $q$  resp.  $q + 1$  and  $M = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{rot} & 0 \end{bmatrix}$ , where following Weyl [55] and to remind of the electro-magnetic background  $\operatorname{rot}$  resp.  $\operatorname{div}$  denotes the exterior derivative resp. co-derivative

$$\operatorname{rot} = d \quad \text{resp.} \quad \operatorname{div} = (-1)^{qN} * d^*$$

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<sup>1</sup>Here ‘generalized’ means the framework of alternating differential forms.

on  $q$ - resp.  $(q+1)$ -forms and  $*$  the usual Hodge star isomorphism. Furthermore, the frequencies  $\omega$  will be taken from the upper half plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z \geq 0\}$ . We define

$$L_s^{2,q,p}(\Omega) := L_s^{2,q}(\Omega) \times L_s^{2,p}(\Omega) \quad , \quad q, p \in \mathbb{Z} \quad , \quad s \in \mathbb{R} \quad ,$$

where  $L_s^{2,q}(\Omega)$  is the Hilbert space of all differential forms  $E \in L_{\text{loc}}^{2,q}(\Omega)$  satisfying  $\rho^s E \in L^{2,q}(\Omega)^2$ , equipped with the scalar product

$$\langle E, H \rangle_{L_s^{2,q}(\Omega)} := \int_{\Omega} \rho^{2s} E \wedge * \bar{H} \quad .$$

Here  $\wedge$  denotes the exterior product and the bar complex conjugation.

We are going to model inhomogeneous, anisotropic and nonsmooth media by  $\Lambda = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}$  and linear  $L^\infty$ -transformations  $\varepsilon$  and  $\mu$  on  $q$ - and  $(q+1)$ -forms, i.e. dielectricity and permeability. Moreover, our right hand sides  $(F, G)$  from  $L_s^{2,q,q+1}(\Omega)$  under consideration do not have to be necessarily compactly supported.

The study of wave scattering at low frequencies was pioneered by Lord Rayleigh [36]. His contributions provide the foundation on which almost all subsequent work is based. Low frequency asymptotics for Maxwell's boundary value problem have been given, for instance, by Müller and Niemeyer [23], Stevenson [37], Kleinman [11], Werner [49, 50, 51, 52, 54], Kress [12], Ramm [33], Kriegsmann and Reiss [13], Ramm, Weaver, Weck and Witsch [35], Athanasiadis, Costakis and Stratis [5] as well as by Picard [30] and Weck and Witsch [40]. We also should mention the book of Dassios and Kleinman [7]. Ramm and Somersalo [34] and Lassas [15] considered the low frequency limit from the point of view of inverse problems.

In none of the works cited above the calculation of the higher order terms in a suitable expansion in terms of the frequency is analyzed. Ammari and Nédélec proved in [2, 4] such expansions. They reformulated the exterior boundary value problem in a truncated bounded domain using an 'exterior electromagnetic operator', called by Monk [19] the 'electric to magnetic Calderon operator' or by Colton and Kress [6] the 'electric to magnetic boundary component map', which is the counterpart of the Dirichlet to Neumann operator for Helmholtz' equation. Unfortunately due to the asymptotic expansion of the exterior electromagnetic operator their method requires exact nonlocal radiation conditions, which leads to nonlocal boundary conditions on a sphere. Thus, it is not possible to identify the expected Neumann series part of the corresponding static solution operator in the solution. Furthermore, they discuss only the case

$$\left( \begin{bmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{bmatrix} - i\omega(\text{Id} + \hat{\Lambda}) \right) (E, H) = -i\omega \hat{\Lambda}(F, G) \quad ,$$

where  $(F, G)$  is the time independent part of a time-harmonic incoming wave and  $\hat{\Lambda}$  some compactly supported perturbation.

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<sup>2</sup>Here  $\rho(r) := (1 + r^2)^{1/2}$  and  $r(x) = |x|$  for  $x \in \mathbb{R}^N$ .

To overcome these problems and limitations and to identify the usual Neumann series part of a static solution operator Weck and Witsch started in [41, 42, 43, 44] an detailed analysis of the low frequency behavior of solutions of Helmholtz' equation. In [45] their new method was completed. They identified degenerate correction operators in terms of special 'polynomially growing' solutions to a corresponding static problem, which must be added to the 'usual' Neumann series in order to describe the low frequency asymptotic adequately. With the help of [46], where a calculus for spherical coordinates suited for differential forms has been established, they were able to apply and extend their methods to the case of generalized linear elasticity.

Now in this paper we transfer their 'Weck-Witsch-method' to the case of generalized Maxwell equations. Hereby we again utilize [46] as an important tool and also the preliminary works [25, 26, 27]. Thus, throughout this paper we will use the notations introduced in these papers.

All these low frequency investigations are not only motivated by the problem in its own right, but also by its applications to the large time behavior of solutions to the initial boundary value problems for the (generalized) wave equation and to the existence proofs for nonlinear (generalized) wave equations. In this context we may refer to Eidus' principle of limiting amplitude [9] and, for instance, the papers of Werner [53] as well as Morgenröther and Werner [20, 21].

In [25] we studied the time-harmonic solutions of (1.1). Since the linear operator

$$\begin{aligned} \mathcal{M} : \mathring{\mathbf{R}}^q(\Omega) \times \mathbf{D}^{q+1}(\Omega) \subset {}_{\Lambda}L^{2,q,q+1}(\Omega) &\longrightarrow {}_{\Lambda}L^{2,q,q+1}(\Omega) \\ (E, H) &\longmapsto i\Lambda^{-1}M(E, H) \end{aligned}$$

is selfadjoint, we were able to obtain radiation solutions  $(E, H)$  for nonvanishing real frequencies and right hand sides  $(F, G) \in L_{>\frac{1}{2}}^{2,q,q+1}(\Omega)$  by means of Eidus' limiting absorption principle [8] (approaching from the upper half plane  $\mathbb{C}_+$ ). These solutions are elements of  $L_{<-\frac{1}{2}}^{2,q,q+1}(\Omega)$  and satisfy the Maxwell radiation condition, i.e.

$$(\tilde{S} + \text{Id})(E, H) \in L_{>-\frac{1}{2}}^{2,q,q+1}(\Omega) \quad , \quad \tilde{S} = \begin{bmatrix} 0 & (-1)^{qN} * \tilde{R}^* \\ \tilde{R} & 0 \end{bmatrix} \quad , \quad \tilde{R} = dr \wedge$$

(see [25, section 2] for details). In other words the resolvent  $(\mathcal{M} - \omega)^{-1}$  of  $\mathcal{M}$  and hence also  $\mathcal{L}_\omega = i(\mathcal{M} - \omega)^{-1}\Lambda^{-1}$  may be extended continuously to the real axis. Then using the fundamental solution of Helmholtz' equation in the whole space  $\mathbb{R}^N$  we showed that eventually eigenvalues of  $\mathcal{M}$  do not accumulate even at  $\omega = 0$ . This makes  $\mathcal{L}_\omega$  well defined on the whole of  $L_s^{2,q,q+1}(\Omega)$  for small frequencies  $0 \neq \omega \in \mathbb{C}_+$ . Finally we proved in [25, Corollary 4.5] that  $\mathcal{L}_\omega$  restricted to the closed subspace  $\text{Reg}_s^{q,0}(\Omega)$ <sup>3</sup> of  $L_s^{2,q,q+1}(\Omega)$  converges to the static solution operator  $\mathcal{L}_0$  as  $\omega$  tends to zero in the norm of bounded linear operators from  $\text{Reg}_s^{q,0}(\Omega)$  to  $\mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)$  for all  $s \in (1/2, N/2)$  and  $t < s - (N + 1)/2$ . Here  $\text{Reg}_s^{q,0}(\Omega)$  consists of solenoidal resp.

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<sup>3</sup>See Definition 2.1

irrotational forms and

$$\mathcal{L}_0 : \text{Reg}^{q,0}(\Omega) \longrightarrow (\mathring{\text{R}}_{-1}^q(\Omega) \times \text{D}_{-1}^{q+1}(\Omega)) \cap \Lambda^{-1} \text{Reg}_{-1}^{q,0}(\Omega) \quad , \\ (F, G) \longmapsto (E, H)$$

where  $(E, H) \in \text{L}_{-1}^{2,q,q+1}(\Omega)$  is the unique solution of the (decoupled) static Maxwell problem

$$\begin{aligned} \text{rot } E = G \quad , \quad \text{div } \varepsilon E = 0 \quad , \quad \iota^* E = 0 \quad , \quad \varepsilon E \perp \mathring{\text{B}}^q(\Omega) \quad , \\ \text{div } H = F \quad , \quad \text{rot } \mu H = 0 \quad , \quad \iota^* \mu H = 0 \quad , \quad \mu H \perp \text{B}^{q+1}(\Omega) \quad , \end{aligned}$$

which may be written shortly as

$$(E, H) \in (\mathring{\text{R}}_{-1}^q(\Omega) \times \text{D}_{-1}^{q+1}(\Omega)) \cap \Lambda^{-1} \text{Reg}_{-1}^{q,0}(\Omega) \quad \wedge \quad M(E, H) = (F, G) \quad .$$

The special forms from  $\mathring{\text{B}}^q(\Omega)$  resp.  $\text{B}^{q+1}(\Omega)$  possess compact resp. bounded supports in  $\Omega$  and they play the role of the Dirichlet forms  ${}_\varepsilon \mathcal{H}^q(\Omega)$  resp.  ${}_{\mu^{-1}} \mathcal{H}^{q+1}(\Omega)$ , where  ${}_\nu \mathcal{H}^q(\Omega) = {}_\nu \mathcal{H}_0^q(\Omega)$  and for  $t \in \mathbb{R}$  (in classical or strong terms)

$${}_\nu \mathcal{H}_t^q(\Omega) = \{ E \in \text{L}_t^{2,q}(\Omega) : \text{rot } E = 0, \text{div } \nu E = 0, \iota^* E = 0 \} \quad .$$

Due to the existence of a nontrivial kernel of the Maxwell operator these (or other) orthogonality constraints are necessary.

In the bounded domain case it is just an easy exercise to show that  $\mathcal{L}_\omega$  is approximated by Neumann's series of  $\mathcal{L}_0$  or  $\mathcal{L} = \Lambda \mathcal{L}_0$  for small frequencies  $\omega$ , i.e.

$$\mathcal{L}_\omega = -(-i\omega)^{-1} \Pi + \sum_{j=0}^{\infty} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}} \quad , \quad (1.2)$$

where  $\Pi$  and  $\Pi_{\text{reg}} = \text{Id} - \Lambda \Pi$  are projections onto the kernel of  $M$  and its orthogonal complement in  $\text{L}^{2,q,q+1}(\Omega)$ . In the case of an exterior domain this low frequency asymptotic holds no longer true, because due to Poincaré's estimate for Maxwell equations the static solution operator maps data from a polynomially weighted Sobolev space to solutions belonging to a less weighted Sobolev space. So a priori it is not clear, in which way one may define higher powers of a static solution operator.

In [26] we took care of some electro-magneto static problems and were able to prove that an (not obvious and relatively complicated) iteration process of a suitable static solution operator  $\mathcal{L}$  still holds true [26, Theorem 5.10]. This gives meaning to the powers  $\mathcal{L}^j$  of  $\mathcal{L}$  as continuous linear operators on subspaces of  $\text{Reg}_{\text{loc}}^{q,0}(\Omega)$  even for exterior domains. As a byproduct we proved a generalized spherical harmonics expansion suited for Maxwell equations, which will be used frequently in this paper as well.

Now in this paper, which is the third and last one of our little series, we analyze the solution formula (1.2) and try to give meaning to it in exterior domains in the sense of an asymptotic expansion

$$\mathcal{L}_\omega + (-i\omega)^{-1}\Pi - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \quad , \quad \mathbf{J} \in \mathbb{N}_0 \quad . \quad (1.3)$$

Thereby we follow closely the ideas of Weck and Witsch [45] and [47, 48]. Due to our exterior boundary value problems there arise three major complications:

1. With growing  $\mathbf{J}$  we have to use stronger norms for the data and obtain estimates in weaker norms for the solutions.
2. As  $\Pi$  and  $\Pi_{\text{reg}}$  already indicate we need weighted Hodge-Helmholtz decompositions of  $L_s^{2,q,q+1}(\Omega)$  respecting inhomogeneities  $\Lambda$ . In [27] we presented results, which will meet our needs. In fact we proved topological direct decompositions

$$L_s^{2,q,q+1}(\Omega) = (\Lambda \text{Tri}_s^q(\Omega) \dot{+} \text{Reg}_s^{q,-1}(\Omega)) \cap L_s^{2,q,q+1}(\Omega) \quad ,$$

where  $\text{Tri}_s^q(\Omega) = \Pi L_s^{2,q,q+1}(\Omega)$  and  $\text{Reg}_s^{q,-1}(\Omega) = \Pi_{\text{reg}} L_s^{2,q,q+1}(\Omega)$ . We note

$$\begin{aligned} \text{Tri}_s^q(\Omega) &\subset {}_0\mathring{R}_t^q(\Omega) \times {}_0D_t^{q+1}(\Omega) \quad , \\ \text{Reg}_s^{q,-1}(\Omega) &\subset \text{Reg}_t^{q,0}(\Omega) \subset {}_0D_t^q(\Omega) \times {}_0\mathring{R}_t^{q+1}(\Omega) \end{aligned}$$

are only subspaces of  $L_t^{2,q,q+1}(\Omega)$  with  $t \leq s$  and  $t < N/2$  and even not of  $L_s^{2,q,q+1}(\Omega)$  if  $s \geq N/2$ . (See Lemma 3.8, [27])

3. We have to correct (1.3) by special operators  $\Gamma_j$ .

More precisely for  $\mathbf{J} \in \mathbb{N}_0$  and  $s, -t > 1/2$  we shall look for asymptotic estimates like

$$\begin{aligned} &\| \mathcal{L}_\omega(F, G) + (-i\omega)^{-1}\Pi(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}}(F, G) \\ &- \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^{j+N-1} \Gamma_j(F, G) \|_{L_t^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \| (F, G) \|_{L_s^{2,q,q+1}(\Omega)} \quad . \end{aligned} \quad (1.4)$$

Hereby the  $\mathcal{O}$ -symbols are always meant for  $\omega \rightarrow 0$  and uniformly with respect to  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $(F, G)$ , where  $\mathbb{C}_{+,\hat{\omega}} := \{\omega \in \mathbb{C}_+ : |\omega| \leq \hat{\omega}\}$  for some  $\hat{\omega} > 0$ .



Throughout this paper we will make the following

**General Assumptions:**

- We restrict our considerations to ranks of forms

$$1 \leq q \leq N - 2$$

and odd space dimensions  $\mathbb{N} \ni N \geq 3$ . Hence, of course the most interesting case of the classical Maxwell equations,  $N = 3$  and  $q = 1$ , is covered. The treatment of even dimensions (especially  $N = 2$ ) would increase the complexity of our calculations considerably due to the appearance of logarithmic terms in the fundamental solution for Helmholtz' equation (Hankel functions). But there is no reasonable doubt that our methods can be used to obtain similar results in any even dimension as well.

- We fix a radius  $r_0 > 0$  and some radii  $r_n := 2^n r_0$ ,  $n \in \mathbb{N}$ , such that

$$\mathbb{R}^N \setminus \Omega \subset U_{r_0} \quad .$$

Moreover, we remind of the cut-off functions  $\eta$ ,  $\hat{\eta}$  and  $\eta$  from [14, (3.1), (3.2), (3.3)].  $\eta$  satisfies  $\text{supp } \eta = \overline{A_{r_1}}$ ,  $\text{supp } \nabla \eta = \overline{A_{r_1} \cap U_{r_2}}$ . Here for  $r > 0$

$$U_r := \{x \in \mathbb{R}^N : |x| < r\} \quad , \quad A_r := \{x \in \mathbb{R}^N : |x| > r\} \quad .$$

- For simplicity  $\Omega \subset \mathbb{R}^N$  may have a Lipschitz boundary. In fact,  $\Omega$  only needs to have the Maxwell local compactness property MLCP from [14, Definition 3.1, Remark 3.2], i.e. the inclusions

$$\mathring{\mathbf{R}}^q(\Omega) \cap \mathbf{D}^q(\Omega) \hookrightarrow L_{\text{loc}}^{2,q}(\overline{\Omega})$$

have to be compact for all  $q$ , as well as the static Maxwell property SMP from [26, section 4], i.e. the existence of special forms  $\mathring{\mathbf{B}}^q(\Omega)$  and  $\mathbf{B}^{q+1}(\Omega)$  must be guaranteed. Anyhow, Lipschitz domains possess these properties.

- We assume  $\varepsilon = \text{Id} + \hat{\varepsilon}$  and  $\mu = \text{Id} + \hat{\mu}$  to be  $\tau$ - $C^1$ -admissible (see [25, Definitions 2.1, 2.2]) linear transformations on  $q$ - resp.  $(q + 1)$ -forms with some rate of decay  $\tau > 0$ , which will vary throughout this paper. The greek letter  $\tau$  always stands for the order of decay of the perturbations  $\hat{\varepsilon}$  and  $\hat{\mu}$ . Clearly we then have  $\Lambda = \text{Id} + \hat{\Lambda}$ . Hence, our transformations may have  $L^\infty$ -entries, which are only assumed to be  $C^1$  in  $A_{r_0}$  and asymptotically homogeneous, i.e.  $\partial^\alpha \hat{\Lambda}$  decays like  $r^{-\tau-|\alpha|}$  for all multi-indices  $\alpha$  with  $|\alpha| \leq 1$  with some order  $\tau$  at infinity.

We may describe our results shortly as follows:

1. We shall identify degenerate correction operators  $\Gamma_j$  by a recursion which involves only special solutions  $E_{\sigma,m}^+$  and  $H_{\sigma,n}^+$  as well as their powers

$$E_{\sigma,m}^{+,k} = \mathcal{L}^k \Lambda(E_{\sigma,m}^+, 0) \quad , \quad H_{\sigma,n}^{+,k} = \mathcal{L}^k \Lambda(0, H_{\sigma,n}^+)$$

of  $\mathcal{L}$  of our homogeneous static boundary value problems

$$\begin{aligned} \operatorname{rot} E_{\sigma,m}^+ = 0 \quad , \quad \operatorname{div} \varepsilon E_{\sigma,m}^+ = 0 \quad , \quad \iota^* E_{\sigma,m}^+ = 0 \quad , \quad \varepsilon E_{\sigma,m}^+ \perp \mathring{B}^q(\Omega) \quad , \\ \operatorname{div} H_{\sigma,n}^+ = 0 \quad , \quad \operatorname{rot} \mu H_{\sigma,n}^+ = 0 \quad , \quad \iota^* \mu H_{\sigma,n}^+ = 0 \quad , \quad \mu H_{\sigma,n}^+ \perp B^{q+1}(\Omega) \quad , \end{aligned}$$

but with inhomogeneities at infinity, namely

$$\begin{aligned} E_{\sigma,m}^+ - {}^+D_{\sigma,m}^{q,0} \quad \text{'decays', i.e. belongs to} \quad L_{>-\frac{N}{2}}^{2,q}(\Omega) \quad , \\ H_{\sigma,n}^+ - {}^+R_{\sigma,n}^{q+1,0} \quad \text{'decays', i.e. belongs to} \quad L_{>-\frac{N}{2}}^{2,q+1}(\Omega) \quad , \end{aligned}$$

where the special growing tower forms  ${}^+D_{\sigma,m}^{q,k}$  and  ${}^+R_{\sigma,n}^{q+1,k}$  from [26, section 2] behave like  $r^{k+\sigma}$  at infinity. (See Lemma 2.4, Remark 2.5, Lemma 2.6, Definition 3.4, Definition 3.12)

2. On the 'trivial' subspace  $\operatorname{Tri}_s^q(\Omega)$  the solution operator  $\mathcal{L}_\omega$  behaves like the division by the frequency, i.e.

$$i\omega \mathcal{L}_\omega \Lambda(F, G) = (F, G) \quad , \quad \forall (F, G) \in \operatorname{Tri}_s^q(\Omega) \quad .$$

(See (3.40))

3. We shall identify closed subspaces  $\operatorname{Reg}_s^{q,\mathbf{J}}(\Omega)$  of  $L_s^{2,q,q+1}(\Omega)$  (and of  $\operatorname{Reg}_s^{q,0}(\Omega)$ ), the 'spaces of regular convergence', for whose elements  $(F, G)$  the 'usual' Neumann expansion

$$\left\| \mathcal{L}_\omega(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(F, G) \right\|_{L_t^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^\mathbf{J}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$$

holds true. We are also able to characterize the spaces of regular convergence by orthogonality relations with the aid of the special static solutions  $E_{\sigma,m}^{+,k}$  and  $H_{\sigma,n}^{+,k}$ . (See Theorem 2.3, Lemma 2.15)

4. For  $(F, G) \in \operatorname{Reg}_s^{q,-1}(\Omega)$  we obtain the corrected Neumann expansion

$$\begin{aligned} \left\| \mathcal{L}_\omega(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(F, G) - \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^{j+N-1} \Gamma_j(F, G) \right\|_{L_t^{2,q,q+1}(\Omega)} \\ = \mathcal{O}(|\omega|^\mathbf{J}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \end{aligned}$$

and for general  $(F, G) \in L_s^{2,q,q+1}(\Omega)$  we get the fully corrected Neumann expansion (1.4). (See Main Theorem)

5. Concerning our media  $\Lambda = \text{Id} + \hat{\Lambda}$  we shall distinguish between two kinds of assumptions on our inhomogeneities:

(a)  $\hat{\Lambda}$  has compact support. Then we may always choose  $r_0$ , such that

$$\text{supp } \hat{\Lambda} \subset U_{r_0} \quad .$$

(b)  $\hat{\Lambda}$  ‘decays’ with a rate  $\tau > 0$  at infinity in the sense of  $\Lambda$  is  $\tau$ - $C^1$ -admissible.

In the first case our results will hold for any  $\mathbf{J}$ , whereas in the second case only  $\mathbf{J} \leq \hat{\mathbf{J}}$  with some  $\hat{\mathbf{J}}$  depending on  $\tau$  are allowed.

Due to [26], [46] and originally [18] we have to exclude a discrete set of ‘bad’ weights, namely

$$\mathbb{I} := (\mathbb{N}_0 + N/2) \cup (1 - N/2 - \mathbb{N}_0) = \{n + N/2, 1 - n - N/2 : n \in \mathbb{N}_0\} \quad .$$

Our main result of this paper reads as follows:

**Main Theorem** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in \mathbb{R} \setminus \mathbb{I}$  as well as*

$$\begin{aligned} s &> \mathbf{J} + 1/2 \quad , \\ t &< \min\{N/2 - \mathbf{J} - 2, -1/2\} \quad , \\ \tau &> \max\{(N + 1)/2, s - t\} \quad . \end{aligned}$$

*Then there exists some  $\hat{\omega} > 0$ , such that the asymptotic*

$$\mathcal{L}_\omega + (-i\omega)^{-1}\Pi - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}} - \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^{j+N-1} \Gamma_j = \mathcal{O}(|\omega|^{\mathbf{J}})$$

*holds uniformly with respect to  $\mathbb{C}_{+, \hat{\omega}} \setminus \{0\} \ni \omega \rightarrow 0$  in the norm of bounded linear operators from  $L_s^{2,q,q+1}(\Omega)$  to  $L_t^{2,q,q+1}(\Omega)$ . This asymptotic also holds true for  $\mathbf{J} = 0$ , if we replace the assumptions on  $t$  and  $\tau$  by*

$$t \leq s - (N + 1)/2 \quad , \quad t < -1/2$$

*and*

$$\tau > \max\{(N + 1)/2, s + 1 - N/2\} \quad .$$

**Remark A** *These asymptotics remain valid even in stronger norms. In the norm of bounded linear operators from  $L_s^{2,q,q+1}(\Omega)$  to  $\mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)$  we obtain the estimate if we assume additionally  $t \leq s - (N + 1)/2$  in the case  $\mathbf{J} = 1$ . In the (strongest) norm of bounded linear operators from  $L_s^{2,q,q+1}(\Omega)$  to  $\mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)$  the estimate holds true if we assume additionally  $t < -3/2$  and moreover  $t \leq s - (N + 3)/2$  if  $\mathbf{J} \in \{0, 1\}$ .*

**Remark B** *Using the estimate (ii) instead of (i) from Theorem 2.3 during our considerations in section 3 we would achieve asymptotics with the small  $o$ -symbol instead of  $\mathcal{O}$ . As an example we obtain (for nearly the same  $s, t$  and  $\tau$ )*

$$\mathcal{L}_\omega + (-i\omega)^{-1}\Pi - \sum_{j=0}^{\mathbf{J}} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}} - \sum_{j=0}^{\mathbf{J}-N+1} (-i\omega)^{j+N-1} \Gamma_j = o(|\omega|^{\mathbf{J}}) \quad .$$

Choosing here  $\mathbf{J} = 1$  we may easily conclude the differentiability of  $\mathcal{L}_\omega$  in  $\omega = 0$  as an operator acting on  $\text{Reg}_s^{q,-1}(\Omega) = \Pi_{\text{reg}} L_s^{2,q,q+1}(\Omega)$ . We obtain

**Corollary** *Let  $s \in (3/2, \infty) \setminus \mathbb{I}$  and  $t < \min\{N/2 - 3, -1/2\}$  as well as*

$$\tau > \max\{(N + 1)/2, s - t\} \quad .$$

*Then*

$$\mathbb{C}_{+, \hat{\omega}} \ni \omega \longmapsto i\mathcal{L}_\omega \in B(\text{Reg}_s^{q,-1}(\Omega), \mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega))$$

*is differentiable in  $\omega = 0$  with derivative  $\Lambda^{-1} \mathcal{L}^2$ .*

**Remark C** *Formally our solution of (1.1) satisfies the perturbed Helmholtz type equation*

$$\left( \begin{bmatrix} \varepsilon^{-1} \text{div } \mu^{-1} \text{rot} & 0 \\ 0 & \mu^{-1} \text{rot } \varepsilon^{-1} \text{div} \end{bmatrix} + \omega^2 \right) (E, H) = (\Lambda^{-1} M - i\omega) \Lambda^{-1} (F, G)$$

*and*

$$i\omega \text{div } \varepsilon E = \text{div } F \quad , \quad i\omega \text{rot } \mu H = \text{rot } G \quad ,$$

*which imply*

$$\begin{aligned} & \left( \begin{bmatrix} \varepsilon^{-1} \text{div } \mu^{-1} \text{rot} + \text{rot } \text{div } \varepsilon & 0 \\ 0 & \mu^{-1} \text{rot } \varepsilon^{-1} \text{div} + \text{div } \text{rot } \mu \end{bmatrix} + \omega^2 \right) (E, H) \\ & = \left( \frac{1}{i\omega} \begin{bmatrix} \text{rot } \text{div} & 0 \\ 0 & \text{div } \text{rot} \end{bmatrix} \Lambda + \Lambda^{-1} M - i\omega \right) \Lambda^{-1} (F, G) \quad . \end{aligned}$$

*We note  $\Delta = \text{div } \text{rot} + \text{rot } \text{div}$ . Due to this formula and under certain regularity restrictions on  $(F, G)$  the cases  $q = 0$  and  $q = N - 1$  are equivalent to scalar (perturbed) Helmholtz problems for  $E$  and  $H$ , since  $E$  for  $q = 0$  and  $H$  for  $q = N - 1$  are scalar functions. The first*

case  $q = 0$ , i.e. a Helmholtz equation with homogeneous Dirichlet boundary condition for  $E$ , has already been discussed in [45] or [48] and even for the most complicated case  $N = 2$  in [29]. The other case  $q = N - 1$  corresponds to a Helmholtz equation with homogeneous or inhomogeneous Dirichlet boundary condition  $i\omega\mu H = G$  on  $\partial\Omega$  for  $H$  and can be handled analogously to the case  $q = 0$  using an adequate extension operator.

However, also in the case  $q = N - 1$  our techniques work. The only difference is that now some exceptional ‘tower forms’ occur. Due to their appearance we have to tackle some additional difficulties and the correction operators occur already at the power  $\omega^{N-2}$  instead of  $\omega^{N-1}$  in the case  $1 \leq q \leq N - 2$ . At this point we note in passing that for more regular data from  $\text{Reg}_s^{q,0}(\Omega)$  the correction operators appear primarily at the power  $\omega^N$  for  $1 \leq q \leq N - 2$  and at  $\omega^{N-1}$  if  $q = N - 1$ .

## 2 The spaces of regular convergence

First let us remind of the special tower forms

$$\pm D_{\sigma,m}^{q,k} \quad , \quad \pm R_{\sigma,m}^{q,k}$$

and their properties from [26, section 2], which will be used frequently throughout this paper. The main tool for their construction is the spherical coordinate calculus developed in [46]. Hence, we shall use also many notations and results from this paper. From this point of view the paper at hand demonstrates also an application of [46].

Utilizing [26, Corollary 5.12] we define some special data spaces recursively by

**Definition 2.1** Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in (\mathbf{J} - N/2, \infty) \setminus \mathbb{I}$  as well as  $\tau > \max\{0, s - N/2\}$  and  $\tau \geq \mathbf{J} - s - 1$ . For  $j = 1, \dots, \mathbf{J}$  we define the ‘spaces of regular convergence’ via

$$\begin{aligned} \text{Reg}_s^{q,0}(\Omega) &:= {}_0\mathbb{D}_s^q(\Omega) \times {}_0\mathring{\mathbb{R}}_s^{q+1}(\Omega) \quad , \\ \text{Reg}_s^{q,j}(\Omega) &:= \{(F, G) \in \text{Reg}_s^{q,j-1}(\Omega) : \mathcal{L}^j(F, G) \in \text{Reg}_{s-j}^{q,0}(\Omega)\} \quad . \end{aligned}$$

We will denote  $\text{Reg}_s^{q,j}(\Omega)$  the ‘space of regular convergence of order  $j$ ’.

**Remark 2.2** We have

$$\text{Reg}_s^{q,j}(\Omega) = \{(F, G) \in \text{Reg}_s^{q,0}(\Omega) : \mathcal{L}^j(F, G) \in L_{s-j}^{2,q,q+1}(\Omega)\} \quad .$$

In words, the space of regular convergence  $\text{Reg}_s^{q,\mathbf{J}}(\Omega)$  is characterized by the following property: For  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$  and  $j = 0, \dots, \mathbf{J}$  no tower-forms  $\eta D_I^q$  or  $\eta R_J^{q+1}$  appear in the powers  $\mathcal{L}^j(F, G)$ .

Clearly for the selfadjoint operator  $\mathcal{M}$  introduced in [25] the resolvent-formula hold for nonreal frequencies. Our next step is to show that this formula still holds true for real frequencies and  $\mathcal{L}_\omega$  acting on  $\text{Reg}_s^{q,\mathbf{J}}(\Omega)$  up to the order  $\mathbf{J}$ . Then  $\mathcal{L}_\omega$  is approximated by the usual Neumann sum up to the order  $\mathbf{J}$ .

For the purpose of a short notation let us put for  $J \in \mathbb{N}_0$

$$\mathcal{L}_{\omega,J} := \mathcal{L}_\omega - \sum_{j=0}^J (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \quad , \quad \mathcal{L}_{\omega,-1} := \mathcal{L}_\omega \quad .$$

**Theorem 2.3** *Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  as well as*

$$\tau > \max \{ (N+1)/2, s - N/2 \} \quad .$$

*Moreover, let  $\hat{\omega}$  be as in [25, Lemma 4.2]. Then for all  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  on  $\text{Reg}_s^{q,\mathbf{J}}(\Omega)$*

$$\mathcal{L}_{\omega,\mathbf{J}-1} = (-i\omega)^{\mathbf{J}} \mathcal{L}_\omega \mathcal{L}^{\mathbf{J}} \quad , \quad \mathcal{L}_{\omega,\mathbf{J}} = (-i\omega)^{\mathbf{J}} (\mathcal{L}_\omega - \mathcal{L}_0) \mathcal{L}^{\mathbf{J}} \quad .$$

*Furthermore, for  $s \in (\mathbf{J} + 1/2, \mathbf{J} + N/2) \setminus \mathbb{I}$  and  $\tilde{t} < t := s - \mathbf{J} - (N+1)/2$*

$$\begin{aligned} \text{(i)} \quad & \left\| \mathcal{L}_{\omega,\mathbf{J}-1}(F, G) \right\|_{L_{\tilde{t}}^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \left\| (F, G) \right\|_{L_s^{2,q,q+1}(\Omega)} \quad , \\ \text{(ii)} \quad & \left\| \mathcal{L}_{\omega,\mathbf{J}}(F, G) \right\|_{L_{\tilde{t}}^{2,q,q+1}(\Omega)} = o(|\omega|^{\mathbf{J}}) \left\| (F, G) \right\|_{L_s^{2,q,q+1}(\Omega)} \end{aligned}$$

*hold uniformly with respect to  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$ .*

**Proof:**  $\mathcal{L}_\omega \mathcal{L}^{\mathbf{J}}(F, G)$  is well defined by [25, Theorem 2.17], since

$$\mathcal{L}^j(F, G) \in \text{Reg}_{s-j}^{q,0}(\Omega) \subset L_{>\frac{1}{2}}^{2,q,q+1}(\Omega)$$

holds for  $j = 0, \dots, \mathbf{J}$ . Thus, also

$$(E, H) := \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(F, G) + (-i\omega)^{\mathbf{J}} \mathcal{L}_\omega \mathcal{L}^{\mathbf{J}}(F, G)$$

is well defined. Because of  $s > \mathbf{J} + 1/2 > \mathbf{J} + 1 - N/2$  even  $\mathbf{J} + 1$  powers of  $\mathcal{L}$  may be applied to  $(F, G)$  by [26, Corollary 5.12]. We get

$$(E, H) = \sum_{j=0}^{\mathbf{J}} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(F, G) + (-i\omega)^{\mathbf{J}} (\mathcal{L}_\omega - \mathcal{L}_0) \mathcal{L}^{\mathbf{J}}(F, G) \quad .$$

Furthermore,  $(E, H) \in \mathring{\mathbf{R}}_{<-\frac{1}{2}}^q(\Omega) \times \mathbf{D}_{<-\frac{1}{2}}^{q+1}(\Omega)$  satisfies the radiation condition. Since  $M \mathcal{L}_0 = \text{Id}$  and  $(M + i\omega\Lambda) \mathcal{L}_\omega = \text{Id}$  we obtain  $(M + i\omega\Lambda)(E, H) = (F, G)$ , which yields  $(E, H) = \mathcal{L}_\omega(F, G)$ .

Noting  $s - \mathbf{J} \in (1/2, N/2)$  we may apply [25, Lemma 4.2 (iv)] to

$$\mathcal{L}^{\mathbf{J}}(F, G) \in \text{Reg}_{s-\mathbf{J}}^{q,0}(\Omega)$$

and obtain uniformly in  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$  the estimate

$$\left\| \mathcal{L}_\omega \mathcal{L}^{\mathbf{J}}(F, G) \right\|_{L_{\tilde{t}}^{2,q,q+1}(\Omega)} \leq c \left\| \mathcal{L}^{\mathbf{J}}(F, G) \right\|_{L_{s-\mathbf{J}}^{2,q,q+1}(\Omega)} \leq c \left\| (F, G) \right\|_{L_s^{2,q,q+1}(\Omega)} \quad .$$

(We observe that  $\mathcal{L}^{\mathbf{J}}$  is continuous by [26, Corollary 5.12].) Analogously using [25, Corollary 4.5] we may estimate

$$\begin{aligned} \|(\mathcal{L}_\omega - \mathcal{L}_0) \mathcal{L}^{\mathbf{J}}(F, G)\|_{L_{\tilde{t}}^{2,q,q+1}(\Omega)} &\leq \| \mathcal{L}_\omega - \mathcal{L}_0 \|_{B_{s-\mathbf{J},\tilde{t}}} \| \mathcal{L}^{\mathbf{J}}(F, G) \|_{L_{s-\mathbf{J}}^{2,q,q+1}(\Omega)} \\ &\leq c \| \mathcal{L}_\omega - \mathcal{L}_0 \|_{B_{s-\mathbf{J},\tilde{t}}} \| (F, G) \|_{L_s^{2,q,q+1}(\Omega)} \end{aligned}$$

again uniformly in  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$ , which completes the proof since  $\| \mathcal{L}_\omega - \mathcal{L}_0 \|_{B_{s-\mathbf{J},\tilde{t}}} \xrightarrow{\omega \rightarrow 0} 0$ .  $\blacksquare$

Now our aim is to characterize  $\text{Reg}_s^{q,\mathbf{J}}(\Omega)$  utilizing orthogonality constraints. To realize this we need special growing Dirichlet-forms, which will be defined in the following lemma. Let us remind of the topological isomorphisms

$$\mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon := {}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q \quad , \quad \widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu := {}^\mu \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^{q+1}$$

for some  $s \in (1 - N/2, \infty) \setminus \mathbb{I}$  introduced in [26, Theorem 4.6].

**Lemma 2.4** *Let  $\sigma \in \mathbb{N}_0$  as well as  $\tau > \sigma$  and  $\tau \geq N/2 - 1$ . Then for the counting indices  $(m, n) \in \{1, \dots, \mu_\sigma^q\} \times \{1, \dots, \mu_\sigma^{q+1}\}$  the ‘special growing Dirichlet-forms’*

$$\begin{aligned} E_{\sigma,m}^+ &:= (\text{Id} - \mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon^{-1} \mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon) \eta^+ D_{\sigma,m}^{q,0} \quad , \\ H_{\sigma,n}^+ &:= (\text{Id} - \widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu^{-1} \widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu) \eta^+ R_{\sigma,n}^{q+1,0} \end{aligned}$$

are well defined and belong to  $L_{<-\frac{N}{2}-\sigma}^{2,q}(\Omega)$  resp.  $L_{<-\frac{N}{2}-\sigma}^{2,q+1}(\Omega)$ . These are the unique solutions of the electro-magneto static problems

$$\begin{aligned} \mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon E_{\sigma,m}^+ &= (0, 0, 0) \quad , & E_{\sigma,m}^+ - {}^+ D_{\sigma,m}^{q,0} &\text{decays} \quad , \\ \widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu H_{\sigma,n}^+ &= (0, 0, 0) \quad , & H_{\sigma,n}^+ - {}^+ R_{\sigma,n}^{q+1,0} &\text{decays} \quad . \end{aligned}$$

**Remark 2.5** *To be more precise:  $E_{\sigma,m}^+$  and  $H_{\sigma,n}^+$  are the unique solutions of*

$$\begin{aligned} E_{\sigma,m}^+ &\in {}_\varepsilon \mathcal{H}_{\text{loc}}^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp_\varepsilon} \quad , & E_{\sigma,m}^+ - {}^+ D_{\sigma,m}^{q,0} &\in L_{>-\frac{N}{2}}^{2,q}(\Omega) \quad , \\ \mu H_{\sigma,n}^+ &\in {}_{\mu^{-1}} \mathcal{H}_{\text{loc}}^{q+1}(\Omega) \cap B^{q+1}(\Omega)^\perp \quad , & H_{\sigma,n}^+ - {}^+ R_{\sigma,n}^{q+1,0} &\in L_{>-\frac{N}{2}}^{2,q+1}(\Omega) \quad . \end{aligned}$$

Here on one hand we used  $\mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon$  resp.  $\widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu$  as formal mappings, e.g.

$$\mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon = \left( \text{div } \varepsilon \cdot , \text{rot } \cdot , \langle \varepsilon \cdot , \mathring{B}_1^q \rangle , \dots , \langle \varepsilon \cdot , \mathring{B}_{d^q}^q \rangle \right) \quad ,$$

and on the other hand  $\mathfrak{M}\mathfrak{a}\mathfrak{x}_\varepsilon^{-1}$  resp.  $\widetilde{\mathfrak{M}\mathfrak{a}\mathfrak{x}}_\mu^{-1}$  as the inverse operators.

**Proof:** Uniqueness is clear by [26, Lemma 3.8] and the properties of  $\mathring{B}^q(\Omega)$ ,  $B^{q+1}(\Omega)$ , see [26, section 4]. Let us assume the well definedness of  $E_{\sigma,m}^+$  for a moment. Since

$$\mathfrak{Max}_{\varepsilon} E_{\sigma,m}^+ = (0, 0, 0)$$

holds we obtain  $E_{\sigma,m}^+ \in {}_{\varepsilon}\mathcal{H}_{\text{loc}}^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp\varepsilon}$ . Moreover,  $E_{\sigma,m}^+ - {}^+D_{\sigma,m}^{q,0} \in L_{>-\frac{N}{2}}^{2,q}(\Omega)$  because  $\mathfrak{Max}_{\varepsilon}^{-1}$  maps in fact to  $L_{>-\frac{N}{2}}^{2,q}(\Omega)$  and finally the integrability of  $E_{\sigma,m}^+$  is determined by the form  $\eta^+ D_{\sigma,m}^{q,0}$ , which belongs to  $L_{<-\frac{N}{2}-\sigma}^{2,q}(\Omega)$  by [26, Remark 2.5]. Analogously we may handle  $H_{\sigma,n}^+$ , which would prove the lemma.

So it remains to show that  $E_{\sigma,m}^+$  is well defined. (Then surely  $H_{\sigma,n}^+$  is well defined as well by similar arguments.) With  $\text{supp } \eta \cap \text{supp } \mathring{b}_{\ell}^q = \emptyset$  and [26, Remark 2.5]

$$\begin{aligned} \mathfrak{Max}_{\varepsilon} \eta^+ D_{\sigma,m}^{q,0} &= (\text{div}(\varepsilon \eta^+ D_{\sigma,m}^{q,0}), \text{rot}(\eta^+ D_{\sigma,m}^{q,0}), 0) \\ &= (C_{\text{div},\eta} {}^+D_{\sigma,m}^{q,0} + \text{div}(\hat{\varepsilon} \eta^+ D_{\sigma,m}^{q,0}), C_{\text{rot},\eta} {}^+D_{\sigma,m}^{q,0}, 0) \\ &\in {}_0\mathbb{D}_{<-\sigma-\frac{N}{2}+\tau+1}^{q-1}(\Omega) \times {}_0\mathring{\mathbb{R}}_{\text{vox}}^{q+1}(\Omega) \times \mathbb{C}^{d^q} \end{aligned}$$

holds. Now  $-\sigma - N/2 + \tau + 1 > 1 - N/2$  since  $\tau > \sigma$  and thus

$$\mathfrak{Max}_{\varepsilon} \eta^+ D_{\sigma,m}^{q,0} \in \mathbb{W}_{>1-\frac{N}{2}}^q(\Omega) \quad .$$

Because also  $\tau \geq N/2 - 1$  and using [26, Theorem 4.6] we get the well definedness of  $\mathfrak{Max}_{\varepsilon}^{-1} \mathfrak{Max}_{\varepsilon} \eta^+ D_{\sigma,m}^{q,0}$  and thus of  $E_{\sigma,m}^+$ . We note

$$\mathfrak{Max}_{\varepsilon} \eta^+ D_{\sigma,m}^{q,0} \in \mathbb{W}_s^q(\Omega) \quad , \quad (2.1)$$

if  $\tau > s + \sigma + N/2 - 1$ . ■

Our next step is to define powers of  $\mathcal{L}$  on the special forms

$$\Lambda(E_{\sigma,m}^+, 0) \quad , \quad \Lambda(0, H_{\sigma,n}^+)$$

using [26, Theorem 5.10]. Let us introduce a new notation. For  $k \in \mathbb{N}_0$  we define

$$E_{\sigma,m}^{+,k} := \mathcal{L}^k \Lambda(E_{\sigma,m}^+, 0) \quad , \quad H_{\sigma,m}^{+,k} := \mathcal{L}^k \Lambda(0, H_{\sigma,m}^+) \quad .$$

The next lemma shows that these definitions are well defined.



**Lemma 2.6** Let  $\mathbf{J}, \sigma \in \mathbb{N}_0$ ,  $s \in (\mathbf{J} + 1 - N/2, \infty) \setminus \mathbb{I}$  as well as  $\tau > \sigma + s + N/2 - 1$  and  $\tau \geq \mathbf{J} - s$ . Moreover, let  $(m, n) \in \{1, \dots, \mu_\sigma^q\} \times \{1, \dots, \mu_\sigma^{q+1}\}$  and  $j \in \{0, \dots, \mathbf{J}\}$ . Then  $\mathbf{J}$  powers of  $\mathcal{L}$  on  $E_{\sigma,m}^{+,0}$  and  $H_{\sigma,n}^{+,0}$  are well defined and for even  $j$

$$\begin{aligned} E_{\sigma,m}^{+,j} - \eta({}^+D_{\sigma,m}^{q,j}, 0) &\in (D_{s-j-1}^q(\Omega) \boxplus \eta \mathcal{D}^q(\bar{\mathcal{J}}_{s-j-1}^{q, \leq j})) \times \{0\} \quad , \\ H_{\sigma,n}^{+,j} - \eta(0, {}^+R_{\sigma,n}^{q+1,j}) &\in \{0\} \times (\mathring{R}_{s-j-1}^{q+1}(\Omega) \boxplus \eta \mathcal{R}^{q+1}(\bar{\mathcal{J}}_{s-j-1}^{q+1, \leq j})) \end{aligned}$$

as well as for odd  $j$

$$\begin{aligned} E_{\sigma,m}^{+,j} - \eta(0, {}^+R_{\sigma,m}^{q+1,j}) &\in \{0\} \times (\mathring{R}_{s-j-1}^{q+1}(\Omega) \boxplus \eta \mathcal{R}^{q+1}(\bar{\mathcal{J}}_{s-j-1}^{q+1, \leq j})) \quad , \\ H_{\sigma,n}^{+,j} - \eta({}^+D_{\sigma,n}^{q,j}, 0) &\in (D_{s-j-1}^q(\Omega) \boxplus \eta \mathcal{D}^q(\bar{\mathcal{J}}_{s-j-1}^{q, \leq j})) \times \{0\} \end{aligned}$$

hold. Furthermore,

$$E_{\sigma,m}^{+,j}, H_{\sigma,n}^{+,j} \in L_{<-\sigma-j-\frac{N}{2}}^{2,q,q+1}(\Omega)$$

and thus

$$E_{\sigma,m}^{+,j}, H_{\sigma,n}^{+,j} \in L_{-t}^{2,q,q+1}(\Omega) \iff t > \sigma + j + N/2 \quad .$$

More precisely: There exist unique constants  $\xi^{j,\sigma}, \zeta^{j,\sigma} \in \mathbb{C}$  and unique forms

$$\begin{aligned} e_\sigma^j &\in (\varepsilon \mathring{R}_{s-j-1}^q(\Omega)) \cap D_{s-j-1}^q(\Omega) \cap \mathring{B}^q(\Omega)^\perp \quad , \\ h_\sigma^j &\in (\mu D_{s-j-1}^{q+1}(\Omega)) \cap \mathring{R}_{s-j-1}^{q+1}(\Omega) \cap B^{q+1}(\Omega)^\perp \quad , \end{aligned}$$

such that for even  $j$

$$\begin{aligned} E_{\sigma,m}^{+,j} &= \eta({}^+D_{\sigma,m}^{q,j}, 0) + \sum_{I \in \bar{\mathcal{J}}_{s-j-1}^{q, \leq j}} \xi_I^{j,\sigma,m} \eta(D_I^q, 0) + (e_{\sigma,m}^j, 0) \quad , \\ H_{\sigma,n}^{+,j} &= \eta(0, {}^+R_{\sigma,n}^{q+1,j}) + \sum_{J \in \bar{\mathcal{J}}_{s-j-1}^{q+1, \leq j}} \zeta_J^{j,\sigma,n} \eta(0, R_J^{q+1}) + (0, h_{\sigma,n}^j) \end{aligned}$$

and for odd  $j$

$$\begin{aligned} E_{\sigma,m}^{+,j} &= \eta(0, {}^+R_{\sigma,m}^{q+1,j}) + \sum_{J \in \bar{\mathcal{J}}_{s-j-1}^{q+1, \leq j}} \zeta_J^{j,\sigma,m} \eta(0, R_J^{q+1}) + (0, h_{\sigma,m}^j) \quad , \\ H_{\sigma,n}^{+,j} &= \eta({}^+D_{\sigma,n}^{q,j}, 0) + \sum_{I \in \bar{\mathcal{J}}_{s-j-1}^{q, \leq j}} \xi_I^{j,\sigma,n} \eta(D_I^q, 0) + (e_{\sigma,n}^j, 0) \quad . \end{aligned}$$

**Remark 2.7** By [26, Remark 2.4] we even have for odd  $j \leq J$

$$\begin{aligned} E_{\sigma,m}^{+,j} - \eta(0, {}^+R_{\sigma,m}^{q+1,j}) &\in \{0\} \times {}_0\mathring{\mathbb{R}}_{s-j-1}^{q+1}(\bar{\mathcal{J}}_{s-j-1}^{q+1,\leq j}, \Omega) \quad , \\ H_{\sigma,n}^{+,j} - \eta({}^+D_{\sigma,n}^{q,j}, 0) &\in {}_0\mathbb{D}_{s-j-1}^q(\bar{\mathcal{J}}_{s-j-1}^{q,\leq j}, \Omega) \times \{0\} \quad , \end{aligned}$$

since then also  $\eta^+ R_{\sigma,m}^{q+1,j}$  is irrotational and  $\eta^+ D_{\sigma,n}^{q,j}$  solenoidal. Moreover, the coefficients satisfy the following recursion:

$$\begin{aligned} \zeta_{1I}^{j+1,\sigma,\ell} &= \zeta_I^{j,\sigma,\ell} \quad , \quad I \in \bar{\mathcal{J}}_{s-j-1}^{q,\leq j} \quad , \quad \ell = m, n \quad , \\ \zeta_{1J}^{j+1,\sigma,\ell} &= \zeta_J^{j,\sigma,\ell} \quad , \quad J \in \bar{\mathcal{J}}_{s-j-1}^{q+1,\leq j} \quad , \quad \ell = m, n \end{aligned}$$

Then clearly the next recursion holds as well:

$$\begin{aligned} \zeta_{2I}^{j+2,\sigma,\ell} &= \zeta_I^{j,\sigma,\ell} \quad , \quad I \in \bar{\mathcal{J}}_{s-j-1}^{q,\leq j} \quad , \quad \ell = m, n \quad , \\ \zeta_{2J}^{j+2,\sigma,\ell} &= \zeta_J^{j,\sigma,\ell} \quad , \quad J \in \bar{\mathcal{J}}_{s-j-1}^{q+1,\leq j} \quad , \quad \ell = m, n \end{aligned}$$

**Proof:** We only have to show that  $E_{\sigma,m}^{+,0}$  and  $H_{\sigma,n}^{+,0}$  are elements of the domain of definition of  $\mathcal{L}$  (and then clearly of  $\mathcal{L}^j$ ) from [26, Theorem 5.10]. Then all our assertions follow by [26, Theorem 5.10], [26, Remark 5.11] and [26, Remark 2.5]. We note that the integrability of the forms is always determined by the integrability of the tower forms with positive sign. Again we only discuss  $E_{\sigma,m}^{+,0} = (\varepsilon E_{\sigma,m}^+, 0)$ , for example.

By Lemma 2.4 and (2.1) as well as [26, Theorem 4.6] we observe

$$\varepsilon E_{\sigma,m}^+ \in {}_0\mathbb{D}_{s-1}^q(\{I\} \cup \bar{\mathcal{J}}_{s-1}^{q,0}, \Omega) \quad , \quad I := (+, 0, \sigma, m) \quad (2.2)$$

by [26, Remark 2.5] since  $\tau > \sigma + s + N/2 - 1$ . Hence, utilizing [26, Theorem 5.10]  $\mathcal{L}^j$  may be applied to  $E_{\sigma,m}^{+,0}$  and the lemma would be proved.

Unfortunately we ignored a trifle in this argument. Here the same problem occurs as in [26], namely the appearance of the exceptional tower forms, which was solved by a second order approach in this paper. A similar approach will help here. The point is that (2.2) only holds true for  $q \neq 1$  in the first sight. In fact for  $q = 1$  and  $s \geq N/2$  we have to deal with the exceptional tower form  $\check{D}_{s-1}^{1,1} = {}^-R_{0,1}^{1,1} = R_{\check{I}}^1$  with  $\check{I} := (-, 1, 0, 1)$ , which would cancel our iteration process in the case of appearance. Now in this special case (2.2) reads correctly as:  $\varepsilon E_{\sigma,m}^+$  is an element of  ${}_0\mathbb{D}_{\text{loc}}^1(\Omega)$  and contained in

$$(\varepsilon^{-1} \mathring{R}_{s-1}^1(\Omega) \cap D_{s-1}^1(\Omega)) \boxplus \eta \mathcal{D}^1(\{I\} \cup \bar{\mathcal{J}}_{s-1}^{q,0}) \boxplus \eta \mathcal{R}^1(\{\check{I}\}) \quad .$$

It remains to show that  $R_{\check{I}}^1$  does not occur even in the exceptional case. We try the ansatz

$$U_{\sigma,m} := \eta^+ R_{\sigma,m}^{2,1} + u_{\sigma,m} \quad ,$$

to find a solution of the problem

$$\text{rot } \varepsilon^{-1} \text{div } U_{\sigma,m} = 0 \quad , \quad U_{\sigma,m} - {}^+R_{\sigma,m}^{2,1} \in L_{>-\frac{N}{2}}^{2,2}(\Omega) \quad .$$

Thus, we are led to search a solution of

$$\operatorname{rot} \varepsilon^{-1} \operatorname{div} u_{\sigma,m} = -\operatorname{rot} \varepsilon^{-1} \operatorname{div} \eta^+ R_{\sigma,m}^{2,1} \quad , \quad u_{\sigma,m} \in L_{>-\frac{N}{2}}^{2,2}(\Omega) \quad . \quad (2.3)$$

Using once more  $\tau > \sigma + s + N/2 - 1$  and [26, Remark 2.5] we obtain that

$$\operatorname{rot} \varepsilon^{-1} \operatorname{div} \eta^+ R_{\sigma,m}^{2,1} = C_{\operatorname{rot} \operatorname{div}, \eta^+} R_{\sigma,m}^{2,1} + \operatorname{rot} \hat{\varepsilon} \operatorname{div} \eta^+ R_{\sigma,m}^{2,1}$$

is an element of  $L_{<-\sigma-\frac{N}{2}+\tau+1}^{2,2}(\Omega) \subset L_s^{2,2}(\Omega)$ , where  $\varepsilon^{-1} = \operatorname{Id} + \hat{\varepsilon}$  is  $\tau$ - $C^1$ -admissible as well. Therefore,  $\operatorname{rot} \varepsilon^{-1} \operatorname{div} \eta^+ R_{\sigma,m}^{2,1} \in \mathring{0}\mathbb{R}_s^2(\Omega)$  lies in the range of  $\operatorname{rot} \Delta_{s-2}^2$  from [26, Lemma 7.1] and we get some  $u_{\sigma,m} \in D(\operatorname{rot} \Delta_{s-2}^2)$  solving (2.3). But then

$$\begin{aligned} \tilde{E}_{\sigma,m} &:= \operatorname{div} U_{\sigma,m} \in \left( D_{s-1}^1(\Omega) \boxplus \eta \mathcal{D}^1(\{I\} \cup \bar{\mathcal{J}}_{s-1}^{q,0}) \right) \cap \mathring{0}\mathbb{D}_{\operatorname{loc}}^1(\Omega) \\ &= \mathring{0}\mathbb{D}_{s-1}^1(\{I\} \cup \bar{\mathcal{J}}_{s-1}^{q,0}, \Omega) \end{aligned}$$

(Compare to [26, Remark 7.4].) and

$$\varepsilon^{-1} \tilde{E}_{\sigma,m} \in \varepsilon \mathcal{H}_{<-\frac{N}{2}-\sigma}^1(\Omega) \cap \mathring{B}^1(\Omega)^{\perp \varepsilon} \quad , \quad \varepsilon^{-1} \tilde{E}_{\sigma,m} - {}^+D_{\sigma,m}^{1,0} \in L_{>-\frac{N}{2}}^{2,1}(\Omega) \quad ,$$

i.e.  $\varepsilon^{-1} \tilde{E}_{\sigma,m} = E_{\sigma,m}^+$  by Lemma 2.4 and Remark 2.5. So in fact  $E_{\sigma,m}^+$  and  $\varepsilon E_{\sigma,m}^+$  do not feature exceptional tower forms.  $\blacksquare$

## 2.1 Compactly supported inhomogeneities

In this subsection we develop some results especially for compactly supported inhomogeneities  $\Lambda$ . In fact we assume  $r_0$  to be so large, such that

$$\operatorname{supp} \hat{\Lambda} \subset U_{r_0} \quad (2.4)$$

holds. Then in particular  $\Lambda = \operatorname{Id}$  on  $\operatorname{supp} \eta$ .

**Corollary 2.8** *Let  $\mathbf{J}, \sigma \in \mathbb{N}_0$  as well as  $(m, n) \in \{1, \dots, \mu_\sigma^q\} \times \{1, \dots, \mu_\sigma^{q+1}\}$  and moreover  $j \in \{0, \dots, \mathbf{J}\}$ . Then there exist unique constants  $\xi^{j,\sigma}, \zeta^{j,\sigma} \in \mathbb{C}$ , such that in  $\text{supp } \eta$  for even  $j$*

$$\begin{aligned} E_{\sigma,m}^{+,j} &= ({}^+D_{\sigma,m}^{q,j}, 0) + \sum_{I \in \bar{\mathcal{J}}^{q,\leq j}} \xi_I^{j,\sigma,m} (D_I^q, 0) \quad , \\ H_{\sigma,n}^{+,j} &= (0, {}^+R_{\sigma,n}^{q+1,j}) + \sum_{J \in \bar{\mathcal{J}}^{q+1,\leq j}} \zeta_J^{j,\sigma,n} (0, R_J^{q+1}) \end{aligned}$$

and for odd  $j$

$$\begin{aligned} E_{\sigma,m}^{+,j} &= (0, {}^+R_{\sigma,m}^{q+1,j}) + \sum_{J \in \bar{\mathcal{J}}^{q+1,\leq j}} \zeta_J^{j,\sigma,m} (0, R_J^{q+1}) \quad , \\ H_{\sigma,n}^{+,j} &= ({}^+D_{\sigma,n}^{q,j}, 0) + \sum_{I \in \bar{\mathcal{J}}^{q,\leq j}} \xi_I^{j,\sigma,n} (D_I^q, 0) \end{aligned}$$

hold. These series converge uniformly in  $\text{supp } \eta$  together with all their derivatives even after multiplication by arbitrary powers of  $r$ . (Compare with [46, p. 1033], [48, Theorem 1] and [26, Theorem 2.6].) The constants  $\xi^{j,\sigma}$  and  $\zeta^{j,\sigma}$  coincide with those of Lemma 2.6, whenever they co-exist.

**Proof:** We show the representation for some even  $j$  and  $(E, 0) := E_{\sigma,m}^{+,j}$ . The other representations may be proved in a similar way.

We have  $\text{div } E = 0$  and  $(M\Lambda^{-1})^{j+1}(E, 0) = (0, 0)$  in  $\Omega$ . Hence,  $M^{j+1}(E, 0)$  vanishes in  $\text{supp } \eta$ . For  $\mathbf{J} + N/2 \leq s \notin \mathbb{I}$  we see by Lemma 2.6

$$\tilde{E} := E - {}^+D_{\sigma,m}^{q,j} - \sum_{I \in \bar{\mathcal{J}}_{s-j-1}^{q,\leq j}} \xi_I^{j,\sigma,m} D_I^q \in L_{s-j-1}^{2,q}(\text{supp } \eta) \subset L_{\frac{N}{2}-1}^{2,q}(\text{supp } \eta)$$

and

$$\text{div } \tilde{E} \Big|_{\text{supp } \eta} = 0 \quad , \quad M^{j+1}(\tilde{E}, 0) \Big|_{\text{supp } \eta} = (0, 0) \quad .$$

Now the generalized spherical harmonics expansion [26, Theorem 2.6] yields with unique constants  $\xi^{j,\sigma,m} \in \mathbb{C}$  the representation

$$\tilde{E} \Big|_{\text{supp } \eta} = \sum_{I \in \bar{\mathcal{J}}^{q,\leq j} \setminus \bar{\mathcal{J}}_{s-j-1}^{q,\leq j}} \xi_I^{j,\sigma,m} D_I^q$$

and therefore the assertion follows immediately. ■

Next we want to characterize

$$\text{Reg}_s^{q,\mathbf{J}}(\Omega)$$

by orthogonality constraints using the growing Dirichlet-forms. For this we need some special properties of the tower-forms. As in [26] we introduce the first order differential operator with compactly supported coefficients  $C := C_{\Delta,\eta} := \Delta\eta - \eta\Delta$ , the commutator of  $\Delta$  and the multiplication by  $\eta$ .

**Lemma 2.9** *Let  $u$  and  $v$  be regular tower-forms corresponding to some finite set of indices. Then the cut-off function  $\eta$  may be chosen, such that*

$$\langle Cu, v \rangle_{L^{2,q}} = 0$$

holds, except for the special cases

$$\begin{aligned} \langle C^\theta D_{\sigma,m}^{q,k}, {}^\vartheta D_{\gamma,n}^{q,\ell} \rangle_{L^{2,q}} \neq 0 &\Leftrightarrow \langle C^\theta R_{\sigma,m}^{q,k}, {}^\vartheta R_{\gamma,n}^{q,\ell} \rangle_{L^{2,q}} \neq 0 \\ \Leftrightarrow \sigma = \gamma, m = n, \theta\vartheta = - , (k, \ell) \in \{(0, 2), (1, 1), (2, 0)\} \end{aligned}$$

and

$$\begin{aligned} \langle C^\theta D_{\sigma,m}^{q,k}, {}^\vartheta R_{\gamma,n}^{q,\ell} \rangle_{L^{2,q}} \neq 0 \\ \Leftrightarrow \sigma = \gamma, m = n, \theta\vartheta = - , (k, \ell) \in \{(0, 2), (2, 0)\} . \end{aligned}$$

**Remark 2.10** *In the special cases we have*

$$\begin{aligned} \langle C^- D_{\sigma,m}^{q,k}, {}^+ D_{\sigma,m}^{q,\ell} \rangle_{L^{2,q}} &= -\langle C^+ D_{\sigma,m}^{q,\ell}, {}^- D_{\sigma,m}^{q,k} \rangle_{L^{2,q}} = \langle C^- R_{\sigma,m}^{q,\ell}, {}^+ R_{\sigma,m}^{q,k} \rangle_{L^{2,q}} \\ &= -\langle C^+ R_{\sigma,m}^{q,k}, {}^- R_{\sigma,m}^{q,\ell} \rangle_{L^{2,q}} = \begin{cases} -\frac{q+\sigma}{N+2\sigma} & , (k, \ell) = (0, 2) \\ 1 & , (k, \ell) = (1, 1) \\ -\frac{q'+\sigma}{N+2\sigma} & , (k, \ell) = (2, 0) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \langle C^- D_{\sigma,m}^{q,k}, {}^+ R_{\sigma,m}^{q,\ell} \rangle_{L^{2,q}} &= \langle C^+ D_{\sigma,m}^{q,k}, {}^- R_{\sigma,m}^{q,\ell} \rangle_{L^{2,q}} = \langle C^- R_{\sigma,m}^{q,\ell}, {}^+ D_{\sigma,m}^{q,k} \rangle_{L^{2,q}} \\ &= \langle C^+ R_{\sigma,m}^{q,\ell}, {}^- D_{\sigma,m}^{q,k} \rangle_{L^{2,q}} = i \frac{\omega_\sigma^{q-1}}{N+2\sigma} \begin{cases} -1 & , (k, \ell) = (0, 2) \\ 1 & , (k, \ell) = (2, 0) \end{cases} . \end{aligned}$$

**Proof:** From [48, (31)] with  $a = b = 1$  we have  $\langle Cu, v \rangle_{L^{2,q}} = -\langle u, Cv \rangle_{L^{2,q}}$  for suitable  $q$ -forms  $u, v$ . Using the spherical calculus presented in [46] we compute for tower-forms  $u, v$

$$\begin{aligned} \langle Cu, v \rangle_{L^{2,q}} &= \int_{\mathbb{R}_+} r^{N-1} \langle Cu, v \rangle_{(r)} dr \\ &= \int_{\mathbb{R}_+} r^{N-1} \left( \langle \rho C \check{\rho} \rho u(r), \rho v(r) \rangle_{L^{2,q-1}(S^{N-1})} + \langle \tau C \check{\tau} \tau u(r), \tau v(r) \rangle_{L^{2,q}(S^{N-1})} \right) dr \\ &= \int_{\mathbb{R}_+} r^{N-1} (\Gamma_{\hat{\eta}} r^\alpha) r^\beta \left( \langle \rho u(1), \rho v(1) \rangle_{L^{2,q-1}(S^{N-1})} + \langle \tau u(1), \tau v(1) \rangle_{L^{2,q}(S^{N-1})} \right) dr \\ &= \langle u, v \rangle_{(1)} \int_{\mathbb{R}_+} r^{N-1+\beta} \Gamma_{\hat{\eta}} r^\alpha dr = (\alpha - \beta) \langle u, v \rangle_{(1)} \int_{\mathbb{R}_+} r^{N-2+\alpha+\beta} \hat{\eta}'(r) dr . \end{aligned} \tag{2.5}$$

Here we put

$$\langle u, v \rangle_{(r)} := \langle \rho u(r), \rho v(r) \rangle_{L^{2,q-1}(S^{N-1})} + \langle \tau u(r), \tau v(r) \rangle_{L^{2,q}(S^{N-1})}$$

and  $\alpha := \text{hom}(u)$ ,  $\beta := \text{hom}(v)$ . We note that  $\rho u$  and  $\tau u$  are homogeneous of degree  $\alpha$  as functions of  $r$ , i.e.  $\rho u(r) = r^\alpha \rho u(1)$  and  $\tau u(r) = r^\beta \tau u(1)$ . Furthermore, we denote by  $\Gamma_\phi$  for suitable  $\phi, \varphi$  the first order ordinary differential operator

$$\Gamma_\phi \varphi(t) := 2\phi'(t)\varphi'(t) + \phi'(t) + (N-1)t^{-1}\phi'(t) \quad .$$

Since the spherical eigenforms  $T_{\sigma,m}^q$  and  $S_{\sigma,m}^q$  present an orthonormal system in  $L^{2,q}(S^{N-1})$ , the expression  $\langle u, v \rangle_{(1)}$  only may differ from zero in the cases

$$\begin{array}{llll} u = {}^\theta D_{\sigma,m}^{q,k} & , & v = {}^\vartheta D_{\sigma,m}^{q,\ell} & , & k - \ell \text{ even} & , \\ u = {}^\theta R_{\sigma,m}^{q,k} & , & v = {}^\vartheta R_{\sigma,m}^{q,\ell} & , & k - \ell \text{ even} & , \\ u = {}^\theta D_{\sigma,m}^{q,k} & , & v = {}^\vartheta R_{\sigma,m}^{q,\ell} & , & k, \ell \text{ even} & , \\ u = {}^\theta R_{\sigma,m}^{q,k} & , & v = {}^\vartheta D_{\sigma,m}^{q,\ell} & , & k, \ell \text{ even} & \end{array}$$

with  $\theta, \vartheta \in \{+, -\}$ . We may assume additionally that our tower forms under consideration are at most of height  $K$  and index  $Z$ . According to [45, Lemma 2 (i)] we may choose the cut-off function  $\eta$  (resp.  $\hat{\eta}, \boldsymbol{\eta}$ ), such that for given  $\hat{j} \in \mathbb{N}_0$

$$\int_{\mathbb{R}} \hat{\eta}'(r) r^j dr = \delta_{0,j} \quad , \quad -\hat{j} \leq j \leq \hat{j} \quad , \quad j \in \mathbb{Z} \quad , \quad (2.6)$$

holds. Let us pick some  $\hat{j} \geq N + 2(1 + K + Z)$ . In the four cases above we have degrees of homogeneities  $\alpha = {}^\theta h_\sigma^k$  and  $\beta = {}^\vartheta h_\sigma^\ell$ . Because of

$$N - 2 + \alpha + \beta \in [-\hat{j}, \hat{j}]$$

and (2.6) the integral (2.5) can only differ from zero, if  $N - 2 + \alpha + \beta = 0$ . But if  $\theta\vartheta = +$ , then either

$$N - 2 + {}^+h_\sigma^k + {}^+h_\sigma^\ell = N - 2 + k + \ell + 2\sigma \neq 0$$

or

$$N - 2 + {}^-h_\sigma^k + {}^-h_\sigma^\ell = N - 2 + k + \ell - 2\sigma - 2N \neq 0$$

since  $k + \ell$  is even and  $N$  odd. So only  $\theta\vartheta = -$  is possible and we get

$$\begin{aligned} N - 2 + {}^-h_\sigma^k + {}^+h_\sigma^\ell &= N - 2 + {}^+h_\sigma^k + {}^-h_\sigma^\ell = -2 + k + \ell = 0 \\ \Leftrightarrow (k, \ell) &\in \{(0, 2), (1, 1), (2, 0)\} \quad , \end{aligned}$$

where the possibility of  $k = \ell = 1$  has to be excluded in the two last cases, where  $k, \ell$  are even. Thus, we have proved the essential assertions of the lemma.

Let us calculate one of the special integrals as an example:

$$\begin{aligned}
& \langle C^- D_{\sigma,m}^{q,k}, {}^+ D_{\sigma,m}^{q,\ell} \rangle_{L^2,q} = ({}^- h_{\sigma}^k - {}^+ h_{\sigma}^{\ell}) \langle {}^- D_{\sigma,m}^{q,k}, {}^+ D_{\sigma,m}^{q,\ell} \rangle_{(1)} \\
& = (k - \ell - 2\sigma - N) \\
& \quad \begin{cases} -\alpha_{\sigma}^{q,0} + \alpha_{\sigma}^{q,1} ((\omega_{\sigma}^{q-1})^2 + (q' + {}^- h_{\sigma}^0)(q' + {}^+ h_{\sigma}^2)) & , (k, \ell) = (0, 2) \\ -\alpha_{\sigma}^{q+1,0} + \alpha_{\sigma}^{q+1,0} & , (k, \ell) = (1, 1) \\ -\alpha_{\sigma}^{q,1} + \alpha_{\sigma}^{q,0} ((\omega_{\sigma}^{q-1})^2 + (q' + {}^- h_{\sigma}^2)(q' + {}^+ h_{\sigma}^0)) & , (k, \ell) = (2, 0) \end{cases} \\
& = \begin{cases} \frac{(-2-2\sigma-N)((\omega_{\sigma}^{q-1})^2 + (q'-\sigma-N)(q'+2+\sigma))}{2(2+2\sigma+N)(-2\sigma-N)} = -\frac{q+\sigma}{N+2\sigma} & , (k, \ell) = (0, 2) \\ -\frac{-2\sigma-N}{2\sigma+N} = 1 & , (k, \ell) = (1, 1) \\ \frac{(2-2\sigma-N)((\omega_{\sigma}^{q-1})^2 + (q'+2-\sigma-N)(q'+\sigma))}{2(2-2\sigma-N)(-2\sigma-N)} = -\frac{q'+\sigma}{N+2\sigma} & , (k, \ell) = (2, 0) \end{cases}
\end{aligned}$$

■

**Lemma 2.11** *In the same sense Lemma 2.9 holds for all tower-forms, if one pays attention to  ${}^- D_{0,1}^{0,0} = 0$  and  ${}^- R_{0,1}^{N,0} = 0$ . Besides in the special cases we get for the exceptional tower-forms*

$$\begin{aligned}
& \langle C^- D_{0,1}^{0,2}, {}^+ D_{0,1}^{0,0} \rangle_{L^2} = -\langle C^+ D_{0,1}^{0,0}, {}^- D_{0,1}^{0,2} \rangle_{L^2} = \langle C^- R_{0,1}^{N,2}, {}^+ R_{0,1}^{N,0} \rangle_{L^{2,N}} \\
& = -\langle C^+ R_{0,1}^{N,0}, {}^- R_{0,1}^{N,2} \rangle_{L^{2,N}} = -\langle C^- R_{0,1}^{1,1}, {}^+ R_{0,1}^{1,1} \rangle_{L^{2,1}} = \langle C^+ R_{0,1}^{1,1}, {}^- R_{0,1}^{1,1} \rangle_{L^{2,1}} \\
& = -\langle C^- D_{0,1}^{N-1,1}, {}^+ D_{0,1}^{N-1,1} \rangle_{L^{2,N-1}} = \langle C^+ D_{0,1}^{N-1,1}, {}^- D_{0,1}^{N-1,1} \rangle_{L^{2,N-1}} = 1 \quad .
\end{aligned}$$

Summing up we obtain

**Remark 2.12** *For tower-forms  $u, v$  the scalar product  $\langle Cu, v \rangle_{L^2,q}$  can only differ from zero, if  $u$  and  $v$  possess different signs  $\pm$  as well as equal eigenvalue and counting indices  $\sigma$  and  $m$ . Additionally in the cases  $u = \pm D_{\sigma,m}^{q,k}, v = \mp D_{\sigma,m}^{q,\ell}$  or  $u = \pm R_{\sigma,m}^{q,k}, v = \mp R_{\sigma,m}^{q,\ell}$  the heights  $(k, \ell)$  must belong to  $\{(0, 2), (1, 1), (2, 0)\}$  as well as in the cases  $u = \pm D_{\sigma,m}^{q,k}, v = \mp R_{\sigma,m}^{q,\ell}$  or in reverse order even to  $\{(0, 2), (2, 0)\}$ .*

Now let us return to our static solutions. We put

$$\tilde{L}^{2,q,q+1}(\Omega) := {}_{\Lambda^{-1}}L^{2,q,q+1}(\Omega) := {}_{\varepsilon^{-1}}L^{2,q}(\Omega) \times {}_{\mu^{-1}}L^{2,q+1}(\Omega)$$

with scalar product  $\langle \cdot, \cdot \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle \Lambda^{-1} \cdot, \cdot \rangle_{L^{2,q,q+1}(\Omega)}$ , see [25, section 2].

**Lemma 2.13** Let  $s \in (2 - N/2, \infty) \setminus \mathbb{I}$  and  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$  with representation

$$\mathcal{L}_0(F, G) = (E, H) + \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q,0}} \mathbf{e}_I \eta(D_I^q, 0) + \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathbf{h}_J \eta(0, R_J^{q+1}) \quad ,$$

where  $(E, H) \in (\mathring{\mathbb{R}}_{s-1}^q(\Omega) \cap \varepsilon^{-1} \mathbb{D}_{s-1}^q(\Omega)) \times (\mu^{-1} \mathring{\mathbb{R}}_{s-1}^{q+1}(\Omega) \cap \mathbb{D}_{s-1}^{q+1}(\Omega))$  and  $\mathbf{e}_I, \mathbf{h}_J \in \mathbb{C}$ . Then for all  $I = (-, 0, \sigma, m) \in \bar{\mathcal{J}}_{s-1}^{q,0}$  and  $J = (-, 0, \gamma, n) \in \bar{\mathcal{J}}_{s-1}^{q+1,0}$

- (i)  $\langle (F, G), E_{\sigma,m}^{+,0} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \langle (F, G), H_{\gamma,n}^{+,0} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = 0 \quad ,$
- (ii)  $\langle (F, G), E_{\sigma,m}^{+,1} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \mathbf{e}_I \quad ,$
- (iii)  $\langle (F, G), H_{\gamma,n}^{+,1} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \mathbf{h}_J \quad .$

**Remark 2.14** It is sufficient to choose  $\hat{j} \geq 2(s+1)$  in (2.6).

**Proof:** We set

$$(0, \tilde{H}_{\sigma,m}^+) := \Lambda^{-1} E_{\sigma,m}^{+,1} = \mathcal{L}_0 E_{\sigma,m}^{+,0} \quad , \quad (\tilde{E}_{\gamma,n}^+, 0) := \Lambda^{-1} H_{\gamma,n}^{+,1} = \mathcal{L}_0 H_{\gamma,n}^{+,0} \quad .$$

Let us look at  $E_{\sigma,m}^+$  and  $\tilde{H}_{\sigma,m}^+$ . For  $(-, 0, \sigma, m) \in \bar{\mathcal{J}}_{s-1}^{q,0}$  we have  $s > \sigma + 1 + N/2$  and thus only weights  $s$  larger than  $1 + N/2$  have to be considered. According to Lemma 2.6

$$E_{\sigma,m}^+ \in {}_\varepsilon \mathcal{H}_{-s+1}^q(\Omega) \subset {}_0 \mathring{\mathbb{R}}_{-s+1}^q(\Omega) \quad , \quad \tilde{H}_{\sigma,m}^+ \in \mathbb{D}_{-s}^{q+1}(\Omega) \quad . \quad (2.7)$$

Therefore all scalar products under consideration are well defined. By [25, Lemma 2.13, Remark 2.14] and Lemma 2.4 we get

$$\langle M(E, H), E_{\sigma,m}^{+,0} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \langle \text{div } H, E_{\sigma,m}^+ \rangle_{\mathbb{L}^{2,q}(\Omega)} = 0$$

and thus with  $(0, R_J^{q+1}) = M(D_{1J}^q, 0)$

$$\begin{aligned} & \langle (F, G), E_{\sigma,m}^{+,0} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \langle M \mathcal{L}_0(F, G), (E_{\sigma,m}^+, 0) \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)} \\ = & \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q,0}} \mathbf{e}_I \underbrace{\langle M \eta(D_I^q, 0), (E_{\sigma,m}^+, 0) \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)}}_{= \langle (0, \text{rot } \eta D_I^q), (E_{\sigma,m}^+, 0) \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)} = 0} \\ & + \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathbf{h}_J \langle M^2 \eta(D_{1J}^q, 0), (E_{\sigma,m}^+, 0) \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)} \\ & - \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathbf{h}_J \underbrace{\langle M C_{M,\eta}(D_{1J}^q, 0), (E_{\sigma,m}^+, 0) \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)}}_{= \langle \text{div } C_{\text{rot},\eta} D_{1J}^q, E_{\sigma,m}^+ \rangle_{\mathbb{L}^{2,q}(\Omega)} = 0} \quad . \end{aligned}$$

For  $J = (-, 0, \gamma, n) \in \bar{\mathcal{J}}_{s-1}^{q+1,0}$  we have  $\text{div } \eta D_{1J}^q = \text{div } \eta^- D_{\gamma,n}^{q,1} = 0$  by [26, Remark 2.4] and therefore  $\text{div } \text{rot } \eta D_{1J}^q = \Delta \eta D_{1J}^q = C D_{1J}^q$ . This shows

$$\langle (F, G), E_{\sigma,m}^{+,0} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} = \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathbf{h}_J \langle C D_{1J}^q, E_{\sigma,m}^+ \rangle_{\mathbb{L}^{2,q}(\Omega)}$$



recalling  $M^2(e, h) = (\operatorname{div} \operatorname{rot} e, \operatorname{rot} \operatorname{div} h)$ .

According to [26, Theorem 5.10, Remark 5.11, Corollary 5.12] under the present assumptions  $\mathcal{L}_0 \mathcal{L}(F, G)$  is also well defined and has the representation

$$\mathcal{L}_0 \mathcal{L}(F, G) = (\tilde{E}, \tilde{H}) + \sum_{I \in \bar{\mathcal{J}}_{s-2}^{q, \leq 1}} \tilde{e}_I \eta(D_I^q, 0) + \sum_{J \in \bar{\mathcal{J}}_{s-2}^{q+1, \leq 1}} \tilde{h}_J \eta(0, R_J^{q+1})$$

with  $(\tilde{E}, \tilde{H}) \in (\mathring{R}_{s-2}^q(\Omega) \cap \varepsilon^{-1} D_{s-2}^q(\Omega)) \times (\mu^{-1} \mathring{R}_{s-2}^{q+1}(\Omega) \cap D_{s-2}^{q+1}(\Omega))$  and  $\tilde{e}_I, \tilde{h}_J \in \mathbb{C}$ . Moreover,  $\tilde{h}_{1I} = e_I$  and  $\tilde{e}_{1J} = h_J$  hold for  $I \in \bar{\mathcal{J}}_{s-1}^{q, 0}$  and  $J \in \bar{\mathcal{J}}_{s-1}^{q+1, 0}$ . From

$$\Lambda^{-1} M(\tilde{E}, \tilde{H}) \in (\mathring{R}_{s-1}^q(\Omega) \cap \varepsilon^{-1} D_{s-1}^q(\Omega)) \times (\mu^{-1} \mathring{R}_{s-1}^{q+1}(\Omega) \cap D_{s-1}^{q+1}(\Omega))$$

as well as by (2.7) and [25, Lemma 2.13, Remark 2.14] we get

$$\begin{aligned} & \langle M \Lambda^{-1} M(\tilde{E}, \tilde{H}), \mathcal{L}_0 \Lambda(E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= - \langle M(\tilde{E}, \tilde{H}), (E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} = 0 \end{aligned}$$

Using this and  $(F, G) = M \Lambda^{-1} M \mathcal{L}_0 \mathcal{L}(F, G)$  we derive

$$\begin{aligned} & \langle (F, G), E_{\sigma, m}^{+, 1} \rangle_{L^{2, q, q+1}(\Omega)} \\ &= \sum_{I \in \bar{\mathcal{J}}_{s-2}^{q, \leq 1}} \tilde{e}_I \underbrace{\langle M^2 \eta(D_I^q, 0), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)}}_{=0} \\ & \quad + \sum_{J \in \bar{\mathcal{J}}_{s-2}^{q+1, \leq 1}} \tilde{h}_J \langle M^2 \eta(0, R_J^{q+1}), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= \sum_{J \in \bar{\mathcal{J}}_{s-2}^{q+1, 0}} \tilde{h}_J \langle M^2 \eta(0, R_J^{q+1}), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)} \\ & \quad + \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q, 0}} \tilde{h}_{1I} \langle M^2 \eta(0, R_{1I}^{q+1}), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)} \end{aligned}$$

because  $\bar{\mathcal{J}}_{s-2}^{q+1, \leq 1} = \bar{\mathcal{J}}_{s-2}^{q+1, 0} \dot{\cup} \bar{\mathcal{J}}_{s-2}^{q+1, 1} = \bar{\mathcal{J}}_{s-2}^{q+1, 0} \dot{\cup} \mathbb{1}(\bar{\mathcal{J}}_{s-1}^{q, 0})$ . Applying once more [26, Remark 2.4] we obtain  $M^2 \eta(0, R_{1I}^{q+1}) = \Delta \eta(0, R_{1I}^{q+1}) = C(0, R_{1I}^{q+1})$  as well as

$$M^2 \eta(0, R_J^{q+1}) = M^2 \eta M(D_{1J}^q, 0) = M M^2 \eta(D_{1J}^q, 0) - M^2 C_{M, \eta}(D_{1J}^q, 0)$$

and thus

$$M^2 \eta(0, R_J^{q+1}) = M C(D_{1J}^q, 0) - M^2 C_{M, \eta}(D_{1J}^q, 0)$$

Partial integration yields

$$\langle M C(D_{1J}^q, 0), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)} = - \langle C(D_{1J}^q, 0), (E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)}$$

and clearly all terms of the sum like

$$\langle M^2 C_{M, \eta}(D_{1J}^q, 0), (0, \tilde{H}_{\sigma, m}^+) \rangle_{L^{2, q, q+1}(\Omega)}$$

vanish by (two times) partial integration. Finally we get for  $(-, 0, \sigma, m) \in \bar{\mathcal{J}}_{s-1}^{q,0}$

$$\begin{aligned} \langle (F, G), E_{\sigma, m}^{+,0} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} &= \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathfrak{h}_J \langle CD_{1J}^q, E_{\sigma, m}^+ \rangle_{L^{2,q}(\Omega)} \quad , \\ \langle (F, G), E_{\sigma, m}^{+,1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} &= - \sum_{J \in \bar{\mathcal{J}}_{s-2}^{q+1,0}} \tilde{\mathfrak{h}}_J \langle CD_{1J}^q, E_{\sigma, m}^+ \rangle_{L^{2,q}(\Omega)} \\ &\quad + \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q,0}} \mathfrak{e}_I \langle CR_{1I}^{q+1}, \tilde{H}_{\sigma, m}^+ \rangle_{L^{2,q+1}(\Omega)} \quad . \end{aligned}$$

Analogously for  $(-, 0, \gamma, n) \in \bar{\mathcal{J}}_{s-1}^{q+1,0}$  one sees

$$\begin{aligned} \langle (F, G), H_{\gamma, n}^{+,0} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} &= \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q,0}} \mathfrak{e}_I \langle CR_{1I}^{q+1}, H_{\gamma, n}^+ \rangle_{L^{2,q+1}(\Omega)} \quad , \\ \langle (F, G), H_{\gamma, n}^{+,1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} &= - \sum_{I \in \bar{\mathcal{J}}_{s-2}^{q,0}} \tilde{\mathfrak{e}}_I \langle CR_{1I}^{q+1}, H_{\gamma, n}^+ \rangle_{L^{2,q+1}(\Omega)} \\ &\quad + \sum_{J \in \bar{\mathcal{J}}_{s-1}^{q+1,0}} \mathfrak{h}_J \langle CD_{1J}^q, \tilde{E}_{\gamma, n}^+ \rangle_{L^{2,q}(\Omega)} \quad . \end{aligned}$$

Now all integrals on the right hand sides only extend over  $\text{supp } \nabla \eta$ . Thus, we may insert the expansions from Corollary 2.8 for

$$E_{\sigma, m}^+, \tilde{H}_{\sigma, m}^+, H_{\gamma, n}^+, \tilde{E}_{\gamma, n}^+ \quad .$$

Using the orthogonality properties from Lemma 2.9 and Remark 2.10 we finally obtain

$$\langle CD_{1J}^q, E_{\sigma, m}^+ \rangle_{L^{2,q}(\Omega)} = \langle CR_{1I}^{q+1}, H_{\gamma, n}^+ \rangle_{L^{2,q+1}(\Omega)} = 0$$

and

$$\langle CR_{1I}^{q+1}, \tilde{H}_{\sigma, m}^+ \rangle_{L^{2,q+1}(\Omega)} = \delta_{I,(-,0,\sigma,m)} \quad , \quad \langle CD_{1J}^q, \tilde{E}_{\gamma, n}^+ \rangle_{L^{2,q}(\Omega)} = \delta_{J,(-,0,\gamma,n)} \quad .$$

We note that by [26, Remark 2.5] we only have to consider tower-forms with maximal heights  $K = 1$  and maximal eigenvalue index  $Z \leq s - 1 - N/2$ . Thus,

$$\hat{j} \geq 2(s+1) \geq N + 2(1 + K + Z)$$

is sufficient according to (2.6). ■

Now we are ready to characterize the spaces of regular convergence by orthogonality constraints.

**Lemma 2.15** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in (\mathbf{J} + 1 - N/2, \infty) \setminus \mathbb{I}$  as well as  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$ . Then  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$ , if and only if*

$$\langle (F, G), E_{\sigma,m}^{+,k+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle (F, G), H_{\gamma,n}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0$$

holds for all  $(k, \sigma, m) \in \Theta_s^{q,\mathbf{J}}$  and  $(\ell, \gamma, n) \in \Theta_s^{q+1,\mathbf{J}}$ , where

$$\Theta_s^{q,\mathbf{J}} := \{(k, \sigma, m) \in \mathbb{N}_0^3 : k \leq \mathbf{J} - 1 \wedge \sigma < s - N/2 - k - 1 \wedge 1 \leq m \leq \mu_\sigma^q\} \quad .$$

Moreover,  $\text{Reg}_s^{q,\mathbf{J}}(\Omega)$  is a closed subspace of  $\text{Reg}_s^{q,0}(\Omega)$  and  $L_s^{2,q,q+1}(\Omega)$ .

**Remark 2.16** *We have the characterizations*

$$\begin{aligned} \Theta_s^{q,\mathbf{J}} &= \{(k, \sigma, m) \in \{0, \dots, \mathbf{J} - 1\} \times \mathbb{N}_0 \times \mathbb{N} : E_{\sigma,m}^{+,k+1} \in L_{-s}^{2,q,q+1}(\Omega)\} \quad , \\ \Theta_s^{q+1,\mathbf{J}} &= \{(\ell, \gamma, n) \in \{0, \dots, \mathbf{J} - 1\} \times \mathbb{N}_0 \times \mathbb{N} : H_{\gamma,n}^{+,\ell+1} \in L_{-s}^{2,q,q+1}(\Omega)\} \quad . \end{aligned}$$

**Proof:** The assertions of the remark follow by Lemma 2.6. The proof of the lemma is a straightforward induction over  $\mathbf{J}$ . The start is given by Lemma 2.13 since

$$\begin{aligned} \Theta_s^{q,1} &= \{(0, \sigma, m) : (-, 0, \sigma, m) \in \bar{\mathcal{J}}_{s-1}^{q,0}\} \quad , \\ \Theta_s^{q+1,1} &= \{(0, \gamma, n) : (-, 0, \gamma, n) \in \bar{\mathcal{J}}_{s-1}^{q+1,0}\} \quad . \end{aligned}$$

For the step we note by definition

$$\begin{aligned} \text{Reg}_s^{q,\mathbf{J}+1}(\Omega) &= \{(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega) : \mathcal{L}^{\mathbf{J}+1}(F, G) \in \text{Reg}_{s-\mathbf{J}-1}^{q,0}(\Omega)\} \\ &= \{(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega) : \mathcal{L}^{\mathbf{J}}(F, G) \in \text{Reg}_{s-\mathbf{J}}^{q,1}(\Omega)\} \end{aligned}$$

and according to the start we obtain  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}+1}(\Omega)$ , if and only if

$$(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$$

and

$$\langle \mathcal{L}^{\mathbf{J}}(F, G), E_{\sigma,m}^{+,1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle \mathcal{L}^{\mathbf{J}}(F, G), H_{\gamma,n}^{+,1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0$$

holds for all  $(0, \sigma, m) \in \Theta_{s-\mathbf{J}}^{q,1}$  and  $(0, \gamma, n) \in \Theta_{s-\mathbf{J}}^{q+1,1}$ . Since  $\mathcal{L}^{\mathbf{J}}(F, G) \in \text{Reg}_{s-\mathbf{J}}^{q,1}(\Omega)$  we get

$$\mathcal{L}_0 \mathcal{L}^{\mathbf{J}-1}(F, G) \in \mathring{R}_{s-\mathbf{J}}^q(\Omega) \times D_{s-\mathbf{J}}^{q+1}(\Omega)$$

and with Lemma 2.6

$$\Lambda^{-1} \mathcal{L}^2 \Lambda E_{\sigma,m}^{+,0} \in \mathring{R}_{-s-1+\mathbf{J}}^q(\Omega) \times \{0\} \quad ,$$

because  $(0, \sigma, m) \in \Theta_{s-\mathbf{J}}^{q,1}$  implies  $\sigma < s - 1 - \mathbf{J} - N/2$ , i.e.  $s + 1 - \mathbf{J} > \sigma + 2 + N/2$ . Using partial integration, i.e. [25, Lemma 2.13, Remark 2.14], we compute

$$\begin{aligned} & \langle \mathcal{L}^{\mathbf{J}}(F, G), E_{\sigma, m}^{+, 1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \\ &= \langle \mathcal{L}_0 \mathcal{L}^{\mathbf{J}^{-1}}(F, G), M\Lambda^{-1} \mathcal{L}^2 E_{\sigma, m}^{+, 0} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = -\langle \mathcal{L}^{\mathbf{J}^{-1}}(F, G), E_{\sigma, m}^{+, 2} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \end{aligned}$$

and therefore repeating this argument

$$\langle \mathcal{L}^{\mathbf{J}}(F, G), E_{\sigma, m}^{+, 1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = (-1)^{\mathbf{J}} \langle (F, G), E_{\sigma, m}^{+, \mathbf{J}+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \quad .$$

Analogously we conclude

$$\langle \mathcal{L}^{\mathbf{J}}(F, G), H_{\gamma, n}^{+, 1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = (-1)^{\mathbf{J}} \langle (F, G), H_{\gamma, n}^{+, \mathbf{J}+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \quad .$$

Finally we obtain  $(F, G) \in \text{Reg}_s^{q, \mathbf{J}+1}(\Omega)$ , if and only if  $(F, G) \in \text{Reg}_s^{q, \mathbf{J}}(\Omega)$  and

$$\langle (F, G), E_{\sigma, m}^{+, \mathbf{J}+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = \langle (F, G), H_{\gamma, n}^{+, \mathbf{J}+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = 0$$

holds for all  $(0, \sigma, m) \in \Theta_{s-\mathbf{J}}^{q, 1}$  and  $(0, \gamma, n) \in \Theta_{s-\mathbf{J}}^{q+1, 1}$  and the induction hypothesis for  $\text{Reg}_s^{q, \mathbf{J}}(\Omega)$  completes the proof.  $\blacksquare$

We are looking for projectors onto  $\text{Reg}_s^{q, \mathbf{J}}(\Omega)$  and thus for a dual basis of

$$E_{\sigma, m}^{+, k} \quad , \quad H_{\sigma, m}^{+, k} \quad .$$

For  $\ell, \sigma \in \mathbb{N}_0$  and  $(m, n) \in \{1, \dots, \mu_\sigma^q\} \times \{1, \dots, \mu_\sigma^{q+1}\}$  let us define

$$\begin{aligned} e_{\sigma, n}^\pm &:= \eta^\pm D_{\sigma, n}^{q, 1} & , & & h_{\sigma, m}^\pm &:= \eta^\pm R_{\sigma, m}^{q+1, 1} & , \\ e_{\sigma, n}^{\pm, \ell} &:= M^\ell(e_{\sigma, n}^\pm, 0) & , & & h_{\sigma, m}^{\pm, \ell} &:= M^\ell(0, h_{\sigma, m}^\pm) & . \end{aligned}$$

**Lemma 2.17** *Let  $\ell, k \in \mathbb{N}_0$ . Then  $e_{\sigma, n}^{\pm, \ell}$  and  $h_{\sigma, m}^{\pm, \ell}$  are  $C^\infty$ -forms on  $\mathbb{R}^N$  and belong to  $\text{Reg}_{< \mp(\frac{N}{2} + \sigma) - 1 + \ell}^{q, 0}(\Omega)$ . Furthermore,  $e_{\sigma, n}^{\pm, \ell+2}$  and  $h_{\sigma, m}^{\pm, \ell+2}$  are compactly supported and thus elements of  $\text{Reg}_{\text{vox}}^{q, 0}(\Omega)$  as well as  $\text{Reg}_s^{q, \ell}(\Omega)$  for  $s \in (\ell - N/2, \infty) \setminus \mathbb{I}$ . Moreover, for  $\ell \geq 2$*

$$\mathcal{L}^k e_{\sigma, n}^{\pm, k+\ell} = e_{\sigma, n}^{\pm, \ell} \quad , \quad \mathcal{L}^k h_{\sigma, m}^{\pm, k+\ell} = h_{\sigma, m}^{\pm, \ell}$$

*hold and these equations even remain valid for the negative forms  $e_{\sigma, n}^{-, \ell}$  and  $h_{\sigma, m}^{-, \ell}$  if  $\ell = 0, 1$ .*

**Proof:** According to [26, Remark 2.4] we have  $\text{div } e_{\sigma, n}^\pm = 0$  and  $\text{rot } h_{\sigma, m}^\pm = 0$  and hence by [26, Remark 2.5]  $e_{\sigma, n}^{\pm, 0}, h_{\sigma, m}^{\pm, 0} \in \text{Reg}_{< \mp(\frac{N}{2} + \sigma) - 1}^{q, 0}(\Omega)$ . Furthermore,

$$e_{\sigma, n}^{\pm, 1} = \eta(0, \pm R_{\sigma, n}^{q+1, 0}) + C_{M, \eta}(\pm D_{\sigma, n}^{q, 1}, 0) \quad , \quad (2.8)$$

$$e_{\sigma, n}^{\pm, 2} = C(\pm D_{\sigma, n}^{q, 1}, 0) = C_{M, \eta}(0, \pm R_{\sigma, n}^{q+1, 0}) + MC_{M, \eta}(\pm D_{\sigma, n}^{q, 1}, 0) \quad , \quad (2.9)$$

$$h_{\sigma, m}^{\pm, 1} = \eta(\pm D_{\sigma, m}^{q, 0}, 0) + C_{M, \eta}(0, \pm R_{\sigma, m}^{q+1, 1}) \quad , \quad (2.10)$$

$$h_{\sigma, m}^{\pm, 2} = C(0, \pm R_{\sigma, m}^{q+1, 1}) = C_{M, \eta}(\pm D_{\sigma, m}^{q, 0}, 0) + MC_{M, \eta}(0, \pm R_{\sigma, m}^{q+1, 1}) \quad . \quad (2.11)$$

Thus,  $e_{\sigma,n}^{\pm,1}, h_{\sigma,m}^{\pm,1} \in \text{Reg}_{<\mp(\frac{N}{2}+\sigma)}^{q,0}(\Omega)$  and for all  $\ell \in \mathbb{N}_0$

$$\text{supp } e_{\sigma,n}^{\pm,\ell+2} \cup \text{supp } h_{\sigma,m}^{\pm,\ell+2} \subset \text{supp } \nabla \eta \quad ,$$

i.e.  $e_{\sigma,n}^{\pm,\ell+2}, h_{\sigma,m}^{\pm,\ell+2} \in \text{Reg}_{\text{vox}}^{q,0}(\Omega)$ . By [26, Theorem 5.10] resp. [26, Corollary 5.12] any power of  $\mathcal{L}$  is well defined on  $e_{\sigma,n}^{\pm,\ell+2}, h_{\sigma,m}^{\pm,\ell+2}$ . Because of the compact supports we obtain for  $\ell \geq 2$

$$\mathcal{L} e_{\sigma,n}^{\pm,\ell+1} = e_{\sigma,n}^{\pm,\ell} \quad , \quad \mathcal{L} h_{\sigma,m}^{\pm,\ell+1} = h_{\sigma,m}^{\pm,\ell} \quad (2.12)$$

and a short induction shows

$$\mathcal{L}^k e_{\sigma,n}^{\pm,k+\ell} = e_{\sigma,n}^{\pm,\ell} \quad , \quad \mathcal{L}^k h_{\sigma,m}^{\pm,k+\ell} = h_{\sigma,m}^{\pm,\ell} \quad (2.13)$$

for all  $k \in \mathbb{N}_0$ . The forms  $e_{\sigma,n}^{-,0}, h_{\sigma,m}^{-,0}$  and by (2.8), (2.10) also  $e_{\sigma,n}^{-,1}, h_{\sigma,m}^{-,1}$  possess the ‘right shape’, such that for the negative forms according to [26, Corollary 5.12] the equations (2.12) and (2.13) hold true for  $\ell = 0, 1$  as well. Once more taking into account the compact supports of  $e_{\sigma,n}^{\pm,2}$  and  $h_{\sigma,m}^{\pm,2}$  we get

$$e_{\sigma,n}^{\pm,\ell+2}, h_{\sigma,m}^{\pm,\ell+2} \in \text{Reg}_s^{q,\ell}(\Omega)$$

for all  $\ell \in \mathbb{N}_0$  and  $s \in (\ell - N/2, \infty) \setminus \mathbb{I}$ . ■

**Lemma 2.18** *Let  $K, Z \in \mathbb{N}_0$ . Then for all  $\sigma \in \{0, \dots, Z\}$  and  $k \in \{-1, \dots, K\}$  as well as all  $(\ell, \gamma) \in \mathbb{N}_0^2$  and appropriate  $m, n$*

$$\begin{aligned} \langle e_{\gamma,n}^{-,\ell+2}, E_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} &= \langle h_{\gamma,n}^{-,\ell+2}, H_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} = 0 \quad , \\ \langle e_{\gamma,n}^{-,\ell+2}, H_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} &= \langle h_{\gamma,n}^{-,\ell+2}, E_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} = (-1)^\ell \delta_{k,\ell} \delta_{\sigma,\gamma} \delta_{m,n} \quad . \end{aligned}$$

**Remark 2.19** *It suffices to choose  $\hat{j} \geq N + 2(2 + K + Z)$  in (2.6).*

**Proof:** We note again that  $e_{\gamma,n}^{-,2}$  and  $h_{\gamma,n}^{-,2}$  have compact supports. For all  $\ell \in \mathbb{N}_0$  partial integration and (2.9) yield

$$\begin{aligned} S_{\sigma,\gamma}^{k,\ell} &:= \langle e_{\gamma,n}^{-,\ell+2}, E_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} = \langle M^\ell e_{\gamma,n}^{-,2}, E_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} \\ &= (-1)^\ell \langle C(-D_{\gamma,n}^{q,1}, 0), M^\ell E_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)} \quad . \end{aligned}$$

Since  $M\mathcal{L} = M\mathcal{L}_0 = \text{Id}$  on  $\text{supp } \eta$  and  $M(E_{\sigma,m}^+, 0) = (0, 0)$  the scalar products  $S_{\sigma,\gamma}^{k,\ell}$  vanish for  $\ell \geq k + 2$ . However, for  $\ell \leq k + 1$  we get

$$S_{\sigma,\gamma}^{k,\ell} = (-1)^\ell \langle C(-D_{\gamma,n}^{q,1}, 0), E_{\sigma,m}^{+,k+1-\ell} \rangle_{L^{2,q,q+1}(\Omega)}$$

and these scalar products can only differ from zero if  $k + 1 - \ell$  is even. The integrals range only over  $\text{supp } \nabla \eta$ . Thus, we may insert the representations from Corollary

2.8 for  $E_{\sigma,m}^{+,k+1-\ell}$  and see that  $S_{\sigma,\gamma}^{k,\ell} = 0$  holds by Lemma 2.9 even in these cases. The same arguments force

$$\tilde{S}_{\sigma,\gamma}^{k,\ell} := \langle e_{\gamma,n}^{-,\ell+2}, H_{\sigma,m}^{+,k+1} \rangle_{L^{2,q,q+1}(\Omega)}$$

to vanish for  $\ell \geq k + 2$ . If  $\ell \leq k + 1$  we get

$$\tilde{S}_{\sigma,\gamma}^{k,\ell} = (-1)^\ell \langle C(-D_{\gamma,n}^{q,1}, 0), H_{\sigma,m}^{+,k+1-\ell} \rangle_{L^{2,q,q+1}(\Omega)}$$

These scalar products can only differ from zero, if  $k + 1 - \ell$  is odd. Again we insert the representations from Corollary 2.8 for  $H_{\sigma,m}^{+,k+1-\ell}$ . But now in the case  $k = \ell$  we get a term  $+D_{\sigma,m}^{q,1}$ , whose scalar product with  $C^{-}D_{\gamma,n}^{q,1}$  does not vanish if  $(\sigma, m) = (\gamma, n)$  according to Lemma 2.9. Therefore, we obtain

$$\tilde{S}_{\sigma,\gamma}^{k,\ell} = (-1)^\ell \langle C(-D_{\gamma,n}^{q,1}, 0), (+D_{\sigma,m}^{q,k+1-\ell}, 0) \rangle_{L^{2,q,q+1}(\Omega)} = (-1)^\ell \delta_{k,\ell} \delta_{\sigma,\gamma} \delta_{m,n}$$

Similarly the assertions about the remaining two scalar products may be shown. ■

We have found our projections.

**Theorem 2.20** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in (\mathbf{J} + 1 - N/2, \infty) \setminus \mathbb{I}$ . Then*

$$\text{Reg}_s^{q,0}(\Omega) = \text{Reg}_s^{q,\mathbf{J}}(\Omega) \dot{+} \Upsilon_s^{q,\mathbf{J}},$$

where  $\Upsilon_s^{q,\mathbf{J}} := \text{Lin} \{ e_{\sigma,m}^{-,k+2}, h_{\gamma,n}^{-,\ell+2} : (k, \sigma, m) \in \Theta_s^{q+1,\mathbf{J}}, (\ell, \gamma, n) \in \Theta_s^{q,\mathbf{J}} \}$ .

More precisely: Each  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$  can be decomposed uniquely as

$$(F, G) = (F_{\text{reg}}, G_{\text{reg}}) + (F_\Upsilon, G_\Upsilon),$$

where  $(F_{\text{reg}}, G_{\text{reg}}) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$  and  $(F_\Upsilon, G_\Upsilon) \in \Upsilon_s^{q,\mathbf{J}}$  are defined by

$$\begin{aligned} (F_\Upsilon, G_\Upsilon) := & \sum_{(k,\sigma,m) \in \Theta_s^{q,\mathbf{J}}} (-1)^k \langle (F, G), E_{\sigma,m}^{+,k+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} h_{\sigma,m}^{-,k+2} \\ & + \sum_{(k,\sigma,m) \in \Theta_s^{q+1,\mathbf{J}}} (-1)^k \langle (F, G), H_{\sigma,m}^{+,k+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} e_{\sigma,m}^{-,k+2}. \end{aligned}$$

**Remark 2.21**  $\Upsilon_s^{q,\mathbf{J}}$  are finite dimensional subspaces of  $(\mathring{C}^{\infty,q}(\Omega) \times \mathring{C}^{\infty,q+1}(\Omega)) \cap \text{Reg}_{\text{vox}}^{q,0}(\Omega)$  and the projections  $(F, G) \mapsto (F_\Upsilon, G_\Upsilon)$  resp.  $(F, G) \mapsto (F_{\text{reg}}, G_{\text{reg}})$  are continuous. Moreover, the choice  $\hat{j} \geq 2(s + \mathbf{J} + 1)$  in (2.6) is sufficient.

**Proof:** According to Lemma 2.17 we have the inclusion  $\Upsilon_s^{q,\mathbf{J}} \subset \text{Reg}_{\text{vox}}^{q,0}(\Omega)$ . Thus,  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$  implies  $(F_{\text{reg}}, G_{\text{reg}}), (F_\Upsilon, G_\Upsilon) \in \text{Reg}_s^{q,0}(\Omega)$ . Applying Lemma 2.18 we obtain for all  $(k, \sigma, m) \in \Theta_s^{q,\mathbf{J}}$  and  $(\ell, \gamma, n) \in \Theta_s^{q+1,\mathbf{J}}$

$$\langle (F_{\text{reg}}, G_{\text{reg}}), E_{\sigma,m}^{+,k+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle (F_{\text{reg}}, G_{\text{reg}}), H_{\gamma,n}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0$$

and therefore  $(F_{\text{reg}}, G_{\text{reg}}) \in \text{Reg}_s^{q, \mathbf{J}}(\Omega)$  by Lemma 2.15, which yields

$$\text{Reg}_s^{q,0}(\Omega) \subset \text{Reg}_s^{q, \mathbf{J}}(\Omega) + \Upsilon_s^{q, \mathbf{J}} \subset \text{Reg}_s^{q,0}(\Omega) \quad .$$

So it remains to show the directness of the sum. Let us pick an element

$$(F, G) = \sum_{(k, \sigma, m) \in \Theta_s^{q+1, \mathbf{J}}} \mathbf{f}_{k, \sigma, m} e_{\sigma, m}^{-, k+2} + \sum_{(k, \sigma, m) \in \Theta_s^{q, \mathbf{J}}} \mathbf{g}_{k, \sigma, m} h_{\sigma, m}^{-, k+2}$$

of the intersection  $\text{Reg}_s^{q, \mathbf{J}}(\Omega) \cap \Upsilon_s^{q, \mathbf{J}}$ . Applying  $\mathcal{L}$  yields that

$$\mathcal{L}(F, G) = \sum_{(k, \sigma, m) \in \Theta_s^{q+1, \mathbf{J}}} \mathbf{f}_{k, \sigma, m} e_{\sigma, m}^{-, k+1} + \sum_{(k, \sigma, m) \in \Theta_s^{q, \mathbf{J}}} \mathbf{g}_{k, \sigma, m} h_{\sigma, m}^{-, k+1}$$

belongs to  $\text{Reg}_{s-1}^{q, \mathbf{J}-1}(\Omega) \subset L_{s-1}^{2, q, q+1}(\Omega)$  by Lemma 2.17. If  $k > 0$  the forms  $e_{\sigma, m}^{-, k+1}$  resp.  $h_{\sigma, m}^{-, k+1}$  have compact supports. But for  $k = 0$  with (2.8), (2.10) the forms  $e_{\sigma, m}^{-, 1}$ ,  $h_{\sigma, m}^{-, 1}$  are no longer compactly supported. However, they belong to  $L_{< \frac{N}{2} + \sigma}^{2, q, q+1}(\Omega)$  but even not to  $L_{\frac{N}{2} + \sigma}^{2, q, q+1}(\Omega)$ . Thus,  $e_{\sigma, m}^{-, 1}$ ,  $h_{\sigma, m}^{-, 1}$  are not elements of  $L_{s-1}^{2, q, q+1}(\Omega)$  since  $(0, \sigma, m) \in \Theta_s^{q+1, \mathbf{J}}$  resp.  $(0, \sigma, m) \in \Theta_s^{q, \mathbf{J}}$  implies  $N/2 + \sigma < s - 1$ . The forms  $e_{\sigma, m}^{-, 1}$  and  $h_{\sigma, m}^{-, 1}$  are linear independent. Consequently the coefficients  $\mathbf{f}_{0, \sigma, m}$ ,  $\mathbf{g}_{0, \sigma, m}$  have to vanish. Repeating this argument with  $\mathcal{L}^j(F, G)$  for  $j = 2, \dots, \mathbf{J}$  finally shows  $\mathbf{f}_{k, \sigma, m} = \mathbf{g}_{k, \sigma, m} = 0$  for all  $(k, \sigma, m) \in \Theta_s^{q+1, \mathbf{J}}$  and  $(k, \sigma, m) \in \Theta_s^{q, \mathbf{J}}$ .  $\blacksquare$

We are ready to approach our desired low frequency asymptotics.

### 3 Low frequency asymptotics

We will prove the desired asymptotic expansion in four steps, which are:

step one: proof in the reduced case, i.e.:

- compactly supported perturbations  $\hat{\Lambda}$ ;
- right hand sides from  $\text{Reg}_s^{q,0}(\Omega)$ ;
- estimates in local norms

step two: replacing  $\text{Reg}_s^{q,0}(\Omega)$  by  $L_s^{2, q, q+1}(\Omega)$

step three: replacing local norms by weighted norms

step four: replacing compactly supported perturbations  $\hat{\varepsilon}$ ,  $\hat{\mu}$  by asymptotically vanishing perturbations

Following this program we only drop the assumption of compactly supported perturbations of the medium in the last step. Thus, (2.4) may be assumed during the first three steps.

### 3.1 First step

**Lemma 3.1** *Let  $\mathbf{J} \in \mathbb{N}_0$ ,  $s \in (\mathbf{J} + 1/2, \mathbf{J} + N/2) \setminus \mathbb{I}$  and  $t := s - \mathbf{J} - (N + 1)/2$ . Then*

$$\begin{aligned} & \left\| \mathcal{L}_{\omega, \mathbf{J}-1}(F, G) - \sum_{(k, \sigma, m) \in \Theta_s^{q, \mathbf{J}}} (i\omega)^k \langle (F, G), E_{\sigma, m}^{+, k+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \mathcal{L}_{\omega, \mathbf{J}-1-k} h_{\sigma, m}^{-, 2} \right. \\ & \quad - \sum_{(k, \sigma, m) \in \Theta_s^{q+1, \mathbf{J}}} (i\omega)^k \langle (F, G), H_{\sigma, m}^{+, k+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \mathcal{L}_{\omega, \mathbf{J}-1-k} e_{\sigma, m}^{-, 2} \left. \right\|_{L_t^{2, q, q+1}(\Omega)} \\ & \quad = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2, q, q+1}(\Omega)} \end{aligned}$$

holds uniformly in  $\omega \in \mathbb{C}_{+, \tilde{\omega}} \setminus \{0\}$  and  $(F, G) \in \text{Reg}_s^{q, 0}(\Omega)$ .

**Proof:** According to Theorem 2.20 we decompose  $(F, G) \in \text{Reg}_s^{q, 0}(\Omega)$

$$(F, G) = (F_{\text{reg}}, G_{\text{reg}}) + (F_{\Upsilon}, G_{\Upsilon}) \in \text{Reg}_s^{q, \mathbf{J}}(\Omega) \dot{+} \Upsilon_s^{q, \mathbf{J}}$$

and obtain by Theorem 2.3 uniformly in  $\omega$  and  $(F_{\text{reg}}, G_{\text{reg}})$

$$\left\| \mathcal{L}_{\omega, \mathbf{J}-1}(F_{\text{reg}}, G_{\text{reg}}) \right\|_{L_t^{2, q, q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F_{\text{reg}}, G_{\text{reg}})\|_{L_s^{2, q, q+1}(\Omega)} .$$

By Remark 2.21 the projections are continuous and thus

$$\left\| \mathcal{L}_{\omega, \mathbf{J}-1}(F, G) - \mathcal{L}_{\omega, \mathbf{J}-1}(F_{\Upsilon}, G_{\Upsilon}) \right\|_{L_t^{2, q, q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2, q, q+1}(\Omega)} .$$

This shows that we only have to determine the asymptotics of the special forms  $e_{\sigma, m}^{-, k+2}$ ,  $h_{\sigma, m}^{-, k+2}$  for  $k \leq \mathbf{J} - 1$ , which belong to  $\text{Reg}_s^{q, k}(\Omega)$  using Lemma 2.17. Theorem 2.3 and Lemma 2.17 yield

$$\begin{aligned} \mathcal{L}_{\omega, k-1} e_{\sigma, m}^{-, k+2} &= (-i\omega)^k \mathcal{L}_{\omega} \mathcal{L}^k e_{\sigma, m}^{-, k+2} = (-i\omega)^k \mathcal{L}_{\omega} e_{\sigma, m}^{-, 2} , \\ \mathcal{L}_{\omega, k-1} h_{\sigma, m}^{-, k+2} &= (-i\omega)^k \mathcal{L}_{\omega} \mathcal{L}^k h_{\sigma, m}^{-, k+2} = (-i\omega)^k \mathcal{L}_{\omega} h_{\sigma, m}^{-, 2} . \end{aligned}$$

Then for  $1 \leq k \leq \mathbf{J} - 1$  we obtain

$$\begin{aligned} \mathcal{L}_{\omega, \mathbf{J}-1} e_{\sigma, m}^{-, k+2} &= \mathcal{L}_{\omega, k-1} e_{\sigma, m}^{-, k+2} - \sum_{j=k}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_{\omega} \mathcal{L}^j e_{\sigma, m}^{-, k+2} \\ &= (-i\omega)^k \mathcal{L}_{\omega, \mathbf{J}-1-k} e_{\sigma, m}^{-, 2} \end{aligned}$$

since  $\mathcal{L}^j e_{\sigma, m}^{-, k+2} = \mathcal{L}^{j-k} e_{\sigma, m}^{-, 2}$ . Analogously we compute

$$\mathcal{L}_{\omega, \mathbf{J}-1} h_{\sigma, m}^{-, k+2} = (-i\omega)^k \mathcal{L}_{\omega, \mathbf{J}-1-k} h_{\sigma, m}^{-, 2} .$$

■



According to the latter lemma we only have to calculate the asymptotics of the special forms

$$\mathcal{L}_{\omega, \mathbf{J}-1-k} e_{\sigma, m}^{-,2} \quad , \quad \mathcal{L}_{\omega, \mathbf{J}-1-k} h_{\sigma, m}^{-,2}$$

for  $\omega \in \mathbb{C}_{+, \hat{\omega}} \setminus \{0\}$ ,  $0 \leq k \leq \mathbf{J} - 1$  and  $\sigma < s - N/2 - 1$ .

For this we will use a technique introduced by Weck and Witsch in [42, 43, 44], which was completed in [45] resp. [48]. The idea is to compare the forms

$$\mathcal{L}_{\omega} e_{\sigma, m}^{-,2} \quad , \quad \mathcal{L}_{\omega} h_{\sigma, m}^{-,2}$$

with special radiating solutions of the homogeneous problem in  $\mathbb{R}^N \setminus \{0\}$  and then to identify the proper static terms in their asymptotic expansions. For this procedure it is essential that the perturbation  $\hat{\Lambda}$  has got a compact support.

Let us define for  $q \in \{0, \dots, N - 1\}$ ,  $\sigma \in \mathbb{N}_0$ ,  $m = 1, \dots$  as well as  $\omega \in \mathbb{C}_+ \setminus \{0\}$  and  $\nu_{\sigma} := N/2 + \sigma$

$$\mathbb{E}_{\sigma, m}^{1, \omega} := \sum_{k=0}^{\infty} (-i\omega)^{2k} - D_{\sigma, m}^{q, 2k+1} + \kappa_{\sigma}^{q+1} \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k} + D_{\sigma, m}^{q, 2k+1} \quad , \quad (3.1)$$

$$\begin{aligned} \mathbb{H}_{\sigma, m}^{1, \omega} &:= \frac{i}{\omega} \operatorname{rot} \mathbb{E}_{\sigma, m}^{1, \omega} \\ &= \sum_{k=0}^{\infty} (-i\omega)^{2k-1} - R_{\sigma, m}^{q+1, 2k} + \kappa_{\sigma}^{q+1} \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k-1} + R_{\sigma, m}^{q+1, 2k} \end{aligned} \quad (3.2)$$

and

$$\mathbb{H}_{\sigma, m}^{2, \omega} := \sum_{k=0}^{\infty} (-i\omega)^{2k} - R_{\sigma, m}^{q+1, 2k+1} + \kappa_{\sigma}^q \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k} + R_{\sigma, m}^{q+1, 2k+1} \quad , \quad (3.3)$$

$$\begin{aligned} \mathbb{E}_{\sigma, m}^{2, \omega} &:= \frac{i}{\omega} \operatorname{div} \mathbb{H}_{\sigma, m}^{2, \omega} \\ &= \sum_{k=0}^{\infty} (-i\omega)^{2k-1} - D_{\sigma, m}^{q, 2k} + \kappa_{\sigma}^q \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k-1} + D_{\sigma, m}^{q, 2k} \quad , \end{aligned} \quad (3.4)$$

where  $\kappa_{\sigma}^q := 2i\nu_{\sigma} 4^{-\nu_{\sigma}} \frac{\Gamma(1-\nu_{\sigma})}{\Gamma(1+\nu_{\sigma})} (-1)^{\nu_{\sigma}+1/2+\delta_{q,0}+\delta_{q,N}}$  and  $\Gamma$  denotes the gamma-function.

These series of  $q$ - resp.  $(q+1)$ -forms converge uniformly on compact subsets of  $\mathbb{R}^N \setminus \{0\}$  and there they define  $C^{\infty}$ -forms. Moreover, they solve

$$(M + i\omega)(E, H) = (0, 0) \quad \text{and hence} \quad (\Delta + \omega^2)(E, H) = (0, 0)$$

in  $\mathbb{R}^N \setminus \{0\}$  since clearly  $(\operatorname{div} E, \operatorname{rot} H) = (0, 0)$ . For real frequencies  $\omega \neq 0$  they fulfill Sommerfeld's (componentwise for the Helmholtz' equation) and Maxwell's radiation condition and for nonreal frequencies  $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$  they decay exponentially at infinity. Moreover,  $(\mathbb{E}_{\sigma, m}^{n, \omega}, \mathbb{H}_{\sigma, m}^{n, \omega})$ ,  $n = 1, 2$ , belong to the Sobolev spaces  $\mathbf{H}_{<-\frac{1}{2}}^{k, q, q+1}(A(1))$  for any  $k \in \mathbb{N}_0$  as well as

$$\begin{aligned}
\mathbb{E}_{\sigma,m}^{1,\omega} &= \frac{\omega^{\nu_\sigma}}{\beta_\sigma} r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\tau} T_{\sigma,m}^q \quad , \\
\mathbb{H}_{\sigma,m}^{1,\omega} &= \frac{\omega^{\nu_\sigma-1}}{\beta_\sigma} r^{-\frac{N}{2}} \left( -\omega_\sigma^q H_{\nu_\sigma}^1(\omega r) \check{\tau} S_{\sigma,m}^{q+1} \right. \\
&\quad \left. + i \left( (N/2 - (q+1)') H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \check{\rho} T_{\sigma,m}^q \right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\mathbb{E}_{\sigma,m}^{2,\omega} &= \frac{\omega^{\nu_\sigma-1}}{\beta_\sigma} r^{-\frac{N}{2}} \left( \omega_\sigma^{q-1} H_{\nu_\sigma}^1(\omega r) \check{\rho} T_{\sigma,m}^{q-1} \right. \\
&\quad \left. + i \left( (N/2 - q) H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \check{\tau} S_{\sigma,m}^q \right) \quad , \\
\mathbb{H}_{\sigma,m}^{2,\omega} &= \frac{\omega^{\nu_\sigma}}{\beta_\sigma} r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\rho} S_{\sigma,m}^q
\end{aligned} \tag{3.6}$$

hold, where  $H_{\nu_\sigma}^1$  denotes Hankel's first function and  $\beta_\sigma := i \frac{2^{\nu_\sigma}}{\Gamma(1-\nu_\sigma)} (-1)^{\nu_\sigma+1/2}$ . For details and proofs we refer to [24, Sektion 5.5]. Compare also with [45, (84)] and [48, section 4].

Now let us turn to the calculation of the asymptotics of  $\mathcal{L}_\omega e_{\sigma,m}^{-,2}$ .

$$\eta(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \in \mathring{\mathbf{H}}_{<-\frac{1}{2}}^{\infty,q,q+1}(\Omega)$$

(using an obvious notation) fulfills the radiation condition and solves

$$(M + i\omega\Lambda)\eta(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) = (M + i\omega)\eta(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) = C_{M,\eta}(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \quad .$$

Hence,

$$\mathcal{L}_\omega C_{M,\eta}(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) = \eta(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \quad . \tag{3.7}$$

Since  $C_{M,\eta}$  has compactly supported coefficients

$$\mathcal{L}_\omega(M + i\omega\Lambda)C_{M,\eta}(-D_{\sigma,m}^{q,1}, 0) = C_{M,\eta}(-D_{\sigma,m}^{q,1}, 0)$$

holds and therefore

$$\mathcal{L}_\omega MC_{M,\eta}(-D_{\sigma,m}^{q,1}, 0) = (\text{Id} - i\omega \mathcal{L}_\omega)C_{M,\eta}(-D_{\sigma,m}^{q,1}, 0) \quad . \tag{3.8}$$

With (2.9) we compute

$$\begin{aligned}
\mathcal{L}_\omega e_{\sigma,m}^{-,2} &= \mathcal{L}_\omega M C_{M,\eta}(-D_{\sigma,m}^{q,1}, 0) + \mathcal{L}_\omega C_{M,\eta}(0, -R_{\sigma,m}^{q+1,0}) \\
&\stackrel{(3.8)}{=} C_{M,\eta}(-D_{\sigma,m}^{q,1}, 0) + \mathcal{L}_\omega C_{M,\eta}((0, -R_{\sigma,m}^{q+1,0}) - i\omega(-D_{\sigma,m}^{q,1}, 0)) \\
&\stackrel{(2.8),(3.7)}{=} e_{\sigma,m}^{-,1} - \eta(0, -R_{\sigma,m}^{q+1,0}) - i\omega\eta(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \\
&\quad + i\omega \mathcal{L}_\omega C_{M,\eta}((\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) - \frac{i}{\omega}(0, -R_{\sigma,m}^{q+1,0}) - (-D_{\sigma,m}^{q,1}, 0)) \\
&= e_{\sigma,m}^{-,1} - i\omega\eta(-D_{\sigma,m}^{q,1}, 0) \\
&\quad + i\omega(\mathcal{L}_\omega C_{M,\eta} - \eta)((\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \\
&\quad - \frac{i}{\omega}(0, -R_{\sigma,m}^{q+1,0}) - (-D_{\sigma,m}^{q,1}, 0)) \quad .
\end{aligned}$$

According to Lemma 2.17 we may write

$$e_{\sigma,m}^{-,1} = \mathcal{L}_0 e_{\sigma,m}^{-,2} \quad , \quad \eta(-D_{\sigma,m}^{q,1}, 0) = e_{\sigma,m}^{-,0} = \mathcal{L}_0 \mathcal{L} e_{\sigma,m}^{-,2}$$

and obtain

$$\begin{aligned}
&\mathcal{L}_{\omega,1} e_{\sigma,m}^{-,2} \\
&= i\omega(\mathcal{L}_\omega C_{M,\eta} - \eta)((\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) - \frac{i}{\omega}(0, -R_{\sigma,m}^{q+1,0}) - (-D_{\sigma,m}^{q,1}, 0)) \quad .
\end{aligned}$$

Now inserting the expansions (3.1) and (3.2) in each case the first term of the  $(-)$ -series of  $\mathbb{E}_{\sigma,m}^{1,\omega}$  resp.  $\mathbb{H}_{\sigma,m}^{1,\omega}$  is killed and we achieve

$$\begin{aligned}
\mathcal{L}_{\omega,1} e_{\sigma,m}^{-,2} &= (\eta - \mathcal{L}_\omega C_{M,\eta}) \left( \sum_{k=1}^{\infty} (-i\omega)^{2k} (M - i\omega)(-D_{\sigma,m}^{q,2k+1}, 0) \right. \\
&\quad \left. + \kappa_\sigma \omega^{N+2\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} (M - i\omega)(+D_{\sigma,m}^{q,2k+1}, 0) \right) \quad (3.9)
\end{aligned}$$

with  $\kappa_\sigma := \kappa_\sigma^{q+1} = 2i\nu_\sigma 4^{-\nu_\sigma} \frac{\Gamma(1-\nu_\sigma)}{\Gamma(1+\nu_\sigma)} (-1)^{\nu_\sigma+1/2}$ .

Since the series converge locally uniformly for  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  in  $\mathbb{R}^N \setminus \{0\}$ , they converge in particular in  $L_{\text{loc}}^{2,q,q+1}(\bar{\Omega})$ . Consequently the series  $C_{M,\eta} \sum \dots$  converge in  $L_s^{2,q,q+1}(\Omega)$  for all  $s \in \mathbb{R}$  because of the compact support of  $C_{M,\eta}$ . The continuity of  $\mathcal{L}_\omega$  yields the convergence of the series  $\mathcal{L}_\omega C_{M,\eta} \sum \dots = \sum \mathcal{L}_\omega C_{M,\eta} \dots$  in  $L_{<-\frac{1}{2}}^{2,q,q+1}(\Omega)$ .

Let  $\Omega_b$  denote a bounded subdomain of  $\Omega$  with  $\text{supp } \nabla \eta \subset \Omega_b$ . We look at

$$(f, g) := C_{M,\eta}(M - i\omega)(\pm D_{\sigma,m}^{q,2k+1}, 0) = (C_{\text{div},\eta} \pm R_{\sigma,m}^{q+1,2k}, -i\omega C_{\text{rot},\eta} \pm D_{\sigma,m}^{q,2k+1}) \quad .$$

By [26, Remark 2.4] we get

$$\text{div } f = -\text{div } \eta \pm D_{\sigma,m}^{q,2k-1} = 0 \quad , \quad \text{rot } g = i\omega \text{rot } \eta \pm R_{\sigma,m}^{q+1,2k} = i\omega C_{\text{rot},\eta} \pm R_{\sigma,m}^{q+1,2k}$$

and moreover  $(f, g)$  is perpendicular to  $\mathring{B}^q(\Omega) \times B^{q+1}(\Omega)$  because  $\text{supp}(f, g) \subset \text{supp } \nabla \eta$ . Furthermore, every  $\|\cdot\|_{L_s^{2,q,q+1}(\Omega)}$ -norm is equivalent to the  $\|\cdot\|_{L^{2,q,q+1}(\Omega_b)}$ -norm for

$(f, g)$ . [26, Remark 2.2 (v)] and [25, Lemma 4.2 (iii)] yield (with generic constants  $c > 0$ )

$$\begin{aligned} \|\mathcal{L}_\omega(f, g)\|_{L^{2,q,q+1}(\Omega_b)} &\leq c \left( \|(f, g)\|_{L_s^{2,q,q+1}(\Omega)} + \|C_{\text{rot},\eta} \pm R_{\sigma,m}^{q+1,2k}\|_{L_s^{2,q,q+1}(\Omega)} \right) \\ &\leq c \left( \|\pm D_{\sigma,m}^{q,2k+1}\|_{L^{2,q,q+1}(\Omega_b)} + \|\pm R_{\sigma,m}^{q+1,2k}\|_{L^{2,q,q+1}(\Omega_b)} \right) \leq c \end{aligned}$$

and thus also

$$\|(\eta - \mathcal{L}_\omega C_{M,\eta})(M - i\omega)(\pm D_{\sigma,m}^{q,2k+1}, 0)\|_{L^{2,q,q+1}(\Omega_b)} \leq c$$

all uniformly in  $k$  and  $\sigma, m$  as well as  $\omega$  (See [26, Remark 2.2 (v)]). For  $K > J$  we obtain by (3.9)

$$\begin{aligned} &\left\| \mathcal{L}_{\omega,1} e_{\sigma,m}^{-,2} - (\eta - \mathcal{L}_\omega C_{M,\eta}) \left( \sum_{k=1}^{K-1} (-i\omega)^{2k} (M - i\omega)(-D_{\sigma,m}^{q,2k+1}, 0) \right. \right. \\ &\quad \left. \left. + \kappa_\sigma \omega^{N+2\sigma} \sum_{k=0}^{K-1} (-i\omega)^{2k} (M - i\omega)(+D_{\sigma,m}^{q,2k+1}, 0) \right) \right\|_{L^{2,q,q+1}(\Omega_b)} \\ &\leq c \sum_{k=K}^{\infty} |\omega|^{2k} \leq c |\omega|^{2K} \quad . \end{aligned}$$

Once again let us introduce a new short notation:

$$u \stackrel{\ell}{\sim} v \quad :\Leftrightarrow \quad \|u - v\|_{L^{2,q,q+1}(\Omega_b)} \leq c |\omega|^\ell \quad \text{uniformly w. r. t. } \omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$$

Using this new notation we have shown so far

$$\begin{aligned} \mathcal{L}_{\omega,1} e_{\sigma,m}^{-,2} &\stackrel{2K}{\sim} \sum_{k=1}^{K-1} (-i\omega)^{2k} (\eta - \mathcal{L}_\omega C_{M,\eta})(M - i\omega)(-D_{\sigma,m}^{q,2k+1}, 0) \\ &\quad + \kappa_\sigma \omega^{N+2\sigma} \sum_{k=0}^{K-1} (-i\omega)^{2k} (\eta - \mathcal{L}_\omega C_{M,\eta})(M - i\omega)(+D_{\sigma,m}^{q,2k+1}, 0) \end{aligned} \quad (3.10)$$

and the only unknown  $\omega$ -behavior is hidden in the terms

$$\mathcal{L}_\omega C_{M,\eta}(M - i\omega)(\pm D_{\sigma,m}^{q,2k+1}, 0) \quad .$$

Using  $C_{M^2,\eta} = MC_{M,\eta} + C_{M,\eta}M$  and [26, Remark 2.4] we compute

$$\begin{aligned} &\mathcal{L}_\omega C_{M,\eta}(M - i\omega)(\pm D_{\sigma,m}^{q,2k+1}, 0) \\ &= \mathcal{L}_\omega C_{M^2,\eta}(\pm D_{\sigma,m}^{q,2k+1}, 0) - \mathcal{L}_\omega(M + i\omega)C_{M,\eta}(\pm D_{\sigma,m}^{q,2k+1}, 0) \\ &= \mathcal{L}_\omega C(\pm D_{\sigma,m}^{q,2k+1}, 0) - \mathcal{L}_\omega(M + i\omega\Lambda)C_{M,\eta}(\pm D_{\sigma,m}^{q,2k+1}, 0) \\ &= \mathcal{L}_\omega C(\pm D_{\sigma,m}^{q,2k+1}, 0) - C_{M,\eta}(\pm D_{\sigma,m}^{q,2k+1}, 0) \end{aligned}$$

and then for  $k \in \mathbb{N}_0$

$$\begin{aligned} & (\eta - \mathcal{L}_\omega C_{M,\eta})(M - i\omega)(\pm D_{\sigma,m}^{q,2k+1}, 0) \\ &= -\mathcal{L}_\omega C(\pm D_{\sigma,m}^{q,2k+1}, 0) + (M - i\omega)\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) \end{aligned} \quad (3.11)$$

If  $k \geq 1$  we have

$$\begin{aligned} M^2\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) &= C_{M^2,\eta}(\pm D_{\sigma,m}^{q,2k+1}, 0) + \eta(\pm D_{\sigma,m}^{q,2k-1}, 0) \\ &= C(\pm D_{\sigma,m}^{q,2k+1}, 0) + \eta(\pm D_{\sigma,m}^{q,2k-1}, 0) \in \text{Reg}_{\text{loc}}^{q,0}(\Omega) \end{aligned} .$$

We note once more

$$\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) \in \text{Reg}_{\text{loc}}^{q,0}(\Omega) \quad (3.12)$$

by [26, Remark 2.4]. Thus, according to [26, Theorem 5.10]  $\mathcal{L}^2$  may be applied to  $M^2\eta(\pm D_{\sigma,m}^{q,2k+1}, 0)$  and we obtain

$$\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) = \mathcal{L}^2 (C(\pm D_{\sigma,m}^{q,2k+1}, 0) + \eta(\pm D_{\sigma,m}^{q,2k-1}, 0)) .$$

By (3.12)  $\mathcal{L}$  and  $\mathcal{L}^2$  are even well defined on  $\eta(\pm D_{\sigma,m}^{q,2k-1}, 0)$ , such that

$$\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) = \mathcal{L}^2 C(\pm D_{\sigma,m}^{q,2k+1}, 0) + \mathcal{L}^2 \eta(\pm D_{\sigma,m}^{q,2k-1}, 0)$$

holds. A short induction shows

$$\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) = \sum_{\ell=1}^k \mathcal{L}^{2\ell} C(\pm D_{\sigma,m}^{q,2k+3-2\ell}, 0) + \mathcal{L}^{2k} \eta(\pm D_{\sigma,m}^{q,1}, 0) ,$$

and hence using  $\Lambda^{-1} = \text{Id}$  on  $\text{supp } \eta$

$$\begin{aligned} \eta(\pm D_{\sigma,m}^{q,2k+1}, 0) &= \sum_{\ell=1}^k \mathcal{L}_0 \mathcal{L}^{2\ell-1} C(\pm D_{\sigma,m}^{q,2k+3-2\ell}, 0) + \mathcal{L}_0 \mathcal{L}^{2k-1} \eta(\pm D_{\sigma,m}^{q,1}, 0) , \\ M\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) &= \sum_{\ell=1}^k \mathcal{L}_0 \mathcal{L}^{2\ell-2} C(\pm D_{\sigma,m}^{q,2k+3-2\ell}, 0) + \mathcal{L}_0 \mathcal{L}^{2k-2} \eta(\pm D_{\sigma,m}^{q,1}, 0) . \end{aligned}$$

We remind of  $\eta(\pm D_{\sigma,m}^{q,1}, 0) = e_{\sigma,m}^{\pm,0}$  and  $C(\pm D_{\sigma,m}^{q,1}, 0) = e_{\sigma,m}^{\pm,2}$ . Putting all together yields for  $k \geq 1$

$$\begin{aligned} & (M - i\omega)\eta(\pm D_{\sigma,m}^{q,2k+1}, 0) \\ &= (\mathcal{L}_0 \mathcal{L}^{2k-2} - i\omega \mathcal{L}_0 \mathcal{L}^{2k-1})e_{\sigma,m}^{\pm,0} \\ & \quad + \sum_{\ell=1}^k (\mathcal{L}_0 \mathcal{L}^{2\ell-2} - i\omega \mathcal{L}_0 \mathcal{L}^{2\ell-1})C(\pm D_{\sigma,m}^{q,2k+3-2\ell}, 0) . \end{aligned} \quad (3.13)$$

Inserting this formula into (3.11) and all this together into (3.10) we obtain

$$\begin{aligned} & \mathcal{L}_{\omega,1} e_{\sigma,m}^{-,2} \\ \approx & {}^{2K}S^- + \kappa_\sigma \omega^{N+2\sigma} S^+ + \kappa_\sigma \omega^{N+2\sigma} \left( -\mathcal{L}_\omega e_{\sigma,m}^{\pm,2} + (M - i\omega) e_{\sigma,m}^{\pm,0} \right) \end{aligned} \quad , \quad (3.14)$$

where  $S^\pm := -S_I^\pm + S_{II}^\pm + S_{III}^\pm$  and

$$\begin{aligned} S_I^\pm &:= \sum_{k=1}^{K-1} (-i\omega)^{2k} \mathcal{L}_\omega C(\pm D_{\sigma,m}^{q,2k+1}, 0) \quad , \\ S_{II}^\pm &:= \sum_{k=1}^{K-1} (-i\omega)^{2k} (\mathcal{L}_0 \mathcal{L}^{2k-2} - i\omega \mathcal{L}_0 \mathcal{L}^{2k-1}) e_{\sigma,m}^{\pm,0} \quad , \\ S_{III}^\pm &:= \sum_{k=1}^{K-1} (-i\omega)^{2k} \sum_{\ell=1}^k (\mathcal{L}_0 \mathcal{L}^{2\ell-2} - i\omega \mathcal{L}_0 \mathcal{L}^{2\ell-1}) C(\pm D_{\sigma,m}^{q,2k+3-2\ell}, 0) \quad . \end{aligned}$$

Obviously

$$S_{II}^\pm = \sum_{k=2}^{2K-1} (-i\omega)^k \mathcal{L}_0 \mathcal{L}^{k-2} e_{\sigma,m}^{\pm,0} \quad . \quad (3.15)$$

In the double sums  $S_{III}^\pm$  we substitute  $\ell$  by  $j(\ell) := k - \ell + 1$ , interchange the sums and again substitute  $k$  by  $i(k) := k - j$ . Then we denote the pair  $(j, i)$  again by  $(k, \ell)$ . We get

$$S_{III}^\pm = \sum_{k=1}^{K-1} (-i\omega)^{2k} \sum_{\ell=0}^{2K-2k-1} (-i\omega)^\ell \mathcal{L}_0 \mathcal{L}^\ell C(\pm D_{\sigma,m}^{q,2k+1}, 0)$$

and thus

$$S_I^\pm - S_{III}^\pm = \sum_{k=1}^{K-1} (-i\omega)^{2k} \mathcal{L}_{\omega,2K-2k-1} C(\pm D_{\sigma,m}^{q,2k+1}, 0) \quad .$$

We have  $C(\pm D_{\sigma,m}^{q,2k+1}, 0) \in \text{Reg}_{\text{vox}}^{q,0}(\Omega)$  and also for all  $k \geq 1$  and  $j \leq 2K$  as well as  $\tilde{s} \in (2K - N/2, \infty) \setminus \mathbb{I}$  according to Lemma 2.15

$$C(\pm D_{\sigma,m}^{q,2k+1}, 0) \in \text{Reg}_{\tilde{s}}^{q,j}(\Omega) \quad ,$$

because for all  $(\ell, \gamma, n)$  with  $\ell \leq 2K$  we may compute

$$\langle C(\pm D_{\sigma,m}^{q,2k+1}, 0), E_{\gamma,n}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle C(\pm D_{\sigma,m}^{q,2k+1}, 0), H_{\gamma,n}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0$$

using Lemma 2.9, the expansions from Corollary 2.8 and observing  $2k + 1 \geq 3$ . In particular for  $1 \leq k \leq K - 1$

$$C(\pm D_{\sigma,m}^{q,2k+1}, 0) \in \text{Reg}_{\tilde{s}}^{q,2K-2k}(\Omega)$$

holds. Therefore, Theorem 2.3 (i) yields uniformly in  $\omega$  (and  $k, \sigma, m$  by [26, Remark 2.2 (v)])

$$\begin{aligned} & \left\| \mathcal{L}_{\omega, 2K-2k-1} C(\pm D_{\sigma, m}^{q, 2k+1}, 0) \right\|_{L^{2, q, q+1}(\Omega_b)} \\ & \leq c |\omega|^{2K-2k} \left\| C(\pm D_{\sigma, m}^{q, 2k+1}, 0) \right\|_{L_s^{2, q, q+1}(\Omega)} \leq c |\omega|^{2K-2k} \end{aligned}$$

and we obtain

$$S_I^\pm - S_{III}^\pm \stackrel{2K}{\sim} (0, 0) \quad .$$

Since  $e_{\sigma, m}^{-, 0} = \mathcal{L}^2 e_{\sigma, m}^{-, 2}$  by Lemma 2.17 we see from (3.15)

$$S_{II}^- = \sum_{k=2}^{2K-1} (-i\omega)^k \mathcal{L}_0 \mathcal{L}^k e_{\sigma, m}^{-, 2}$$

and inserting all in (3.14) (using (3.15) again) we arrive at

$$\begin{aligned} & \mathcal{L}_{\omega, 2K-1} e_{\sigma, m}^{-, 2} = \mathcal{L}_{\omega, 1} e_{\sigma, m}^{-, 2} - S_{II}^- \\ & \stackrel{2K}{\sim} \kappa_\sigma \omega^{N+2\sigma} \left( \left( \sum_{k=2}^{2K-1} (-i\omega)^k \mathcal{L}_0 \mathcal{L}^{k-2} + (M - i\omega) \right) e_{\sigma, m}^{+, 0} - \mathcal{L}_\omega e_{\sigma, m}^{+, 2} \right) \quad . \end{aligned}$$

For each  $\mathbb{N} \ni j \leq \mathbf{J} + 1$  choosing some  $K \in \mathbb{N}_0$  with  $2K \geq j$  we finally obtain

$$\begin{aligned} & \mathcal{L}_{\omega, j-1} e_{\sigma, m}^{-, 2} \\ & \stackrel{j}{\sim} \kappa_\sigma \omega^{N+2\sigma} \left( \left( \sum_{k=2}^{j-N-2\sigma-1} (-i\omega)^k \mathcal{L}_0 \mathcal{L}^{k-2} + (M - i\omega) \right) e_{\sigma, m}^{+, 0} - \mathcal{L}_\omega e_{\sigma, m}^{+, 2} \right) \quad . \end{aligned} \quad (3.16)$$

It remains to identify the terms. The equation

$$e_{\sigma, m}^{+, 1} = M\eta(+D_{\sigma, m}^{q, 1}, 0) = C_{M, \eta}(+D_{\sigma, m}^{q, 1}, 0) + \eta(0, +R_{\sigma, m}^{q+1, 0}) \quad ,$$

(3.12) and [26, Theorem 5.10] show

$$\mathcal{L}_0 e_{\sigma, m}^{+, 1} = e_{\sigma, m}^{+, 0} \quad . \quad (3.17)$$

Using (3.17) in (3.16) we get

$$\begin{aligned} & \mathcal{L}_{\omega, j-1} e_{\sigma, m}^{-, 2} \\ & \stackrel{j}{\sim} \kappa_\sigma \omega^{N+2\sigma} \left( \sum_{k=0}^{j-N-2\sigma-1} (-i\omega)^k \Lambda^{-1} \mathcal{L}^k e_{\sigma, m}^{+, 1} - \mathcal{L}_\omega e_{\sigma, m}^{+, 2} \right) \quad . \end{aligned} \quad (3.18)$$

Since  $e_{\sigma,m}^{+,2} \in \text{Reg}_{\text{vox}}^{q,0}(\Omega)$  we may look at

$$(0, h) := e_{\sigma,m}^{+,1} - \mathcal{L}_0 e_{\sigma,m}^{+,2}$$

utilizing [26, Corollary 5.12]. With  $M(0, h) = (0, 0)$  and  $\text{rot } \mu h = 0$  we have

$$\mu h \in \mu^{-1} \mathcal{H}_{<-\frac{N}{2}-\sigma}^{q+1}(\Omega) \cap \mathbb{B}^{q+1}(\Omega)^\perp, \quad h - {}^+ R_{\sigma,m}^{q+1,0} \in \mathbb{L}_{>-\frac{N}{2}}^{2,q+1}(\Omega).$$

Hence,  $H_{\sigma,m}^+ = h$  by Lemma 2.4. Finally (3.18) turns to

$$\begin{aligned} & \mathcal{L}_{\omega,j-1} e_{\sigma,m}^{-,2} \\ & \stackrel{j}{\sim} \kappa_\sigma \omega^{N+2\sigma} \left( \sum_{k=0}^{j-N-2\sigma-1} (-i\omega)^k \Lambda^{-1} H_{\sigma,m}^{+,k} - \mathcal{L}_{\omega,j-N-2\sigma-1} e_{\sigma,m}^{+,2} \right) \\ & =: \kappa_\sigma \omega^{N+2\sigma} \mathcal{A}_{\omega,\sigma,m}^{j-N-2\sigma-1}. \end{aligned} \quad (3.19)$$

Similar calculations using the forms  $(\mathbb{E}_{\sigma,m}^{2,\omega}, \mathbb{H}_{\sigma,m}^{2,\omega})$  from (3.4), (3.3) and looking at  $(0, \pm R_{\sigma,m}^{q+1,2k+1})$  yield a corresponding estimate for  $\mathcal{L}_\omega h_{\sigma,m}^{-,2}$ , i.e.

$$\begin{aligned} & \mathcal{L}_{\omega,j-1} h_{\sigma,m}^{-,2} \\ & \stackrel{j}{\sim} \kappa_\sigma \omega^{N+2\sigma} \left( \sum_{k=0}^{j-N-2\sigma-1} (-i\omega)^k \Lambda^{-1} E_{\sigma,m}^{+,k} - \mathcal{L}_{\omega,j-N-2\sigma-1} h_{\sigma,m}^{+,2} \right) \\ & =: \kappa_\sigma \omega^{N+2\sigma} \mathcal{B}_{\omega,\sigma,m}^{j-N-2\sigma-1}. \end{aligned} \quad (3.20)$$

**Lemma 3.2** *Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s \in (\mathbf{J} + 1/2, \mathbf{J} + N/2) \setminus \mathbb{I}$ . Then for all bounded subdomains  $\Omega_b$  of  $\Omega$*

$$\begin{aligned} & \left\| \mathcal{L}_{\omega,\mathbf{J}-1}(F, G) \right. \\ & - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\mathbf{J}-1-N}^q} (-i\omega)^{N+k} \kappa_{k,\sigma} \langle (F, G), E_{\sigma,m}^{+,k-2\sigma+1} \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)} \mathcal{B}_{\omega,\sigma,m}^{\mathbf{J}-1-N-k} \\ & - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\mathbf{J}-1-N}^{q+1}} (-i\omega)^{N+k} \kappa_{k,\sigma} \langle (F, G), H_{\sigma,m}^{+,k-2\sigma+1} \rangle_{\mathbb{L}^{2,q,q+1}(\Omega)} \mathcal{A}_{\omega,\sigma,m}^{\mathbf{J}-1-N-k} \left. \right\|_{\mathbb{L}^{2,q,q+1}(\Omega_b)} \\ & = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{\mathbb{L}_s^{2,q,q+1}(\Omega)} \end{aligned}$$

holds uniformly with respect to  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$ .

Here  $\kappa_{k,\sigma} := i^{2k-2\sigma+N} \kappa_\sigma$  and

$$\tilde{\Theta}_j^q := \{(k, \sigma, m) \in \mathbb{N}_0^3 : 2\sigma \leq k \leq j \wedge 1 \leq m \leq \mu_\sigma^q\}.$$

In particular for  $j \leq \min\{\mathbf{J}, N\}$

$$\left\| \mathcal{L}_{\omega,j-1}(F, G) \right\|_{\mathbb{L}^{2,q,q+1}(\Omega_b)} = \mathcal{O}(|\omega|^j) \cdot \|(F, G)\|_{\mathbb{L}_s^{2,q,q+1}(\Omega)}.$$



**Proof:** We insert the asymptotics of (3.19), (3.20) in the estimates of Lemma 3.1 with  $j := \mathbf{J} - k$ . The sums range over  $\Theta_s^{q,\mathbf{J}}$  and  $\Theta_s^{q+1,\mathbf{J}}$ . In particular for  $(k, \sigma, m) \in \Theta_s^{q,\mathbf{J}}$  we get

$$0 \leq k \leq \mathbf{J} - 1 \quad , \quad 0 \leq \sigma < s - N/2 - k - 1 \quad , \quad 1 \leq m \leq \mu_\sigma^q \quad .$$

Additionally we have the condition  $k + 2\sigma + N \leq \mathbf{J} - 1$  since higher order terms may be swallowed by the  $\mathcal{O}$ -term. Because  $\mathbf{J} + 1/2 < s < \mathbf{J} + N/2$  we only sum over

$$\begin{aligned} 0 \leq k \leq \mathbf{J} - 1 - N \quad , \\ 0 \leq \sigma \leq \min \left\{ s - \frac{N}{2} - k - 1, \frac{\mathbf{J} - 1 - N - k}{2} \right\} = \frac{\mathbf{J} - 1 - N - k}{2} \quad , \\ 1 \leq m \leq \mu_\sigma^q \quad . \end{aligned}$$

We interchange the sums over  $k$  and  $\sigma$ , set  $\ell(k) := k + 2\sigma$ , interchange  $\sigma$  and  $\ell$  and finally denote  $\ell$  again by  $k$ . This proves the first assertion. Once more recalling

$$e_{\sigma,m}^{+,2}, h_{\sigma,m}^{+,2} \in \text{Reg}_{\text{vox}}^{q,0}(\Omega) \quad (3.21)$$

we apply [25, Lemma 4.2 (iv)] and get

$$\mathcal{L}_\omega e_{\sigma,m}^{+,2}, \mathcal{L}_\omega h_{\sigma,m}^{+,2} \overset{0}{\sim} (0, 0) \quad .$$

Thus,  $\mathcal{A}_{\omega,\sigma,m}^\ell, \mathcal{B}_{\omega,\sigma,m}^\ell \overset{0}{\sim} (0, 0)$ , which yields the second assertion. ■

In the following we often use without further reference an uniqueness result for asymptotic expansions.

**Lemma 3.3** *Let  $\tilde{L}, L \in \mathbb{N}_0$  and  $x_{-\tilde{L}}, \dots, x_L$  be elements of some normed space  $X$ . Moreover, let*

$$\left\| \sum_{\ell=-\tilde{L}}^L \omega^\ell x_\ell \right\|_X = o(|\omega|^L)$$

*hold uniformly with respect to  $\mathbb{C}_+ \setminus \{0\} \ni \omega \rightarrow 0$ . Then all  $x_\ell$  vanish.*

According to (3.19), (3.20) we have

$$\mathcal{A}_{\omega,\sigma,m}^k = \mathbb{X}_{\sigma,m}^k(\omega) - \mathcal{L}_{\omega,k} e_{\sigma,m}^{+,2} \quad , \quad \mathcal{B}_{\omega,\sigma,m}^k = \mathbb{Y}_{\sigma,m}^k(\omega) - \mathcal{L}_{\omega,k} h_{\sigma,m}^{+,2}$$

with polynomials

$$\mathbb{X}_{\sigma,m}^k(\omega) := \sum_{\ell=0}^k (-i\omega)^\ell \Lambda^{-1} H_{\sigma,m}^{+,\ell} \quad , \quad \mathbb{Y}_{\sigma,m}^k(\omega) := \sum_{\ell=0}^k (-i\omega)^\ell \Lambda^{-1} E_{\sigma,m}^{+,\ell} \quad (3.22)$$

of degree  $k$  in  $\omega$ . By (3.21) we may apply Lemma 3.2 with  $\mathbf{J} := j$  to

$$(F, G) := e_{\gamma,n}^{+,2}, h_{\gamma,\nu}^{+,2} \quad ,$$

which yields the asymptotics

$$\begin{aligned}
& \mathcal{L}_{\omega, j-1} e_{\gamma, n}^{+,2} \\
& \stackrel{j}{\sim} (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-1-N}^q} (-i\omega)^k \beta_{e, \gamma, n}^{k, \sigma, m} \left( \mathbb{Y}_{\sigma, m}^{j-1-N-k}(\omega) - \mathcal{L}_{\omega, j-1-N-k} h_{\sigma, m}^{+,2} \right) \\
& + (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-1-N}^{q+1}} (-i\omega)^k \alpha_{e, \gamma, n}^{k, \sigma, m} \left( \mathbb{X}_{\sigma, m}^{j-1-N-k}(\omega) - \mathcal{L}_{\omega, j-1-N-k} e_{\sigma, m}^{+,2} \right)
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
& \mathcal{L}_{\omega, j-1} h_{\gamma, \nu}^{+,2} \\
& \stackrel{j}{\sim} (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-1-N}^q} (-i\omega)^k \beta_{h, \gamma, \nu}^{k, \sigma, m} \left( \mathbb{Y}_{\sigma, m}^{j-1-N-k}(\omega) - \mathcal{L}_{\omega, j-1-N-k} h_{\sigma, m}^{+,2} \right) \\
& + (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-1-N}^{q+1}} (-i\omega)^k \alpha_{h, \gamma, \nu}^{k, \sigma, m} \left( \mathbb{X}_{\sigma, m}^{j-1-N-k}(\omega) - \mathcal{L}_{\omega, j-1-N-k} e_{\sigma, m}^{+,2} \right) ,
\end{aligned} \tag{3.24}$$

where

$$\beta_{e, \gamma, n}^{k, \sigma, m} := \kappa_{k, \sigma} \langle e_{\gamma, n}^{+,2}, E_{\sigma, m}^{+, k-2\sigma+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} , \tag{3.25}$$

$$\alpha_{e, \gamma, n}^{k, \sigma, m} := \kappa_{k, \sigma} \langle e_{\gamma, n}^{+,2}, H_{\sigma, m}^{+, k-2\sigma+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} , \tag{3.26}$$

$$\beta_{h, \gamma, \nu}^{k, \sigma, m} := \kappa_{k, \sigma} \langle h_{\gamma, \nu}^{+,2}, E_{\sigma, m}^{+, k-2\sigma+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} , \tag{3.27}$$

$$\alpha_{h, \gamma, \nu}^{k, \sigma, m} := \kappa_{k, \sigma} \langle h_{\gamma, \nu}^{+,2}, H_{\sigma, m}^{+, k-2\sigma+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} . \tag{3.28}$$

Thus, there exist polynomials  $\tilde{\mathbb{X}}_{\gamma, n}^\ell(\omega)$  and  $\tilde{\mathbb{Y}}_{\gamma, \nu}^\ell(\omega)$  of degree  $\ell$  in  $\omega$ , such that

$$\mathcal{L}_{\omega, j-1} e_{\gamma, n}^{+,2} \stackrel{j}{\sim} \tilde{\mathbb{X}}_{\gamma, n}^{j-1}(\omega) , \quad \mathcal{L}_{\omega, j-1} h_{\gamma, \nu}^{+,2} \stackrel{j}{\sim} \tilde{\mathbb{Y}}_{\gamma, \nu}^{j-1}(\omega)$$

hold. Since

$$\mathcal{L}_{\omega, j} e_{\gamma, n}^{+,2} \stackrel{j}{\sim} \mathcal{L}_{\omega, j-1} e_{\gamma, n}^{+,2} , \quad \mathcal{L}_{\omega, j} h_{\gamma, \nu}^{+,2} \stackrel{j}{\sim} \mathcal{L}_{\omega, j-1} h_{\gamma, \nu}^{+,2}$$

the coefficients of  $\tilde{\mathbb{X}}_{\gamma, n}^\ell(\omega)$  and  $\tilde{\mathbb{Y}}_{\gamma, \nu}^\ell(\omega)$  do not depend on  $\ell$ . Consequently there exist forms

$$X_{\gamma, n}^\ell, Y_{\gamma, \nu}^\ell \in L_{\text{loc}}^{2, q, q+1}(\Omega) ,$$

such that

$$\mathcal{A}_{\omega, \gamma, n}^{j-1} \stackrel{j}{\sim} \mathbb{X}_{\gamma, n}^{j-1}(\omega) - \tilde{\mathbb{X}}_{\gamma, n}^{j-1}(\omega) =: \sum_{\ell=0}^{j-1} (-i\omega)^\ell X_{\gamma, n}^\ell , \tag{3.29}$$

$$\mathcal{B}_{\omega, \gamma, \nu}^{j-1} \stackrel{j}{\sim} \mathbb{Y}_{\gamma, \nu}^{j-1}(\omega) - \tilde{\mathbb{Y}}_{\gamma, \nu}^{j-1}(\omega) =: \sum_{\ell=0}^{j-1} (-i\omega)^\ell Y_{\gamma, \nu}^\ell . \tag{3.30}$$

We obtain immediately

$$\begin{aligned} \mathcal{L}_{\omega, j-1} e_{\sigma, m}^{-, 2} \overset{j}{\sim} \kappa_{\sigma} \omega^{N+2\sigma} \sum_{\ell=0}^{j-N-2\sigma-1} (-i\omega)^{\ell} X_{\sigma, m}^{\ell} \quad , \\ \mathcal{L}_{\omega, j-1} h_{\sigma, m}^{-, 2} \overset{j}{\sim} \kappa_{\sigma} \omega^{N+2\sigma} \sum_{\ell=0}^{j-N-2\sigma-1} (-i\omega)^{\ell} Y_{\sigma, m}^{\ell} \end{aligned}$$

and

$$\mathcal{L}_{\omega, j-1} e_{\gamma, n}^{+, 2} \overset{j}{\sim} \tilde{X}_{\gamma, n}^{j-1}(\omega) \quad , \quad \mathcal{L}_{\omega, j-1} h_{\gamma, \nu}^{+, 2} \overset{j}{\sim} \tilde{Y}_{\gamma, \nu}^{j-1}(\omega) \quad .$$

(3.23) and (3.24) yield for  $1 \leq j \leq N$

$$\mathcal{L}_{\omega, j-1} e_{\gamma, n}^{+, 2} \quad , \quad \mathcal{L}_{\omega, j-1} h_{\gamma, \nu}^{+, 2} \overset{j}{\sim} (0, 0) \quad (3.31)$$

and therefore we get for  $\ell = 0, \dots, N-1$

$$X_{\gamma, n}^{\ell} = \Lambda^{-1} H_{\gamma, n}^{+, \ell} \quad , \quad Y_{\gamma, \nu}^{\ell} = \Lambda^{-1} E_{\gamma, \nu}^{+, \ell} \quad . \quad (3.32)$$

The higher order coefficients  $X_{\gamma, n}^{\ell}, Y_{\gamma, \nu}^{\ell}$  may be computed recursively utilizing (3.23), (3.24). In particular we have for  $j \geq N$

$$\begin{aligned} \tilde{X}_{\gamma, n}^j(\omega) = (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-N}^q} (-i\omega)^k \beta_{e, \gamma, n}^{k, \sigma, m} \sum_{\ell=0}^{j-N-k} (-i\omega)^{\ell} Y_{\sigma, m}^{\ell} \\ + (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-N}^{q+1}} (-i\omega)^k \alpha_{e, \gamma, n}^{k, \sigma, m} \sum_{\ell=0}^{j-N-k} (-i\omega)^{\ell} X_{\sigma, m}^{\ell} \quad , \end{aligned} \quad (3.33)$$

$$\begin{aligned} \tilde{Y}_{\gamma, \nu}^j(\omega) = (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-N}^q} (-i\omega)^k \beta_{h, \gamma, \nu}^{k, \sigma, m} \sum_{\ell=0}^{j-N-k} (-i\omega)^{\ell} Y_{\sigma, m}^{\ell} \\ + (-i\omega)^N \sum_{(k, \sigma, m) \in \tilde{\Theta}_{j-N}^{q+1}} (-i\omega)^k \alpha_{h, \gamma, \nu}^{k, \sigma, m} \sum_{\ell=0}^{j-N-k} (-i\omega)^{\ell} X_{\sigma, m}^{\ell} \quad . \end{aligned} \quad (3.34)$$

Using (3.22) and (3.29), (3.30) we get the following recursion for the forms  $X_{\gamma,n}^\ell$ ,  $Y_{\gamma,\nu}^\ell$  and  $\ell \geq N$

$$X_{\gamma,n}^\ell = \Lambda^{-1} H_{\gamma,n}^{+,\ell} - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^q} \beta_{e,\gamma,n}^{k,\sigma,m} Y_{\sigma,m}^{\ell-N-k} - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^{q+1}} \alpha_{e,\gamma,n}^{k,\sigma,m} X_{\sigma,m}^{\ell-N-k} , \quad (3.35)$$

$$Y_{\gamma,\nu}^\ell = \Lambda^{-1} E_{\gamma,\nu}^{+,\ell} - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^q} \beta_{h,\gamma,\nu}^{k,\sigma,m} Y_{\sigma,m}^{\ell-N-k} - \sum_{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^{q+1}} \alpha_{h,\gamma,\nu}^{k,\sigma,m} X_{\sigma,m}^{\ell-N-k} . \quad (3.36)$$

Additionally we obtain

$$\begin{aligned} X_{\gamma,n}^\ell, Y_{\gamma,\nu}^\ell &\in \text{Lin}\{\Lambda^{-1} H_{\gamma,n}^{+,\ell}, \Lambda^{-1} E_{\gamma,\nu}^{+,\ell}\} \\ &\quad + \text{Lin}\{X_{\sigma,m}^{\ell-N-k}, Y_{\tilde{\sigma},\tilde{m}}^{\ell-N-\tilde{k}} : (k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^{q+1} \wedge (\tilde{k},\tilde{\sigma},\tilde{m}) \in \tilde{\Theta}_{\ell-N}^q\} \\ &\subset \text{Lin}\{\Lambda^{-1} H_{\gamma,n}^{+,\ell}, \Lambda^{-1} E_{\gamma,\nu}^{+,\ell}\} + \text{Lin}\{X_{\sigma,m}^k, Y_{\sigma,\tilde{m}}^k : k + 2\sigma \leq \ell - N\} \end{aligned}$$

and a short induction shows

$$\begin{aligned} X_{\gamma,n}^\ell, Y_{\gamma,\nu}^\ell &\in \text{Lin}\{\Lambda^{-1} H_{\gamma,n}^{+,\ell}, \Lambda^{-1} E_{\gamma,\nu}^{+,\ell}\} \\ &\quad + \text{Lin}\{\Lambda^{-1} E_{\sigma,m}^{+,k}, \Lambda^{-1} H_{\sigma,\tilde{m}}^{+,k} : k + 2\sigma \leq \ell - N\} . \end{aligned} \quad (3.37)$$

Moreover, our coefficient forms satisfy

$$MX_{\gamma,n}^0 = MY_{\gamma,\nu}^0 = (0, 0) \quad (3.38)$$

and for  $\ell \leq N - 1$

$$\Lambda^{-1} MX_{\gamma,n}^\ell = X_{\gamma,n}^{\ell-1} , \quad \Lambda^{-1} MY_{\gamma,\nu}^\ell = Y_{\gamma,\nu}^{\ell-1} . \quad (3.39)$$

Once again by induction these equations hold true for all  $\ell \geq N$ .

We may formulate the main result of step one. For this let the coefficients  $X_{\gamma,n}^\ell$  and  $Y_{\gamma,\nu}^\ell$  be defined recursively by (3.32), (3.35), (3.36) and

**Definition 3.4** Let  $\mathbf{J} \in \mathbb{N}_0$ ,  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  and  $(F, G) \in L_s^{2,q,q+1}(\Omega)$ . Then for  $j = 0, \dots, \mathbf{J} - 1 - N$  we define the ‘correction operators’

$$\begin{aligned} \hat{\Gamma}_j(F, G) := & \sum_{(k,\sigma,m) \in \tilde{\Theta}_j^q} \kappa_{k,\sigma} \langle (F, G), E_{\sigma,m}^{+,k-2\sigma+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} Y_{\sigma,m}^{j-k} \\ & + \sum_{(k,\sigma,m) \in \tilde{\Theta}_j^{q+1}} \kappa_{k,\sigma} \langle (F, G), H_{\sigma,m}^{+,k-2\sigma+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} X_{\sigma,m}^{j-k} . \end{aligned}$$

**Theorem 3.5** Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$ . Then for all bounded subdomains  $\Omega_b \subset \Omega$  the asymptotic

$$\begin{aligned} & \left\| \mathcal{L}_{\omega, \mathbf{J}-1}(F, G) - (-i\omega)^N \sum_{j=0}^{\mathbf{J}-1-N} (-i\omega)^j \hat{\Gamma}_j(F, G) \right\|_{L^{2,q,q+1}(\Omega_b)} \\ & = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \end{aligned}$$

holds uniformly with respect to  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$  and  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$ .

**Remark 3.6**

- (i) The coefficients  $X_{\gamma,n}^\ell, Y_{\gamma,\nu}^\ell$  have to be computed only for  $\ell, 2\gamma \leq \mathbf{J} - 1 - N$  and  $n = 1, \dots, \mu_\gamma^{q+1}, \nu = 1, \dots, \mu_\gamma^q$ .
- (ii) Because of (3.38), (3.39) the correction operators satisfy

$$\Lambda^{-1} M \hat{\Gamma}_j = \hat{\Gamma}_{j-1} ,$$

where  $\hat{\Gamma}_{-1} := 0$ .

- (iii) By (3.37) we have

$$\hat{\Gamma}_j(F, G) \in \text{Cor}^{q,j}(\Omega) := \text{Lin}\{\Lambda^{-1} E_{\sigma,m}^{+,k}, \Lambda^{-1} H_{\sigma,n}^{+,k} : k + 2\sigma \leq j\} .$$

Hence, the correction operators

$$\hat{\Gamma}_j : L_s^{2,q,q+1}(\Omega) \longrightarrow \text{Cor}^{q,j}(\Omega)$$

are degenerated (and clearly continuous).

- (iv)  $\hat{\Gamma}_j(F, G) = (0, 0)$  for  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$ .

**Remark 3.7** Utilizing the representations of  $E_{\sigma,m}^{+,k}$ ,  $H_{\sigma,m}^{+,k}$  from Corollary 2.8 and the orthogonality properties from Lemma 2.9 we may obtain a more detailed recursive definition of the coefficient forms  $X_{\gamma,n}^\ell$ ,  $Y_{\gamma,\nu}^\ell$ . Namely looking at (3.25) and keeping in mind  $k - 2\sigma \geq 0$  we see that  $\beta_{e,\gamma,n}^{k,\sigma,m}$  vanishes for odd  $k - 2\sigma + 1$ , i.e. even  $k$ . However, for even  $k - 2\sigma + 1$ , i.e. odd  $k$ , we get by Corollary 2.8 and Lemma 2.9

$$\beta_{e,\gamma,n}^{k,\sigma,m} = -\kappa_{k,\sigma} \zeta_{(1,\gamma,n,-)}^{k-2\sigma+1,\sigma,m} .$$

Accordingly we achieve for odd  $k$

$$\alpha_{h,\gamma,\nu}^{k,\sigma,m} = -\kappa_{k,\sigma} \zeta_{(1,\gamma,\nu,-)}^{k-2\sigma+1,\sigma,m} , \quad \alpha_{e,\gamma,n}^{k,\sigma,m} = \beta_{h,\gamma,\nu}^{k,\sigma,m} = 0$$

and for even  $k$

$$\alpha_{h,\gamma,\nu}^{k,\sigma,m} = 0 , \quad \alpha_{e,\gamma,n}^{k,\sigma,m} = -\kappa_{k,\sigma} \zeta_{(1,\gamma,n,-)}^{k-2\sigma+1,\sigma,m} , \quad \beta_{h,\gamma,\nu}^{k,\sigma,m} = -\kappa_{k,\sigma} \zeta_{(1,\gamma,\nu,-)}^{k-2\sigma+1,\sigma,m} .$$

Now our recursion (3.35), (3.36) appears in a more explicit shape

$$\begin{aligned} X_{\gamma,n}^\ell &:= \Lambda^{-1} H_{\gamma,n}^{+, \ell} + \sum_{\substack{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^q \\ k \text{ odd}}} \kappa_{k,\sigma} \zeta_{(1,\gamma,n,-)}^{k-2\sigma+1,\sigma,m} Y_{\sigma,m}^{\ell-N-k} \\ &\quad + \sum_{\substack{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^{q+1} \\ k \text{ even}}} \kappa_{k,\sigma} \zeta_{(1,\gamma,n,-)}^{k-2\sigma+1,\sigma,m} X_{\sigma,m}^{\ell-N-k} , \\ Y_{\gamma,\nu}^\ell &:= \Lambda^{-1} E_{\gamma,\nu}^{+, \ell} + \sum_{\substack{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^q \\ k \text{ even}}} \kappa_{k,\sigma} \zeta_{(1,\gamma,\nu,-)}^{k-2\sigma+1,\sigma,m} Y_{\sigma,m}^{\ell-N-k} \\ &\quad + \sum_{\substack{(k,\sigma,m) \in \tilde{\Theta}_{\ell-N}^{q+1} \\ k \text{ odd}}} \kappa_{k,\sigma} \zeta_{(1,\gamma,\nu,-)}^{k-2\sigma+1,\sigma,m} X_{\sigma,m}^{\ell-N-k} . \end{aligned}$$

**Proof:** W. l. o. g. let  $s \in (\mathbf{J} + 1/2, \mathbf{J} + N/2) \setminus \mathbb{I}$ . We insert the asymptotics (3.29), (3.30) into the estimates of Lemma 3.2. Introducing the new variable  $j(\ell) := \ell + k$  and ordering the sums according to  $j, k, \sigma, m$  we have proved the theorem. The assertions of the two remarks are easy consequences of the definition of the correction operators and (3.37), (3.38), (3.39).  $\blacksquare$

The first step is completed.

### 3.2 Second step

By the results obtained in [27, section 3] we get the following essential decomposition of  $L_s^{2,q,q+1}(\Omega)$ :

**Lemma 3.8** *Let  $s > 1 - N/2$  and  $s + 1 \notin \mathbb{I}$  as well as  $t \leq s$  and  $t < N/2$ . Then every*

$$(F, G) \in L_s^{2,q,q+1}(\Omega)$$

*may be uniquely decomposed into*

$$(F, G) = \Lambda(F_r, G_d) + (F_d, G_r) \quad ,$$

*where  $(F_r, G_d)$  and  $(F_d, G_r)$  are uniquely decomposed into*

$$\begin{aligned} (F_r, G_d) &= (\mathring{b}, b) + (\tilde{F}_r, \tilde{G}_d) + \left( \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \varphi_I (\vartheta_I^q)^{-1} \eta R_I^q, \sum_{I \in \bar{\mathcal{J}}_s^{q+1,0}} \psi_I \vartheta_I^{q+1} \eta D_I^{q+1} \right) \quad , \\ (F_d, G_r) &= (\tilde{F}_d, \tilde{G}_r) + \left( \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \varphi_I \operatorname{div} \eta R_{1I}^{q+1}, \sum_{I \in \bar{\mathcal{J}}_s^{q+1,0}} \psi_I \operatorname{rot} \eta D_{1I}^q \right) \end{aligned}$$

*with constants  $\varphi_I, \psi_I \in \mathbb{C}$ , where  $\vartheta_I^q := i(q + \mathbf{e}(I))^{1/2} (q + \mathbf{e}(I))^{-1/2}$ . Moreover*

$$\begin{aligned} (\mathring{b}, b) &\in \operatorname{Lin} \mathring{B}^q \times \operatorname{Lin} B^{q+1} \subset {}_0\mathring{R}_{\operatorname{vox}}^q(\Omega) \times {}_0D_{\operatorname{vox}}^{q+1}(\Omega) \quad , \\ (\tilde{F}_r, \tilde{G}_d) &\in \mathring{R}_s^q(\Omega) \times D_s^{q+1}(\Omega) \quad , \\ (F_r, G_d) &\in \operatorname{Tri}_s^q(\Omega) := (\operatorname{Lin} \mathring{B}^q \times \operatorname{Lin} B^{q+1}) \dot{+} ({}_0\mathring{R}_s^q(\bar{\mathcal{J}}_s^{q,0}, \Omega) \times {}_0D_s^{q+1}(\bar{\mathcal{J}}_s^{q+1,0}, \Omega)) \\ &\subset {}_0\mathring{R}_t^q(\Omega) \times {}_0D_t^{q+1}(\Omega) \quad , \\ (\tilde{F}_d, \tilde{G}_r) &\in \operatorname{Reg}_s^{q,0}(\Omega) \quad , \\ (F_d, G_r) &\in \operatorname{Reg}_s^{q,-1}(\Omega) := {}_0D_s^q(\bar{\mathcal{J}}_s^{q,0}, \Omega) \times {}_0\mathring{R}_s^{q+1}(\bar{\mathcal{J}}_s^{q+1,0}, \Omega) \\ &= \operatorname{Reg}_s^{q,0}(\Omega) \dot{+} (\operatorname{div} \eta \mathcal{R}^{q+1}({}_1\bar{\mathcal{J}}_s^{q,0}) \times \operatorname{rot} \eta \mathcal{D}^q({}_1\bar{\mathcal{J}}_s^{q+1,0})) \\ &\subset \operatorname{Reg}_t^{q,0}(\Omega) \end{aligned}$$

*and all projections are continuous. We denote the projection  $(F, G) \rightarrow (F_r, G_d)$  by  $\Pi$  and the projection  $(F, G) \rightarrow (F_d, G_r)$  by  $\Pi_{\operatorname{reg}} = \operatorname{Id} - \Lambda\Pi$ .*

*Shortly written we get*

$$L_s^{2,q,q+1}(\Omega) = (\Lambda \operatorname{Tri}_s^q(\Omega) \dot{+} \operatorname{Reg}_s^{q,-1}(\Omega)) \cap L_s^{2,q,q+1}(\Omega) \quad ,$$

*where  $\operatorname{Tri}_s^q(\Omega) = \Pi L_s^{2,q,q+1}(\Omega)$  and  $\operatorname{Reg}_s^{q,-1}(\Omega) = \Pi_{\operatorname{reg}} L_s^{2,q,q+1}(\Omega)$ .*

**Remark 3.9** *This lemma still holds true for  $\tau$ - $C^1$ -admissible transformations  $\Lambda$ , if  $\tau > 0$ ,  $\tau > s + 1 - N/2$  and  $\tau \geq -s - 1$ .*

Let us consider for  $\mathbf{J} \in \mathbb{N}_0$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  some  $(F, G) \in L_s^{2,q,q+1}(\Omega)$ . We decompose  $(F, G)$  according to the latter lemma.  $(F_r, G_d)$  solves

$$(M + i\omega\Lambda)(F_r, G_d) = i\omega\Lambda(F_r, G_d)$$

and satisfies the boundary, integrability and radiation condition since  $t > -1/2$ . Thus, for the ‘trivial projection’ we have

$$i\omega \mathcal{L}_\omega \Lambda(F_r, G_d) = (F_r, G_d) \quad , \text{i.e.} \quad i\omega \mathcal{L}_\omega \Lambda \Pi = \Pi \quad . \quad (3.40)$$

Looking at the ‘regular projection’ we see that Theorem 3.5 determines the asymptotic of  $(\tilde{F}_d, \tilde{G}_r)$  completely. Therefore, it remains to compute the asymptotics of

$$(f_d, g_r) := \left( \sum_{I \in \tilde{\mathcal{J}}_s^{q,0}} \varphi_I \operatorname{div} \eta R_{1I}^{q+1}, \sum_{J \in \tilde{\mathcal{J}}_s^{q+1,0}} \psi_J \operatorname{rot} \eta D_{1J}^q \right) \quad .$$

We note for  $I = (-, 0, \sigma, m)$  and  $J = (-, 0, \sigma, n)$

$$\eta R_{1I}^{q+1} = h_{\sigma,m}^- \quad , \quad \eta D_{1J}^q = e_{\sigma,n}^- \quad .$$

Thus, we have to calculate the asymptotics of

$$(\operatorname{div} \eta R_{1I}^{q+1}, 0) = M h_{\sigma,m}^{-,0} = h_{\sigma,m}^{-,1} \quad , \quad (0, \operatorname{rot} \eta D_{1J}^q) = M e_{\sigma,n}^{-,0} = e_{\sigma,n}^{-,1} \quad .$$

But this is quite easy since  $e_{\sigma,n}^{-,1}$  and  $h_{\sigma,m}^{-,1}$  inherit the asymptotics of  $e_{\sigma,n}^{-,2}$ ,  $h_{\sigma,m}^{-,2}$  derived in the first step. We discuss for example  $e_{\sigma,n}^{-,1}$ .  $e_{\sigma,n}^{-,1}$  satisfies the boundary, integrability and radiation condition as well as solves

$$(M + i\omega\Lambda)e_{\sigma,n}^{-,1} = e_{\sigma,n}^{-,2} + i\omega e_{\sigma,n}^{-,1} \quad .$$

This yields

$$e_{\sigma,n}^{-,1} = \mathcal{L}_\omega e_{\sigma,n}^{-,2} + i\omega \mathcal{L}_\omega e_{\sigma,n}^{-,1} \quad .$$

By Lemma 2.17 we get

$$\begin{aligned} \mathcal{L}_\omega e_{\sigma,n}^{-,1} &= \frac{1}{i\omega} (e_{\sigma,n}^{-,1} - \mathcal{L}_\omega e_{\sigma,n}^{-,2}) = \frac{1}{-i\omega} \mathcal{L}_{\omega,0} e_{\sigma,n}^{-,2} \\ &= \frac{1}{-i\omega} (\mathcal{L}_{\omega,j} e_{\sigma,n}^{-,2} + \sum_{\ell=1}^j (-i\omega)^\ell \mathcal{L}_0 \mathcal{L}^\ell e_{\sigma,n}^{-,2}) \\ &= \frac{1}{-i\omega} (\mathcal{L}_{\omega,j} e_{\sigma,n}^{-,2} + \sum_{\ell=0}^{j-1} (-i\omega)^{\ell+1} \mathcal{L}_0 \mathcal{L}^\ell e_{\sigma,n}^{-,1}) \end{aligned}$$

and hence

$$\mathcal{L}_{\omega,j-1} e_{\sigma,n}^{-,1} = \frac{1}{-i\omega} \mathcal{L}_{\omega,j} e_{\sigma,n}^{-,2} \quad .$$



By (3.19) and (3.29) we obtain

$$\begin{aligned} \mathcal{L}_{\omega, j-1} e_{\sigma, n}^{-, 1} &\stackrel{j}{\sim} i \kappa_{\sigma} \omega^{N+2\sigma-1} \sum_{\ell=0}^{j-N-2\sigma} (-i\omega)^{\ell} X_{\sigma, n}^{\ell} \\ &= \kappa_{0, \sigma} (-i\omega)^{N-1} \sum_{\ell=0}^{j-N-2\sigma} (-i\omega)^{\ell+2\sigma} X_{\sigma, n}^{\ell} \end{aligned} \quad (3.41)$$

and similarly

$$\mathcal{L}_{\omega, j-1} h_{\sigma, m}^{-, 1} = \kappa_{0, \sigma} (-i\omega)^{N-1} \sum_{\ell=0}^{j-N-2\sigma} (-i\omega)^{\ell+2\sigma} Y_{\sigma, m}^{\ell} \quad . \quad (3.42)$$

In order to compute the coefficients  $\varphi_I, \psi_J$  in terms of  $(F, G)$  we have the following lemma. Please compare to Lemma 2.13.

**Lemma 3.10** *Let  $s \in (1 - N/2, \infty) \setminus \mathbb{I}$  and  $(F, G) \in L_s^{2, q, q+1}(\Omega)$ . Then for all appropriate  $\sigma, m$  and  $\ell \geq 1$*

$$\begin{aligned} \text{(i)} \quad &\langle (F, G), E_{\sigma, m}^{+, 0} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = \langle F, E_{\sigma, m}^+ \rangle_{L^{2, q}(\Omega)} = -\varphi_I \quad , \\ &\langle (F, G), H_{\sigma, m}^{+, 0} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = \langle G, H_{\sigma, m}^+ \rangle_{L^{2, q+1}(\Omega)} = -\psi_J \quad , \\ \text{(ii)} \quad &\langle (F, G), E_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = \langle (\tilde{F}_d, \tilde{G}_r), E_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \quad , \\ &\langle (F, G), H_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = \langle (\tilde{F}_d, \tilde{G}_r), H_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} \end{aligned}$$

hold, where  $I = (-, 0, \sigma, m)$  resp.  $J = (-, 0, \sigma, m)$ .

**Remark 3.11** *Here of course ‘appropriate’ means that all  $E_{\sigma, m}^{+, 0}, H_{\sigma, m}^{+, 0}, E_{\sigma, m}^{+, \ell}, H_{\sigma, m}^{+, \ell}$  are elements of  $L_{-s}^{2, q, q+1}(\Omega)$ . More precisely we may pick indices  $I = (-, 0, \sigma, m) \in \bar{\mathcal{J}}_s^{q, 0}$  resp.  $J = (-, 0, \sigma, m) \in \bar{\mathcal{J}}_s^{q+1, 0}$  and  $(\ell, \sigma, m) \in \Theta_s^{q, \ell}$  resp.  $(\ell, \sigma, m) \in \Theta_s^{q+1, \ell}$ .*

**Proof:** Let us first discuss **(i)** and  $\varphi_I$ . During the proof we denote  $I = (-, 0, \sigma, m)$ . The representation of  $F$  in Lemma 3.8 may be written as (see [27, Theorem 3.2 (iv)])

$$F = \varepsilon \overset{\circ}{b} + \varepsilon \hat{F}_r + \hat{F}_d - i \sum_{I \in \bar{\mathcal{J}}_s^{q, 0}} \varphi_I (q' + \sigma)^{-1/2} \varepsilon \Delta_{\varepsilon} \eta P_I^q \quad , \quad (3.43)$$

where  $P_I^q := (q + \sigma)^{1/2} R_{2I}^q + i(q' + \sigma)^{1/2} D_{2I}^q$  is a potential form and by [27, section 3]

$$\hat{F}_r \in {}_0\mathring{\mathbb{R}}_s^q(\Omega) = \text{rot } \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \quad , \quad \hat{F}_d \in {}_0\mathbb{D}_s^q(\Omega) = \text{div } D_{s-1}^{q+1}(\Omega) \quad .$$

Since  $\varepsilon \Delta_{\varepsilon} = \varepsilon \text{rot div} + \text{div rot} = \Delta$  on  $\text{supp } \eta$  and  $\Delta P_I^q = 0$  we obtain

$$\varepsilon \Delta_{\varepsilon} \eta P_I^q = C P_I^q \quad .$$

Clearly we have

$$\langle F, E_{\gamma,n}^+ \rangle_{L^2,q(\Omega)} = -i \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \varphi_I (q' + \sigma)^{-1/2} \langle CP_I^q, E_{\gamma,n}^+ \rangle_{L^2,q(\Omega)}$$

and utilizing the expansion of  $E_{\sigma,m}^+$  from Corollary 2.8 as well as Lemma 2.9 and Remark 2.10 we get

$$\begin{aligned} \langle F, E_{\gamma,n}^+ \rangle_{L^2,q(\Omega)} &= \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \varphi_I \left( (\vartheta_\sigma^q)^{-1} \langle CR_{2I}^q, E_{\gamma,n}^+ \rangle_{L^2,q(\Omega)} + \langle CD_{2I}^q, E_{\gamma,n}^+ \rangle_{L^2,q(\Omega)} \right) \\ &= -\varphi_{(-,0,\gamma,n)} \quad , \end{aligned}$$

since the sums vanish except for  $R_{2I}^q = -R_{\gamma,n}^{q,2}$  resp.  $D_{2I}^q = -D_{\gamma,n}^{q,2}$ . The other assertion of **(i)** for  $\psi_J$  may be shown in a similar way.

To prove **(ii)** we write

$$F = \varepsilon \overset{\circ}{b} + \varepsilon \check{F}_r + \check{F}_d + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \varphi_I \left( (\vartheta_\sigma^q)^{-1} \operatorname{rot} \eta D_{1I}^{q-1} + \operatorname{div} \eta R_{1I}^{q+1} \right)$$

with some  $\check{F}_r \in {}_0\check{\mathbb{R}}_s^q(\Omega)$ . For any  $\ell$  we have

$$\langle (\varepsilon \overset{\circ}{b}, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = \langle (\varepsilon \check{F}_r, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0 \quad .$$

Thus, it remains to show for all  $I$

$$\langle (Q_I^q, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0 \quad ,$$

where  $Q_I^q := (\vartheta_\sigma^q)^{-1} \operatorname{rot} \eta D_{1I}^{q-1} + \operatorname{div} \eta R_{1I}^{q+1}$ . In order to prove this we compute

$$i(q' + \sigma)^{1/2} Q_I^q = CP_I^q - \operatorname{div} C_{\operatorname{rot},\eta} P_I^q - \operatorname{rot} C_{\operatorname{div},\eta} P_I^q$$

and obtain directly  $\langle (\operatorname{rot} C_{\operatorname{div},\eta} P_I^q, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} = 0$ . With the second term on the right hand side we proceed as follows. We write  $E_{\sigma,m}^{+,\ell} = M\Lambda^{-1} E_{\sigma,m}^{+,\ell+1}$  and since  $C_{\operatorname{rot},\eta}$  is compactly supported partial integration yields

$$\begin{aligned} &\langle (\operatorname{div} C_{\operatorname{rot},\eta} P_I^q, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \\ &= -\langle (0, \operatorname{rot} \operatorname{div} C_{\operatorname{rot},\eta} P_I^q), E_{\sigma,m}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \\ &= -\langle (0, \operatorname{rot} CP_I^q), E_{\sigma,m}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \\ &\quad + i(q' + \sigma)^{1/2} \langle (0, \operatorname{rot} \operatorname{div} \eta R_{1I}^{q+1}), E_{\sigma,m}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \\ &= \langle (CP_I^q, 0), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \\ &\quad + i(q' + \sigma)^{1/2} \langle (0, CR_{1I}^{q+1}), E_{\sigma,m}^{+,\ell+1} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \quad , \end{aligned}$$

where the last equation follows once more by [26, Remark 2.4]. Consequently

$$\langle (Q_I^q, 0), E_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} = -\langle (0, CR_{1I}^{q+1}), E_{\sigma, m}^{+, \ell+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} .$$

Once again by Corollary 2.8, Lemma 2.9 and Remark 2.10 the scalar products

$$\langle (0, CR_{1I}^{q+1}), E_{\sigma, m}^{+, \ell+1} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)}$$

vanish because  $\ell \geq 1$ . Besides also the scalar products  $\langle (CP_I^q, 0), E_{\sigma, m}^{+, \ell} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)}$  vanish since  $\ell \geq 1$ , though this is not necessary for the proof. The other assertions of **(ii)** are shown analogously.  $\blacksquare$

Putting all together we obtain the main result of step two.

**Definition 3.12** Let  $\mathbf{J} \in \mathbb{N}_0$ ,  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  and  $(F, G) \in L_s^{2, q, q+1}(\Omega)$ . For  $j = 0, \dots, \mathbf{J} - N$  we define the ‘correction operators’

$$\begin{aligned} \tilde{\Gamma}_j(F, G) := & - \sum_{2\sigma \leq j, m} \kappa_{0, \sigma} \langle (F, G), E_{\sigma, m}^{+, 0} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} Y_{\sigma, m}^{j-2\sigma} \\ & - \sum_{2\sigma \leq j, m} \kappa_{0, \sigma} \langle (F, G), H_{\sigma, m}^{+, 0} \rangle_{\tilde{L}^{2, q, q+1}(\Omega)} X_{\sigma, m}^{j-2\sigma} \end{aligned}$$

as well as  $\Gamma_0 := \tilde{\Gamma}_0$  and  $\Gamma_j := \hat{\Gamma}_{j-1} + \tilde{\Gamma}_j$  for  $j = 1, \dots, \mathbf{J} - N$ .

**Theorem 3.13** Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$ . Then for all bounded subdomains  $\Omega_b \subset \Omega$  the asymptotic

$$\begin{aligned} & \left\| \mathcal{L}_\omega(F, G) + (-i\omega)^{-1} \Pi(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}}(F, G) \right. \\ & \left. - (-i\omega)^{N-1} \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^j \Gamma_j(F, G) \right\|_{L^{2, q, q+1}(\Omega_b)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2, q, q+1}(\Omega)} \end{aligned}$$

holds uniformly with respect to  $(F, G) \in L_s^{2, q, q+1}(\Omega)$  and  $\omega \in \mathbb{C}_{+, \hat{\omega}} \setminus \{0\}$ .

**Remark 3.14**

- (i) By Lemma 2.13 on  $\text{Reg}_s^{q,0}(\Omega)$  we have  $\tilde{\Gamma}_j = 0$  and thus  $\Gamma_j = \hat{\Gamma}_{j-1}$  for  $j \geq 1$  as well as  $\Gamma_0 = 0$ .
- (ii)  $\Pi(F, G) = 0$  and  $\Pi_{\text{reg}}(F, G) = (\tilde{F}_d, \tilde{G}_r) = (F, G)$  hold for  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$ .
- (iii) Because of (3.38), (3.39) the correction operators satisfy  $\Lambda^{-1}M\tilde{\Gamma}_j = \tilde{\Gamma}_{j-1}$ , where  $\tilde{\Gamma}_{-1} := 0$ . Thus, we have  $\Lambda^{-1}M\Gamma_j = \Gamma_{j-1}$ .
- (iv) From Definition 3.12 we get

$$\tilde{\Gamma}_j(F, G) \in \text{Lin}\{X_{\sigma,m}^{j-2\sigma}, Y_{\sigma,m}^{j-2\sigma} : 2\sigma \leq j\}$$

and thus the correction operators

$$\tilde{\Gamma}_j, \Gamma_j : L_s^{2,q,q+1}(\Omega) \longrightarrow \text{Cor}^{q,j}(\Omega)$$

are degenerated (and clearly continuous).

**Proof:** From the arguments above and the continuity of the projections from Lemma 3.8

$$\begin{aligned} & \mathcal{L}_\omega(F, G) + (-i\omega)^{-1}(F_r, G_d) \\ & - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(\tilde{F}_d, \tilde{G}_r) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(f_d, g_r) \\ & - (-i\omega)^N \sum_{j=0}^{\mathbf{J}-1-N} (-i\omega)^j \hat{\Gamma}_j(\tilde{F}_d, \tilde{G}_r) \\ & + (-i\omega)^{N-1} \sum_{\sigma \leq s-N/2, m} \kappa_{0,\sigma} \langle (F, G), E_{\sigma,m}^{+,0} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \sum_{\ell=0}^{\mathbf{J}-N-2\sigma} (-i\omega)^{\ell+2\sigma} Y_{\sigma,n}^\ell \\ & + (-i\omega)^{N-1} \sum_{\sigma \leq s-N/2, m} \kappa_{0,\sigma} \langle (F, G), H_{\sigma,m}^{+,0} \rangle_{\tilde{L}^{2,q,q+1}(\Omega)} \sum_{\ell=0}^{\mathbf{J}-N-2\sigma} (-i\omega)^{\ell+2\sigma} X_{\sigma,n}^\ell \end{aligned}$$

may be estimated by  $c|\omega|^{\mathbf{J}} \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$  in the  $L^{2,q,q+1}(\Omega_b)$ -norm uniformly in  $\omega$  and  $(F, G)$ . We note  $I = (-, 0, \sigma, m) \in \bar{\mathcal{J}}_s^{q,0}$ , if and only if  $J = (-, 0, \sigma, m) \in \bar{\mathcal{J}}_s^{q+1,0}$ , if and only if  $\sigma \leq s - N/2$  and  $m = 1, \dots$ . By Lemma 3.10 and Definition 3.4 we have  $\hat{\Gamma}_j(\tilde{F}_d, \tilde{G}_r) = \hat{\Gamma}_j(F, G)$ . Rearranging the latter two terms we obtain

$$\begin{aligned} & \left\| \mathcal{L}_\omega(F, G) + (-i\omega)^{-1}(F_r, G_d) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j(F_d, G_r) \right. \\ & \left. - (-i\omega)^{N-1} \left( \sum_{j=1}^{\mathbf{J}-N} (-i\omega)^j \hat{\Gamma}_{j-1}(F, G) + \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^j \tilde{\Gamma}_j(F, G) \right) \right\|_{L^{2,q,q+1}(\Omega_b)} \\ & = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}, \end{aligned}$$

which completes the proof. ■

### 3.3 Third step

Now we approach estimates in weighted norms. For this we compare our solutions with the solutions of the homogeneous whole space case. Let us denote

$$L_\omega := \mathcal{L}_\omega$$

in the special case  $\Omega = \mathbb{R}^N$  and  $\Lambda = \text{Id}$ .  $L_\omega$  is well defined on  $L_{>\frac{1}{2}}^{2,q,q+1}$  for all  $\mathbb{C}_+ \setminus \{0\}$  by [25, Theorem 2.17] since there are no eigensolutions in this case. We obtain

**Lemma 3.15** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s > \mathbf{J} - 1/2$  as well as  $t < \min\{s, N/2\} - \mathbf{J} - 1$ . Then for  $j = 0, \dots, \mathbf{J} - 1$  there exist bounded linear operators*

$$\Phi_j \in B(L_s^{2,q,q+1}, L_t^{2,q,q+1}) \quad , \quad \Psi_j \in B(L_s^{2,q-1,q+2}, L_t^{2,q,q+1})$$

and a constant  $c > 0$ , such that

$$\begin{aligned} & \left\| L_\omega(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j (\Phi_j(F, G) + (-i\omega)^{-1} \Psi_j(\text{div } F, \text{rot } G)) \right\|_{L_t^{2,q,q+1}} \\ & \leq c|\omega|^{\mathbf{J}} \left( \|(F, G)\|_{L_s^{2,q,q+1}} + \frac{1}{|\omega|} \|(\text{div } F, \text{rot } G)\|_{L_s^{2,q-1,q+2}} \right) \end{aligned}$$

holds uniformly with respect to  $\omega \in \mathbb{C}_+ \setminus \{0\}$  and  $(F, G) \in \mathbf{D}_s^q \times \mathbf{R}_s^{q+1}$ . The assertion holds also true for  $\mathbf{J} = 0$  and  $s, -t > 1/2, t \leq s - (N + 1)/2$ .

**Proof:** Using the fundamental solution for the scalar Helmholtz equation in  $\mathbb{R}^N$

$$\Phi_{\omega,\nu}(x) = \varphi_{\omega,\nu}(|x|) \quad \text{with} \quad \varphi_{\omega,\nu}(t) = c_N \omega^\nu t^{-\nu} H_\nu^1(\omega t) \quad ,$$

where the constant  $c_N$  only depends on the dimension  $N$  and  $H_\nu^1(z)$  denotes Hankel's first function of index  $\nu := (N - 2)/2$ , see [25, section 4], we may represent  $(E, H) := L_\omega(F, G)$  by [25, Theorem 4.1], i.e.

$$E_I = G \star \text{rot } \Phi_{\omega,\nu}^I + (-i\omega)F \star \Phi_{\omega,\nu}^I - (-i\omega)^{-1} \text{div } F \star \text{div } \Phi_{\omega,\nu}^I \quad , \quad (3.44)$$

$$H_J = F \star \text{div } \Phi_{\omega,\nu}^J + (-i\omega)G \star \Phi_{\omega,\nu}^J - (-i\omega)^{-1} \text{rot } G \star \text{rot } \Phi_{\omega,\nu}^J \quad , \quad (3.45)$$

if  $E = E_I dx^I$  and  $H = H_J dx^J$  as well as  $\Phi_{\omega,\nu}^I := \Phi_{\omega,\nu} dx^I$ . Here  $\star$  is the convolution in  $\mathbb{R}^N$  for forms, which simply is the sum of the scalar convolutions of their Euclidean components. Utilizing Taylor's expansion theorem we get constants  $c_j, c'_j \in \mathbb{C}$  and functions  $\text{rem}_{\mathbf{J}}, \widetilde{\text{rem}}_{\mathbf{J}}$ , such that for  $t \in \mathbb{R}_+$  and  $\omega \in \mathbb{C}_+$  the expansions

$$\begin{aligned} \varphi_{\omega,\nu}(t) &= t^{2-N} \sum_{j=0}^{\mathbf{J}-2} c_j (\omega t)^j + \text{rem}_{\mathbf{J}}(\omega t) t^{1-N+\mathbf{J}} \omega^{\mathbf{J}-1} \quad , \\ \varphi'_{\omega,\nu}(t) &= t^{1-N} \sum_{j=0}^{\mathbf{J}-1} c'_j (\omega t)^j + \widetilde{\text{rem}}_{\mathbf{J}}(\omega t) t^{1-N+\mathbf{J}} \omega^{\mathbf{J}} \end{aligned}$$

hold. The remainder functions  $\text{rem}_{\mathbf{J}}(z)$  and  $\widetilde{\text{rem}}_{\mathbf{J}}(z)$  are uniformly bounded with respect to  $z \in \mathbb{C}_+$  and the bounds only depend on  $N$  and  $\mathbf{J}$ . Inserting these Taylor representations into (3.44), (3.45) we obtain

$$(E, H) = \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j (\Phi_j(F, G) + (-i\omega)^{-1} \Psi_j(\text{div } F, \text{rot } G)) \\ + \omega^{\mathbf{J}} (\text{Rem}_{\omega, \mathbf{J}}(F, G) + \frac{1}{\omega} \widetilde{\text{Rem}}_{\omega, \mathbf{J}}(\text{div } F, \text{rot } G)) \quad , \quad (3.46)$$

where  $\Phi_j$  and  $\Psi_j$  resp.  $\text{Rem}_{\omega, \mathbf{J}}$  and  $\widetilde{\text{Rem}}_{\omega, \mathbf{J}}$  are convolution operators with integral kernels of shape  $b_j(x, y)|x-y|^{j+1-N}$  for  $j = 0, \dots, \mathbf{J}-1$  resp.  $b_{\text{Rem}}(x, y, \omega)|x-y|^{\mathbf{J}+1-N}$ . The kernel parts  $b_j(x, y)$  are uniformly bounded with respect to  $x, y \in \mathbb{R}^N$  and independent of  $\omega$ . Moreover, the kernel parts  $b_{\text{Rem}}(x, y, \omega)$  are uniformly bounded with respect to  $x, y \in \mathbb{R}^N$  and  $\omega \in \mathbb{C}_+$ . Thus, it remains to show that the kernels

$$|x - y|^{j+1-N} \quad , \quad j = 0, \dots, \mathbf{J} \quad ,$$

generate bounded linear operators from  $L_s^2$  to  $L_t^2$ . All kernels belong to  $L_{\text{loc}}^1$  and grow with  $j$  if  $|x - y| > 1$ . Therefore, we only have to discuss the worst kernel  $|x - y|^{\mathbf{J}+1-N}$ . The assertion follows now by [18, Lemma 1] and [48, Lemma 13] as well as some case studies. For a more detailed proof we refer to [24, Sektionen 5.1-5.3].  $\blacksquare$

According to [46, Theorem 4] there exist continuous projections

$$\begin{aligned} \pi &: L_s^{2,q,q+1} \rightarrow {}_0R_s^q \times {}_0D_s^{q+1} & ; & \quad (F, G) \mapsto (F_R, G_D) & , \\ \pi_{\text{reg}} &: L_s^{2,q,q+1} \rightarrow {}_0D_s^q \times {}_0R_s^{q+1} & ; & \quad (F, G) \mapsto (F_D, G_R) & , \\ \pi_S &: L_s^{2,q,q+1} \rightarrow S_s^q \times S_s^{q+1} & ; & \quad (F, G) \mapsto (F_S, G_S) & , \end{aligned}$$

such that each  $(F, G) \in L_s^{2,q,q+1}$  may be uniquely decomposed as

$$(F, G) = (F_R, G_D) + (F_D, G_R) + (F_S, G_S) \quad .$$

**Corollary 3.16** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s > \mathbf{J} - 1/2$  as well as  $t < \min\{s, N/2 - 1\} - \mathbf{J} - 1$ . Then*

$$\begin{aligned} & \left\| L_\omega(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \Phi_j(F_D + F_S, G_R + G_S) + (-i\omega)^{-1} (F_R, G_D) \right. \\ & \left. - \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Psi_{j+1}(\text{div } F_S, \text{rot } G_S) \right\|_{L_t^{2,q,q+1}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}} \end{aligned}$$

holds uniformly with respect to  $\omega \in \mathbb{C}_+ \setminus \{0\}$  and  $(F, G) \in L_s^{2,q,q+1}$ , if  $\omega$  ranges in a bounded set. The estimate remains valid even for  $\mathbf{J} = 0$  and  $s, -t > 1/2, t \leq s - (N + 1)/2$ .

**Proof:** We easily see  $L_\omega(F_R, G_D) = -(-i\omega)^{-1}(F_R, G_D)$ . (Compare to (3.40).) Moreover, Lemma 3.15 may be applied to  $(F_D, G_R)$  and we get

$$\|L_\omega(F_D, G_R) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \Phi_j(F_D, G_R)\|_{L_t^{2,q,q+1}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}} \quad .$$

Furthermore, Lemma 3.15 may also be applied to  $(F_S, G_S)$  but with  $\mathbf{J} + 1$  instead of  $\mathbf{J}$  as well as  $\tilde{s} := s + 1$  and  $\tilde{t} := t$ . We achieve

$$\begin{aligned} & \|L_\omega(F_S, G_S) - \sum_{j=0}^{\mathbf{J}} (-i\omega)^j \Phi_j(F_S, G_S) - \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Psi_{j+1}(\operatorname{div} F_S, \operatorname{rot} G_S)\|_{L_t^{2,q,q+1}} \\ & \leq c|\omega|^{\mathbf{J}+1} \left( \|(F_S, G_S)\|_{L_s^{2,q,q+1}} + \frac{1}{|\omega|} \|(\operatorname{div} F_S, \operatorname{rot} G_S)\|_{L_s^{2,q-1,q+2}} \right) \\ & \leq c|\omega|^{\mathbf{J}} \|(F_S, G_S)\|_{D_s^q \times R_s^{q+1}} \quad . \end{aligned}$$

Now the  $L_t^{2,q,q+1}$ -norm of the term  $(-i\omega)^{\mathbf{J}} \Phi_{\mathbf{J}}(F_S, G_S)$  may be swallowed by the right hand side, which itself can be further estimated by  $\mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}}$  since  $\mathbb{S}_s^q$  are finite dimensional subspaces of  $\mathring{C}^{\infty,q}$  for all  $q$  and  $s$ . Putting all together yields the desired assertion.  $\blacksquare$

We are able to formulate the main result of this section:

**Theorem 3.17** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  as well as  $t < \min\{s, N/2 - 1\} - \mathbf{J} - 1$ . Then the asymptotic*

$$\begin{aligned} & \|\mathcal{L}_\omega(F, G) + (-i\omega)^{-1} \Pi(F, G) - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\operatorname{reg}}(F, G) \\ & - (-i\omega)^{N-1} \sum_{j=0}^{\mathbf{J}-N} (-i\omega)^j \Gamma_j(F, G)\|_{L_t^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \end{aligned}$$

holds uniformly with respect to  $(F, G) \in L_s^{2,q,q+1}(\Omega)$  and  $\omega \in \mathbb{C}_{+,\tilde{\omega}} \setminus \{0\}$ .

This asymptotic holds for  $\mathbf{J} = 0$  as well, if we replace the assumptions on  $t$  by  $t < -1/2$  and  $t \leq s - (N + 1)/2$ .

**Proof:** Let us define operators  $\mathcal{K}_j$  via

$$\begin{aligned} \mathcal{K}_{-1} & := -\Pi \quad , \\ \mathcal{K}_j & := \mathcal{L}_0 \mathcal{L}^j \Pi_{\operatorname{reg}} \quad , \quad j = 0, \dots, N-2 \quad , \\ \mathcal{K}_j & := \mathcal{L}_0 \mathcal{L}^j \Pi_{\operatorname{reg}} + \Gamma_{j-N+1} \quad , \quad j = N-1, \dots, \mathbf{J}-1 \end{aligned}$$

and for  $J \leq \mathbf{J} - 1$

$$\mathcal{L}_{\omega, J}^{\mathcal{K}} := \mathcal{L}_{\omega} - \sum_{j=-1}^J (-i\omega)^j \mathcal{K}_j \quad . \quad (3.47)$$

Then we have to show uniformly with respect to  $\omega$  and  $(F, G)$

$$\|\mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L_t^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \quad .$$

From now on all estimates are to be understood uniformly with respect to  $\omega$  and  $(F, G)$ . We want to combine the asymptotics in local norms proved in the second step with the whole space asymptotics in weighted norms from the latter corollary. Since we have by Theorem 3.13 for every bounded subdomain  $\Omega_b$  of  $\Omega$

$$\|\mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L^{2,q,q+1}(\Omega_b)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \quad , \quad (3.48)$$

we get immediately

$$\|(1 - \eta) \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L_t^{2,q,q+1}(\Omega)} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$$

and it remains to estimate  $\|\eta \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L_t^{2,q,q+1}}$ .

To do so let  $\omega \in \mathbb{C}_{+, \hat{\omega}} \setminus \{0\}$  and  $(F, G) \in L_s^{2,q,q+1}(\Omega)$ . According to [27, Theorem 3.2 (iv)] (compare with Lemma 3.8 and (3.43)) we decompose

$$\begin{aligned} (F, G) &= \Lambda(\underbrace{(\hat{b}, b)}_{=: \hat{\Pi}(F, G)} + \underbrace{(\hat{F}_r, \hat{G}_d)}_{=: \hat{\Pi}_{\text{reg}}(F, G)}) + \underbrace{\left( \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \hat{\varphi}_I CP_I^q, \sum_{J \in \bar{\mathcal{J}}_s^{q+1,0}} \hat{\psi}_J CP_J^{q+1} \right)}_{=: \hat{\Pi}_C(F, G)} \\ &=: \hat{\Pi}(F, G) + \hat{\Pi}_{\text{reg}}(F, G) + \hat{\Pi}_C(F, G) \end{aligned}$$

with continuous projections, where  $CP_I^q \in \mathring{C}^{\infty, q}$  and  $(\hat{b}, b) \in \text{Lin } \mathring{B}^q \times \text{Lin } B^{q+1}$  as well as

$$(\hat{F}_r, \hat{G}_d) \in {}_0\mathring{\mathbb{R}}_s^q(\Omega) \times {}_0\mathbb{D}_s^{q+1}(\Omega) \quad , \quad (\hat{F}_d, \hat{G}_r) \in \text{Reg}_s^{q,0}(\Omega) \quad .$$

By Lemma 3.10 we have

$$|\hat{\varphi}_I| + |\hat{\psi}_J| \leq c \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \quad .$$

The trick is to apply this decomposition using another cut-off function  $\tilde{\eta}$ , which satisfies  $\text{supp } \nabla \tilde{\eta} \subset \overline{Z_{r_3, r_4}}$ , whereas  $\text{supp } \nabla \eta \subset \overline{Z_{r_1, r_2}}$ . More precisely we set

$$\tilde{\eta} := \tilde{\eta} \circ r \quad , \quad \tilde{\eta}(t) := \eta\left(1 + \frac{t - r_3}{r_4 - r_3}\right)$$

and note  $C = C_{\Delta, \tilde{\eta}}$  in this case. Since

$$\mathcal{L}_{\omega} \Lambda \hat{\Pi}(F, G) = -(-i\omega)^{-1} \hat{\Pi}(F, G)$$



it suffices to discuss  $\eta \mathcal{L}_\omega(\hat{F}, \hat{G})$  with  $(\hat{F}, \hat{G}) := (\hat{\Pi}_{\text{reg}} + \hat{\Pi}_C)(F, G)$ .

$\eta \mathcal{L}_\omega(\hat{F}, \hat{G}) \in \mathbf{R}_{<-\frac{1}{2}}^q \times \mathbf{D}_{<-\frac{1}{2}}^{q+1}$  satisfies the radiation condition and solves

$$(M + i\omega)\eta \mathcal{L}_\omega(\hat{F}, \hat{G}) = (M + i\omega\Lambda)\eta \mathcal{L}_\omega(\hat{F}, \hat{G}) = (f, g) \quad (3.49)$$

with  $(f, g) := (\eta + C_{M,\eta} \mathcal{L}_\omega)(\hat{F}, \hat{G}) \in L_s^{2,q,q+1}$ . (Without further comments here and in the following we often identify forms with their extensions by zero to  $\mathbb{R}^N$ .) Thus,  $\eta \mathcal{L}_\omega(\hat{F}, \hat{G}) = L_\omega(f, g)$  or in another notation

$$\eta \mathcal{L}_\omega = L_\omega(\eta \text{Id} + C_{M,\eta} \mathcal{L}_\omega) \quad (3.50)$$

holds even on  $L_s^{2,q,q+1}(\Omega)$ . By Corollary 3.16 there exist bounded linear operators  $\Xi_{-1}, \dots, \Xi_{\mathbf{J}-1}$  mapping  $L_s^2$  to  $L_t^2$ , which satisfy

$$\|L_{\omega, \mathbf{J}-1}(f, g)\|_{L_t^{2,q,q+1}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \|(f, g)\|_{L_s^{2,q,q+1}}, \quad (3.51)$$

where

$$L_{\omega, \mathbf{J}-1} := L_\omega - \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Xi_j. \quad (3.52)$$

Moreover,  $\|(f, g)\|_{L_s^{2,q,q+1}}$  can be further estimated by

$$\|(f, g)\|_{L_s^{2,q,q+1}} \leq c \left( \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} + \|C_{M,\eta} \mathcal{L}_\omega(\hat{F}, \hat{G})\|_{L^{2,q,q+1}(\text{supp } \nabla \eta)} \right).$$

Using [25, Lemma 4.2 (iv)] we can estimate

$$\|\mathcal{L}_\omega \hat{\Pi}_{\text{reg}}(F, G)\|_{L^{2,q,q+1}(\text{supp } \nabla \eta)} \leq c \|\hat{\Pi}_{\text{reg}}(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \leq c \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}.$$

Looking at some term of  $\hat{\Pi}_C(F, G)$  we see

$$C_{M,\eta} \mathcal{L}_\omega(CP_I^q, CP_J^{q+1}) = C_{M,\eta} \mathcal{L}_\omega(\text{div rot } \tilde{\eta} P_I^q, \text{rot div } \tilde{\eta} P_J^{q+1})$$

because

$$\mathcal{L}_\omega(\text{rot div } \tilde{\eta} P_I^q, \text{div rot } \tilde{\eta} P_J^{q+1}) = \frac{1}{i\omega} (\text{rot div } \tilde{\eta} P_I^q, \text{div rot } \tilde{\eta} P_J^{q+1}) \in L_{<\frac{N}{2}}^{2,q,q+1}$$

and thus  $C_{M,\eta} \mathcal{L}_\omega(\text{rot div } \tilde{\eta} P_I^q, \text{div rot } \tilde{\eta} P_J^{q+1}) = 0$  since  $\text{supp } \nabla \eta \cap \text{supp } \tilde{\eta} = \emptyset$ . With some  $\tilde{s} \in (1/2, N/2)$  we have  $(\text{div rot } \tilde{\eta} P_I^q, \text{rot div } \tilde{\eta} P_J^{q+1}) \in \text{Reg}_{\tilde{s}}^{q,0}(\Omega)$  and therefore we may utilize [25, Lemma 4.2 (iv)] once more to estimate

$$\begin{aligned} & \|C_{M,\eta} \mathcal{L}_\omega(CP_I^q, CP_J^{q+1})\|_{L^{2,q,q+1}(\text{supp } \nabla \eta)} \\ & \leq c \|(\text{div rot } \tilde{\eta} P_I^q, \text{rot div } \tilde{\eta} P_J^{q+1})\|_{L_s^{2,q,q+1}(\Omega)} \leq c. \end{aligned}$$

Hence, we obtain

$$\|C_{M,\eta} \mathcal{L}_\omega \hat{\Pi}_C(F, G)\|_{L^{2,q,q+1}(\text{supp } \nabla \eta)} \leq c \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}.$$

Putting all together yields

$$\|(f, g)\|_{L_s^{2,q,q+1}} \leq c\|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$$

and after inserting in (3.51)

$$\|L_{\omega, \mathbf{J}-1}(f, g)\|_{L_t^{2,q,q+1}} = \mathcal{O}(|\omega|^{\mathbf{J}})\|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} .$$

Now using (3.52) and (3.49), (3.50) we obtain

$$\eta \mathcal{L}_{\omega}(\hat{F}, \hat{G}) = L_{\omega, \mathbf{J}-1}(f, g) + \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Xi_j(\eta + C_{M,\eta} \mathcal{L}_{\omega})(\hat{F}, \hat{G}) .$$

Collecting terms and utilizing (3.47) gives

$$\begin{aligned} \eta \mathcal{L}_{\omega}(F, G) + (-i\omega)^{-1} \hat{\Pi}(F, G) - \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Xi_j \eta (\hat{\Pi}_{\text{reg}} + \hat{\Pi}_C)(F, G) \\ - \sum_{j=-1}^{\mathbf{J}-1} \sum_{k=-1}^{\mathbf{J}-1-j} (-i\omega)^{j+k} \Xi_j C_{M,\eta} \mathcal{K}_k (\hat{\Pi}_{\text{reg}} + \hat{\Pi}_C)(F, G) \\ = L_{\omega, \mathbf{J}-1}(f, g) + \sum_{j=-1}^{\mathbf{J}-1} (-i\omega)^j \Xi_j C_{M,\eta} \mathcal{L}_{\omega, \mathbf{J}-1-j}^{\mathcal{K}}(\hat{F}, \hat{G}) . \end{aligned} \quad (3.53)$$

Moreover, the continuity of the operators  $\Xi_j$  from  $L_s^2$  to  $L_t^2$  as well as Theorem 3.13 yield

$$\begin{aligned} \|\Xi_j C_{M,\eta} \mathcal{L}_{\omega, \mathbf{J}-1-j}^{\mathcal{K}}(\hat{F}, \hat{G})\|_{L_t^{2,q,q+1}} &\leq c \|\mathcal{L}_{\omega, \mathbf{J}-1-j}^{\mathcal{K}}(\hat{F}, \hat{G})\|_{L^{2,q,q+1}(\text{supp } \nabla \eta)} \\ &\leq c |\omega|^{\mathbf{J}-j} \|(\hat{F}, \hat{G})\|_{L_s^{2,q,q+1}(\Omega)} . \end{aligned}$$

Therefore, the right hand side of (3.53) behaves in the  $L_t^{2,q,q+1}$ -norm like

$$\mathcal{O}(|\omega|^{\mathbf{J}})\|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$$

and thus so does the left hand side. By (3.48)

$$\|\eta \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L_t^{2,q,q+1}(\Omega_b)} = \mathcal{O}(|\omega|^{\mathbf{J}})\|(F, G)\|_{L_s^{2,q,q+1}(\Omega)}$$

holds also for every bounded domain  $\Omega_b \subset \mathbb{R}^N$ . Applying Lemma 3.3 we find that the left hand side of (3.53) equals  $\eta \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)$  and this yields finally

$$\|\eta \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}(F, G)\|_{L_t^{2,q,q+1}} = \mathcal{O}(|\omega|^{\mathbf{J}})\|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} ,$$

which completes our proof. ■

### 3.4 Fourth step

We are ready to face the proof of the Main Theorem. This last step may be done by an abstract argument similar to [45, Lemma 12]. Thus, our aim is to identify two Banach spaces  $X$  and  $Y$ , such that all operators involved in our asymptotic belong to  $B(X, Y)$ , which denotes the space of all bounded linear operators from  $X$  to  $Y$ . Good candidates are  $X = \mathbb{L}_s^{2,q,q+1}(\Omega)$  and  $Y = \mathbb{L}_t^{2,q,q+1}(\Omega)$  as well as for some  $s > t$

$$B(X, Y) = B_{s,t} := B(\mathbb{L}_s^{2,q,q+1}(\Omega), \mathbb{L}_t^{2,q,q+1}(\Omega)) \quad .$$

By Theorem 3.17 and (3.47) the ingredients of our asymptotic are the linear operators  $\mathcal{L}_\omega$  and  $\mathcal{K}_j, j = -1, \dots, \mathbf{J} - 1$ , i.e. the operators

$$\Lambda \quad , \quad \Pi \quad , \quad \mathcal{L}_\omega \quad \text{and} \quad \mathcal{L}^j \quad , \quad j = 0, \dots, \mathbf{J}$$

as well as the correction operators

$$\Gamma_j \quad , \quad j = 0, \dots, \mathbf{J} - N \quad .$$

Moreover, the correction operators  $\Gamma_j$  map to  $\text{Cor}^{q,j}(\Omega)$  and their coefficients are given by the scalar products

$$s_{\sigma,m}^{E,\ell}(F, G) := \langle (F, G), E_{\sigma,m}^{+,\ell} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)} \quad , \quad s_{\sigma,m}^{H,\ell}(F, G) := \langle (F, G), H_{\sigma,m}^{+,\ell} \rangle_{\tilde{\mathbb{L}}^{2,q,q+1}(\Omega)}$$

and the numbers  $\xi_{1,\gamma,n,-}^{\ell,\sigma,m}, \zeta_{1,\gamma,n,-}^{\ell,\sigma,m}$ . (See the definitions and remarks around Theorem 3.5 and Theorem 3.13.)

Let us assume  $\omega$  to be small enough and still for the moment the perturbation  $\hat{\Lambda}$  to be compactly supported. Then clearly  $\Lambda \in B_{s,t}$  for all  $s \geq t$ . By Lemma 3.8

$$\begin{aligned} \Pi &\in B(\mathbb{L}_s^{2,q,q+1}(\Omega), \mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)) \subset B_{s,t} \quad , \\ \Pi_{\text{reg}} &\in B(\mathbb{L}_s^{2,q,q+1}(\Omega), \text{Reg}_s^{q,-1}(\Omega)) \subset B_{s,t} \end{aligned} \quad (3.54)$$

for all  $s \in (1 - N/2, \infty) \setminus \mathbb{I}$  and  $t \leq s, t < N/2$ , since the embedding

$$\text{Reg}_s^{q,-1}(\Omega) \subset \text{Reg}_t^{q,0}(\Omega)$$

is continuous. By [25, Theorem 2.17] (see also [25, Theorem 2.24]) for  $s, -t > 1/2$

$$\mathcal{L}_\omega \in B(\mathbb{L}_s^{2,q,q+1}(\Omega), \mathring{\mathbf{R}}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)) \subset B_{s,t} \quad . \quad (3.55)$$

By [26, Theorem 5.10]

$$\mathcal{L}^j \in B(\text{Reg}_s^{q,-1}(\Omega), \text{Reg}_t^{q,0}(\Omega)) \quad (3.56)$$

and therefore

$$\mathcal{L}^j \Pi_{\text{reg}} \in B_{s,t} \quad (3.57)$$

for  $s \in (j - N/2, \infty) \setminus \mathbb{I}$  and  $t \leq s - j, t < N/2 - j$ . By Lemma 2.6

$$E_{\sigma,m}^{+,k}, H_{\sigma,m}^{+,k} \in L_t^{2,q,q+1}(\Omega) \quad (3.58)$$

for  $t < -k - \sigma - N/2$  and thus

$$\text{Cor}^{q,j}(\Omega) = \text{Lin}\{\Lambda^{-1}E_{\sigma,m}^{+,k}, \Lambda^{-1}H_{\sigma,n}^{+,k} : k + 2\sigma \leq j\} \subset L_t^{2,q,q+1}(\Omega) \quad (3.59)$$

for all  $t < -j - N/2$ . Moreover,  $s_{\sigma,m}^{E,k-2\sigma+1}, s_{\sigma,m}^{H,k-2\sigma+1}$  and  $s_{\sigma,m}^{E,0}, s_{\sigma,m}^{H,0}$  are continuous linear functionals on  $L_s^{2,q,q+1}(\Omega)$  for  $s \in (j + 1 + N/2, \infty) \setminus \mathbb{I}$  since again by Lemma 2.6

$$E_{\sigma,m}^{+,k-2\sigma+1}, H_{\sigma,m}^{+,k-2\sigma+1}, E_{\sigma,m}^{+,0}, H_{\sigma,m}^{+,0} \in L_{<\sigma-k-1-\frac{N}{2}}^{2,q,q+1}(\Omega) \subset L_{-s}^{2,q,q+1}(\Omega)$$

for  $0 \leq 2\sigma \leq k \leq j$ . This yields for  $s \in (j + 1 + N/2, \infty) \setminus \mathbb{I}$  and  $t < -j - N/2$

$$\Gamma_j \in B(L_s^{2,q,q+1}(\Omega), \text{Cor}^{q,j}(\Omega)) \subset B_{s,t} \quad . \quad (3.60)$$

Now we weaken our assumptions on the perturbations  $\hat{\Lambda}$ , such that they do not have to be compactly supported anymore. Thus, let us assume  $\Lambda$  to be  $\tau$ - $C^1$ -admissible. By [25, Lemma 4.2]  $\mathbb{P}$ , the generalized point spectrum of  $\mathcal{M}$ , does not accumulate at zero for

$$\tau > (N + 1)/2 \quad ,$$

i.e.  $\mathcal{L}_\omega$  is well defined for small  $\omega$ . Furthermore, the following assertions still hold true:

$$(3.54) \text{ for } \tau > 0, \tau > s + 1 - N/2 \text{ and } \tau \geq -s - 1$$

$$(3.55) \text{ for } \tau > 1$$

$$(3.56) \text{ for } \tau > 0, \tau > s - N/2 \text{ and } \tau \geq j - s - 1$$

$$(3.57) \text{ if } (3.54) \text{ and } (3.56)$$

$$(3.58) \text{ for } \tau > k + \sigma \text{ and } \tau \geq N/2 - 1$$

$$(3.59) \text{ for } \tau > j \text{ and } \tau \geq N/2 - 1$$

$$(3.60) \text{ for } \tau > j + 1 \text{ and } \tau \geq N/2 - 1$$

Collecting the values for  $s, t$  and  $\tau$  we obtain

**Lemma 3.18** *Let  $\mathbf{J} \in \mathbb{N}_0$  and  $\mathbb{I} \not\ni s > \max\{1/2, \mathbf{J} + 1 - N/2\}$  as well as  $t \leq s - \mathbf{J}$  and  $t < \min\{-1/2, N/2 - \mathbf{J}\}$ . Moreover, let  $\tau > \max\{(N + 1)/2, s + 1 - N/2\}$ . Then for  $j = 0, \dots, \mathbf{J} - 1$  and  $i = 0, \dots, \mathbf{J} - N$*

$$\mathcal{L}_\omega, \Pi, \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}}, \Gamma_i \in B_{s,t}$$

and thus also for  $j = -1, \dots, \mathbf{J} - 1$

$$\mathcal{K}_j, \mathcal{L}_{\omega,j}^{\mathcal{K}} \in B_{s,t} \quad .$$

Now we approximate  $\hat{\Lambda}$  by compactly supported perturbations. For this purpose we define  $\varphi_n := (1 - \eta)(r/n)$ ,  $n \in \mathbb{N}$  and

$$\Lambda_n := \text{Id} + \varphi_n \hat{\Lambda} \quad .$$

(We note  $\varphi_n|_{U_n} \equiv 1$  and  $\varphi_n|_{A_{2n}} \equiv 0$ .) Then  $\Lambda_n$  converges to  $\Lambda$  for  $n \rightarrow \infty$  pointwise a.e. and also in the operator norm of  $B_{t,t}$  for all  $t \in \mathbb{R}$  since  $\tau > 0$ . Moreover, if  $\tau > s - t$  this convergence also holds true in  $B_{t,s}$ .

From now on all operators, forms and numbers carrying an index  $n$  correspond to the truncated transformation  $\Lambda_n$ .

By a short calculation and a regularity result, e.g. [14, Corollary 3.8 (ii)], we obtain

**Lemma 3.19** *Let  $\tau > 0$ . For all  $s \in \mathbb{R}$  and  $r_2 \leq n \in \mathbb{N}$*

$$\begin{aligned} \mathring{\mathbf{R}}_s^q(\Omega) \cap \varepsilon^{-1} \mathbf{D}_s^q(\Omega) &\subset \mathring{\mathbf{R}}_s^q(\Omega) \cap \varepsilon_n^{-1} \mathbf{D}_s^q(\Omega) \quad , \\ \mu^{-1} \mathring{\mathbf{R}}_s^{q+1}(\Omega) \cap \mathbf{D}_s^{q+1}(\Omega) &\subset \mu_n^{-1} \mathring{\mathbf{R}}_s^{q+1}(\Omega) \cap \mathbf{D}_s^{q+1}(\Omega) \end{aligned}$$

hold with continuous embeddings, whose norms do not depend on  $n$ .

Let  $s, t$  and  $\tau$  satisfy the assumptions of Lemma 3.18 and  $\tau > s - t$ . Applying the latter lemma our static operators  ${}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q$  and  ${}_{\varepsilon_n} \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q$  from [26, Theorem 4.6] are well defined on their common domain of definition  $D({}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q)$ . A long but straight forward computation shows that  ${}_{\varepsilon_n} \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q$  converges to  ${}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q$  in the operator norm of  $B(D({}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q), \mathbb{W}_s^q(\Omega))$ . Therefore, also the inverse operators converge in the operator norm and clearly the same holds true for  ${}^\mu \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^{q+1}$  and  ${}^{\mu_n} \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^{q+1}$ .

Since  $\mathcal{L}_0$  consists of the inverses of  ${}_\varepsilon \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^q$  and  ${}^\mu \mathfrak{M}\mathfrak{a}\mathfrak{x}_{s-1}^{q+1}$  also  ${}_n \mathcal{L}_0$  converges to  $\mathcal{L}_0$  in the operator norm. Thus,  ${}_n \mathcal{L}$  converges to  $\mathcal{L}$  in the operator norm and the same holds true for their powers. By Lemma 2.4 and Lemma 2.6  ${}_n E_{\sigma,m}^{+,k}$  resp.  ${}_n H_{\sigma,m}^{+,k}$  converge to  $E_{\sigma,m}^{+,k}$  resp.  $H_{\sigma,m}^{+,k}$  in the corresponding  $L_t^{2,q,q+1}(\Omega)$ . Looking at the representations in Lemma 2.6 the coefficients  ${}_n \xi^{k,\sigma}$  and  ${}_n \zeta^{k,\sigma}$  of  ${}_n E_{\sigma,m}^{+,k}$  and  ${}_n H_{\sigma,m}^{+,k}$  converge to  $\xi^{k,\sigma}$  and  $\zeta^{k,\sigma}$  in  $\mathbb{C}$ . Hence, also the correction operators  ${}_n \Gamma_j$  converge to  $\Gamma_j$  in the operator norm. Furthermore, it follows that the projections  $\Pi_n$  and  $\Pi_{\text{reg},n}$  converge to  $\Pi$  and  $\Pi_{\text{reg}}$  in the respective operator norms.

It remains to discuss the time-harmonic solution operator. For  $\omega$  small enough and  $(F, G) \in L_{>\frac{1}{2}}^{2,q,q+1}(\Omega)$

$$(E, H) := \mathcal{L}_\omega(F, G), (E_n, H_n) := {}_n \mathcal{L}_\omega(F, G) \in \mathring{\mathbf{R}}_{<-\frac{1}{2}}^q(\Omega) \times \mathbf{D}_{<-\frac{1}{2}}^{q+1}(\Omega) \quad .$$

Consequently the difference form  $(e, h) := (E, H) - (E_n, H_n)$  satisfies the radiation condition and solves

$$(M + i\omega\Lambda)(e, h) = i\omega(\Lambda_n - \Lambda)(E_n, H_n) \quad .$$

For  $\tau > 1$  we have  $(\Lambda_n - \Lambda)(E_n, H_n) \in L_{>\frac{1}{2}}^{2,q,q+1}(\Omega)$  and thus

$$(e, h) = i\omega \mathcal{L}_\omega(\Lambda_n - \Lambda)(E_n, H_n) \quad ,$$

i.e.

$$\mathcal{L}_\omega - {}_n\mathcal{L}_\omega = i\omega \mathcal{L}_\omega(\Lambda_n - \Lambda) {}_n\mathcal{L}_\omega \quad .$$

Interchanging  $\Lambda$  and  $\Lambda_n$  yields also

$$\mathcal{L}_\omega(\Lambda_n - \Lambda) {}_n\mathcal{L}_\omega = {}_n\mathcal{L}_\omega(\Lambda_n - \Lambda) \mathcal{L}_\omega \quad .$$

Since  $\Lambda_n \rightarrow \Lambda$  in  $B_{t,s}$  we obtain  ${}_n\mathcal{L}_\omega \rightarrow \mathcal{L}_\omega$  in  $B_{s,t}$ .

Summing up and using Theorem 3.17 we finally achieve

**Lemma 3.20** *Let  $\mathbf{J} \in \mathbb{N}$  and  $s \in (\mathbf{J} + 1/2, \infty) \setminus \mathbb{I}$  as well as  $t < \min\{-1/2, N/2 - \mathbf{J} - 2\}$ . Moreover, let  $\tau > \max\{(N+1)/2, s-t\}$ . Then for  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $j = -1, \dots, \mathbf{J} - 1$*

$$\begin{aligned} {}_n\mathcal{L}_\omega &\xrightarrow{n \rightarrow \infty} \mathcal{L}_\omega & \text{in } & B_{s,t} & , & \quad \Lambda_n &\xrightarrow{n \rightarrow \infty} \Lambda & \text{in } & B_{t,s} & , \\ {}_n\mathcal{K}_j &\xrightarrow{n \rightarrow \infty} \mathcal{K}_j & \text{in } & B_{s,t} & & & & & & \end{aligned}$$

as well as for all  $n$

$$\mathcal{L}_\omega - {}_n\mathcal{L}_\omega = i\omega \mathcal{L}_\omega(\Lambda_n - \Lambda) {}_n\mathcal{L}_\omega$$

and

$$\|{}_n\mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}}\|_{B_{s,t}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \quad .$$

Now we have to modify the result [45, Lemma 12] slightly.

**Lemma 3.21** *Let  $\mathbf{J} \in \mathbb{N}_0$ ,  $\hat{\omega} > 0$ ,  $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$  and  $X, Y$  be Banach spaces. Moreover, let for  $n \in \mathbb{N}$*

$$\mathcal{L}_\omega \quad , \quad \mathcal{L}_\omega^{(n)}$$

and

$$\mathcal{K}_j \quad , \quad \mathcal{K}_j^{(n)} \quad , \quad j = -1, \dots, \mathbf{J} - 1$$

resp.

$$\mathcal{N}^{(n)}$$

be families of bounded linear operators from  $X$  to  $Y$  resp.  $Y$  to  $X$ . Furthermore, let

$$\mathcal{N}^{(n)} \rightarrow 0 \quad , \quad \mathcal{K}_j^{(n)} \rightarrow \mathcal{K}_j \quad , \quad j = -1, \dots, \mathbf{J} - 1$$

with convergence in the respective operator norms and

$$\mathcal{L}_\omega - \mathcal{L}_\omega^{(n)} = \omega \mathcal{L}_\omega \mathcal{N}^{(n)} \mathcal{L}_\omega^{(n)} \tag{3.61}$$

as well as for all  $n$

$$\mathcal{L}_\omega^{(n)} - \sum_{j=-1}^{\mathbf{J}-1} \mathcal{K}_j^{(n)} = \mathcal{O}(|\omega|^{\mathbf{J}})$$

as  $\omega \rightarrow 0$  with respect to the operator norm in  $B(X, Y)$ . Then also

$$\mathcal{L}_\omega - \sum_{j=-1}^{\mathbf{J}-1} \mathcal{K}_j = \mathcal{O}(|\omega|^{\mathbf{J}})$$

holds as  $\omega \rightarrow 0$  with respect to the operator norm in  $B(X, Y)$ .

**Proof:** The proof is quite similar to the one of [45, Lemma 12] and hence may be omitted here. We just note that in the Maxwell case it is indispensable that (3.61) contains a term  $\omega \mathcal{N}^{(n)}$  contrary to just  $\mathcal{N}^{(n)}$  in the case of the (generalized) perturbed Helmholtz equation or the (generalized) perturbed equations of linear elasticity. ■

We are ready for the

**Proof of the Main Theorem:** If we set

$$X := L_s^{2,q,q+1}(\Omega) \quad , \quad Y := L_t^{2,q,q+1}(\Omega) \quad ,$$

then a combination of Lemma 3.20 and Lemma 3.21 yields our desired asymptotic

$$\| \mathcal{L}_{\omega, \mathbf{J}-1}^{\mathcal{K}} \|_{B_{s,t}} = \mathcal{O}(|\omega|^{\mathbf{J}}) \quad ,$$

which proves the main theorem for  $\mathbf{J} \geq 1$ . If  $\mathbf{J} = 0$  we have by Lemma 3.8 and (3.40)

$$\mathcal{L}_{\omega, -1}^{\mathcal{K}} = \mathcal{L}_{\omega} + (-i\omega)^{-1} \Pi = \mathcal{L}_{\omega} \Pi_{\text{reg}} \quad .$$

Hence, [25, Lemma 4.2 (iv)] yields the stated assertion.

To prove the first remark we compute

$$M \mathcal{L}_{\omega, j}^{\mathcal{K}} = -i\omega \Lambda \mathcal{L}_{\omega, j-1}^{\mathcal{K}}$$

for  $0 \leq j \leq \mathbf{J} - 1$  and

$$M \mathcal{L}_{\omega, -1}^{\mathcal{K}} = M \mathcal{L}_{\omega} = M \mathcal{L}_{\omega} \Pi_{\text{reg}} = \Pi_{\text{reg}} - i\omega \Lambda \mathcal{L}_{\omega} \Pi_{\text{reg}} \quad .$$

Moreover, we set  $\tilde{s} := s, \tilde{t} := t$  or  $\tilde{t} := t + 1$ . Then for  $\mathbf{J} \geq 2$  we may utilize the main theorem with  $\mathbf{J} - 1, \tilde{s}, \tilde{t}$  and for  $\mathbf{J} \in \{0, 1\}$  once more [25, Lemma 4.2 (iv)], which completes the proof of Remark A. ■

Using the Main Theorem as well as Definitions 3.4 and 3.12 and the corresponding remarks we note two final observations concerning the correction operators.

**Remark 3.22** Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s, t$  as well as  $\tau$  be as in the Main Theorem. Moreover, let  $(F, G) \in \text{Reg}_s^{q,0}(\Omega)$ .

(i) For  $\mathbf{J} \leq N$  we have the noncorrected asymptotic

$$\begin{aligned} & \left\| \left( \mathcal{L}_{\omega} - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \right) (F, G) \right\|_{L_t^{2,q,q+1}(\Omega)} \\ &= \mathcal{O}(|\omega|^{\mathbf{J}}) \left\| (F, G) \right\|_{L_s^{2,q,q+1}(\Omega)} \quad . \end{aligned} \quad (3.62)$$

(ii) For  $\mathbf{J} \geq N + 1$  we know by Theorem 2.3 and Lemma 2.15 that the noncorrected asymptotic (3.62) holds true for all  $(F, G) \in \text{Reg}_s^{q,\mathbf{J}}(\Omega)$ , i.e. for  $(F, G)$  perpendicular (in  $L_s^{2,q,q+1}(\Omega)$ ) to all special growing forms  $\Lambda^{-1} E_{\sigma,m}^{+,k}, \Lambda^{-1} H_{\sigma,n}^{+,k} \in L_{-s}^{2,q,q+1}(\Omega)$  with  $1 \leq k \leq \mathbf{J}$ . Albeit this condition is sufficient for (3.62) to hold it is not sharp. Of course, the noncorrected asymptotic (3.62) holds, if and only if the correction operators  $\Gamma_j(F, G) = \hat{\Gamma}_{j-1}(F, G)$  vanish for all  $j = 1, \dots, \mathbf{J} - N$ , i.e.  $(F, G)$  must be perpendicular only to all  $\Lambda^{-1} E_{\sigma,m}^{+,k}, \Lambda^{-1} H_{\sigma,n}^{+,k} \in L_{-s}^{2,q,q+1}(\Omega)$  with  $1 \leq k \leq \mathbf{J} - N$ .

**Remark 3.23** Let  $\mathbf{J} \in \mathbb{N}_0$  and  $s, t$  as well as  $\tau$  be as in the Main Theorem. Moreover, let  $(F, G) \in L_s^{2,q,q+1}(\Omega)$ .

(i) For  $\mathbf{J} \leq N - 1$  we have the noncorrected asymptotic

$$\begin{aligned} & \left\| \left( \mathcal{L}_\omega + (-i\omega)^{-1} \Pi - \sum_{j=0}^{\mathbf{J}-1} (-i\omega)^j \mathcal{L}_0 \mathcal{L}^j \Pi_{\text{reg}} \right) (F, G) \right\|_{L_t^{2,q,q+1}(\Omega)} \\ &= \mathcal{O}(|\omega|^{\mathbf{J}}) \|(F, G)\|_{L_s^{2,q,q+1}(\Omega)} \end{aligned} \quad (3.63)$$

(ii) For  $\mathbf{J} \geq N$  the noncorrected asymptotic (3.63) holds true, if and only if  $\Gamma_j(F, G) = 0$  for all  $j = 0, \dots, \mathbf{J} - N$ , i.e. if and only if  $(F, G)$  is perpendicular to all special forms  $\Lambda^{-1} E_{\sigma,m}^{+,k}, \Lambda^{-1} H_{\sigma,n}^{+,k} \in L_{-s}^{2,q,q+1}(\Omega)$  with  $0 \leq k \leq \mathbf{J} - N$ .

## 4 Inhomogeneous boundary data

Let us finish this report by briefly pointing out how one may easily obtain a low frequency asymptotic for inhomogeneous boundary data as well. For this purpose let  $\Omega$  be a  $C^3$ -domain. Then we may utilize the linear and continuous tangential trace and extension operators  $\Gamma_t$  and  $\check{\Gamma}_t$  from [14, section 3.3].

Let us remind of the continuous solution operator

$$\begin{aligned} \mathcal{S}_\omega : L_s^{2,q,q+1}(\Omega) \times \mathcal{R}^q(\partial\Omega) &\longrightarrow \mathbf{R}_t^q(\Omega) \times \mathbf{D}_t^{q+1}(\Omega) \subset L_t^{2,q,q+1}(\Omega) \\ (F, G, \lambda) &\longmapsto (E, H) \end{aligned}$$

for  $s, -t > 1/2$  from [25, section 5] of the Maxwell system

$$(M + i\omega\Lambda)(E, H) = (F, G) \in L_s^{2,q,q+1}(\Omega) \quad , \quad \Gamma_t E = \lambda \in \mathcal{R}^q(\partial\Omega) \quad ,$$

where

$$\mathcal{R}^q(\partial\Omega) = \{ \varkappa \in \mathbf{H}^{-\frac{1}{2},q}(\partial\Omega) : \text{Rot } \varkappa \in \mathbf{H}^{-\frac{1}{2},q+1}(\partial\Omega) \}$$

and  $\text{Rot} := d$  denotes the exterior derivative on the boundary manifold  $\partial\Omega$ . Again  $\mathcal{S}_\omega$  is well defined for small frequencies  $\omega$  and connected to  $\mathcal{L}_\omega$  via

$$\mathcal{S}_\omega(F, G, \lambda) = (\check{\Gamma}_t \lambda, 0) + \mathcal{L}_\omega(F, G) - i\omega \mathcal{L}_\omega(\varepsilon \check{\Gamma}_t \lambda, 0) - \mathcal{L}_\omega(0, \text{rot } \check{\Gamma}_t \lambda) \quad .$$

Thus,  $\mathcal{S}_\omega$  inherits its low frequency behavior from  $\mathcal{L}_\omega$ .

By [14, Theorem 3.14, Remark 3.15] for slightly more regular coefficients, i.e.  $\varepsilon \in C^1(\overline{A_{r_1}} \cap \overline{U_{r_2}})$ , the extension

$$\check{\Gamma}_t \lambda \in \mathring{R}_{\text{vox}}^q(\Omega) \cap \varepsilon^{-1} D_{\text{vox}}^q(\Omega)$$

may be chosen in a way, such that (for the first component!)

$$(\varepsilon \check{\Gamma}_t \lambda, \text{rot } \check{\Gamma}_t \lambda) \in ({}_0 D_{\text{vox}}^q(\Omega) \cap \mathring{B}^q(\Omega)^\perp) \times ({}_0 R_{\text{vox}}^{q+1}(\Omega) \cap \check{\mathcal{H}}^{q+1}(\Omega)^\perp)$$



holds, where  $\tilde{\mathcal{H}}^q(\Omega) := {}_0\mathbf{R}^q(\Omega) \cap \mathring{\mathbf{D}}^q(\Omega)$  denotes the finite dimensional vector space of (generalized) Neumann forms. Hence, in any case we have

$$(\varepsilon \check{\Gamma}_t \lambda, 0) \in \text{Reg}_{\text{vox}}^{q,0}(\Omega) \quad .$$

Due to  $\Gamma_t \text{rot} = \text{Rot} \Gamma_t$  the condition  $\text{Rot} \lambda = 0$  would imply homogeneous boundary data for  $\text{rot} \check{\Gamma}_t \lambda$ , i.e.  $\text{rot} \check{\Gamma}_t \lambda \in \mathring{\mathbf{R}}_{\text{vox}}^{q+1}(\Omega)$ . Furthermore, for  $b \in \mathbf{B}^{q+1}(\Omega)$

$$\langle \text{rot} \check{\Gamma}_t \lambda, b \rangle_{L^2, q+1(\Omega)} = \langle \lambda, \gamma_n b \rangle_{L^2, q(\partial\Omega)}$$

holds by Stokes' theorem in the sense of the  $\mathbf{H}^{-\frac{1}{2}, q}(\partial\Omega)$ - $\mathbf{H}^{\frac{1}{2}, q}(\partial\Omega)$ -duality. Here we denote by  $\gamma_t = \iota^*$  and  $\gamma_n := \pm \circledast \gamma_t^*$  the usual tangential and normal traces on  $\mathbf{H}^{1, q}(\Omega)$ , by  $\iota : \partial\Omega \rightarrow \bar{\Omega}$  the natural embedding of the boundary and by  $\circledast$  the Hodge star operator on the boundary manifold  $\partial\Omega$ . By the regularity assumptions on the boundary we have  $b \in \mathbf{B}^{q+1}(\Omega) \subset \mathbf{H}^{1, q+1}(\Omega)$  and thus  $\gamma_n b \in \mathbf{H}^{\frac{1}{2}, q}(\partial\Omega)$ . (For this  $\partial\Omega \in C^2$  would be enough!) Hence,  $\text{rot} \check{\Gamma}_t \lambda \in \mathbf{B}^{q+1}(\Omega)^\perp$ , if  $\lambda \perp \gamma_n \mathbf{B}^{q+1}(\Omega)$ . Thus, for  $\lambda \in \mathcal{R}^q(\partial\Omega)$  perpendicular to  $\gamma_n \mathbf{B}^{q+1}(\Omega)$  and with vanishing rotation we also have  $\text{rot} \check{\Gamma}_t \lambda \in \mathring{\mathbf{R}}_{\text{vox}}^{q+1}(\Omega) \cap \mathbf{B}^{q+1}(\Omega)^\perp$ , i.e.  $(0, \text{rot} \check{\Gamma}_t \lambda) \in \text{Reg}_{\text{vox}}^{q,0}(\Omega)$ . Consequently for this extension operator  $\check{\Gamma}_t$  and  $\lambda \in \mathcal{R}^q(\partial\Omega)$  we get

$$(\varepsilon \check{\Gamma}_t \lambda, \text{rot} \check{\Gamma}_t \lambda) \in \text{Reg}_{\text{vox}}^{q,0}(\Omega) \quad \Leftrightarrow \quad \lambda \in {}_0\mathcal{R}^q(\partial\Omega) \cap (\gamma_n \mathbf{B}^{q+1}(\Omega))^{\perp_{L^2, q(\partial\Omega)}},$$

which would enhance the asymptotic since then  $(\varepsilon \check{\Gamma}_t \lambda, \text{rot} \check{\Gamma}_t \lambda)$  would be an element of the kernels of  $\Pi$  and  $\tilde{\Gamma}$  (See Definition 3.12 and Remark 3.14).

Moreover, it may be of interest to have conditions on the extension operator  $\check{\Gamma}_t$  and  $\lambda$ , which also would imply

$$(\varepsilon \check{\Gamma}_t \lambda, \text{rot} \check{\Gamma}_t \lambda) \in \text{Reg}_{\text{vox}}^{q, \mathbf{J}}(\Omega) \quad .$$

That would enhance the asymptotic of  $\mathcal{S}_\omega$  once more because then all correction operators would vanish on the extended boundary forms.

To reach this aim we have to modify our extension operator again. For this we recall from [14, section 3.3] the constructions of the extension operators  $\check{\Gamma}_t$  and  ${}_0\check{\Gamma}_t$ . Their construction is based on the extension operator  $\check{\Gamma}_t^{\text{b}}$  mapping to forms defined in the bounded domain

$$\Omega_{\text{b}} = \Omega \cap U_{r_3} \quad .$$

Using the notations from there we have for some  $\lambda \in \mathcal{R}^q(\partial\Omega)$

$$E_\lambda^{\text{b}} := \check{\Gamma}_t^{\text{b}} \tilde{\lambda} \in \mathbf{R}^q(\Omega_{\text{b}}) \cap \varepsilon^{-1} \text{div} \mathbf{D}^{q+1}(\Omega_{\text{b}}) \quad ,$$

where  $\tilde{\lambda} \in \mathcal{R}^q(\partial\Omega_{\text{b}})$ ,  $\partial\Omega_{\text{b}} = \partial\Omega \dot{\cup} S_{r_3}$ , is the extension by zero of  $\lambda$  as a linear and continuous functional. Let us introduce the linear and continuous solution operator

$$\mathcal{L}_{\text{b},0} : \text{Reg}^{q,0}(\Omega_{\text{b}}) \longrightarrow \left( \mathring{\mathbf{R}}^q(\Omega_{\text{b}}) \times \mathbf{D}^{q+1}(\Omega_{\text{b}}) \right) \cap \Lambda^{-1} \text{Reg}^{q,0}(\Omega_{\text{b}}) \quad ,$$

$$(f, g) \longmapsto (e, h)$$

where

$$\text{Reg}^{q,0}(\Omega_b) := \text{div } \mathbf{D}^{q+1}(\Omega_b) \times \text{rot } \mathring{\mathbf{R}}^q(\Omega_b)$$

and  $(e, h)$  denotes the unique solution of the static Maxwell problem

$$M(e, h) = (f, g) \quad , \quad (e, h) \in \left( \mathring{\mathbf{R}}^q(\Omega_b) \times \mathbf{D}^{q+1}(\Omega_b) \right) \cap \Lambda^{-1} \text{Reg}^{q,0}(\Omega_b) \quad .$$

Then  $\mathcal{L}_b := \Lambda \mathcal{L}_{b,0}$  may be iterated on  $(\varepsilon E_\lambda^b, 0)$ . This defines arbitrary powers of  $\mathcal{L}_b$  on  $(\varepsilon E_\lambda^b, 0)$ . We note  $\mathcal{M} \mathcal{L}_b = \text{Id}$  on  $\text{Reg}^{q,0}(\Omega_b)$ , where  $\mathcal{M} := M \Lambda^{-1}$ . Then for any  $\ell \in \mathbb{N}_0$  and appropriate  $\Lambda$  an extension operator  $\check{\Gamma}_t^\ell$  may also be defined by

$$\left( \check{\Gamma}_t^\ell \lambda, 0 \right) := \Lambda^{-1} \mathcal{M}^\ell \varphi \mathcal{L}_b^\ell (\varepsilon E_\lambda^b, 0) = \Lambda^{-1} \mathcal{M}^\ell \varphi \mathcal{L}_b^\ell (\varepsilon \check{\Gamma}_t^b \tilde{\lambda}, 0) \quad ,$$

where  $\varphi := \hat{\varphi} \circ r$  and  $\hat{\varphi} := 1 - \hat{\eta}$ . We note  $\text{supp } \varphi \subset \overline{\Omega \cap U_{r_2}}$  and  $\text{supp } \nabla \varphi \subset \overline{A_{r_1} \cap U_{r_2}}$ . Of course then  $\check{\Gamma}_t^\ell \lambda$  has support in  $\overline{\Omega \cap U_{r_2}}$  and  $\varepsilon \check{\Gamma}_t^\ell \lambda \in \text{div } \mathbf{D}_{\text{vox}}^{q+1}(\Omega)$  holds for  $\ell \geq 1$ . For  $\ell = 0$  and  $\ell = 1$  we get back our old extensions.

Before we proceed we have to make sure that  $\check{\Gamma}_t^\ell$  is well defined, i.e. that  $\check{\Gamma}_t^\ell \lambda$  has got a rotation and that its trace equals  $\lambda$ .

**Lemma 4.1** *Let  $\ell \in \mathbb{N}_0$  and  $\Lambda \in C^\ell(Z_\delta)$  be 0-admissible with  $Z_\delta := A_{r_1-\delta} \cap U_{r_2+\delta}$  for some  $\delta > 0$ . Then for all  $\lambda \in \mathcal{R}^q(\partial \Omega)$*

$$\check{\Gamma}_t^\ell \lambda \in \mathbf{R}_{\text{vox}}^q(\Omega) \cap \varepsilon^{-1} \mathbf{D}_{\text{vox}}^q(\Omega) \quad , \quad \Gamma_t \check{\Gamma}_t^\ell \lambda = \lambda \quad .$$

Moreover,  $\check{\Gamma}_t^\ell \lambda \in \varepsilon^{-1} \text{div } \mathbf{D}_{\text{vox}}^{q+1}(\Omega)$ , if  $\ell \geq 1$ .

We note that for  $\ell = 0$  the additional assumption  $\Lambda \in C^0(Z_\delta)$  is not needed. Furthermore,  $Z_\delta$  may be replaced by an arbitrarily thin shell  $A_{\rho-\delta} \cap U_{\rho+\delta}$  using an appropriate cut-off function  $\varphi$ .

**Proof:** First we note that the case  $\ell = 0$  has already been proved in [14, Theorem 3.13], since in fact  $\check{\Gamma}_t^0$  is our old extension operator.

The form  $(e, h) := \Lambda^{-1} \mathcal{L}_b^\ell (\varepsilon E_\lambda^b, 0) = \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0)$ , which is even well defined for 0-admissible  $\varepsilon$ , belongs to  $\left( \mathring{\mathbf{R}}^q(\Omega_b) \times \mathbf{D}^{q+1}(\Omega_b) \right) \cap \Lambda^{-1} \text{Reg}^{q,0}(\Omega_b)$  and solves

$$M(e, h) = \mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0) \quad .$$

Thus, if  $\mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0) \in \mathbf{H}^{\ell-1, q, q+1}(Z_\delta)$  for any  $\delta > 0$  small enough we get by inner regularity (for instance [14, Theorem 2.8] or [14, Theorem 3.6, Remark 3.7] in combination with a usual cutting technique)

$$(e, h), \mathcal{L}_b^\ell (\varepsilon E_\lambda^b, 0) \in \mathbf{H}^{\ell, q, q+1}(Z_\delta) \quad (4.1)$$

for any  $\delta > 0$  small enough. Hence, a tiny induction argument shows (4.1). Now

$$\begin{aligned} \mathcal{M} \varphi \mathcal{L}_b^\ell (\varepsilon E_\lambda^b, 0) &= M \varphi \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0) \\ &= \varphi \mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0) + \check{\varphi}'(r) r^{-1} S \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-1} (\varepsilon E_\lambda^b, 0) \quad . \end{aligned} \quad (4.2)$$

Since the second term in the sum of the right hand side belongs to  $\mathbf{H}^{\ell,q,q+1}(Z_\delta)$  for  $\delta > 0$  small enough, we obtain  $\mathcal{M}^{\ell+1}\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0)$  is well defined, if and only if  $\mathcal{M}^\ell\varphi \mathcal{L}_b^{\ell-1}(\varepsilon E_\lambda^b, 0)$  is well defined. Another tiny induction shows  $\mathcal{M}^{\ell+1}\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0)$  is well defined, if and only if  $\mathcal{M}\varphi(\varepsilon E_\lambda^b, 0) = M\varphi(E_\lambda^b, 0) = M(\check{\Gamma}_t^0\lambda, 0) = (0, \text{rot } \check{\Gamma}_t^0\lambda)$  is well defined. But this has been proved in [14, Theorem 3.13]. Thus,

$$\check{\Gamma}_t^\ell\lambda \quad , \quad \text{rot } \check{\Gamma}_t^\ell\lambda$$

are well defined. Furthermore, we compute

$$\begin{aligned} (\check{\Gamma}_t^\ell\lambda, 0) &= \Lambda^{-1}\mathcal{M}^{\ell-1}M\varphi \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-1}(\varepsilon E_\lambda^b, 0) \\ &= (\check{\Gamma}_t^{\ell-1}\lambda, 0) + \hat{\varphi}'(r)r^{-1}\Lambda^{-1}\mathcal{M}^{\ell-1}S \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-1}(\varepsilon E_\lambda^b, 0) \quad , \end{aligned}$$

which shows  $\Gamma_t\check{\Gamma}_t^\ell\lambda = \Gamma_t\check{\Gamma}_t^{\ell-1}\lambda$  since the second term of the sum on the right hand side has compact support in  $\Omega_b$ . Finally again by [14, Theorem 3.13] we get

$$\Gamma_t\check{\Gamma}_t^\ell\lambda = \Gamma_t\check{\Gamma}_t^0\lambda = \lambda \quad ,$$

which completes the proof. ■

**Remark 4.2** Repeating (4.2) shows for all  $k \leq \ell$

$$\Lambda^{-1}\mathcal{M}^k\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) = \Lambda^{-1}\varphi \mathcal{L}_b^{\ell-k}(\varepsilon E_\lambda^b, 0) + (\tilde{E}_k, \tilde{H}_k)$$

and therefore for all  $k \leq \ell - 1$

$$\Lambda^{-1}\mathcal{M}^k\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) = \varphi \mathcal{L}_{b,0} \mathcal{L}_b^{\ell-k-1}(\varepsilon E_\lambda^b, 0) + (\tilde{E}_k, \tilde{H}_k) \quad ,$$

where the  $(\tilde{E}_k, \tilde{H}_k)$  are compactly supported. This shows for all  $1 \leq k \leq \ell - 1$

$$\Lambda^{-1}\mathcal{M}^k\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \in (\mathring{\mathbb{R}}_{\text{vox}}^q(\Omega) \times D_{\text{vox}}^{q+1}(\Omega)) \cap \Lambda^{-1}\text{Reg}_{\text{Svox}}^{q,0}(\Omega)$$

and

$$\Lambda^{-1}\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \in (\mathring{\mathbb{R}}_{\text{vox}}^q(\Omega) \times D_{\text{vox}}^{q+1}(\Omega)) \cap \Lambda^{-1}(D_{\text{vox}}^q(\Omega) \times \mathring{\mathbb{R}}_{\text{vox}}^{q+1}(\Omega)) \quad .$$

Now let

$$\check{\Gamma}_t := \check{\Gamma}_t^\ell$$

for some  $\ell \geq \mathbf{J} + 1$ , i.e.  $\ell := \mathbf{J} + 1$ , and  $\mathbf{J} \geq 1$ . Then since  $1 \leq \ell - \mathbf{J} \leq \ell - 1$  we obtain by the latter remark

$$\begin{aligned} \mathcal{L}^{\mathbf{J}} \Lambda(\check{\Gamma}_t\lambda, 0) &= \mathcal{L}^{\mathbf{J}} \mathcal{M}^\ell\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \\ &= \mathcal{M}^{\ell-\mathbf{J}}\varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \in \text{Reg}_{\text{Svox}}^{q,0}(\Omega) \cap \Lambda(\mathring{\mathbb{R}}_{\text{vox}}^q(\Omega) \times D_{\text{vox}}^{q+1}(\Omega)) \quad . \end{aligned}$$

Thus,

$$(\varepsilon \check{\Gamma}_t \lambda, 0) \in \text{Reg}_{\text{vox}}^{q, \mathbf{J}}(\Omega)$$

by Remark 2.2. Unfortunately (or maybe not) this argument holds not for the other part  $(0, \text{rot } \check{\Gamma}_t \lambda) = M(\check{\Gamma}_t \lambda, 0)$  because

$$\mathcal{L}^{\mathbf{J}} M(\check{\Gamma}_t \lambda, 0) = \mathcal{L}^{\mathbf{J}} \mathcal{M} \Lambda(\check{\Gamma}_t \lambda, 0) = \mathcal{L}^{\mathbf{J}} \mathcal{M}^{\ell+1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0)$$

and of course  $\mathcal{M}^\ell \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) = \Lambda(\check{\Gamma}_t \lambda, 0)$  does not satisfy the homogeneous boundary condition as long as  $\lambda \neq 0$  and thus

$$\mathcal{L} \mathcal{M}^{\ell+1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \neq \mathcal{M}^\ell \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0) \quad .$$

Hence, we have to use the other characterization of  $\text{Reg}_{\text{vox}}^{q, \mathbf{J}}(\Omega)$  using orthogonality constraints. Since  $M(\check{\Gamma}_t \lambda, 0)$  is already an element of  $\text{Reg}_{\text{vox}}^{q, 0}(\Omega)$ , the correction operators  $\check{\Gamma}_j$  vanish on  $M(\check{\Gamma}_t \lambda, 0)$ . Thus, by Definition 3.12 and Definition 3.4, Remark 3.6 we have to show that also the correction operators  $\hat{\Gamma}_j$ ,  $j = 0, \dots, \mathbf{J} - N - 1$ , vanish on  $M(\check{\Gamma}_t \lambda, 0)$ . Hence, we have to prove

$$\Lambda^{-1} M(\check{\Gamma}_t \lambda, 0) \perp E_{\sigma, m}^{+, j}, H_{\sigma, m}^{+, j}$$

for all  $j = 1, \dots, \mathbf{J}$  and appropriate  $\sigma, m$ . Of course for even  $j$  we have

$$\langle \Lambda^{-1} M(\check{\Gamma}_t \lambda, 0), E_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} = 0 \quad .$$

For odd  $j$  since  $\check{\Gamma}_t \lambda$  is compactly supported we compute

$$\begin{aligned} & \langle \Lambda^{-1} M(\check{\Gamma}_t \lambda, 0), E_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} \\ &= - \langle (\check{\Gamma}_t \lambda, 0), \mathcal{M} E_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} + \langle \lambda, \gamma_n \pi_2 \Lambda^{-1} E_{\sigma, m}^{+, j} \rangle_{L^{2, q}(\partial \Omega)} \quad , \end{aligned}$$

where  $\pi_2$  denotes the projection onto the second component. Moreover, since the first partners have homogeneous boundary conditions respectively

$$\begin{aligned} & \langle (\check{\Gamma}_t \lambda, 0), \mathcal{M} E_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} \\ &= \langle \Lambda^{-1} \mathcal{M}^\ell \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \mathcal{L}^{j-1} \Lambda(E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= (-1)^\ell \langle \Lambda^{-1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \mathcal{M}^\ell \mathcal{L}^{j-1} \Lambda(E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= (-1)^\ell \langle \Lambda^{-1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \mathcal{M}^{\ell-j+1} \Lambda(E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= (-1)^\ell \langle \Lambda^{-1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \underbrace{\mathcal{M}^{\ell-j} M(E_{\sigma, m}^+, 0)}_{=0} \rangle_{L^{2, q, q+1}(\Omega)} = 0 \end{aligned}$$

since  $\ell - j \geq \ell - \mathbf{J} \geq 1 \geq 0$ . We note that for this argument  $\ell \geq \mathbf{J}$ , i.e.  $\ell := \mathbf{J}$ , is sufficient.

Analogously we achieve for odd  $j$

$$\langle \Lambda^{-1} M(\check{\Gamma}_t \lambda, 0), H_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} = 0$$

and for even  $j$

$$\begin{aligned} & \langle \Lambda^{-1} M(\check{\Gamma}_t \lambda, 0), H_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} \\ &= - \underbrace{\langle (\check{\Gamma}_t \lambda, 0), \mathcal{M} H_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)}}_{=0} + \langle \lambda, \gamma_n \pi_2 \Lambda^{-1} H_{\sigma, m}^{+, j} \rangle_{L^{2, q}(\partial \Omega)} \quad . \end{aligned}$$

Of course these arguments also work for  $\Lambda(\check{\Gamma}_t \lambda, 0) = (\varepsilon \check{\Gamma}_t \lambda, 0)$ . Again we have to show

$$(\check{\Gamma}_t \lambda, 0) \perp E_{\sigma, m}^{+, j}, H_{\sigma, m}^{+, j}$$

for all  $j = 1, \dots, \mathbf{J}$  and appropriate  $\sigma, m$ . But now

$$\begin{aligned} & \langle (\check{\Gamma}_t \lambda, 0), E_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} \\ &= \langle \Lambda^{-1} \mathcal{M}^\ell \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \mathcal{L}^j \Lambda(E_{\sigma, m}^+, 0) \rangle_{L^{2, q, q+1}(\Omega)} \\ &= (-1)^\ell \langle \Lambda^{-1} \varphi \mathcal{L}_b^\ell(\varepsilon E_\lambda^b, 0), \underbrace{\mathcal{M}^{\ell-j-1} M(E_{\sigma, m}^+, 0)}_{=0} \rangle_{L^{2, q, q+1}(\Omega)} = 0 \quad , \end{aligned}$$

since  $\ell - j - 1 \geq \ell - \mathbf{J} - 1 \geq 0$ , and analogously

$$\langle (\check{\Gamma}_t \lambda, 0), H_{\sigma, m}^{+, j} \rangle_{L^{2, q, q+1}(\Omega)} = 0 \quad .$$

This shows once again  $(\varepsilon \check{\Gamma}_t \lambda, 0) \in \text{Reg}_{\text{vox}}^{q, \mathbf{J}}(\Omega)$ .

Summing up we receive the following result: For 0-admissible  $\Lambda \in C^{\mathbf{J}+1}(Z_\delta)$  with some  $\delta > 0$  we may choose our extension operator  $\check{\Gamma}_t$  in a way, such that

$$(\varepsilon \check{\Gamma}_t \lambda, \text{rot } \check{\Gamma}_t \lambda) \in \text{Reg}_{\text{vox}}^{q, \mathbf{J}}(\Omega) \quad ,$$

if and only if

$$\lambda \in {}_0\mathcal{R}^q(\partial \Omega) \cap (\gamma_n B^{q+1}(\Omega))^{\perp_{L^{2, q}(\partial \Omega)}}$$

as well as for all  $j = 1, \dots, \mathbf{J}$  and all appropriate  $\sigma, m$

$$\lambda \perp_{L^{2, q}(\partial \Omega)} \gamma_n \pi_2 \Lambda^{-1} E_{\sigma, m}^{+, j}, \gamma_n \pi_2 \Lambda^{-1} H_{\sigma, m}^{+, j} \quad .$$

**Remark 4.3** *It is an interesting and open question if these considerations may hold for Lipschitz boundaries as well.*

## 5 Appendix: Special radiating solutions in the whole space case

In this appendix we give an exact and easy derivation of the special radiating solutions

$$(\mathbb{E}_{\sigma,m}^{1,\omega}, \mathbb{H}_{\sigma,m}^{1,\omega}) \quad , \quad (\mathbb{E}_{\sigma,m}^{2,\omega}, \mathbb{H}_{\sigma,m}^{2,\omega})$$

presented in (3.5), (3.6). This means we try to find special solutions of the homogeneous Maxwell equation in  $\mathbb{R}^N \setminus \{0\}$

$$(M + i\omega)(E, H) = (0, 0) \quad , \quad \omega \in \mathbb{C}_+ \setminus \{0\} \quad (5.1)$$

which satisfy the radiation condition. Using a separation of variables technique, i.e. our spherical calculus from [46], and our well known eigenforms  $S_{\sigma,m}^q$  and  $T_{\sigma,m}^q$  presented in [46] and further discussed in [26, section 2] we will reduce the calculation to the solution of the well known Bessel differential equation. Since

$$(\operatorname{div} E, \operatorname{rot} H) = (0, 0)$$

$(E, H)$  are  $C^\infty$ -forms by regularity [14]. Applying  $M - i\omega$  to (5.1) we get

$$(\Delta + \omega^2)(E, H) = (0, 0) \quad .$$

Hence, first let us try to find a solution of the system

$$\operatorname{div} E = 0 \quad \text{and} \quad (\Delta + \omega^2)E = 0 \quad (5.2)$$

which will be translated utilizing [46, (21)] and [46, (25)] into the system

$$\begin{aligned} [\check{\rho} \quad \check{\tau}]r^{-1} \begin{bmatrix} -\operatorname{Div} & 0 \\ r^{-q'+1} \operatorname{D} r^{q'} & \operatorname{Div} \end{bmatrix} \begin{bmatrix} \rho E \\ \tau E \end{bmatrix} &= 0 \quad , \\ [\check{\rho} \quad \check{\tau}]r^{-2} \begin{bmatrix} \operatorname{B} + r^2 \operatorname{R}_1 + r^2 \omega^2 & -2 \operatorname{Div} \\ 2 \operatorname{Rot} & \operatorname{B} + r^2 \operatorname{R}_2 + r^2 \omega^2 \end{bmatrix} \begin{bmatrix} \rho E \\ \tau E \end{bmatrix} &= 0 \quad , \end{aligned}$$

i.e. into the system

$$\operatorname{Div} \rho E = 0 \quad , \quad (5.3)$$

$$r^{1-q'} \operatorname{D} r^{q'} \rho E + \operatorname{Div} \tau E = 0 \quad , \quad (5.4)$$

$$(\operatorname{B} + r^2(\operatorname{R}_1 + \omega^2))\rho E - 2 \operatorname{Div} \tau E = 0 \quad , \quad (5.5)$$

$$2 \operatorname{Rot} \rho E + (\operatorname{B} + r^2(\operatorname{R}_2 + \omega^2))\tau E = 0 \quad . \quad (5.6)$$

The first two equations suggest the following two 'ansätze':

1. ansatz:  $\rho E := 0$  ,  $\tau E := e(r) T_{\sigma,m}^q$
2. ansatz:  $\rho E := e_\rho(r) T_{\sigma,m}^{q-1}$  ,  $\tau E := e_\tau(r) S_{\sigma,m}^q$

Let us discuss the first ansatz. Then (5.3), (5.4) and (5.5) are automatically fulfilled. However (5.6) turns to

$$(B + r^2(R_2 + \omega^2))e(r)T_{\sigma,m}^q = 0 \quad \Longleftrightarrow \quad (-\lambda_\sigma^q + r^2(R_2 + \omega^2))e(r)T_{\sigma,m}^q = 0$$

using [46, (33)], i.e.  $\text{Div } T_{\sigma,m}^q = 0$  and  $(B + \lambda_\sigma^q)T_{\sigma,m}^q = 0$ , and  $R_2 := R_2$  from [46, p. 1024]. But this is equivalent to the second order ordinary differential equation

$$r^2 e''(r) + (N - 1)re'(r) + (r^2\omega^2 + q(q' - 2) - \lambda_\sigma^q)e(r) = 0 \quad ,$$

which will be transformed by the substitution

$$e(r) := r^\ell \varphi(\omega r)$$

into the equation

$$\begin{aligned} & \omega^2 r^2 \varphi''(\omega r) + (2\ell + N - 1)\omega r \varphi'(\omega r) \\ & + (r^2\omega^2 + q(q' - 2) - \lambda_\sigma^q + \ell(\ell + N - 2))\varphi(\omega r) = 0 \quad . \end{aligned}$$

Setting  $t := \omega r$  and

$$2\ell + N - 1 := 1 \quad \Leftrightarrow \quad \ell = 1 - N/2$$

we obtain Bessel's differential equation

$$t^2 \varphi''(t) + t\varphi'(t) + (t^2 - \nu_\sigma^2)\varphi(t) = 0 \quad , \quad \nu_\sigma := N/2 + \sigma \quad . \quad (5.7)$$

Consequently  $E$  fulfills Sommerfeld's radiation condition for Helmholtz' equation, if  $\varphi$  is a multiple of  $H_{\nu_\sigma}^1$ , Hankel's function of first kind for  $\nu_\sigma$ . Hence we get a first solution

$$E_{\sigma,m}^{1,\omega} := r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\tau} T_{\sigma,m}^q \quad .$$

Utilizing the second ansatz (5.3) is trivially fulfilled and (5.4), (5.5), (5.6) are translated into the system

$$\begin{aligned} & r^{1-q'} \frac{d}{dr} (r^{q'} e_\rho(r) T_{\sigma,m}^{q-1}) + e_\tau(r) \text{Div } S_{\sigma,m}^q = 0 \quad , \\ & (B + r^2(R_1 + \omega^2))e_\rho(r) T_{\sigma,m}^{q-1} - 2 \text{Div } e_\tau(r) S_{\sigma,m}^q = 0 \quad , \\ & 2 \text{Rot } e_\rho(r) T_{\sigma,m}^{q-1} + (B + r^2(R_2 + \omega^2))e_\tau(r) S_{\sigma,m}^q = 0 \quad , \end{aligned}$$

which turns to the system

$$r e'_\rho(r) + q' e_\rho(r) + i\omega_\sigma^{q-1} e_\tau(r) = 0 \quad , \quad (5.8)$$

$$r^2(R_1 + \omega^2)e_\rho(r) - \lambda_\sigma^{q-1} e_\rho(r) - 2i\omega_\sigma^{q-1} e_\tau(r) = 0 \quad , \quad (5.9)$$

$$r^2(R_2 + \omega^2)e_\tau(r) - \kappa_\sigma^q e_\tau(r) + 2i\omega_\sigma^{q-1} e_\rho(r) = 0 \quad (5.10)$$

using [46, (32), (33)], i.e.

$$\begin{aligned} \text{Rot } S_{\sigma,m}^q &= 0 & , & & \text{Div } T_{\sigma,m}^{q-1} &= 0 & , \\ (B + \kappa_\sigma^q) S_{\sigma,m}^q &= 0 & , & & (B + \lambda_\sigma^{q-1}) T_{\sigma,m}^{q-1} &= 0 & , \end{aligned}$$

and [46, (34), (35)], i.e.

$$\text{Rot } T_{\sigma,m}^{q-1} = i \omega_\sigma^{q-1} S_{\sigma,m}^q \quad , \quad \text{Div } S_{\sigma,m}^q = i \omega_\sigma^{q-1} T_{\sigma,m}^{q-1} \quad ,$$

and again  $R_i := R_i$  from [46, p. 1024]. Inserting (5.8) in (5.9) we obtain the equation

$$r^2 e_\rho''(r) + (N+1) r e_\rho'(r) + (r^2 \omega^2 - \sigma(\sigma+N)) e_\rho(r) = 0 \quad ,$$

which again with the substitution

$$e_\rho(r) := r^k \varphi(\omega r) \quad , \quad t := \omega r$$

is carried over to

$$t^2 \varphi''(t) + (2k + N + 1) t \varphi'(t) + (t^2 + k(k+N) - \sigma(\sigma+N)) \varphi(t) = 0 \quad .$$

The choice

$$2k + N + 1 := 1 \quad \Leftrightarrow \quad k = -N/2$$

yields once more Bessel's differential equation (5.7) with the same  $\nu_\sigma$ . From (5.8) we get

$$e_\tau(r) = \frac{i}{\omega_\sigma^{q-1}} r^k ((q' + k) \varphi(\omega r) + \omega r \varphi'(\omega r)) \quad .$$

A few lines later we will see by an easy argument that  $E$  fulfills Sommerfeld's radiation condition for Helmholtz' equation, if  $\varphi$  is a multiple of  $H_{\nu_\sigma}^1$ . Please compare also to Weck and Witsch [48, p. 1520]. Thus we obtain a second solution

$$E_{\sigma,m}^{2,\omega} := r^{-\frac{N}{2}} \left( H_{\nu_\sigma}^1(\omega r) \check{\rho} T_{\sigma,m}^{q-1} + \frac{i}{\omega_\sigma^{q-1}} ((N/2 - q) H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r)) \check{\tau} S_{\sigma,m}^q \right) \quad .$$

Then defining

$$H_{\sigma,m}^{n,\omega} := \frac{i}{\omega} \text{rot } E_{\sigma,m}^{n,\omega} \quad , \quad n = 1, 2$$

we have

$$\text{rot } E_{\sigma,m}^{n,\omega} + i \omega H_{\sigma,m}^{n,\omega} = 0$$

by definition and moreover

$$\text{div } H_{\sigma,m}^{n,\omega} = \frac{i}{\omega} \text{div rot } E_{\sigma,m}^{n,\omega} = \frac{i}{\omega} \Delta E_{\sigma,m}^{n,\omega} = -i \omega E_{\sigma,m}^{n,\omega}$$

since  $\text{div } E_{\sigma,m}^{n,\omega} = 0$  and  $(\Delta + \omega^2) E_{\sigma,m}^{n,\omega} = 0$ . Hence the forms

$$(E_{\sigma,m}^{n,\omega}, H_{\sigma,m}^{n,\omega}) \quad , \quad n = 1, 2$$



are indeed solutions of (5.1). Let us calculate  $H_{\sigma,m}^{n,\omega}$  more explicit:

$$\begin{aligned}
H_{\sigma,m}^{1,\omega} &= \frac{i}{\omega} \text{rot } E_{\sigma,m}^{1,\omega} = \frac{i}{\omega} [\check{\rho} \ \check{\tau}] r^{-1} \begin{bmatrix} -\text{Rot} & r^{1-q} D r^q \\ 0 & \text{Rot} \end{bmatrix} \begin{bmatrix} \rho E_{\sigma,m}^{1,\omega} \\ \tau E_{\sigma,m}^{1,\omega} \end{bmatrix} \\
&= \frac{i}{\omega} [\check{\rho} \ \check{\tau}] r^{-1} \begin{bmatrix} -\text{Rot} & r^{1-q} D r^q \\ 0 & \text{Rot} \end{bmatrix} \begin{bmatrix} 0 \\ r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) T_{\sigma,m}^q \end{bmatrix} \\
&= \frac{i}{\omega} [\check{\rho} \ \check{\tau}] \begin{bmatrix} r^{-q} D r^{q+1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) T_{\sigma,m}^q \\ i \omega_\sigma^q r^{-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) S_{\sigma,m}^{q+1} \end{bmatrix} \\
&= -\frac{\omega_\sigma^q}{\omega} r^{-\frac{N}{2}} \left( H_{\nu_\sigma}^1(\omega r) \check{\tau} S_{\sigma,m}^{q+1} \right. \\
&\quad \left. + \frac{i}{\omega_\sigma^q} \left( (N/2 - q - 1) H_{\nu_\sigma}^1(\omega r) - \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \check{\rho} T_{\sigma,m}^q \right)
\end{aligned}$$

$$\begin{aligned}
H_{\sigma,m}^{2,\omega} &= \frac{i}{\omega} \text{rot } E_{\sigma,m}^{2,\omega} = \frac{i}{\omega} [\check{\rho} \ \check{\tau}] r^{-1} \begin{bmatrix} -\text{Rot} & r^{1-q} D r^q \\ 0 & \text{Rot} \end{bmatrix} \begin{bmatrix} \rho E_{\sigma,m}^{2,\omega} \\ \tau E_{\sigma,m}^{2,\omega} \end{bmatrix} \\
&= \frac{i}{\omega} [\check{\rho} \ \check{\tau}] r^{-1} \begin{bmatrix} -\text{Rot} & r^{1-q} D r^q \\ 0 & \text{Rot} \end{bmatrix} \begin{bmatrix} r^{-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) T_{\sigma,m}^{q-1} \\ \dots S_{\sigma,m}^q \end{bmatrix} \\
&= \frac{i}{\omega} \left( -i \omega_\sigma^{q-1} r^{-1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \right. \\
&\quad \left. + r^{-q} \frac{d}{dr} \left( r^{q-\frac{N}{2}} \frac{i}{\omega_\sigma^{q-1}} \left( (N/2 - q) H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \right) \right) \check{\rho} S_{\sigma,m}^q \\
&= \frac{i}{\omega} \left( - \left( i \omega_\sigma^{q-1} + \frac{i}{\omega_\sigma^{q-1}} (N/2 - q)^2 \right) r^{-1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \right. \\
&\quad + \frac{i}{\omega_\sigma^{q-1}} (N/2 - q + q - N/2 + 1) \omega r^{-\frac{N}{2}} (H_{\nu_\sigma}^1)'(\omega r) \\
&\quad \left. + \frac{i}{\omega_\sigma^{q-1}} \omega^2 r^{1-\frac{N}{2}} (H_{\nu_\sigma}^1)''(\omega r) \right) \check{\rho} S_{\sigma,m}^q \\
&= -\frac{1}{\omega \omega_\sigma^{q-1}} r^{-1-\frac{N}{2}} \left( - \left( (\omega_\sigma^{q-1})^2 + (N/2 - q)^2 \right) H_{\nu_\sigma}^1(\omega r) \right. \\
&\quad \left. + \omega r (H_{\nu_\sigma}^1)'(\omega r) + \omega^2 r^2 (H_{\nu_\sigma}^1)''(\omega r) \right) \check{\rho} S_{\sigma,m}^q
\end{aligned}$$

Here we insert Bessel's differential equation (5.7) and obtain

$$\begin{aligned}
H_{\sigma,m}^{2,\omega} &= -\frac{r^{-1-\frac{N}{2}}}{\omega \omega_\sigma^{q-1}} \left( -\omega^2 r^2 + \underbrace{\left( \nu_\sigma^2 - (N/2 - q)^2 - (q + \sigma)(q' + \sigma) \right)}_{=0} \right) H_{\nu_\sigma}^1(\omega r) \check{\rho} S_{\sigma,m}^q \\
&= \frac{\omega}{\omega_\sigma^{q-1}} r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\rho} S_{\sigma,m}^q .
\end{aligned}$$

Now also  $H_{\sigma,m}^{2,\omega}$  fulfills Sommerfeld's radiation condition for Helmholtz' equation as the latter explicit formula shows. Therefore the choice of Hankel's function is also legitimated in the second case.

Another way to compute the forms  $(E_{\sigma,m}^{n,\omega}, H_{\sigma,m}^{n,\omega})$  is to solve the related dual problem

$$\operatorname{rot} H = 0 \quad \text{and} \quad (\Delta + \omega^2)H = 0 \quad . \quad (5.11)$$

Analogously to (5.2) this would lead to the system

$$\begin{aligned} -\operatorname{Rot} \rho H + r^{-q} \operatorname{D} r^{q+1} \tau H &= 0 \quad , \\ \operatorname{Rot} \tau H &= 0 \quad , \\ (\operatorname{B} + r^2(\operatorname{R}_1 + \omega^2)) \rho H - 2 \operatorname{Div} \tau H &= 0 \quad , \\ 2 \operatorname{Rot} \rho H + (\operatorname{B} + r^2(\operatorname{R}_2 + \omega^2)) \tau H &= 0 \end{aligned}$$

for the  $(q+1)$ -form  $H$ , which can be handled similar to (5.3)-(5.6) utilizing the following two 'ansätze':

$$\begin{aligned} \text{1. ansatz:} \quad \rho H &:= h(r) S_{\sigma,m}^q \quad , \quad \tau H := 0 \\ \text{1. ansatz:} \quad \rho H &:= h_\rho(r) T_{\sigma,m}^q \quad , \quad \tau H := h_\tau(r) S_{\sigma,m}^{q+1} \end{aligned}$$

Let us summarize: Defining for  $\sigma \in \mathbb{N}_0$  and  $m = 1, \dots$  as well as  $\nu_\sigma = N/2 + \sigma$  the forms

$$\begin{aligned} \tilde{\mathbb{E}}_{\sigma,m}^{1,\omega} &:= \omega r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\tau} T_{\sigma,m}^q \quad , \\ \tilde{\mathbb{H}}_{\sigma,m}^{1,\omega} &:= r^{-\frac{N}{2}} \left( -\omega_\sigma^q H_{\nu_\sigma}^1(\omega r) \check{\tau} S_{\sigma,m}^{q+1} \right. \\ &\quad \left. + i \left( (N/2 - (q+1)') H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \check{\rho} T_{\sigma,m}^q \right) \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} \tilde{\mathbb{E}}_{\sigma,m}^{2,\omega} &:= r^{-\frac{N}{2}} \left( \omega_\sigma^{q-1} H_{\nu_\sigma}^1(\omega r) \check{\rho} T_{\sigma,m}^{q-1} \right. \\ &\quad \left. + i \left( (N/2 - q) H_{\nu_\sigma}^1(\omega r) + \omega r (H_{\nu_\sigma}^1)'(\omega r) \right) \check{\tau} S_{\sigma,m}^q \right) \quad , \\ \tilde{\mathbb{H}}_{\sigma,m}^{2,\omega} &:= \omega r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \check{\rho} S_{\sigma,m}^q \end{aligned} \quad (5.13)$$

the forms

$$c_n (\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in C^{\infty, q, q+1}(\mathbb{R}^N \setminus \{0\}) \quad , \quad n = 1, 2$$

are solutions of (5.1) for all constants  $c_1, c_2 \in \mathbb{C}$ . For  $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$  these forms decay exponentially and for  $\omega \in \mathbb{R} \setminus \{0\}$  they satisfy the radiation condition for Helmholtz' equation. In particular we have

$$\exp(-i\omega r) (\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in H_{>-\frac{3}{2}}^{1, q, q+1}(A(1))$$

and thus

$$(M - i\omega r^{-1}S)(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in L_{>-\frac{1}{2}}^{2,q,q+1}(A(1))$$

$$\stackrel{(5.1)}{=} (-i\omega - i\omega r^{-1}S)(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \quad ,$$

i.e.

$$(r^{-1}S + \text{Id})(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in L_{>-\frac{1}{2}}^{2,q,q+1}(A(1)) \quad .$$

Hence these forms satisfy Maxwell's radiation condition as well. Furthermore we derive by the properties of Hankel's function ( $N$  is odd!) uniformly with respect to  $z \in \mathbb{C}_+$  the estimate

$$|H_\nu^1(z)| \leq c(|z|^{-\frac{1}{2}} + |z|^{-\nu}) \quad , \quad \nu := \nu_{-1} = N/2 - 1 \quad .$$

For this see for example [17, p. 72] or more detailed [24, Sektion 5.1] and compare with [25, (4.1)]. Because  $\nu \geq 1/2$  we obtain for  $\omega \in \mathbb{C}_+$  and uniformly with respect to  $r \in (1, \infty)$

$$|H_\nu^1(\omega r)| \leq cr^{-\frac{1}{2}} \quad , \text{ i.e. } \quad |\tilde{\mathbb{E}}_{\sigma,m}^{1,\omega}|_q, |\tilde{\mathbb{H}}_{\sigma,m}^{2,\omega}|_{q+1} \leq cr^{\frac{1-N}{2}} \quad .$$

Since the first derivative of Hankel's function shows the same behaviour at infinity as Hankel's function itself we obtain

$$|(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega})|_{q,q+1} \leq cr^{\frac{1-N}{2}} \quad , \quad n = 1, 2$$

and hence

$$(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in L_{<-\frac{1}{2}}^{2,q,q+1}(A(1)) \quad .$$

By regularity, for example [14, Corollary 3.8 (i)], we finally get for all  $k \in \mathbb{N}$

$$(\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in \mathbf{H}_{<-\frac{1}{2}}^{k,q,q+1}(A(1)) \quad . \quad (5.14)$$

Our next aim is to expand both solutions into power series with respect to  $\omega$ . Using for example [17, p. 66] we have

$$H_\nu^1(z) = \frac{-i}{\sin(\pi\nu)} (J_{-\nu}(z) - e^{-i\pi\nu} J_\nu(z))$$

with Bessel's function

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+1+\nu)} (z/2)^{2k+\nu} \quad .$$

This yields with the coefficients from [26, Remark 2.2 (v)]

$$H_{\nu_\sigma}^1(z) = i(-1)^{\nu_\sigma+1/2} \frac{2^{\nu_\sigma}}{\Gamma(1-\nu_\sigma)} \left( \sum_{k=0}^{\infty} (-1)^k - \alpha_\sigma^{q,k} z^{2k-\nu_\sigma} \right) \\ + i(-1)^{\nu_\sigma-1/2} 4^{-\nu_\sigma} \frac{\Gamma(1-\nu_\sigma)}{\Gamma(1+\nu_\sigma)} (2\sigma+N) (-1)^{1+\delta_{q,0}+\delta_{q,N}} \sum_{k=0}^{\infty} (-1)^k + \alpha_\sigma^{q,k} z^{2k+\nu_\sigma} \quad .$$

We obtain the representation

$$r^{1-\frac{N}{2}} H_{\nu_\sigma}^1(\omega r) \\ = \beta_\sigma \omega^{-\nu_\sigma} \left( \sum_{k=0}^{\infty} (-i\omega)^{2k} - \alpha_\sigma^{q,k} r^{2k+1-N-\sigma} + \kappa_\sigma^q \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} + \alpha_\sigma^{q,k} r^{2k+1+\sigma} \right)$$

with constants

$$\beta_\sigma := i \frac{2^{\nu_\sigma}}{\Gamma(1-\nu_\sigma)} (-1)^{\nu_\sigma+1/2} \quad , \quad \kappa_\sigma^q := i 2\nu_\sigma 4^{-\nu_\sigma} \frac{\Gamma(1-\nu_\sigma)}{\Gamma(1+\nu_\sigma)} (-1)^{\nu_\sigma+1/2+\delta_{q,0}+\delta_{q,N}} \quad .$$

Looking at [26, Definition 2.1] we finally get the series representations

$$\tilde{\mathbb{E}}_{\sigma,m}^{1,\omega} = \beta_\sigma \omega^{1-\nu_\sigma} \left( \sum_{k=0}^{\infty} (-i\omega)^{2k} - \alpha_\sigma^{q+1,k} r^{2k+1-N-\sigma} \check{\tau} T_{\sigma,m}^q \right. \\ \left. + \kappa_\sigma^{q+1} \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} + \alpha_\sigma^{q+1,k} r^{2k+1+\sigma} \check{\tau} T_{\sigma,m}^q \right) \\ = \beta_\sigma \omega^{1-\nu_\sigma} \left( \sum_{k=0}^{\infty} (-i\omega)^{2k} - D_{\sigma,m}^{q,2k+1} + \kappa_\sigma^{q+1} \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} + D_{\sigma,m}^{q,2k+1} \right) \quad , \\ \tilde{\mathbb{H}}_{\sigma,m}^{2,\omega} = \beta_\sigma \omega^{1-\nu_\sigma} \left( \sum_{k=0}^{\infty} (-i\omega)^{2k} - \alpha_\sigma^{q,k} r^{2k+1-N-\sigma} \check{\rho} S_{\sigma,m}^q \right. \\ \left. + \kappa_\sigma^q \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} + \alpha_\sigma^{q,k} r^{2k+1+\sigma} \check{\rho} S_{\sigma,m}^q \right) \\ = \beta_\sigma \omega^{1-\nu_\sigma} \left( \sum_{k=0}^{\infty} (-i\omega)^{2k} - R_{\sigma,m}^{q+1,2k+1} + \kappa_\sigma^q \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-i\omega)^{2k} + R_{\sigma,m}^{q+1,2k+1} \right) \quad .$$

Thus defining for  $q \in \{0, \dots, N - 1\}$  the series

$$\mathbb{E}_{\sigma,m}^{1,\omega} := \sum_{k=0}^{\infty} (-i\omega)^{2k} D_{\sigma,m}^{q,2k+1} + \kappa_{\sigma}^{q+1} \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k+1} D_{\sigma,m}^{q,2k+1} ,$$

$$\mathbb{H}_{\sigma,m}^{1,\omega} := \frac{i}{\omega} \operatorname{rot} \mathbb{E}_{\sigma,m}^{1,\omega} = \sum_{k=0}^{\infty} (-i\omega)^{2k-1} R_{\sigma,m}^{q+1,2k} + \kappa_{\sigma}^{q+1} \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k-1+} R_{\sigma,m}^{q+1,2k}$$

and

$$\mathbb{H}_{\sigma,m}^{2,\omega} := \sum_{k=0}^{\infty} (-i\omega)^{2k} R_{\sigma,m}^{q+1,2k+1} + \kappa_{\sigma}^q \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k+1} R_{\sigma,m}^{q+1,2k+1} ,$$

$$\mathbb{E}_{\sigma,m}^{2,\omega} := \frac{i}{\omega} \operatorname{div} \mathbb{H}_{\sigma,m}^{2,\omega} = \sum_{k=0}^{\infty} (-i\omega)^{2k-1} D_{\sigma,m}^{q,2k} + \kappa_{\sigma}^q \omega^{2\nu_{\sigma}} \sum_{k=0}^{\infty} (-i\omega)^{2k-1+} D_{\sigma,m}^{q,2k}$$

we can summarize the discussion about the solutions of (5.1) in the following way:

**Remark 5.1** *Let  $q \in \{0, \dots, N - 1\}$ ,  $\sigma \in \mathbb{N}_0$ ,  $m = 1, \dots$  and  $\omega \in \mathbb{C}_+ \setminus \{0\}$  as well as  $\nu_{\sigma} := N/2 + \sigma$ . The series  $(\mathbb{E}_{\sigma,m}^{n,\omega}, \mathbb{H}_{\sigma,m}^{n,\omega})$ ,  $n = 1, 2$ , converge by the convergence properties of Hankel's function uniformly on compact subsets of  $\mathbb{R}^N \setminus \{0\}$  and there they define  $C^{\infty}$ -forms. Moreover they are solutions of (5.1), which satisfy for  $\omega \in \mathbb{R} \setminus \{0\}$  both Maxwell's radiation condition and Sommerfeld's radiation condition for Helmholtz' equation and decay exponentially for  $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$ . Particularly we have  $\operatorname{div} \mathbb{E}_{\sigma,m}^{n,\omega} = 0$  and  $\operatorname{rot} \mathbb{H}_{\sigma,m}^{n,\omega} = 0$  as well as with (5.12), (5.13) and (5.14) for all  $k \in \mathbb{N}$*

$$(\mathbb{E}_{\sigma,m}^{n,\omega}, \mathbb{H}_{\sigma,m}^{n,\omega}) = \frac{\omega^{\nu_{\sigma}-1}}{\beta_{\sigma}} (\tilde{\mathbb{E}}_{\sigma,m}^{n,\omega}, \tilde{\mathbb{H}}_{\sigma,m}^{n,\omega}) \in \mathbf{H}_{<-\frac{1}{2}}^{k,q,q+1}(A(1))$$

holds.

## Acknowledgement

The author is grateful to Norbert Weck and Karl-Josef Witsch for teaching.

## References

- [1] Ammari, H., Laouadi, M., Nédélec, J.-C., 'Low frequency behavior of solutions to electromagnetic scattering problems in chiral media', *SIAM J. Appl. Math.*, 58 (3), (1998), 1022-1042.
- [2] Ammari, H., Nédélec, J.-C., 'Propagation d'ondes électromagnétiques à basses fréquences', *J. Math. Pures Appl.*, 77 (4), (1998), 839-849.

- [3] Ammari, H., Nédélec, J.-C., 'Full Low-Frequency Asymptotics for the Reduced Wave Equation', *Appl. Math. Letters*, 12, (1999), 127-131.
- [4] Ammari, H., Nédélec, J.-C., 'Low-frequency electromagnetic scattering', *SIAM J. Math. Anal.*, 31 (4), (2000), 836-861.
- [5] Athanasiadis, C., Costakis, G., Stratis, I. G., 'Electromagnetic scattering by a perfectly conducting obstacle in a homogeneous chiral environment: solvability and low-frequency theory', *Math. Meth. Appl. Sci.*, 25, (2002), 927-944.
- [6] Colton, D., Kress, R., *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, Berlin, (1992).
- [7] Dassios, G., Kleinman, R., *Low Frequency Scattering*, Clarendon Press, Oxford, (2000).
- [8] Eidus, D. M., 'The Principle of Limiting Absorption', *Math. Sb.*, 57 (99) (1962), 13-44, and *AMS Transl.*, 47 (2), (1965), 157-191.
- [9] Eidus, D. M., 'The Principle of Limiting Amplitude', *Russ. Math. Surv.*, 24 (3), (1969), 97-167.
- [10] Ihlenburg, F., *Finite element analysis of acoustic scattering*, Springer, New York, (1998).
- [11] Kleinman, R. E., *Low frequency electromagnetic scattering*, in *Electromagnetic Scattering*, P. L. E. Uslenghi, ed., Academic Press, New York, (1978), 1-28.
- [12] Kress, R., 'On the limiting behavior of solutions to boundary integral equations associated with time-harmonic wave equations for small frequencies', *Math. Meth. Appl. Sci.*, 1, (1979), 89-100.
- [13] Kriegsmann, G. A., Reiss, E. L., 'Low frequency scattering by local inhomogeneities', *SIAM. J. Appl. Math.*, 43, (1983), 923-934.
- [14] Kuhn, P., Pauly, D., 'Generalized Maxwell Equations in Exterior Domains I: Regularity Results, Trace Theorems and Static Solution Theory', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 7/2007, ISBN 978-951-39-2832-2, ISSN 1456-436X.
- [15] Lassas, M., 'The impedance imaging as a low frequency limit', *Inverse Problems*, 13 (6), (1997), 1503-1518.
- [16] Leis, R., *Initial Boundary Value Problems in Mathematical Physics*, Teubner, Stuttgart, (1986).
- [17] Magnus, W., Oberhettinger, F., Soni, R. P., *Formulas and theorems for the special functions of mathematical physics*, Springer, Berlin - Heidelberg - New York, (1966).

- [18] McOwen, R. C., 'Behavior of the Laplacian in weighted Sobolev spaces', *Comm. Pure Appl. Math.*, 32, (1979), 783-795.
- [19] Monk, P., *Finite Element Methods for Maxwell's Equations*, Oxford University Press, New York, (2003).
- [20] Morgenröther, K., Werner, P., 'On the Principles of Limiting Absorption and Limit Amplitude for a Class of Locally Perturbed Waveguides. Part 1: Time-independent Theory', *Math. Meth. Appl. Sci.*, 10 (2), (1988), 125-144.
- [21] Morgenröther, K., Werner, P., 'On the Principles of Limiting Absorption and Limit Amplitude for a Class of Locally Perturbed Waveguides. Part 2: Time-dependent Theory', *Math. Meth. Appl. Sci.*, 11 (1), (1989), 1-26.
- [22] Müller, C., *Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen*, Springer, Berlin - Heidelberg - New York, (1957).
- [23] Müller, C., Niemyer, H., 'Greensche Tensoren und asymptotische Gesetze der elektromagnetischen Hohlraumschwingungen', *Arch. Rat. Mech. Anal.*, 7, (1961), 305-348.
- [24] Pauly, D., 'Niederfrequenzasymptotik der Maxwell-Gleichung im inhomogenen und anisotropen Außengebiet', <http://duepublico.uni-duisburg-essen.de>, **Dissertation, Duisburg-Essen**, (2003).
- [25] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains II: Radiation Problems and Low Frequency Behavior', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 8/2007, ISBN 978-951-39-2833-9, ISSN 1456-436X.
- [26] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains III: Electromagneto Statics', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 9/2007, ISBN 978-951-39-2834-6, ISSN 1456-436X.
- [27] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains IV: Hodge-Helmholtz Decompositions', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 10/2007, ISBN 978-951-39-2835-3, ISSN 1456-436X.
- [28] Pepperl, T., 'Niederfrequenzasymptotik und Wirbelstrom-Approximation der verallgemeinerten dissipativen Maxwell-Gleichung', <http://duepublico.uni-duisburg-essen.de>, **Dissertation, Duisburg-Essen**, (2006).
- [29] Peter, B., 'Die Lösungen der Helmholtzschen Schwingungsgleichung in Außengebieten und die Asymptotik ihrer Frequenzableitungen bei hohen und niedrigen Frequenzen', *Shaker Verlag*, **Dissertation, Essen**, (1998).

- [30] Picard, R., 'On the low frequency asymptotics in electromagnetic theory', *J. Reine Angew. Math.*, 354, (1984), 50-73.
- [31] Picard, R., 'The low frequency limit for the time-harmonic acoustic waves', *Math. Meth. Appl. Sci.*, 8, (1986), 436-450.
- [32] Picard, R., Weck, N., Witsch, K. J., 'Time-Harmonic Maxwell Equations in the Exterior of Perfectly Conducting, Irregular Obstacles', *Analysis*, 21, (2001), 231-263.
- [33] Ramm, A. G., 'Behavior of the solutions to the exterior boundary value problems at low frequencies', *J. Math. Anal. Appl.*, 117, (1986), 561-569.
- [34] Ramm, A. G., Somersalo E., 'Electromagnetic inverse problems with surface measurements at low frequencies', *Inverse Problems*, 5, (1989), 1107-1116.
- [35] Ramm, A. G., Weaver O. L., Weck, N., Witsch, K. J., 'Dissipative Maxwell's Equations at Low Frequencies', *Math. Meth. Appl. Sci.*, 13, (1990), 305-322.
- [36] Lord Rayleigh, 'On the electromagnetic theory of light', *Phil. Mag.*, 12, (1881), 81-101.
- [37] Stevenson, A., 'Solution of electromagnetic scattering problems as power series in the ratio (dimension of scatter/wave length)', *J. Appl. Phys.*, 24, (1953), 1134-1142.
- [38] Weck, N., 'Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries', *J. Math. Anal. Appl.*, 46, (1974), 410-437.
- [39] Weck, N., 'Traces of Differential Forms on Lipschitz Boundaries', *Analysis*, 24, (2004), 147-169.
- [40] Weck, N., Witsch, K. J., 'Low frequency asymptotics for dissipative Maxwell's equations in bounded domains', *Math. Meth. Appl. Sci.*, 13, (1990), 81-93.
- [41] Weck, N., Witsch, K. J., 'The low frequency limit of the exterior Dirichlet problem for the reduced wave equation', *Appl. Anal.*, 38, (1990), 33-43.
- [42] Weck, N., Witsch, K. J., 'Exterior Dirichlet problem for the reduced wave equation: asymptotic analysis of low frequencies', *Comm. PDE*, 16, (1991), 173-195.
- [43] Weck, N., Witsch, K. J., 'Exact Low Frequency Analysis for a Class of Exterior Boundary Value Problems for the Reduced Wave Equation in Two Dimensions', *J. Diff. Equ.*, 100, (1992), 312-340.
- [44] Weck, N., Witsch, K. J., 'Exact Low Frequency Analysis for a Class of Exterior Boundary Value Problems for the Reduced Wave Equation in Higher Dimensions', *Asymptotic Analysis*, 6, (1992), 161-172.



- [45] Weck, N., Witsch, K. J., 'Complete Low Frequency Analysis for the Reduced Wave Equation with Variable Coefficients in Three Dimensions', *Comm. PDE*, 17, (1992), 1619-1663.
- [46] Weck, N., Witsch, K. J., 'Generalized Spherical Harmonics and Exterior Differentiation in Weighted Sobolev Spaces', *Math. Meth. Appl. Sci.*, 17, (1994), 1017-1043.
- [47] Weck, N., Witsch, K. J., 'Generalized Linear Elasticity in Exterior Domains I: Radiation Problems', *Math. Meth. Appl. Sci.*, 20, (1997), 1469-1500.
- [48] Weck, N., Witsch, K. J., 'Generalized Linear Elasticity in Exterior Domains II: Low-Frequency Asymptotics', *Math. Meth. Appl. Sci.*, 20, (1997), 1501-1530.
- [49] Werner, P., 'Randwertprobleme für die zeitabhängigen Maxwellschen Gleichungen mit variablen Koeffizienten', *Arch. Rational Mech. Anal.*, 18, (1965), 167-195.
- [50] Werner, P., 'On the behaviour of stationary electromagnetic wave fields for small frequencies', *J. Math. Anal. Appl.*, 15, (1966), 447-496.
- [51] Werner, P., 'Über das Verhalten elektromagnetischer Felder für kleine Frequenzen in mehrfach zusammenhängenden Gebieten I', *J. Reine Angew. Math.*, 278, (1975), 365-397.
- [52] Werner, P., 'Über das Verhalten elektromagnetischer Felder für kleine Frequenzen in mehrfach zusammenhängenden Gebieten II', *J. Reine Angew. Math.*, 280, (1976), 98-125.
- [53] Werner, P., 'Zur Asymptotik der Wellengleichung und der Wärmeleitungsgleichung in zweidimensionalen Außenräumen', *Math. Meth. Appl. Sci.*, 7 (2), (1985), 170-201.
- [54] Werner, P., 'Low Frequency Asymptotics for the Reduced Wave Equation in Two-dimensional Exterior Spaces', *Math. Meth. Appl. Sci.*, 8 (1), (1986), 134-156.
- [55] Weyl, H., 'Die natürlichen Randwertaufgaben im Außenraum für Strahlungsfelder beliebiger Dimension und beliebigen Ranges', *Math. Z.*, 56, (1952), 105-119.