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Generalized Maxwell Equations in Exterior Domains II: Radiation Problems and Low Frequency Behavior

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Abstract

We discuss the radiation problem of total reflection for a time-harmonic generalized Maxwell system in an exterior domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, with nonsmooth inhomogeneous, anisotropic coefficients converging near infinity with a rate $r^{-\tau}$, $\tau > 1$, towards the identity. By means of the limiting absorption principle we prove for real frequencies that a Fredholm alternative holds true, that eigensolutions decay polynomially resp. exponentially at infinity and that the corresponding eigenvalues do not accumulate even at zero. Then we show the convergence of the time-harmonic solutions to a solution of an electro-magneto static Maxwell system as the frequency tends to zero. Finally we are able to generalize these results to the corresponding Maxwell system with inhomogeneous boundary data.

Key words: Maxwell's equations, exterior boundary value problems, radiating solutions, polynomial and exponential decay of eigensolutions, variable coefficients, electro-magneto static, low frequency scattering, low frequency asymptotics, inhomogeneous boundary data, electro-magnetic theory

AMS MSC-classifications: 35Q60, 78A25, 78A30

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1 Introduction

If we choose a time-harmonic ansatz (resp. Fourier transform with respect to time) for the classical time dependent Maxwell system in $\mathbb{R}^3$

$$-\operatorname{curl}\mathbf{H} + \partial_t\,\mathbf{D} = \mathbf{I} \quad , \qquad \operatorname{curl}\mathbf{E} + \partial_t\,\mathbf{B} = \mathbf{0} \quad ,$$
$$\operatorname{div}\mathbf{D} = \boldsymbol{\rho} \quad , \qquad \operatorname{div}\mathbf{B} = \mathbf{0} \quad ,$$

we are led to consider the time-harmonic Maxwell system with non zero complex frequency ω and complex valued data ε , μ , I and ρ

$$-\operatorname{curl} H + \mathrm{i}\,\omega\varepsilon E = I \quad , \qquad \operatorname{curl} E + \mathrm{i}\,\omega\mu H = 0 \quad , \tag{1.1}$$
$$\operatorname{div}\varepsilon E = \rho \quad , \qquad \operatorname{div}\mu H = 0 \quad . \tag{1.2}$$

Here we denote the electric resp. magnetic field by E resp. H , the displacement current resp. magnetic induction by $D = \varepsilon E$ resp. $B = \mu H$ and the current resp. charge density by I resp. ρ . The matrix valued functions ε and μ are assumed to be time independent and describe material properties, i.e. the dielectricity and permeability of the medium. $\operatorname{curl} = \nabla\times$ (rotation) and $\operatorname{div} = \nabla\,\cdot$ (divergence) mark the usual differential operators from classical vector analysis. By differentiation we get

$$\operatorname{div}\varepsilon E = -\frac{\mathrm{i}}{\omega}\operatorname{div} I \quad , \qquad \operatorname{div}\mu H = 0$$

from (1.1), such that we can neglect (for $\omega \neq 0$) the equations (1.2). To formulate these equations as a boundary value problem in a domain $\Omega \subset \mathbb{R}^3$ we need a boundary condition at $\partial\,\Omega$. Modeling total reflection of the electric field at the boundary, i.e. $\mathbb{R}^N \setminus \Omega$ is a perfect conductor, we impose the homogeneous boundary condition (assuming sufficient smoothness of the boundary for the purpose of these introductory remarks)

$$\nu \times E = 0 \qquad \text{on} \qquad \partial\,\Omega \qquad , \tag{1.3}$$

which means that E possesses vanishing tangential components at $\partial\,\Omega$. Here ν denotes the outward unit normal on $\partial\,\Omega$ and $\times$ the vector product in $\mathbb{R}^3$. We are interested in the case of an exterior domain Ω, i.e. a connected open set with compact complement. Therefore, we have to impose an additional condition like

$$\xi \times H + E\,,\, \xi \times E - H \in \mathrm{L}^2(\Omega) \tag{1.4}$$

$\big(\xi(x) := x/|x|\big)$ the classical so called outgoing Silver-Müller radiation condition, which allows to separate outgoing from incoming waves. Interchanging $+$ and $-$ in (1.4) would yield incoming waves. We call the problem of finding E and H with (1.1), (1.3) and (1.4) the radiation problem of total reflection for the time-harmonic Maxwell system.

In 1952 Hermann Weyl [28] suggested a generalization of the system (1.1) and (1.3) on Riemannian manifolds Ω of arbitrary dimension N with the aid of alternating differential forms. If E is a form of rank q (q-form) and H a $(q+1)$-form and if we denote the exterior differential d resp. the codifferential δ $\big($acting on q- resp. $(q+1)$-forms$\big)$ by

$$\mathrm{rot} := \mathrm{d} \qquad \text{resp.} \qquad \mathrm{div} := \delta = (-1)^{qN} * \mathrm{d} *$$

($*$: Hodge star operator), the generalization of our system (1.1) and (1.3) reads

$$\mathrm{div}\, H + \mathrm{i}\,\omega\varepsilon E = F \qquad , \qquad \mathrm{rot}\, E + \mathrm{i}\,\omega\mu H = G \qquad , \tag{1.5}$$

$$\iota^* E = 0 \tag{1.6}$$

and we call it the generalized time-harmonic Maxwell system of total reflection. Now F (former I) is a q-form, G (former 0) a $(q+1)$-form, ε resp. μ a linear transformation on q- resp. $(q+1)$-forms, $\iota : \partial\,\Omega \hookrightarrow \overline{\Omega}$ the natural embedding and ι^* the pull-back of ι. In the case $N = 3$ and $q = 1$, i.e. E is a 1-form and H a 2-form, the generalized Maxwell system is equivalent to the classical Maxwell system of a perfect conductor, since the operators rot and div acting on q-forms are nothing else than the classical differential operators curl and div if $q = 1$ resp. div and $-\mathrm{curl}$ if $q = 2$. Moreover, for $N = 3$ and 1- resp. 2-forms E we observe that the boundary condition (1.6) means in the classical language $\nu \times E = 0$ resp. $\nu \cdot E = 0$ on the boundary, i.e. vanishing tangential resp. normal components of the considered fields. We remark that another classical case is discussed by this generalization. If $N = 3$ and $q = 0$ resp. $q = 2$, i.e. E resp. H are scalar valued, we get the equations of linear acoustics with homogeneous Dirichlet- resp. Neumann boundary condition, because rot resp. div turns out to be the classical gradient ∇ on 0- resp. 3-forms. Moreover, rot resp. div is the zero-mapping on 3- resp. 0-forms. In the case of an exterior domain $\Omega \subset \mathbb{R}^N$, which we want to treat in this paper, we give a generalization of the radiation condition (1.4) later. For a short notation we introduce the formal matrix operators

$$M := \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{rot} & 0 \end{bmatrix} \quad , \quad \Lambda := \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix} \tag{1.7}$$

and write our problem (1.5), (1.6) shortly as

$$(M + \mathrm{i}\,\omega\Lambda)(E,H) = (F,G) \quad , \quad \iota^* E = 0 \quad . \tag{1.8}$$

$\big($For typographical reasons we write form-pairs as (E,H), although the matrix calculus would expect the notation $\begin{bmatrix} E \\ H \end{bmatrix}$.$\big)$

Time-harmonic exterior boundary value problems concerning the classical vector-valued Maxwell equations, i.e. $N = 3$ and $q = 1$, have been studied by Müller [12] in domains with smooth boundaries and homogeneous, isotropic media, i.e. $\varepsilon = \mu = \mathrm{Id}$, with integral equation methods and by Leis [7] $\big($see also [9]$\big)$ with the aid of the limiting absorption principle for media, which are inhomogeneous and anisotropic within a bounded subset of Ω. The generalized time-harmonic Maxwell system has been treated by Weck [25] and Picard [18].

In this paper we want to discuss the time-harmonic radiation boundary value problem of total reflection for the generalized Maxwell equation (1.8) in an exterior domain $\Omega \subset \mathbb{R}^N$ for arbitrary dimensions N and ranks q. A main goal of our investigations is to treat data (F,G) in weighted $\mathrm{L}^2(\Omega)$-spaces and inhomogeneous, anisotropic and irregular $\big(\mathrm{L}^\infty(\Omega)$-$\big)$ coefficients ε, μ only converging near infinity with a rate $r^{-\tau}$, $\tau > 0$, towards the identity. $\big(r(x) := |x|$ the 'radius'$\big)$ We follow in close lines the papers of Weck and Witsch [27] and Picard, Weck and Witsch $\big[$[22], part 1$\big]$, which deal with the system of generalized linear elasticity and the classical Maxwell equations. In particular we generalize the results obtained in the second paper to arbitrary dimensions N and ranks of forms q. To present a time-harmonic solution theory we prove that for nonzero frequencies ω and data $(F,G) \in \mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$[1] and L^∞-coefficients ε, μ a Fredholm alternative holds true. The main tool to handle irregular coefficients is a decomposition lemma, Lemma 2.6, which allows us to prove the polynomial decay of eigensolutions as well as an a-priori estimate by reduction to the similar results known for the scalar Helmholtz equation. The key to this decomposition lemma are weighted Hodge-Helmholtz decompositions, i.e. decompositions in irrotational and solenoidal fields, in the whole space case.

The idea of the decomposition lemma is to use a well known procedure to decouple the electric and magnetic field by discussing a second order elliptic system. To illustrate this calculation let us look at (1.8) in the homogeneous case $\Lambda = \mathrm{Id}$. Applying $M - \mathrm{i}\,\omega$ yields

$$(M^2 + \omega^2)(E,H) = (M - \mathrm{i}\,\omega)(F,G) \quad . \tag{1.9}$$

[1] Throughout this paper we will use the notations from [6] and [14].

If we choose F solenoidal, i.e. $\operatorname{div} F = 0$, and G irrotational, i.e. $\operatorname{rot} G = 0$, these properties will be transfered to E, i.e. $\operatorname{div} E = 0$, and H, i.e. $\operatorname{rot} H = 0$, by (1.8) because of

$$\operatorname{div}\operatorname{div} = 0 \qquad \text{and} \qquad \operatorname{rot}\operatorname{rot} = 0 \qquad .$$

From $\Delta = \operatorname{rot}\operatorname{div} + \operatorname{div}\operatorname{rot}$ (The Laplacian acts on each component.) we get the identity $M^2(E,H) = (\operatorname{div}\operatorname{rot} E, \operatorname{rot}\operatorname{div} H) = \Delta(E,H)$ and finally (1.9) turns to the (componentwise) Helmholtz equation

$$(\Delta + \omega^2)(E,H) = (M - \mathrm{i}\,\omega)(F,G) \qquad . \tag{1.10}$$

Armed with the polynomial decay of eigensolutions and an a-priori estimate for the solutions corresponding to non real frequencies (We get these solutions from the existence of a selfadjoint realization of M.) we obtain our radiating solutions to frequencies $\omega \in \mathbb{R} \setminus \{0\}$ with the method of limiting absorption invented by Eidus [2] as limits of solutions to frequencies $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$[2]. We have to admit finite dimensional eigenspaces for certain eigenvalues and show that these eigenvalues do not accumulate in $\mathbb{R} \setminus \{0\}$. Proving an estimate for the solutions of the homogeneous, isotropic whole space problem with the aid of a representation formula and studying some special convolution kernels (Hankel functions) we even can exclude 0 as an accumulation point of eigenvalues. Thus the time-harmonic solution operator $\mathcal{L}_\omega$ is well defined on $\mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$ for small frequencies $\omega \neq 0$. To reach this aim we have to increase the order of decay τ of the coefficients ε, μ and assume that they are C^1 in the outside of an arbitrary compact set. With stronger differentiability assumptions on ε and μ, i.e. C^2 in the outside of a ball, we are able to show the exponential decay of eigensolutions as well. To the best of our knowledge it is an open question whether there exist such eigenvalues in this general case. Recently under comparable stronger assumptions on the coefficients Bauer [1] was able prove that no eigenvalues occur in the classical case of Maxwell's equations ($N = 3$, $q = 1$). Unfortunately his methods are not applicable in our general case. It seems to be the same problem that rises up trying to prove the principle of unique continuation for the generalized Maxwell equation. In the classical case the principle of unique continuation was shown by Leis [8] or $\big[$[9], p. 168, Theorem 8.17$\big]$. However, in the case of homogeneous, isotropic coefficients, i.e. $\varepsilon = \mathrm{Id}$, $\mu = \mathrm{Id}$, in the outside of a ball all components of a possible eigensolution solve the homogeneous Helmholtz equation $\big($compare (1.10)$\big)$ near infinity and therefore by Rellich's estimate [23] must have compact support. With the validity of the principle of unique continuation for our Maxwell system this eigensolution must vanish. In the general case the principle of unique continuation is given for scalar valued C^2-functions ε, μ and in the classical case for matrices ε, μ with entries in C^2. (See the citation above from Leis.)

Having established the time-harmonic solution theory in section 2 we approach the low frequency asymptotics of our time-harmonic solution operator. To this end first we have to provide a static solution theory. This one is more complicated than

[2]The Definitions will be supplied in section 2.1.

for example the static solution theory for Helmholtz' equation. The first reason is that for $\omega = 0$ the system (1.5) resp. (1.8), i.e.

$$\operatorname{rot} E = G \quad , \quad \operatorname{div} H = F \quad ,$$

is no longer coupled and that we have to add two more equations to determine E and H, i.e.

$$\operatorname{div} \varepsilon E = f \quad , \quad \operatorname{rot} \mu H = g \quad , \tag{1.11}$$

which in the case $\omega \neq 0$ automatically follow by differentiation from (1.5) as mentioned before. $\big(f = -\frac{\mathrm{i}}{\omega} \operatorname{div} F$ and $g = -\frac{\mathrm{i}}{\omega} \operatorname{rot} G$, if $\operatorname{div} F$ and $\operatorname{rot} G$ exist.$\big)$ Furthermore, we need a boundary condition for the magnetic field (form). Because the exterior derivative $\operatorname{rot} = \mathrm{d}$ and the pull-back ι^* commute we derive $\mathrm{i}\,\omega\iota^*\mu H = \iota^* G$ for $\omega \neq 0$ from (1.5). This suggests to impose a condition on the term $\iota^*\mu H$, and for example we can choose the homogeneous boundary condition

$$\iota^*\mu H = 0$$

for our magnetic field. The second reason is that this static Maxwell boundary value problem

$$\begin{aligned} \operatorname{rot} E &= G \quad , & \operatorname{div} H &= F \quad , \\ \operatorname{div} \varepsilon E &= f \quad , & \operatorname{rot} \mu H &= g \quad , \\ \iota^* E &= 0 \quad , & \iota^*\mu H &= 0 \end{aligned} \tag{1.12}$$

has got a nontrivial kernel ${}_\varepsilon\mathcal{H}^q(\Omega) \times \mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}(\Omega)$ consisting of harmonic Dirichlet forms. So we are forced to work with orthogonality constraints on the static solutions to achieve uniqueness. For the static system (1.12) a solution theory was given by Kress [5] and Picard [17] for the homogeneous, isotropic case, i.e. $\varepsilon = \mathrm{Id}$, $\mu = \mathrm{Id}$, by Picard [21] for the inhomogeneous, anisotropic case (Here ε and μ even are allowed to be nonlinear transformations.) as well as by Picard [19] for the inhomogeneous, anisotropic classical case. For our purpose we need a result like that given by Picard in [17]. In [6] and [14] we already discussed the electro-magneto static problem with inhomogeneous, anisotropic coefficients ε, μ, and the results obtained there will meet our needs. We shortly present these results and introduce our static solution concept in section 3.

Then in section 4, the main section of this paper, we prove the convergence of the time-harmonic solutions to a special static solution of (1.12). This result generalizes the paper of Picard [20], which considers the classical Maxwell equations, to arbitrary odd dimensions N and ranks $q \neq 0$ as well as to coefficients and right hand side data, which necessarily do not have to be compactly supported. We note that similar results hold true for even dimensions. Since the complexity of the calculations increases considerably due to the appearance of logarithmic terms in the fundamental solution of the Helmholtz operator $\Delta + \omega^2$ (Hankel's function), we restrict our considerations for simplicity to odd dimensions.

The last section 5 deals with inhomogeneous boundary conditions. Utilizing the trace and extension operators from [6] we discuss the time-harmonic problem

$$(M + \mathrm{i}\,\omega\Lambda)(E,H) = (F,G) \qquad , \qquad \iota^* E = \lambda$$

and the static problem

$$M(E,H) = (F,G) \qquad , \qquad (\iota^* E, \iota^* \mu H) = (\lambda, \varkappa) \qquad ,$$
$$(\operatorname{div} \varepsilon E, \operatorname{rot} \mu H) = (f,g) \qquad .$$

It turns out that the solution theories as well as the low frequency asymptotics for these problems are easy consequences of the results for homogeneous boundary conditions and the existence of an adequate extension operator for our traces.

Of course, by the Hodge star operator we always get easily the corresponding dual results, but we renounce them to shorten this paper.

Essentially, this is the first part of the authors ph.d. thesis. Thus sometimes we only sketch or neglect some proofs and do not mention all results obtained in [13]. To get more details on the proofs or some additional results we refer the interested reader to [13].

The report at hand is the second one of a series of five reports having the aim to determine the low frequency asymptotics of the time-harmonic Maxwell equations completely. In the third report [14] we will discuss the corresponding electro-magneto static equations in detail and demonstrate, how one may iterate a static solution operator in weighted Sobolev spaces. This will allow us to write down a generalized Neumann sum, which is a good candidate for the asymptotic series approaching the time-harmonic solution operator for small frequencies. The fourth report [15] deals with Hodge-Helmholtz decompositions in weighted Sobolev spaces, which are necessary, since the Maxwell operator possesses a non trivial kernel. In the fifth report [16] we finally present the complete low frequency asymptotics in the operator norm of weighted Sobolev spaces up to arbitrary orders in powers of the frequency.

2 Time-harmonic scattering theory

2.1 Formulation of the time-harmonic boundary value problem

Throughout this paper we use the notations from [6] and [14]. We consider an exterior domain, i.e. a domain with compact complement,

$$\Omega \subset \mathbb{R}^N \qquad , \qquad 3 \leq N \in \mathbb{N} \qquad ,$$

and fix some radius r_0 and radii $r_n := 2^n r_0$, $n \in \mathbb{N}$, such that

$$\mathbb{R}^N \setminus \Omega \Subset U_{r_0} \qquad .$$

Moreover, we use the cut-off functions $\boldsymbol{\eta}$, $\hat{\eta}$ and η introduced in $\big[$[6], (3.1), (3.2), (3.3)$\big]$. We will consider the following kinds of transformations:

Definition 2.1 *Let $\tau \geq 0$. We call a transformation ν τ-*admissible*, if*

- *$\nu(x)$ is a linear transformation on q-forms for all $x \in \Omega$,*
- *ν possesses $\mathrm{L}^\infty(\Omega)$-coefficients, i.e. the matrix representation of ν corresponding to the canonical basis $\big($and then for every chart basis $\{\mathrm{d}h^I\}\big)$ has $\mathrm{L}^\infty(\Omega)$-entries,*
- *ν is symmetric, i.e. for all $E, H \in \mathrm{L}^{2,q}(\Omega)$*

$$\langle \nu E, H \rangle_{\mathrm{L}^{2,q}(\Omega)} = \langle E, \nu H \rangle_{\mathrm{L}^{2,q}(\Omega)}$$

holds, and uniformly positive definite, i.e.

$$\exists \quad c > 0 \qquad \forall \quad E \in \mathrm{L}^{2,q}(\Omega) \qquad \langle \nu E, E \rangle_{\mathrm{L}^{2,q}(\Omega)} \geq c \cdot \|E\|^2_{\mathrm{L}^{2,q}(\Omega)} \qquad ,$$

- *ν is asymptotically the identity, i.e. $\nu = \nu_0 \operatorname{Id} + \hat{\nu}$ with $\nu_0 \in \mathbb{R}_+$ and $\hat{\nu} = \mathcal{O}(r^{-\tau})$ as $r \to \infty$. We call τ the* order of decay *of the perturbation $\hat{\nu}$.*

For some results obtained in this paper we need one more additional assumption on the perturbations $\hat{\nu}$ of our transformations. That is, $\hat{\nu}$ has to be differentiable in the outside of an arbitrarily large ball. More precisely:

Definition 2.2 *Let $\tau \geq 0$. We call a transformation ν τ-C^1-*admissible*, if*

- *ν is τ-admissible,*
- *$\hat{\nu} \in \mathrm{C}^1(A_{r_0})$, which means that the matrix representation of $\hat{\nu}$ corresponding to the canonical basis $\big($and then for every chart basis $\{\mathrm{d}h^I\}\big)$ has $\mathrm{C}^1(A_{r_0})$-entries, with the additional asymptotic*

$$\partial_n \hat{\nu} = \mathcal{O}(r^{-1-\tau}) \qquad \text{as} \qquad r \to \infty \qquad , \qquad n = 1, \ldots, N \qquad .$$

Now let ε and μ be two τ-admissible transformations on q- resp. $(q+1)$-forms with order of decay $\tau \geq 0$ and M, Λ as in (1.7).

As mentioned above we want to treat the (generalized) time-harmonic inhomogeneous, anisotropic Maxwell equation

$$(M + \mathrm{i}\,\omega\Lambda)(E, H) = (F, G) \qquad ,$$

with frequencies taken from the upper half plane

$$\omega \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \qquad .$$

A substitution like $\tilde{x} := \alpha x$, $\tilde{H} := \beta H$ allows us to suppose w.l.o.g.

$$\varepsilon_0 = \mu_0 = 1 \qquad \text{and thus} \qquad \Lambda = \operatorname{Id} + \hat{\Lambda} \qquad . \tag{2.1}$$

To shorten and simplify the formulas we always want to assume (2.1) through this paper.

Now let us introduce our time-harmonic solution concept. From the adjointness of the two operators

$$\begin{aligned} \mathrm{i}\,\mathrm{rot} : \overset{\circ}{\mathbf{R}}{}^q(\Omega) \subset \mathrm{L}^{2,q}(\Omega) &\longrightarrow \mathrm{L}^{2,q+1}(\Omega) \qquad , \\ \mathrm{i}\,\mathrm{div} : \mathbf{D}^{q+1}(\Omega) \subset \mathrm{L}^{2,q+1}(\Omega) &\longrightarrow \mathrm{L}^{2,q}(\Omega) \end{aligned}$$

to each other we obtain the selfadjointness of

$$\mathcal{M} : D(\mathcal{M}) := \overset{\circ}{\mathbf{R}}{}^q(\Omega) \times \mathbf{D}^{q+1}(\Omega) \subset {}_{\varepsilon}\mathrm{L}^{2,q}(\Omega) \times {}_{\mu}\mathrm{L}^{2,q+1}(\Omega) \longrightarrow {}_{\varepsilon}\mathrm{L}^{2,q}(\Omega) \times {}_{\mu}\mathrm{L}^{2,q+1}(\Omega)$$

with

$$\mathcal{M}(E,H) := \mathrm{i}\,\Lambda^{-1} M(E,H) = \mathrm{i}(\varepsilon^{-1}\,\mathrm{div}\,H, \mu^{-1}\,\mathrm{rot}\,E) \qquad .$$

Here ${}_{\nu}\mathrm{L}^{2,q}(\Omega) := \mathrm{L}^{2,q}(\Omega)$ equipped with the scalar product $\langle \nu\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)}$. This suggests

Definition 2.3 *Let $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $(F,G) \in \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{loc}}(\Omega)$. Then (E,H) solves the problem $\mathsf{Max}(\Lambda,\omega,F,G)$, if and only if*

(i) $\quad (E,H) \in \overset{\circ}{\mathbf{R}}{}^q(\Omega) \times \mathbf{D}^{q+1}(\Omega) \qquad ,$

(ii) $\quad (M + \mathrm{i}\,\omega\Lambda)(E,H) = (F,G) \qquad .$

The selfadjointness of $\mathcal{M}$ yields the unique solvability of $\mathsf{Max}(\Lambda,\omega,F,G)$ for non real frequencies $\omega \in \mathbb{C} \setminus \mathbb{R}$ and right hand sides $(F,G) \in \mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)$. We denote the continuous solution operator by

$$\mathcal{L}_\omega := \mathrm{i}(\mathcal{M} - \omega)^{-1}\Lambda^{-1} \qquad .$$

It can be seen easily that the spectrum of $\mathcal{M}$ is the whole real axis. Thus we expect from Helmholtz' equation that we have to work in weighted L^2-spaces and with radiating solutions to get a solution theory for real frequencies.

Reminding of the operators

$$R := r\,\mathrm{d}r \wedge \,\cdot = x_n\,\mathrm{d}x^n \wedge \,\cdot \qquad , \qquad T := (-1)^{(q-1)N} * R *$$

from $\big[$[6], (2.20), (2.21)$\big]$ acting pointwise on q-forms we present with

$$S := \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix}$$

our solution concept for real frequencies:

Definition 2.4 *Let $\omega \in \mathbb{R} \setminus \{0\}$ and $(F, G) \in \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{loc}}(\Omega)$. Then (E, H) solves the problem* $\mathsf{Max}(\Lambda, \omega, F, G)$*, if and only if*

(i) $(E, H) \in \mathring{\mathbf{R}}^{q}_{<-\frac{1}{2}}(\Omega) \times \mathbf{D}^{q+1}_{<-\frac{1}{2}}(\Omega)$,

(ii) $(M + \mathrm{i}\,\omega\Lambda)(E, H) = (F, G)$,

(iii) $(r^{-1}S + \mathrm{Id})(E, H) \in \mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>-\frac{1}{2}}(\Omega)$.

Here once again we introduced a new notation. For some weighted Sobolev spaces V_t, $t \in \mathbb{R}$, we define

$$V_{<s} := \bigcap_{t<s} V_t \qquad , \qquad V_{>s} := \bigcup_{t>s} V_t \qquad , \qquad s \in \mathbb{R} \qquad .$$

Remark 2.5 *We call condition (iii) 'Maxwell radiation condition' or 'radiation condition'. This condition generalizes the classical ($N = 3$, $q = 1$) Silver-Müller incoming radiation condition for Maxwell's equations* (*see* (1.4))

$$\xi \times H - E \in \mathrm{L}^{2}_{>-\frac{1}{2}}(\Omega) \qquad , \qquad \xi \times E + H \in \mathrm{L}^{2}_{>-\frac{1}{2}}(\Omega) \qquad .$$

We note that the radiation condition reads

$$(r^{-1}TH + E, r^{-1}RE + H) \in \mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>-\frac{1}{2}}(\Omega) \qquad .$$

2.2 A decomposition lemma

To use the results from Weck and Witsch [26] we put

$$\mathbb{I} := \big\{ n + N/2 : n \in \mathbb{N}_0 \big\} \cup \big\{ 1 - n - N/2 : n \in \mathbb{N}_0 \big\} \qquad .$$

The following decomposition lemma suited for our electric and magnetic fields will be essential and allows us to transfer results known from Helmholtz' equation to Maxwell's equation:

Lemma 2.6 *Let $\omega \in K \Subset \mathbb{C} \setminus \{0\}$, $t, s \in \mathbb{R}$ with $0 \leq s \in \mathbb{R} \setminus \mathbb{I}$, $t \leq s \leq t+\tau$, $\rho \geq r_0$ and $\varphi := \boldsymbol{\eta}(r/\rho)$. Moreover, let $(E,H) \in \mathbf{R}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$ be a solution of*

$$(M + \mathrm{i}\,\omega\Lambda)(E,H) =: (F,G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \qquad .$$

Then

$$(\hat{F},\hat{G}) := \varphi(F,G) + (C_{M,\varphi} - \mathrm{i}\,\omega\hat{\Lambda}\varphi)(E,H) \in \mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s$$

and by decomposing

$$\begin{aligned}(\hat{F},\hat{G}) &=: (F_R, G_D) + (F_D, G_R) + (F_{\mathcal{S}}, G_{\mathcal{S}}) \\ &\in \ ({}_0\mathrm{R}^q_s \times {}_0\mathrm{D}^{q+1}_s) \dot{+} ({}_0\mathrm{D}^q_s \times {}_0\mathrm{R}^{q+1}_s) \dot{+} (\mathcal{S}^q_s \times \mathcal{S}^{q+1}_s)\end{aligned}$$

according to $\big[$[26], Theorem 4$\big]$

$$(\tilde{F},\tilde{G}) := (F_D, G_R) + \frac{\mathrm{i}}{\omega} M(F_{\mathcal{S}}, G_{\mathcal{S}}) \in {}_0\mathrm{D}^q_s \times {}_0\mathrm{R}^{q+1}_s$$

holds. Then (E,H) may be decomposed as

$$(E,H) = (1-\varphi)(E,H) + (E_s, H_s) + (E_{\mathcal{F}}, H_{\mathcal{F}}) + (E_\Delta, H_\Delta)$$

and there exists a constant $c > 0$ independent of (E,H), (F,G) or ω, such that

- **(i)** $(1-\varphi)(E,H) \in \mathrm{R}^q_{\mathrm{vox}}(\Omega) \times \mathrm{D}^{q+1}_{\mathrm{vox}}(\Omega)$ *and for all $\tilde{t} \in \mathbb{R}$*

$$\begin{aligned}&\big\|(1-\varphi)(E,H)\big\|_{\mathrm{R}^q_{\tilde{t}}(\Omega)\times\mathrm{D}^{q+1}_{\tilde{t}}(\Omega)} \\ &\leq c \cdot \Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)} \Big) \qquad ,\end{aligned}$$

- **(ii)** $(E_s, H_s) := -\dfrac{\mathrm{i}}{\omega}\big((F_R, G_D) + (F_{\mathcal{S}}, G_{\mathcal{S}})\big) \in \mathrm{R}^q_s \times \mathrm{D}^{q+1}_s$ *and*

$$\big\|(E_s,H_s)\big\|_{\mathrm{R}^q_s\times\mathrm{D}^{q+1}_s} \leq c \cdot \big\|(\hat{F},\hat{G})\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s} \qquad ,$$

- **(iii)** $(E_{\mathcal{F}}, H_{\mathcal{F}}) := \mathcal{F}^{-1}\big((1+r^2)^{-1}(\mathrm{Id} - \mathrm{i}\,S)\mathcal{F}(\tilde{F},\tilde{G})\big) \in (\mathbf{H}^{1,q}_s \cap {}_0\mathrm{D}^q_s) \times (\mathbf{H}^{1,q+1}_s \cap {}_0\mathrm{R}^{q+1}_s)$ *and*

$$\big\|(E_{\mathcal{F}},H_{\mathcal{F}})\big\|_{\mathbf{H}^{1,q}_s\times\mathbf{H}^{1,q+1}_s} \leq c \cdot \big\|(\tilde{F},\tilde{G})\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s} \qquad ,$$

- **(iv)** $(E_\Delta, H_\Delta) := (\tilde{E},\tilde{H}) - (E_{\mathcal{F}}, H_{\mathcal{F}}) \in (\mathbf{H}^{2,q}_t \cap {}_0\mathrm{D}^q_t) \times (\mathbf{H}^{2,q+1}_t \cap {}_0\mathrm{R}^{q+1}_t)$ *and*

$$\begin{aligned}&\big\|(E_\Delta,H_\Delta)\big\|_{\mathbf{H}^{2,q}_{\tilde{t}}\times\mathbf{H}^{2,q+1}_{\tilde{t}}} \\ &\leq c \cdot \Big(\big\|(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{\tilde{t}}\times\mathrm{L}^{2,q+1}_{\tilde{t}}} + \big\|(E_{\mathcal{F}},H_{\mathcal{F}})\big\|_{\mathbf{H}^{1,q}_{\tilde{t}}\times\mathbf{H}^{1,q+1}_{\tilde{t}}} \Big)\end{aligned}$$

  for all $\tilde{t} \leq t$ with

$$(\tilde{E},\tilde{H}) := -\frac{\mathrm{i}}{\omega}\big((F_D, G_R) - M\varphi(E,H)\big) \in (\mathbf{H}^{1,q}_t \cap {}_0\mathrm{D}^q_t) \times (\mathbf{H}^{1,q+1}_t \cap {}_0\mathrm{R}^{q+1}_t) \,.$$

These forms solve the following equations:

- $$(M+\mathrm{i}\,\omega)\varphi(E,H)=(\hat{F},\hat{G})$$
- $$(M+\mathrm{i}\,\omega)(\tilde{E},\tilde{H})=(\tilde{F},\tilde{G})$$
- $$(M+\mathrm{i}\,\omega)(E_\Delta,H_\Delta)=(1-\mathrm{i}\,\omega)(E_{\mathfrak{F}},H_{\mathfrak{F}})$$
- $$(M+1)(E_{\mathfrak{F}},H_{\mathfrak{F}})=(\tilde{F},\tilde{G})$$
- $$(\Delta+\omega^2)(E_\Delta,H_\Delta)=-(1+\omega^2)(E_{\mathfrak{F}},H_{\mathfrak{F}})+(1-\mathrm{i}\,\omega)(\tilde{F},\tilde{G})$$

Moreover, the following estimates hold for all $\tilde{t}\leq t$ and uniformly in $\lambda\in K$, (E,H) and (F,G):

- $$\big\|(\tilde{F},\tilde{G})\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s}\leq c\cdot\big\|(\hat{F},\hat{G})\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s}$$
- $$\begin{aligned}\big\|(\hat{F},\hat{G})\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s}\leq c\cdot\Big(&\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}\\&+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big)\end{aligned}$$
- $$\begin{aligned}\big\|(E,H)\big\|_{\mathbf{R}^q_{\tilde{t}}(\Omega)\times\mathbf{D}^{q+1}_{\tilde{t}}(\Omega)}\leq c\cdot\Big(&\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}\\&+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\\&+\big\|(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{\tilde{t}}\times\mathrm{L}^{2,q+1}_{\tilde{t}}}\Big)\end{aligned}$$
- $$\begin{aligned}\big\|(\Delta+\omega^2)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s}\leq c\cdot\Big(&\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}\\&+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big)\end{aligned}$$
- $$\begin{aligned}&\big\|(M-\mathrm{i}\,\lambda r^{-1}S)(E,H)\big\|_{\mathrm{L}^{2,q}_{\tilde{t}}(\Omega)\times\mathrm{L}^{2,q+1}_{\tilde{t}}(\Omega)}\\&\leq c\cdot\Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\\&\qquad+\big\|(M-\mathrm{i}\lambda r^{-1}S)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{\tilde{t}}\times\mathrm{L}^{2,q+1}_{\tilde{t}}}\Big)\end{aligned}$$

Proof: $\varphi(E,H)\in\mathbf{R}^q_t\times\mathbf{D}^{q+1}_t$ yields $(E,H)=(1-\varphi)(E,H)+\varphi(E,H)$ and

$$\begin{aligned}M\varphi(E,H)&=\varphi M(E,H)+C_{M,\varphi}(E,H)\\&=-\mathrm{i}\,\omega\Lambda\varphi(E,H)+\varphi(F,G)+C_{M,\varphi}(E,H)\qquad.\end{aligned}$$

With the commutator $C_{M,\varphi}=M(\varphi\,\cdot\,)-\varphi M=\boldsymbol{\eta}'(\rho^{-1}\cdot r)\rho^{-1}r^{-1}S$ we get

$$(M+\mathrm{i}\,\omega)\varphi(E,H)=(\hat{F},\hat{G})\in\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s\qquad,\tag{2.2}$$

since $\operatorname{supp}C_{M,\varphi}(E,H)$ is compact and $t+\tau\geq s$. We write (2.2) in the form

$$\mathrm{i}\,\omega\varphi(E,H)=(F_{\mathrm{D}},G_{\mathrm{R}})-M\varphi(E,H)+(F_{\mathrm{R}},G_{\mathrm{D}})+(F_{\mathrm{S}},G_{\mathrm{S}})$$

and note

$$(\tilde{E},\tilde{H})=-\frac{\mathrm{i}}{\omega}\big((F_{\mathrm{D}},G_{\mathrm{R}})-M\varphi(E,H)\big)\in(\mathbf{R}^q_t\cap{}_0\mathrm{D}^q_t)\times(\mathbf{D}^{q+1}_t\cap{}_0\mathrm{R}^{q+1}_t)\quad,$$

$$(E_s, H_s) = -\frac{\mathrm{i}}{\omega}\big((F_{\mathrm{R}}, G_{\mathrm{D}}) + (F_{\mathbb{S}}, G_{\mathbb{S}})\big) \in \mathrm{R}^q_s \times \mathrm{D}^{q+1}_s$$

with $\varphi(E, H) = (\tilde{E}, \tilde{H}) + (E_s, H_s)$ and $(\tilde{E}, \tilde{H}) \in \mathbf{H}^{1,q}_t \times \mathbf{H}^{1,q+1}_t$ by $\big[$[6], Theorem 3.6 (i)$\big]$. $\big($For $s < N/2$ even $(F_{\mathbb{S}}, G_{\mathbb{S}}) = (0, 0)$ holds.$\big)$ Moreover, $(\tilde{E}, \tilde{H})$ solves

$$\begin{aligned} M(\tilde{E}, \tilde{H}) &= M\varphi(E, H) - M(E_s, H_s) \\ &= -\mathrm{i}\,\omega(\tilde{E}, \tilde{H}) + (F_{\mathrm{D}}, G_{\mathrm{R}}) + \frac{\mathrm{i}}{\omega} M(F_{\mathbb{S}}, G_{\mathbb{S}}) \qquad , \end{aligned}$$

i.e. $(M + \mathrm{i}\,\omega)(\tilde{E}, \tilde{H}) = (\tilde{F}, \tilde{G}) \in {}_0\mathrm{D}^q_s \times {}_0\mathrm{R}^{q+1}_s$.

To define $(M + 1)^{-1}(\tilde{F}, \tilde{G})$ by the Fourier transformation $\mathcal{F}$ (the componentwise scalar Fourier transformation w.r.t. Euclidean coordinates) we put

$$(E_{\mathcal{F}}, H_{\mathcal{F}}) = \mathcal{F}^{-1}\big((1 + r^2)^{-1}(\mathrm{Id} - \mathrm{i}\,S)\mathcal{F}(\tilde{F}, \tilde{G})\big) \qquad .$$

Then $(E_{\mathcal{F}}, H_{\mathcal{F}}) \in \mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1}$ as well as $\mathcal{F}(E_{\mathcal{F}}, H_{\mathcal{F}}) \in \mathrm{L}^{2,q}_1 \times \mathrm{L}^{2,q+1}_1$ are implied by $\mathcal{F}(\tilde{F}, \tilde{G}) \in \mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1}$. Thus by $\big[$[6], (3.6)$\big]$ we have

$$(E_{\mathcal{F}}, H_{\mathcal{F}}) \in \mathbf{H}^{1,q} \times \mathbf{H}^{1,q+1} \qquad .$$

From $(\tilde{F}, \tilde{G}) \in \mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s$ we get $\mathcal{F}(\tilde{F}, \tilde{G}) \in \mathbf{H}^{s,q} \times \mathbf{H}^{s,q+1}$. The components of $\mathcal{F}(E_{\mathcal{F}}, H_{\mathcal{F}})$ arise from those of $\mathcal{F}(\tilde{F}, \tilde{G})$ by multiplication with bounded C^{∞}-functions. Hence

$$\mathcal{F}(E_{\mathcal{F}}, H_{\mathcal{F}}) \in \mathbf{H}^{s,q} \times \mathbf{H}^{s,q+1}$$

follows $\Big($see e.g. Wloka $\big[$[29], p. 71, Lemma 3.2$\big]\Big)$, i.e. $(E_{\mathcal{F}}, H_{\mathcal{F}}) \in \mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s$, and we obtain the estimate

$$\big\|(E_{\mathcal{F}}, H_{\mathcal{F}})\big\|_{\mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s} \le c \cdot \big\|(\tilde{F}, \tilde{G})\big\|_{\mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s}$$

as well. From $\big[$[6], (2.27), (2.28)$\big]$ we derive $\mathcal{F}M = \mathrm{i}\,S\mathcal{F}$ and using this formula we compute

$$\begin{aligned} \mathcal{F}(M + 1)(E_{\mathcal{F}}, H_{\mathcal{F}}) &= (1 + r^2)^{-1}(\mathrm{Id} + \mathrm{i}\,S)(\mathrm{Id} - \mathrm{i}\,S)\mathcal{F}(\tilde{F}, \tilde{G}) \\ &= (1 + r^2)^{-1}(\mathrm{Id} + S^2)\mathcal{F}(\tilde{F}, \tilde{G}) \qquad . \end{aligned}$$

By $\big[$[6], (2.28), (2.27)$\big]$ $\operatorname{div}\tilde{F} = 0$ and $\operatorname{rot}\tilde{G} = 0$ imply $T\mathcal{F}\tilde{F} = 0$ and $R\mathcal{F}\tilde{G} = 0$. Therefore, applying $\big[$[6], (2.22)$\big]$, i.e. $RT + TR = r^2$,

$$S^2\mathcal{F}(\tilde{F}, \tilde{G}) = \begin{bmatrix} TR & 0 \\ 0 & RT \end{bmatrix} \mathcal{F}(\tilde{F}, \tilde{G}) = r^2\mathcal{F}(\tilde{F}, \tilde{G})$$

holds and we obtain

$$\mathcal{F}(M + 1)(E_{\mathcal{F}}, H_{\mathcal{F}}) = \mathcal{F}(\tilde{F}, \tilde{G}) \qquad \text{or} \qquad (M + 1)(E_{\mathcal{F}}, H_{\mathcal{F}}) = (\tilde{F}, \tilde{G}) \qquad .$$

Besides we have $(E_{\mathcal{F}}, H_{\mathcal{F}}) \in (\mathbf{R}_s^q \cap {}_0\mathrm{D}_s^q) \times (\mathbf{D}_s^{q+1} \cap {}_0\mathrm{R}_s^{q+1})$ and thus

$$(E_{\mathcal{F}}, H_{\mathcal{F}}) \in (\mathbf{H}_s^{1,q} \cap {}_0\mathrm{D}_s^q) \times (\mathbf{H}_s^{1,q+1} \cap {}_0\mathrm{R}_s^{q+1})$$

by $\big[$[6], Theorem 3.6 (i)$\big]$. Looking at

$$(E_\Delta, H_\Delta) = (\tilde{E}, \tilde{H}) - (E_{\mathcal{F}}, H_{\mathcal{F}}) \in (\mathbf{H}_t^{1,q} \cap {}_0\mathrm{D}_t^q) \times (\mathbf{H}_t^{1,q+1} \cap {}_0\mathrm{R}_t^{q+1})$$

we calculate

$$(M + \mathrm{i}\,\omega)(E_\Delta, H_\Delta) = (1 - \mathrm{i}\,\omega)(E_{\mathcal{F}}, H_{\mathcal{F}}) \qquad .$$

Again $\big[$[6], Theorem 3.6 (i)$\big]$ yields $(E_\Delta, H_\Delta) \in (\mathbf{H}_t^{2,q} \cap {}_0\mathrm{D}_t^q) \times (\mathbf{H}_t^{2,q+1} \cap {}_0\mathrm{R}_t^{q+1})$ and we compute

$$\begin{aligned}(\Delta + \omega^2)(E_\Delta, H_\Delta) &= (M - \mathrm{i}\,\omega)(M + \mathrm{i}\,\omega)(E_\Delta, H_\Delta)\\ &= (1 - \mathrm{i}\,\omega)(M - \mathrm{i}\,\omega)(E_{\mathcal{F}}, H_{\mathcal{F}})\\ &= -(1 + \omega^2)(E_{\mathcal{F}}, H_{\mathcal{F}}) + (1 - \mathrm{i}\,\omega)(\tilde{F}, \tilde{G}) \qquad .\end{aligned}$$

Finally we achieve the asserted estimates from $\big[$[6], Theorem 3.6 (i)$\big]$ and the continuity of the projections in $\mathrm{L}_s^{2,q}$ onto ${}_0\mathrm{R}_s^q$, ${}_0\mathrm{D}_s^q$ resp. $\mathcal{S}_s^q$ mentioning that

$$\big\|M(F_{\mathcal{S}}, G_{\mathcal{S}})\big\| \leq c \cdot \big\|(F_{\mathcal{S}}, G_{\mathcal{S}})\big\|$$

holds in every norm, since $\mathcal{S}_s^q \times \mathcal{S}_s^{q+1}$ is finite dimensional and M linear. ■

2.3 The polynomial and exponential decay

First we need a technical lemma:

Lemma 2.7 *For all $t, \tilde{t} \in \mathbb{R}$ with $\tilde{t} < t$ and all $\vartheta > 0$ there exist a constant $c > 0$ and a compact set $K \subset \overline{\Omega}$, such that*

$$\|u\|_{\mathrm{L}_{\tilde{t}}^{2,q}(\Omega)} \leq c \cdot \|u\|_{\mathrm{L}^{2,q}(K)} + \vartheta \cdot \|u\|_{\mathrm{L}_t^{2,q}(\Omega)}$$

holds for all $u \in \mathrm{L}_t^{2,q}(\Omega)$.

Proof: For sufficient large $\delta > 0$ we get from $\tilde{t} - t < 0$

$$\begin{aligned}\|u\|^2_{\mathrm{L}_{\tilde{t}}^{2,q}(\Omega)} &= \|\rho^{\tilde{t}} u\|^2_{\mathrm{L}^{2,q}(\Omega \cap U_\delta)} + \|\rho^{\tilde{t}-t} u\|^2_{\mathrm{L}_t^{2,q}(A_\delta)}\\ &\leq c(\Omega, \tilde{t}, \delta) \cdot \|u\|^2_{\mathrm{L}^{2,q}(\Omega \cap U_\delta)} + (1 + \delta^2)^{\tilde{t}-t} \cdot \|u\|^2_{\mathrm{L}_t^{2,q}(A_\delta)} \qquad .\end{aligned}$$

Thus $\lim\limits_{\delta \to \infty} (1 + \delta^2)^{\tilde{t}-t} = 0$ completes the proof. ■

From now on we may assume generally ε and μ, i.e. Λ, to be τ-admissible with order of decay

$$\tau > 1 \qquad .$$

With our decomposition lemma we receive

Theorem 2.8 *Let $\omega \in I \Subset \mathbb{R} \setminus \{0\}$ be some interval and $1/2 < s \in \mathbb{R} \setminus \mathbb{I}$. If*

$$(E, H) \in \mathbf{R}^q_{>-\frac{1}{2}}(\Omega) \times \mathbf{D}^{q+1}_{>-\frac{1}{2}}(\Omega)$$

is a solution of Maxwell's equation

$$(M + \mathrm{i}\,\omega\Lambda)(E, H) =: (F, G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \qquad ,$$

then $(E, H) \in \mathbf{R}^q_{s-1}(\Omega) \times \mathbf{D}^{q+1}_{s-1}(\Omega)$ and there exist constants $c, \delta > 0$ independent of (E, H), (F, G) or ω, such that

$$\begin{aligned} &\big\|(E, H)\big\|_{\mathbf{R}^q_{s-1}(\Omega)\times\mathbf{D}^{q+1}_{s-1}(\Omega)} \\ &\leq c \cdot \Big(\big\|(F, G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E, H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_\delta)\times\mathrm{L}^{2,q+1}(\Omega\cap U_\delta)}\Big) \qquad . \end{aligned}$$

Proof: Let $t > -1/2$ and $(E, H) \in \mathbf{R}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$ with $t < s - 1$. W.l.o.g. we may assume $t + 1 < s < t + \tau$. Otherwise we replace t and s by $t_k := t + k\alpha$ and $s_k := t+1+(k+1)\alpha \leq s$, $k = 0, \dots$, with $\alpha := (\tau-1)/2 > 0$ and obtain the assertions after finitely many α-steps.

Decomposing the electric and magnetic field by Lemma 2.6 we get solutions $(E_\Delta, H_\Delta) \in \mathbf{H}^{2,q}_t \times \mathbf{H}^{2,q+1}_t$ of Helmholtz' equation in $\mathbb{R}^N$

$$(\Delta + \omega^2)(E_\Delta, H_\Delta) \in {}_0\mathrm{D}^q_s \times {}_0\mathrm{R}^{q+1}_s \qquad .$$

A componentwise application of $\big[$[27], Lemma 5$\big]$ yields $(E_\Delta, H_\Delta) \in \mathbf{H}^{2,q}_{s-1} \times \mathbf{H}^{2,q+1}_{s-1}$ and with a constant $c > 0$ independent of (E_Δ, H_Δ), $(\Delta + \omega^2)(E_\Delta, H_\Delta)$ or ω

$$\begin{aligned} &\big\|(E_\Delta, H_\Delta)\big\|_{\mathbf{H}^{2,q}_{s-1}\times\mathbf{H}^{2,q+1}_{s-1}} \\ &\leq c \cdot \Big(\big\|(\Delta + \omega^2)(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s} + \big\|(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-2}\times\mathrm{L}^{2,q+1}_{s-2}}\Big) \qquad . \end{aligned}$$

Moreover, from Lemma 2.6 we have $(E, H) \in \mathbf{R}^q_{s-1}(\Omega) \times \mathbf{D}^{q+1}_{s-1}(\Omega)$ and the estimate (w.l.o.g. $1 < \tau < 2$)

$$\begin{aligned} &\big\|(E, H)\big\|_{\mathbf{R}^q_{s-1}(\Omega)\times\mathbf{D}^{q+1}_{s-1}(\Omega)} \\ &\leq c \cdot \Big(\big\|(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}} + \big\|(F, G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} \\ &\qquad\qquad + \big\|(E, H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big) \\ &\leq c \cdot \Big(\big\|(\Delta + \omega^2)(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_s\times\mathrm{L}^{2,q+1}_s} + \big\|(F, G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} \\ &\qquad\qquad + \big\|(E, H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big) \\ &\leq c \cdot \Big(\big\|(F, G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E, H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big) \qquad . \end{aligned}$$

Because $s - \tau < s - 1$ the assertion follows by Lemma 2.7. ∎

Remark 2.9 *If* $(F, G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$ *for all* $s \in \mathbb{R}$ *, we get*

$$(E, H) \in \mathbf{R}^q_s(\Omega) \times \mathbf{D}^{q+1}_s(\Omega)$$

for all $s \in \mathbb{R}$ *by Theorem 2.8. This holds, for instance, if* $(F, G) \in \mathrm{L}^{2,q}_{\mathrm{vox}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{vox}}(\Omega)$ *.*

Although we do not need the exponential decay to establish our solution theory we note that also this property can be shown. Using the 'partial integration' technique introduced by Eidus [3] for the classical Maxwell equations we obtain the exponential decay in our generalized case as well. To this end we need some additional assumptions concerning the regularity of our data ε , μ and (F, G) . We get

Theorem 2.10 *Let* $\Xi \subset \Omega$ *be some other exterior domain and* $\omega \in \mathbb{R} \setminus \{0\}$ *. Additionally let*

$$\exp(t\, r) \cdot (F, G) \in \mathbf{H}^{2,q}(\Xi) \times \mathbf{H}^{2,q+1}(\Xi)$$

for all $t \in \mathbb{R}$ *and the coefficients*

$$(\varepsilon, \mu) \in \mathrm{C}^{2,q}(\Xi) \times \mathrm{C}^{2,q+1}(\Xi)$$

with bounded derivatives up to second order. If $(E, H) \in \mathbf{R}^q_{>-\frac{1}{2}}(\Omega) \times \mathbf{D}^{q+1}_{>-\frac{1}{2}}(\Omega)$ *solves* $(M + \mathrm{i}\,\omega\Lambda)(E, H) = (F, G)$ *, then*

$$\exp(t\, r) \cdot (E, H) \in \big(\mathbf{R}^q(\Omega) \cap \mathbf{H}^{2,q}(\tilde{\Xi})\big) \times \big(\mathbf{D}^{q+1}(\Omega) \cap \mathbf{H}^{2,q+1}(\tilde{\Xi})\big)$$

holds for all $t \in \mathbb{R}$ *and any exterior domain* $\tilde{\Xi} \subset \Xi$ *with* $\mathrm{dist}(\tilde{\Xi}, \partial\,\Xi) > 0$ *. Especially this assertion is valid, if* $(F, G) \in \mathrm{L}^{2,q}_{\mathrm{vox}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{vox}}(\Omega)$ *.*

We note that we only need the decay of $\hat{\varepsilon}$ and $\hat{\mu}$, but not of their derivatives. For a proof we refer the interested reader to $\big[$[13], Kapitel 4.6, Satz 4.19$\big]$.

2.4 An a-priori estimate

We prove an a-priori estimate for our Maxwell operator:

Lemma 2.11 *Let* $I \Subset \mathbb{R} \setminus \{0\}$ *be a compact interval and* $-t, s > 1/2$ *. Then there exist constants* $c, \delta > 0$ *and some* $\hat{t} > -1/2$ *, such that for all* $\omega \in \mathbb{C}_+$ *with* $\omega^2 = \lambda^2 + \mathrm{i}\,\sigma\lambda$ *,* $\lambda \in I$ *,* $\sigma \in (0, 1]$ *and* $(F, G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$ *the following estimate holds true:*

$$\begin{aligned} &\|\, \mathcal{L}_\omega(F, G)\|_{\mathbf{R}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)} + \|(r^{-1}S + \mathrm{Id})\, \mathcal{L}_\omega(F, G)\|_{\mathrm{L}^{2,q}_{\hat{t}}(\Omega) \times \mathrm{L}^{2,q+1}_{\hat{t}}(\Omega)} \\ &\leq c \cdot \Big(\|(F, G)\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + \|\, \mathcal{L}_\omega(F, G)\|_{\mathrm{L}^{2,q}(\Omega \cap U_\delta) \times \mathrm{L}^{2,q+1}(\Omega \cap U_\delta)} \Big) \end{aligned}$$

Proof: W.l.o.g. let $s \in (1/2, 1)$. We note $(1/2,1) \cap \mathbb{I} = \emptyset$ and decompose (F, G), $(E, H) := \mathcal{L}_\omega(F, G)$ using Lemma 2.6 with $s = s$, $t = 0$ and $(F_\mathbb{S}, G_\mathbb{S}) = (0, 0)$, since $s < N/2$. We obtain $(E_\Delta, H_\Delta) \in \mathbf{H}^{2,q} \times \mathbf{H}^{2,q+1}$ with

$$
\begin{aligned}
(\Delta + \omega^2)(E_\Delta, H_\Delta) = & -(1+\omega^2)(E_\mathfrak{F}, H_\mathfrak{F}) + (1 - \mathrm{i}\,\omega)(\tilde{F}, \tilde{G}) \\
& =: (F_\Delta, G_\Delta) \in \mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s \qquad .
\end{aligned}
$$

The selfadjointness of $\Delta : \mathbf{H}^{2,q} \times \mathbf{H}^{2,q+1} \subset \mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1} \longrightarrow \mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1}$ yields

$$
(\Delta + \omega^2)^{-1}(F_\Delta, G_\Delta) = (E_\Delta, H_\Delta) \qquad .
$$

Applying $\big[$[27], Lemma 7$\big]$ $\Big($a well known a-priori estimate for the scalar Helmholtz equation in $\mathbb{R}^N$; see also Ikebe and Saito [4] or Vogelsang $\big[$[24], section 2$\big]\Big)$ componentwise to (E_Δ, H_Δ) and by Lemma 2.6 with

$$
M\big(\exp(-\mathrm{i}\,\lambda r)(E_\Delta, H_\Delta)\big) = \exp(-\mathrm{i}\,\lambda r)(M - \mathrm{i}\,\lambda r^{-1} S)(E_\Delta, H_\Delta)
$$

we get the estimate

$$
\begin{aligned}
& \big\|(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_t \times \mathrm{L}^{2,q+1}_t} + \big\|(M - \mathrm{i}\,\lambda r^{-1} S)(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1} \times \mathrm{L}^{2,q+1}_{s-1}} \\
\leq\ & c \cdot \Big(\big\|(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_t \times \mathrm{L}^{2,q+1}_t} + \big\|\exp(-\mathrm{i}\,\lambda r)(E_\Delta, H_\Delta)\big\|_{\mathrm{H}^{1,q}_{s-2} \times \mathrm{H}^{1,q+1}_{s-2}}\Big) \\
\leq\ & c \cdot \big\|(F_\Delta, G_\Delta)\big\|_{\mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s} \\
\leq\ & c \cdot \Big(\big\|(F, G)\big\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E, H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega) \times \mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big) \qquad ,
\end{aligned}
\tag{2.3}
$$

which holds uniformly in (E_Δ, H_Δ), (F_Δ, G_Δ) and ω. But actually we would like to estimate the term $\big\|(M - \mathrm{i}\,\omega r^{-1} S)(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1} \times \mathrm{L}^{2,q+1}_{s-1}}$. This needs an additional argument. The resolvent estimate yields

$$
\sigma |\lambda| \cdot \big\|(E_\Delta, H_\Delta)\big\|_{\mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1}} \leq \big\|(F_\Delta, G_\Delta)\big\|_{\mathrm{L}^{2,q} \times \mathrm{L}^{2,q+1}}
\tag{2.4}
$$

and because of

$$
\omega = |\lambda| \cdot \big(1 + (\sigma/\lambda)^2\big)^{1/4} \cdot \begin{cases} \exp(\mathrm{i}\,\varphi/2) & , \lambda > 0 \\ \exp\big(\mathrm{i}(\varphi/2 + \pi)\big) & , \lambda < 0 \end{cases} \quad , \quad \varphi := \arctan(\sigma/\lambda)
$$

we have $|\operatorname{Re}\omega| \geq |\lambda|\sqrt{2}/2$ and thus $|\omega + \lambda| \geq |\lambda|\sqrt{3/2}$. From this, (2.4) and

$$
\omega - \lambda = \frac{\omega^2 - \lambda^2}{\omega + \lambda} = \frac{\mathrm{i}\,\sigma\lambda}{\omega + \lambda}
$$

we achieve uniformly in ω

$$
\begin{aligned}
&\big\|(M-\mathrm{i}\,\omega r^{-1}S)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}}\\
&\leq \big\|(M-\mathrm{i}\,\lambda r^{-1}S)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}}+c\cdot|\omega-\lambda|\cdot\big\|(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}}\\
&\leq \big\|(M-\mathrm{i}\,\lambda r^{-1}S)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}}+c\cdot|\lambda|^{-1}\cdot\big\|(F_\Delta,G_\Delta)\big\|_{\mathrm{L}^{2,q}\times\mathrm{L}^{2,q+1}}\quad .
\end{aligned}
$$

A combination of the latter estimate with (2.3) and Lemma 2.6 yield

$$
\begin{aligned}
&\big\|(E,H)\big\|_{\mathbf{R}^q_t(\Omega)\times\mathbf{D}^{q+1}_t(\Omega)}+\big\|(M-\mathrm{i}\,\omega r^{-1}S)(E,H)\big\|_{\mathrm{L}^{2,q}_{s-1}(\Omega)\times\mathrm{L}^{2,q+1}_{s-1}(\Omega)}\\
&\leq c\cdot\Big(\big\|(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{t}\times\mathrm{L}^{2,q+1}_{t}}+\big\|(M-\mathrm{i}\,\omega r^{-1}S)(E_\Delta,H_\Delta)\big\|_{\mathrm{L}^{2,q}_{s-1}\times\mathrm{L}^{2,q+1}_{s-1}}\\
&\qquad+\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_{s}(\Omega)\times\mathrm{L}^{2,q+1}_{s}(\Omega)}+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big)\\
&\leq c\cdot\Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_{s}(\Omega)\times\mathrm{L}^{2,q+1}_{s}(\Omega)}+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big)
\end{aligned}
$$

uniformly in (E,H), (F,G) and ω. Noting

$$
(M-\mathrm{i}\,\omega r^{-1}S)(E,H)=-\,\mathrm{i}\,\omega(E,H)-\mathrm{i}\,\omega\hat{\Lambda}(E,H)+(F,G)-\mathrm{i}\,\omega\,r^{-1}S(E,H)
$$

we finally arrive at

$$
\begin{aligned}
&\big\|(E,H)\big\|_{\mathbf{R}^q_t(\Omega)\times\mathbf{D}^{q+1}_t(\Omega)}+\big\|(r^{-1}S+\mathrm{Id})(E,H)\big\|_{\mathrm{L}^{2,q}_{s-1}(\Omega)\times\mathrm{L}^{2,q+1}_{s-1}(\Omega)}\\
&\leq c\cdot\Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_{s}(\Omega)\times\mathrm{L}^{2,q+1}_{s}(\Omega)}+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\Big)\quad .
\end{aligned}
$$

Because of the monotone dependence of the weighted L^2-norms on the weights we may assume w.l.o.g. t near to $-1/2$ and s near to $1/2$, such that $1<s-t<\tau$ holds. Then Lemma 2.7 completes the proof. ■

2.5 Fredholm theory

To establish the time-harmonic solution theory we now follow in close lines the first part of [22]. Thus we only sketch some similar proofs.

First we present two more technical lemmas:

Lemma 2.12 *Let $\alpha,\beta\in\mathbb{R}$ with $0\leq\alpha<\beta$ and $\mathbb{R}^N\setminus\Omega\subset U_\alpha$. Moreover, for some $t\in\mathbb{R}$ let $(E,H)\in\overset{\circ}{\mathbf{R}}{}^q_t(\Omega)\times\mathbf{D}^{q+1}_t(\Omega)$ and $\varphi\in\mathrm{C}^0\big([\alpha,\beta],\mathbb{C}\big)$. Then with*

$$
\begin{aligned}
\psi\ :\ [0,\beta]\ &\longrightarrow\ \mathbb{C}\\
\sigma\ &\longmapsto\ \int_{\max\{\alpha,\sigma\}}^{\beta}\varphi(s)\,ds
\end{aligned}
$$

and $\Phi:=\varphi\circ r$, $\Psi:=\psi\circ r$

$$
\begin{aligned}
&\langle\Phi\,r^{-1}RE,H\rangle_{\mathrm{L}^{2,q+1}(Z_{\alpha,\beta})}\\
&=\langle\Psi\,\mathrm{rot}\,E,H\rangle_{\mathrm{L}^{2,q+1}(\Omega\cap U_\beta)}+\langle\Psi E,\mathrm{div}\,H\rangle_{\mathrm{L}^{2,q}(\Omega\cap U_\beta)}
\end{aligned}
$$

holds. (Here as before $Z_{\alpha,\beta}=A_\alpha\cap U_\beta$.)

Proof: Assume $(E,H) \in \overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega) \times \mathrm{C}^{\infty,q+1}(\Omega)$. With $\gamma := \displaystyle\int_\alpha^\beta \varphi(s)\,ds$ we have $\psi|_{[0,\alpha]} = \gamma$, $\psi(\beta) = 0$ and $\psi \in \mathrm{C}^1\big((\alpha,\beta)\big)$ with $\psi' = -\varphi$. By Stokes' theorem we compute

$$
\begin{aligned}
&\langle \Psi \operatorname{rot} E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega\cap U_\beta)} + \langle \Psi E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(\Omega\cap U_\beta)}\\
&= \gamma\cdot\langle \operatorname{rot} E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega\cap U_\alpha)} + \gamma\cdot\langle E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(\Omega\cap U_\alpha)}\\
&\qquad + \langle \Psi \operatorname{rot} E, H\rangle_{\mathrm{L}^{2,q+1}(Z_{\alpha,\beta})} + \langle \Psi E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(Z_{\alpha,\beta})}\\
&= \gamma\cdot\int_{S_\alpha} \iota_\alpha^*(E\wedge *\bar{H}) + \langle \Phi\, r^{-1} R E, H\rangle_{\mathrm{L}^{2,q+1}(Z_{\alpha,\beta})}\\
&\qquad - \gamma\cdot\int_{S_\alpha} \iota_\alpha^*(E\wedge *\bar{H}) + \psi(\beta)\cdot\int_{S_\beta}\iota_\beta^*(E\wedge *\overline{H}) \qquad .
\end{aligned}
$$

Here we denote by $\iota_r : S_r \longrightarrow \mathbb{R}^N$ the natural embedding. With the aid of mollifiers we get the desired formula for all $(E,H) \in \overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega) \times \mathbf{D}_t^{q+1}(\Omega)$. Since $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ is dense in $\overset{\circ}{\mathbf{R}}{}_t^q(\Omega)$ the assertion holds as stated. ■

We also need the rule of partial integration for weighted forms.

Lemma 2.13 *Let $(E,H) \in \overset{\circ}{\mathrm{R}}{}^q_{\mathrm{loc}}(\Omega) \times \mathrm{D}^{q+1}_{\mathrm{loc}}(\overline{\Omega})$, $\varrho \in \mathbb{R}_+$ as well as $\varphi_\varrho := \mathbf{1} - \boldsymbol{\eta}(\varrho^{-1}\,\cdot\,)$, $\Phi_\varrho := \varphi_\varrho \circ r$. Then*

$$\langle \operatorname{rot} E, \Phi_\varrho H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \Phi_\varrho E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(\Omega)} = -\big\langle \varphi_\varrho'(r) r^{-1} R E, H\big\rangle_{\mathrm{L}^{2,q+1}(\Omega)}$$

holds. If additionally $(E,H) \in \overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_s(\Omega)$ resp. $(E,H) \in \overset{\circ}{\mathrm{R}}{}^q_t(\Omega) \times \mathrm{D}^{q+1}_s(\Omega)$ with $t,s \in \mathbb{R}$ and $t+s \geq 0$ resp. $t+s \geq -1$, then

$$\langle \operatorname{rot} E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(\Omega)} = 0 \qquad .$$

The proof is quite standard and may be omitted.

Remark 2.14 *If $(E,H) \in \overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$ resp. $\overset{\circ}{\mathrm{R}}{}^q_t(\Omega) \times \mathrm{D}^{q+1}_t(\Omega)$ and furthermore $(e,h) \in \overset{\circ}{\mathbf{R}}{}^q_s(\Omega) \times \mathbf{D}^{q+1}_s(\Omega)$ resp. $\overset{\circ}{\mathrm{R}}{}^q_s(\Omega) \times \mathrm{D}^{q+1}_s(\Omega)$ with $t+s \geq 0$ resp. $t+s \geq -1$, then*

$$\big\langle M(E,H), (e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} + \big\langle (E,H), M(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} = 0 \quad .$$

Another essential ingredient of the solution theory generating convergence in the limiting absorption argument is the so called Maxwell local compactness property MLCP, i.e. the embeddings

$$\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{D}^q(\Omega) \hookrightarrow \mathrm{L}^{2,q}_{\mathrm{loc}}(\overline{\Omega})$$

have to be compact for all q. For details we refer the reader to $\big[$[6], Definition 3.1 and Definition 2.5$\big]$ and the papers cited there.

To formulate our first main result we need one more

Definition 2.15 *We define*

$$\mathbb{P} := \big\{ \omega \in \mathbb{C} \setminus \{0\} : \mathsf{Max}(\Lambda, \omega, 0, 0) \text{ has a nontrivial solution.} \big\}$$

and for $\omega \in \mathbb{C} \setminus \{0\}$

$$\mathcal{N}(\mathsf{Max}, \Lambda, \omega) := \big\{ (E, H) : (E, H) \text{ is a solution of } \mathsf{Max}(\Lambda, \omega, 0, 0)\,. \big\} \quad .$$

Remark 2.16 *We have* $\mathbb{P} \subset \mathbb{R} \setminus \{0\}$ *and*

$$\mathcal{N}(\mathsf{Max}, \Lambda, \omega) = N(\mathcal{M} - \omega) = \big\{ (0, 0) \big\} \qquad , \qquad \omega \in \mathbb{C} \setminus \mathbb{R} \qquad .$$

Now we are ready to prove our first main result:

Theorem 2.17 *Let* $\omega \in \mathbb{R} \setminus \{0\}$ *.*

(i) *For all* $t \in \mathbb{R}$

$$\begin{aligned} \mathcal{N}(\mathsf{Max}, \Lambda, \omega) &= N(\mathcal{M} - \omega) \\ &\subset \big(\mathring{\mathbf{R}}^q_t(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q_t(\Omega) \big) \times \big(\mathbf{D}^{q+1}_t(\Omega) \cap \mu^{-1}{}_0\mathring{\mathbf{R}}^{q+1}_t(\Omega) \big) \quad , \end{aligned}$$

i.e. eigensolutions decay polynomially.

Additionally let Ω *have the* MLCP*. Then*

(ii) $\mathcal{N}(\mathsf{Max}, \Lambda, \omega)$ *is finite dimensional;*

(iii) $\mathbb{P}$ *has no accumulation point in* $\mathbb{R} \setminus \{0\}$ *;*

(iv) *for every* $(F, G) \in \mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$ *there exists a solution* (E, H) *of the problem* $\mathsf{Max}(\Lambda, \omega, F, G)$ *, if and only if*

$$\forall \quad (e, h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega) \qquad \big\langle (F, G), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} = 0 \qquad . \tag{2.5}$$

The solution can be chosen, such that

$$\big\langle \Lambda(E, H), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} = 0 \tag{2.6}$$

holds for all $(e, h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega)$ *. By this condition* (E, H) *is uniquely determined;*

(v) *the solution operator introduced in* **(iv)***, which we will also call* $\mathcal{L}_\omega$ *, maps*

$$\big(\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \big) \cap \mathcal{N}(\mathsf{Max}, \Lambda, \omega)^\perp$$

to

$$\big(\mathring{\mathbf{R}}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega) \big) \cap \mathcal{N}(\mathsf{Max}, \Lambda, \omega)^{\perp_\Lambda}$$

continuously for all $s, -t > 1/2$ *. Here we denote the orthogonality corresponding to the* $\langle \Lambda \,\cdot\, , \,\cdot\, \rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)}$*-scalar product by* $\perp_\Lambda$ *and we put* $\perp := \perp_{\mathrm{Id}}$ *.*

Proof: The proof follows in close lines the proof of $\big[$[22], Theorem 2.10$\big]$.

To show **(i)**, i.e. the polynomial decay of any eigensolution (E,H), we only have to prove

$$(E,H)\in \mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega)\times \mathrm{L}^{2,q+1}_{>-\frac{1}{2}}(\Omega)$$

because of Theorem 2.8, Remark 2.9, the equation $M(E,H)=-\mathrm{i}\,\omega\Lambda(E,H)$ and the inclusions

$$\overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q}(\Omega)}\subset {}_0\overset{\circ}{\mathbf{R}}{}^{q+1}(\Omega)\qquad,\qquad \overline{\operatorname{div}\mathbf{D}^{q+1}(\Omega)}\subset {}_0\mathbf{D}^{q}(\Omega)\qquad.$$

Using the second part of the radiation condition we obtain some $t>-1/2$, such that

$$\lim_{\beta\to\infty}\|r^{-1}RE+H\|_{\mathrm{L}^{2,q+1}_t(Z_{r_0,\beta})}<\infty$$

holds true. We calculate

$$\begin{aligned}&\|r^{-1}RE+H\|^2_{\mathrm{L}^{2,q+1}_t(Z_{r_0,\beta})}\\=&\|r^{-1}RE\|^2_{\mathrm{L}^{2,q+1}_t(Z_{r_0,\beta})}+\|H\|^2_{\mathrm{L}^{2,q+1}_t(Z_{r_0,\beta})}+2\operatorname{Re}\langle\Phi r^{-1}RE,H\rangle_{\mathrm{L}^{2,q+1}(Z_{r_0,\beta})}\end{aligned}$$

with $\varphi(\sigma):=(1+\sigma^2)^t$ and $\Phi:=\varphi\circ r$. Lemma 2.12, the differential equation and the symmetry of ε, μ yield

$$\begin{aligned}&\langle\Phi r^{-1}RE,H\rangle_{\mathrm{L}^{2,q+1}(Z_{r_0,\beta})}\\=&\langle\Psi\operatorname{rot}E,H\rangle_{\mathrm{L}^{2,q+1}(\Omega\cap U_\beta)}+\langle\Psi E,\operatorname{div}H\rangle_{\mathrm{L}^{2,q}(\Omega\cap U_\beta)}\\=&-\mathrm{i}\,\omega\underbrace{\langle\Psi\mu H,H\rangle_{\mathrm{L}^{2,q+1}(\Omega\cap U_\beta)}}_{\in\mathbb{R}}+\mathrm{i}\,\omega\underbrace{\langle\Psi E,\varepsilon E\rangle_{\mathrm{L}^{2,q}(\Omega\cap U_\beta)}}_{\in\mathbb{R}}\quad\in\mathrm{i}\cdot\mathbb{R}\qquad.\end{aligned}$$

Thus $\operatorname{Re}\langle\Phi r^{-1}RE,H\rangle_{\mathrm{L}^{2,q+1}(Z_{r_0,\beta})}=0$ and by means of the monotone convergence theorem

$$H\in\mathrm{L}^{2,q+1}_t(\Omega)$$

follows for $\beta\to\infty$. Finally we get $E\in\mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega)$ using the first part of the radiation condition.

If **(ii)** or **(iii)** would be wrong, then there would exist a sequence of eigenvalues $(\omega_\ell)_{\ell\in\mathbb{N}}\subset\mathbb{R}\setminus\{0\}$ and a sequence of eigenforms $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}}\subset N(\mathcal{M}-\omega_\ell)$, such that $\omega_\ell\xrightarrow{\ell\to\infty}\omega$ and $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}}$ is an orthonormal system with respect to the $\langle\Lambda\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}$-scalar product. As an orthonormal system $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}}$ converges in $\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)$ weakly to zero. Moreover, by the differential equation $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}}$ is bounded in

$$\big(\overset{\circ}{\mathbf{R}}{}^{q}(\Omega)\cap\varepsilon^{-1}{}_0\mathbf{D}^{q}(\Omega)\big)\times\big(\mathbf{D}^{q+1}(\Omega)\cap\mu^{-1}{}_0\overset{\circ}{\mathbf{R}}{}^{q+1}(\Omega)\big)\qquad.$$

Hence from the MLCP we get a subsequence $\big((E_{\pi\ell},H_{\pi\ell})\big)_{\ell\in\mathbb{N}}$ ($\pi:\mathbb{N}\to\mathbb{N}$ strictly monotone) converging in $\mathrm{L}^{2,q}_{\mathrm{loc}}(\overline{\Omega})\times\mathrm{L}^{2,q+1}_{\mathrm{loc}}(\overline{\Omega})$ to $(0,0)$ because of the weak convergence. For $1\leq s\in\mathbb{R}\setminus\mathbb{I}$ Theorem 2.8 yields uniformly in $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}}$ and $(\omega_\ell)_{\ell\in\mathbb{N}}$

the estimate

$$
\begin{aligned}
1 &= \big\langle \Lambda(E_{\pi\ell}, H_{\pi\ell}), (E_{\pi\ell}, H_{\pi\ell}) \big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} \\
&\leq c \cdot \big\| (E_{\pi\ell}, H_{\pi\ell}) \big\|^2_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} \leq c \cdot \big\| (E_{\pi\ell}, H_{\pi\ell}) \big\|^2_{\mathrm{L}^{2,q}_{s-1}(\Omega)\times\mathrm{L}^{2,q+1}_{s-1}(\Omega)} \\
&\leq c \cdot \big\| (E_{\pi\ell}, H_{\pi\ell}) \big\|^2_{\mathrm{L}^{2,q}(\Omega\cap U_\delta)\times\mathrm{L}^{2,q+1}(\Omega\cap U_\delta)} \xrightarrow{\ell\to\infty} 0 \qquad ,
\end{aligned}
$$

which is a contradiction.

We prove **(iv)** and **(v)**: First of all (2.5) is necessary, because for all eigenforms $(e,h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega)$ we get by the polynomial decay of eigensolutions and Remark 2.14

$$
\begin{aligned}
\big\langle (F,G), (e,h) \big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} &= \big\langle (M + \mathrm{i}\,\omega\Lambda)(E,H), (e,h) \big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} \\
&= -\big\langle (E,H), \underbrace{(M + \mathrm{i}\,\omega\Lambda)(e,h)}_{=0} \big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} = 0 \quad .
\end{aligned}
$$

Now we present an existence proof using Eidus' principle of limiting absorption. For that purpose let $(F,G) \in \mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$ with (2.5). Moreover, let $(\sigma_\ell)_{\ell\in\mathbb{N}}$ be a positive sequence tending to zero and $\big((F_\ell, G_\ell)\big)_{\ell\in\mathbb{N}} \subset \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$ with some $s > 1/2$ be a sequence satisfying

$$
\forall \quad (e,h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega) \qquad \big\langle (F_\ell, G_\ell), (e,h) \big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)} = 0 \qquad ,
$$

such that (F_ℓ, G_ℓ) converges to (F,G) in $\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$ as ℓ tends to infinity. Defining non real frequencies

$$
\omega_\ell := \sqrt{\omega^2 + \mathrm{i}\,\sigma_\ell\omega} \in \mathbb{C}_+ \setminus \mathbb{R}
$$

with $\omega_\ell^2 = \omega^2 + \mathrm{i}\,\sigma_\ell\omega$ and $\omega_\ell \xrightarrow{\ell\to\infty} \omega$ we obtain L^2-solutions

$$
(E_\ell, H_\ell) := \mathcal{L}_{\omega_\ell}(F_\ell, G_\ell) \in \mathring{\mathbf{R}}^q(\Omega) \times \mathbf{D}^{q+1}(\Omega)
$$

solving the problem $\mathsf{Max}(\Lambda, \omega_\ell, F_\ell, G_\ell)$, i.e.

$$
(M + \mathrm{i}\,\omega_\ell\Lambda)(E_\ell, H_\ell) = (F_\ell, G_\ell) \tag{2.7}
$$

holds. Applying the L^2-decompositions $\big[$[6], (3.5)$\big]$ we decompose

$$
(E_\ell, H_\ell) = (E_\ell^1, H_\ell^1) + (E_\ell^2, H_\ell^2) \qquad \text{and} \qquad (F_\ell, G_\ell) = (F_\ell^1, G_\ell^1) + (F_\ell^2, G_\ell^2)
$$

orthogonal with

$$
\begin{aligned}
\varepsilon E_\ell^1, F_\ell^1 &\in \overline{\varepsilon \operatorname{rot} \mathring{\mathbf{R}}^{q-1}(\Omega)} \subset \varepsilon\, {}_0\mathring{\mathbf{R}}^q(\Omega) \qquad , \qquad \varepsilon E_\ell^2, F_\ell^2 \in {}_0\mathbf{D}^q(\Omega) \qquad , \\
\mu H_\ell^1, G_\ell^1 &\in \overline{\mu \operatorname{div} \mathbf{D}^{q+2}(\Omega)} \subset \mu\, {}_0\mathbf{D}^q(\Omega) \qquad , \qquad \mu H_\ell^2, G_\ell^2 \in {}_0\mathring{\mathbf{R}}^{q+1}(\Omega) \qquad .
\end{aligned}
$$

Inserting this decompositions in (2.7) we receive the equation

$$\underline{M(E_\ell^2,H_\ell^2)}+\mathrm{i}\,\omega_\ell\Lambda(E_\ell^1,H_\ell^1)+\underline{\mathrm{i}\,\omega_\ell\Lambda(E_\ell^2,H_\ell^2)}=(F_\ell^1,G_\ell^1)+\underline{(F_\ell^2,G_\ell^2)}\qquad,$$

where the underlined terms belong to ${}_0\mathbf{D}^q(\Omega)\times{}_0\mathring{\mathbf{R}}^{q+1}(\Omega)$ and the others to its orthogonal complement $\Lambda\big(\operatorname{rot}\mathring{\mathbf{R}}^{q-1}(\Omega)\times\operatorname{div}\mathbf{D}^{q+2}(\Omega)\big)$. Thus we get by orthogonality the two equations

$$\begin{aligned}\mathrm{i}\,\omega_\ell\Lambda(E_\ell^1,H_\ell^1)&=(F_\ell^1,G_\ell^1)\qquad,\\(M+\mathrm{i}\,\omega_\ell\Lambda)(E_\ell^2,H_\ell^2)&=(F_\ell^2,G_\ell^2)\qquad,\end{aligned}\tag{2.8}$$

noting that the first one is trivial. As orthogonal projections the forms (F_ℓ^k,G_ℓ^k), $k=1,2$, converge in $\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)$ and so does (E_ℓ^1,H_ℓ^1).

We need an additional assumption, namely

$$\forall\quad t<-1/2\qquad\exists\quad c>0\qquad\forall\quad\ell\in\mathbb{N}\qquad\big\|(E_\ell,H_\ell)\big\|_{\mathrm{L}_t^{2,q}(\Omega)\times\mathrm{L}_t^{2,q+1}(\Omega)}\le c\qquad.\tag{2.9}$$

At the end of the proof we will show by contradiction that in fact (2.9) holds.

Let t' be such a t with (2.9). Then $\big((E_\ell^2,H_\ell^2)\big)_{\ell\in\mathbb{N}}$ is bounded in $\mathrm{L}_{t'}^{2,q}(\Omega)\times\mathrm{L}_{t'}^{2,q+1}(\Omega)$ and by (2.8) even in $\big(\mathring{\mathbf{R}}_{t'}^q(\Omega)\cap\varepsilon^{-1}{}_0\mathbf{D}_{t'}^q(\Omega)\big)\times\big(\mathbf{D}_{t'}^{q+1}(\Omega)\cap\mu^{-1}{}_0\mathring{\mathbf{R}}_{t'}^{q+1}(\Omega)\big)$. Hence the MLCP yields for an arbitrary $\tilde{t}<t'$ a subsequence $\big((E_{\pi\ell}^2,H_{\pi\ell}^2)\big)_{\ell\in\mathbb{N}}$ converging in $\mathrm{L}_{\tilde{t}}^{2,q}(\Omega)\times\mathrm{L}_{\tilde{t}}^{2,q+1}(\Omega)$ and even in $\mathring{\mathbf{R}}_{\tilde{t}}^q(\Omega)\times\mathbf{D}_{\tilde{t}}^{q+1}(\Omega)$ because of (2.8). Therefore, the entire sequence $\big((E_{\pi\ell},H_{\pi\ell})\big)_{\ell\in\mathbb{N}}$ converges in $\mathring{\mathbf{R}}_{\tilde{t}}^q(\Omega)\times\mathbf{D}_{\tilde{t}}^{q+1}(\Omega)$ to, let us say,

$$(E,H)\in\mathring{\mathbf{R}}_{\tilde{t}}^q(\Omega)\times\mathbf{D}_{\tilde{t}}^{q+1}(\Omega)\qquad,$$

which solves

$$(M+\mathrm{i}\,\omega\Lambda)(E,H)=(F,G)\qquad.$$

With the polynomial decay of eigensolutions and Remark 2.14 we compute for all eigenforms $(e,h)\in\mathcal{N}(\mathsf{Max},\Lambda,\omega)$ and all $\ell\in\mathbb{N}$

$$\begin{aligned}0&=\big\langle(F_{\pi\ell},G_{\pi\ell}),(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}\\&=-\big\langle(E_{\pi\ell},H_{\pi\ell}),(M+\mathrm{i}\,\omega_{\pi\ell}\Lambda)(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}\\&=\mathrm{i}\,\underbrace{(\overline{\omega}_{\pi\ell}-\omega)}_{\neq0}\big\langle\Lambda(E_{\pi\ell},H_{\pi\ell}),(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}\qquad.\end{aligned}$$

Consequently $\big\langle\Lambda(E_{\pi\ell},H_{\pi\ell}),(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}=0$. The observation that the mapping $\big\langle\,\cdot\,,\Lambda(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}$ is a continuous linear functional on $\mathrm{L}_{\tilde{t}}^{2,q}(\Omega)\times\mathrm{L}_{\tilde{t}}^{2,q+1}(\Omega)$ for all $(e,h)\in\mathcal{N}(\mathsf{Max},\Lambda,\omega)$ yields

$$\forall\quad(e,h)\in\mathcal{N}(\mathsf{Max},\Lambda,\omega)\qquad\big\langle\Lambda(E,H),(e,h)\big\rangle_{\mathrm{L}^{2,q}(\Omega)\times\mathrm{L}^{2,q+1}(\Omega)}=0\qquad.\tag{2.10}$$

Now we pick some $t < -1/2$. Then we obtain by Lemma 2.11 some constants $\hat{t} > -1/2$ and $c, \delta > 0$, such that

$$\begin{aligned}&\big\|(E,H)\big\|_{\mathbf{R}^q_t(\Omega)\times\mathbf{D}^{q+1}_t(\Omega)} + \big\|(r^{-1}S+\mathrm{Id})(E,H)\big\|_{\mathrm{L}^{2,q}_{\hat{t}}(\Omega)\times\mathrm{L}^{2,q+1}_{\hat{t}}(\Omega)}\\&\leq c\cdot\Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E,H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_\delta)\times\mathrm{L}^{2,q+1}(\Omega\cap U_\delta)}\Big)\end{aligned}$$

holds by the monotone convergence theorem. Thus $(E,H) \in \overset{\circ}{\mathbf{R}}{}^q_{<-\frac{1}{2}}(\Omega) \times \mathbf{D}^{q+1}_{<-\frac{1}{2}}(\Omega)$ and it satisfies the radiation condition $(r^{-1}S+\mathrm{Id})(E,H) \in \mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>-\frac{1}{2}}(\Omega)$, i.e. (E,H) solves $\mathsf{Max}(\Lambda,\omega,F,G)$.

By the way, this shows that the principle of limiting absorption holds. The choice $(F_\ell,G_\ell) := (F,G)$ for all $\ell \in \mathbb{N}$ yields the existence of a solution of $\mathsf{Max}(\Lambda,\omega,F,G)$ and this one is unique because of (2.10).

Moreover, for $-t, s > 1/2$ the solution operator $\mathcal{L}_\omega$ maps

$$D_s(\mathcal{L}_\omega) := \big(\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)\big) \cap \mathcal{N}(\mathsf{Max},\Lambda,\omega)^\perp$$

to

$$W_t(\mathcal{L}_\omega) := \big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)\big) \cap \mathcal{N}(\mathsf{Max},\Lambda,\omega)^{\perp_\Lambda}$$

continuously. This follows by the closed graph theorem because $D_s(\mathcal{L}_\omega)$ and $W_t(\mathcal{L}_\omega)$ are Hilbert spaces by the polynomial decay of eigensolutions and $\mathcal{L}_\omega$ is closed, which is a consequence of Lemma 2.11 and the monotone convergence theorem.

It remains to contradict the contrary assumption to (2.9). To this end let $t < -1/2$ and $\big((E_\ell,H_\ell)\big)_{\ell\in\mathbb{N}} \subset \overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$ be a sequence with

$$\big\|(E_\ell,H_\ell)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)} \xrightarrow{\ell\to\infty} \infty \qquad .$$

Defining

$$\begin{aligned}(\tilde{E}_\ell,\tilde{H}_\ell) &:= \big\|(E_\ell,H_\ell)\big\|^{-1}_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)} \cdot (E_\ell,H_\ell) \qquad ,\\(\tilde{F}_\ell,\tilde{G}_\ell) &:= \big\|(E_\ell,H_\ell)\big\|^{-1}_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)} \cdot (F_\ell,G_\ell)\end{aligned}$$

we have

$$\begin{aligned}\big\|(\tilde{E}_\ell,\tilde{H}_\ell)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)} &= 1 \qquad \forall \quad \ell \in \mathbb{N} \qquad ,\\\lim_{\ell\to\infty}\big\|(\tilde{F}_\ell,\tilde{G}_\ell)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} &= 0\end{aligned}$$

as well as

$$(M + \mathrm{i}\,\omega_\ell\Lambda)(\tilde{E}_\ell,\tilde{H}_\ell) = (\tilde{F}_\ell,\tilde{G}_\ell) \qquad .$$

Following the arguments above we obtain a subsequence $\big((\tilde E_{\pi\ell},\tilde H_{\pi\ell})\big)_{\ell\in\mathbb N}$ converging in $\mathrm{L}^{2,q}_{\tilde t}(\Omega)\times\mathrm{L}^{2,q+1}_{\tilde t}(\Omega)$, $\tilde t<t$, towards $(\tilde E,\tilde H)\in\mathcal N(\mathsf{Max},\Lambda,\omega)^{\perp_\Lambda}$, which solves $\mathsf{Max}(\Lambda,\omega,0,0)$. Hence $(\tilde E,\tilde H)=(0,0)$ and Lemma 2.11 yields constants $c,\delta>0$ independent of $\sigma_{\pi\ell}$, $(\tilde F_{\pi\ell},\tilde G_{\pi\ell})$ or $(\tilde E_{\pi\ell},\tilde H_{\pi\ell})$, such that

$$\begin{aligned}1&=\big\|(\tilde E_{\pi\ell},\tilde H_{\pi\ell})\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)}\\&\le c\cdot\Big(\underbrace{\big\|(\tilde F_{\pi\ell},\tilde G_{\pi\ell})\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}}_{\xrightarrow{\ell\to\infty}0}+\underbrace{\big\|(\tilde E_{\pi\ell},\tilde H_{\pi\ell})\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_\delta)\times\mathrm{L}^{2,q+1}(\Omega\cap U_\delta)}}_{\xrightarrow{\ell\to\infty}0}\Big)\end{aligned}$$

holds true, a contradiction. ■

The polynomial decay of eigensolutions proved above and Theorem 2.10 yield

Corollary 2.18 *Let $\omega\in\mathbb R\setminus\{0\}$ and $(E,H)\in\mathcal N(\mathsf{Max},\Lambda,\omega)$. If additionally*

$$(\varepsilon,\mu)\in\mathrm{C}^{2,q}(\Xi)\times\mathrm{C}^{2,q+1}(\Xi)$$

with bounded derivatives for some exterior domain $\Xi\subset\Omega$, then

$$\begin{aligned}&\exp(t\,r)\cdot(E,H)\in\big(\overset{\circ}{\mathbf R}{}^q(\Omega)\cap\varepsilon^{-1}\mathbf D^q(\Omega)\big)\times\big(\mathbf D^{q+1}(\Omega)\cap\mu^{-1}\overset{\circ}{\mathbf R}{}^{q+1}(\Omega)\big)\quad,\\&\exp(t\,r)\cdot(E,H)\in\mathbf H^{2,q}(\tilde\Xi)\times\mathbf H^{2,q+1}(\tilde\Xi)\end{aligned}$$

hold for all $t\in\mathbb R$ and for all exterior domains $\tilde\Xi\subset\Xi$ with $\operatorname{dist}(\tilde\Xi,\partial\Xi)>0$, i.e. eigensolutions decay exponentially.

Remark 2.19 *The polynomial resp. exponential decay of eigensolutions holds for arbitrary exterior domains Ω , i.e. Ω does not need to have the* MLCP.

Remark 2.20 *If the media are homogeneous and isotropic in the outside of some ball, i.e.*

$$\operatorname{supp}\hat\Lambda\cup(\mathbb R^N\setminus\Omega)\Subset U_\rho$$

for some $\rho>0$, then

$$\operatorname{supp}(E,H)\subset\overline\Omega\cap K_\rho\qquad,\textit{ i.e.}\qquad(E,H)=(0,0)\quad\textit{in}\quad A_\rho\qquad,$$

for all $\omega\in\mathbb R\setminus\{0\}$ and $(E,H)\in\mathcal N(\mathsf{Max},\Lambda,\omega)$, since in this case (E,H) solves Helmholtz' equation

$$(\Delta+\omega^2)(E,H)=(0,0)$$

in A_ρ and therefore, by Rellich's estimate [23] or $\big[$[9], p. 59$\big]$ must vanish in A_ρ . If the principle of unique continuation holds for our Maxwell system, then even

$$\mathcal N(\mathsf{Max},\Lambda,\omega)=\big\{(0,0)\big\}\qquad.$$

Remark 2.21 *Let $\omega \in \mathbb{P} \neq \emptyset$, $d^q(\omega) := \dim \mathcal{N}(\mathsf{Max}, \Lambda, \omega)$ and $\big\{(e_\ell, h_\ell)\big\}_{\ell=1}^{d^q(\omega)}$ be some basis of $\mathcal{N}(\mathsf{Max}, \Lambda, \omega)$. Then for any $\gamma \in \mathbb{C}^{d^q(\omega)}$ we can choose a unique solution (E, H) of $\mathsf{Max}(\Lambda, \omega, F, G)$ in Theorem 2.17 (iv), such that*

$$\big\langle \Lambda(E, H), (e_\ell, h_\ell) \big\rangle = \gamma_\ell \qquad , \qquad \ell = 1, \ldots, d^q(\omega) \qquad .$$

Moreover, using the a priori estimate of the limiting absorption principle and some indirect arguments followed by the (trivial) decomposition of $\mathrm{L}^{2,q}_s(\Omega)$ from $\big[$[15], Lemma 4.1$\big]$ we are able to prove stronger estimates for the solution operator $\mathcal{L}_\omega$ as the ones given in Theorem 2.17 (v). By referring to $\big[$[13], Kapitel 4.9$\big]$ for the proofs we only present the results here.

Let Ω have the MLCP and $s, -t > 1/2$ as well as $K \Subset \mathbb{C}_+ \setminus \{0\}$.

Lemma 2.22 *There exist constants $c, \delta > 0$ and some $\hat{t} > -1/2$, such that*

$$\begin{aligned} &\big\| \mathcal{L}_\omega(F, G) \big\|_{\mathbf{R}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)} + \big\| (r^{-1} S + \mathrm{Id})\, \mathcal{L}_\omega(F, G) \big\|_{\mathrm{L}^{2,q}_{\hat{t}}(\Omega) \times \mathrm{L}^{2,q+1}_{\hat{t}}(\Omega)} \\ &\leq c \cdot \Big(\big\| (F, G) \big\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + \big\| \mathcal{L}_\omega(F, G) \big\|_{\mathrm{L}^{2,q}(\Omega \cap U_\delta) \times \mathrm{L}^{2,q+1}(\Omega \cap U_\delta)} \Big) \end{aligned}$$

holds for all $\omega \in \overline{K}$ and $(F, G) \in \big(\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)\big) \cap \mathcal{N}(\mathsf{Max}, \Lambda, \omega)^\perp$.

Corollary 2.23 *Let $\overline{K} \cap \mathbb{P} = \emptyset$. Then there exist constants $c > 0$ and $\hat{t} > -1/2$, such that the estimate*

$$\begin{aligned} &\big\| \mathcal{L}_\omega(F, G) \big\|_{\mathbf{R}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)} + \big\| (r^{-1} S + \mathrm{Id})\, \mathcal{L}_\omega(F, G) \big\|_{\mathrm{L}^{2,q}_{\hat{t}}(\Omega) \times \mathrm{L}^{2,q+1}_{\hat{t}}(\Omega)} \\ &\qquad\qquad \leq c \cdot \big\| (F, G) \big\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} \end{aligned}$$

holds true for all $\omega \in \overline{K}$ and $(F, G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$. Especially the solution operator

$$\mathcal{L}_\omega \;:\; \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \;\longrightarrow\; \mathring{\mathbf{R}}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$$

is equicontinuous w. r. t. $\omega \in \overline{K}$.

Theorem 2.24 *Let $\overline{K} \cap \mathbb{P} = \emptyset$. Then the mapping*

$$\begin{array}{rccc} \mathcal{L} & : & \overline{K} & \longrightarrow & B\big(\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega), \mathring{\mathbf{R}}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)\big) \\ & & \omega & \longmapsto & \mathcal{L}_\omega \end{array}$$

is (uniformly) continuous. $\big($Here we denote the bounded linear operators from some normed space X to some normed space Y by $B(X, Y)$.$\big)$

3 The static problem

To introduce our static solution concept we remind of the special forms $\mathring{\mathrm{B}}^q(\Omega)$, $\mathrm{B}^{q+1}(\Omega)$ from $\big[$[14], section 4$\big]$ and the 'static Maxwell property' (SMP), which guarantees their existence and also implies the MLCP. For instance, if Ω is Lipschitz homeomorphic to a smooth exterior domain, then Ω possesses the SMP. To be able to work with these forms we may assume that Ω has got the SMP and restrict our considerations to ranks $1 \leq q \leq N$.

Definition 3.1 *(E,H) is a solution of* $\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi)$ *with data*

$$(f, F, G, g) \in \mathrm{L}^{2,q-1}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q+2}_{\mathrm{loc}}(\Omega)$$

and $(\zeta, \xi) \in \mathbb{C}^{d^q} \times \mathbb{C}^{d^{q+1}}$, if and only if

$$(E,H) \in \big(\mathrm{L}^{2,q}_{>-\frac{N}{2}}(\Omega) \cap \mathring{\mathrm{R}}^q_{\mathrm{loc}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{loc}}(\Omega)\big) \\ \times \big(\mathrm{L}^{2,q+1}_{>-\frac{N}{2}}(\Omega) \cap \mu^{-1}\mathring{\mathrm{R}}^{q+1}_{\mathrm{loc}}(\Omega) \cap \mathrm{D}^{q+1}_{\mathrm{loc}}(\Omega)\big)$$

solves the electro-magneto static system

$$\begin{array}{cccccccc} \operatorname{rot} E = G & , & \operatorname{div} \varepsilon E = f & , & \langle \varepsilon E, \mathring{b}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)} = \zeta_\ell & , & \ell = 1, \ldots, d^q & , \\ \operatorname{div} H = F & , & \operatorname{rot} \mu H = g & , & \langle \mu H, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \xi_k & , & k = 1, \ldots, d^{q+1} & . \end{array}$$

Now we want to use $\big[$[14], Theorem 4.6$\big]$ in the special case $s = 0$ to solve $\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi)$. For this let ε, μ be τ-C^1-admissible with some order of decay $\tau > 0$ as well as

$${}_0\mathbb{D}^q_s(\Omega) := {}_0\mathrm{D}^q_s(\Omega) \cap \mathring{\mathrm{B}}^q(\Omega)^\perp \qquad , \qquad {}_0\mathring{\mathbb{R}}^q_s(\Omega) := {}_0\mathring{\mathrm{R}}^q_s(\Omega) \cap \mathrm{B}^q(\Omega)^\perp \qquad ,$$

where the latter is defined for $q \neq 1$. Moreover, for $q \neq 0$ we define the 'range'

$$\mathbb{W}^q_s(\Omega) := {}_0\mathbb{D}^{q-1}_s(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_s(\Omega) \times \mathbb{C}^{d^q}$$

and for $s = 0$ we put as usual ${}_0\mathbb{D}^q(\Omega) := {}_0\mathbb{D}^q_0(\Omega)$, ${}_0\mathring{\mathbb{R}}^q(\Omega) := {}_0\mathring{\mathbb{R}}^q_0(\Omega)$ as well as $\mathbb{W}^q(\Omega) := \mathbb{W}^q_0(\Omega)$.

Theorem 3.2 *For every data $(f, G, \zeta) \in \mathbb{W}^q(\Omega)$ and $(F, g, \xi) \in \mathbb{W}^{q+1}(\Omega)$ there exists a unique solution*

$$(E,H) \in \big(\mathring{\mathrm{R}}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega)\big) \times \big(\mathrm{D}^{q+1}_{-1}(\Omega) \cap \mu^{-1}\mathring{\mathrm{R}}^{q+1}_{-1}(\Omega)\big)$$

of the electro-magneto static problem $\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi)$ *and the corresponding solution operator is continuous.*

Remark 3.3 *For special data* $(0, G, 0) \in \mathbb{W}^q(\Omega)$ *,* $(F, 0, 0) \in \mathbb{W}^{q+1}(\Omega)$ *, i.e.*

$$(F, G) \in {}_0\mathbb{D}^q(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}(\Omega) \qquad ,$$

we will denote the corresponding continuous solution operator by

$$\mathcal{L}_0 : {}_0\mathbb{D}^q(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}(\Omega) \to \big(\mathring{\mathrm{R}}^q_{-1}(\Omega) \times \mathrm{D}^{q+1}_{-1}(\Omega)\big) \cap \Lambda^{-1}\big({}_0\mathbb{D}^q_{-1}(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_{-1}(\Omega)\big) \quad .$$

We note that $\mathcal{L}_0$ *even maps* ${}_0\mathbb{D}^q_s(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_s(\Omega)$ *to*

$$\big(\mathring{\mathrm{R}}^q_{s-1}(\Omega) \times \mathrm{D}^{q+1}_{s-1}(\Omega)\big) \cap \Lambda^{-1}\big({}_0\mathbb{D}^q_{s-1}(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_{s-1}(\Omega)\big)$$

continuously for all $1 - N/2 < s < N/2$ *.*

4 Low frequency scattering

To approach the low frequency asymptotics of $\mathcal{L}_\omega$ we first have to be sure that $\mathbb{P}$ does not accumulate at zero. To this end first of all we derive a representation formula for the solutions of the homogeneous, isotropic whole space problem, i.e. $\Omega = \mathbb{R}^N$ and $\Lambda = \mathrm{Id}$. Utilizing this we obtain an estimate for the whole space solution uniformly for small frequencies. Then we are able to prove a similar estimate for the solution of the general problem.

4.1 An estimate in the whole space case

Let $\Phi_{\omega,\nu}$ be the fundamental solution to the scalar Helmholtz operator in $\mathbb{R}^N$

$$\Delta + \omega^2 \qquad , \qquad \omega \in \mathbb{C}_+ \setminus \{0\} \qquad .$$

This one can be written as

$$\Phi_{\omega,\nu}(x) = \varphi_{\omega,\nu}\big(|x|\big) \qquad \text{with} \qquad \varphi_{\omega,\nu}(t) = c_N \omega^\nu t^{-\nu} H^1_\nu(\omega t) \qquad ,$$

where the constant c_N only depends on the dimension N and $H^1_\nu(z)$ represents Hankel's function of first kind to the index $\nu := (N-2)/2$. By the properties of the Hankel function $\Big($e.g. see [10] or $\big[$[9], p. 76$\big]\Big)$ $\varphi_{\omega,\nu}$ and its first derivative can be estimated by

$$\big|\varphi_{\omega,\nu}(t)\big| \leq c \cdot (t^{2-N} + t^{\frac{1-N}{2}}) \qquad , \qquad \big|\varphi'_{\omega,\nu}(t)\big| \leq c \cdot (t^{1-N} + t^{\frac{1-N}{2}}) \tag{4.1}$$

uniformly in $t \in \mathbb{R}_+$ and $\omega \in K \Subset \mathbb{C}_+$ with some constant $c > 0$ depending only on N and K .

From Remark 2.20 we have (in the case $\Omega = \mathbb{R}^N$)

$$\mathcal{N}(\mathsf{Max}, \mathrm{Id}, \omega) = \big\{(0, 0)\big\} \qquad .$$

Thus L_ω is well defined on the entire space $\mathrm{L}^{2,q}_{>\frac{1}{2}} \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}$, if we denote $\mathcal{L}_\omega$ in the special case $\Omega = \mathbb{R}^N$ and $\Lambda = \mathrm{Id}$ by L_ω. Let $\omega \in \mathbb{C}_+ \setminus \{0\}$ and $(F, G) \in \overset{\circ}{\mathrm{C}}{}^{\infty,q} \times \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}$. Looking at $(E, H) := L_\omega(F, G)$ we get

$$(E, H) \in (\mathbf{H}^{2,q}_{<-\frac{1}{2}} \cap \mathrm{C}^{\infty,q}) \times (\mathbf{H}^{2,q+1}_{<-\frac{1}{2}} \cap \mathrm{C}^{\infty,q+1})$$

by $\big[$[6], Theorem 3.6 (i)$\big]$. Applying $(M - \mathrm{i}\,\omega)$ to $(M + \mathrm{i}\,\omega)(E, H) = (F, G)$ and using $\mathrm{i}\,\omega(\operatorname{div} E, \operatorname{rot} H) = (\operatorname{div} F, \operatorname{rot} G)$ we observe, that (E, H) satisfies

$$(\Delta + \omega^2)(E, H) = (M - \mathrm{i}\,\omega - \frac{\mathrm{i}}{\omega}\Box)(F, G) =: (f, g) \in \overset{\circ}{\mathrm{C}}{}^{\infty,q} \times \overset{\circ}{\mathrm{C}}{}^{\infty,q+1} \qquad (4.2)$$

with $\Box := \Delta - M^2 = \begin{bmatrix} \operatorname{rot}\operatorname{div} & 0 \\ 0 & \operatorname{div}\operatorname{rot} \end{bmatrix}$. We obtain $(E, H) = (e, h)$, where (e, h) is the unique radiating solution of the (componentwise) problem

$$\begin{aligned} (\Delta + \omega^2)(e, h) &= (f, g) \qquad , \\ (e, h) &\in \mathbf{H}^{2,q}_{<-\frac{1}{2}} \times \mathbf{H}^{2,q+1}_{<-\frac{1}{2}} \qquad , \\ \exp(-\mathrm{i}\,\omega r) \cdot (e, h) &\in \mathrm{H}^{1,q}_{>-\frac{3}{2}} \times \mathrm{H}^{1,q+1}_{>-\frac{3}{2}} \qquad . \end{aligned}$$

For nonreal frequencies $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$ this is trivial, because $\big[$[6], Theorem 3.6 (i)$\big]$ yields $(E, H) \in \mathbf{H}^{2,q} \times \mathbf{H}^{2,q+1}$. But then $(E, H) = (e, h)$ holds for real frequencies $\omega \in \mathbb{R} \setminus \{0\}$ as well, since one receives the solutions of both radiating problems with the principle of limiting absorption.

Using the representation formula for the solutions of the scalar Helmholtz equation, which e.g. can be found in $\big[$[9], p. 78/79, Remark 4.28$\big]$, we can represent the Euclidean components of our forms $E = E_I\,\mathrm{d}x^I$ and $H = H_J\,\mathrm{d}x^J$ by

$$E_I = f_I \star \Phi_{\omega,\nu} \qquad , \qquad H_J = g_J \star \Phi_{\omega,\nu} \qquad .$$

Here we denote the scalar convolution in $\mathbb{R}^N$ by $\star$, i.e. with $\vartheta_x\psi(y) := \psi(x - y)$ we have $E_I(x) = \langle f_I, \vartheta_x\overline{\Phi_{\omega,\nu}}\rangle_{\mathrm{L}^2} = \int_{\mathbb{R}^N} f_I \cdot \vartheta_x\Phi_{\omega,\nu}\, d\lambda$ for all $x \in \mathbb{R}^N$. (λ: Lebesgue's measure in $\mathbb{R}^N$)

Defining the convolution for suitable q-forms $e = e_I\,\mathrm{d}x^I$ and $h = h_I\,\mathrm{d}x^I$ by

$$e \star h(x) := \langle e, \vartheta_x\bar{h}\rangle_{\mathrm{L}^{2,q}} \qquad \text{with} \qquad \vartheta_x h(y) := \vartheta_x h_I(y)\,\mathrm{d}y^I = (-1)^q\vartheta_x^* h(y)$$

we see that this gives nothing else than the sum of the componentwise scalar convolutions of the Euclidean coordinates

$$e \star h = e_I \star h_I \qquad .$$

Furthermore, we have for suitable forms the rule of partial integration

$$\operatorname{rot} e \star h(x) = \langle \operatorname{rot} e, \vartheta_x\bar{h}\rangle_{\mathrm{L}^{2,q+1}} = \langle e, \vartheta_x \operatorname{div}\bar{h}\rangle_{\mathrm{L}^{2,q}} = e \star \operatorname{div} h(x) \qquad . \qquad (4.3)$$

With the special forms

$$\Phi^I_{\omega,\nu} := \Phi_{\omega,\nu} \cdot \mathrm{d}x^I$$

we get the representations

$$E_I = f \star \Phi^I_{\omega,\nu} \qquad , \qquad H_J = g \star \Phi^J_{\omega,\nu} \qquad ,$$

i.e. reminding of (4.2)

$$E_I = (\operatorname{div} G - \mathrm{i}\,\omega F - \frac{\mathrm{i}}{\omega} \operatorname{rot} \operatorname{div} F) \star \Phi^I_{\omega,\nu} \qquad , \tag{4.4}$$

$$H_J = (\operatorname{rot} F - \mathrm{i}\,\omega G - \frac{\mathrm{i}}{\omega} \operatorname{div} \operatorname{rot} G) \star \Phi^J_{\omega,\nu} \qquad . \tag{4.5}$$

Our next goal is to use the partial integration formula (4.3) to remove the second derivatives from F and G. Let us look at

$$(\operatorname{div} G) \star \Phi^I_{\omega,\nu}$$

for example. Because of the compact support of (F, G) we do not have to pay attention to the integrability of $\Phi_{\omega,\nu}$ at infinity. By (4.1) we can estimate $\Phi_{\omega,\nu}$ and $\nabla\Phi_{\omega,\nu}$ in U_1 by $|\Phi_{\omega,\nu}| \le c \cdot r^{2-N}$, $|\nabla\Phi_{\omega,\nu}| \le c \cdot r^{1-N}$ and thus we have $\Phi_{\omega,\nu}$, $\nabla\Phi_{\omega,\nu} \in \mathrm{L}^1(U_1)$. With the cut-off functions

$$\psi_n(y) := \boldsymbol{\eta}\big(n \cdot |x - y|\big) \qquad , \qquad n \in \mathbb{N} \qquad ,$$

which satisfy $\big|\nabla\psi_n(y)\big| \le c \cdot |x - y|^{-1}$ uniformly in n, we have

$$\psi_n \cdot \vartheta_x \Phi_{\omega,\nu} \, , \; \nabla\psi_n \cdot \vartheta_x \Phi_{\omega,\nu} \, , \; \psi_n \cdot \nabla(\vartheta_x \Phi_{\omega,\nu}) \in \mathrm{L}^1\big(U_1(x)\big) \qquad .$$

Therefore, (4.3) yields

$$(\operatorname{div} G_n) \star \Phi^I_{\omega,\nu} = G_n \star \operatorname{rot} \Phi^I_{\omega,\nu}$$

with $G_n := \psi_n \cdot G$ and we obtain

$$(\operatorname{div} G) \star \Phi^I_{\omega,\nu} = G \star \operatorname{rot} \Phi^I_{\omega,\nu}$$

passing to the limit $n \to \infty$ by Lebesgue's dominated convergence theorem. Using these partial integrations in (4.4) and (4.5) we finally get the representations

$$E_I = G \star (\operatorname{rot} \Phi^I_{\omega,\nu}) - \mathrm{i}\,\omega F \star \Phi^I_{\omega,\nu} - \frac{\mathrm{i}}{\omega}(\operatorname{div} F) \star (\operatorname{div} \Phi^I_{\omega,\nu}) \qquad , \tag{4.6}$$

$$H_J = F \star (\operatorname{div} \Phi^J_{\omega,\nu}) - \mathrm{i}\,\omega G \star \Phi^J_{\omega,\nu} - \frac{\mathrm{i}}{\omega}(\operatorname{rot} G) \star (\operatorname{rot} \Phi^J_{\omega,\nu}) \tag{4.7}$$

for any $(F, G) \in \overset{\circ}{\mathrm{C}}{}^{\infty,q} \times \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}$ and $(E, H) = L_\omega(F, G)$. We get

Theorem 4.1 *Let $0 \neq \omega \in K \Subset \mathbb{C}_+$ and*

$$s \in (1/2, N/2) \qquad , \qquad t := s - (N+1)/2 \in (-N/2, -1/2)$$

as well as

$$(F, G) \in \mathbf{D}^q_s \times \mathbf{R}^{q+1}_s \qquad .$$

Then for $(E, H) := L_\omega(F, G)$ the representation formulas

$$\begin{aligned} E &= \big(G \star (\operatorname{rot} \Phi^I_{\omega,\nu}) - \mathrm{i}\,\omega F \star \Phi^I_{\omega,\nu} - \frac{\mathrm{i}}{\omega}(\operatorname{div} F) \star (\operatorname{div} \Phi^I_{\omega,\nu})\big) \cdot \mathrm{d}x^I \qquad , \\ H &= \big(F \star (\operatorname{div} \Phi^J_{\omega,\nu}) - \mathrm{i}\,\omega G \star \Phi^J_{\omega,\nu} - \frac{\mathrm{i}}{\omega}(\operatorname{rot} G) \star (\operatorname{rot} \Phi^J_{\omega,\nu})\big) \cdot \mathrm{d}x^J \end{aligned}$$

hold in the sense of $\mathrm{L}^{2,q}_t$ resp. $\mathrm{L}^{2,q+1}_t$. Furthermore, there exists a constant $c > 0$, such that the estimate

$$\begin{aligned} \big\| L_\omega(F, G) \big\|_{\mathbf{R}^q_t \times \mathbf{D}^{q+1}_t} \leq c \cdot \Big(&\big\| (F, G) \big\|_{\mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s} \\ &+ \frac{1}{|\omega|} \cdot \big\| (\operatorname{div} F, \operatorname{rot} G) \big\|_{\mathrm{L}^{2,q-1}_s \times \mathrm{L}^{2,q+2}_s} \Big) \end{aligned}$$

holds true uniformly in ω and (F, G).

Proof: We choose a sequence $\big((F_n, G_n)\big)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathrm{C}}{}^{\infty,q} \times \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}$ converging to (F, G) in $\mathbf{D}^q_s \times \mathbf{R}^{q+1}_s$ as $n \to \infty$. Then Theorem 2.17 (v) yields the convergence of the forms $(E_n, H_n) := L_\omega(F_n, G_n)$ to $(E, H) \in \mathbf{R}^q_t \times \mathbf{D}^{q+1}_t$ in $\mathbf{R}^q_t \times \mathbf{D}^{q+1}_t$ since $t < -1/2$.

By (4.6) and (4.7) we may represent the forms (E_n, H_n) and observe that the involved convolution kernels essentially consist of $\varphi_{\omega,\nu} \circ r$ and $\varphi'_{\omega,\nu} \circ r$. Using (4.1) these functions can be estimated by

$$\big|\varphi_{\omega,\nu}(r)\big| \, , \, \big|\varphi'_{\omega,\nu}(r)\big| \leq c \cdot (r^{2-N} + r^{1-N} + r^{\frac{1-N}{2}}) \leq c \cdot (r^{1-N} + r^{\frac{1-N}{2}})$$

uniformly in r and ω. From McOwen $\big[$[11], Lemma 1$\big]$ we obtain that integral operators with kernels like $|x - y|^{s-t-N}$ map L^2_s continuously into L^2_t, if

$$-N/2 < t < s < N/2 \qquad .$$

As a direct consequence the right hand sides of (4.6) and (4.7) define continuous linear operators from L^2_s into L^2_t. This proves the asserted representation formulas.

By the differential equation it is sufficient to estimate $\big\| L_\omega(F, G) \big\|_{\mathrm{L}^{2,q}_t \times \mathrm{L}^{2,q+1}_t}$. By the uniform boundedness of the convolution operators w.r.t. $0 \neq \omega \in K$ in the representation formulas we get the desired estimate. ■

4.2 Low frequency asymptotics

For $\gamma \in \mathbb{R}_+$ we put

$$\mathbb{C}_{+,\gamma} := \big\{ \omega \in \mathbb{C}_+ : |\omega| \leq \gamma \big\} \qquad .$$

From now on we assume that our exterior domain Ω possesses the SMP, $q \neq 0$ and ε , μ are τ-C^1-admissible with order of decay

$$\tau > (N+1)/2 \qquad .$$

We note that here it would be sufficient to demand the asymptotics

$$\hat{\varepsilon}, \hat{\mu}, \partial_n \hat{\varepsilon}, \partial_n \hat{\mu} = \mathcal{O}(r^{-\tau}) \qquad \text{as} \qquad r \to \infty \qquad , \qquad n = 1, \ldots, N \qquad .$$

Lemma 4.2 *Let $s \in (1/2, N/2)$ and $t := s - (N+1)/2 \in (-N/2, -1/2)$.*

(i) *$\mathbb{P}$ does not accumulate at zero. In particular, $\mathbb{P}$ has no accumulation point in $\mathbb{C}$ and there exists some $\tilde{\omega} > 0$, such that $\mathbb{P} \cap \mathbb{C}_{+,\tilde{\omega}} = \emptyset$.*

(ii) *$\mathcal{L}_\omega$ is well defined on the entire space $\mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$ for all $\omega \in \mathbb{C}_{+,\tilde{\omega}} \setminus \{0\}$.*

(iii) *There exist constants $c > 0$ and $0 < \hat{\omega} \leq \tilde{\omega}$, such that the estimate*

$$\begin{aligned} &\| \mathcal{L}_\omega(F,G) \|_{\mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega)} \\ &\leq c \cdot \Big(\|(F,G)\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + |\omega|^{-1} \cdot \|(\operatorname{div} F, \operatorname{rot} G)\|_{\mathrm{L}^{2,q-1}_s(\Omega) \times \mathrm{L}^{2,q+2}_s(\Omega)} \\ &\qquad + |\omega|^{-1} \cdot \sum_{\ell=1}^{d^q} \big| \langle F, \mathring{b}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)} \big| + |\omega|^{-1} \cdot \sum_{\ell=1}^{d^{q+1}} \big| \langle G, b^{q+1}_\ell \rangle_{\mathrm{L}^{2,q+1}(\Omega)} \big| \Big) \end{aligned}$$

holds true for all $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$ and $(F,G) \in \mathbf{D}^q_s(\Omega) \times \mathring{\mathbf{R}}^{q+1}_s(\Omega)$.

(iv) *Especially there exists a constant $c > 0$, such that*

$$\| \mathcal{L}_\omega(F,G) \|_{\mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega)} \leq c \cdot \|(F,G)\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)}$$

holds for all $(F,G) \in {}_0\mathbb{D}^q_s(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_s(\Omega)$ and $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$.

The $\| \cdot \|_{\mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega)}$-norms on the left hand sides of **(iii)** *and* **(iv)** *can be replaced by the natural norms in $\big(\mathring{\mathbf{R}}^q_t(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_t(\Omega)\big) \times \big(\mu^{-1}\mathring{\mathbf{R}}^{q+1}_t(\Omega) \cap \mathbf{D}^{q+1}_t(\Omega)\big)$.*

Proof: First we prove the following:

For all $\check{\omega} > 0$, $s \in (1/2, N/2)$ and $t := s - (N+1)/2$ there exist constants $c, \varrho > 0$, such that the estimate

$$\begin{aligned} &\|(E,H)\|_{\mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega)} \\ &\leq c \cdot \Big(\|(F,G)\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + \|(E,H)\|_{\mathrm{L}^{2,q}(\Omega \cap U_\varrho) \times \mathrm{L}^{2,q+1}(\Omega \cap U_\varrho)} \\ &\qquad + |\omega|^{-1} \cdot \|(\operatorname{div} F, \operatorname{rot} G)\|_{\mathrm{L}^{2,q-1}_s(A_{r_0}) \times \mathrm{L}^{2,q+2}_s(A_{r_0})} \Big) \end{aligned} \tag{4.8}$$

holds for all $\omega \in \mathbb{C}_{+,\tilde{\omega}} \setminus \{0\}$, all

$$(F,G) \in \big(\mathrm{L}^{2,q}_s(\Omega) \cap \mathbf{D}^q_s(A_{r_0})\big) \times \big(\mathrm{L}^{2,q+1}_s(\Omega) \cap \mathbf{R}^{q+1}_s(A_{r_0})\big)$$

and all solutions (E,H) of $\mathsf{Max}(\Lambda,\omega,F,G)$.

Let (E,H) be a solution to $\mathsf{Max}(\Lambda,\omega,F,G)$ and let $(\tilde{E},\tilde{H})$ be the extension by zero of $\eta(E,H)$ to $\mathbb{R}^N$. This one satisfies the radiation condition, is an element of $\mathbf{R}^q_{<-\frac{1}{2}} \times \mathbf{D}^{q+1}_{<-\frac{1}{2}}$, even of $\mathbf{H}^{1,q}_{<-\frac{1}{2}} \times \mathbf{H}^{1,q+1}_{<-\frac{1}{2}}$ by $\big[$[6], Theorem 3.6 (i)$\big]$, and solves

$$(M + \mathrm{i}\,\omega)(\tilde{E},\tilde{H}) = \eta(F,G) + C_{M,\eta}(E,H) - \mathrm{i}\,\omega\hat{\Lambda}(\tilde{E},\tilde{H}) =: (\tilde{F},\tilde{G}) \in \mathbf{D}^q_s \times \mathbf{R}^{q+1}_s$$

in $\mathbb{R}^N$ since $\tau > (N+1)/2 > s + 1/2$. Thus we obtain $(\tilde{E},\tilde{H}) = L_\omega(\tilde{F},\tilde{G})$ and Theorem 4.1 yields a constant $c > 0$ independent of ω, $(\tilde{F},\tilde{G})$ or $(\tilde{E},\tilde{H})$ with

$$\begin{aligned}
&\big\|(\tilde{E},\tilde{H})\big\|_{\mathrm{L}^{2,q}_t \times \mathrm{L}^{2,q+1}_t} \\
&\leq c \cdot \Big(\big\|(\tilde{F},\tilde{G})\big\|_{\mathrm{L}^{2,q}_s \times \mathrm{L}^{2,q+1}_s} + |\omega|^{-1} \cdot \big\|(\operatorname{div}\tilde{F},\operatorname{rot}\tilde{G})\big\|_{\mathrm{L}^{2,q-1}_s \times \mathrm{L}^{2,q+2}_s}\Big) \quad .
\end{aligned} \tag{4.9}$$

Furthermore, by the differential equations we get

$$\mathrm{i}\,\omega \operatorname{div}\varepsilon E = \operatorname{div} F \qquad , \qquad \mathrm{i}\,\omega \operatorname{rot}\mu H = \operatorname{rot} G \tag{4.10}$$

in A_{r_0} and

$$\mathrm{i}\,\omega \operatorname{div}\tilde{E} = \operatorname{div}\tilde{F} \qquad , \qquad \mathrm{i}\,\omega \operatorname{rot}\tilde{H} = \operatorname{rot}\tilde{G} \tag{4.11}$$

in $\mathbb{R}^N$. Combining (4.9) and (4.11) we have

$$\begin{aligned}
&\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega)} \\
&\leq c \cdot \Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)} + \big\|(E,H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_{r_2}) \times \mathrm{L}^{2,q+1}(\Omega\cap U_{r_2})} \\
&\qquad + \big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega) \times \mathrm{L}^{2,q+1}_{s-\tau}(\Omega)} + \big\|(\operatorname{div}\tilde{E},\operatorname{rot}\tilde{H})\big\|_{\mathrm{L}^{2,q-1}_s \times \mathrm{L}^{2,q+2}_s}\Big)
\end{aligned} \tag{4.12}$$

and using (4.10) we can estimate the last term on the right hand side by

$$\begin{aligned}
&\big\|(\operatorname{div}\tilde{E},\operatorname{rot}\tilde{H})\big\|_{\mathrm{L}^{2,q-1}_s \times \mathrm{L}^{2,q+2}_s} \\
&\leq c \cdot \Big(\big\|(E,H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_{r_2}) \times \mathrm{L}^{2,q+1}(\Omega\cap U_{r_2})} + \big\|(\operatorname{div} E,\operatorname{rot} H)\big\|_{\mathrm{L}^{2,q-1}_s(\operatorname{supp}\eta) \times \mathrm{L}^{2,q+2}_s(\operatorname{supp}\eta)}\Big) \\
&\leq c \cdot \Big(\big\|(E,H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_{r_2}) \times \mathrm{L}^{2,q+1}(\Omega\cap U_{r_2})} + \big\|(\operatorname{div}\hat{\varepsilon}E,\operatorname{rot}\hat{\mu}H)\big\|_{\mathrm{L}^{2,q-1}_s(\operatorname{supp}\eta) \times \mathrm{L}^{2,q+2}_s(\operatorname{supp}\eta)} \\
&\qquad + |\omega|^{-1} \cdot \big\|(\operatorname{div} F,\operatorname{rot} G)\big\|_{\mathrm{L}^{2,q-1}_s(A_{r_0}) \times \mathrm{L}^{2,q+2}_s(A_{r_0})}\Big) \\
&\leq c \cdot \Big(\big\|(E,H)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_{r_2}) \times \mathrm{L}^{2,q+1}(\Omega\cap U_{r_2})} + \big\|(E,H)\big\|_{\mathbf{H}^{1,q}_{s-\tau}(\operatorname{supp}\eta) \times \mathbf{H}^{1,q+1}_{s-\tau}(\operatorname{supp}\eta)} \\
&\qquad + |\omega|^{-1} \cdot \big\|(\operatorname{div} F,\operatorname{rot} G)\big\|_{\mathrm{L}^{2,q-1}_s(A_{r_0}) \times \mathrm{L}^{2,q+2}_s(A_{r_0})}\Big) \quad .
\end{aligned}$$

Inserting this estimate into (4.12), using the regularity result $\big[$[6], Corollary 3.8 (i)$\big]$, the differential equation as well as (4.10) we finally get

$$
\begin{aligned}
&\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)}\\
&\leq c\cdot\Big(\big\|(F,G)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}+\big\|(E,H)\big\|_{\mathrm{L}^{2,q}_{s-\tau}(\Omega)\times\mathrm{L}^{2,q+1}_{s-\tau}(\Omega)}\\
&\qquad\qquad+|\omega|^{-1}\cdot\big\|(\operatorname{div}F,\operatorname{rot}G)\big\|_{\mathrm{L}^{2,q-1}_s(A_{r_0})\times\mathrm{L}^{2,q+2}_s(A_{r_0})}\Big)\qquad .
\end{aligned}
$$

By $\tau>(N+1)/2$ we have $s-\tau<t$ and thus (4.8) follows by Lemma 2.7.

If we now assume that 0 is an accumulation point of $\mathbb{P}$ or the estimate in **(iii)** is false, then there would exist a sequence $(\omega_n)_{n\in\mathbb{N}}\subset\mathbb{C}_+\setminus\{0\}$ tending to zero and a data sequence $\big((F_n,G_n)\big)_{n\in\mathbb{N}}\subset\big(\mathbf{D}^q_s(\Omega)\times\overset{\circ}{\mathbf{R}}{}^{q+1}_s(\Omega)\big)\cap\mathcal{N}(\mathsf{Max},\Lambda,\omega_n)^{\perp}$ as well as a sequence of normed solutions (E_n,H_n) to $(M+\mathrm{i}\,\omega_n\Lambda)(E_n,H_n)=(F_n,G_n)$ with $\big\|(E_n,H_n)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)}=1$ and

$$
\begin{aligned}
\big\|(F_n,G_n)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)}&\xrightarrow{n\to\infty}0\quad ,\\
|\omega_n|^{-1}\cdot\big\|(\operatorname{div}F_n,\operatorname{rot}G_n)\big\|_{\mathrm{L}^{2,q-1}_s(\Omega)\times\mathrm{L}^{2,q+2}_s(\Omega)}&\xrightarrow{n\to\infty}0\quad ,\\
|\omega_n|^{-1}\cdot\big|\langle F_n,\overset{\circ}{b}{}^q_\ell\rangle_{\mathrm{L}^{2,q}(\Omega)}\big|&\xrightarrow{n\to\infty}0\qquad ,\qquad \ell=1,\dots,d^q\quad ,\\
|\omega_n|^{-1}\cdot\big|\langle G_n,b^{q+1}_k\rangle_{\mathrm{L}^{2,q+1}(\Omega)}\big|&\xrightarrow{n\to\infty}0\qquad ,\qquad k=1,\dots,d^{q+1}\quad .
\end{aligned}
$$

$\big($In the case of **(iii)** we have of course $(E_n,H_n)=\mathcal{L}_{\omega_n}(F_n,G_n)\big)$. By the differential equation we get $\mathrm{i}\,\omega_n(\operatorname{div}\varepsilon E_n,\operatorname{rot}\mu H_n)=(\operatorname{div}F_n,\operatorname{rot}G_n)$ and thus

$$
\big\|M(E_n,H_n)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)}+\big\|(\operatorname{div}\varepsilon E_n,\operatorname{rot}\mu H_n)\big\|_{\mathrm{L}^{2,q-1}_s(\Omega)\times\mathrm{L}^{2,q+2}_s(\Omega)}\xrightarrow{n\to\infty}0\quad . \tag{4.13}
$$

Consequently (E_n,H_n) is bounded in

$$
\big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega)\cap\varepsilon^{-1}\mathbf{D}^q_t(\Omega)\big)\times\big(\mu^{-1}\overset{\circ}{\mathbf{R}}{}^{q+1}_t(\Omega)\cap\mathbf{D}^{q+1}_t(\Omega)\big)
$$

and thus the MLCP yields a subsequence, which we also denote by $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$, converging for every $\tilde{t}<t$ in $\mathrm{L}^{2,q}_{\tilde{t}}(\Omega)\times\mathrm{L}^{2,q+1}_{\tilde{t}}(\Omega)$. Because of (4.13) this sequence even converges in

$$
\big(\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega)\cap\varepsilon^{-1}\mathbf{D}^q_{\tilde{t}}(\Omega)\big)\times\big(\mu^{-1}\overset{\circ}{\mathbf{R}}{}^{q+1}_{\tilde{t}}(\Omega)\cap\mathbf{D}^{q+1}_{\tilde{t}}(\Omega)\big)
$$

to a Dirichlet form, let us say

$$
(E,H)\in{}_{\varepsilon}\mathcal{H}^q_{\tilde{t}}(\Omega)\times\mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}_{\tilde{t}}(\Omega)\qquad .
$$

Since $t=s-(N+1)/2\in(-N/2,-1/2)$ we may assume w.l.o.g. $\tilde{t}\geq-N/2$. Therefore, by $\big[$[14], Lemma 3.8$\big]$ we obtain

$$
(E,H)\in{}_{\varepsilon}\mathcal{H}^q(\Omega)\times\mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}(\Omega)\qquad .
$$

For $\ell = 1, \dots, d^q$ we compute

$$0 \xleftarrow{n\to\infty} |\omega_n|^{-1} \cdot \big|\langle F_n, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}\big| = |\omega_n|^{-1} \cdot \big| \underbrace{\langle \operatorname{div} H_n, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}}_{=0} + \mathrm{i}\,\omega_n \langle \varepsilon E_n, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)} \big|$$
$$= \big|\langle \varepsilon E_n, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}\big| \xrightarrow{n\to\infty} \big|\langle \varepsilon E, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}\big| \qquad ,$$

i.e. $E \in \overset{\circ}{\mathrm{B}}{}^q(\Omega)^{\perp_\varepsilon}$. Analogously we see $H \in \mathrm{B}^{q+1}(\Omega)^{\perp_\mu}$. Thus (E, H) must vanish and finally (4.8) yields constants $c, \varrho > 0$ independent of n with

$$\begin{aligned} 1 &= \big\|(E_n, H_n)\big\|_{\mathrm{L}^{2,q}_t(\Omega)\times\mathrm{L}^{2,q+1}_t(\Omega)} \\ &\leq c \cdot \Big(\big\|(F_n, G_n)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} + |\omega_n|^{-1} \cdot \big\|(\operatorname{div} F_n, \operatorname{rot} G_n)\big\|_{\mathrm{L}^{2,q-1}_s(A_{r_0})\times\mathrm{L}^{2,q+2}_s(A_{r_0})} \\ &\qquad\quad + \big\|(E_n, H_n)\big\|_{\mathrm{L}^{2,q}(\Omega\cap U_\varrho)\times\mathrm{L}^{2,q+1}(\Omega\cap U_\varrho)} \Big) \xrightarrow{n\to\infty} 0 \qquad , \end{aligned}$$

a contradiction. ∎

We are ready to prove our second main result:

Theorem 4.3 *Let $s \in (1/2, N/2)$, $t := s - (N+1)/2 \in (-N/2, -1/2)$ and $\hat\omega$ from Lemma 4.2. Furthermore, let $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{C}_{+,\hat\omega} \setminus \{0\}$ be a sequence tending to 0 and*

$$\big((F_n, G_n)\big)_{n\in\mathbb{N}} \subset \mathbf{D}^q_s(\Omega) \times \overset{\circ}{\mathbf{R}}{}^{q+1}_s(\Omega)$$

be a data sequence, such that

$$\begin{aligned} (F_n, G_n) &\xrightarrow{n\to\infty} (F, G) && \text{in} \quad \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \qquad , \\ -\mathrm{i}\,\omega_n^{-1}(\operatorname{div} F_n, \operatorname{rot} G_n) &\xrightarrow{n\to\infty} (f, g) && \text{in} \quad \mathrm{L}^{2,q-1}_s(\Omega) \times \mathrm{L}^{2,q+2}_s(\Omega) \qquad , \\ -\mathrm{i}\,\omega_n^{-1}\langle F_n, \overset{\circ}{b}{}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)} &\xrightarrow{n\to\infty} \zeta_\ell && \text{in} \quad \mathbb{C} \quad , \quad \ell = 1, \dots, d^q \qquad , \\ -\mathrm{i}\,\omega_n^{-1}\langle G_n, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)} &\xrightarrow{n\to\infty} \xi_k && \text{in} \quad \mathbb{C} \quad , \quad k = 1, \dots, d^{q+1} \end{aligned}$$

hold. Then $\big((E_n, H_n)\big)_{n\in\mathbb{N}} := \big(\mathcal{L}_{\omega_n}(F_n, G_n)\big)_{n\in\mathbb{N}}$ converges for all $\tilde t < t$ in

$$\big(\overset{\circ}{\mathbf{R}}{}^q_{\tilde t}(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_{\tilde t}(\Omega)\big) \times \big(\mu^{-1}\overset{\circ}{\mathbf{R}}{}^{q+1}_{\tilde t}(\Omega) \cap \mathbf{D}^{q+1}_{\tilde t}(\Omega)\big)$$

to (E, H), the unique solution of the static problem $\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi)$.

Proof: From Lemma 4.2 we get the boundedness of $\big((E_n, H_n)\big)_{n\in\mathbb{N}}$ in

$$\big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_t(\Omega)\big) \times \big(\mu^{-1}\overset{\circ}{\mathbf{R}}{}^{q+1}_t(\Omega) \cap \mathbf{D}^{q+1}_t(\Omega)\big) \qquad .$$

Thus by the MLCP we can extract a subsequence, which also will be denoted by $\big((E_n, H_n)\big)_{n\in\mathbb{N}}$, such that

$$(E_n, H_n) \xrightarrow{n\to\infty} : (\tilde E, \tilde H) \qquad \text{in} \qquad \mathrm{L}^{2,q}_{\tilde t}(\Omega) \times \mathrm{L}^{2,q+1}_{\tilde t}(\Omega)$$

holds for all $\tilde{t} \in (-N/2, t)$. The differential equation

$$M(E_n, H_n) + \mathrm{i}\,\omega_n \Lambda(E_n, H_n) = (F_n, G_n)$$

and the assumptions yield

$$\begin{array}{rclcl} M(E_n, H_n) & \xrightarrow{n\to\infty} & (F, G) & \text{in} & \mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega) \quad , \\ (\operatorname{div} \varepsilon E_n, \operatorname{rot} \mu H_n) & \xrightarrow{n\to\infty} & (f, g) & \text{in} & \mathrm{L}^{2,q-1}_s(\Omega) \times \mathrm{L}^{2,q+2}_s(\Omega) \quad . \end{array}$$

For $k = 1, \ldots, d^{q+1}$ we compute

$$\langle \mu H_n, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \frac{\mathrm{i}}{\omega_n} \underbrace{\langle \operatorname{rot} E_n, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)}}_{=0} - \frac{\mathrm{i}}{\omega_n} \langle G_n, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)} \xrightarrow{n\to\infty} \xi_k$$

and analogously $\langle \varepsilon E_n, \mathring{b}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)} \xrightarrow{n\to\infty} \zeta_\ell$ for $\ell = 1, \ldots, d^q$. Thus $(\tilde{E}, \tilde{H})$ is an element of

$$\big(\mathring{\mathrm{R}}^q_{>-\frac{N}{2}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{>-\frac{N}{2}}(\Omega)\big) \times \big(\mu^{-1}\mathring{\mathrm{R}}^{q+1}_{>-\frac{N}{2}}(\Omega) \cap \mathrm{D}^{q+1}_{>-\frac{N}{2}}(\Omega)\big)$$

solving the electro-magneto static system

$$\begin{array}{rclcrcl} \operatorname{rot} \tilde{E} &=& G \quad , & & \operatorname{div} \tilde{H} &=& F \quad , \\ \operatorname{div} \varepsilon \tilde{E} &=& f \quad , & & \operatorname{rot} \mu \tilde{H} &=& g \quad , \\ \big[\langle \varepsilon \tilde{E}, \mathring{b}^q_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}\big]_{\ell=1}^{d^q} &=& \zeta \quad , & & \big[\langle \mu \tilde{H}, b^{q+1}_k \rangle_{\mathrm{L}^{2,q+1}(\Omega)}\big]_{k=1}^{d^{q+1}} &=& \xi \quad . \end{array}$$

For the difference $(e, h) := (E, H) - (\tilde{E}, \tilde{H})$ we obtain

$$(e, h) \in \big({}_{\varepsilon}\mathcal{H}^q_{>-\frac{N}{2}}(\Omega) \cap \mathring{\mathrm{B}}^q(\Omega)^{\perp_\varepsilon}\big) \times \big(\mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}_{>-\frac{N}{2}}(\Omega) \cap \mathrm{B}^{q+1}(\Omega)^{\perp_\mu}\big)$$

and even $(e, h) \in \mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)$ again by $\big[$[14], Lemma 3.8$\big]$. Thus (e, h) must vanish and because of the uniqueness of the limit $(\tilde{E}, \tilde{H}) = (E, H)$ even the whole sequence $\big((E_n, H_n)\big)_{n\in\mathbb{N}}$ must converge to (E, H) in $\mathrm{L}^{2,q}_{<t}(\Omega) \times \mathrm{L}^{2,q+1}_{<t}(\Omega)$. ■

Corollary 4.4 *Let s, t, $\hat{\omega}$ be as in Theorem 4.3 and*

$$(F, G) \in {}_0\mathbb{D}^q_s(\Omega) \times {}_0\mathring{\mathbb{R}}^{q+1}_s(\Omega) \quad .$$

Then the solutions $\mathcal{L}_\omega(F, G)$ of the time-harmonic problem $\mathsf{Max}(\Lambda, \omega, F, G)$ converge for all $\tilde{t} < t$ in $\mathring{\mathbf{R}}^q_{\tilde{t}}(\Omega) \times \mathbf{D}^{q+1}_{\tilde{t}}(\Omega)$ to $\mathcal{L}_0(F, G)$, the unique solution of the static problem $\mathsf{Max}(\Lambda, 0, 0, F, G, 0, 0, 0)$, as $\omega \in \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$ tends to zero.

By a similar indirect argument we obtain

Corollary 4.5 *Let $s \in (1/2, N/2)$, $t := s - (N+1)/2 \in (-N/2, -1/2)$, $\hat{\omega}$ from Lemma 4.2 and $B_{s,t}$ be the Banach space of bounded linear operators from the Hilbert spaces*

$$ {}_0\mathbb{D}^q_s(\Omega) \times {}_0\overset{\circ}{\mathbb{R}}{}^{q+1}_s(\Omega) \qquad \textit{to} \qquad \overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega) \qquad . $$

Then $\| \mathcal{L}_\omega \|_{B_{s,t}}$ is uniformly bounded w.r.t. $\omega \in \mathbb{C}_{+,\hat{\omega}}$ (even for $\omega = 0$!). Moreover, the mapping

$$ \begin{array}{rcccl} \mathcal{L} & : & \mathbb{C}_{+,\hat{\omega}} & \longrightarrow & B_{s,\tilde{t}} \\ & & \omega & \longmapsto & \mathcal{L}_\omega \end{array} $$

is (uniformly) continuous for all $\tilde{t} < t$.

Remark 4.6 *Clearly $\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)$ resp. $\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega) \times \mathbf{D}^{q+1}_{\tilde{t}}(\Omega)$ may be replaced by its closed subspace*

$$ \big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \times \mathbf{D}^{q+1}_t(\Omega)\big) \cap \Lambda^{-1}\big({}_0\mathbb{D}^q_t(\Omega) \times {}_0\overset{\circ}{\mathbb{R}}{}^{q+1}_t(\Omega)\big) $$

resp.

$$ \big(\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega) \times \mathbf{D}^{q+1}_{\tilde{t}}(\Omega)\big) \cap \Lambda^{-1}\big({}_0\mathbb{D}^q_{\tilde{t}}(\Omega) \times {}_0\overset{\circ}{\mathbb{R}}{}^{q+1}_{\tilde{t}}(\Omega)\big) \qquad . $$

Proof: By Lemma 4.2 (ii) and Remark 3.3 $\mathcal{L}$ is well defined. Furthermore, by Lemma 4.2 (iv) and Remark 3.3 the boundedness of $\mathcal{L}$ is clear. Theorem 2.24 yields the continuity of $\mathcal{L}$ in $\mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$. Hence we only have to prove that $\mathcal{L}$ is continuous in zero, i.e.

$$ \forall\, \epsilon > 0 \quad \exists\, 0 < \hat{\hat{\omega}} \leq \hat{\omega} \quad \forall\, \omega \in \mathbb{C}_{+,\hat{\hat{\omega}}} \setminus \{0\} \qquad \| \mathcal{L}_\omega - \mathcal{L}_0 \|_{B_{s,\tilde{t}}} < \epsilon \qquad . $$

The contrary assumption yields some $\epsilon > 0$, a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{C}_{+,\hat{\omega}} \setminus \{0\}$ tending to zero and a data sequence $\big((F_n, G_n)\big)_{n\in\mathbb{N}} \subset {}_0\mathbb{D}^q_s(\Omega) \times {}_0\overset{\circ}{\mathbb{R}}{}^{q+1}_s(\Omega)$ with norm 1, i.e. $\big\|(F_n, G_n)\big\|_{\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)} = 1$, such that

$$ \big\|(\mathcal{L}_{\omega_n} - \mathcal{L}_0)(F_n, G_n)\big\|_{\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega)\times\mathbf{D}^{q+1}_{\tilde{t}}(\Omega)} \geq \epsilon/2 \qquad . \tag{4.14} $$

Defining the forms $(e_n, h_n) := \mathcal{L}_{\omega_n}(F_n, G_n)$ and $(\hat{e}_n, \hat{h}_n) := \mathcal{L}_0(F_n, G_n)$ as well as $(E_n, H_n) := (e_n, h_n) - (\hat{e}_n, \hat{h}_n)$ we obtain that (e_n, h_n) by Lemma 4.2 (iv) and $(\hat{e}_n, \hat{h}_n)$ by Remark 3.3 and thus also (E_n, H_n) are uniformly bounded w.r.t. n in

$$ \big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_t(\Omega)\big) \times \big(\mu^{-1}\overset{\circ}{\mathbf{R}}{}^{q+1}_t(\Omega) \cap \mathbf{D}^{q+1}_t(\Omega)\big) \qquad . $$

In fact each one of the three pairs of forms is even an element of

$$ \big(\overset{\circ}{\mathbf{R}}{}^q_t(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q_t(\Omega)\big) \times \big(\mu^{-1}{}_0\overset{\circ}{\mathbf{R}}{}^{q+1}_t(\Omega) \cap \mathbf{D}^{q+1}_t(\Omega)\big) \qquad . $$

Once again by the MLCP we may extract a subsequence, which we denote also by $\big((E_n, H_n)\big)_{n\in\mathbb{N}}$, converging in $\mathrm{L}^{2,q}_{<t}(\Omega) \times \mathrm{L}^{2,q+1}_{<t}(\Omega)$. We have

$$ M(E_n, H_n) = -\,\mathrm{i}\,\omega_n \Lambda(e_n, h_n) \xrightarrow{n\to\infty} 0 \qquad \text{in} \qquad \mathrm{L}^{2,q}_t(\Omega) \times \mathrm{L}^{2,q+1}_t(\Omega) $$

and therefore $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ converges in

$$\big(\overset{\circ}{\mathbf{R}}{}^q_{<t}(\Omega)\cap\varepsilon^{-1}{}_0\mathbf{D}^q_{<t}(\Omega)\big)\times\big(\mu^{-1}{}_0\overset{\circ}{\mathbf{R}}{}^{q+1}_{<t}(\Omega)\cap\mathbf{D}^{q+1}_{<t}(\Omega)\big)$$

to some

$$(E,H)\in{}_\varepsilon\mathcal{H}^q_{\geq-\frac{N}{2}}(\Omega)\times\mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}_{\geq-\frac{N}{2}}(\Omega)={}_\varepsilon\mathcal{H}^q(\Omega)\times\mu^{-1}{}_{\mu^{-1}}\mathcal{H}^{q+1}(\Omega)\qquad.$$

Since $\mathrm{i}\,\omega_n\Lambda(e_n,h_n)=(F_n,G_n)-M(e_n,h_n)\in\overset{\circ}{\mathrm{B}}{}^q(\Omega)^\perp\times\mathrm{B}^{q+1}(\Omega)^\perp$ and by definition we get $(e_n,h_n)\,,\,(\hat{e}_n,\hat{h}_n)\,,\,(E_n,H_n)\in\overset{\circ}{\mathrm{B}}{}^q(\Omega)^{\perp_\varepsilon}\times\mathrm{B}^{q+1}(\Omega)^{\perp_\mu}$. Thus (E,H) belongs to $\overset{\circ}{\mathrm{B}}{}^q(\Omega)^{\perp_\varepsilon}\times\mathrm{B}^{q+1}(\Omega)^{\perp_\mu}$ as well and $(E,H)=(0,0)$ follows. Because $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ converges to zero in $\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega)\times\mathbf{D}^{q+1}_{\tilde{t}}(\Omega)$ for $\tilde{t}<t$ we have got a contradiction to (4.14). ■

Corollary 4.7 *Let s, t, $\hat{\omega}$, (ω_n) be as in Theorem 4.3 as well as*

$$\big((F_n,G_n)\big)_{n\in\mathbb{N}}\subset\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)\qquad,$$

which may be decomposed by $\big[$[15], Theorem 3.2 (iv)$\big]$, such that

$$(F_n,G_n)=\Lambda(F^r_n,G^d_n)+(F^d_n,G^r_n)$$

with

$$\begin{aligned}&(F^r_n,G^d_n)\in\big({}_0\overset{\circ}{\mathbb{R}}{}^q_s(\Omega)\dotplus\operatorname{Lin}\overset{\circ}{\mathrm{B}}{}^q(\Omega)\big)\times\big({}_0\mathbb{D}^{q+1}_s(\Omega)\dotplus\operatorname{Lin}\mathrm{B}^{q+1}(\Omega)\big)\qquad,\\&(F^d_n,G^r_n)\in{}_0\mathbb{D}^q_s(\Omega)\times{}_0\overset{\circ}{\mathbb{R}}{}^{q+1}_s(\Omega)\qquad.\end{aligned}$$

Moreover, let $\big((F^d_n,G^r_n)\big)_{n\in\mathbb{N}}$ converge to some (F^d,G^r) in $\mathrm{L}^{2,q}_s(\Omega)\times\mathrm{L}^{2,q+1}_s(\Omega)$ as well as $\big(-\frac{\mathrm{i}}{\omega_n}(F^r_n,G^d_n)\big)_{n\in\mathbb{N}}$ converge to some (E^r,H^d) in $\mathrm{L}^{2,q}_{\tilde{t}}(\Omega)\times\mathrm{L}^{2,q+1}_{\tilde{t}}(\Omega)$ for all $\tilde{t}<t$. Then

$$\big((E_n,H_n)\big)_{n\in\mathbb{N}}:=\big(\mathcal{L}_{\omega_n}(F_n,G_n)\big)_{n\in\mathbb{N}}$$

converges for all $\tilde{t}<t$ in $\mathrm{L}^{2,q}_{\tilde{t}}(\Omega)\times\mathrm{L}^{2,q+1}_{\tilde{t}}(\Omega)$ resp. $\overset{\circ}{\mathbf{R}}{}^q_{\tilde{t}}(\Omega)\times\mathbf{D}^{q+1}_{\tilde{t}}(\Omega)$ to

$$(E,H)=(E^r,H^d)+\mathcal{L}_0(F^d,G^r)\qquad.$$

Proof: $\big(\mathcal{L}_{\omega_n}(F^d_n,G^r_n)\big)_{n\in\mathbb{N}}$ converges to $\mathcal{L}_0(F^d,G^r)$ by Corollary 4.4. Moreover,

$$\mathcal{L}_{\omega_n}\Lambda(F^r_n,G^d_n)=-\frac{\mathrm{i}}{\omega_n}(F^r_n,G^d_n)$$

holds. ■

5 Inhomogeneous boundary data

We conclude this report with a discussion of inhomogeneous boundary data. Let ε and μ be τ-admissible with some $\tau \geq 0$ for a moment. If additionally Ω possesses a C^3-boundary (Of course, this implies the SMP.), then we know from $\big[$[6], section 3.3$\big]$ the existence of a linear and continuous tangential trace operator

$$\Gamma_t \ : \ \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega}) \ \longrightarrow \ \mathcal{R}^q(\partial\,\Omega) = \big\{ \lambda \in \mathbf{H}^{-\frac{1}{2},q}(\partial\,\Omega) : \mathrm{Rot}\,\lambda \in \mathbf{H}^{-\frac{1}{2},q+1}(\partial\,\Omega) \big\}$$

and corresponding linear and continuous tangential extension operator

$$\check{\Gamma}_t \ : \ \mathcal{R}^q(\partial\,\Omega) \ \longrightarrow \ \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega)$$

satisfying $\Gamma_t\check{\Gamma}_t = \mathrm{Id}$ on $\mathcal{R}^q(\partial\,\Omega)$. Let us also remind of the tangential and normal trace operators γ_t and γ_n mapping $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^{\frac{1}{2}}(\partial\,\Omega)$ for appropriate values of q as well as their right inverses, i.e. the extension operators $\check{\gamma}_t$ and $\check{\gamma}_n$.

Let $(F,G) \in \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) \times \mathrm{L}^{2,q+1}_{\mathrm{loc}}(\Omega)$ and $\lambda \in \mathcal{R}^q(\partial\,\Omega)$ be some boundary data. We want to discuss the solvability of the time-harmonic Maxwell system

$$(M + \mathrm{i}\,\omega\Lambda)(E,H) = (F,G) \qquad , \qquad \Gamma_t E = \lambda \tag{5.1}$$

using the results obtained so far. By definition we have

$$E_\lambda := \check{\Gamma}_t\lambda \in \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega)$$

and with the ansatz

$$(E,H) := (\tilde{E},\tilde{H}) + (E_\lambda, 0) \tag{5.2}$$

the equations (5.1) turn to

$$(M + \mathrm{i}\,\omega\Lambda)(\tilde{E},\tilde{H}) = (\tilde{F},\tilde{G}) \qquad , \qquad \Gamma_t \tilde{E} = 0 \tag{5.3}$$

with $(\tilde{F},\tilde{G}) := (F,G) - (\mathrm{i}\,\omega\varepsilon E_\lambda, \mathrm{rot}\,E_\lambda)$. Thus we are looking for $\tilde{E} \in \mathring{\mathrm{R}}^q_{\mathrm{loc}}(\overline{\Omega})$ and we can use the results from the previous sections. Moreover, for any $s \in \mathbb{R}$ we clearly have

$$(F,G) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) \quad \Longleftrightarrow \quad (\tilde{F},\tilde{G}) \in \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega)$$

and for nonreal frequencies $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $(F,G) \in \mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)$ we easily get unique square integrable time-harmonic solutions

$$\begin{aligned} (E,H) :=& \,\mathcal{L}_\omega(\tilde{F},\tilde{G}) + (E_\lambda, 0) \\ =& \,\mathcal{L}_\omega(F,G) - \mathcal{L}_\omega(\mathrm{i}\,\omega\varepsilon\check{\Gamma}_t\lambda, \mathrm{rot}\,\check{\Gamma}_t\lambda) + (\check{\Gamma}_t\lambda, 0) \in \mathbf{R}^q(\Omega) \times \mathbf{D}^{q+1}(\Omega) \quad . \end{aligned}$$

We denote the continuous solution operator by $\mathcal{S}_\omega \ : \ (F,G,\lambda) \mapsto (E,H)$ and note $\mathcal{L}_\omega = \mathcal{S}_\omega(\,\cdot\,,\,\cdot\,,0)$.

To establish a solution theory for non vanishing real frequencies $\omega \in \mathbb{R} \setminus \{0\}$ and data $(F, G) \in \mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$ with our Fredholm theory from Theorem 2.17 we consider ε and μ to be τ-admissible with order of decay $\tau > 1$. Using the ansatz (5.2) we only have to guarantee

$$(\tilde{F}, \tilde{G}) \quad \perp \quad \mathcal{N}(\mathsf{Max}, \Lambda, \omega) \qquad .$$

Let $(e, h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega)$. With $\big[$[6], Theorem 3.11 (i)$\big]$ we compute

$$\begin{aligned}
&\big\langle (\tilde{F}, \tilde{G}), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} \\
&= \big\langle (F, G), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} - \langle \operatorname{rot} E_\lambda, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E_\lambda, \mathrm{i}\,\omega \varepsilon e \rangle_{\mathrm{L}^{2,q}(\Omega)} \\
&= \big\langle (F, G), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} - \langle \operatorname{rot} E_\lambda, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle E_\lambda, \operatorname{div} h \rangle_{\mathrm{L}^{2,q}(\Omega)} \\
&= \big\langle (F, G), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} - \langle \Gamma_t E_\lambda, \gamma_n h \rangle_{\mathbf{H}^{-\frac{1}{2},q}(\partial \Omega)} \qquad .
\end{aligned}$$

Here we needed some additional regularity of h, i.e. $h \in \mathbf{H}^{1,q+1}(\Omega)$, which implies $\gamma_n h \in \mathbf{H}^{\frac{1}{2},q}(\partial \Omega)$. To guarantee this we assume additionally $\mu \in \mathrm{C}^{1,q+1}(\Omega)$ with bounded derivatives. Then we even obtain $h \in \mathbf{H}^{1,q+1}_s(\Omega)$ for all $s \in \mathbb{R}$ by $\big[$[6], Theorem 3.9 (i)$\big]$. Again we denote the linear and continuous solution operator by $\mathcal{S}_\omega : (F, G, \lambda) \mapsto (E, H)$.

This considerations yield the following solution concept for $\omega \in \mathbb{R} \setminus \{0\}$: We call (E, H) a solution to the radiation problem $\mathsf{Max}(\Lambda, \omega, F, G, \lambda)$, if and only if

- $(E, H) \in \mathbf{R}^q_{<-\frac{1}{2}}(\Omega) \times \mathbf{D}^{q+1}_{<-\frac{1}{2}}(\Omega)$,
- $(M + \mathrm{i}\,\omega \Lambda)(E, H) = (F, G)$ and $\Gamma_t E = \lambda$,
- $(r^{-1} S + \mathrm{Id})(E, H) \in \mathrm{L}^{2,q}_{>-\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>-\frac{1}{2}}(\Omega)$.

Theorem 5.1 *Let ε and μ be τ-admissible with order of decay $\tau > 1$ and $\mu \in \mathrm{C}^{1,q+1}(\Omega)$ with bounded derivatives. Then for all $\omega \in \mathbb{R} \setminus \{0\}$ and all*

$$\lambda \in \mathcal{R}^q(\partial \Omega) \qquad , \qquad (F, G) \in \mathrm{L}^{2,q}_{>\frac{1}{2}}(\Omega) \times \mathrm{L}^{2,q+1}_{>\frac{1}{2}}(\Omega)$$

there exists a solution (E, H) to $\mathsf{Max}(\Lambda, \omega, F, G, \lambda)$, if and only if

$$\big\langle (F, G), (e, h) \big\rangle_{\mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)} = \langle \lambda, \gamma_n h \rangle_{\mathbf{H}^{-\frac{1}{2},q}(\partial \Omega)}$$

holds for all $(e, h) \in \mathcal{N}(\mathsf{Max}, \Lambda, \omega)$. The solution can be chosen in a way, such that

$$(E, H) \quad \perp_\Lambda \quad \mathcal{N}(\mathsf{Max}, \Lambda, \omega) \qquad .$$

Then by this condition the solution (E, H) is uniquely determined and the corresponding solution operator

$$\mathcal{S}_\omega : (F, G, \lambda) \mapsto (E, H)$$

is continuous in the sense of Theorem 2.17 (v).

Now we need an adequate static solution theory to describe the asymptotic behaviour of $\mathcal{S}_\omega$. We call (E,H) a solution of $\mathsf{Max}(\Lambda,0,f,F,G,g,\zeta,\xi,\lambda,\varkappa)$, if and only if

$$(E,H)\in\big(\mathrm{R}^q_{>-\frac{N}{2}}(\Omega)\cap\varepsilon^{-1}\mathrm{D}^q_{>-\frac{N}{2}}(\Omega)\big)\times\big(\mu^{-1}\mathrm{R}^{q+1}_{>-\frac{N}{2}}(\Omega)\cap\mathrm{D}^{q+1}_{>-\frac{N}{2}}(\Omega)\big)$$

and

- $\operatorname{rot}E=G \quad , \qquad \operatorname{div}H=F \quad ,$
- $\operatorname{div}\varepsilon E=f \quad , \qquad \operatorname{rot}\mu H=g \quad ,$
- $\big[\langle\varepsilon E,\mathring{b}^q_\ell\rangle_{\mathrm{L}^{2,q}(\Omega)}\big]_{\ell=1}^{d^q}=\zeta \quad , \qquad \big[\langle\mu H,b^{q+1}_k\rangle_{\mathrm{L}^{2,q+1}(\Omega)}\big]_{k=1}^{d^{q+1}}=\xi \quad ,$
- $\Gamma_t E=\lambda \quad , \qquad \Gamma_t\mu H=\varkappa$

hold.

Now and for the rest of this report let $q\neq 0$ as well as ε and μ be τ-C^1-admissible with order of decay $\tau>1$ and additionally $\mu\in\mathrm{C}^{1,q+1}(\Omega)$ with bounded derivatives. From $\big[$[14], Theorem 6.1 and Remark 6.2$\big]$ we get

Theorem 5.2 *For all $f\in{}_0\mathbb{D}^{q-1}(\Omega)$, $F\in{}_0\mathbb{D}^q(\Omega)$, $\zeta\in\mathbb{C}^{d^q}$, $\xi\in\mathbb{C}^{d^{q+1}}$ and all forms $G\in{}_0\mathrm{R}^{q+1}(\Omega)$, $g\in{}_0\mathrm{R}^{q+2}(\Omega)$, $\lambda\in\mathcal{R}^q(\partial\,\Omega)$, $\varkappa\in\mathcal{R}^{q+1}(\partial\,\Omega)$ satisfying*

$$\begin{aligned}
& & \operatorname{Rot}\lambda&=\Gamma_t G \quad ,\\
& & \operatorname{Rot}\varkappa&=\Gamma_t g \quad ,\\
&\forall\,\Phi\in\mathrm{B}^{q+1}(\Omega) & \langle G,\Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}&=\langle\lambda,\gamma_n\Phi\rangle_{\mathbf{H}^{-\frac12,q}(\partial\,\Omega)} \quad ,\\
&\forall\,\Psi\in\mathrm{B}^{q+2}(\Omega) & \langle g,\Psi\rangle_{\mathrm{L}^{2,q+2}(\Omega)}&=\langle\varkappa,\gamma_n\Psi\rangle_{\mathbf{H}^{-\frac12,q+1}(\partial\,\Omega)}
\end{aligned}$$

there exists a unique solution

$$(E,H)\in\big(\mathrm{R}^q_{-1}(\Omega)\cap\varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega)\big)\times\big(\mu^{-1}\mathrm{R}^{q+1}_{-1}(\Omega)\cap\mathrm{D}^{q+1}_{-1}(\Omega)\big)$$

of $\mathsf{Max}(\Lambda,0,f,F,G,g,\zeta,\xi,\lambda,\varkappa)$. *The solution depends continuously on the data.*

Finally we are ready to prove our last result:

Theorem 5.3 *Let $\tau > (N+1)/2$ and $s \in (1/2, N/2)$, $t := s-(N+1)/2 \in (-N/2, -1/2)$ as well as $\hat\omega$ be as in Lemma 4.2. Moreover, let $(\omega_m)_{m\in\mathbb{N}} \subset \mathbb{C}_{+,\hat\omega} \setminus \{0\}$ be a sequence tending to zero and*

$$\big((F_m, G_m)\big)_{m\in\mathbb{N}} \subset \mathbf{D}^q_s(\Omega) \times \mathbf{R}^{q+1}_s(\Omega) \qquad , \qquad (\lambda_m)_{m\in\mathbb{N}} \subset \mathcal{R}^q(\partial\Omega)$$

be some data sequences with

$$\Gamma_t G_m = \operatorname{Rot}\lambda_m \qquad ,$$

such that

$$\begin{aligned}
\lambda_m &\xrightarrow{m\to\infty} \lambda && \text{in} \quad \mathcal{R}^q(\partial\Omega) && ,\\
(F_m, G_m) &\xrightarrow{m\to\infty} (F, G) && \text{in} \quad \mathrm{L}^{2,q}_s(\Omega) \times \mathrm{L}^{2,q+1}_s(\Omega) && ,\\
-\mathrm{i}\,\omega_m^{-1}(\operatorname{div} F_m, \operatorname{rot} G_m) &\xrightarrow{m\to\infty} (f, g) && \text{in} \quad \mathrm{L}^{2,q-1}_s(\Omega) \times \mathrm{L}^{2,q+2}_s(\Omega) && ,\\
-\mathrm{i}\,\omega_m^{-1}\langle F_m, \mathring{b}^q_\ell\rangle_{\mathrm{L}^{2,q}(\Omega)} &\xrightarrow{m\to\infty} \zeta_\ell && \text{in} \quad \mathbb{C} \quad , \quad \ell = 1, \dots, d^q && ,\\
-\mathrm{i}\,\omega_m^{-1}\big(\langle G_m, b^{q+1}_k\rangle_{\mathrm{L}^{2,q+1}(\Omega)} & && && \\
-\langle \lambda_m, \gamma_n b^{q+1}_k\rangle_{\mathbf{H}^{-\frac12,q}(\partial\Omega)}\big) &\xrightarrow{m\to\infty} \xi_k && \text{in} \quad \mathbb{C} \quad , \quad k = 1, \dots, d^{q+1} &&
\end{aligned}$$

hold. Then $\big((E_m, H_m)\big)_{m\in\mathbb{N}} := \big(\mathcal{S}_{\omega_m}(F_m, G_m, \lambda_m)\big)_{m\in\mathbb{N}}$ converges for all $\tilde t < t$ in

$$\big(\mathbf{R}^q_{\tilde t}(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_{\tilde t}(\Omega)\big) \times \big(\mu^{-1}\mathbf{R}^{q+1}_{\tilde t}(\Omega) \cap \mathbf{D}^{q+1}_{\tilde t}(\Omega)\big)$$

to (E, H), the unique solution of the static problem $\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi, \lambda, 0)$.

Proof: From Theorem 5.1 and (5.2) we have $(E_m, H_m) = (\tilde E_m, \tilde H_m) + (E_{\lambda_m}, 0)$ with $E_{\lambda_m} := \check\Gamma_t\lambda_m$, $(\tilde E_m, \tilde H_m) := \mathcal{L}_{\omega_m}(\tilde F_m, \tilde G_m)$ and

$$\tilde F_m := F_m - \mathrm{i}\,\omega_m\varepsilon E_{\lambda_m} \qquad , \qquad \tilde G_m := G_m - \operatorname{rot} E_{\lambda_m} \qquad .$$

Because of the compact supports of E_{λ_m} and the continuity of $\check\Gamma_t$ we have

$$E_{\lambda_m} \xrightarrow{m\to\infty} E_\lambda := \check\Gamma_t\lambda \qquad \text{in} \qquad \mathrm{R}^q_s(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_s(\Omega)$$

for all $s \in \mathbb{R}$. Moreover, $(\tilde F_m, \tilde G_m)$ fulfills the assumptions of Theorem 4.3. Thus $(\tilde E_m, \tilde H_m)$ converges for all $\tilde t < t$ in $\big(\mathring{\mathbf{R}}^q_{\tilde t}(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_{\tilde t}(\Omega)\big) \times \big(\mu^{-1}\mathring{\mathbf{R}}^{q+1}_{\tilde t}(\Omega) \cap \mathbf{D}^{q+1}_{\tilde t}(\Omega)\big)$ to $(\tilde E, \tilde H)$, the unique solution of $\mathsf{Max}(\Lambda, 0, \tilde f, \tilde F, \tilde G, \tilde g, \tilde\zeta, \tilde\xi)$ with $\tilde F = F$, $\tilde g = g$, $\tilde \xi = \xi$ and

$$\tilde G = G - \operatorname{rot} E_\lambda \quad , \quad \tilde f = f - \operatorname{div}\varepsilon E_\lambda \quad , \quad \tilde\zeta = \zeta - \big[\langle \varepsilon E_\lambda, \mathring{b}^q_\ell\rangle_{\mathrm{L}^{2,q}(\Omega)}\big]_{\ell=1}^{d^q} \quad .$$

We obtain $(E_m, H_m) \xrightarrow{m\to\infty} (E, H) := (\tilde E, \tilde H) + (E_\lambda, 0)$ with the asserted mode of convergence and clearly (E, H) is the unique solution of the static problem

$$\mathsf{Max}(\Lambda, 0, f, F, G, g, \zeta, \xi, \lambda, 0) \qquad ,$$

which completes the proof. ∎

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References

[1] Bauer, S., 'On the absence of eigenvalues of Maxwell and Lamé Systems', *Math. Meth. Appl. Sci.*, 27, (2004), 313-327.

[2] Eidus, D. M., 'The Principle of Limiting Absorption', *Math. Sb.*, 57 (99) (1962), 13-44, and *AMS Transl.*, 47 (2), (1965), 157-191.

[3] Eidus, D. M., 'On the spectra and eigenfunctions of the Schrödinger and Maxwell operators', *J. Math. Anal. Appl.*, 106, (1985), 540-568.

[4] Ikebe, T., Saito, Y., 'Limiting absorption method and absolute continuity for the Schrödinger operators', *J. Math. Kyoto Univ.*, 12, (1972), 513-542.

[5] Kress, R., 'Potentialtheoretische Randwertprobleme bei Tensorfeldern beliebiger Dimension und beliebigen Ranges', *Arch. Rational Mech. Anal.*, 47, (1972), 59-80.

[6] Kuhn, P., Pauly, D., 'Generalized Maxwell Equations in Exterior Domains I: Regularity Results, Trace Theorems and Static Solution Theory', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 7/2007, ISBN 978-951-39-2832-2, ISSN 1456-436X.

[7] Leis, R., 'Zur Theorie elektromagnetischer Schwingungen in anisotropen inhomogenen Medien', *Math. Z.*, 106, (1968), 213-224.

[8] Leis, R., 'Über die eindeutige Fortsetzbarkeit der Lösungen der Maxwellschen Gleichungen in anisotropen inhomogenen Medien', *Bull. Polyt. Inst. Jassy*, XIV (VIII), Fasc. 3-4, (1968), 119-124.

[9] Leis, R., *Initial Boundary Value Problems in Mathematical Physics*, Teubner, Stuttgart, (1986).

[10] Magnus, W., Oberhettinger, F., Soni, R. P., *Formulas and theorems for the special functions of mathematical physics*, Springer, Berlin - Heidelberg - New York, (1966).

[11] McOwen, R. C., 'Behavior of the Laplacian in weighted Sobolev spaces', *Comm. Pure Appl. Math.*, 32, (1979), 783-795.

[12] Müller, C., *Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen*, Springer, Berlin - Heidelberg - New York, (1957).

[13] Pauly, D., 'Niederfrequenzasymptotik der Maxwell-Gleichung im inhomogenen und anisotropen Außengebiet', `http://duepublico.uni-duisburg-essen.de`, Dissertation, Duisburg-**Essen**, (2003).

[14] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains III: Electro-Magneto Statics', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 9/2007, ISBN 978-951-39-2834-6, ISSN 1456-436X.

[15] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains IV: Hodge-Helmholtz Decompositions', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 10/2007, ISBN 978-951-39-2835-3, ISSN 1456-436X.

[16] Pauly, D., 'Generalized Maxwell Equations in Exterior Domains V: Low Frequency Asymptotics', *Reports of the Department of Mathematical Information Technology*, University of Jyväskylä, Series B. Scientific Computing, No. B. 11/2007, ISBN 978-951-39-2836-0, ISSN 1456-436X.

[17] Picard, R., 'Randwertaufgaben der verallgemeinerten Potentialtheorie', *Math. Meth. Appl. Sci.*, 3, (1981), 218-228.

[18] Picard, R., 'Ein vereinheitlichter Zugang für eine Klasse linearer Wellenausbreitungs-Phänomene', *Habilitationsschrift*, Bonn, (1981).

[19] Picard, R., 'On the boundary value problems of electro- and magnetostatics', *Proc. Roy. Soc. Edin.*, 92 (A), (1982), 165-174.

[20] Picard, R., 'On the low frequency asymptotics in electromagnetic theory', *J. Reine Angew. Math.*, 354, (1984), 50-73.

[21] Picard, R., 'Some decomposition theorems and their applications to non-linear potential theory and Hodge theory', *Math. Meth. Appl. Sci.*, 12, (1990), 35-53.

[22] Picard, R., Weck, N., Witsch, K. J., 'Time-Harmonic Maxwell Equations in the Exterior of Perfectly Conducting, Irregular Obstacles', *Analysis*, 21, (2001), 231-263.

[23] Rellich, F., 'Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten', *Jber. Dt. Math.-Verein.*, 53, (1943), 57-65.

[24] Vogelsang, V., 'Die absolute Stetigkeit des positiven Spektrums der Schwingungsgleichung mit oszillierendem Hauptteil', *Math. Z.*, 181, (1982), 201-213.

[25] Weck, N., 'Eine Lösungstheorie für die Maxwellschen Gleichungen auf Riemannschen Mannigfaltigkeiten mit nicht-glattem Rand', *Habilitationsschrift*, Bonn, (1972).

[26] Weck, N., Witsch, K. J., 'Generalized Spherical Harmonics and Exterior Differentiation in Weighted Sobolev Spaces', *Math. Meth. Appl. Sci.*, 17, (1994), 1017-1043.

[27] Weck, N., Witsch, K. J., 'Generalized Linear Elasticity in Exterior Domains I', *Math. Meth. Appl. Sci.*, 20, (1997), 1469-1500.

[28] Weyl, H., 'Die natürlichen Randwertaufgaben im Außenraum für Strahlungsfelder beliebiger Dimension und beliebigen Ranges', *Math. Z.*, 56, (1952), 105-119.

[29] Wloka, J., *Partielle Differentialgleichungen*, Teubner, Stuttgart, (1982).