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**Generalized Maxwell Equations**  
**in**  
**Exterior Domains IV:**  
**Hodge-Helmholtz Decompositions**

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# Generalized Maxwell Equations in Exterior Domains IV: Hodge-Helmholtz Decompositions\*

Dirk Pauly<sup>†</sup>

## Abstract

We study in detail Hodge-Helmholtz decompositions in nonsmooth exterior domains  $\Omega \subset \mathbb{R}^N$  filled with inhomogeneous and anisotropic media. We show decompositions of alternating differential forms belonging to the weighted  $L^2$ -space  $L_s^{2,q}(\Omega)$ ,  $s \in \mathbb{R}$ , into irrotational and solenoidal  $q$ -forms. These decompositions are essential tools in electro-magnetic theory for exterior domains. To the best of our knowledge these decompositions in exterior domains with nonsmooth boundaries and inhomogeneous and anisotropic media are fully new results. In the appendix we translate our results to the classical framework of vector analysis  $N = 3$  and  $q = 1, 2$ .

**Key words:** Hodge-Helmholtz decompositions, Maxwell equations, Electro-magnetic theory, Weighted Sobolev spaces

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## 1 Introduction

Hodge-Helmholtz decompositions of square integrable fields, i.e. decompositions into irrotational and solenoidal fields, are important and strong tools for solving partial differential equations, e.g. the electro-magneto static or time-harmonic system, called the static or time-harmonic Maxwell equations. Since formally  $\text{grad}$  and  $\text{div}$  resp.  $\text{curl}$  and  $\text{curl}$  are adjoint to each other and  $\text{curl grad} = 0$  and  $\text{div curl} = 0$  holds as well, the  $\varepsilon$ -orthogonal decompositions

$$\begin{aligned} L^2(\Omega) &= \mathbb{H}(\overset{\circ}{\text{curl}}_0, \Omega) \oplus_{\varepsilon} \varepsilon^{-1} \mathbb{H}(\text{div}_0, \Omega) \oplus_{\varepsilon} \varepsilon \mathcal{H}(\Omega) \quad , \\ L^2(\Omega) &= \tilde{\mathbb{H}}(\text{curl}_0, \Omega) \oplus_{\varepsilon} \varepsilon^{-1} \tilde{\mathbb{H}}(\overset{\circ}{\text{div}}_0, \Omega) \oplus_{\varepsilon} \varepsilon \tilde{\mathcal{H}}(\Omega) \quad , \end{aligned} \tag{1.1}$$

where  $\Omega \subset \mathbb{R}^3$  is a domain, are easy consequences of the projection theorem in Hilbert space. Here  $\varepsilon : \Omega \rightarrow \mathbb{R}^{3 \times 3}$  is a symmetric and uniformly bounded and positive definite matrix, which models material properties, such as the dielectricity or the permeability of the medium. Moreover,

$$\begin{aligned} \mathbb{H}(\overset{\circ}{\text{curl}}_0, \Omega) &= \{E \in L^2(\Omega) : \text{curl } E = 0 \wedge \nu \times E|_{\partial\Omega} = 0\} \cap \mathcal{H}(\Omega)^{\perp} \quad , \\ \mathbb{H}(\text{div}_0, \Omega) &= \{E \in L^2(\Omega) : \text{div } E = 0\} \cap \mathcal{H}(\Omega)^{\perp} \quad , \\ \tilde{\mathbb{H}}(\text{curl}_0, \Omega) &= \{E \in L^2(\Omega) : \text{curl } E = 0\} \cap \tilde{\mathcal{H}}(\Omega)^{\perp} \quad , \\ \tilde{\mathbb{H}}(\overset{\circ}{\text{div}}_0, \Omega) &= \{E \in L^2(\Omega) : \text{div } E = 0 \wedge \nu \cdot E|_{\partial\Omega} = 0\} \cap \tilde{\mathcal{H}}(\Omega)^{\perp} \end{aligned}$$

and

$$\begin{aligned} \varepsilon \mathcal{H}(\Omega) &= \{E \in L^2(\Omega) : \text{curl } E = 0 \wedge \text{div } \varepsilon E = 0 \wedge \nu \times E|_{\partial\Omega} = 0\} \quad , \\ \varepsilon \tilde{\mathcal{H}}(\Omega) &= \{E \in L^2(\Omega) : \text{curl } E = 0 \wedge \text{div } \varepsilon E = 0 \wedge \nu \cdot \varepsilon E|_{\partial\Omega} = 0\} \end{aligned}$$

denote the spaces of Dirichlet resp. Neumann fields, where we set  $\mathcal{H}(\Omega) = \text{Id}\mathcal{H}(\Omega)$  resp.  $\tilde{\mathcal{H}}(\Omega) = \text{Id}\tilde{\mathcal{H}}(\Omega)$ . The symbol  $\perp$  means orthogonality with respect to the  $L^2(\Omega)$ -scalar product. (See section 6.2 for further definitions of all these spaces.)

This problem may be generalized if we formulate Maxwell's equations in the framework of alternating differential forms of order  $q$ , short  $q$ -forms. This generalization not only makes things much easier, but also provides a deeper insight into the structure of the underlying problems. To remind of the electro-magnetic background it has become customary to denote the exterior derivative  $d$  by  $\text{rot}$  and the co-derivative  $\delta = \pm * d*$  by  $\text{div}$ , where  $*$  denotes the Hodge star operator. Since  $\text{rot}$  and  $\text{div}$  are (formally) skew adjoint to each other as well as  $\text{rot rot} = 0$  and  $\text{div div} = 0$  hold, the corresponding Hodge-Helmholtz decompositions of  $L^2$ - $q$ -forms read

$$L^{2,q}(\Omega) = \overset{\circ}{\mathbb{R}}^q(\Omega) \oplus_{\varepsilon} \varepsilon^{-1} \mathbb{D}^q(\Omega) \oplus_{\varepsilon} \mathcal{H}^q(\Omega) \quad , \quad (1.2)$$

where  $\Omega$  is a domain in  $\mathbb{R}^N$  or more general a Riemannian manifold of dimension  $N$ , and again are easy consequences of the projection theorem. Here  $\varepsilon$  maps  $\Omega$  to the real, linear, symmetric and uniformly bounded and positive definite transformations on  $q$ -forms. Furthermore, we denote by  $\oplus_{\varepsilon}$  the orthogonal sum with respect to the  $\langle \varepsilon \cdot, \cdot \rangle_{L^{2,q}(\Omega)}$ -scalar product, where for  $E, H \in L^{2,q}(\Omega)$

$$\langle E, H \rangle_{L^{2,q}(\Omega)} = \int_{\Omega} E \wedge * \bar{H}$$

and the bar denotes complex conjugation. Here we have

$$\begin{aligned} \overset{\circ}{\mathbb{R}}^q(\Omega) &= \{ E \in L^{2,q}(\Omega) : \text{rot } E = 0 \wedge \iota^* E = 0 \} \cap \mathcal{H}^q(\Omega)^{\perp} \quad , \\ \mathbb{D}^q(\Omega) &= \{ E \in L^{2,q}(\Omega) : \text{div } E = 0 \} \cap \mathcal{H}^q(\Omega)^{\perp} \end{aligned}$$

and the Dirichlet forms

$$\varepsilon \mathcal{H}^q(\Omega) = \{ E \in L^{2,q}(\Omega) : \text{rot } E = 0 \wedge \text{div } \varepsilon E = 0 \wedge \iota^* E = 0 \} \quad .$$

and the symbol  $\perp$  means orthogonality with respect to the  $L^{2,q}(\Omega)$ -scalar product. (See section 2 for more definitions.) For  $N = 3$  and  $q = 1$  or  $q = 2$  we obtain the two classical decompositions (1.1).

In the case of unbounded domains it is often necessary and useful to work with weighted Sobolev spaces. Especially in our efforts to determine the low frequency asymptotics of the solutions of time-harmonic Maxwell equations in exterior domains [8, 9, 10] it turned out, that decompositions of weighted  $L^2$ -spaces are essential tools. Thus, motivated by this in the present paper we want to answer the question, how the weighted  $L^2$ -space

$$L_s^{2,q}(\Omega) := \{ F \in L_{\text{loc}}^{2,q}(\Omega) : \rho^s F \in L^{2,q}(\Omega) \} \quad , \quad s \in \mathbb{R} \quad ,$$

where  $\Omega \subset \mathbb{R}^N$  is an exterior domain and  $\rho := (1 + r^2)^{1/2}$ ,  $r(x) := |x|$ ,  $x \in \mathbb{R}^N$ , may be decomposed into irrotational and solenoidal forms.

For the special case  $s = 0$  Picard showed (1.2) in [13], [11] and (in the classical framework) in [12], [6]. Moreover, for domains  $\Omega$  possessing the local Maxwell compactness property MLCP (See section 2.) he proved the representations

$$\begin{aligned} {}_0\mathring{\mathbb{R}}^q(\Omega) &= \text{rot } \mathring{\mathbb{R}}_{-1}^{q-1}(\Omega) = \text{rot} \left( \mathring{\mathbb{R}}_{-1}^{q-1}(\Omega) \cap {}_0D_{-1}^{q-1}(\Omega) \right) \quad , \\ {}_0\mathbb{D}^q(\Omega) &= \text{div } D_{-1}^{q+1}(\Omega) = \text{div} \left( D_{-1}^{q+1}(\Omega) \cap {}_0\mathring{\mathbb{R}}_{-1}^{q+1}(\Omega) \right) \quad , \end{aligned}$$

i.e. any form from  ${}_0\mathring{\mathbb{R}}^q(\Omega)$  may be represented as a rotation of a solenoidal form and any form from  ${}_0\mathbb{D}^q(\Omega)$  may be represented as a divergence of a irrotational form.

Now one may expect for arbitrary  $s \in \mathbb{R}$  the direct decompositions

$$L_s^{2,q}(\Omega) = {}_0\mathring{\mathbb{R}}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \dot{+} {}_\varepsilon\mathcal{H}_s^q(\Omega) \quad .$$

But we will see that these only hold in a small interval around zero, i.e.

$$1 - N/2 < s < N/2 - 1 \quad ,$$

since for small  $s$  we lose the directness of the decomposition and for large  $s$  the right hand side is too small.

For general  $s \in \mathbb{R} \setminus \mathbb{I}$  with the discrete set of weights

$$\mathbb{I} = \{N/2 + n : n \in \mathbb{N}_0\} \cup \{1 - N/2 - n : n \in \mathbb{N}_0\}$$

Weck and Witsch [15] showed in the special case  $\Omega = \mathbb{R}^N$  and  $\varepsilon = \text{Id}$ , where no Dirichlet forms and no boundary exist, the decompositions

$$L_s^{2,q} = \begin{cases} {}_0\mathbb{R}_s^q + {}_0\mathbb{D}_s^q & , \quad s \in (-\infty, -N/2) \\ {}_0\mathbb{R}_s^q \dot{+} {}_0\mathbb{D}_s^q & , \quad s \in (-N/2, N/2) \\ {}_0\mathbb{R}_s^q \dot{+} {}_0\mathbb{D}_s^q \dot{+} \mathcal{S}_s^q & , \quad s \in (N/2, \infty) \end{cases}$$

with

$${}_0\mathbb{R}_s^q = \text{rot } \mathbb{R}_{s-1}^{q-1} \quad , \quad {}_0\mathbb{D}_s^q = \text{div } D_{s-1}^{q+1} \quad .$$

Thereby  $\mathcal{S}_s^q$  is a finite dimensional subspace of  $C^{\infty,q}(\mathbb{R}^N \setminus \{0\})$ , the vector space of infinitely differentiable  $q$ -forms with compact supports in  $\mathbb{R}^N \setminus \{0\}$ , generated by the action of the commutator of the Laplacian and a cut-off function on the linear hull of some finitely many decaying potential forms in  $\mathbb{R}^N \setminus \{0\}$ , i.e. spherical harmonics multiplied by a negative power of  $r$  solving Laplace's equation. (Here we omit the dependence on the domain  $\mathbb{R}^N$  and denote the direct sum by  $\dot{+}$ .) We note that Weck and Witsch even decomposed the Lebesgue-Banach spaces  $L_s^{p,q}$  with  $p \in (1, \infty)$  instead of  $p = 2$ . The proof of their results uses heavily the corresponding results for the scalar Laplacian in  $\mathbb{R}^N$  developed by McOwen in [5]. For the Hilbert space case  $p = 2$  these results have been generalized to smooth ( $C^3$ ) exterior domains  $\Omega \subset \mathbb{R}^N$  by Bauer in [1]. Unfortunately by their second order approach these techniques can

not be applied to handle inhomogeneities  $\varepsilon$  and the smoothness of  $\Omega$  is essential as well.

Results in the classical case  $q = N - 1$  have been given by Specovius-Neugebauer in [14] for  $\varepsilon = \text{Id}$  and a smooth ( $C^2$ ) exterior domain  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 3$ . She considered the weaker version of (1.2), which reads as

$$L^{2,N-1}(\Omega) = {}_0\mathring{R}^{N-1}(\Omega) \oplus \text{div } D_{-1}^N(\Omega)$$

resp. in the classical language

$$L^2(\Omega) = \text{grad } H_{-1}(\text{grad}, \Omega) \oplus H(\text{div}_0, \Omega) \quad .$$

She was able to show for  $s \in \mathbb{R} \setminus \mathbb{I}$

$$L_s^2(\Omega) = \begin{cases} \text{grad } H_{s-1}(\text{grad}, \Omega) + H_s(\text{div}_0, \Omega) & , \quad s \in (-\infty, -N/2) \\ \text{grad } H_{s-1}(\text{grad}, \Omega) \dot{+} H_s(\text{div}_0, \Omega) & , \quad s \in (-N/2, N/2) \\ \text{grad } H_{s-1}(\text{grad}, \Omega) \dot{+} H_s(\text{div}_0, \Omega) \dot{+} \mathcal{S}_s & , \quad s \in (N/2, \infty) \end{cases} \quad ,$$

where  $\mathcal{S}_s$  corresponds to  $\mathcal{S}_s^{N-1}$ . We note that she proved the corresponding decompositions even for the Banach spaces  $L_s^p(\Omega)$  with  $1 < p < \infty$ . Since she used heavily trace operators and convolution techniques, her results can not be generalized to nonsmooth boundaries or inhomogeneities  $\varepsilon$ . Moreover, she showed no further decomposition of  $H_s(\text{div}_0, \Omega)$  in Neumann fields and images of curl-terms (for  $N = 3$ ), which is highly important in electro-magnetic theory.

Our main focus is to treat nonsmooth boundaries, i.e. Lipschitz boundaries or even weaker assumptions, and most of all nonsmooth inhomogeneities corresponding to anisotropic media, which are only asymptotically homogeneous. To the best of our knowledge it was an open question, if those weighted  $L^2$ -decompositions hold for inhomogeneous and anisotropic media or for nonsmooth boundaries. We will allow our transformations on  $q$ -forms  $\varepsilon$  to be  $L^\infty$ -perturbations of the identity, i.e.  $\varepsilon = \text{Id} + \hat{\varepsilon}$ , where  $\hat{\varepsilon}$  needs not to be compactly supported but decays at infinity. Moreover,  $\hat{\varepsilon}$  is not assumed to be smooth. We only require  $\hat{\varepsilon} \in C^1$  outside an arbitrary large ball. Omitting some details for this introductory remarks we will show for small  $s \in (-\infty, -N/2) \setminus \tilde{\mathbb{I}}$

$$L_s^{2,q}(\Omega) = {}_0\mathring{R}_s^q(\Omega) + \varepsilon^{-1} {}_0D_s^q(\Omega)$$

and for large  $s \in (-N/2, \infty) \setminus \tilde{\mathbb{I}}$

$$L_s^{2,q}(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp_\varepsilon} = \begin{cases} {}_0\mathring{R}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0D_s^q(\Omega) & , \quad s < N/2 \\ {}_0\mathring{R}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0D_s^q(\Omega) \dot{+} \Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q & , \quad s > N/2 \end{cases} \quad .$$

Here  $\Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q$  is a finite dimensional subspace of  $\mathring{H}_s^{1,q}(\Omega) \cap C^{1,q}(\Omega)$ , whose elements have supports outside of an arbitrary large ball. The forms from  $\bar{\mathcal{P}}_{s-2}^q$  are potential forms, i.e. solve Laplace's equation in  $\mathbb{R}^N \setminus \{0\}$ , and

$$\Delta_\varepsilon = \text{rot div} + \varepsilon^{-1} \text{div rot} \quad .$$

In the case  $\varepsilon = \text{Id}$  since  $\Delta = \text{rot div} + \text{div rot}$  (Here  $\Delta$  is to be understood componentwise.) we have

$$\Delta \eta \bar{\mathcal{P}}_{s-2}^q = C_{\Delta, \eta} \bar{\mathcal{P}}_{s-2}^q = \mathcal{S}_s^q \subset \mathring{C}^{\infty,q}(\mathbb{R}^N \setminus \{0\}) \quad ,$$

where  $C_{A,B} := AB - BA$  denotes the commutator of the operators  $A$  and  $B$ . (For details see Theorem 3.2.) Furthermore,  $L_s^{2,q}(\Omega)$  decomposes for large  $s$  into the closed subspace  $L_s^{2,q}(\Omega) \cap \varepsilon \mathcal{H}^q(\Omega)^{\perp_\varepsilon}$  and the linear hull of finitely many smooth forms, which have bounded supports. We note that for all  $t \in [-N/2, N/2 - 1)$  the spaces of Dirichlet forms  ${}_\varepsilon \mathcal{H}_t^q(\Omega)$  coincide. Moreover, for all  $s \in \mathbb{R} \setminus \mathbb{I}$  the irrotational forms in  ${}_0 \mathring{\mathbb{R}}_s^q(\Omega)$  resp. the solenoidal forms in  ${}_0 \mathbb{D}_s^q(\Omega)$  can be represented as rotations resp. divergences, i.e.

$${}_0 \mathring{\mathbb{R}}_s^q(\Omega) = \text{rot } \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \quad , \quad {}_0 \mathbb{D}_s^q(\Omega) = \text{div } \mathbb{D}_{s-1}^{q+1}(\Omega)$$

hold except of some special values of  $s$  or  $q$ . But contrarily to the case  $s = 0$  for large  $s > 1 + N/2$  we lose integrability properties, if we want to represent forms in  ${}_0 \mathring{\mathbb{R}}_s^q(\Omega)$  resp.  ${}_0 \mathbb{D}_s^q(\Omega)$  by rotations of solenoidal resp. divergences of irrotational forms. Looking at Theorem 3.5 we obtain

$$\begin{aligned} {}_0 \mathring{\mathbb{R}}_s^q(\Omega) &= \text{rot} \left( (\mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{q-1}) \cap {}_0 \mathbb{D}_{< \frac{N}{2}}^{q-1}(\Omega) \right) \quad , \\ {}_0 \mathbb{D}_s^q(\Omega) &= \text{div} \left( (\mathbb{D}_{s-1}^{q+1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{q+1}) \cap {}_0 \mathring{\mathbb{R}}_{< \frac{N}{2}}^{q+1}(\Omega) \right) \quad , \end{aligned}$$

i.e. the representing solenoidal resp. irrotational forms no longer belong to  $L_{s-1}^{2,q\mp 1}(\Omega)$  but only to  $L_t^{2,q\mp 1}(\Omega)$  for all  $t < N/2$ . (For details see Theorems 3.4, 3.5 and 3.8.)

If we project onto the orthogonal complement of  ${}_\varepsilon \mathcal{H}_{-s}^q(\Omega)$ , i.e. of more Dirichlet forms, we finally obtain even for large  $s > N/2$

$$L_s^{2,q}(\Omega) \cap {}_\varepsilon \mathcal{H}_{-s}^q(\Omega)^{\perp_\varepsilon} = {}_0 \mathring{\mathbb{R}}_s^q(\Omega) \oplus_\varepsilon \varepsilon^{-1} {}_0 \mathbb{D}_s^q(\Omega) \quad .$$

In Appendix A we give a explicit representation of the low weighted Dirichlet spaces  ${}_\varepsilon \mathcal{H}_{-s}^q(\Omega)$  and compute their finite dimensions.

## 2 Definitions and preliminaries

We consider an exterior domain  $\Omega \subset \mathbb{R}^N$ , i.e.  $\mathbb{R}^N \setminus \Omega$  is compact, as a special smooth Riemannian manifold of dimension  $3 \leq N \in \mathbb{N}$ , and fix a radius  $r_0$  and some radii



$r_n := 2^n r_0$ ,  $n \in \mathbb{N}$ , such that  $\mathbb{R}^N \setminus \Omega$  is a compact subset of the open ball around zero  $U_{r_0} := \{x \in \mathbb{R}^N : |x| < r_0\}$ . Moreover, we choose a cut-off function  $\eta$ , such that [4, (3.1), (3.2), (3.3)] hold. Putting  $A_r := \{x \in \mathbb{R}^N : |x| > r\}$  we note that  $\text{supp } \nabla \eta \subset \overline{A_{r_1}} \cap \overline{U_{r_2}}$ .

Throughout this paper we will use the notations from [4, 8, 9]. Considering alternating differential forms of rank  $q \in \mathbb{Z}$  (short  $q$ -forms) we denote the exterior derivative d by  $\text{rot}$  and the co-derivative  $\delta = (-1)^{(q-1)N} * d *$  ( $*$ : Hodge star operator) by  $\text{div}$ , where both of them are applied to  $q$ -forms, to remind of the electro-magnetic background. On  $\mathring{C}^{\infty,q}(\Omega)$  (the vector space of all  $C^\infty$ - $q$ -forms with compact supports in  $\Omega$ ) we have a scalar product

$$\langle \Phi, \Psi \rangle_{L^{2,q}(\Omega)} := \int_{\Omega} \Phi \wedge * \bar{\Psi} \quad \forall \quad \Phi, \Psi \in \mathring{C}^{\infty,q}(\Omega)$$

and hence an induced norm  $\| \cdot \|_{L^{2,q}(\Omega)} := \langle \cdot, \cdot \rangle_{L^{2,q}(\Omega)}^{1/2}$ . Thus, we may define (taking the closure in the latter norm)

$$L^{2,q}(\Omega) := \overline{\mathring{C}^{\infty,q}(\Omega)} \quad ,$$

the Hilbert space of all square integrable  $q$ -forms on  $\Omega$ . Moreover, on  $\mathring{C}^{\infty,q}(\Omega)$  the linear operators  $\text{rot}$  and  $\text{div}$  are formally skew adjoint to each other due to Stokes' theorem, i.e.

$$\langle \text{rot } \Phi, \Psi \rangle_{L^{2,q+1}(\Omega)} = -\langle \Phi, \text{div } \Psi \rangle_{L^{2,q}(\Omega)}$$

for all  $(\Phi, \Psi) \in \mathring{C}^{\infty,q}(\Omega) \times \mathring{C}^{\infty,q+1}(\Omega)$ , which gives rise to weak formulations of  $\text{rot}$  and  $\text{div}$ . Using these and the weight function  $\rho(r) := (1 + r^2)^{1/2}$  for  $s \in \mathbb{R}$  we introduce the following weighted Hilbert spaces (endowed with their natural norms) of  $q$ -forms

$$\begin{aligned} L_s^{2,q}(\Omega) &:= \{E \in L_{\text{loc}}^{2,q}(\Omega) : \rho^s E \in L^{2,q}(\Omega)\} \quad , \\ R_s^q(\Omega) &:= \{E \in L_s^{2,q}(\Omega) : \text{rot } E \in L_{s+1}^{2,q+1}(\Omega)\} \quad , \\ D_s^q(\Omega) &:= \{H \in L_s^{2,q}(\Omega) : \text{div } H \in L_{s+1}^{2,q-1}(\Omega)\} \quad . \end{aligned}$$

(All these spaces equal zero if  $q \notin \{0, \dots, N\}$ .) Furthermore, we introduce the Hilbert space

$$\mathring{R}_s^q(\Omega) := \overline{\mathring{C}^{\infty,q}(\Omega)} \quad ,$$

(The closure is taken in  $R_s^q(\Omega)$ .) which generalizes the boundary condition of vanishing tangential components of the forms at the boundary  $\partial \Omega$ . More precisely this generalizes the boundary condition  $\iota^* E = 0$ , which means that the pull-back of  $E$  on the boundary of  $\Omega$  (considered as a  $(N - 1)$ -dimensional Riemannian submanifold of  $\overline{\Omega}$ ) vanishes. Here  $\iota : \partial \Omega \hookrightarrow \overline{\Omega}$  denotes the natural embedding.

A lower left corner index 0 indicates vanishing rotation resp. divergence.

For weighted Sobolev spaces  $V_s$ ,  $s \in \mathbb{R}$ , we define

$$V_{<t} := \bigcap_{s < t} V_s \quad .$$

We only consider exterior domains  $\Omega$ , which possess the local Maxwell compactness property MLCP, i.e. for all  $q$  and all  $t < s$  the embeddings

$$\mathring{R}_s^q(\Omega) \cap D_s^q(\Omega) \hookrightarrow L_t^{2,q}(\Omega)$$

are compact. (See [4, Definition 3.1, Remark 3.2] and the literature cited there.)

We assume our transformations to be  $\tau$ -admissible resp.  $\tau$ - $C^1$ -admissible as defined in [8, Definition 2.1 and 2.2].

Let  $\varepsilon$  be a  $\tau$ - $C^1$ -admissible transformation on  $q$ -forms with some  $\tau > 0$ . We need the finite dimensional vector space of weighted Dirichlet forms

$${}_\varepsilon \mathcal{H}_t^q(\Omega) = {}_0 \mathring{R}_t^q(\Omega) \cap \varepsilon^{-1} {}_0 D_t^q(\Omega) \quad , \quad t \in \mathbb{R} \quad .$$

(Here we neglect the indices  $\varepsilon$  or  $t$  in the cases  $\varepsilon = \text{Id}$  or  $t = 0$ .) Citing [9, Lemma 3.8] we have

$${}_\varepsilon \mathcal{H}_{-\frac{N}{2}}^q(\Omega) = {}_\varepsilon \mathcal{H}^q(\Omega) = {}_\varepsilon \mathcal{H}_{<\frac{N}{2}-1}^q(\Omega)$$

and even  ${}_\varepsilon \mathcal{H}^q(\Omega) = {}_\varepsilon \mathcal{H}_{<\frac{N}{2}}^q(\Omega)$  if  $q \notin \{1, N-1\}$ . Thus,

$${}_\varepsilon \mathcal{H}^q(\Omega) \subset L_{-s}^{2,q}(\Omega)$$

for  $s > 1 - N/2$  and even for  $s > -N/2$  if  $q \notin \{1, N-1\}$ .

Furthermore, for  $\mathbb{R} \ni s > 1 - N/2$  we introduce the (then well defined) Hilbert spaces

$${}_0 \mathbb{D}_s^q(\Omega) = {}_0 D_s^q(\Omega) \cap {}_\varepsilon \mathcal{H}^q(\Omega)^\perp \quad , \quad {}_0 \mathring{\mathbb{R}}_s^q(\Omega) = {}_0 \mathring{R}_s^q(\Omega) \cap {}_\varepsilon \mathcal{H}^q(\Omega)^{\perp_\varepsilon} \quad , \quad (2.1)$$

where we denote by  $\perp_\varepsilon$  the orthogonality with respect to the  $\langle \varepsilon \cdot, \cdot \rangle_{L^{2,q}(\Omega)}$ -scalar product, i.e. the duality between  $L_t^{2,q}(\Omega)$  and  $L_{-t}^{2,q}(\Omega)$ ,  $t \in \mathbb{R}$ . If  $\varepsilon = \text{Id}$  we simply write  $\perp := \perp_{\text{Id}}$ . The restrictions upon the weights  $s$  guarantee  ${}_\varepsilon \mathcal{H}^q(\Omega) \subset L_{-s}^{2,q}(\Omega)$ . But if  $q \notin \{1, N-1\}$  also  ${}_\varepsilon \mathcal{H}^q(\Omega) \subset L_{<\frac{N}{2}}^{2,q}(\Omega)$  holds and these definitions extend to  $s > -N/2$ . Since there are no Dirichlet forms (with  $t = 0$ !) for  $q \in \{0, N\}$  in these special cases the definitions (2.1) may be extended to all  $s \in \mathbb{R}$  and we have

$$\begin{aligned} {}_0 \mathring{\mathbb{R}}_s^0(\Omega) = {}_0 \mathring{R}_s^0(\Omega) = \{0\} \quad , \quad {}_0 \mathring{\mathbb{R}}_s^N(\Omega) = {}_0 \mathring{R}_s^N(\Omega) = L_s^{2,N}(\Omega) \quad , \\ {}_0 \mathbb{D}_s^0(\Omega) = {}_0 D_s^0(\Omega) = L_s^{2,0}(\Omega) \quad , \quad {}_0 \mathbb{D}_s^N(\Omega) = {}_0 D_s^N(\Omega) = \begin{cases} \{0\} & , s \geq -N/2 \\ \text{Lin}\{*\mathbf{1}\} & , s < -N/2 \end{cases} \quad . \end{aligned}$$

Moreover, there are some other characterizations of these spaces. We remind of the finitely many special smooth forms

$$\mathring{B}^q(\Omega) \subset {}_0 \mathring{R}^q(\Omega) \quad \text{and} \quad B^q(\Omega) \subset {}_0 D^q(\Omega)$$

presented in [9, section 4], which have compact resp. bounded supports in  $\Omega$  and the properties

$${}_{\varepsilon}\mathcal{H}^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp\varepsilon} = {}_{\varepsilon}\mathcal{H}^q(\Omega) \cap B^q(\Omega)^{\perp} = \{0\} \quad . \quad (2.2)$$

We note in passing

$$\dim {}_{\varepsilon}\mathcal{H}^q(\Omega) = \dim \mathcal{H}^q(\Omega) = \#\mathring{B}^q(\Omega) = \#B^q(\Omega) = d^q \in \mathbb{N}_0 \quad .$$

Using [9, Corollary 4.4] we see in fact that

$$\begin{aligned} {}_0\mathbb{D}_s^q(\Omega) &= {}_0D_s^q(\Omega) \cap \mathcal{H}^q(\Omega)^{\perp} = {}_0D_s^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp} \quad , \\ {}_0\mathring{\mathbb{R}}_s^q(\Omega) &= {}_0\mathring{R}_s^q(\Omega) \cap \mathcal{H}^q(\Omega)^{\perp} = {}_0\mathring{R}_s^q(\Omega) \cap B^q(\Omega)^{\perp} \end{aligned}$$

do not depend on the transformation  $\varepsilon$ . Since  $B^q(\Omega)$  is only defined for  $q \neq 1$  the last characterization in the second equation holds only for  $q \neq 1$ . Now the definitions of  ${}_0\mathbb{D}_s^q(\Omega)$  and  ${}_0\mathring{\mathbb{R}}_s^q(\Omega)$  extend to arbitrary weights  $s \in \mathbb{R}$  because the forms  $\mathring{B}^q(\Omega)$ ,  $B^q(\Omega)$  have bounded supports. We say  $\Omega$  possesses the **SMP**, if and only if  $\Omega$  has the **MLCP** and the forms  $\mathring{B}^q(\Omega)$ ,  $B^q(\Omega)$  exist, which is guaranteed for Lipschitz domains  $\Omega$ , for instance. (See [9, section 4] and the literature cited there.) We may choose  $r_0$ , such that  $\text{supp } \eta \cap \text{supp } b = \emptyset$  for all  $b \in \mathring{B}^q(\Omega) \cup B^q(\Omega)$  and all  $q$ .

Finally for  $s > 1 - N/2$  or  $s > -N/2$  and  $q \notin \{1, N-1\}$  we put

$${}_{\varepsilon}\mathbb{L}_s^{2,q}(\Omega) := L_s^{2,q}(\Omega) \cap {}_{\varepsilon}\mathcal{H}^q(\Omega)^{\perp\varepsilon} \quad .$$

We also need the negative ‘tower forms’

$${}^{-}D_{\sigma,m}^{q,\ell} \quad , \quad {}^{-}R_{\sigma,m}^{q,\ell}$$

for the values  $\ell = 0, 1, 2$  and  $\sigma \in \mathbb{N}_0$ ,  $m \in \{1, \dots, \mu_{\sigma}^q\}$  from [9, section 2], which are essentially harmonic polynomials except of a multiplication by some negative integer power of  $r$ . These forms are homogeneous of degree  ${}^{-}h_{\sigma}^{\ell} := \ell - \sigma - N$ , belong to  $C^{\infty,q}(\mathbb{R}^N \setminus \{0\})$  and satisfy

$$\begin{aligned} \text{rot } {}^{-}D_{\sigma,m}^{q,0} &= 0 \quad , & \text{div } {}^{-}R_{\sigma,m}^{q+1,0} &= 0 \quad , \\ \text{div } {}^{-}D_{\sigma,m}^{q,\ell} &= 0 \quad , & \text{rot } {}^{-}R_{\sigma,m}^{q+1,\ell} &= 0 \quad , \\ \text{rot } {}^{-}D_{\sigma,m}^{q,k} &= {}^{-}R_{\sigma,m}^{q+1,k-1} \quad , & \text{div } {}^{-}R_{\sigma,m}^{q+1,k} &= {}^{-}D_{\sigma,m}^{q,k-1} \quad , \end{aligned}$$

where  $\ell = 0, 1, 2$  and  $k = 1, 2$ . (We note briefly that we need the positive tower forms of height zero  ${}^{+}D_{\sigma,m}^{q,0}$ ,  ${}^{+}R_{\sigma,m}^{q,0}$  in our proofs as well. But they are not required to formulate our results.) From [9, Remark 2.5] we have for all  $\sigma \in \mathbb{N}_0$ ,  $m \in \{1, \dots, \mu_{\sigma}^q\}$  and all  $\ell = 0, 1, 2$  as well as  $k \in \mathbb{N}_0$

$${}^{-}D_{\sigma,m}^{q,\ell} \in L_s^{2,q}(A_1) \quad \Leftrightarrow \quad {}^{-}D_{\sigma,m}^{q,\ell} \in H_s^{k,q}(A_1) \quad \Leftrightarrow \quad s < N/2 + \sigma - \ell \quad ,$$

which completely determines the integrability properties of our tower forms at infinity. The same integrability holds true for  ${}^-R_{\sigma,m}^{q,\ell}$ . Moreover, the ground forms (forms of height 0), which only occur for  $1 \leq q \leq N-1$ , are linearly dependent, i.e. we have

$$\alpha_\sigma^q \cdot {}^-R_{\sigma,m}^{q,0} + i \alpha_\sigma^{q'} \cdot {}^-D_{\sigma,m}^{q,0} = 0 \quad , \quad (2.3)$$

where  $\alpha_\sigma^q := (q + \sigma)^{1/2}$  and  $q' := N - q$ . This motivates to define the harmonic tower forms

$$H_{\sigma,m}^q := \alpha_\sigma^q \cdot {}^-R_{\sigma,m}^{q,0} = -i \alpha_\sigma^{q'} \cdot {}^-D_{\sigma,m}^{q,0} \quad (2.4)$$

and moreover the potential tower forms

$$P_{\sigma,m}^q := \alpha_\sigma^q \cdot {}^-R_{\sigma,m}^{q,2} + i \alpha_\sigma^{q'} \cdot {}^-D_{\sigma,m}^{q,2} \quad , \quad (2.5)$$

since the tower forms of height 0 and 1 (and then clearly the harmonic tower forms as well) and then also the potential tower forms  $P_{\sigma,m}^q$  (of height 2) satisfy

$$\Delta {}^-D_{\sigma,m}^{q,\ell} = \Delta {}^-R_{\sigma,m}^{q,\ell} = \Delta H_{\sigma,m}^q = \Delta P_{\sigma,m}^q = 0 \quad , \quad \ell = 0, 1 \quad ,$$

because  $\Delta = \text{rot div} + \text{div rot}$ . Here  $\Delta$  denotes the componentwise scalar Laplacian in Euclidean coordinates. We note that  $P_{\sigma,m}^q = 0$  if  $q \in \{0, N\}$ . Furthermore, for  $s \in \mathbb{R}$  and  $\ell = 0, 1, 2$  we introduce the finite dimensional vector spaces

$$\begin{aligned} \bar{\mathcal{D}}_s^{q,\ell} &:= \text{Lin} \left\{ {}^-D_{\sigma,m}^{q,\ell} : {}^-D_{\sigma,m}^{q,\ell} \notin L_s^{2,q}(A_1) \right\} = \text{Lin} \left\{ {}^-D_{\sigma,m}^{q,\ell} : \sigma \leq s - N/2 + \ell \right\} \quad , \\ \bar{\mathcal{R}}_s^{q,\ell} &:= \text{Lin} \left\{ {}^-R_{\sigma,m}^{q,\ell} : {}^-R_{\sigma,m}^{q,\ell} \notin L_s^{2,q}(A_1) \right\} = \text{Lin} \left\{ {}^-R_{\sigma,m}^{q,\ell} : \sigma \leq s - N/2 + \ell \right\} \quad , \\ \bar{\mathcal{H}}_s^q &:= \text{Lin} \left\{ H_{\sigma,m}^q : H_{\sigma,m}^q \notin L_s^{2,q}(A_1) \right\} = \text{Lin} \left\{ H_{\sigma,m}^q : \sigma \leq s - N/2 \right\} \quad , \\ \bar{\mathcal{P}}_s^q &:= \text{Lin} \left\{ P_{\sigma,m}^q : P_{\sigma,m}^q \notin L_s^{2,q}(A_1) \right\} = \text{Lin} \left\{ P_{\sigma,m}^q : \sigma \leq s - N/2 + 2 \right\} \quad . \end{aligned}$$

(Here we set  $\text{Lin } \emptyset := \{0\}$ .) We note

$$\bar{\mathcal{H}}_s^q = \bar{\mathcal{P}}_{s-2}^q = \bar{\mathcal{D}}_{s-\ell}^{q,\ell} = \bar{\mathcal{R}}_{s-\ell}^{q,\ell} = \{0\} \quad \iff \quad s < N/2 \quad .$$

Unfortunately due to the fact that  $\text{rot } r^{2-N}$  (the gradient of  $r^{2-N}$  in classical terms) is irrotational and solenoidal but is itself no divergence, there exist four exceptional tower forms. These are (up to constants)

$$\check{P}^0 := r^{2-N} = {}^-D_{0,1}^{0,2} \quad , \quad \check{H}^1 := \text{rot } \check{P}^0 = r^{1-N} dr = {}^-R_{0,1}^{1,1}$$

and their duals  $\check{P}^N := * \check{P}^0 = {}^-R_{0,1}^{N,2}$ ,  $\check{H}^{N-1} := * \check{H}^1 = {}^-D_{0,1}^{N-1,1}$ . Of course we have  $\text{div } \check{P}^N = \check{H}^{N-1}$ . Following the construction of the regular tower forms we define for  $s \geq N/2 - 2$

$$\check{\mathcal{P}}_s^0 := \check{\mathcal{P}}_s^0 := \text{Lin} \{ \check{P}^0 \} \quad , \quad \check{\mathcal{P}}_s^N := \check{\mathcal{P}}_s^N := \text{Lin} \{ \check{P}^N \}$$

and for  $s \geq N/2 - 1$

$$\check{\mathcal{H}}_s^1 := \check{\mathcal{H}}_s^1 := \text{Lin} \{ \check{H}^1 \} \quad , \quad \check{\mathcal{H}}_s^{N-1} := \check{\mathcal{H}}_s^{N-1} := \text{Lin} \{ \check{H}^{N-1} \} \quad .$$

For all other values of  $s$  and  $q$  we put  $\check{\mathcal{P}}_s^q := \{0\}$  and  $\check{\mathcal{H}}_s^q := \{0\}$ .

As described, for instance, in [9, section 3] for  $s \in \mathbb{R}$  we consider vector spaces

$$V_s^q \dot{+} \eta \mathcal{V}_s^q \quad , \quad (\dot{+} : \text{direct sum})$$

where  $V_s^q \subset L_s^{2,q}(\Omega)$  is some Hilbert space and  $\mathcal{V}_s^q$  is some finite subset of the tower forms, e.g.  $V_s^q = \mathring{R}_s^q(\Omega) \cap D_s^q(\Omega)$  and  $\mathcal{V}_s^q = \bar{\mathcal{H}}_s^q$ . On  $V_s^q \dot{+} \eta \mathcal{V}_s^q$  we define a scalar product, such that

- in  $V_s^q$  the original scalar product is kept,
- $\eta \mathcal{V}_s^q$  is an orthonormal system,
- the sum  $V_s^q \dot{+} \eta \mathcal{V}_s^q = V_s^q \boxplus \eta \mathcal{V}_s^q$  is orthogonal.

As already indicated we denote the orthogonal sum with respect to this new inner product by  $\boxplus$  and clearly  $V_s^q \boxplus \eta \mathcal{V}_s^q$  is a Hilbert space since  $\mathcal{V}_s^q$  is finite.

### 3 Results

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) be an exterior domain as introduced in the last section with the SMP or the MLCP depending on whether the forms  $\mathring{B}^q(\Omega), B^q(\Omega)$  are involved in our considerations or not. Recalling from [9, section 3] the set of special weights  $\mathbb{I}$  we put

$$\tilde{\mathbb{I}} := \mathbb{I} - 1 = \{N/2 + n - 1 : n \in \mathbb{N}_0\} \cup \{-N/2 - n : n \in \mathbb{N}_0\}$$

and from now on we make the following general assumptions:

- $s \in \mathbb{R} \setminus \tilde{\mathbb{I}}$ , i.e.  $s + 1 \in \mathbb{R} \setminus \mathbb{I}$ , i.e. for all  $n \in \mathbb{N}_0$

$$s \neq n + N/2 - 1 \quad \text{and} \quad s \neq -n - N/2 \quad .$$

- $\varepsilon$ :  $\tau$ - $C^1$ -admissible transformation on  $q$ -forms with some  $\tau = \tau_{s+1}$  satisfying

$$\tau > \max\{0, s + 1 - N/2\} \quad \text{and} \quad \tau \geq -s - 1 \quad ,$$

i.e.

$$\tau \begin{cases} \geq -s - 1 & , s \in (-\infty, -1) \\ > 0 & , s \in [-1, N/2 - 1] \\ > s + 1 - N/2 & , s \in (N/2 - 1, \infty) \end{cases} .$$

- $\nu, \mu$ :  $\tilde{\tau}$ - $C^1$ -admissible transformation on  $(q - 1)$ - resp  $(q + 1)$ -forms with some  $\tilde{\tau} = \tau_s$  satisfying

$$\tilde{\tau} > \max\{0, s - N/2\} \quad \text{and} \quad \tilde{\tau} \geq -s \quad ,$$

i.e.

$$\tilde{\tau} \begin{cases} \geq -s & , s \in (-\infty, 0) \\ > 0 & , s \in [0, N/2] \\ > s - N/2 & , s \in (N/2, \infty) \end{cases} .$$

If  $1 - N/2 < s < N/2 - 1$  or  $-N/2 < s < N/2$  and  $q \notin \{1, N - 1\}$  the 'trivial' orthogonal decomposition

$$L_s^{2,q}(\Omega) = {}_\varepsilon L_s^{2,q}(\Omega) \oplus_\varepsilon {}_\varepsilon \mathcal{H}^q(\Omega) \quad (3.1)$$

holds. We will denote the orthogonality with respect to the  $\langle \varepsilon \cdot, \cdot \rangle_{L^{2,q}(\Omega)}$ -scalar product or  $L_s^{2,q}(\Omega)$ - $L_{-s}^{2,q}(\Omega)$ -duality by  $\oplus_\varepsilon$  and put  $\oplus = \oplus_{\text{Id}}$ .

The first lemma shows how one may get rid of Dirichlet forms even for larger weights.

**Lemma 3.1** *Let  $s > 1 - N/2$ . Then the direct decompositions*

$$L_s^{2,q}(\Omega) = {}_\varepsilon L_s^{2,q}(\Omega) \dot{+} \text{Lin } \mathring{B}^q(\Omega) \quad , \quad L_s^{2,q}(\Omega) = {}_\varepsilon L_s^{2,q}(\Omega) \dot{+} \varepsilon^{-1} \text{Lin } B^q(\Omega)$$

hold, where the latter is only defined for  $q \neq 1$ . If  $q \notin \{1, N - 1\}$  this decompositions hold for  $s > -N/2$  as well.

To formulate our main decomposition result we need the operator (a perturbation of the Laplacian  $\Delta$ )

$$\Delta_\varepsilon := \text{rot div} + \varepsilon^{-1} \text{div rot} = \Delta + \check{\varepsilon} \text{div rot} \quad ,$$

where  $\varepsilon^{-1} =: \text{Id} + \check{\varepsilon}$  is also  $\tau$ - $C^1$ -admissible. We obtain

**Theorem 3.2** *The following decompositions hold:*

(i) *If  $s < -N/2$ , then*

$$L_s^{2,q}(\Omega) = {}_0 \mathring{R}_s^q(\Omega) + \varepsilon^{-1} {}_0 D_s^q(\Omega)$$

*and the intersection equals the finite dimensional space of Dirichlet forms  ${}_\varepsilon \mathcal{H}_s^q(\Omega)$ . Moreover,*

$$L_s^{2,q}(\Omega) = {}_0 \mathring{R}_s^q(\Omega) + \varepsilon^{-1} {}_0 \mathbb{D}_s^q(\Omega)$$

*and for  $q \neq 1$  even*

$$L_s^{2,q}(\Omega) = {}_0 \mathring{\mathbb{R}}_s^q(\Omega) + \varepsilon^{-1} {}_0 \mathbb{D}_s^q(\Omega) \quad .$$

*In both cases the intersection equals the finite dimensional space of Dirichlet forms  ${}_\varepsilon \mathcal{H}_s^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp_\varepsilon}$ .*

(ii) If  $-N/2 < s \leq 1 - N/2$ , then

$$\begin{aligned} L_s^{2,1}(\Omega) &= {}_0\mathring{R}_s^1(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^1(\Omega) \quad , \\ L_s^{2,N-1}(\Omega) &= {}_0\mathring{R}_s^{N-1}(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^{N-1}(\Omega) \dot{+} {}_\varepsilon\mathcal{H}^{N-1}(\Omega) \quad . \end{aligned}$$

(iii) If  $1 - N/2 < s < N/2$  or  $-N/2 < s \leq 1 - N/2$  and  $q \notin \{1, N - 1\}$ , then

$${}_\varepsilon L_s^{2,q}(\Omega) = {}_0\mathring{R}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \quad .$$

For  $s \geq 0$  this decomposition is even  $\langle \varepsilon \cdot, \cdot \rangle_{L^{2,q}(\Omega)}$ -orthogonal.

(iv) If  $s > N/2$ , then

$$\begin{aligned} {}_\varepsilon L_s^{2,q}(\Omega) &= \left( ([L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q] \cap {}_0\mathring{R}_{<\frac{N}{2}}^q(\Omega)) \right. \\ &\quad \left. \oplus_\varepsilon \varepsilon^{-1} ([L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q] \cap {}_0\mathbb{D}_{<\frac{N}{2}}^q(\Omega)) \right) \cap L_s^{2,q}(\Omega) \end{aligned}$$

and

$${}_\varepsilon L_s^{2,q}(\Omega) = {}_0\mathring{R}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \dot{+} \Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q \quad ,$$

where the first two terms in the second decomposition are  $\langle \varepsilon \cdot, \cdot \rangle_{L^{2,q}(\Omega)}$ -orthogonal as well. Furthermore,

$$L_s^{2,q}(\Omega) \cap {}_\varepsilon \mathcal{H}_{-s}^q(\Omega)^{\perp_\varepsilon} = {}_0\mathring{R}_s^q(\Omega) \oplus_\varepsilon \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \quad .$$

### Remark 3.3

- The decompositions in **(ii)**-**(iv)** are direct and define continuous projections.
- In **(ii)** we are forced to use the forms  $B^q(\Omega)$  and  $\mathring{B}^q(\Omega)$  in the definitions of  ${}_0\mathring{R}_s^q(\Omega)$  and  ${}_0\mathbb{D}_s^q(\Omega)$ .
- To prove the last equation in **(iv)**, we additionally assume  $\tau \geq N/2 - 1$ .
- The coefficients of the tower forms in the first equation of **(iv)** are related in the following way: If

$$F_{r,s} + \sum_\ell h_{r,\ell} \cdot \eta H_\ell + \varepsilon^{-1} (F_{d,s} + \sum_\ell h_{d,\ell} \cdot \eta H_\ell) = F \in {}_\varepsilon L_s^{2,q}(\Omega)$$

with  $F_{r,s} \in \mathring{R}_s^q(\Omega)$ ,  $F_{d,s} \in \mathbb{D}_s^q(\Omega)$  and  $H_\ell \in \bar{\mathcal{H}}_s^q$  as well as  $h_{r,\ell}, h_{d,\ell} \in \mathbb{C}$ , then

$$h_{r,\ell} + h_{d,\ell} = 0 \quad ,$$

since the  $H_\ell$  are linear independent and do not belong to  $L_s^{2,q}(\Omega)$ .

- $\Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q$  is a finite dimensional subspace of  $\mathring{H}_s^{1,q}(\Omega) \cap C^{1,q}(\Omega)$ , whose elements have supports in  $\overline{A_{r_1}}$ .
- For  $s < -N/2$  (and  $\tau \geq N/2 - 1$ ) we have

$${}_\varepsilon \mathcal{H}_s^q(\Omega) = {}_\varepsilon \mathcal{H}^q(\Omega) \dot{+} {}_\varepsilon \mathcal{H}_s^q(\Omega) \cap \mathring{B}^q(\Omega)^{\perp_\varepsilon} .$$

- Clearly the transformation  $\varepsilon$  may be moved to the irrotational terms in our decompositions as well.

Our decompositions and representations may be refined. For small weights we get

**Theorem 3.4** *Let  $s < N/2 + 1 - \delta_{q,0} - \delta_{q,N}$ . Then  ${}_0\mathbb{D}_s^q(\Omega)$  and  ${}_0\mathring{\mathbb{R}}_s^q(\Omega)$  are closed subspaces of  $L_s^{2,q}(\Omega)$  whenever they exist and*

$$\begin{aligned} \text{(i)} \quad {}_0\mathring{\mathbb{R}}_s^q(\Omega) &= \text{rot} \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \cap \nu^{-1} {}_0\mathbb{D}_{s-1}^{q-1}(\Omega) \right) \\ &= \text{rot} \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \cap \nu^{-1} {}_0\mathbb{D}_{s-1}^{q-1}(\Omega) \right) = \text{rot} \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \end{aligned}$$

holds for  $2 \leq q \leq N$  as well as for  $q = 1$  and  $s > 1 - N/2$ ,

$$\text{(ii)} \quad {}_0\mathbb{D}_s^q(\Omega) = \text{div} \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{s-1}^{q+1}(\Omega) \right)$$

holds for  $1 \leq q \leq N - 1$  as well as for  $q = 0$  and  $s > 2 - N/2$ ,

$$\text{(iii)} \quad {}_0\mathbb{D}_s^q(\Omega) = \text{div} \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{s-1}^{q+1}(\Omega) \right) = \text{div} \mathbb{D}_{s-1}^{q+1}(\Omega)$$

holds for  $0 \leq q \leq N - 1$ .



For large weights we have

**Theorem 3.5** *Let  $1 \leq q \leq N - 1$ . Then for  $s > N/2 + 1$*

$$\begin{aligned}
\text{(i)} \quad {}_0\mathring{\mathbb{R}}_s^q(\Omega) &= \text{rot} \left( \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{q-1} \right) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}}^{q-1}(\Omega) \right) \\
&= \text{rot} \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{q-1}(\Omega) \cap \mathring{\mathbb{B}}^{q-1}(\Omega)^{\perp\nu} \right) = \text{rot} \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \quad , \\
\text{(ii)} \quad {}_0\mathbb{D}_s^q(\Omega) &= \text{div} \left( \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{q+1} \right) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}}^{q+1}(\Omega) \right) \\
&= \text{div} \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \cap \mu^{-1} \mathring{\mathbb{R}}_{s-1}^{q+1}(\Omega) \cap \mathbb{B}^{q+1}(\Omega)^{\perp\mu} \right) = \text{div} \mathbb{D}_{s-1}^{q+1}(\Omega)
\end{aligned}$$

are closed subspaces of  $L_s^{2,q}(\Omega)$  and for  $s > N/2$

$$\begin{aligned}
\text{(iii)} \quad & \left( L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q \right) \cap {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}}^q(\Omega) \\
&= \text{rot} \left( \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \boxplus \eta \bar{\mathcal{D}}_{s-1}^{q-1,0} \boxplus \eta \bar{\mathcal{D}}_{s-1}^{q-1,1} \right) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}-1}^{q-1}(\Omega) \right) \\
&= {}_0\mathring{\mathbb{R}}_s^q(\Omega) \dot{+} \text{rot} \nu^{-1} \eta \bar{\mathcal{D}}_{s-1}^{q-1,1} \quad ,
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \left( L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q \right) \cap {}_0\mathbb{D}_{<\frac{N}{2}}^q(\Omega) \\
&= \text{div} \left( \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \boxplus \eta \bar{\mathcal{R}}_{s-1}^{q+1,0} \boxplus \eta \bar{\mathcal{R}}_{s-1}^{q+1,1} \right) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}-1}^{q+1}(\Omega) \right) \\
&= {}_0\mathbb{D}_s^q(\Omega) \dot{+} \text{div} \mu^{-1} \eta \bar{\mathcal{R}}_{s-1}^{q+1,1}
\end{aligned}$$

are closed subspaces of  $L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q$ .

**Remark 3.6** *We note  $\text{div} \eta^- D_{\sigma,m}^{q-1,1} = 0$  and  $\text{rot} \eta^- R_{\sigma,m}^{q+1,1} = 0$  by [9, Remark 2.4] and thus*

$$\eta \bar{\mathcal{D}}_{s-1}^{q-1,1} \subset {}_0\mathbb{D}_{<\frac{N}{2}-1}^{q-1}(\Omega) \quad , \quad \eta \bar{\mathcal{R}}_{s-1}^{q+1,1} \subset {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}-1}^{q+1}(\Omega) \quad .$$

**Remark 3.7** *Since there are no regular harmonic tower forms in the cases  $q \in \{0, N\}$ , i.e.  $\bar{\mathcal{H}}_s^0 = \{0\}$ ,  $\bar{\mathcal{H}}_s^N = \{0\}$ , and because of  $\eta \bar{\mathcal{H}}_s^q \subset L_{<\frac{N}{2}}^{2,q}(\Omega)$  the first equations in (iii) and (iv) simplify:*

*If  $s > N/2$ , then*

$$\begin{aligned}
& \left( L_s^{2,1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^1 \right) \cap {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}}^1(\Omega) = \text{rot} \left( \mathring{\mathbb{R}}_{s-1}^0(\Omega) \boxplus \eta \bar{\mathcal{D}}_{s-1}^{0,1} \right) \quad , \\
& \left( L_s^{2,N-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^{N-1} \right) \cap {}_0\mathbb{D}_{<\frac{N}{2}}^{N-1}(\Omega) = \text{div} \left( \mathbb{D}_{s-1}^N(\Omega) \boxplus \eta \bar{\mathcal{R}}_{s-1}^{N,1} \right) \quad .
\end{aligned}$$

*If  $N/2 < s < N/2 + 1$ , then*

$$\begin{aligned}
& \left( L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q \right) \cap {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}}^q(\Omega) = \text{rot} \left( \left( \mathring{\mathbb{R}}_{s-1}^{q-1}(\Omega) \boxplus \eta \bar{\mathcal{D}}_{s-1}^{q-1,1} \right) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}-1}^{q-1}(\Omega) \right) \quad , \\
& \left( L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q \right) \cap {}_0\mathbb{D}_{<\frac{N}{2}}^q(\Omega) = \text{div} \left( \left( \mathbb{D}_{s-1}^{q+1}(\Omega) \boxplus \eta \bar{\mathcal{R}}_{s-1}^{q+1,1} \right) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}-1}^{q+1}(\Omega) \right) \quad .
\end{aligned}$$

As mentioned above there are no harmonic tower forms in the remaining cases  $q \in \{0, N\}$ . Thus, the equations in (i), (ii) and (iii), (iv) of the latter theorem would coincide for these values. Furthermore, in these special cases there occur the exceptional tower forms. We obtain

**Theorem 3.8** *Let  $s > N/2$ . Then*

$$\begin{aligned}
\text{(i)} \quad L_s^{2,N}(\Omega) &= {}_0\mathring{\mathbb{R}}_s^N(\Omega) \\
&= \text{rot} \left( \left( \mathring{\mathbb{R}}_{s-1}^{N-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{N-1} \boxplus \eta \check{\mathcal{H}}^{N-1} \right) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}-1}^{N-1}(\Omega) \right) \\
&= \text{rot} \left( \left( \mathring{\mathbb{R}}_{s-1}^{N-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{N-1}(\Omega) \cap \mathring{\mathbb{B}}^{N-1}(\Omega)^{\perp\nu} \right) \boxplus \eta \check{\mathcal{H}}^{N-1} \right) \\
&= \text{rot} \left( \mathring{\mathbb{R}}_{s-1}^{N-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{N-1}(\Omega) \cap \mathring{\mathbb{B}}^{N-1}(\Omega)^{\perp\nu} \right) \dot{+} \Delta \eta \check{\mathcal{P}}^N \\
&= \text{rot} \mathring{\mathbb{R}}_{s-1}^{N-1}(\Omega) \dot{+} \Delta \eta \check{\mathcal{P}}^N \quad ,
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad L_s^{2,0}(\Omega) &= {}_0\mathbb{D}_s^0(\Omega) \\
&= \text{div} \left( \left( \mathbb{D}_{s-1}^1(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^1 \boxplus \eta \check{\mathcal{H}}^1 \right) \cap \mu^{-1} {}_0\mathring{\mathbb{R}}_{<\frac{N}{2}-1}^1(\Omega) \right) \\
&= \text{div} \left( \left( \mathbb{D}_{s-1}^1(\Omega) \cap \mu^{-1} \mathring{\mathbb{R}}_{s-1}^1(\Omega) \right) \boxplus \eta \check{\mathcal{H}}^1 \right) \\
&= \text{div} \left( \mathbb{D}_{s-1}^1(\Omega) \cap \mu^{-1} \mathring{\mathbb{R}}_{s-1}^1(\Omega) \right) \dot{+} \Delta \eta \check{\mathcal{P}}^0 \\
&= \text{div} \mathbb{D}_{s-1}^1(\Omega) \dot{+} \Delta \eta \check{\mathcal{P}}^0 \quad .
\end{aligned}$$

Finally we want to note that we always get dual results using the Hodge star operator. (This will change the homogeneous boundary condition from the electric to the magnetic one.) Since this will multiply the number of results by two we let their formulation to the interested reader.

## 4 Proofs

Let  $\varepsilon, \nu$  and  $\mu$  be as in section 3. We start with the

**Proof of Lemma 3.1:** Let  $E \in L_s^{2,q}(\Omega)$ . Looking at [9, section 4] and using the Helmholtz decompositions [4, (3.5)] we may choose  $b_\ell \in \text{Lin } \mathring{\mathbb{B}}^q(\Omega)$ ,  $\ell = 1, \dots, d^q$ , with  $b_\ell = \Phi_\ell + H_\ell \in \text{rot } \mathring{\mathbb{R}}^{q-1}(\Omega) \oplus_{\varepsilon\varepsilon} \mathcal{H}^q(\Omega)$ , where  $\{H_\ell\}$  is a  $\oplus_\varepsilon$ -ONB of  ${}_\varepsilon\mathcal{H}^q(\Omega)$ . Then  $e := E - \sum_\ell \langle E, \varepsilon H_\ell \rangle_{L^2,q(\Omega)} b_\ell \in {}_\varepsilon L_s^{2,q}(\Omega)$ . This proves one inclusion and the other one is trivial, because the forms of  $\mathring{\mathbb{B}}^q(\Omega)$  are smooth and compactly supported. Moreover, if  $E \in \text{Lin } \mathring{\mathbb{B}}^q(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp\varepsilon}$ , then  $\varepsilon E = \sum_\ell e_\ell \varepsilon b_\ell \in {}_\varepsilon\mathcal{H}^q(\Omega)^\perp$  and thus

$$0 = \langle \varepsilon E, H_k \rangle_{L^2,q(\Omega)} = \sum_\ell e_\ell \langle \varepsilon H_\ell, H_k \rangle_{L^2,q(\Omega)} = e_k \quad ,$$

which proves the directness of the sum. The other direct decomposition may be shown in a similar way.  $\blacksquare$

We introduce the Hilbert spaces (closed subspaces of  $L_s^{2,q}(\Omega)$ )

$$\begin{aligned}\mathring{\underline{R}}_s^q(\Omega) &:= \{E \in \mathring{R}_s^q(\Omega) : \text{rot}(\rho^{2s}E) = 0\} \\ &= \{E \in \mathring{R}_s^q(\Omega) : \text{rot} E = -2s\rho^{-2}RE\} \quad , \\ \underline{D}_s^q(\Omega) &:= \{E \in D_s^q(\Omega) : \text{div}(\rho^{2s}E) = 0\} \\ &= \{E \in D_s^q(\Omega) : \text{div} E = -2s\rho^{-2}TE\}\end{aligned}$$

and note that the  $\|\cdot\|_{\mathring{R}_s^q(\Omega)}$ ,  $\|\cdot\|_{R_s^q(\Omega)}$  and  $\|\cdot\|_{L_s^{2,q}(\Omega)}$ -norms resp. the  $\|\cdot\|_{D_s^q(\Omega)}$ ,  $\|\cdot\|_{\underline{D}_s^q(\Omega)}$  and  $\|\cdot\|_{L_s^{2,q}(\Omega)}$ -norms are equivalent on  $\mathring{R}_s^q(\Omega)$  resp.  $\underline{D}_s^q(\Omega)$ . Here we used the operators  $R, T$  from [4, (2.20) and (2.21)] and the commutator formulas [4, (2.24)]. First we need an easy consequence of the projection theorem:

**Lemma 4.1** *Let  $s \in \mathbb{R}$ . Then the orthogonal decompositions*

$$\begin{aligned}\text{(i)} \quad L_s^{2,q}(\Omega) &= \overline{\text{rot } \mathring{R}_{s-1}^{q-1}(\Omega)} \oplus_{s,\varepsilon} \varepsilon^{-1} \underline{D}_s^q(\Omega) = \varepsilon^{-1} \overline{\text{rot } \mathring{R}_{s-1}^{q-1}(\Omega)} \oplus_{s,\varepsilon} \underline{D}_s^q(\Omega) \quad , \\ \text{(ii)} \quad L_s^{2,q}(\Omega) &= \overline{\text{div } D_{s-1}^{q+1}(\Omega)} \oplus_{s,\varepsilon} \varepsilon^{-1} \mathring{\underline{R}}_s^q(\Omega) = \varepsilon^{-1} \overline{\text{div } D_{s-1}^{q+1}(\Omega)} \oplus_{s,\varepsilon} \mathring{\underline{R}}_s^q(\Omega)\end{aligned}$$

hold with continuous projections. Here we denote by  $\oplus_{s,\varepsilon}$  the orthogonal sum with respect to the  $\langle \varepsilon \rho^{2s} \cdot, \cdot \rangle_{L_s^{2,q}(\Omega)}$ -scalar product and the closures are taken in  $L_s^{2,q}(\Omega)$ . The space  $\overline{\text{rot } \mathring{R}_{s-1}^{q-1}(\Omega)}$  resp.  $\overline{\text{div } D_{s-1}^{q+1}(\Omega)}$  may be replaced by  $\overline{\text{rot } \mathring{C}^{\infty,q-1}(\Omega)}$  resp.  $\overline{\text{div } D_{\text{vox}}^{q+1}(\Omega)}$ .

**Proof:** Since  $\mathring{C}^{\infty,q-1}(\Omega)$  is dense in  $\mathring{R}_{s-1}^{q-1}(\Omega)$  we have  $E \in L_s^{2,q}(\Omega) \cap (\text{rot } \mathring{R}_{s-1}^{q-1}(\Omega))^{\perp_{s,\varepsilon}}$ , if and only if  $E \in L_s^{2,q}(\Omega) \cap (\text{rot } \mathring{C}^{\infty,q-1}(\Omega))^{\perp_{s,\varepsilon}}$ , which means  $\rho^{2s}\varepsilon E \in {}_0D_{-s}^q(\Omega)$ . Thus,  $E \in \underline{D}_s^q(\Omega)$ , because  $0 = \text{div}(\rho^{2s}\varepsilon E) = \rho^{2s} \text{div } \varepsilon E + 2s\rho^{2s-2}T\varepsilon E$ . This shows (i) and (ii) follows analogously.  $\blacksquare$

**Remark 4.2** *Clearly this lemma holds for 0-admissible transformations  $\varepsilon$  as well.*

We need two important results, which may be formulated as follows: Defining the Hilbert space

$${}_\varepsilon X_t^q(\Omega) := \left( \mathring{R}_t^q(\Omega) \cap \varepsilon^{-1} D_t^q(\Omega) \right) \boxplus \eta \bar{\mathcal{H}}_t^q \boxplus \eta \check{\mathcal{H}}_t^q \quad , \quad t \in \mathbb{R} \quad ,$$

we have

**Lemma 4.3** *Let  $s \in \mathbb{R} \setminus \tilde{\mathbb{I}}$  and  $q \neq 0$  or  $q = 0$  and  $s > -N/2$ . Then*

$$\begin{aligned} \varepsilon \text{ROT}_s^q &: \varepsilon X_s^q(\Omega) \cap \varepsilon^{-1} {}_0\mathbb{D}_{\text{loc}}^q(\Omega) &\longrightarrow & {}_0\mathring{\mathbb{R}}_{s+1}^{q+1}(\Omega) \\ &E &\longmapsto & \text{rot } E \end{aligned} ,$$

$$\begin{aligned} \varepsilon \text{DIV}_s^q &: \varepsilon X_s^q(\Omega) \cap {}_0\mathring{\mathbb{R}}_{\text{loc}}^q(\Omega) &\longrightarrow & {}_0\mathbb{D}_{s+1}^{q-1}(\Omega) \\ &H &\longmapsto & \text{div } \varepsilon H \end{aligned}$$

*are continuous and surjective Fredholm operators with kernels*

$$N(\varepsilon \text{ROT}_s^q) = N(\varepsilon \text{DIV}_s^q) = \begin{cases} \varepsilon \mathcal{H}_s^q(\Omega) & , \text{ if } s < -N/2 \\ \varepsilon \mathcal{H}^q(\Omega) & , \text{ if } s > -N/2 \end{cases} .$$

**Remark 4.4** *By adding suitable Dirichlet forms from  $\varepsilon \mathcal{H}^q(\Omega)$  we always may obtain the constraints  $H \perp B^q(\Omega)$ ,  $q \neq 1$ , and  $\varepsilon E \perp B^q(\Omega)$  or  $\varepsilon H, \varepsilon E \perp \varepsilon \mathcal{H}^q(\Omega)$ , if  $s > -N/2$ . Then for  $s > -N/2$  the operators*

$$\begin{aligned} \varepsilon \text{ROT}_s^q &: \varepsilon X_s^q(\Omega) \cap \varepsilon^{-1} {}_0\mathbb{D}_{\text{loc}}^q(\Omega) &\longrightarrow & {}_0\mathring{\mathbb{R}}_{s+1}^{q+1}(\Omega) \\ \varepsilon \text{DIV}_s^q &: \varepsilon X_s^q(\Omega) \cap {}_0\mathring{\mathbb{R}}_{\text{loc}}^q(\Omega) &\longrightarrow & {}_0\mathbb{D}_{s+1}^{q-1}(\Omega) \end{aligned} ,$$

*are also injective and hence topological isomorphisms by the bounded inverse theorem. Furthermore, we note (using the notations from [9])*

$$\bar{\mathcal{H}}_t^q = \mathcal{D}^q(\bar{\mathcal{J}}_t^{q,0}) = \mathcal{R}^q(\bar{\mathcal{J}}_t^{q,0}) \quad , \quad \check{\mathcal{H}}_t^q = \check{\mathcal{D}}_t^{q,1} = \check{\mathcal{R}}_t^{q,1}$$

*and for  $t < N/2$*

$$\bar{\mathcal{H}}_t^q = \check{\mathcal{H}}_{t-1}^q = \{0\} .$$

**Proof:** This lemma has been proved in [9, Corollary 3.13, Lemma 3.14] for  $s > -N/2$ . So we only have to discuss the cases of small weights  $s < -N/2$  and ranks of forms  $1 \leq q \leq N$ . Welldefinedness, continuity and the assertion about the kernels are trivial in these cases. So for example to show surjectivity for  $\varepsilon \text{DIV}_s^q$ , let us pick some  $F \in {}_0\mathbb{D}_{s+1}^{q-1}(\Omega)$ . Following the proofs of [9, Lemma 3.5, Lemma 3.12] and using [15, Theorem 4] we represent the extension by zero of  $F$  to  $\mathbb{R}^N$

$$\hat{F} =: F_{\mathbb{D}} + F_{\mathbb{R}} \in {}_0\mathbb{D}_{s+1}^{q-1}(\mathbb{R}^N) + {}_0\mathbb{R}_{s+1}^{q-1}(\mathbb{R}^N) .$$

Now  $F_{\mathbb{D}}$  is contained in the range of the operator  $B$  from [15, Theorem 7] and thus we get some

$$h \in \mathbb{D}_s^q(\mathbb{R}^N) \cap {}_0\mathbb{R}_s^q(\mathbb{R}^N) \quad \text{solving} \quad \text{div } h = F_{\mathbb{D}} .$$

Applying [4, Theorem 3.6 (ii)] we even have  $h \in \mathbb{H}_s^{1,q}(\mathbb{R}^N)$ . Since

$$F_{\mathbb{R}} \in {}_0\mathbb{D}_{s+1}^{q-1}(\Omega) \cap {}_0\mathbb{R}_{s+1}^{q-1}(\Omega)$$

we may represent this form in terms of a spherical harmonics expansion

$$F_{\mathbb{R}}|_{A_{r_0}} = \sum_{I \in \bar{\mathcal{J}}^{q-1,0}} f_I D_I^{q-1} + f \hat{D}^{q-1,1} + \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I D_I^{q-1}$$

with uniquely determined  $f_I, f \in \mathbb{C}$  using [9, Theorem 2.6], where

$${}_s\bar{\mathcal{J}}^{q-1,0} := \{I \in \mathcal{J}^{q-1,0} : \mathbf{s}(I) = + \wedge \mathbf{e}(I) < -s - 1 - N/2\} \quad .$$

Now looking at [9, Remark 2.5] the first term of the sum on the right hand side belongs to  $L_{<\frac{N}{2}}^{2,q-1}(A_{r_0})$ , the second to  $L_{<\frac{N}{2}-1}^{2,q-1}(A_{r_0})$  and the third to  $L_{s+1}^{2,q-1}(A_{r_0})$ . Therefore,

$$F_{\mathbb{R}} - \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \eta D_I^{q-1} \in L^{2,q-1}(\mathbb{R}^N) \quad .$$

This suggests the ansatz

$$H := \eta h + \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \eta R_{1I}^q + \Phi$$

to solve  $H \in {}_0\mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega)$  and  $\operatorname{div} \varepsilon H = F$ . We note that the middle term of the right hand side of the ansatz belongs to  $\mathring{R}_s^q(\Omega)$  and even to  ${}_0\mathring{R}_s^q(\Omega)$ , for instance. Thus, we are searching for some  $\Phi \in \mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega)$  satisfying

$$\begin{aligned} \operatorname{rot} \Phi &= -\operatorname{rot}(\eta h) = -C_{\operatorname{rot},\eta} h =: \tilde{G} \quad , \\ \operatorname{div} \varepsilon \Phi &= F - \operatorname{div}(\eta \varepsilon h) - \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \operatorname{div}(\eta \varepsilon R_{1I}^q) =: \tilde{F} \quad , \end{aligned} \quad (4.1)$$

since  $\operatorname{rot}(\eta R_{1I}^q) = 0$ . Clearly we have  $\tilde{G} \in {}_0\mathring{\mathbb{R}}_{\operatorname{vox}}^{q+1}(\Omega) \subset {}_0\mathring{\mathbb{R}}^{q+1}(\Omega)$ . Moreover, not only  $\tilde{F} \in {}_0\mathbb{D}_{s+1}^{q-1}(\Omega)$ , but also  $\tilde{F} \in {}_0\mathbb{D}^{q-1}(\Omega)$  holds, because

$$\begin{aligned} \tilde{F} &= \operatorname{div}(1 - \eta)h - \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I C_{\operatorname{div},\eta} R_{1I}^q + F_{\mathbb{R}} - \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \eta D_I^{q-1} \\ &\quad - \operatorname{div}(\hat{\varepsilon} \eta (h + \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \eta R_{1I}^q)) \quad , \end{aligned}$$

where the first two terms of the sum on the right hand side lie in  $L_{\operatorname{vox}}^{2,q-1}(\Omega)$ , the sum of the third and fourth terms in  $L^{2,q-1}(\Omega)$  and the last one in  $L_{s+1+\tau}^{2,q-1}(\Omega) \subset L^{2,q-1}(\Omega)$ , since

$$\eta (h + \sum_{I \in {}_s\bar{\mathcal{J}}^{q-1,0}} f_I \eta R_{1I}^q) \in H_s^{1,q}(\mathbb{R}^N)$$

and  $\tau \geq -s - 1$ . Now we are able to apply [4, Theorem 3.21] and get some  $\Phi \in \mathring{R}_{-1}^q(\Omega) \cap \varepsilon^{-1}D_{-1}^q(\Omega)$  solving the system (4.1). Since  $s < -N/2 < -3/2$  we have  $\Phi \in \mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega)$ , which completes the proof. The assertion about  ${}_{\varepsilon}\text{ROT}_s^q$  follows analogously.  $\blacksquare$

**Proof of Theorem 3.4:** Apply Lemma 4.3 and Remark 4.4 with the modified values of  $q, s$  and  $\varepsilon$ .  $\blacksquare$

**Proof of Theorem 3.5:** The first equations in (i)-(iv) have already been proved in [9, Theorem 5.8]. Let

$$G \in {}_0\mathring{R}_s^q(\Omega) = \text{rot} \left( \left( \mathring{R}_{s-1}^{q-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{q-1} \right) \cap \nu^{-1} {}_0\mathbb{D}_{< \frac{N}{2}}^{q-1}(\Omega) \right),$$

i.e.

$$G = \text{rot} G_{s-1} + \sum_{I \in \bar{\mathcal{J}}_{s-1}^{q-1,0}} g_I \text{rot} \eta D_I^{q-1}$$

with  $G_{s-1} \in \mathring{R}_{s-1}^{q-1}(\Omega) \cap \nu^{-1}D_{s-1}^{q-1}(\Omega) \cap \mathring{B}^{q-1}(\Omega)^{\perp\nu}$  and  $g_I \in \mathbb{C}$ . Since

$$\text{rot} \eta D_I^{q-1} = \text{rot} \Delta \eta D_{2I}^{q-1} - \text{rot} C_{\text{div rot}, \eta} D_{2I}^{q-1}$$

and clearly

$$\text{rot} \Delta \eta R_{2I}^{q-1} = \text{rot} \text{div} C_{\text{rot}, \eta} R_{2I}^{q-1}$$

we get

$$\begin{aligned} i \alpha_{\mathbf{e}(I)}^{q'} \text{rot} \eta D_I^{q-1} &= \text{rot} \Delta \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^{q-1} - i \alpha_{\mathbf{e}(I)}^{q'} \text{rot} C_{\text{div rot}, \eta} D_{2I}^{q-1} \\ &\quad - \alpha_{\mathbf{e}(I)}^q \text{rot} \text{div} C_{\text{rot}, \eta} R_{2I}^{q-1}. \end{aligned}$$

Now  $\Delta \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^{q-1} = C_{\Delta, \eta} P_{\mathbf{e}(I), \mathbf{c}(I)}^{q-1}$  has compact support and therefore

$$G \in \text{rot} \left( \mathring{R}_{s-1}^{q-1}(\Omega) \cap \nu^{-1}D_{s-1}^{q-1}(\Omega) \cap \mathring{B}^{q-1}(\Omega)^{\perp\nu} \right),$$

which proves (i). (ii) is shown analogously. The last equation in (iii) follows from (i), since we can split off a term

$$\eta^- D_{\sigma, m}^{q-1, 1} = \nu^{-1} \nu \eta^- D_{\sigma, m}^{q-1, 1} = \nu^{-1} \eta^- D_{\sigma, m}^{q-1, 1} + \nu^{-1} \hat{\nu} \eta^- D_{\sigma, m}^{q-1, 1}$$

and the tower forms are smooth and  $\hat{\nu}$  decays as well as  $\text{div} \eta^- D_{\sigma, m}^{q-1, 1} = 0$  holds by Remark 3.6. Finally the last equation in (iv) is a direct consequence of (ii) and a similar argument like the latter one.  $\blacksquare$

**Proof of Theorem 3.8:** Lemma 4.3 yields

$$L_s^{2,N}(\Omega) = \text{rot} \left( (\mathring{R}_{s-1}^{N-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{N-1} \boxplus \eta \check{\mathcal{H}}^{N-1}) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}-1}^{N-1}(\Omega) \right)$$

and the same arguments used in the latter proof show

$$\begin{aligned} & \text{rot} \left( (\mathring{R}_{s-1}^{N-1}(\Omega) \boxplus \eta \bar{\mathcal{H}}_{s-1}^{N-1}) \cap \nu^{-1} {}_0\mathbb{D}_{<\frac{N}{2}-1}^{N-1}(\Omega) \right) \\ &= \text{rot} \left( \mathring{R}_{s-1}^{N-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{N-1}(\Omega) \cap \mathring{B}^{N-1}(\Omega)^{\perp\nu} \right) . \end{aligned}$$

Because  $\text{div} \eta \check{H}^{N-1} = 0$  we have

$$L_s^{2,N}(\Omega) = \text{rot} \left( (\mathring{R}_{s-1}^{N-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{N-1}(\Omega) \cap \mathring{B}^{N-1}(\Omega)^{\perp}) \boxplus \eta \check{\mathcal{H}}^{N-1} \right) .$$

Now  $\text{div} \check{P}^N = \check{H}^{N-1}$  and thus  $\text{rot} \eta \check{H}^{N-1} = \Delta \eta \check{P}^N - \text{rot} C_{\text{div},\eta} \check{P}^N$ , which proves

$$L_s^{2,N}(\Omega) = \text{rot} \left( \mathring{R}_{s-1}^{N-1}(\Omega) \cap \nu^{-1} \mathbb{D}_{s-1}^{N-1}(\Omega) \cap \mathring{B}^{N-1}(\Omega)^{\perp\nu} \right) \dot{+} \Delta \eta \check{P}^N .$$

Finally we have to show that the sum is direct. To do this, let  $F = f \Delta \eta \check{P}^N = \text{rot} E$  with some  $f \in \mathbb{C}$  and  $E \in \mathring{R}_{s-1}^{N-1}(\Omega)$ . We want to use the notations from [15], i.e. the forms  $P_{0,1}^{N,4}$  and  $Q_{0,1}^{N,4}$ , as well. By definition there exists a constant  $c \neq 0$ , such that  $\check{P}^N = c^- R_{0,1}^{N,2} = \frac{c}{2-N} Q_{0,1}^{N,4}$  using [9, Remark 2.3]. Since  $s > N/2$  and  $P_{0,1}^{N,4} \in L_{<-\frac{N}{2}}^{2,N}(\Omega)$  partial integration yields

$$\langle \text{rot} E, P_{0,1}^{N,4} \rangle_{L^{2,q}(\Omega)} = 0 ,$$

because  $P_{0,1}^{N,4} \in \text{Lin}\{*\mathbf{1}\}$  is constant. But on the other hand we obtain

$$\langle F, P_{0,1}^{N,4} \rangle_{L^{2,q}(\Omega)} = f \langle \Delta \eta \check{P}^N, P_{0,1}^{N,4} \rangle_{L^{2,q}(\Omega)} = \frac{c f}{2-N} \langle C_{\Delta,\eta} Q_{0,1}^{N,4}, P_{0,1}^{N,4} \rangle_{L^{2,q}(\Omega)} = c f$$

by [15, (73)], i.e.  $f = 0$ . ■

Now we turn to the main idea of our decompositions and the

**Proof of Theorem 3.2:** Let  $1 \leq q \leq N-1$  and  $s$  be as in section 3 as well as

$$F \in L_s^{2,q}(\Omega) .$$

Using Lemma 4.1 we decompose

$$F = F_r + \varepsilon^{-1} \hat{F}_d$$

with  $F_r \in \text{rot} \mathring{C}^{\infty,q-1}(\Omega)$  and  $\hat{F}_d \in \underline{D}_s^q(\Omega)$ . A second application of this lemma yields the decomposition

$$\varepsilon^{-1} \hat{F}_d = \varepsilon^{-1} F_d + \tilde{F}$$

with  $F_d \in \overline{\text{div } D_{\text{vox}}^{q+1}(\Omega)}$  and  $\tilde{F} \in \mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega)$ . Furthermore, there exists a constant  $c > 0$  independent of  $F$ , such that

$$\|F_r\|_{L_s^{2,q}(\Omega)} + \|F_d\|_{L_s^{2,q}(\Omega)} + \|\tilde{F}\|_{R_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega)} \leq c \cdot \|F\|_{L_s^{2,q}(\Omega)} .$$

Now  $\tilde{F}$  is more regular than  $F$  and this enables us to solve

$$\text{div } \varepsilon H = \text{div } \varepsilon \tilde{F} \in {}_0\mathbb{D}_{s+1}^{q-1}(\Omega) \quad , \quad \text{rot } E = \text{rot } \tilde{F} \in {}_0\mathring{\mathbb{R}}_{s+1}^{q+1}(\Omega)$$

with some  $H \in {}_\varepsilon X_s^q(\Omega) \cap {}_0\mathring{R}_{\text{loc}}^q(\Omega)$  and  $E \in {}_\varepsilon X_s^q(\Omega) \cap \varepsilon^{-1}{}_0D_{\text{loc}}^q(\Omega)$  by Lemma 4.3. We note  $E, H \in L_t^{2,q}(\Omega)$  for all  $t$  with  $t \leq s$  and  $t < N/2 - 1$ . Now

$$\hat{F} := \tilde{F} - E - H \in {}_\varepsilon \mathcal{H}_t^q(\Omega)$$

and

$$F = F_r + H + \varepsilon^{-1}(F_d + \varepsilon E) + \hat{F} \quad , \quad (4.2)$$

where  $F_r + H \in {}_0\mathring{R}_t^q(\Omega)$  and  $F_d + \varepsilon E \in {}_0D_t^q(\Omega)$ .

**Remark 4.5** *This short and simple argument is the main idea. The rest is just refinement utilizing the result from [9]. We note that the idea of using the more regular  $\tilde{F}$  replaces the old idea of working in  $H^{-1}(\Omega)$ , the dual of  $\mathring{H}^1(\Omega)$ , in the case of the classical Helmholtz decomposition.*

For  $s > -N/2$  we may refine the representation (4.2) of  $F$ . For these  $s$  we have  $\hat{F} \in {}_\varepsilon \mathcal{H}_{>-\frac{N}{2}}^q(\Omega) = {}_\varepsilon \mathcal{H}^q(\Omega)$  by [9, Lemma 3.8]. Using Remark 4.4 or [9, Theorem 5.1] additionally we may obtain  $\varepsilon E \perp \mathring{B}^q(\Omega)$  and  $H \perp B^q(\Omega)$ , if  $q \neq 1$ , or  $\varepsilon E \perp {}_\varepsilon \mathcal{H}^q(\Omega)$  and  $\varepsilon H \perp {}_\varepsilon \mathcal{H}^q(\Omega)$ , if  $s > 1 - N/2$ . Therefore,  $F_d + \varepsilon E \in {}_0\mathbb{D}_t^q(\Omega)$  holds for  $s > -N/2$  and  $F_r + H \in {}_0\mathring{R}_t^q(\Omega)$  for  $s > 1 - N/2$  or  $1 - N/2 > s > -N/2$  and  $q \neq 1$ . Moreover, [9, Theorem 5.1] yields not only  $E, H \in {}_\varepsilon X_s^q(\Omega)$  but also

$$\begin{aligned} E &\in \left( \mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega) \right) \boxplus \eta \mathcal{D}^q(\bar{\mathcal{J}}_s^{q,0}) & , \text{ if } q \neq N - 1, \\ H &\in \left( \mathring{R}_s^q(\Omega) \cap \varepsilon^{-1}D_s^q(\Omega) \right) \boxplus \eta \mathcal{R}^q(\bar{\mathcal{J}}_s^{q,0}) & , \text{ if } q \neq 1, \end{aligned}$$

i.e. the exceptional forms do not appear in these cases.

The stronger assumption on  $F \in {}_\varepsilon \mathbb{L}_s^{2,q}(\Omega)$  for weights  $s > 1 - N/2$  or  $s > -N/2$  and  $2 \leq q \leq N - 2$  implies  $\hat{F} \in {}_\varepsilon \mathcal{H}^q(\Omega)^{\perp_\varepsilon}$  and thus  $\hat{F} = 0$ . So in these cases (4.2) turns to

$$F = F_r + H + \varepsilon^{-1}(F_d + \varepsilon E) \quad . \quad (4.3)$$

Until now we have shown the assertions of Theorem 3.2 (i), (ii) and also (iii) for  $s < N/2 - 1$ . Considering larger weights  $s > N/2 - 1$  and  $F \in {}_\varepsilon \mathbb{L}_s^{2,q}(\Omega)$  the tower forms occur in the representation (4.3). More precisely we have



$$\begin{aligned}
E &= E_s + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \eta D_I^q + e \eta \begin{cases} -D_{0,1}^{N-1,1} & , q = N-1 \\ 0 & , \text{otherwise} \end{cases} , \\
H &= H_s + \sum_{J \in \bar{\mathcal{J}}_s^{q,0}} h_J \eta R_J^q + h \eta \begin{cases} -R_{0,1}^{1,1} & , q = 1 \\ 0 & , \text{otherwise} \end{cases}
\end{aligned}$$

with uniquely determined  $E_s, H_s \in \mathring{\mathbb{R}}_s^q(\Omega) \cap \varepsilon^{-1} \mathbb{D}_s^q(\Omega)$  and  $e_I, e, h_J, h \in \mathbb{C}$ . We note  $\bar{\mathcal{J}}_s^{q,0} = \bar{\mathcal{J}}_s^{q,0}$ . For  $s < N/2$  we have  $\bar{\mathcal{J}}_s^{q,0} = \emptyset$  and for  $s > N/2$  we see

$$\alpha_{\mathbf{e}(I)}^q \eta R_I^q = -i \alpha_{\mathbf{e}(I)}^{q'} \eta D_I^q \in L_{< \frac{N}{2}}^{2,q}(\Omega)$$

for all  $I \in \bar{\mathcal{J}}_s^{q,0}$  by (2.3). Thus, we obtain

$$\begin{aligned}
F &= F_r + H_s + \varepsilon^{-1} F_d + E_s \\
&+ \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} (h_I - \tilde{e}_I) \eta R_I^q + \eta \begin{cases} h^- R_{0,1}^{1,1} & , q = 1 \\ e^- D_{0,1}^{N-1,1} & , q = N-1 \\ 0 & , \text{otherwise} \end{cases} ,
\end{aligned}$$

where  $\tilde{e}_I := -i e_I \alpha_{\mathbf{e}(I)}^q / \alpha_{\mathbf{e}(I)}^{q'}$ . Now looking, for example, at the case  $q = 1$  we see  $\eta^- R_{0,1}^{1,1} \notin L_{\geq \frac{N}{2}-1}^{2,q}(\Omega)$ . Then for integrability reasons we get  $h = 0$ , such that the exceptional tower form does not appear. Clearly also  $e = 0$  holds true for  $q = N-1$ . Moreover,  $h_I = \tilde{e}_I$  since  $R_I^q$  are linear independent and  $\eta R_I^q \notin L_s^{2,q}(\Omega)$  for all  $I \in \bar{\mathcal{J}}_s^{q,0}$  and  $s > N/2$ .

By the smoothness of  $\eta D_I^q$  as well as the decay and differentiability properties of  $\hat{\varepsilon}$  we obtain furthermore  $\hat{\varepsilon} \eta D_I^q \in H_s^{1,q}(\Omega)$ . Thus, for all  $s > 1 - N/2$  we get the representation

$$F = \tilde{F}_r + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} h_I \eta R_I^q + \varepsilon^{-1} (\tilde{F}_d + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \eta D_I^q) ,$$

where  $\tilde{F}_r := F_r + H_s$  and  $\tilde{F}_d := F_d + \varepsilon E_s + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \hat{\varepsilon} \eta D_I^q$  as well as

$$\begin{aligned}
\tilde{F}_r + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} h_I \eta R_I^q &\in (L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q) \cap {}_0 \mathring{\mathbb{R}}_{< \frac{N}{2}}^q(\Omega) , \\
\tilde{F}_d + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \eta D_I^q &\in (L_s^{2,q}(\Omega) \boxplus \eta \bar{\mathcal{H}}_s^q) \cap {}_0 \mathbb{D}_{< \frac{N}{2}}^q(\Omega)
\end{aligned}$$

with  $\alpha_{\mathbf{e}(I)}^{q'} h_I + i \alpha_{\mathbf{e}(I)}^q e_I = 0$  for all  $I \in \bar{\mathcal{J}}_s^{q,0}$ .

This proves the remaining assertions of (iii) and the first equation in (iv). To show the second equation in (iv) we observe

$$\begin{aligned}\eta D_I^q &= \eta \operatorname{div} \operatorname{rot} D_{2I}^q = \operatorname{div} \operatorname{rot} \eta D_{2I}^q - C_{\operatorname{div} \operatorname{rot}, \eta} D_{2I}^q \quad , \\ \eta R_I^q &= \eta \operatorname{rot} \operatorname{div} R_{2I}^q = \operatorname{rot} \operatorname{div} \eta R_{2I}^q - C_{\operatorname{rot} \operatorname{div}, \eta} R_{2I}^q\end{aligned}$$

and therefore

$$F = \tilde{F}_r + \varepsilon^{-1} \tilde{F}_d + \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \frac{h_I}{\alpha_{\mathbf{e}(I)}^q} \Delta_\varepsilon \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q \quad ,$$

where

$$\begin{aligned}\tilde{F}_r &:= \tilde{F}_r - \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} h_I C_{\operatorname{rot} \operatorname{div}, \eta} R_{2I}^q - \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \operatorname{rot} C_{\operatorname{div}, \eta} D_{2I}^q \in {}_0\mathring{\mathbb{R}}_s^q(\Omega) \quad , \\ \tilde{F}_d &:= \tilde{F}_d - \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I C_{\operatorname{div} \operatorname{rot}, \eta} D_{2I}^q - \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} h_I \operatorname{div} C_{\operatorname{rot}, \eta} R_{2I}^q \in {}_0\mathbb{D}_s^q(\Omega) \quad , \\ &\quad \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} \frac{h_I}{\alpha_{\mathbf{e}(I)}^q} \Delta_\varepsilon \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q \in \Delta_\varepsilon \eta \bar{\mathcal{J}}_{s-2}^q \quad .\end{aligned}$$

Clearly all sums are direct resp. orthogonal as stated. Only in the second equation of (iv) one may see this not directly. So for example if

$$E = \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \Delta_\varepsilon \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q = G + \varepsilon^{-1} F \in {}_0\mathring{\mathbb{R}}_s^q(\Omega) \dot{+} \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega)$$

with some  $s > N/2$ , then

$$H := G - \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \operatorname{rot} \operatorname{div} \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q = \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \varepsilon^{-1} \operatorname{div} \operatorname{rot} \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q - \varepsilon^{-1} F$$

is a Dirichlet form, i.e.  $H \in {}_\varepsilon \mathcal{H}^q(\Omega)$ , but also an element of  ${}_0\mathring{\mathbb{B}}^q(\Omega)^{\perp_\varepsilon}$ . Hence,  $H$  must vanish and thus

$$G = \sum_{I \in \bar{\mathcal{J}}_s^{q,0}} e_I \operatorname{rot} \operatorname{div} \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q \in L_s^{2,q}(\Omega) \quad ,$$

which is only possible, if  $e_I = 0$  for all  $I \in \bar{\mathcal{J}}_s^{q,0}$ , since  $\operatorname{rot} \operatorname{div} \eta P_{\mathbf{e}(I), \mathbf{c}(I)}^q \notin L_s^{2,q}(\Omega)$  are linear independent.

It remains to prove the last equation of (iv). Before we start with this, we observe, that by the closed graph theorem all projections in (ii)-(iv) are continuous.

To show the last equation of (iv) we note

$${}_0\mathring{\mathbb{R}}_s^q(\Omega), \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \subset L_s^{2,q}(\Omega) \cap {}_\varepsilon \mathcal{H}_{-s}^q(\Omega)^{\perp_\varepsilon} =: \mathbb{Y}_s^q(\Omega)$$

and thus

$$\mathbb{X}_s^q(\Omega) := {}_0\mathring{\mathbb{R}}_s^q(\Omega) \oplus_\varepsilon \varepsilon^{-1} {}_0\mathbb{D}_s^q(\Omega) \subset \mathbb{Y}_s^q(\Omega) \quad .$$

Furthermore,  $\mathbb{X}_s^q(\Omega)$  and  $\mathbb{Y}_s^q(\Omega)$  are closed subsets of  $L_s^{2,q}(\Omega)$ . By the first equation of (iv) and Lemma 3.1 we have

$$\text{codim } \mathbb{X}_s^q(\Omega) = \dim {}_\varepsilon\mathcal{H}^q(\Omega) + \dim \Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q = d^q + \sum_{0 \leq \sigma < s - \frac{N}{2}} \mu_\sigma^q \quad ,$$

since  $\dim \Delta_\varepsilon \eta \bar{\mathcal{P}}_{s-2}^q = \dim \bar{\mathcal{P}}_{s-2}^q = \dim \bar{\mathcal{H}}_s^q$  and  $s - N/2 \notin \mathbb{N}_0$  because  $s \notin \tilde{\mathbb{I}}$ . With  $\text{codim } \mathbb{Y}_s^q(\Omega) = \dim {}_\varepsilon\mathcal{H}_{-s}^q(\Omega)$  we get by Appendix A that  $\mathbb{X}_s^q(\Omega)$  and  $\mathbb{Y}_s^q(\Omega)$  possess the same finite codimension in  $L_s^{2,q}(\Omega)$ . Consequently we obtain  $\mathbb{X}_s^q(\Omega) = \mathbb{Y}_s^q(\Omega)$ . ■

## 5 Appendix A: Weighted Dirichlet forms

Let  $\tau > 0$ . As already mentioned in section 2 for the space of Dirichlet forms we have

$${}_\varepsilon\mathcal{H}_{-\frac{N}{2}}^q(\Omega) = {}_\varepsilon\mathcal{H}^q(\Omega) = {}_\varepsilon\mathcal{H}_{<t}^q(\Omega) \quad , \quad t := N/2 - \delta_{q,1} - \delta_{q,N-1} \quad ,$$

and its dimension equals  $d^q = \beta_{q'}$ , the  $q'$ -th Betti number of  $\Omega$ . Furthermore, we have for all  $t \in \mathbb{R}$

$${}_\varepsilon\mathcal{H}_t^0(\Omega) = \{0\} \quad , \quad {}_\varepsilon\mathcal{H}_t^N(\Omega) = \begin{cases} \{0\} & , t \geq -N/2 \\ \varepsilon^{-1} \text{Lin}\{*\mathbf{1}\} & , t < -N/2 \end{cases} \quad .$$

(This holds even for  $\tau = 0$ .) We repeat some notations and results from [7], [9] and [15]. Let us introduce the ‘special growing Dirichlet forms’  $E_{\sigma,m}^+$  from [10, Lemma 2.4 and Remark 2.5] as the unique solutions of the problems

$$E_{\sigma,m}^+ \in {}_\varepsilon\mathcal{H}_{<-\frac{N}{2}-\sigma}^q(\Omega) \cap \mathring{\mathbb{B}}^q(\Omega)^{\perp_\varepsilon} \quad , \quad E_{\sigma,m}^+ - {}^+D_{\sigma,m}^{q,0} \in L_{>-\frac{N}{2}}^{2,q}(\Omega) \quad ,$$

where  $\sigma \in \mathbb{N}_0$  and  $1 \leq m \leq \mu_\sigma^q$  with

$$\mu_\sigma^q = \binom{N}{q} \binom{N-1+\sigma}{\sigma} \frac{qq'(N+2\sigma)}{N(q+\sigma)(q'+\sigma)}$$

from [15, Theorem 1 (iii)]. To guarantee their existence we have to impose the decay conditions  $\tau > \sigma$  and  $\tau \geq N/2 - 1$ . We note  $\mu_0^q = \binom{N}{q}$  and thus  $\mu_0^0 = \mu_0^N = 1$ . Moreover,  ${}^+D_{0,1}^{0,0}$  resp.  ${}^+D_{0,1}^{N,0}$  is a multiple of  $\mathbf{1}$  resp.  $*\mathbf{1}$ .

**Lemma 5.1** *Let  $1 \leq q \leq N - 1$  and  $s \in (-\infty, -N/2) \setminus \tilde{\mathbb{I}}$  as well as  $\tau > -s - N/2$  and  $\tau \geq N/2 - 1$ . Then*

$${}_{\varepsilon}\mathcal{H}_s^q(\Omega) = {}_{\varepsilon}\mathcal{H}^q(\Omega) \dot{+} {}_{\varepsilon}\mathcal{H}_s^q(\Omega) \cap \mathring{\mathbb{B}}^q(\Omega)^{\perp\varepsilon}$$

holds. Moreover,

$${}_{\varepsilon}\mathcal{H}_s^q(\Omega) \cap \mathring{\mathbb{B}}^q(\Omega)^{\perp\varepsilon} = \text{Lin}\{E_{\sigma,m}^q : \sigma < -s - N/2\} \quad .$$

**Corollary 5.2** *The dimension  $d_s^q$  of  ${}_{\varepsilon}\mathcal{H}_s^q(\Omega)$  is finite and independent of  $\varepsilon$ . More precisely*

$$d_s^q = d^q + \sum_{\sigma < -s - \frac{N}{2}} \mu_{\sigma}^q \quad .$$

Furthermore, the mapping

$$\begin{aligned} d^q(\cdot) : (-\infty, -N/2) \setminus \tilde{\mathbb{I}} \cup (-N/2, N/2 - 1) &\longrightarrow \mathbb{N}_0 \\ s &\longmapsto d_s^q \end{aligned}$$

is locally constant and monotone decreasing. It jumps exactly at the points  $s \in \tilde{\mathbb{I}}$ , i.e.  $-s - N/2 \in \mathbb{N}_0$ .

**Proof:** The directness of the sum follows by (2.2) and the inclusions

$$\begin{aligned} {}_{\varepsilon}\mathcal{H}^q(\Omega) \dot{+} {}_{\varepsilon}\mathcal{H}_s^q(\Omega) \cap \mathring{\mathbb{B}}^q(\Omega)^{\perp\varepsilon} &\subset {}_{\varepsilon}\mathcal{H}_s^q(\Omega) \quad , \\ \text{Lin}\{E_{\sigma,m}^q : \sigma < -s - N/2\} &\subset {}_{\varepsilon}\mathcal{H}_s^q(\Omega) \cap \mathring{\mathbb{B}}^q(\Omega)^{\perp\varepsilon} \end{aligned}$$

are trivial. So it remains to prove

$${}_{\varepsilon}\mathcal{H}_s^q(\Omega) \subset {}_{\varepsilon}\mathcal{H}^q(\Omega) \dot{+} \text{Lin}\{E_{\sigma,m}^q : \sigma < -s - N/2\} \quad .$$

Therefore, we pick some  $E \in {}_{\varepsilon}\mathcal{H}_s^q(\Omega)$ . We observe  $E \in H_s^{1,q}(A_{\rho})$  by the regularity result [4, Corollary 3.8 (ii)] and even

$$\text{rot } E = 0 \quad , \quad \text{div } E|_{A_{\rho}} = -\text{div } \hat{\varepsilon}E|_{A_{\rho}} \in L_{s+\tau+1}^{2,q}(A_{\rho})$$

for all  $r_0 < \rho < r_1$ . Thus, we have  $\eta E \in \mathring{\mathbb{R}}_s^q(\Omega) \cap D_s^q(\Omega)$  with

$$\begin{aligned} \text{rot } \eta E &= C_{\text{rot},\eta} E \in {}_0\mathring{\mathbb{R}}_{\text{vox}}^{q+1}(\Omega) \cap B^{q+1}(\Omega)^{\perp} \quad , \\ \text{div } \eta E &= C_{\text{div},\eta} E - \eta \text{div } E \in {}_0D_{s+\tau+1}^{q-1}(\Omega) \cap \mathring{\mathbb{B}}^{q-1}(\Omega)^{\perp} \quad . \end{aligned}$$

The assumptions on  $\tau$  yield  $s + \tau + 1 > 1 - N/2$ . Thus, by Lemma 4.3 there exists  $e \in \mathring{\mathbb{R}}_t^q(\Omega) \cap D_t^q(\Omega)$  with some  $t > -N/2$  solving

$$\text{rot } e = \text{rot } \eta E \quad , \quad \text{div } e = \text{div } \eta E \quad .$$

Therefore,  $H := \eta E - e \in \mathcal{H}_s^q(\Omega)$ . Thus,  $\operatorname{rot} H = 0$  and  $\operatorname{div} H = 0$  in  $A_{r_0}$  and we may represent  $H$  in terms of a spherical harmonics expansion

$$H|_{A_{r_0}} = \sum_{\gamma,n} h_{\gamma,n}^- D_{\gamma,n}^{q,0} + \hat{h} \hat{D}^{q,1} + \sum_{\sigma < -s - \frac{N}{2}} \sum_{m=1}^{\mu_\sigma^q} h_{\sigma,m}^+ D_{\sigma,m}^{q,0}$$

with uniquely determined  $h_{\gamma,n}^-, \hat{h}, h_{\sigma,m}^+ \in \mathbb{C}$  using [9, Theorem 2.6]. By [9, Remark 2.5] the first term of the sum on the right hand side belongs to  $L_{< \frac{N}{2}}^{2,q}(A_{r_0})$ , the second to  $L_{< \frac{N}{2}-1}^{2,q}(A_{r_0})$  and the third to  $L_s^{2,q}(A_{r_0})$ . We get

$$H - \sum_{\sigma < -s - \frac{N}{2}} \sum_{m=1}^{\mu_\sigma^q} e_{\sigma,m}^+ D_{\sigma,m}^{q,0} \in L_{< \frac{N}{2}-1}^{2,q}(A_{r_0}) \quad ,$$

which yields

$$h := H - \sum_{\sigma < -s - \frac{N}{2}} \sum_{m=1}^{\mu_\sigma^q} h_{\sigma,m}^+ E_{\sigma,m}^+ \in L_{> -\frac{N}{2}}^{2,q}(\Omega) \quad .$$

Finally we obtain

$$E - \sum_{\sigma < -s - \frac{N}{2}} \sum_{m=1}^{\mu_\sigma^q} h_{\sigma,m}^+ E_{\sigma,m}^+ = (1 - \eta)E + e + h \in \epsilon \mathcal{H}_{> -\frac{N}{2}}^q(\Omega) = \epsilon \mathcal{H}^q(\Omega) \quad .$$

■

## 6 Appendix B: Vector fields in three dimensions

Now we will translate our results to the classical framework of vector analysis. Thus, we switch to some (maybe) more common notations.

Let  $N := 3$ . We identify 1-forms with vector fields via Riesz' representation theorem and 2-forms with 1-forms via the Hodge star operator and thus with vector fields as well. Using Euclidean coordinates  $\{x_1, x_2, x_3\}$  this means in detail we identify the vector field

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

with the 1-form

$$E_1 dx_1 + E_2 dx_2 + E_3 dx_3$$

resp. with the 2-form

$$E_1 * dx_1 + E_2 * dx_2 + E_3 * dx_3 = E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2 \quad .$$

Moreover, we identify the 3-form  $E dx_1 \wedge dx_2 \wedge dx_3$  with the 0-form and/or function  $E$ . We will denote these identification isomorphisms by  $\cong$ . Then the exterior derivative and co-derivative turn to the classical differential operators

$$\text{grad} = \nabla = \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \quad , \quad \text{curl} = \nabla \times \quad , \quad \text{div} = \nabla \cdot$$

from vector analysis, where  $\times$  resp.  $\cdot$  denotes the vector resp. scalar product in  $\mathbb{R}^3$ . In particular we have the following identification table:

	$q = 0$	$q = 1$	$q = 2$	$q = 3$
rot = d	grad	curl	div	0
div = $\delta$	0	div	$-\text{curl}$	grad

## 6.1 Tower functions and fields

Let us briefly construct our tower forms once more in classical terms. Using polar coordinates  $\{r, \varphi, \vartheta\}$ , i.e.

$$x = \Phi(r, \varphi, \vartheta) = r \begin{bmatrix} \cos \varphi \cos \vartheta \\ \sin \varphi \cos \vartheta \\ \sin \vartheta \end{bmatrix} \quad ,$$

we have with an obvious notation

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = J_\Phi \begin{bmatrix} dr \\ d\varphi \\ d\vartheta \end{bmatrix} = Q \begin{bmatrix} dr \\ r \cos \vartheta d\varphi \\ r d\vartheta \end{bmatrix} \quad ,$$

where  $Q := [\mathbf{e}_r \ \mathbf{e}_\varphi \ \mathbf{e}_\vartheta]$  is an orthonormal matrix and

$$\mathbf{e}_r := \begin{bmatrix} \cos \vartheta \cos \varphi \\ \cos \vartheta \sin \varphi \\ \sin \vartheta \end{bmatrix} \quad , \quad \mathbf{e}_\varphi := \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} \quad , \quad \mathbf{e}_\vartheta := \begin{bmatrix} -\sin \vartheta \cos \varphi \\ -\sin \vartheta \sin \varphi \\ \cos \vartheta \end{bmatrix}$$

the corresponding orthonormal basis of  $\mathbb{R}^3$ . Since  $\{dx_1, dx_2, dx_3\}$  is an orthonormal basis of 1-forms,  $\{dr, r \cos \vartheta d\varphi, r d\vartheta\}$  is an orthonormal basis as well. Moreover, we have again with an obvious notation

$$\begin{bmatrix} dr \\ r \cos \vartheta d\varphi \\ r d\vartheta \end{bmatrix} = Q^t \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_r^t \\ \mathbf{e}_\varphi^t \\ \mathbf{e}_\vartheta^t \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_r \cdot dx \\ \mathbf{e}_\varphi \cdot dx \\ \mathbf{e}_\vartheta \cdot dx \end{bmatrix} \quad ,$$

which shows

$$dr \cong \mathbf{e}_r \quad , \quad r \cos \vartheta d\varphi \cong \mathbf{e}_\varphi \quad , \quad r d\vartheta \cong \mathbf{e}_\vartheta \quad .$$

We will denote the representations of  $\text{grad}$ ,  $\text{curl}$  and  $\text{div}$  in polar coordinates by  $\mathbf{grad}$ ,  $\mathbf{curl}$  and  $\mathbf{div}$  as well as their realizations on  $S := S^2$  by  $\mathbf{grad}_S$ ,  $\mathbf{curl}_S$  and  $\mathbf{div}_S$ . These may be derived by the formula

$$(\nabla_x u) \circ \Phi = J_\Phi^{-t} \nabla_{r,\varphi,\vartheta}(u \circ \Phi) = [\mathbf{e}_r \quad (r \cos \vartheta)^{-1} \mathbf{e}_\varphi \quad r^{-1} \mathbf{e}_\vartheta] \nabla_{r,\varphi,\vartheta}(u \circ \Phi)$$

and we then have the following representations:

$$\begin{aligned} \mathbf{grad} u &= \mathbf{e}_r \partial_r u + \frac{1}{r} \mathbf{grad}_S u \quad , & \mathbf{grad}_S u &= \frac{1}{\cos \vartheta} \mathbf{e}_\varphi \partial_\varphi u + \mathbf{e}_\vartheta \partial_\vartheta u \\ \mathbf{curl} v &= \mathbf{e}_r \times \partial_r v + \frac{1}{r} \mathbf{curl}_S v \quad , & \mathbf{curl}_S v &= \frac{1}{\cos \vartheta} \mathbf{e}_\varphi \times \partial_\varphi v + \mathbf{e}_\vartheta \times \partial_\vartheta v \\ \mathbf{div} v &= \mathbf{e}_r \cdot \partial_r v + \frac{1}{r} \mathbf{div}_S v \quad , & \mathbf{div}_S v &= \frac{1}{\cos \vartheta} \mathbf{e}_\varphi \cdot \partial_\varphi v + \mathbf{e}_\vartheta \cdot \partial_\vartheta v \end{aligned}$$

We note that we do not distinguish between  $u$  resp.  $v$  and  $u \circ \Phi$  resp.  $v \circ \Phi$  anymore. Moreover, in polar coordinates the Laplacian reads

$$\Delta = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta_S \quad ,$$

where

$$\Delta_S = \frac{1}{\cos^2 \vartheta} \partial_\varphi^2 + \partial_\vartheta^2 - \frac{\sin \vartheta}{\cos \vartheta} \partial_\vartheta$$

is the Laplace-Beltrami operator.

Let us introduce the classical spherical harmonics of order  $n$

$$y_{n,m} \quad , \quad n \in \mathbb{N}_0, m = 1, \dots, 2n+1 \quad ,$$

which form a complete orthonormal system in  $L^2(S)$ , i.e.  $\langle y_{n,m}, y_{\ell,k} \rangle_{L^2(S)} = \delta_{n,\ell} \delta_{m,k}$ , and satisfy

$$(\Delta_S + \lambda_n) y_{n,m} = 0 \quad , \quad \lambda_n := n(n+1) \quad ,$$

as well as the corresponding potential functions

$$z_{\pm,n,m} := r^{\theta_{\pm,n}} y_{n,m} \quad , \quad n \in \mathbb{N}_0, m = 1, \dots, 2n+1 \quad ,$$

which are homogeneous of degree

$$\theta_{\pm,n} := \begin{cases} n & , \text{ if } \pm = + \\ -n-1 & , \text{ if } \pm = - \end{cases}$$

and solve

$$\Delta z_{\pm,n,m} = \Delta_S z_{\pm,n,m} = 0 \quad .$$

(See for example [3, Kapitel VII, 4] or [2, chapter 2.3].)

Moreover, for  $k, n \in \mathbb{N}_0, m = 1, \dots, 2n + 1$  we define

$$z_{\pm, n, m}^k := \xi_{\pm, n}^k r^{2k} z_{\pm, n, m} = \xi_{\pm, n}^k r^{\theta_{\pm, n}^{2k}} y_{n, m} \quad ,$$

where  $\xi_{\pm, n}^k := \frac{\Gamma(1 \pm n \pm 1/2)}{4^k \cdot k! \cdot \Gamma(k + 1 \pm n \pm 1/2)}$  and  $\Gamma$  denotes the gamma-function. The  $z_{\pm, n, m}^k$  are homogeneous of degree  $\theta_{\pm, n}^{2k}$  with

$$\theta_{\pm, n}^\ell := \ell + \theta_{\pm, n}$$

and satisfy

$$\Delta z_{\pm, n, m}^k = z_{\pm, n, m}^{k-1} \quad ,$$

where  $z_{\pm, n, m}^{-1} := 0$ . With the aid of these functions, which we will call a 'Δ-tower', we construct for  $k \in \mathbb{N}_0$  the functions and fields

$$\begin{aligned} U_{\pm, n, m}^{2k} &:= z_{\pm, n, m}^k \quad , \\ U_{\pm, n, m}^{2k-1} &:= \text{grad } U_{\pm, n, m}^{2k} \quad , \end{aligned}$$

which we will call a 'div grad-tower', as well as the fields

$$\begin{aligned} V_{\pm, n, m}^{2k} &:= r \mathbf{e}_r \times \text{grad } z_{\pm, n, m}^k = \mathbf{e}_r \times \mathbf{grad}_S z_{\pm, n, m}^k = \xi_{\pm, n}^k r^{\theta_{\pm, n}^{2k}} \mathbf{e}_r \times Y_{n, m} \quad , \\ V_{\pm, n, m}^{2k-1} &:= -\text{curl } V_{\pm, n, m}^{2k} \quad , \end{aligned}$$

which we will call a '- curl curl-tower'. Here  $Y_{n, m} := \mathbf{grad}_S y_{n, m}$ . The fields  $U_{\pm, n, m}^{2k-1}$  are irrotational and the fields  $V_{\pm, n, m}^\ell$  solenoidal. Moreover, we have

$$\text{div } U_{\pm, n, m}^{2k+1} = U_{\pm, n, m}^{2k} \quad , \quad \text{curl } V_{\pm, n, m}^{2k+1} = V_{\pm, n, m}^{2k}$$

as well as  $\text{div } U_{\pm, n, m}^{-1} = 0$  and  $\text{curl } V_{\pm, n, m}^{-1} = 0$  and thus

$$\begin{aligned} \Delta U_{\pm, n, m}^{2k} &= \text{div grad } U_{\pm, n, m}^{2k} = U_{\pm, n, m}^{2k-2} \quad , \\ \Delta U_{\pm, n, m}^{2k+1} &= \text{grad div } U_{\pm, n, m}^{2k+1} = U_{\pm, n, m}^{2k-1} \quad , \\ \Delta V_{\pm, n, m}^\ell &= -\text{curl curl } V_{\pm, n, m}^\ell = V_{\pm, n, m}^{\ell-2} \quad , \end{aligned}$$

where  $U_{\pm, n, m}^{-2} := 0, V_{\pm, n, m}^{-2} := 0$ . We mention that  $U_{\pm, n, m}^\ell$  and  $V_{\pm, n, m}^\ell$  are homogeneous of degree  $\theta_{\pm, n}^\ell$ . Moreover,

$$U_{\pm, n, m}^{-1} = V_{\pm, n, m}^{-1} \quad .$$

Thus, we define

$$P_{\pm, n, m} := U_{\pm, n, m}^1 - V_{\pm, n, m}^1 \quad .$$

Then  $U_{\pm, n, m}^\ell, V_{\pm, n, m}^\ell, \ell = -1, 0$  and even  $P_{\pm, n, m}$  are potential fields resp. functions.



The next picture may illustrate the denotations tower:

	...		...		...
		div ↓		↓ curl	
3. floor		$U_{\pm,n,m}^2$		$V_{\pm,n,m}^2$	$\xrightarrow{\text{div}} 0$
		grad ↓		↓ -curl	
2. floor	0 $\xleftarrow{\text{curl}}$	$U_{\pm,n,m}^1$		$V_{\pm,n,m}^1$	$\xrightarrow{\text{div}} 0$
		div ↓		↓ curl	
1. floor		$U_{\pm,n,m}^0$		$V_{\pm,n,m}^0$	$\xrightarrow{\text{div}} 0$
		grad ↓		↓ -curl	
ground	0 $\xleftarrow{\text{curl}}$	$U_{\pm,n,m}^{-1}$	=	$V_{\pm,n,m}^{-1}$	$\xrightarrow{\text{div}} 0$
		div ↓		↓ curl,div	
		0		0	
		div grad-tower		- curl curl-tower	

In the exceptional case  $(n, m) = (0, 1)$  the function  $z_{\pm,0,1}^k$  is a multiple of  $r^{2k+\theta_{\pm,0}}$  and thus we get  $V_{\pm,0,1}^\ell = 0$  for all  $\ell$  as well as an exceptional div grad-tower

$$U_{\pm,0,1}^{2k} = \xi_{\pm,0}^k r^{2k-\delta_{-, \pm}} \quad ,$$

$$U_{\pm,0,1}^{2k-1} = \text{grad } U_{\pm,0,1}^{2k} = \xi_{\pm,0}^k \partial_r r^{2k-\delta_{-, \pm}} \mathbf{e}_r = \xi_{\pm,0}^k (2k - \delta_{-, \pm}) r^{2k-1-\delta_{-, \pm}} \mathbf{e}_r \quad ,$$

where  $U_{+,0,1}^{-1} = 0$ .

Let us briefly compare these classical towers with the  $q$ -form towers: On  $S$  we identify 1-forms with linear combinations of  $\mathbf{e}_\varphi$  and  $\mathbf{e}_\vartheta$  as well as 2-forms with scalar functions. More precisely a 1-form  $\omega_\varphi \cos \vartheta d\varphi + \omega_\vartheta d\vartheta$  will be identified with the tangential vector field  $\omega_\varphi \mathbf{e}_\varphi + \omega_\vartheta \mathbf{e}_\vartheta$  and a 2-form  $\omega \cos \vartheta d\varphi \wedge d\vartheta$  with the function  $\omega$ . Then our operators  $\rho, \tau$  and  $\check{\rho}, \check{\tau}$  from [15] turn to

	$q = 0$	$q = 1$	$q = 2$	$q = 3$
$\rho v \cong$	0	$v \cdot \mathbf{e}_r$	$v \times \mathbf{e}_r$	$v$
$\tau v \cong$	$v$	$-(v \times \mathbf{e}_r) \times \mathbf{e}_r$	$v \cdot \mathbf{e}_r$	0
$\check{\rho} v \cong$	$v \mathbf{e}_r$	$-v \times \mathbf{e}_r$	$v$	—
$\check{\tau} v \cong$	$v$	$v$	$v \mathbf{e}_r$	—

where

$$-(v \times \mathbf{e}_r) \times \mathbf{e}_r = v \cdot \mathbf{e}_\varphi \mathbf{e}_\varphi + v \cdot \mathbf{e}_\vartheta \mathbf{e}_\vartheta = v - v \cdot \mathbf{e}_r \mathbf{e}_r \quad ,$$

$$v \times \mathbf{e}_r = v \cdot \mathbf{e}_\vartheta \mathbf{e}_\varphi - v \cdot \mathbf{e}_\varphi \mathbf{e}_\vartheta \quad .$$

We note

$$\begin{aligned} y_{0,1} &\cong S_{0,1}^0 \quad \text{is constant,} \\ y_{n,m} &\cong T_{n-1,m}^0 \quad , \\ Y_{n,m} &= \mathbf{grad}_S y_{n,m} \cong i n^{1/2} (n+1)^{1/2} S_{n-1,m}^1 \end{aligned}$$

for  $n \in \mathbb{N}$  in the terminology of [15]. Then for  $n \in \mathbb{N}$  we get up to constants

$$\begin{aligned} U_{\pm,n,m}^{2k} &\cong D_{(\pm,2k+1,n-1,m)}^0 = *R_{(\pm,2k+1,n-1,m)}^3 \quad , \\ U_{\pm,n,m}^{2k-1} &\cong R_{(\pm,2k,n-1,m)}^1 = *D_{(\pm,2k,n-1,m)}^2 \quad , \\ V_{\pm,n,m}^\ell &\cong D_{(\pm,\ell+1,n-1,m)}^1 = *R_{(\pm,\ell+1,n-1,m)}^2 \end{aligned}$$

and for  $n = 0$

$$\begin{aligned} U_{+,0,1}^{2k} &\cong D_{(+,2k,0,1)}^0 = *R_{(+,2k,0,1)}^3 \quad , \\ U_{+,0,1}^{2k-1} &\cong R_{(+,2k-1,0,1)}^1 = *D_{(+,2k-1,0,1)}^2 \quad , \\ U_{-,0,1}^{2k} &\cong D_{(-,2k+2,0,1)}^0 = *R_{(-,2k+2,0,1)}^3 \quad , \\ U_{-,0,1}^{2k-1} &\cong R_{(-,2k+1,0,1)}^1 = *D_{(-,2k+1,0,1)}^2 \quad . \end{aligned}$$

Finally for  $s \in \mathbb{R}$  and  $\ell = -1, 0$  we put (with  $\text{Lin } \emptyset := \{0\}$ )

$$\begin{aligned} \bar{V}_s^\ell &:= \text{Lin} \{V_{-,n,m}^\ell : V_{-,n,m}^\ell \notin L_s^2(A_1)\} = \text{Lin} \{V_{-,n,m}^\ell : n \leq \ell + s + 1/2\} \quad , \\ \bar{U}_s^\ell &:= \text{Lin} \{U_{-,n,m}^\ell : U_{-,n,m}^\ell \notin L_s^2(A_1)\} = \text{Lin} \{U_{-,n,m}^\ell : n \leq \ell + s + 1/2\} \quad , \\ \bar{P}_s &:= \text{Lin} \{P_{-,n,m} : P_{-,n,m} \notin L_s^2(A_1)\} = \text{Lin} \{P_{-,n,m} : n \leq s + 3/2\} \quad , \\ \check{U}^\ell &:= \check{U}_s^\ell := \text{Lin} \{U_{-,0,1}^\ell : U_{-,0,1}^\ell \notin L_s^2(A_1)\} = \text{Lin} \{U_{-,0,1}^\ell : 0 \leq \ell + s + 1/2\} \quad . \end{aligned}$$

## 6.2 Results for vector fields

For every operator  $\diamond \in \{\text{grad}, \text{curl}, \text{div}\}$  and  $s \in \mathbb{R}$  we define the Hilbert spaces

$$\begin{aligned} H_s(\diamond, \Omega) &:= \{u \in L_s^2(\Omega) : \diamond u \in L_{s+1}^2(\Omega)\} \quad , \quad H_s(\overset{\circ}{\diamond}, \Omega) := \overline{\overset{\circ}{C}^\infty(\Omega)} \quad , \\ H_s(\diamond_0, \Omega) &:= \{H_s(\diamond, \Omega) : \diamond u = 0\} \quad , \quad H_s(\overset{\circ}{\diamond}_0, \Omega) := \{H_s(\overset{\circ}{\diamond}, \Omega) : \diamond u = 0\} \quad , \end{aligned}$$

where the closure is taken in  $H_s(\diamond, \Omega)$ . Then the spaces  $\overset{\circ}{R}_s^q(\Omega)$  and  $D_s^q(\Omega)$  turn to the usual Sobolev spaces, i.e.:

	$q = 0$	$q = 1$	$q = 2$	$q = 3$
$\overset{\circ}{R}_s^q(\Omega)$	$H_s(\overset{\circ}{\text{grad}}, \Omega) = \overset{\circ}{H}_s^1(\Omega)$	$H_s(\overset{\circ}{\text{curl}}, \Omega)$	$H_s(\overset{\circ}{\text{div}}, \Omega)$	$L_s^2(\Omega)$
$D_s^q(\Omega)$	$L_s^2(\Omega)$	$H_s(\text{div}, \Omega)$	$H_s(\text{curl}, \Omega)$	$H_s(\text{grad}, \Omega) = H_s^1(\Omega)$

For every two operators  $\diamond, \square \in \{\overset{(\circ)}{\text{grad}}_{(0)}, \overset{(\circ)}{\text{curl}}_{(0)}, \overset{(\circ)}{\text{div}}_{(0)}\}$  we define

$$H_s(\diamond, \square, \Omega) := H_s(\diamond, \Omega) \cap H_s(\square, \Omega) \quad .$$

The operator  $\Delta_\varepsilon$  reads as:

$q$	$\Delta_\varepsilon$
0	$\varepsilon^{-1} \text{div grad}$
1	$\text{grad div} - \varepsilon^{-1} \text{curl curl}$
2	$-\text{curl curl} + \varepsilon^{-1} \text{grad div}$
3	$\text{div grad}$

The generalized boundary condition  $\iota^* E = 0$  for a  $q$ -form  $E$  from  $\overset{\circ}{R}_{\text{loc}}^q(\overline{\Omega})$  turns to the usual boundary conditions

$$\gamma E = E|_{\partial\Omega} = 0 \quad , \quad \gamma_t E = \nu \times E|_{\partial\Omega} = 0 \quad , \quad \gamma_n E = \nu \cdot E|_{\partial\Omega} = 0$$

(for  $q = 0, 1, 2$ ) weakly formulated in the spaces  $H(\overset{\circ}{\text{grad}}, \Omega)$ ,  $H(\overset{\circ}{\text{curl}}, \Omega)$  and  $H(\overset{\circ}{\text{div}}, \Omega)$ , where  $\nu$  denotes the outward unit normal at  $\partial\Omega$  and  $\gamma$  the trace as well as  $\gamma_t$  resp.  $\gamma_n$  the tangential resp. normal trace of the vector field  $E$ . The linear transformations  $\varepsilon, \nu, \mu$  ( $\nu$  and  $\mu$  may be identified!) can be considered as variable, real, symmetric and uniformly positive definite matrices with  $L^\infty(\Omega)$ -entries, which satisfy the asymptotics at infinity assumed in sections 2 and 3.

Moreover, for any two operators  $\diamond, \square \in \{\overset{(\circ)}{\text{curl}}_{(0)}, \overset{(\circ)}{\text{div}}_{(0)}\}$  we define

$$H_s(\square\varepsilon, \Omega) := \varepsilon^{-1} H_s(\square, \Omega) \quad , \quad H_s(\diamond, \square\varepsilon, \Omega) := H_s(\diamond, \Omega) \cap H_s(\square\varepsilon, \Omega) \quad .$$

Now we have two kinds of Dirichlet fields. The first ones, the classical Dirichlet fields,

$${}_\varepsilon \mathcal{H}_s(\Omega) := H_s(\overset{\circ}{\text{curl}}_0, \overset{\circ}{\text{div}}_0 \varepsilon, \Omega) \cong {}_\varepsilon \mathcal{H}_s^1(\Omega) \quad , \quad s \in \mathbb{R}$$

correspond to  $q = 1$  and the second ones, the classical Neumann fields,

$${}_\varepsilon \tilde{\mathcal{H}}_s(\Omega) := H_s(\overset{\circ}{\text{curl}}_0, \overset{\circ}{\text{div}}_0 \varepsilon, \Omega) \cong \varepsilon^{-1} {}_{\varepsilon^{-1}} \mathcal{H}_s^2(\Omega) \quad , \quad s \in \mathbb{R}$$

correspond to  $q = 2$ . Moreover, we have the compactly supported fields

$$\overset{\circ}{B}^1(\Omega) \cong: \overset{\circ}{\mathcal{B}}^1(\Omega) \subset H(\overset{\circ}{\text{curl}}_0, \Omega) \quad , \quad \overset{\circ}{B}^2(\Omega) \cong: \overset{\circ}{\mathcal{B}}^2(\Omega) \subset H(\overset{\circ}{\text{div}}_0, \Omega)$$

and the fields with bounded supports

$$B^2(\Omega) \cong: \mathcal{B}(\Omega) \subset H(\text{curl}_0, \Omega) \quad .$$

Let  $s > -1/2$ . Using the Dirichlet and Neumann fields we put

$$\mathbb{H}_s(\diamond, \Omega) := H_s(\diamond, \Omega) \cap \mathcal{H}(\Omega)^\perp \quad , \quad \tilde{\mathbb{H}}_s(\diamond, \Omega) := H_s(\diamond, \Omega) \cap \tilde{\mathcal{H}}(\Omega)^\perp$$

and define in the same way  $\mathbb{H}_s(\diamond, \square, \Omega)$ ,  $\mathbb{H}_s(\diamond, \square_\varepsilon, \Omega)$  and  $\tilde{\mathbb{H}}_s(\diamond, \square, \Omega)$ ,  $\tilde{\mathbb{H}}_s(\diamond, \square_\varepsilon, \Omega)$ . Then we have

$$\begin{aligned}\mathbb{H}_s(\operatorname{div}_0, \Omega) &= H_s(\operatorname{div}_0, \Omega) \cap {}_\varepsilon\mathcal{H}(\Omega)^\perp = H_s(\operatorname{div}_0, \Omega) \cap \mathring{\mathcal{B}}^1(\Omega)^\perp, \\ \tilde{\mathbb{H}}_s(\operatorname{div}_0, \Omega) &= H_s(\operatorname{div}_0, \Omega) \cap {}_\varepsilon\tilde{\mathcal{H}}(\Omega)^\perp = H_s(\operatorname{div}_0, \Omega) \cap \mathcal{B}(\Omega)^\perp, \\ \tilde{\mathbb{H}}_s(\operatorname{curl}_0, \Omega) &= H_s(\operatorname{curl}_0, \Omega) \cap {}_\varepsilon\tilde{\mathcal{H}}(\Omega)^{\perp_\varepsilon} = H_s(\operatorname{curl}_0, \Omega) \cap \mathring{\mathcal{B}}^2(\Omega)^\perp, \\ \mathbb{H}_s(\operatorname{curl}_0, \Omega) &= H_s(\operatorname{curl}_0, \Omega) \cap {}_\varepsilon\mathcal{H}(\Omega)^{\perp_\varepsilon},\end{aligned}$$

and thus except of the last one the definitions of these spaces extend to all  $s \in \mathbb{R}$ . Moreover, we set for  $s > -1/2$

$${}_\varepsilon\mathbb{L}_s^2(\Omega) := L_s^2(\Omega) \cap {}_\varepsilon\mathcal{H}(\Omega)^{\perp_\varepsilon}, \quad {}_\varepsilon\tilde{\mathbb{L}}_s^2(\Omega) := L_s^2(\Omega) \cap {}_\varepsilon\tilde{\mathcal{H}}(\Omega)^{\perp_\varepsilon}.$$

We get

**Lemma 6.1** *Let  $s > -1/2$ . Then the direct decompositions*

$$\begin{aligned}L_s^2(\Omega) &= {}_\varepsilon\mathbb{L}_s^2(\Omega) \dot{+} \operatorname{Lin} \mathring{\mathcal{B}}^1(\Omega), \\ L_s^2(\Omega) &= {}_\varepsilon\tilde{\mathbb{L}}_s^2(\Omega) \dot{+} \varepsilon^{-1} \operatorname{Lin} \mathring{\mathcal{B}}^2(\Omega) = {}_\varepsilon\tilde{\mathbb{L}}_s^2(\Omega) \dot{+} \operatorname{Lin} \mathcal{B}(\Omega)\end{aligned}$$

hold. If additionally  $s < 1/2$ , then

$$L_s^2(\Omega) = {}_\varepsilon\mathbb{L}_s^2(\Omega) \oplus_\varepsilon {}_\varepsilon\mathcal{H}(\Omega), \quad L_s^2(\Omega) = {}_\varepsilon\tilde{\mathbb{L}}_s^2(\Omega) \oplus_\varepsilon {}_\varepsilon\tilde{\mathcal{H}}(\Omega).$$

With

$$\square_\varepsilon := \operatorname{grad} \operatorname{div} - \varepsilon^{-1} \operatorname{curl} \operatorname{curl} = \Delta - \check{\varepsilon} \operatorname{curl} \operatorname{curl}$$

and always assuming  $s \notin \tilde{\mathbb{I}}$ , i.e. for all  $n \in \mathbb{N}_0$

$$s \neq n + 1/2 \quad \text{and} \quad s \neq -n - 3/2,$$

we obtain

**Theorem 6.2** *The following decompositions hold:*

(i) *If  $s < -3/2$ , then*

$$\begin{aligned}L_s^2(\Omega) &= H_s(\operatorname{curl}_0, \Omega) + \varepsilon^{-1} H_s(\operatorname{div}_0, \Omega) = H_s(\operatorname{curl}_0, \Omega) + \varepsilon^{-1} H_s(\operatorname{div}_0, \Omega) \\ &= H_s(\operatorname{curl}_0, \Omega) + \varepsilon^{-1} \mathbb{H}_s(\operatorname{div}_0, \Omega) = \tilde{\mathbb{H}}_s(\operatorname{curl}_0, \Omega) + \varepsilon^{-1} \tilde{\mathbb{H}}_s(\operatorname{div}_0, \Omega).\end{aligned}$$

*In the first line the intersections equal the finite dimensional space of Dirichlet resp. Neumann fields  ${}_\varepsilon\mathcal{H}_s(\Omega)$  resp.  ${}_\varepsilon\tilde{\mathcal{H}}_s(\Omega)$  and in the second line the intersections equal the finite dimensional space of Dirichlet resp. Neumann fields  ${}_\varepsilon\mathcal{H}_s(\Omega) \cap \mathring{\mathcal{B}}^1(\Omega)^{\perp_\varepsilon}$  resp.  ${}_\varepsilon\tilde{\mathcal{H}}_s(\Omega) \cap \mathring{\mathcal{B}}^2(\Omega)^\perp$ .*

(ii) If  $-3/2 < s \leq -1/2$ , then

$$\begin{aligned} L_s^2(\Omega) &= \mathbb{H}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \mathbb{H}_s(\text{div}_0, \Omega) \\ &= \tilde{\mathbb{H}}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \tilde{\mathbb{H}}_s(\text{div}_0, \Omega) \dot{+} \varepsilon \tilde{\mathcal{H}}(\Omega) \quad . \end{aligned}$$

(iii) If  $-1/2 < s < 3/2$ , then

$$\begin{aligned} \varepsilon L_s^2(\Omega) &= \mathbb{H}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \mathbb{H}_s(\text{div}_0, \Omega) \quad , \\ \varepsilon \tilde{L}_s^2(\Omega) &= \tilde{\mathbb{H}}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \tilde{\mathbb{H}}_s(\text{div}_0, \Omega) \quad . \end{aligned}$$

For  $s \geq 0$  this decomposition is even  $\langle \varepsilon \cdot, \cdot \rangle_{L^2(\Omega)}$ -orthogonal.

(iv) If  $s > 3/2$ , then

$$\begin{aligned} \varepsilon L_s^2(\Omega) &= \left( ([L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}] \cap \mathbb{H}_{< \frac{3}{2}}(\text{curl}_0, \Omega)) \right. \\ &\quad \left. \oplus_\varepsilon \varepsilon^{-1} ([L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}] \cap \mathbb{H}_{< \frac{3}{2}}(\text{div}_0, \Omega)) \right) \cap L_s^2(\Omega) \quad , \\ \varepsilon \tilde{L}_s^2(\Omega) &= \left( ([L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}] \cap \tilde{\mathbb{H}}_{< \frac{3}{2}}(\text{curl}_0, \Omega)) \right. \\ &\quad \left. \oplus_\varepsilon \varepsilon^{-1} ([L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}] \cap \tilde{\mathbb{H}}_{< \frac{3}{2}}(\text{div}_0, \Omega)) \right) \cap L_s^2(\Omega) \end{aligned}$$

and

$$\begin{aligned} \varepsilon L_s^2(\Omega) &= \mathbb{H}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \mathbb{H}_s(\text{div}_0, \Omega) \dot{+} \square_\varepsilon \eta \bar{\mathcal{P}}_{s-2} \quad , \\ \varepsilon \tilde{L}_s^2(\Omega) &= \tilde{\mathbb{H}}_s(\text{curl}_0, \Omega) \dot{+} \varepsilon^{-1} \tilde{\mathbb{H}}_s(\text{div}_0, \Omega) \dot{+} \square_\varepsilon \eta \bar{\mathcal{P}}_{s-2} \quad , \end{aligned}$$

where the first two terms in the latter two decomposition are  $\langle \varepsilon \cdot, \cdot \rangle_{L^2(\Omega)}$ -orthogonal as well. Furthermore,

$$\begin{aligned} L_s^2(\Omega) \cap \varepsilon \mathcal{H}_{-s}(\Omega)^{\perp_\varepsilon} &= \mathbb{H}_s(\text{curl}_0, \Omega) \oplus_\varepsilon \varepsilon^{-1} \mathbb{H}_s(\text{div}_0, \Omega) \quad , \\ L_s^2(\Omega) \cap \varepsilon \tilde{\mathcal{H}}_{-s}(\Omega)^{\perp_\varepsilon} &= \tilde{\mathbb{H}}_s(\text{curl}_0, \Omega) \oplus_\varepsilon \varepsilon^{-1} \tilde{\mathbb{H}}_s(\text{div}_0, \Omega) \quad . \end{aligned}$$

Here Remark 3.3 holds analogously. In particular the matrix  $\varepsilon$  may be moved to the curl-free terms in our decompositions as well and for  $s < -3/2$  and  $\tau \geq 1/2$  we have

$$\begin{aligned} \varepsilon \mathcal{H}_s(\Omega) &= \varepsilon \mathcal{H}(\Omega) \dot{+} \varepsilon \mathcal{H}_s(\Omega) \cap \mathring{\mathcal{B}}^1(\Omega)^{\perp_\varepsilon} \quad , \\ \varepsilon \tilde{\mathcal{H}}_s(\Omega) &= \varepsilon \tilde{\mathcal{H}}(\Omega) \dot{+} \varepsilon \tilde{\mathcal{H}}_s(\Omega) \cap \mathring{\mathcal{B}}^2(\Omega)^\perp \quad . \end{aligned}$$

**Theorem 6.3** *Let  $s \in \mathbb{R}$ . Then*

$$(i) \quad \tilde{\mathbb{H}}_s(\text{curl}_0, \Omega) = \text{grad } H_{s-1}(\text{grad}, \Omega)$$

and for  $s > -1/2$

$$(i') \quad \mathbb{H}_s(\overset{\circ}{\text{curl}}_0, \Omega) = \text{grad } H_{s-1}(\overset{\circ}{\text{grad}}, \Omega) \quad .$$

If  $s < 5/2$ , then

$$(ii) \quad \begin{aligned} \tilde{\mathbb{H}}_s(\overset{\circ}{\text{div}}_0, \Omega) &= \text{curl } \mathbb{H}_{s-1}(\overset{\circ}{\text{curl}}, \text{div}_0 \mu, \Omega) \\ &= \text{curl } H_{s-1}(\overset{\circ}{\text{curl}}, \text{div}_0 \mu, \Omega) = \text{curl } H_{s-1}(\overset{\circ}{\text{curl}}, \Omega) \quad , \\ \mathbb{H}_s(\text{div}_0, \Omega) &= \text{curl } \tilde{\mathbb{H}}_{s-1}(\text{curl}, \overset{\circ}{\text{div}}_0 \mu, \Omega) \\ &= \text{curl } H_{s-1}(\text{curl}, \overset{\circ}{\text{div}}_0 \mu, \Omega) = \text{curl } H_{s-1}(\text{curl}, \Omega) \quad . \end{aligned}$$

All these spaces are closed subspaces of  $L_s^2(\Omega)$ . For  $s < 3/2$

$$(iii) \quad \begin{aligned} L_s^2(\Omega) &= \text{div } \tilde{\mathbb{H}}_{s-1}(\overset{\circ}{\text{div}}, \text{curl}_0 \mu, \Omega) \\ &= \text{div } H_{s-1}(\overset{\circ}{\text{div}}, \text{curl}_0 \mu, \Omega) = \text{div } H_{s-1}(\overset{\circ}{\text{div}}, \Omega) \quad , \\ L_s^2(\Omega) &= \text{div } \mathbb{H}_{s-1}(\text{div}, \overset{\circ}{\text{curl}}_0 \mu, \Omega) \quad , \text{ if } s > 1/2 \quad , \\ L_s^2(\Omega) &= \text{div } H_{s-1}(\text{div}, \overset{\circ}{\text{curl}}_0 \mu, \Omega) = \text{div } H_{s-1}(\text{div}, \Omega) \quad . \end{aligned}$$

**Theorem 6.4**

(i) For  $s > 5/2$

$$\begin{aligned} \tilde{\mathbb{H}}_s(\overset{\circ}{\text{div}}_0, \Omega) &= \text{curl} \left( (H_{s-1}(\overset{\circ}{\text{curl}}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1}) \cap \mathbb{H}_{< \frac{3}{2}}(\text{div}_0 \mu, \Omega) \right) \\ &= \text{curl} \left( H_{s-1}(\overset{\circ}{\text{curl}}, \text{div } \mu, \Omega) \cap \overset{\circ}{\mathcal{B}}^1(\Omega)^{\perp \mu} \right) = \text{curl } H_{s-1}(\overset{\circ}{\text{curl}}, \Omega) \quad , \\ \mathbb{H}_s(\text{div}_0, \Omega) &= \text{curl} \left( (H_{s-1}(\text{curl}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1}) \cap \tilde{\mathbb{H}}_{< \frac{3}{2}}(\overset{\circ}{\text{div}}_0 \mu, \Omega) \right) \\ &= \text{curl} \left( H_{s-1}(\text{curl}, \text{div } \mu, \Omega) \cap \mathcal{B}(\Omega)^{\perp \mu} \right) = \text{curl } H_{s-1}(\text{curl}, \Omega) \end{aligned}$$

are closed subspaces of  $L_s^2(\Omega)$ .

(ii) For  $s > 3/2$

$$\begin{aligned} & (L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}) \cap \mathbb{H}_{< \frac{3}{2}}(\mathring{\text{curl}}_0, \Omega) \\ &= \text{grad} \left( H_{s-1}(\mathring{\text{grad}}, \Omega) \boxplus \eta \bar{\mathcal{U}}_{s-1}^0 \right) = \mathbb{H}_s(\mathring{\text{curl}}_0, \Omega) \dot{+} \text{grad} \eta \bar{\mathcal{U}}_{s-1}^0 \quad , \end{aligned}$$

$$\begin{aligned} & (L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}) \cap \tilde{\mathbb{H}}_{< \frac{3}{2}}(\mathring{\text{curl}}_0, \Omega) \\ &= \text{grad} \left( H_{s-1}(\mathring{\text{grad}}, \Omega) \boxplus \eta \bar{\mathcal{U}}_{s-1}^0 \right) = \tilde{\mathbb{H}}_s(\mathring{\text{curl}}_0, \Omega) \dot{+} \text{grad} \eta \bar{\mathcal{U}}_{s-1}^0 \quad , \end{aligned}$$

$$\begin{aligned} & (L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}) \cap \tilde{\mathbb{H}}_{< \frac{3}{2}}(\mathring{\text{div}}_0, \Omega) \\ &= \text{curl} \left( (H_{s-1}(\mathring{\text{curl}}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1} \boxplus \eta \bar{\mathcal{V}}_{s-1}^0) \cap \mathbb{H}_{< \frac{1}{2}}(\mathring{\text{div}}_0 \mu, \Omega) \right) \\ &= \tilde{\mathbb{H}}_s(\mathring{\text{div}}_0, \Omega) \dot{+} \text{curl} \mu^{-1} \eta \bar{\mathcal{V}}_{s-1}^0 \quad , \end{aligned}$$

$$\begin{aligned} & (L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}) \cap \mathbb{H}_{< \frac{3}{2}}(\mathring{\text{div}}_0, \Omega) \\ &= \text{curl} \left( (H_{s-1}(\mathring{\text{curl}}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1} \boxplus \eta \bar{\mathcal{V}}_{s-1}^0) \cap \tilde{\mathbb{H}}_{< \frac{1}{2}}(\mathring{\text{div}}_0 \mu, \Omega) \right) \\ &= \mathbb{H}_s(\mathring{\text{div}}_0, \Omega) \dot{+} \text{curl} \mu^{-1} \eta \bar{\mathcal{V}}_{s-1}^0 \end{aligned}$$

are closed subspaces of  $L_s^2(\Omega) \boxplus \eta \bar{\mathcal{V}}_s^{-1}$ . Moreover,  $\text{div} \eta V_{-,n,m}^0 = 0$ .

**Theorem 6.5** *Let  $s > 3/2$ . Then*

$$\begin{aligned} \text{(i)} \quad L_s^2(\Omega) &= \text{div} \left( (H_{s-1}(\mathring{\text{div}}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1} \boxplus \eta \check{\mathcal{U}}^{-1}) \cap \tilde{\mathbb{H}}_{< \frac{1}{2}}(\mathring{\text{curl}}_0 \mu, \Omega) \right) \\ &= \text{div} \left( (H_{s-1}(\mathring{\text{div}}, \mathring{\text{curl}} \mu, \Omega) \cap \mathring{\mathcal{B}}^2(\Omega)^{\perp \mu}) \boxplus \eta \check{\mathcal{U}}^{-1} \right) \\ &= \text{div} \left( H_{s-1}(\mathring{\text{div}}, \mathring{\text{curl}} \mu, \Omega) \cap \mathring{\mathcal{B}}^2(\Omega)^{\perp \mu} \right) \dot{+} \Delta \eta \check{\mathcal{U}}^0 \\ &= \text{div} H_{s-1}(\mathring{\text{div}}, \Omega) \dot{+} \Delta \eta \check{\mathcal{U}}^0 \quad , \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad L_s^2(\Omega) &= \text{div} \left( (H_{s-1}(\mathring{\text{div}}, \Omega) \boxplus \eta \bar{\mathcal{V}}_{s-1}^{-1} \boxplus \eta \check{\mathcal{U}}^{-1}) \cap \mathbb{H}_{< \frac{1}{2}}(\mathring{\text{curl}}_0 \mu, \Omega) \right) \\ &= \text{div} \left( H_{s-1}(\mathring{\text{div}}, \mathring{\text{curl}} \mu, \Omega) \boxplus \eta \check{\mathcal{U}}^{-1} \right) \\ &= \text{div} H_{s-1}(\mathring{\text{div}}, \mathring{\text{curl}} \mu, \Omega) \dot{+} \Delta \eta \check{\mathcal{U}}^0 \\ &= \text{div} H_{s-1}(\mathring{\text{div}}, \Omega) \dot{+} \Delta \eta \check{\mathcal{U}}^0 \quad . \end{aligned}$$

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