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Generalized Maxwell Equations in Exterior Domains I: Regularity Results, Trace Theorems and Static Solution Theory

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Generalized Maxwell Equations in Exterior Domains I: Regularity Results, Trace Theorems and Static Solution Theory*

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Abstract

We discuss the generalized Maxwell equations on Riemannian manifolds with boundary using the calculus of alternating differential forms. We prove regularity results up to the boundary as well as 'polynomially weighted' regularity in exterior domains of $\mathbb{R}^N$. Moreover, trace and corresponding extension theorems will be presented. Finally we use these results to establish an electro-magneto static solution theory for a generalized Maxwell system on Riemannian manifolds with compact closure as well as in exterior domains of $\mathbb{R}^N$.

Key words: boundary value problems for Maxwell equations, electro-magneto static, variable coefficients, regularity results, trace theorems, extension theorems

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary $\partial\Omega$. Then the classical time-dependent Maxwell equations are

$$\operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0} \quad , \quad -\operatorname{curl}\mathbf{H} + \partial_t\mathbf{D} = \mathbf{I} \quad ,$$
$$\operatorname{div}\mathbf{D} = \boldsymbol{\rho} \quad , \quad \operatorname{div}\mathbf{B} = \mathbf{0}$$

in Ω, where $\mathbf{E}$ resp. $\mathbf{H}$ is the electric resp. magnetic field, $\mathbf{D}$ resp. $\mathbf{B}$ the displacement current resp. magnetic induction and $\mathbf{I}$ resp. $\boldsymbol{\rho}$ the current resp. charge density. Here the gradient $\operatorname{grad} = \nabla$ and $\operatorname{curl} = \nabla\times$, $\operatorname{div} = \nabla\cdot$ denote the usual differential operators from vector analysis and $\times$ resp. $\cdot$ the vector resp. scalar product in $\mathbb{R}^3$. A time-harmonic ansatz leads to the time-harmonic Maxwell equations

$$\operatorname{curl}E + \mathrm{i}\,\omega\mu H = 0 \quad , \quad -\operatorname{curl}H + \mathrm{i}\,\omega\varepsilon E = I \quad , \tag{1.1}$$
$$\operatorname{div}\varepsilon E = \rho = -\frac{\mathrm{i}}{\omega}\operatorname{div}I \quad , \quad \operatorname{div}\mu H = 0 \tag{1.2}$$

in Ω with complex frequency $\omega \neq 0$. Here we assumed that the relations $D = \varepsilon E$ and $B = \mu H$ hold, where the matrix valued functions ε and μ, which are supposed to be uniformly positive definite, bounded and symmetric, describe material properties, i.e. the dielectricity and permeability. If we let $\partial\,\Omega$ be a perfect conductor, then the tangential component of the electric field vanishes at the boundary $\partial\,\Omega$ and thus so does the normal component of μH by the first equation in (1.1) and the relation $\nu\cdot\operatorname{curl}|_{\partial\,\Omega} = -\operatorname{div}_{\partial\,\Omega}\,\nu\times|_{\partial\,\Omega}$, where $\operatorname{div}_{\partial\,\Omega}$ denotes the surface divergence and ν the outward unit normal at $\partial\,\Omega$. This motivates to impose boundary conditions like

$$\nu\times E = \lambda \quad , \quad \nu\cdot\mu H = \varkappa \quad \text{on} \quad \partial\,\Omega \tag{1.3}$$

for some given vector resp. scalar valued function λ resp. $\varkappa$. In the case $\omega = 0$ the time-harmonic Maxwell system (1.1), (1.2) turns to the decoupled static Maxwell system

$$\operatorname{curl} E = 0 \quad , \quad -\operatorname{curl} H = I \quad , \tag{1.4}$$

$$\operatorname{div} \varepsilon E = \rho \quad , \quad \operatorname{div} \mu H = 0 \tag{1.5}$$

in Ω.

In 1952 Hermann Weyl [24] suggested a generalization of (1.4), (1.5) and (1.3) on Riemannian manifolds Ω of arbitrary dimension $N \in \mathbb{N}$ within the framework of alternating differential forms. If we let E and $F(= I)$ be differential forms of rank q for some $q \in \mathbb{Z}$, shortly q-forms, H and $G(= 0)$ respectively $(q+1)$-forms, $f(= \rho)$ a $(q-1)$-form and last but not least $g(= 0)$ a $(q+2)$-form, then we call

$$\begin{aligned} \mathrm{d}E &= G \quad , & \delta H &= F \quad , \\ \delta\varepsilon E &= f \quad , & \mathrm{d}\mu H &= g \quad , \\ \iota^* E &= \lambda \quad , & \iota^* \mu H &= \varkappa \end{aligned}$$

on Ω the generalized static Maxwell system, where d denotes the exterior differential, $\delta = \pm * \mathrm{d} *$ the co-differential, $*$ the Hodge star operator and ι^* the pullback of the natural embedding $\iota : \partial\Omega \hookrightarrow \overline{\Omega}$. Moreover, now ε resp. μ is a linear transformation on q- resp. $(q+1)$-forms and λ resp. $\varkappa$ is a q- resp. $(q+1)$-form on the $(N-1)$-dimensional Riemannian submanifold $\partial\Omega$ of $\overline{\Omega}$. For $N = 3$, $q = 1$ and some domain $\Omega \subset \mathbb{R}^3$ interpreted as a Riemannian submanifold of $\mathbb{R}^3$ we get back our classical system (1.4), (1.5), (1.3), if we identify 0- and 3-forms with scalar functions, 1-forms with vector fields via the Riesz representation theorem and 2-forms with 1-forms by the star operator and thus with vector fields as well. Then the exterior differential d acts on 0-, 1-, 2- resp. 3-forms as grad, curl, div resp. the zero mapping and the co-differential δ as the zero mapping, div, $-\operatorname{curl}$ resp. grad.

It is sufficient to study the electro static system for E, since we obtain the magneto static system for H replacing q by $q+1$, εE by H and ε^{-1} by μ.

In this paper we want to establish a solution theory for the electro static Maxwell system

$$\mathrm{d}E = G \quad , \quad \delta\varepsilon E = F \quad , \quad \iota^* E = \lambda$$

on N-dimensional Riemannian manifolds Ω with compact closure in section 2 as well as on exterior domains $\Omega \subset \mathbb{R}^N$ in section 3. In order to use Hilbert space methods we will formulate this system in the usual weak sense. To remind of the electro-magnetic background we denote the operator d resp. δ (acting on smooth forms) by rot resp. div. We use the well known Hodge-Helmholtz decompositions from Picard [12, 16] as well as the compactness results from Weck [21] and Picard [15] to obtain static solutions, which satisfy the homogeneous boundary condition.

To handle the inhomogeneous boundary condition we assume that Ω possesses a C^3-boundary, and characterize the traces of differential forms $E \in \mathbf{R}^q(\Omega)$[1], i.e.

[1] Exact definitions will be supplied in sections 2.1 and 3.1.

$E \in \mathrm{L}^{2,q}(\Omega)$ and E has a weak rotation $\operatorname{rot} E \in \mathrm{L}^{2,q+1}(\Omega)$. We show the existence of a continuous and surjective tangential trace operator

$$\Gamma_t \ : \ \mathbf{R}^q(\Omega) \ \longrightarrow \ \mathcal{R}^q(\partial\Omega) \qquad \text{resp.} \qquad \Gamma_t \ : \ \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega}) \ \longrightarrow \ \mathcal{R}^q(\partial\Omega) \qquad ,$$

which coincides with ι^* on smooth forms and where the latter one acts in exterior domains. By the star operator we easily get the corresponding normal trace operator $\Gamma_n = \pm \circledast \Gamma_t *$ as well. Here $\circledast$ denotes the star operator on $\partial\,\Omega$. The space $\mathcal{R}^q(\partial\Omega)$ is defined as the space of boundary differential forms $\lambda \in \mathbf{H}^{-1/2,q}(\partial\Omega)$ having a weak boundary rotation $\operatorname{Rot} \lambda \in \mathbf{H}^{-1/2,q+1}(\partial\Omega)$. Here $\mathbf{H}^{-1/2,q}(\partial\Omega)$ is the dual space of $\mathbf{H}^{1/2,q}(\partial\Omega)$. For instance, for smooth boundaries such trace results have been proved by Paquet [9]. In [2, 4] one can find corresponding results for the classical Maxwell equation in Lipschitz domains of $\mathbb{R}^3$. Recently Weck [22] generalized the results from [4] to our general setting.

The usage of the Sobolev spaces $\mathbf{H}^{m,q}(\Omega)$ within our trace theory requires regularity results up to the boundary suited for Maxwell equations. To prove these we follow the ideas of Weber, who showed such results in [19] for vector fields in the classical case of $\Omega \subset \mathbb{R}^3$ (see also [8]). The discussion of exterior domains needs similar results for weighted Sobolev spaces.

Finally we present a solution theory for the problem

$$\operatorname{rot} E = G \qquad , \qquad \operatorname{div} \varepsilon E = F \qquad , \qquad \Gamma_t E = \lambda \tag{1.6}$$

with

$$E \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega) \qquad \text{resp.} \qquad E \in \mathrm{R}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega) \qquad ,$$

if Ω resp. $\mathbb{R}^N \setminus \Omega$ is bounded. To achieve uniqueness we additionally have to impose some suitable orthogonality constraints on E, since the problem (1.6) has a nontrivial finite dimensional kernel ${}_\varepsilon\mathcal{H}^q(\Omega)$, the harmonic Dirichlet forms.

The static Maxwell boundary value problem (1.6) has been investigated by Kress [6] and Picard [12] for the homogeneous, isotropic case, i.e. $\varepsilon = \mathrm{id}$, by Picard [16] for the inhomogeneous, anisotropic case as well as by Picard [13] for the inhomogeneous, anisotropic classical case. All these results only cover the homogeneous boundary condition.

Essentially section 2 is the main part of the first authors ph.d. thesis and section 3 contains some results from the second authors ph.d. thesis. Thus we refer the interested reader to [7] and [10] for more details on the proofs or some additional results.

2 Manifolds with compact closure

We will distinguish between two fundamentally different cases, i.e. Ω or its complement is bounded. In this section we consider the first case, i.e. Ω is an open subset with compact closure of some C^∞-Riemannian manifold M of arbitrary dimension N.

2.1 Notations and preliminaries

We denote the sets of positive integers, nonnegative integers, integers, reals, positive reals and complex numbers by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ respectively. If z is a complex number we write $\bar{z}$ for the conjugation. The Euclidean norm in $\mathbb{R}^N$ resp. $\mathbb{C}^N$ is denoted by $r := |\cdot|$. If U, V are subsets of some metric space (X, d), we write $\overline{U}$ or $\overline{U}^d$ for the closure and ∂U for the boundary of U. We say $U \Subset V$, if $\overline{U}$ is compact and $\overline{U} \subset V$. $U_r(x)$, $K_r(x)$ resp. $S_r(x)$ is the open, closed ball resp. sphere with radius r around x. If $x = 0$ we often omit this argument. Furthermore, for $U_r \subset \mathbb{R}^N$ we define

$$U_r^{\pm} := \{x \in U_r : \pm x_N > 0\} \qquad , \qquad U_r^0 := \{x \in U_r : x_N = 0\} \qquad .$$

Let X be some normed vector space. Then $\|\cdot\|_X$ denotes its norm and $\langle\cdot\,,\cdot\rangle_X$ its scalar product with naturally induced norm $\|\cdot\|_X = \big(\langle\cdot\,,\cdot\rangle_X\big)^{1/2}$, if X even has a scalar product. For two subspaces U and V of X with $U \cap V = \emptyset$ we denote the direct sum by $U \dotplus V$ and, if X possesses a scalar product and U, V are orthogonal to each other, we write $U \oplus V$ for the orthogonal sum. The adjoint resp. closure of a densely defined linear operator A is denoted by A^* resp. $\overline{A}$ and the space of bounded linear operators from X into Y by $B(X, Y)$. For the commutator of two operators A, B use the symbol $C_{A,B} := AB - BA$.

Let f be a mapping. We use the notation $D(f)$ for its domain of definition, $W(f)$ for its range and $N(f)$ for its kernel. $f|_U$ is the restriction of f to $U \subset D(f)$. The support of f is denoted by $\operatorname{supp} f$. Let U be an open subset of $\mathbb{R}^N$. For $m \in \mathbb{N}_0 \cup \{\infty\}$ and $p \in [1, \infty]$ we define

$$\begin{aligned}
\mathrm{C}^m(U) &:= \{f : U \to \mathbb{C} : f \text{ is } m\text{-times continuously differentiable.}\} \qquad , \\
\overset{\circ}{\mathrm{C}}{}^m(U) &:= \big\{f \in \mathrm{C}^m(U) : \operatorname{supp} f \Subset U\big\} \qquad , \\
\overset{\circ}{\mathrm{C}}{}^m(\overline{U}) &:= \big\{\, f|_U : f \in \overset{\circ}{\mathrm{C}}{}^m(\mathbb{R}^N)\big\} \qquad , \\
\mathrm{L}^p(U) &:= \big\{f : U \to \mathbb{C} : f \text{ Lebesgue-measurable with } \|f\|_{\mathrm{L}^p(U)} < \infty\big\} \qquad , \\
\mathbf{H}^m(U) &:= \big\{f \in \mathrm{L}^2(U) : \partial^\alpha f \in \mathrm{L}^2(U) \text{ for all } |\alpha| \le m\big\} \qquad .
\end{aligned}$$

Here we have $\|f\|_{\mathrm{L}^p(U)} := \big(\int_U |f|^p \, d\lambda\big)^{1/p}$ for $p \in [1, \infty)$ and $\|f\|_{\mathrm{L}^\infty(U)} := \operatorname{ess\,sup}_U |f|$, where λ is the Lebesgue measure, as well as

$$\langle f, g\rangle_{\mathrm{L}^2(U)} := \int_U f \bar{g} \, d\lambda \qquad , \qquad \langle f, g\rangle_{\mathbf{H}^m(U)} := \sum_{|\alpha| \le m} \langle \partial^\alpha f, \partial^\alpha g\rangle_{\mathrm{L}^2(U)} \qquad .$$

See Agmon $\big[$[1], chapters 2 and 3$\big]$ for an exact definition of the Sobolev spaces $\mathbf{H}^m(U)\big(= W_m(U)$ in his notation$\big)$. We note that the Sobolev spaces $\mathbf{H}^m(U)$ also may be defined for $m \in [0, \infty]$.

We denote the Kronecker symbol by $\delta_{i,j}$. Empty sums or undefined terms will always be set to zero. We often use c as a constant, which may change during a proof. Moreover, we assume the summation convention.

Now let M be a complete N-dimensional, real, C^∞-differentiable manifold with orientation and Riemannian metric, short manifold. First we collect some results from [3] or [5] and [21]: For each $x \in \mathrm{M}$ there exist a chart (V,h) around x, i.e. an open neighbourhood $V \subset \mathrm{M}$ of x and a homeomorphism $h : V \to U$ onto an open subset $U = h(V)$ of $\mathbb{R}^N$. The changing of charts is C^∞. In our notation each diffeomorphism and its inverse is bounded and has bounded derivatives. Let $\mathrm{A}^q(x)$ be the complex linear space of alternating covariant tensors of rank q acting on the tangent space $\mathrm{T}_x(\mathrm{M})$ in x and $\mathrm{A}^q(\mathrm{M})$ its bundle. Elements of the latter space are called q-forms or forms. In the case $q < 0$ or $q > N$ we identify a q-form with the zero mapping and $\mathrm{A}^0(\mathrm{M})$ is the space of the complex valued functions on M. The exterior product

$$\wedge : \mathrm{A}^q(\mathrm{M}) \times \mathrm{A}^p(\mathrm{M}) \to \mathrm{A}^{q+p}(\mathrm{M})$$

acts pointwise and satisfies

$$\bigwedge_{\Phi\in\mathrm{A}^q(\mathrm{M})} \quad \bigwedge_{\Psi\in\mathrm{A}^p(\mathrm{M})} \qquad \Phi \wedge \Psi = (-1)^{qp} \cdot \Psi \wedge \Phi \qquad .$$

Any chart (V,h) induces special tangential vectors $\partial_j^h \in \mathrm{T}_x(\mathrm{M})$ for all $x \in V$ by $\partial_j^h f := \big(\partial_j(f \circ h^{-1})\big) \circ h$. We have $\partial_j^h h_i = \delta_{j,i}$ and thus $\{\partial_1^h, \dots, \partial_N^h\}$ is a basis of $\mathrm{T}_x(\mathrm{M})$ for all $x \in V$. Moreover, the differential

$$\mathrm{d}\tau : \mathrm{T}(\mathrm{M}) \to \mathrm{T}(\tilde{\mathrm{M}})$$

of a differentiable mapping $\tau : \mathrm{M} \to \tilde{\mathrm{M}}$ acts pointwise as

$$\mathrm{d}\tau(\partial)(f) := \partial(f \circ \tau)$$

for all $\partial \in \mathrm{T}_x(\mathrm{M})$ and satisfies the chain rule $\mathrm{d}(\tau_2 \circ \tau_1) = \mathrm{d}\tau_2 \circ \mathrm{d}\tau_1$. Locally using charts h for M and $\tilde{h}$ for $\tilde{\mathrm{M}}$ we have

$$\{\mathrm{d}\tau\}_{\partial^h}^{\partial^{\tilde{h}}} = J_{\tilde{\tau}} \qquad , \qquad \tilde{\tau} := \tilde{h} \circ \tau \circ h^{-1} \qquad ,$$

where $J_{\tilde{\tau}}$ denotes the Jacobian matrix of $\tilde{\tau}$, if we represent the linear mapping $\mathrm{d}\tau$ in the chart bases $\{\partial_1^h, \dots, \partial_N^h\}$ and $\{\partial_1^{\tilde{h}}, \dots, \partial_{\tilde{N}}^{\tilde{h}}\}$. In the special case $\tilde{\mathrm{M}} = \mathbb{R}^\ell$ we note $\mathrm{d}\tau(\partial_j^h) = \partial_j^h \tau$. Hence the chart differentials $\mathrm{d}h_i$ satisfy $\mathrm{d}h_i(\partial_j^h) = \partial_j^h h_i = \delta_{j,i}$ and form a basis of $\mathrm{A}^1(x)$ and $\mathrm{A}^1(V)$. Thus for each $\Phi \in \mathrm{A}^q(V)$ we have an unique representation

$$\Phi = \sum_{I\in\mathcal{S}(q,N)} \Phi_I \, \mathrm{d}h^I \qquad , \tag{2.1}$$

where $\Phi_I := \Phi(\partial_{i_1}^h, \dots, \partial_{i_q}^h) : V \to \mathbb{C}$, $\mathrm{d}h^I := \mathrm{d}h_{i_1} \wedge \cdots \wedge \mathrm{d}h_{i_q}$ and $\mathcal{S}(q,N)$ denotes the set of ordered multi-indices $I := (i_1, \cdots, i_q)$ of length q. Especially for differentiable $f : V \to \mathbb{R}$ the differential $\mathrm{d}f$ is a 1-form and we get the local representation

$$\mathrm{d}f = \sum_{j=1}^N \partial_j^h f \, \mathrm{d}h_j \qquad .$$

The assumptions on M yield an orientation and a scalar product on $\mathrm{T}_x(\mathrm{M})$, which induces in a natural way a scalar product on $\mathrm{A}^1(x)$ and hence on $\mathrm{A}^q(x)$. We introduce the Hodge star operator $*$ on $\mathrm{A}^q(x)$, which acts on every positively oriented orthonormal basis $\{\phi^1, \ldots, \phi^N\}$ of $\mathrm{A}^1(x)$ as

$$*\phi^I = \sigma(I, I') \cdot \phi^{I'} \qquad ,$$

where $I \cup I' = \{1, \ldots, N\}$ and $\sigma(I, I')$ is the sign of that permutation, which carries over the indices $I \cup I'$ to $(1, \ldots, N)$. The Hodge star operator is independent of the orthonormal basis chosen, can be extended to $\mathrm{A}^q(\mathrm{M})$ and thus yields an isomorphism $* : \mathrm{A}^q(\mathrm{M}) \to \mathrm{A}^{N-q}(\mathrm{M})$ satisfying

$$* * \Phi = (-1)^{q(N-q)}\Phi \quad , \quad \Phi \wedge \Psi = *\Phi \wedge *\Psi \quad , \quad *\,(\varphi\Phi) = \varphi * \Phi \tag{2.2}$$

for all $\Phi \in \mathrm{A}^q(\mathrm{M})$, $\Psi \in \mathrm{A}^{N-q}(\mathrm{M})$ and $\varphi \in \mathrm{A}^0(\mathrm{M})$.

Let Ω be some open subset of M and $m \in \mathbb{N}_0 \cup \{\infty\}$. We write $f \in \mathrm{C}^m(\Omega)$ for some function $f : \Omega \to \mathbb{C}$, if $\varphi \circ h^{-1} \in \mathrm{C}^m\big(h(\Omega \cap V)\big)$ for all charts (V, h). We say $\Phi \in \mathrm{C}^{m,q}(\Omega)$, if $\Phi_I \in \mathrm{C}^m(\Omega)$ holds for all component functions Φ_I from (2.1) of a form Φ and all charts h. Moreover, we put

$$\begin{aligned} \overset{\circ}{\mathrm{C}}{}^{m,q}(\Omega) &:= \big\{\Phi \in \mathrm{C}^{m,q}(\Omega) : \operatorname{supp}\Phi \Subset \Omega\big\} \qquad , \\ \mathrm{C}^{m,q}(\overline{\Omega}) &:= \big\{\, \Phi|_{\Omega} : \Phi \in \overset{\circ}{\mathrm{C}}{}^{m,q}(\mathrm{M})\big\} \qquad . \end{aligned}$$

For those and the following spaces of forms we often omit the upper index q in the case $q = 0$.

We introduce the exterior derivative

$$\mathrm{d} : \mathrm{C}^{\infty,q}(\Omega) \to \mathrm{C}^{\infty,q+1}(\Omega)$$

having the properties

$$\mathrm{d}(\Phi \wedge \Psi) = \mathrm{d}\Phi \wedge \Psi + (-1)^q \Phi \wedge \mathrm{d}\Psi \qquad , \tag{2.3}$$

$$\mathrm{dd}\Phi = 0 \tag{2.4}$$

for all $\Phi \in \mathrm{C}^{\infty,q}(\Omega)$, $\Psi \in \mathrm{C}^{\infty,p}(\Omega)$. $\mathrm{d}$is a linear operator and on 0-forms it acts like the differential. Locally it is defined by

$$\begin{aligned} \mathrm{d}\Phi &= \sum_{I \in \mathcal{S}(q,N)} \sum_{j=1}^{N} \partial_j^h \, \Phi_I \, \mathrm{d}h_j \wedge \mathrm{d}h^I \\ &= \sum_{I \in \mathcal{S}(q+1,N)} \sum_{I \ni j=1}^{N} \sigma(j, I-j) \cdot \partial_j^h \, \Phi_{I-j} \, \mathrm{d}h^I \qquad , \end{aligned} \tag{2.5}$$

if Φ is represented by (2.1). Here the ordered index $I \pm j$ is a permutation of $I \cup \{j\}$ resp. $I \setminus \{j\}$. Furthermore, we get the co-derivative

$$\begin{array}{cccc} \delta & : & \mathrm{C}^{\infty,q}(\Omega) & \longrightarrow & \mathrm{C}^{\infty,q-1}(\Omega) \\ & & \Phi & \longmapsto & (-1)^{(q-1)N} * \mathrm{d} * \Phi \end{array} \qquad ,$$

which analogously locally acts as

$$\delta\Phi = \sum_{I\in\mathcal{S}(q-1,N)} \sum_{I\not\ni j=1}^{N} \sigma(j,I)\cdot\partial_j^h\,\Phi_{I+j}\,\mathrm{d}h^I \qquad , \tag{2.6}$$

if $\{\mathrm{d}h_1,\ldots,\mathrm{d}h_N\}$ is a positively oriented orthonormal basis.

Let $\tilde{\Omega}$ be an open subset of another $\tilde{N}$-dimensional manifold $\tilde{\mathrm{M}}$. Then the pull back map

$$\tau^* : \mathrm{A}^q(\tilde{\Omega}) \to \mathrm{A}^q(\Omega)$$

of a C^1-mapping $\tau : \Omega\subset\mathrm{M}\to\tilde{\Omega}\subset\tilde{\mathrm{M}}$ is defined pointwise by

$$\tau^*\Phi(\partial_1,\ldots,\partial_q) := \Phi\big(\mathrm{d}\tau(\partial_1),\ldots,\mathrm{d}\tau(\partial_q)\big)$$

for all $\Phi\in\mathrm{A}^q(\tilde{\Omega})$, $\partial_j\in\mathrm{T}(\Omega)$. We note

$$\tau^*\varphi = \varphi\circ\tau \qquad , \qquad \tau^*(\Phi\wedge\Psi) = \tau^*\Phi\wedge\tau^*\Psi \qquad , \qquad \mathrm{d}\tau^*\phi = \tau^*\mathrm{d}\phi$$

for all $\varphi\in\mathrm{A}^0(\tilde{\Omega})$, $\Phi\in\mathrm{A}^q(\tilde{\Omega})$, $\Psi\in\mathrm{A}^p(\tilde{\Omega})$ and $\phi\in\mathrm{C}^{\infty,q}(\tilde{\Omega})$ as well as the chain rule $(\tau_2\circ\tau_1)^* = \tau_1^*\circ\tau_2^*$. Locally τ^* acts in the following way: Let (V,h), $(\tilde{V},\tilde{h})$ be some charts in Ω, $\tilde{\Omega}$ and $\tau : V\to\tilde{V}$ as well as

$$f := \tilde{h}\circ\tau\circ h^{-1} : h(V)\subset\mathbb{R}^N \to \tilde{h}(\tilde{V})\subset\mathbb{R}^{\tilde{N}} \qquad .$$

Then

$$\tau^*\Phi = \sum_{I\in\mathcal{S}(q,N)} \sum_{|J|=q} \sigma(J)\cdot\big((\partial_I\, f_J)\circ h\big)\cdot(\Phi_{\pi(J)}\circ\tau)\cdot\mathrm{d}h^I \tag{2.7}$$

holds for $\Phi = \sum_{I\in\mathcal{S}(q,\tilde{N})} \Phi_I\mathrm{d}\tilde{h}^I$, where

$$\partial_I\, f_J(x) := \partial_{i_1} f_{j_1}(x)\ldots\partial_{i_q} f_{j_q}(x)$$

and π is the permutation ordering the indices.

For subsets Ξ of $\mathbb{R}^N$ and q-forms

$$\Phi = \phi\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^N \in \mathring{\mathrm{C}}^{\infty,N}(\Xi) \qquad ,$$

where $\{x_1,\ldots,x_N\}$ denote Euclidean coordinates, we define the integral

$$\int_\Xi \Phi := \int_\Xi \phi\; d\lambda \qquad .$$

Using this definition the integral over some chart domain (V,h) of a q-form

$$\Phi = \phi\,\mathrm{d}h^1\wedge\cdots\wedge\mathrm{d}h^N \in \mathring{\mathrm{C}}^{\infty,N}(\mathrm{M})$$

is given by

$$\int_V \Phi := \int_{h(V)} (h^{-1})^*\Phi = \int_{h(V)} \phi \circ h^{-1}\, d\lambda$$

and finally we define $\displaystyle\int_{\mathrm{M}} \Phi$ with a partition of unity. If $\tilde{N} = N$ and $\tau : \Omega \to \tilde{\Omega}$ is a C^1-mapping respecting orientation we have the transformation formula

$$\int_\Omega \tau^*\Phi = \int_{\tilde{\Omega}} \Phi$$

for all $\Phi \in \mathrm{C}^{\infty,N}\big(\overline{\tilde{\Omega}}\big)$. If $\partial\,\Omega$ is a $(N-1)$-dimensional submanifold of $\overline{\Omega}$, then Stokes theorem

$$\int_\Omega \mathrm{d}\Phi = \int_{\partial\,\Omega} \iota^*\Phi \tag{2.8}$$

holds for all $\Phi \in \mathrm{C}^{\infty,N-1}(\overline{\Omega})$, where $\iota : \partial\,\Omega \hookrightarrow \overline{\Omega}$ denotes the natural embedding.

On $\mathrm{A}^q(\mathrm{M})$ we have a pointwise scalar product and induced norm

$$\langle \Phi, \Psi \rangle_q := *(\Phi \wedge *\bar{\Psi}) = \langle *\Phi, *\Psi \rangle_{N-q} \qquad , \qquad |\Phi|_q := \big(\langle \Phi, \Phi \rangle_q\big)^{1/2} \qquad .$$

This yields an inner product and a norm on $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$

$$\langle \Phi, \Psi \rangle_\Omega := \int_\Omega *\langle \Phi, \Psi \rangle_q = \int_\Omega \Phi \wedge *\bar{\Psi} \qquad , \qquad \|\Phi\|_\Omega := \big(\langle \Phi, \Phi \rangle_\Omega\big)^{1/2}$$

and we denote the closure of $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ in this norm by $\mathrm{L}^{2,q}(\Omega)$. Equipped with the scalar product

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)} := \langle\,\cdot\,,\,\cdot\,\rangle_\Omega$$

$\mathrm{L}^{2,q}(\Omega)$ becomes a Hilbert space. By (2.2), (2.3) and Stokes theorem (2.8)

$$\langle \Phi, \delta\Psi \rangle_{\mathrm{L}^{2,q}(\Omega)} + \langle \mathrm{d}\Phi, \Psi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \int_\Omega \mathrm{d}(\Phi \wedge *\bar{\Psi}) = \int_{\partial\,\Omega} \iota^*(\Phi \wedge *\bar{\Psi}) \tag{2.9}$$

holds for all $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ and $\Psi \in \mathrm{C}^{\infty,q+1}(\overline{\Omega})$ and thus

$$\langle \Phi, \delta\Psi \rangle_{\mathrm{L}^{2,q}(\Omega)} + \langle \mathrm{d}\Phi, \Psi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = 0 \qquad , \tag{2.10}$$

if one partner of Φ , Ψ has compact support in Ω , i.e. formally d and δ are skew adjoint to each other. To remind of the electro-magnetic background from now on we denote the exterior derivative d by the rotation rot and the co-derivative δ by the divergence div .

Using (2.10) we say that $E \in \mathrm{L}^{2,q}(\Omega)$ possesses a weak rotation in $\mathrm{L}^{2,q+1}(\Omega)$, if there exists some $G \in \mathrm{L}^{2,q+1}(\Omega)$, such that

$$\langle E, \operatorname{div}\Phi \rangle_{\mathrm{L}^{2,q}(\Omega)} = -\langle G, \Phi \rangle_{\mathrm{L}^{2,q+1}(\Omega)}$$

holds for all $\Phi \in \overset{\circ}{\mathrm{C}}^{\infty,q+1}(\Omega)$, and write $\operatorname{rot} E = G \in \mathrm{L}^{2,q+1}(\Omega)$. Analogously we define the weak divergence. Then

$$\begin{aligned}\mathbf{R}^q(\Omega) &:= \big\{E \in \mathrm{L}^{2,q}(\Omega) : \operatorname{rot} E \in \mathrm{L}^{2,q+1}(\Omega)\big\} \quad ,\\ \mathbf{D}^{q+1}(\Omega) &:= \big\{H \in \mathrm{L}^{2,q+1}(\Omega) : \operatorname{div} H \in \mathrm{L}^{2,q}(\Omega)\big\}\end{aligned}$$

are Hilbert spaces, if we equip them with their natural scalar products

$$\begin{aligned}\langle E, H\rangle_{\mathbf{R}^q(\Omega)} &:= \langle E, H\rangle_{\mathrm{L}^{2,q}(\Omega)} + \langle \operatorname{rot} E, \operatorname{rot} H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} \quad ,\\ \langle E, H\rangle_{\mathbf{D}^{q+1}(\Omega)} &:= \langle E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \operatorname{div} E, \operatorname{div} H\rangle_{\mathrm{L}^{2,q}(\Omega)} \quad .\end{aligned}$$

We introduce the following densely defined linear operators:

$$\begin{aligned}\mathrm{ROT} \ : \ \overset{\circ}{\mathrm{C}}^{\infty,q}(\Omega) \subset \mathrm{L}^{2,q}(\Omega) &\longrightarrow \overset{\circ}{\mathrm{C}}^{\infty,q+1}(\Omega) \subset \mathrm{L}^{2,q+1}(\Omega)\\ \Phi &\longmapsto \mathrm{d}\Phi = \operatorname{rot} \Phi\end{aligned}$$

$$\begin{aligned}\mathrm{DIV} \ : \ \overset{\circ}{\mathrm{C}}^{\infty,q+1}(\Omega) \subset \mathrm{L}^{2,q+1}(\Omega) &\longrightarrow \overset{\circ}{\mathrm{C}}^{\infty,q}(\Omega) \subset \mathrm{L}^{2,q}(\Omega)\\ \Phi &\longmapsto \delta\Phi = \operatorname{div} \Phi\end{aligned}$$

The operators

$$\begin{aligned}\overline{\mathrm{ROT}} &= (\mathrm{ROT}^*)^* \subset -\,\mathrm{DIV}^* \quad ,\\ \overline{\mathrm{DIV}} &= (\mathrm{DIV}^*)^* \subset -\,\mathrm{ROT}^*\end{aligned}$$

are extensions of ROT resp. DIV with domains of definition $D(\mathrm{DIV}^*) = \mathbf{R}^q(\Omega)$ resp. $D(\mathrm{ROT}^*) = \mathbf{D}^{q+1}(\Omega)$ and

$$\begin{aligned}D(\overline{\mathrm{ROT}}) &= \overline{\overset{\circ}{\mathrm{C}}^{\infty,q}(\Omega)}^{\mathbf{R}^q(\Omega)} =: \overset{\circ}{\mathbf{R}}^q(\Omega) \quad ,\\ D(\overline{\mathrm{DIV}}) &= \overline{\overset{\circ}{\mathrm{C}}^{\infty,q+1}(\Omega)}^{\mathbf{D}^{q+1}(\Omega)} =: \overset{\circ}{\mathbf{D}}^{q+1}(\Omega)\end{aligned}$$

(with closures in the graph norms). Therefore $-\,\mathrm{DIV}^*$ resp. $-\,\mathrm{ROT}^*$ is the weak rotation rot resp. divergence div and thus on their domains of definition ROT, $\overline{\mathrm{ROT}}$, $-\,\mathrm{DIV}^*$ resp. DIV, $\overline{\mathrm{DIV}}$, $-\,\mathrm{ROT}^*$ act like the weak rotation resp. divergence. Moreover, $\overset{\circ}{\mathbf{R}}^q(\Omega)$ resp. $\overset{\circ}{\mathbf{D}}^{q+1}(\Omega)$ is a closed subspace of $\mathbf{R}^q(\Omega)$ resp. $\mathbf{D}^{q+1}(\Omega)$ and hence a Hilbert space. Clearly DIV^*, $\overline{\mathrm{ROT}}$, ROT^*, $\overline{\mathrm{DIV}}$ are closed operators and thus the nullspaces or kernels

$$\begin{aligned}{}_0\mathbf{R}^q(\Omega) &:= N(\mathrm{DIV}^*) = \big\{E \in \mathbf{R}^q(\Omega) : \operatorname{rot} E = 0\big\} \quad ,\\ {}_0\overset{\circ}{\mathbf{R}}^q(\Omega) &:= N(\overline{\mathrm{ROT}}) = \big\{E \in \overset{\circ}{\mathbf{R}}^q(\Omega) : \operatorname{rot} E = 0\big\} \quad ,\\ {}_0\mathbf{D}^{q+1}(\Omega) &:= N(\mathrm{ROT}^*) = \big\{H \in \mathbf{D}^{q+1}(\Omega) : \operatorname{div} H = 0\big\} \quad ,\\ {}_0\overset{\circ}{\mathbf{D}}^{q+1}(\Omega) &:= N(\overline{\mathrm{DIV}}) = \big\{H \in \overset{\circ}{\mathbf{D}}^{q+1}(\Omega) : \operatorname{div} H = 0\big\}\end{aligned}$$

are closed subspaces of $\mathrm{L}^{2,q}(\Omega)$ resp. $\mathrm{L}^{2,q+1}(\Omega)$, i .e. Hilbert spaces. The star operator yields

$$*_{(0)}\overset{(\circ)}{\mathbf{D}}{}^{N-q}(\Omega) = {}_{(0)}\overset{(\circ)}{\mathbf{R}}{}^{q}(\Omega) \qquad , \qquad *\,{}_{(0)}\overset{(\circ)}{\mathbf{R}}{}^{N-q}(\Omega) = {}_{(0)}\overset{(\circ)}{\mathbf{D}}{}^{q}(\Omega) \qquad .$$

Because of $\delta\delta = 0$ and $\mathrm{dd} = 0$ we see that

$$\mathrm{rot}\,\mathrm{rot} = 0 \qquad , \qquad \mathrm{div}\,\mathrm{div} = 0$$

still hold in the weak sense. We even obtain

$$\mathrm{rot}\,\overset{(\circ)}{\mathbf{R}}{}^{q}(\Omega) \subset {}_{0}\overset{(\circ)}{\mathbf{R}}{}^{q+1}(\Omega) \qquad , \qquad \mathrm{div}\,\overset{(\circ)}{\mathbf{D}}{}^{q+1}(\Omega) \subset {}_{0}\overset{(\circ)}{\mathbf{D}}{}^{q}(\Omega) \qquad .$$

For $\varphi \in \mathrm{C}^{\infty}(\overline{\Omega})$ and $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ we calculate

$$\mathrm{rot}(\varphi\Phi) = (\mathrm{rot}\varphi) \wedge \Phi + \varphi\mathrm{rot}\Phi \qquad , \tag{2.11}$$

$$\mathrm{div}(\varphi\Phi) = (-1)^{(q-1)N} * \big((\mathrm{rot}\varphi) \wedge *\Phi\big) + \varphi\mathrm{div}\Phi \qquad . \tag{2.12}$$

These formulas imply $\varphi E \in \overset{(\circ)}{\mathbf{D}}{}^{q}(\Omega)$ resp. $\varphi E \in \overset{(\circ)}{\mathbf{R}}{}^{q}(\Omega)$ for $\varphi \in \mathrm{C}^{\infty}(\overline{\Omega})$ and $E \in \overset{(\circ)}{\mathbf{D}}{}^{q}(\Omega)$ resp. $E \in \overset{(\circ)}{\mathbf{R}}{}^{q}(\Omega)$. Furthermore, we obtain $\varphi E \in \overset{\circ}{\mathbf{R}}{}^{q}(\Omega)$ resp. $\varphi E \in \overset{\circ}{\mathbf{D}}{}^{q}(\Omega)$ for all $E \in \mathbf{R}^{q}(\Omega)$ resp. $E \in \mathbf{D}^{q}(\Omega)$, if $\varphi \in \overset{\circ}{\mathrm{C}}{}^{\infty}(\Omega)$. This may be proved using mollifiers $\big($see $\big[$[1], Theorem 1.5$\big]\big)$, i.e. one can show that for any $E \in \mathbf{R}^{q}(\Omega)$ satisfying $\mathrm{supp}\,E \Subset \Omega$ there exists a sequence $(\Phi_n) \subset \overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ with $\Phi_n \to E$ in $\mathbf{R}^{q}(\Omega)$.

We note that we generalize the boundary condition $\iota^* E = 0$ resp. $\iota^* * E = 0$ in the space $\overset{\circ}{\mathbf{R}}{}^{q}(\Omega)$ resp. $\overset{\circ}{\mathbf{D}}{}^{q}(\Omega)$. Namely by $\overline{\mathrm{ROT}} = (\mathrm{ROT}^*)^*$ we observe $E \in \overset{\circ}{\mathbf{R}}{}^{q}(\Omega)$, if and only if $E \in \mathbf{R}^{q}(\Omega)$ and even

$$\langle E, \mathrm{div}\, H\rangle_{\mathrm{L}^{2,q}(\Omega)} + \langle \mathrm{rot}\, E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} = 0$$

holds for all $H \in D(\mathrm{ROT}^*) = \mathbf{D}^{q+1}(\Omega)$. Hence assuming sufficient smoothness of E and the boundary $\partial\,\Omega$ we obtain by (2.9)

$$\int_{\partial\,\Omega} (\iota^* E) \wedge (\iota^* * \bar{H}) = 0$$

for all $H \in \mathrm{C}^{\infty,q+1}(\overline{\Omega})$, i.e. $\iota^* E = 0$.

From now on let Ω denote some connected open subset of M with compact closure in M.

Our next aim is to define Sobolev spaces on our manifold.

Definition 2.1

(i) *Let $m \in \mathbb{N}_0$. We call Ω a 'C^m-region', if $\partial\,\Omega$ is a $(N-1)$-dimensional C^m-submanifold of M, i.e. for each $x \in \partial\,\Omega$ there exists a C^m-boundary chart (V,h) with $h(x)=0$ and $h(\overline{V}) = \overline{U}_1$, such that*

$$h(\partial\,\Omega \cap V) = U_1^0 \quad , \quad h(\Omega \cap V) = U_1^- \quad , \quad h\big((\mathrm{M} \setminus \overline{\Omega}) \cap V\big) = U_1^+$$

and $k \circ h^{-1} \in \mathrm{C}^m\big(U_1^0, \mathbb{R}^N\big)$ hold for all charts (V,k) of $x \in \Omega$. In this case we call $\partial\,\Omega$ a 'C^m-boundary'.

(ii) *We say Ω has the 'segment property', if for each $x \in \partial\,\Omega$ there exist a chart (V,h), some $\varrho \in (0,1)$ and some vector $v \in \mathbb{R}^N$ with $h(x)=0$, $h(\overline{V}) = \overline{U}_1$ and*

$$U_\varrho \cap \overline{h(\Omega \cap V)} + \tau v \subset h(\Omega \cap V)$$

for all $\tau \in (0,1)$. $\big($See $\big[$[1], Definition 2.1$\big]$ for the classical segment property.$\big)$

We note that C^1-regions possess the segment property. Due to the compactness of $\overline{\Omega}$ a finite collection of charts $\big\{(V_k, h_k) : k = 1, \ldots, K\big\}$ is sufficient to cover $\overline{\Omega}$. Let $\{\xi_k : k = 1, \ldots, K\}$ be a corresponding partition of unity. W. l. o. g. we may assume $h_k(V_k) = U_1$ and $\operatorname{supp} \xi_k \circ h_k^{-1} \subset U_{1/3}$ for all k.

Then for $m \in [0, \infty)$ we define the Sobolev spaces

$$\mathbf{H}^{m,q}(\Omega)$$

as the set of forms $E \in \mathrm{A}^q(\Omega)$, whose Cartesian components E_I^k of $(h_k^{-1})^* E = E_I^k \, \mathrm{d}x^I$ are elements of $\mathbf{H}^m\big(h_k(\Omega \cap V_k)\big)$, and put

$$\|E\|_{\mathbf{H}^{m,q}(\Omega)} := \Big(\sum_{k=1}^{K} \sum_{I \in \mathcal{S}(q,N)} \|E_I^k\|^2_{\mathbf{H}^m(h_k(\Omega \cap V_k))} \Big)^{1/2} \qquad .$$

Here and in future we identify a form with its restriction on subsets of its domain of definition. Using transformation theorems, (2.7) and $\big[$[26], Satz 4.1$\big]$ for scalar functions one sees that this definition is independent of the chosen charts and partition of unity. A second covering yields the same Sobolev space but with an equivalent norm. Another consequence of (2.7) is that for $m \in \mathbb{N}_0$ and any C^{m+1}-diffeomorphism $\tau : \tilde{\Omega} \to \Omega$ there exists a constant $c > 0$, such that

$$c^{-1} \cdot \|E\|_{\mathbf{H}^{m,q}(\Omega)} \leq \|\tau^* E\|_{\mathbf{H}^{m,q}(\tilde{\Omega})} \leq c \cdot \|E\|_{\mathbf{H}^{m,q}(\Omega)} \tag{2.13}$$

holds for all $E \in \mathbf{H}^{m,q}(\Omega)$.

Using charts and the completeness of $\mathbf{H}^{m,q}(\Omega)$ the following results may be obtained from the scalar Sobolev spaces:

- $\mathrm{C}^{\infty,q}(\Omega) \cap \mathbf{H}^{m,q}(\Omega)$ is dense in $\mathbf{H}^{m,q}(\Omega)$.
- $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ is dense in $\mathbf{H}^{0,q}(\Omega)$.
- $\displaystyle\bigwedge_{\Phi\in\mathrm{C}^{\infty,p}(\overline{\Omega})} \bigvee_{c>0} \bigwedge_{E\in\mathbf{H}^{m,q}(\Omega)} \qquad \|\Phi\wedge E\|_{\mathbf{H}^{m,q+p}(\Omega)} \leq c\cdot\|E\|_{\mathbf{H}^{m,q}(\Omega)}$
- $\displaystyle\bigvee_{c>0} \bigwedge_{E\in\mathbf{H}^{m,q}(\Omega)} \qquad \|*E\|_{\mathbf{H}^{m,N-q}(\Omega)} \leq c\cdot\|E\|_{\mathbf{H}^{m,q}(\Omega)}$

We note $\mathrm{L}^{2,q}(\Omega) = \mathbf{H}^{0,q}(\Omega)$ with equivalent norms. Furthermore, we define $\overset{\circ}{\mathbf{H}}{}^{m,q}(\Omega)$ as the closure of $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ in the $\mathbf{H}^{m,q}(\Omega)$-norm. If Ω has the segment property we can take over more properties from the scalar case, i.e.

$$\mathrm{C}^{\infty,q}(\overline{\Omega}) \quad \text{is dense in} \quad \mathbf{H}^{m,q}(\Omega) \tag{2.14}$$

as well as $\Phi \in \overset{\circ}{\mathbf{H}}{}^{m,q}(\Omega)$ for some $\Phi \in \mathbf{H}^{m,q}(\Omega)$, if and only if its extension by zero into $\tilde{\Omega}$ is an element of $\mathbf{H}^{m,q}(\tilde{\Omega})$ for an open set $\tilde{\Omega}$ with $\Omega \Subset \tilde{\Omega} \Subset \mathrm{M}$. The first assertion may be proved analogously to $\big[$[26], Theorem 3.6$\big]$ or $\big[$[1], Theorem 2.1$\big]$ and the second analogously to $\big[$[26], Theorem 3.7$\big]$. The same techniques yield

$$\mathbf{R}^q(\Omega) = \overline{\mathrm{C}^{\infty,q}(\overline{\Omega})}^{\mathbf{R}^q(\Omega)} \qquad , \qquad \mathbf{D}^q(\Omega) = \overline{\mathrm{C}^{\infty,q}(\overline{\Omega})}^{\mathbf{D}^q(\Omega)} \qquad . \tag{2.15}$$

Definition 2.2 *We call a transformation ε 'admissible' and write $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$, if and only if*

- *$\varepsilon(x)$ is a linear transformation on $\mathrm{A}^q(\Omega)$ for all $x \in \Omega$,*
- *ε possesses $\mathrm{L}^\infty(\Omega)$-coefficients, i.e. the matrix representation of ε corresponding to an arbitrary chart basis $\{\mathrm{d}h^I\}$ has $\mathrm{L}^\infty(\Omega)$-entries,*
- *ε is symmetric, i.e. for all $E, H \in \mathrm{L}^{2,q}(\Omega)$*

$$\langle\varepsilon E, H\rangle_{\mathrm{L}^{2,q}(\Omega)} = \langle E, \varepsilon H\rangle_{\mathrm{L}^{2,q}(\Omega)}$$

holds, and uniformly positive definite, i.e.

$$\bigvee_{c>0} \bigwedge_{E\in\mathrm{L}^{2,q}(\Omega)} \langle\varepsilon E, E\rangle_{\mathrm{L}^{2,q}(\Omega)} \geq c\cdot\|E\|^2_{\mathrm{L}^{2,q}(\Omega)} \qquad .$$

Let $\ell \in \mathbb{N}_0$. We say $\varepsilon \in \mathrm{C}^{\ell,q}(\Omega)$ resp. $\varepsilon \in \mathrm{C}^{\ell,q}(\overline{\Omega})$, if and only if ε has $\mathrm{C}^\ell(\Omega)$- resp. $\mathrm{C}^\ell(\overline{\Omega})$-entries, and write $\partial^\alpha\varepsilon$ for $|\alpha| \leq \ell$ meaning componentwise differentiation. Moreover, for $\ell \in \mathbb{N}$ we define

$$\mathbb{A}^{\ell,q}(\Omega) := \mathbb{A}^{0,q}(\Omega) \cap \mathrm{C}^{\ell,q}(\Omega) \quad \textit{resp.} \quad \mathbb{A}^{\ell,q}(\overline{\Omega}) := \mathbb{A}^{0,q}(\Omega) \cap \mathrm{C}^{\ell,q}(\overline{\Omega}) \quad .$$

On $\mathrm{L}^{2,q}(\Omega)$ an admissible transformation ε yields an equivalent scalar product $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)}$ and we set ${}_{\varepsilon}\mathrm{L}^{2,q}(\Omega) := \mathrm{L}^{2,q}(\Omega)$ equipped with $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)}$.

If $\tau : \Omega \to \tilde{\Omega}$ is a C^1-diffeomorphism respecting orientation we define a linear transformation $\varepsilon_\tau : \mathrm{A}^q(\Omega) \to \mathrm{A}^q(\Omega)$ by

$$\varepsilon_\tau := \varepsilon_\tau^q := (-1)^{q(N-q)} * \tau^* * (\tau^*)^{-1} \tag{2.16}$$

satisfying $*\varepsilon_\tau\tau^* = \tau^* *$. This transformation ε_τ is admissible. We obtain

Lemma 2.3 *Let $\tau : \Omega \to \tilde{\Omega}$ be a C^2-diffeomorphism respecting orientation, ε_τ from* (2.16) *and $\tilde{\varepsilon} \in \mathbb{A}^{0,q}(\tilde{\Omega})$ an admissible transformation. Then the transformation*

$$\varepsilon := \varepsilon_\tau \tau^* \tilde{\varepsilon} (\tau^*)^{-1} \in \mathbb{A}^{0,q}(\Omega)$$

is admissible. Furthermore,

- **(i)** *if $E \in \overset{(\circ)}{\mathbf{R}}{}^q(\tilde{\Omega})$, then $\tau^* E \in \overset{(\circ)}{\mathbf{R}}{}^q(\Omega)$ and $\mathrm{rot}\tau^* E = \tau^* \mathrm{rot} E$. Moreover, there exists some $c > 0$ independent of E , such that*

$$\|\tau^* E\|_{\mathbf{R}^q(\Omega)} \le c \cdot \|E\|_{\mathbf{R}^q(\tilde{\Omega})} \qquad .$$

- **(ii)** *if $E \in \tilde{\varepsilon}^{-1}\overset{(\circ)}{\mathbf{D}}{}^q(\tilde{\Omega})$, then $\tau^* E \in \varepsilon^{-1}\overset{(\circ)}{\mathbf{D}}{}^q(\Omega)$ and $\mathrm{div}\varepsilon\tau^* E = \varepsilon_\tau \tau^* \mathrm{div}\tilde{\varepsilon} E$. Moreover, there exists a constant $c > 0$ independent of E or $\tilde{\varepsilon}$, such that*

$$\|\tau^* E\|_{\varepsilon^{-1}\mathbf{D}^q(\Omega)} \le c \cdot \|E\|_{\tilde{\varepsilon}^{-1}\mathbf{D}^q(\tilde{\Omega})} \qquad .$$

Proof: Using the transformation theorem and some properties of the exterior product and star operator one easily checks that ε is admissible as well as that for smooth forms $\Psi \in \mathrm{C}^{\infty,q}(\tilde{\Omega})$

$$\mathrm{rot}\tau^*\Psi = \tau^*\mathrm{rot}\Psi \qquad , \qquad \mathrm{div}\varepsilon_\tau\tau^*\Psi = \varepsilon_\tau\tau^*\mathrm{div}\Psi$$

holds. Let $E \in \mathbf{R}^q(\tilde{\Omega})$ and $\Phi \in \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega)$. We calculate

$$\begin{aligned}
\langle \tau^* E, \mathrm{div}\Phi \rangle_{\mathrm{L}^{2,q}(\Omega)} &= (-1)^{q^2} \int_\Omega (\tau^* E) \wedge (\mathrm{rot} *\bar{\Phi}) \\
&= (-1)^{q^2} \int_\Omega (\tau^* E) \wedge \big(\underbrace{\mathrm{rot}\,\tau^*}_{=\tau^*\,\mathrm{rot}} (\tau^*)^{-1} * \bar{\Phi} \big) \\
&= (-1)^{qN+(q+1)(N-q-1)} \int_{\tilde{\Omega}} E \wedge \big(**\mathrm{rot} ** (\tau^*)^{-1} * \bar{\Phi} \big) \\
&= (-1)^{(q+1)(N-q-1)} \big\langle E, \mathrm{div} *(\tau^*)^{-1} * \Phi \big\rangle_{\mathrm{L}^{2,q}(\tilde{\Omega})} \qquad .
\end{aligned}$$

Since by (2.7) $*(\tau^*)^{-1} * \Phi \in \overset{\circ}{\mathrm{C}}{}^{1,q+1}(\tilde{\Omega}) \subset \overset{\circ}{\mathbf{D}}{}^{q+1}(\tilde{\Omega})$ we obtain

$$\begin{aligned}
\langle \tau^* E, \operatorname{div}\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)} &= -(-1)^{(q+1)(N-q-1)}\big\langle \operatorname{rot} E, *(\tau^*)^{-1} * \Phi\big\rangle_{\mathrm{L}^{2,q+1}(\tilde{\Omega})} \\
&= -\int_{\tilde{\Omega}} (\operatorname{rot} E) \wedge \big((\tau^*)^{-1} * \bar{\Phi}\big) \\
&= -\int_{\Omega} (\tau^* \operatorname{rot} E) \wedge (*\bar{\Phi}) = -\langle \tau^* \operatorname{rot} E, \Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} \qquad .
\end{aligned}$$

Thus $\tau^* E \in \mathbf{R}^q(\Omega)$ and $\operatorname{rot} \tau^* E = \tau^* \operatorname{rot} E$. From (2.13) we get the asserted estimate. If $E \in \overset{\circ}{\mathbf{R}}{}^q(\tilde{\Omega})$ and $\Phi \in \mathbf{D}^{q+1}(\Omega)$, then using the results obtained so far we note $(\tau^*)^{-1} * \Phi \in \mathbf{R}^{N-q-1}(\tilde{\Omega})$ and $\tau^* \operatorname{rot}(\tau^*)^{-1} * \Phi = \operatorname{rot} *\Phi$ as well as $*(\tau^*)^{-1} * \Phi \in \mathbf{D}^{q+1}(\tilde{\Omega})$. This shows that the calculation from above still holds true for those E and Φ, i.e. $\tau^* E \in \overset{\circ}{\mathbf{R}}{}^q(\Omega)$. Hence **(i)** is proved and may be used to show **(ii)** as follows:

$$\begin{array}{lccc}
& E \in \tilde{\varepsilon}^{-1}\overset{(\circ)}{\mathbf{D}}{}^q(\tilde{\Omega}) & \iff & *\tilde{\varepsilon}E \in \overset{(\circ)}{\mathbf{R}}{}^{N-q}(\tilde{\Omega}) \\
\iff & \tau^* * \tilde{\varepsilon}E \in \overset{(\circ)}{\mathbf{R}}{}^{N-q}(\Omega) & \text{and} & \operatorname{rot} \tau^* * \tilde{\varepsilon}E = \tau^* \operatorname{rot} *\tilde{\varepsilon}E \\
\iff & \varepsilon\tau^* E \in \overset{(\circ)}{\mathbf{D}}{}^q(\Omega) & \text{and} & \operatorname{div} \varepsilon\tau^* E = \varepsilon_\tau \tau^* \operatorname{div} \tilde{\varepsilon}E
\end{array}$$

Again (2.13) yields the stated estimate. ∎

Let ε be an admissible transformation. We define the '*(harmonic) Dirichlet forms*' by

$$_{\varepsilon}\mathcal{H}^q(\Omega) := {}_0\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$$

and denote them by $\mathcal{H}^q(\Omega)$, if $\varepsilon = \mathrm{id}$. Moreover, we define the dimension of the Dirichlet forms by

$$d^q := \dim {}_{\varepsilon}\mathcal{H}^q(\Omega) \qquad .$$

By the projection theorem and the $\mathrm{L}^{2,q}(\Omega)$-orthogonality of $\overline{\operatorname{rot} \overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)}^{\mathrm{L}^{2,q}(\Omega)}$ and ${}_0\mathbf{D}^q(\Omega)$ resp. $\overline{\operatorname{div} \mathbf{D}^{q+1}(\Omega)}^{\mathrm{L}^{2,q}(\Omega)}$ and ${}_0\overset{\circ}{\mathbf{R}}{}^q(\Omega)$ we get the following Helmholtz decompositions (see [[12], Lemma 1], [[16], Lemma 1] or in the classical case [[13], p. 168], [[17], Lemma 3.13]):

Lemma 2.4 *The following $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)}$-orthogonal (denoted by $\oplus_\varepsilon$) decompositions hold for admissible transformations ε :*

$$\textbf{(i)} \qquad {}_{\varepsilon}\mathrm{L}^{2,q}(\Omega) = \overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega) = {}_0\overset{\circ}{\mathbf{R}}{}^q(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\operatorname{div}\mathbf{D}^{q+1}(\Omega)}$$
$$= \varepsilon^{-1}\overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon {}_0\mathbf{D}^q(\Omega) = \varepsilon^{-1}{}_0\overset{\circ}{\mathbf{R}}{}^q(\Omega) \oplus_\varepsilon \overline{\operatorname{div}\mathbf{D}^{q+1}(\Omega)}$$

$$\textbf{(ii)} \qquad {}_{\varepsilon}\mathrm{L}^{2,q}(\Omega) = \overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon {}_\varepsilon\mathcal{H}^q(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\operatorname{div}\mathbf{D}^{q+1}(\Omega)}$$
$$= \varepsilon^{-1}\overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_{\varepsilon^{-1}}\mathcal{H}^q(\Omega) \oplus_\varepsilon \overline{\operatorname{div}\mathbf{D}^{q+1}(\Omega)}$$

All closures are taken in $\mathrm{L}^{2,q}(\Omega)$.

If ν is another admissible transformation, then an easy application of this lemma shows, that the orthogonal projection

$$\pi\ :\ {}_\nu\mathcal{H}^q(\Omega) \longrightarrow {}_\varepsilon\mathcal{H}^q(\Omega)$$

on $\varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$ along $\overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)}^{\mathrm{L}^{2,q}(\Omega)}$ is well defined, linear, continuous and injective. Therefore by symmetry we obtain $\dim{}_\nu\mathcal{H}^q(\Omega) = \dim{}_\varepsilon\mathcal{H}^q(\Omega)$ and hence d^q is independent of transformations, i.e.

$$d^q = \dim{}_\varepsilon\mathcal{H}^q(\Omega) = \dim\mathcal{H}^q(\Omega) \qquad .$$

Another essential ingredient of our solution theory is the so called Maxwell's compactness property.

Definition 2.5 *Ω possesses the 'Maxwell's compactness property' (*MCP*), if and only if the embeddings*

$$\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{D}^q(\Omega) \hookrightarrow \mathrm{L}^{2,q}(\Omega)$$

are compact for all q .

The MCP is a property of the boundary and there is a large amount of literature about the MCP. The first idea was to use Gaffney's inequality, i.e. to estimate the $\mathbf{H}^{1,q}(\Omega)$-norm by the $\big(\mathbf{R}^q(\Omega)\cap\mathbf{D}^q(\Omega)\big)$-norm, and then Rellich's selection theorem. To do this one needs smooth boundaries, which for instance may be seen in $\big[$[8], p. 157, Theorem 8.6$\big]$. If $q=0$ we even have

$$\overset{\circ}{\mathbf{R}}{}^0(\Omega) \cap \mathbf{D}^0(\Omega) = \overset{\circ}{\mathbf{R}}{}^0(\Omega) = \overset{\circ}{\mathbf{H}}{}^{1,0}(\Omega) \qquad .$$

In 1972 [20] resp. [21] Weck presented for the first time a proof of the MCP for bounded manifolds with nonsmooth boundaries ('cone-property'). More proofs of the MCP were given by Picard [15] ('Lipschitz-domains') and in the classical case by Weber [18] (another 'cone-property') and Witsch [25] ('p-cusp-property'). A proof of the MCP in the classical case for bounded domains handling the largest known class of boundaries was given by Picard, Weck and Witsch in [17]. They combine the techniques from [21], [15] and [25].

We note that the MCP is independent of transformations, i.e. let ε_q admissible transformations for all q, then Ω possesses the MCP, if and only if the embeddings

$$\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon_q^{-1}\mathbf{D}^q(\Omega) \hookrightarrow \mathrm{L}^{2,q}(\Omega)$$

are compact for all q.

For $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$ the MCP implies (by an indirect argument) the existence of a positive constant c, such that the estimate

$$\|E\|_{\mathrm{L}^{2,q}(\Omega)} \leq c \cdot \big(\| \operatorname{rot} E\|_{\mathrm{L}^{2,q+1}(\Omega)} + \| \operatorname{div} \varepsilon E\|_{\mathrm{L}^{2,q-1}(\Omega)} \big) \tag{2.17}$$

holds uniformly in $E \in \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^\perp$.

An application of this estimate yields the finite dimension of the space of Dirichlet forms ${}_\varepsilon\mathcal{H}^q(\Omega)$. In fact the dimension is determined by topological properties of Ω, i.e. $d^q = \dim \mathcal{H}^q(\Omega) = \beta_{N-q}$ is the $(N-q)$-th Betti number of Ω (see [14]). Moreover, from (2.17) the closedness of $\operatorname{rot} \overset{\circ}{\mathbf{R}}{}^q(\Omega)$ resp. $\operatorname{div} \mathbf{D}^q(\Omega)$ in $\mathrm{L}^{2,q+1}(\Omega)$ resp. $\mathrm{L}^{2,q-1}(\Omega)$ follows. We even have (with any $\nu \in \mathbb{A}^{0,q}(\Omega)$)

$$\overline{\operatorname{rot} \overset{\circ}{\mathbf{R}}{}^q(\Omega)} = \operatorname{rot} \overset{\circ}{\mathbf{R}}{}^q(\Omega) = \operatorname{rot} \big(\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp_\nu} \big) \quad , \tag{2.18}$$

$$\overline{\operatorname{div} \mathbf{D}^q(\Omega)} = \operatorname{div} \mathbf{D}^q(\Omega) = \operatorname{div} \big(\mathbf{D}^q(\Omega) \cap \varepsilon^{-1}{}_0\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap {}_{\varepsilon^{-1}}\mathcal{H}^q(\Omega)^{\perp_\nu} \big) \quad , \tag{2.19}$$

which was shown in [12] in the case $\varepsilon = \nu = \mathrm{id}$. Here we denote the orthogonality w. r. t. the $\langle \nu \,\cdot\,, \,\cdot\, \rangle_{\mathrm{L}^{2,q}(\Omega)}$-scalar product by $\perp_\nu$ and put $\perp := \perp_{\mathrm{id}}$.

Let us define the range

$$W^q(\Omega) := \overline{\operatorname{div} \mathbf{D}^q(\Omega)} \times \overline{\operatorname{rot} \overset{\circ}{\mathbf{R}}{}^q(\Omega)} \times \mathbb{C}^{d^q} \qquad .$$

As in [12] a combination of the $\mathrm{L}^{2,q}(\Omega)$-decompositions from Lemma 2.4 and (2.18), (2.19) yields easily

Theorem 2.6 *Let $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$, Ω have the* MCP *and d^q continuous linear functionals Φ_ε^ℓ on $\mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$ with*

$${}_\varepsilon\mathcal{H}^q(\Omega) \cap \bigcap_{\ell=1}^{d^q} N(\Phi_\varepsilon^\ell) = \{0\}$$

be given. Then with $\Phi_\varepsilon := (\Phi_\varepsilon^1 \,\cdot\,, \ldots, \Phi_\varepsilon^{d^q} \,\cdot\,)$

$$\begin{array}{rccc} \mathcal{M}\mathrm{ax}_\varepsilon \ : & \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega) & \longrightarrow & W^q(\Omega) \\ & E & \longmapsto & \big(\operatorname{div} \varepsilon E, \operatorname{rot} E, \Phi_\varepsilon(E) \big) \end{array}$$

is a topological isomorphism.

Remark 2.7

(i) *For any* $\nu \in \mathbb{A}^{0,q}(\Omega)$ *we can choose* $\Phi^{\ell}_{\varepsilon} := \langle \nu \,\cdot\, , h_\ell \rangle_{\mathrm{L}^{2,q}(\Omega)}$ *with an arbitrary basis* $\{h_\ell\}_{\ell=1}^{d^q}$ *of* ${}_{\varepsilon}\mathcal{H}^q(\Omega)$.

(ii) *Let* $(\tilde{\nu}, \hat{\nu}) \in \mathbb{A}^{0,q-1}(\Omega) \times \mathbb{A}^{0,q+1}(\Omega)$ *. By Lemma 2.4 we obtain*

$$W^q(\Omega) = \big({}_0\mathbf{D}^{q-1}(\Omega) \cap {}_{\tilde{\nu}}\mathcal{H}^{q-1}(\Omega)^{\perp} \big) \times \big({}_0\overset{\circ}{\mathbf{R}}{}^{q+1}(\Omega) \cap {}_{\hat{\nu}}\mathcal{H}^{q+1}(\Omega)^{\perp_{\hat{\nu}}} \big) \times \mathbb{C}^{d^q} \ .$$

(iii) *If we replace* ε *by* ε^{-1} *and consider* ${}_{\varepsilon}\mathcal{M}\mathrm{ax} = \mathcal{M}\mathrm{ax}_{\varepsilon^{-1}}\ \varepsilon$ *, then*

$$\begin{array}{rcccc} {}_{\varepsilon}\mathcal{M}\mathrm{ax} & : & \varepsilon^{-1}\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{D}^q(\Omega) & \longrightarrow & W^q(\Omega) \\ & & E & \longmapsto & \big(\operatorname{div} E, \operatorname{rot} \varepsilon E, \Phi_{\varepsilon^{-1}}(\varepsilon E) \big) \end{array}$$

is a topological isomorphism as well.

(iv) *Clearly using the star operator we have the corresponding dual results.*

Finally in the special case $\mathrm{M} = \mathbb{R}^N$ we need some operators from the calculus developed in [23]. Let $\{x_1, \ldots, x_N\}$ denote Euclidean coordinates. We introduce

$$\begin{array}{rcccc} R & : & \mathrm{A}^q(\mathbb{R}^N) & \longrightarrow & \mathrm{A}^{q+1}(\mathbb{R}^N) \\ & & E & \longmapsto & x_n \, \mathrm{d}x^n \wedge E = r \, \mathrm{d}r \wedge E \end{array} \quad , \tag{2.20}$$

$$\begin{array}{rcccc} T & : & \mathrm{A}^{q+1}(\mathbb{R}^N) & \longrightarrow & \mathrm{A}^q(\mathbb{R}^N) \\ & & E & \longmapsto & (-1)^{qN} * R * E \end{array} \tag{2.21}$$

and recall the formulas

$$RR = 0 \quad , \quad TT = 0 \quad , \quad RT + TR = r^2 \tag{2.22}$$

as well as for $E \in \mathrm{A}^q(\mathbb{R}^N)$, $H \in \mathrm{A}^{q+1}(\mathbb{R}^N)$

$$RE \wedge *H = E \wedge *TH \quad , \quad TH \wedge *E = H \wedge *RE \quad , \tag{2.23}$$

i.e. $\langle RE, H \rangle_{q+1} = \langle E, TH \rangle_q$. The operators rot and div correspond to R and T in the sense that

$$C_{\operatorname{rot},\varphi(r)}E = \varphi'(r) r^{-1} RE \qquad \text{resp.} \qquad C_{\operatorname{div},\varphi(r)}E = \varphi'(r) r^{-1} TE \tag{2.24}$$

hold for $\varphi \in \mathrm{C}^1(\mathbb{R})$ and $E \in \mathbf{R}^q(\mathbb{R}^N)$ resp. $E \in \mathbf{D}^q(\mathbb{R}^N)$.

To conclude with this introductory section we present the componentwise (w. r. t. Euclidean coordinates) Fourier transformation on q-forms $\mathcal{F}$, which is a unitary mapping on $\mathrm{L}^{2,q}(\mathbb{R}^N)$. With $\mathcal{X}(x) := x$ and the well known formula

$$\mathcal{F}(\partial^\alpha u) = \mathrm{i}^{|\alpha|} \, \mathcal{X}^\alpha \mathcal{F}(u)$$

for scalar distributions u we get some formulas for $\mathcal{F}$ operating on q-forms E :

$$\mathcal{F} * E = *\mathcal{F}E \tag{2.25}$$

$$\mathcal{F}(\partial^\alpha E) = \mathrm{i}^{|\alpha|}\, \mathcal{X}^\alpha \mathcal{F}(E) \quad , \quad \partial^\alpha \mathcal{F}(E) = (-\mathrm{i})^{|\alpha|} \mathcal{F}(\mathcal{X}^\alpha E) \tag{2.26}$$

$$\mathcal{F}(\operatorname{rot} E) = \mathrm{i}\, R\mathcal{F}(E) \quad , \quad \operatorname{rot} \mathcal{F}(E) = -\mathrm{i}\, \mathcal{F}(RE) \tag{2.27}$$

$$\mathcal{F}(\operatorname{div} E) = \mathrm{i}\, T\mathcal{F}(E) \quad , \quad \operatorname{div} \mathcal{F}(E) = -\mathrm{i}\, \mathcal{F}(TE) \tag{2.28}$$

$$\mathcal{F}(\Delta E) = -r^2 \cdot \mathcal{F}(E) \quad , \quad \Delta \mathcal{F}(E) = -\mathcal{F}(r^2 \cdot E) \tag{2.29}$$

These formulas may be checked for smooth forms from Schwartz' space and hence remain valid for distributional q-forms, i.e. extend to our weak calculus. We note $\operatorname{rot}\operatorname{div} + \operatorname{div}\operatorname{rot} = \Delta$, where the Laplacian Δ acts on each Euclidean component of E.

2.2 Regularity

Theorem 2.8 *Let $m \in \mathbb{N}_0$, Ω be a bounded C^{m+2}-region and $\varepsilon \in \mathbb{A}^{m+1,q}(\overline{\Omega})$. Furthermore, let $E \in \big(\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)\big) \cup \big(\mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\overset{\circ}{\mathbf{D}}{}^q(\Omega)\big)$ with*

$$\operatorname{rot} E \in \mathbf{H}^{m,q+1}(\Omega) \quad , \quad \operatorname{div} \varepsilon E \in \mathbf{H}^{m,q-1}(\Omega) \quad .$$

Then $E \in \mathbf{H}^{m+1,q}(\Omega)$ and there exists a positive constant c independent of E, such that

$$\|E\|_{\mathbf{H}^{m+1,q}(\Omega)} \leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}(\Omega)} + \|\operatorname{rot} E\|_{\mathbf{H}^{m,q+1}(\Omega)} + \|\operatorname{div}\varepsilon E\|_{\mathbf{H}^{m,q-1}(\Omega)}\big) \quad .$$

Remark 2.9 *By the star operator and some transformation $E \rightsquigarrow \varepsilon E$ we get the corresponding theorem for spaces of the form $\varepsilon^{-1}\mathbf{R}^q(\Omega) \cap \mathbf{D}^q(\Omega)$ as well.*

We only prove this theorem in the case $E \in \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$, since the other case follows by $*$-duality. The classical case $N = 3$, $q = 1$ and Ω is an open subset of $\mathbb{R}^3$ has been proved by Weber in [19] using the natural regularity of $(q-1=0)$- resp. $(q+2=3)$-forms, i.e. scalar functions. In the generalized case there occur some additional difficulties.

We need a few preparations:

Lemma 2.10 *Let $r > 0$, $x' := (x_1, \cdots, x_{N-1})$ and*

$$\tau \; : \; \begin{array}{ccc} U_r^+ & \longrightarrow & U_r^- \\ x & \longmapsto & (x', -x_N) \end{array} \quad .$$

Then the mirror operator

$$S_{\mathrm{rot}} : \mathbf{R}^q(U_r^-) \to \mathbf{R}^q(U_r)$$

defined by $S_{\mathrm{rot}}E|_{U_r^-} := E$ and $S_{\mathrm{rot}}E|_{U_r^+} := \tau^* E$ is well defined, linear and continuous. S_{rot} commutates with rot and $\|S_{\mathrm{rot}}E\|_{\mathrm{L}^{2,q+1}(U_r)} = \sqrt{2} \cdot \|E\|_{\mathrm{L}^{2,q+1}(U_r^-)}$ holds. $\big(\sqrt{2}/2 \cdot S_{\mathrm{rot}}$ even is an isometry.$\big)$ Moreover, if $\operatorname{supp} E \subset \overline{U_\varrho^-}$ for some $\varrho < r$, then $\operatorname{supp} S_{\mathrm{rot}}E \subset \overline{U_\varrho}$.

Proof: By (2.15) it is enough to show $S_{\mathrm{rot}}E \in \mathbf{R}^q(U_r)$ and $\operatorname{rot} S_{\mathrm{rot}}E = S_{\mathrm{rot}} \operatorname{rot} E$ for $E \in \mathrm{C}^{\infty,q}(\overline{U_r^-})$. The assertions about the continuity and the support follow directly. Let $\iota : U_r^0 \hookrightarrow \overline{U_r^-}$ denote the natural embedding. Observing that τ changes the orientation, we get from Stokes theorem for $\Phi \in \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(U_r)$ (Clearly we identify Φ with its restriction on $U_r^\pm$.)

$$\begin{aligned}
\langle S_{\mathrm{rot}}E, \operatorname{div}\Phi\rangle_{\mathrm{L}^{2,q}(U_r)} &= (-1)^{q^2}\int_{U_r^-} E \wedge (\mathrm{d} * \bar{\Phi}) + (-1)^{q^2}\int_{U_r^+} (\tau^* E) \wedge (\mathrm{d} * \bar{\Phi}) \\
&= (-1)^q \int_{U_r^-} E \wedge \mathrm{d}\big(* \bar{\Phi} - (\tau^{-1})^* * \bar{\Phi}\big) \\
&= -\int_{U_r^-} (\mathrm{d}E) \wedge \big(* \bar{\Phi} - (\tau^{-1})^* * \bar{\Phi}\big) \\
&\qquad + \int_{U_r^0} (\iota^* E) \wedge \Big(\big(\iota^* - \iota^*(\tau^{-1})^*\big) * \bar{\Phi}\Big) \qquad .
\end{aligned}$$

By $\iota - \tau^{-1} \circ \iota = 0$ the boundary integral vanishes and we obtain

$$\begin{aligned}
\langle S_{\mathrm{rot}}E, \operatorname{div}\Phi\rangle_{\mathrm{L}^{2,q}(U_r)} &= -\int_{U_r^-} (\mathrm{d}E) \wedge *\bar{\Phi} - \int_{U_r^+} (\tau^* \mathrm{d}E) \wedge *\bar{\Phi} \\
&= -\langle G, \Phi\rangle_{\mathrm{L}^{2,q+1}(U_r)} \qquad ,
\end{aligned}$$

where $G = S_{\mathrm{rot}} \operatorname{rot} E$. ■

The mirror operator

$$S_{\mathrm{div}} := (-1)^{q(N-q)} * S_{\mathrm{rot}} * : \mathbf{D}^q(U_r^-) \to \mathbf{D}^q(U_r) \tag{2.30}$$

has the corresponding properties.

Lemma 2.11 *Let $N \geq 3$ and $\varrho > 0$. There exists a constant $c > 0$, such that for all $E \in {}_0\mathbf{D}^q(\mathbb{R}^N)$ with $\operatorname{supp} E \subset U_\varrho$ there exists some $H \in \mathbf{H}^{1,q+1}(\mathbb{R}^N)$ satisfying*

$$\operatorname{div}H = E \qquad , \qquad \|H\|_{\mathbf{H}^{1,q+1}(\mathbb{R}^N)} \leq c \cdot \|E\|_{\mathrm{L}^{2,q}(\mathbb{R}^N)} \qquad .$$

Proof: Let $E \in {}_0\mathbf{D}^q(\mathbb{R}^N)$ with $\operatorname{supp} E \subset U_\varrho$. By the Fourier transformation we get $\big|\mathcal{F}E_I(x)\big| \leq c \cdot \|E\|_{\mathrm{L}^{2,q}(\mathbb{R}^N)}$, i.e. $c = \lambda(U_\varrho)^{1/2}$, and hence all components of $\mathcal{F}E$ are bounded. Let $\hat{H} := r^{-2}R\mathcal{F}E$ $\big(\hat{H}(0) := 0\big)$. The estimate

$$\big|\hat{H}_J(x)\big| \leq c \sum_{I \in \mathcal{S}(q,N)} |x|^{-1} \cdot \big|\mathcal{F}E_I(x)\big| \qquad , \qquad J \in \mathcal{S}(q+1,N) \qquad ,$$

implies $\mathcal{X}_n\hat{H} \in \mathrm{L}^{2,q+1}(\mathbb{R}^N)$ as well as $\hat{H}, \mathcal{F}^{-1}\hat{H} \in \mathrm{L}^{2,q+1}(\mathbb{R}^N)$, since $N \geq 3$. Moreover, we get

$$\|\hat{H}\|_{\mathrm{L}^{2,q+1}(\mathbb{R}^N)} + \|r\hat{H}\|_{\mathrm{L}^{2,q+1}(\mathbb{R}^N)} \leq c \cdot \|E\|_{\mathrm{L}^{2,q}(\mathbb{R}^N)} \qquad .$$

Thus $H := -\mathrm{i}\,\mathcal{F}^{-1}\hat{H} \in \mathbf{H}^{1,q+1}(\mathbb{R}^N)$ with $\|H\|_{\mathbf{H}^{1,q+1}(\mathbb{R}^N)} \leq c \cdot \|E\|_{\mathrm{L}^{2,q}(\mathbb{R}^N)}$ and using (2.28) as well as (2.22) we obtain

$$\operatorname{div} H = \mathcal{F}^{-1} T \hat{H} = \mathcal{F}^{-1} r^{-2} T R \mathcal{F} E = E \qquad ,$$

because $\operatorname{div} E = 0$ yields $T\mathcal{F}E = 0$ again by (2.28). ∎

To prepare the next lemma let $U \subset \mathbb{R}^N$ and

$$\Phi = \sum_{I \in \mathcal{S}(q,N)} \Phi_I \mathrm{d}x^I \in \mathrm{L}^{2,q}(U) \qquad .$$

Then $\Phi = \Phi^\tau + \Phi^\rho$ is an orthogonal decomposition in $\mathrm{L}^{2,q}(U)$ where

$$\Phi^\tau := \sum_{I \in \mathcal{S}(q,N-1)} \Phi_I \mathrm{d}x^I \qquad , \qquad \Phi^\rho := \sum_{N \ni I \in \mathcal{S}(q,N)} \Phi_I \mathrm{d}x^I \qquad .$$

Lemma 2.12 *Let $U \subset \mathbb{R}^N$, $m \in \mathbb{N}$, $\varepsilon \in \mathbb{A}^{m,q}(\overline{U})$ and $E \in \mathrm{L}^{2,q}(\Omega)$. Furthermore, let $E^\tau, (\varepsilon E)^\rho \in \mathbf{H}^{m,q}(U)$. Then $E \in \mathbf{H}^{m,q}(U)$.*

Proof: From $(\varepsilon E^\rho)^\rho = (\varepsilon E)^\rho - (\varepsilon E^\tau)^\rho \in \mathbf{H}^{m,q}(U)$ we get $(\varepsilon E^\rho)^\rho \in \mathbf{H}^{m,q}(U)$. Since the restriction $\varepsilon^{\rho,\rho}$ of ε acting on the normal parts, i.e. $\varepsilon^{\rho,\rho} E^\rho = (\varepsilon E^\rho)^\rho$, is pointwise invertible with $\mathrm{C}^m(\overline{U})$ entries we obtain $E^\rho \in \mathbf{H}^{m,q}(U)$. ∎

Now let us turn to the proof of Theorem 2.8. Using a partition of unity we localize our problem and only consider the more difficult case of boundary charts. By (2.13) and Lemma 2.3 we transform our problem to the special domain U_1^- using a C^{m+2}-boundary chart. Hence we have to show the following assertion: Let $\varepsilon \in \mathbb{A}^{m+1,q}\big(\overline{U_1^-}\big)$ and $E \in \overset{\circ}{\mathbf{R}}{}^q(U_1^-) \cap \varepsilon^{-1}\mathbf{D}^q(U_1^-)$ with $\operatorname{supp} E \subset \overline{U_\varrho^-}$ for some $\varrho \in (0,1)$ as well as

$$\operatorname{rot} E \in \mathbf{H}^{m,q+1}(U_1^-) \qquad , \qquad \operatorname{div} \varepsilon E \in \mathbf{H}^{m,q-1}(U_1^-) \qquad .$$

Then $E \in \mathbf{H}^{m+1,q}(U_1^-)$ and

$$\begin{aligned} &\|E\|_{\mathbf{H}^{m+1,q}(U_1^-)} \\ &\leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}(U_1^-)} + \|\operatorname{rot} E\|_{\mathbf{H}^{m,q+1}(U_1^-)} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{m,q-1}(U_1^-)} \big) \end{aligned} \tag{2.31}$$

holds uniformly in E.

First let us discuss the case $N \geq 3$. We prove (2.31) by induction on q and m. Since $\overset{\circ}{\mathbf{R}}{}^0(U_1^-) = \overset{\circ}{\mathbf{H}}{}^1(U_1^-)$ (and rot acts as ∇) the case $q = 0$ is trivial. Because of $\mathbf{D}^N(U_1^-) = \mathbf{H}^1(U_1^-)$ (and div acts as ∇) the case $q = N$ is trivial as well. Thus we assume that the assertion is valid for $q - 1$. Let $m = 0$. First we take care about the tangential derivatives and show

$$\begin{aligned} \partial_i E &\in \mathrm{L}^{2,q}(U_1^-) \qquad , \\ \|\partial_i E\|_{\mathrm{L}^{2,q}(U_1^-)} &\leq c \cdot \|E\|_{\mathbf{R}^q(U_1^-) \cap \varepsilon^{-1}\mathbf{D}^q(U_1^-)} \end{aligned} \tag{2.32}$$

for $i = 1, \ldots, N-1$. By symmetry it is sufficient to consider $i = 1$. We choose some $\theta \in (0,1)$ satisfying $\varrho + 4\theta < 1$ and put $\varrho_j := \varrho + j\theta$, $j = 1, \ldots, 4$. For $0 < |h| < \theta$ we introduce the mappings

$$\begin{array}{rcccl} \tau_h & : & \mathbb{R}^N_- & \longrightarrow & \mathbb{R}^N_- \\ & & x & \longmapsto & (x_1 + h, x_2, \cdots, x_N) \end{array} \quad , \qquad \delta_h := \frac{1}{h}(\tau_h - \mathrm{id}) \qquad ,$$

where $\mathbb{R}^N_- := \{x \in \mathbb{R}^N : x_N < 0\}$. The pullback δ_h^* of latter operator acts componentwise as the differential quotient and commutates with rot, $*$ and div. For all $F, G \in \mathrm{L}^{2,q}(U_1^-)$ with support in $\overline{U_{\varrho_3}^-}$ we have

$$\begin{aligned} \langle \delta_h^* F, G \rangle_{\mathrm{L}^{2,q}(U_1^-)} &= -\langle F, \delta_{-h}^* G \rangle_{\mathrm{L}^{2,q}(U_1^-)} \qquad , \\ \delta_h^*(\varepsilon F) &= \varepsilon \delta_h^* F + (\delta_h \varepsilon) \tau_h^* F \qquad , \\ \|\tau_h^* F\|_{\mathrm{L}^{2,q}(U_1^-)} &\leq c \cdot \|F\|_{\mathrm{L}^{2,q}(U_1^-)} \qquad , \\ \|(\delta_h \varepsilon) F\|_{\mathrm{L}^{2,q}(U_1^-)} &\leq c \cdot \|F\|_{\mathrm{L}^{2,q}(U_1^-)} \qquad , \end{aligned} \tag{2.33}$$

where $(\delta_h\varepsilon)\Phi(x) := \sum_{I,J \in \mathcal{S}(q,N)} \big(\delta_h \varepsilon_{J,I}(x)\big)\Phi_I(x)\mathrm{d}x^J$ for the matrix entries $\varepsilon_{I,J}$ of ε and $\Phi(x) = \sum_{I \in \mathcal{S}(q,N)} \Phi_I(x)\mathrm{dx}^I$ and c is independent of h or F. From $\big[$[1], Theorem 3.13$\big]$ one obtains for $m \in \mathbb{N}$ and all $F \in \mathbf{H}^{m,q}(U_1^-)$ supported in $\overline{U_{\varrho_3}^-}$

$$\|\delta_h^* F\|_{\mathbf{H}^{m-1,q}(U_1^-)} \leq \|F\|_{\mathbf{H}^{m,q}(U_1^-)} \qquad .$$

By $\big[$[1], Theorem 3.15$\big]$ to show (2.32) it suffices to prove

$$\|\delta_h^* E\|_{\mathrm{L}^{2,q}(U_{\varrho_1}^-)} \leq c \cdot \|E\|_{\mathbf{R}^q(U_1^-) \cap \varepsilon^{-1}\mathbf{D}^q(U_1^-)} \qquad ,$$

where c is independent of h, ϱ or E. Since we have $\delta_h^* E \in \mathring{\mathbf{R}}^q(U_{\varrho_1}^-)$ and moreover $\operatorname{supp} \delta_h^* E \Subset \overline{U_{\varrho_1}^-}$ this estimate follows by a density argument from

$$\big|\langle \varepsilon \delta_h^* E, \Phi \rangle_{\mathrm{L}^{2,q}(U_{\varrho_1}^-)}\big| \leq c \cdot \|E\|_{\mathbf{R}^q(U_1^-) \cap \varepsilon^{-1}\mathbf{D}^q(U_1^-)} \cdot \|\Phi\|_{\mathrm{L}^{2,q}(U_{\varrho_1}^-)} \tag{2.34}$$

for all $\Phi \in \mathring{\mathrm{C}}^{\infty,q}(U_{\varrho_1}^-)$, where c is independent of h, ϱ, E or Φ. Let $\Phi \in \mathring{\mathrm{C}}^{\infty,q}(U_{\varrho_1}^-)$. According to Lemma 2.4 we decompose $\Phi = \Phi_1 + \varepsilon^{-1}\Phi_2$ orthogonally in ${}_\varepsilon\mathrm{L}^{2,q}(\Omega)$, where $\Phi_1 \in \overline{\operatorname{rot} \mathring{\mathbf{R}}^{q-1}(U_1^-)}$ and $\Phi_2 \in \overline{\operatorname{div} \mathbf{D}^{q+1}(U_1^-)}$ $\big($closures in $\mathrm{L}^{2,q}(\Omega)\big)$, since $\mathcal{H}^q(U_1^-)$ vanishes by $\big[$[11], Satz 1, Satz 2$\big]$ and thus ${}_\varepsilon\mathcal{H}^q(U_1^-) = \{0\}$ as well. Moreover, by (2.18), (2.19) we may assume $\Phi_1 = \operatorname{rot} \Psi_1$ and $\Phi_2 = \operatorname{div} \Psi_2$ with some differential forms $\Psi_1 \in \mathring{\mathbf{R}}^{q-1}(U_1^-) \cap {}_0\mathbf{D}^{q-1}(U_1^-)$ and $\Psi_2 \in \mathbf{D}^{q+1}(U_1^-) \cap {}_0\mathring{\mathbf{R}}^{q+1}(U_1^-)$. Furthermore, the estimate (2.17) yields a constant $c > 0$ independent of Φ, Φ_ℓ, Ψ, Ψ_ℓ, such that

$$\|\Psi_1\|_{\mathbf{R}^{q-1}(U_1^-)} + \|\Psi_2\|_{\mathbf{D}^{q+1}(U_1^-)} \leq c \cdot \|\Phi\|_{\mathrm{L}^{2,q}(U_{\varrho_1}^-)}$$

holds. Let $\chi \in \overset{\circ}{\mathrm{C}}{}^{\infty}(U_{\varrho 2})$ with $\chi|_{U^-_{\varrho 1}} = 1$. Then the induction assumption yields $\chi\Psi_1 \in \mathbf{H}^{1,q-1}(U^-_1)$ as well as

$$\|\chi\Psi_1\|_{\mathbf{H}^{1,q-1}(U^-_1)} \le c \cdot \|\Psi_1\|_{\mathbf{R}^{q-1}(U^-_1)} \le c \cdot \|\Phi\|_{\mathrm{L}^{2,q}(U^-_{\varrho 1})} \qquad .$$

Clearly the form $\chi\Psi_2$ possesses compact support in $U^-_{\varrho 2} \cup U^0_{\varrho 2}$ and by Lemma 2.10 and (2.30) the extension by zero of $S_{\mathrm{div}}\chi\Psi_2$ to $\mathbb{R}^N$ is an element of $\mathbf{D}^{q+1}(\mathbb{R}^N)$. Hence we have $\tilde{\Phi}_2 := \mathrm{div} S_{\mathrm{div}}\chi\Psi_2 \in {}_0\mathbf{D}^q(\mathbb{R}^N)$ with $\mathrm{supp}\,\tilde{\Phi}_2 \Subset U_{\varrho 2}$ and $\tilde{\Phi}_2\big|_{U^-_{\varrho 1}} = \Phi_2$. Lemma 2.11 yields some $H \in \mathbf{H}^{1,q+1}(\mathbb{R}^N)$ satisfying $\mathrm{div} H = \tilde{\Phi}_2$ and furthermore the estimate $\|H\|_{\mathbf{H}^{1,q+1}(\mathbb{R}^N)} \le c\|\Phi\|_{\mathrm{L}^{2,q}(U^-_{\varrho 1})}$. Using $\Phi = \mathrm{rot}\,\chi\Psi_1 + \varepsilon^{-1}\,\mathrm{div}\,\chi H$ in $U^-_{\varrho 1}$ and (2.33) as well as $\delta^*_{-h}(\chi\Psi_1) \in \overset{\circ}{\mathbf{R}}{}^{q-1}(U^-_1)$, $E \in \overset{\circ}{\mathbf{R}}{}^{q}(U^-_1)$ we get

$$\begin{aligned}
&\langle \varepsilon\delta^*_h E, \Phi\rangle_{\mathrm{L}^{2,q}(U^-_{\varrho 1})}\\
&= \big\langle \delta^*_h(\varepsilon E), \Phi\big\rangle_{\mathrm{L}^{2,q}(U^-_{\varrho 1})} - \big\langle (\delta_h\varepsilon)\tau^*_h E, \Phi\big\rangle_{\mathrm{L}^{2,q}(U^-_{\varrho 1})}\\
&= -\big\langle \varepsilon E, \mathrm{rot}\delta^*_{-h}(\chi\Psi_1)\big\rangle_{\mathrm{L}^{2,q}(U^-_1)} - \big\langle E, \mathrm{div}\delta^*_{-h}(\chi H)\big\rangle_{\mathrm{L}^{2,q}(U^-_1)}\\
&\qquad - \big\langle \varepsilon E, (\delta_{-h}\varepsilon^{-1})\tau^*_{-h}\mathrm{div}\chi H\big\rangle_{\mathrm{L}^{2,q}(U^-_1)} - \big\langle (\delta_h\varepsilon)\tau^*_h E, \Phi\big\rangle_{\mathrm{L}^{2,q}(U^-_{\varrho 1})}\\
&= \big\langle\, \mathrm{div}\,\varepsilon E, \delta^*_{-h}(\chi\Psi_1)\big\rangle_{\mathrm{L}^{2,q}(U^-_1)} + \big\langle\, \mathrm{rot}\,E, \delta^*_{-h}(\chi H)\big\rangle_{\mathrm{L}^{2,q}(U^-_1)}\\
&\qquad - \big\langle \varepsilon E, (\delta_{-h}\varepsilon^{-1})\tau^*_{-h}\mathrm{div}\chi H\big\rangle_{\mathrm{L}^{2,q}(U^-_1)} - \big\langle (\delta_h\varepsilon)\tau^*_h E, \Phi\big\rangle_{\mathrm{L}^{2,q}(U^-_{\varrho 1})} \qquad ,
\end{aligned}$$

which immediately implies (2.34). Hence (2.32) is proved.

By (2.5) we have

$$\pm\,\partial_N\, E_J = (\mathrm{rot}\,E)_{J+N} - \sum_{J\ni j=1}^{N-1} \sigma(j, J+N-j)\cdot \partial_j\, E_{J+N-j} \in \mathrm{L}^{2,q}(U^-_1) \tag{2.35}$$

for $N \notin J$ and thus $E^\tau \in \mathbf{H}^{1,q}(U^-_1)$. Using $\partial_i(\varepsilon E) = (\partial_i\,\varepsilon)E + \varepsilon\,\partial_i\,E$ we obtain $\partial_i(\varepsilon E) \in \mathrm{L}^{2,q}(U^-_1)$ for $i = 1,\dots,N-1$ and by (2.6)

$$\pm\,\partial_N(\varepsilon E)_J = (\mathrm{div}\,\varepsilon E)_{J-N} - \sum_{J\not\ni j=1}^{N-1} \sigma(j, J)\cdot \partial_j(\varepsilon E)_{J-N+j} \in \mathrm{L}^{2,q}(U^-_1) \tag{2.36}$$

for $N \in J$ and hence $(\varepsilon E)^\rho \in \mathbf{H}^{1,q}(U^-_1)$. Lemma 2.12 yields $E \in \mathbf{H}^{1,q}(U^-_1)$ and the case $m = 0$ is proved.

Let $m \ge 1$ and our assertions be valid for $m-1$ as well as the assumptions be given for m. We consider $E, \varepsilon E \in \mathbf{H}^{m,q}(U^-_1)$ with $E \in \overset{\circ}{\mathbf{R}}{}^{q}(U^-_1) \cap \varepsilon^{-1}\mathbf{D}^q(U^-_1)$, $\mathrm{supp}\,E \subset \overline{U^-_\varrho}$,

$$\mathrm{rot}E \in \mathbf{H}^{m,q+1}(U^-_1) \qquad , \qquad \mathrm{div}\varepsilon E \in \mathbf{H}^{m,q-1}(U^-_1)$$

and the estimate

$$\|E\|_{\mathbf{H}^{m,q}(U^-_1)} \le c\cdot\big(\|E\|_{\mathrm{L}^{2,q}(U^-_1)} + \|\mathrm{rot}E\|_{\mathbf{H}^{m-1,q+1}(U^-_1)} + \|\mathrm{div}\varepsilon E\|_{\mathbf{H}^{m-1,q-1}(U^-_1)}\big) \qquad .$$

For sufficient small h we have $\delta_h^* E \in \overset{\circ}{\mathbf{R}}{}^q(U_1^-)$ and $\delta_h^* E$ resp. $\delta_h^* \operatorname{rot} E$ converges weakly to $\partial_1 E$ resp. $\partial_1 \operatorname{rot} E$ in $\mathrm{L}^{2,q}(U_1^-)$ resp. $\mathrm{L}^{2,q+1}(U_1^-)$ as $h \to 0$. Thus we obtain $\partial_1 E \in \overset{\circ}{\mathbf{R}}{}^q(U_1^-)$ and analogously $\partial_i E \in \overset{\circ}{\mathbf{R}}{}^q(U_1^-)$ for $i = 1, \ldots, N-1$. Hence all tangential derivatives $\partial_i E \in \overset{\circ}{\mathbf{R}}{}^q(U_1^-)$, $i = 1, \ldots, N-1$, satisfy

$$\begin{aligned} \operatorname{rot} \partial_i E = \partial_i \operatorname{rot} E &\in \mathbf{H}^{m-1,q+1}(U_1^-) \qquad , \\ \operatorname{div}\varepsilon\, \partial_i E = \partial_i \operatorname{div}\varepsilon E - \operatorname{div}(\partial_i \varepsilon) E &\in \mathbf{H}^{m-1,q-1}(U_1^-) \qquad , \end{aligned}$$

which implies $\partial_i E \in \mathbf{H}^{m,q}(U_1^-)$ and also $\partial_i(\varepsilon E) \in \mathbf{H}^{m,q}(U_1^-)$ by assumption. By (2.35) and (2.36) we obtain $\partial_N E^\tau, \partial_N(\varepsilon E)^\rho \in \mathbf{H}^{m,q}(U_1^-)$. Therefore we get also $E^\tau, (\varepsilon E)^\rho \in \mathbf{H}^{m+1,q}(U_1^-)$ and finally by Lemma 2.12 $E \in \mathbf{H}^{m+1,q}(U_1^-)$, which completes the proof for $N \geq 3$.

The only non trivial remaining case is $N = 2$, $q = 1$. But this case can be proved similarly to the case $N \geq 3$ without using Lemma 2.11, since even $\Psi_2 \in \mathbf{H}^{1,2}(U_1^-)$ holds. ∎

2.3 Trace and extension theorems

Let Ω be a C^3-region. We provide a 'tangential trace' operator

$$\Gamma_t : \mathbf{R}^q(\Omega) \to \mathcal{R}^q(\partial\Omega)$$

and a 'tangential extension' operator

$$\check{\Gamma}_t : \mathcal{R}^q(\partial\Omega) \to \mathbf{R}^q(\Omega) \qquad ,$$

where the space of tangential traces $\mathcal{R}^q(\partial\Omega)$ will be defined below. The corresponding results for 'normal traces' on $\mathbf{D}^q(\Omega)$ will be achieved using the Hodge star operator.

From now on we will distinguish between rot, div and $*$ on Ω and $\partial\,\Omega$. Keeping the old notation for the operators on Ω we denote the corresponding operators on the boundary $\partial\,\Omega$ by Rot, Div and $\circledast$.

First we need some preparations: For $m \in (0, \infty)$ let $\mathbf{H}^{-m,q}(\partial\Omega)$ denote the dual space of $\overset{\circ}{\mathbf{H}}{}^{m,q}(\partial\Omega) = \mathbf{H}^{m,q}(\partial\Omega)$ and $\langle \lambda, \Phi \rangle_{\mathbf{H}^{-m,q}(\partial\Omega)}$ for $\lambda \in \mathbf{H}^{-m,q}(\partial\Omega)$ and $\Phi \in \mathbf{H}^{m,q}(\partial\Omega)$ the duality. We always demand antilinearity in the second component of the duality. We define rotation, divergence and star operator by

$$\begin{aligned} \langle \operatorname{Rot} \lambda, \Phi \rangle_{\mathbf{H}^{-(m+1),q+1}(\partial\Omega)} &:= -\langle \lambda, \operatorname{Div} \Phi \rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} \qquad , \\ \langle \operatorname{Div} \lambda, \Psi \rangle_{\mathbf{H}^{-(m+1),q-1}(\partial\Omega)} &:= -\langle \lambda, \operatorname{Rot} \Psi \rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} \qquad , \\ \langle \circledast\lambda, \phi \rangle_{\mathbf{H}^{-m,N-1-q}(\partial\Omega)} &:= (-1)^{q(N-1-q)} \langle \lambda, \circledast\phi \rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} \end{aligned} \tag{2.37}$$

for $\Phi \in \mathbf{H}^{m+1,q+1}(\partial\Omega)$, $\Psi \in \mathbf{H}^{m+1,q-1}(\partial\Omega)$ and $\phi \in \mathbf{H}^{m,N-1-q}(\partial\Omega)$. We have

$$\begin{aligned}
\langle \circledast\lambda, \circledast\Phi\rangle_{\mathbf{H}^{-m,N-1-q}(\partial\Omega)} &= (-1)^{q(N-1-q)}\langle \lambda, \circledast\circledast\Phi\rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} \\
&= \langle \lambda, \Phi\rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} \qquad , \\
\langle \circledast\circledast\lambda, \Phi\rangle_{\mathbf{H}^{-m,q}(\partial\Omega)} &= (-1)^{q(N-1-q)}\langle \circledast\lambda, \circledast\Phi\rangle_{\mathbf{H}^{-m,N-1-q}(\partial\Omega)} \\
&= (-1)^{q(N-1-q)}\langle \lambda, \Phi\rangle_{\mathbf{H}^{-m,q}(\partial\Omega)}
\end{aligned}$$

and $\operatorname{Div} = (-1)^{(q-1)(N-1)} \circledast \operatorname{Rot} \circledast$. Moreover, we introduce the spaces

$$\begin{aligned}
\mathcal{R}^q(\partial\Omega) &:= \left\{\lambda \in \mathbf{H}^{-1/2,q}(\partial\Omega) : \operatorname{Rot}\lambda \in \mathbf{H}^{-1/2,q+1}(\partial\Omega)\right\} \qquad , \\
\mathcal{D}^q(\partial\Omega) &:= \left\{\lambda \in \mathbf{H}^{-1/2,q}(\partial\Omega) : \operatorname{Div}\lambda \in \mathbf{H}^{-1/2,q-1}(\partial\Omega)\right\} \qquad ,
\end{aligned}$$

which will be equipped with their canonical norms

$$\begin{aligned}
\|\lambda\|_{\mathcal{R}^q(\partial\Omega)} &:= \left(\|\lambda\|_{\mathbf{H}^{-1/2,q}(\partial\Omega)} + \|\operatorname{Rot}\lambda\|_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)}\right)^{1/2} \qquad , \\
\|\lambda\|_{\mathcal{D}^q(\partial\Omega)} &:= \left(\|\lambda\|_{\mathbf{H}^{-1/2,q}(\partial\Omega)} + \|\operatorname{Div}\lambda\|_{\mathbf{H}^{-1/2,q-1}(\partial\Omega)}\right)^{1/2} \qquad ,
\end{aligned}$$

where we identify $\operatorname{Rot}\lambda \in \mathbf{H}^{-3/2,q+1}(\partial\Omega)$ resp. $\operatorname{Div}\lambda \in \mathbf{H}^{-3/2,q-1}(\partial\Omega)$ with its continuous extension on $\mathbf{H}^{-1/2,q+1}(\partial\Omega)$ resp. $\mathbf{H}^{-1/2,q-1}(\partial\Omega)$. The property

$$\mathcal{R}^q(\partial\Omega) = \circledast\mathcal{D}^{N-1-q}(\partial\Omega)$$

keeps true and the induced mapping is isometric.

To define traces on $\mathbf{R}^q(\Omega)$ we first have to discuss tangential and normal traces on $\mathbf{H}^{m,q}(\Omega)$. Using boundary charts, (2.13) and the corresponding results for scalar Sobolev spaces (see e.g. [[26], Satz 8.7, Satz 8.8]), which componentwise will be applied to q-forms in $\mathbb{R}^N$, we obtain the following two lemmas:

Lemma 2.13 *Let $m \in \mathbb{N}$, Ω be a C^{m+1}-region and $\iota : \partial\,\Omega \hookrightarrow \overline{\Omega} \subset \mathrm{M}$ the natural embedding. Then there exists a linear and continuous tangential trace operator*

$$\gamma_t : \mathbf{H}^{m,q}(\Omega) \to \mathbf{H}^{m-1/2,q}(\partial\Omega)$$

with

$$\gamma_t\Phi = \iota^*\Phi \qquad , \qquad \operatorname{Rot}\gamma_t\Phi = \gamma_t \operatorname{rot}\Phi$$

for all $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$. Moreover, γ_t is surjective, i.e. there exists a linear and continuous tangential extension operator

$$\check{\gamma}_t : \mathbf{H}^{m-1/2,q}(\partial\Omega) \to \mathbf{H}^{m,q}(\Omega)$$

with the property $\gamma_t\check{\gamma}_t = \mathrm{id}$.

Furthermore, using the star operator we define linear and continuous normal trace and extension operators by

$$\begin{array}{rccc} \gamma_n & : & \mathbf{H}^{m,q}(\Omega) & \longrightarrow & \mathbf{H}^{m-1/2,q-1}(\partial\Omega) \\ & & \Psi & \longmapsto & (-1)^{(q-1)N} \circledast \gamma_t * \Psi \end{array} \quad ,$$

$$\begin{array}{rccc} \check{\gamma}_n & : & \mathbf{H}^{m-1/2,q-1}(\partial\Omega) & \longrightarrow & \mathbf{H}^{m,q}(\Omega) \\ & & \lambda & \longmapsto & (-1)^{q(N-q)} * \check{\gamma}_t \circledast \lambda \end{array} \quad ,$$

which possess the corresponding properties, i.e. $\operatorname{Div} \gamma_n \Psi = -\gamma_n \operatorname{div} \Psi$ for all smooth forms $\Psi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ and $\gamma_n \check{\gamma}_n = \mathrm{id}$. In local coordinates we check $\gamma_t * \check{\gamma}_t = 0$ and thus

$$\gamma_n \check{\gamma}_t = 0 \qquad , \qquad \gamma_t \check{\gamma}_n = 0 \qquad . \tag{2.38}$$

By (2.9) and (2.14) we obtain

$$\langle \mathrm{rot}\Phi, \Psi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \Phi, \mathrm{div}\Psi \rangle_{\mathrm{L}^{2,q}(\Omega)} = \langle \gamma_t \Phi, \gamma_n \Psi \rangle_{\mathrm{L}^{2,q}(\partial\Omega)} \tag{2.39}$$

for $\Phi \in \mathbf{H}^{1,q}(\Omega)$, $\Psi \in \mathbf{H}^{1,q+1}(\Omega)$.

This suggests to define the tangential trace

$$\Gamma_t E \in \mathbf{H}^{-1/2,q}(\partial\Omega)$$

of a q-form $E \in \mathbf{R}^q(\Omega)$ by

$$\Gamma_t E(\varphi) = \langle \Gamma_t E, \varphi \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} := \langle \mathrm{rot} E, \check{\gamma}_n \varphi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \mathrm{div} \check{\gamma}_n \varphi \rangle_{\mathrm{L}^{2,q}(\Omega)} \tag{2.40}$$

for all $\varphi \in \mathbf{H}^{1/2,q}(\partial\Omega)$. Clearly acting on $E \in \mathbf{H}^{1,q}(\Omega)$ it satisfies

$$\langle \Gamma_t E, \varphi \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} = \langle \gamma_t E, \varphi \rangle_{\mathrm{L}^{2,q}(\partial\Omega)} \tag{2.41}$$

for all $\varphi \in \mathbf{H}^{1/2,q}(\partial\Omega)$. Hence in this case we have $\Gamma_t E = \langle \gamma_t E, \cdot \rangle_{\mathrm{L}^{2,q}(\partial\Omega)}$ and we identify $\Gamma_t E$ with $\gamma_t E$ as an element in $\mathbf{H}^{1/2,q}(\partial\Omega)$.

Theorem 2.14 *For each $E \in \mathbf{R}^q(\Omega)$ the tangential trace $\Gamma_t E$ is an element of $\mathcal{R}^q(\partial\Omega)$. Moreover, the tangential trace Γ_t has the following properties:*

(i) $$\bigwedge_{\substack{E \in \mathbf{R}^q(\Omega) , \\ \Psi \in \mathbf{H}^{1,q+1}(\Omega)}} \quad \langle \Gamma_t E, \gamma_n \Psi \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} = \langle \mathrm{rot} E, \Psi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \mathrm{div} \Psi \rangle_{\mathrm{L}^{2,q}(\Omega)}$$

(ii) $$\bigwedge_{E \in \mathbf{R}^q(\Omega)} \quad \operatorname{Rot} \Gamma_t E = \Gamma_t \operatorname{rot} E$$

(iii) *The mapping $\Gamma_t : \mathbf{R}^q(\Omega) \to \mathcal{R}^q(\partial\Omega)$ is continuous.*

Proof: By (2.39) we get for $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ and $\Psi \in \mathbf{H}^{1,q+1}(\Omega)$

$$\begin{aligned} &\langle \mathrm{rot}\Phi, \check{\gamma}_n\gamma_n\Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \Phi, \mathrm{div}\check{\gamma}_n\gamma_n\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)} \\ &= \langle \gamma_t\Phi, \gamma_n\Psi\rangle_{\mathrm{L}^{2,q}(\partial\Omega)} = \langle \mathrm{rot}\Phi, \Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \Phi, \mathrm{div}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)} \qquad . \end{aligned}$$

The density argument (2.15) and the definition of Γ_t yield **(i)**. For $E \in \mathbf{R}^q(\Omega)$ we obtain

$$\|\Gamma_t E\|_{\mathbf{H}^{-1/2,q}(\partial\Omega)} \leq c \cdot \|E\|_{\mathbf{R}^q(\Omega)} \qquad ,$$

i.e. $\Gamma_t : \mathbf{R}^q(\Omega) \to \mathbf{H}^{-1/2,q}(\partial\Omega)$ is continuous. Furthermore, for $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ and $\varphi \in \mathbf{H}^{3/2,q+1}(\partial\Omega)$ we calculate

$$\begin{aligned} \langle \Gamma_t\Phi, \mathrm{Div}\,\varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} &= \langle \gamma_t\Phi, \mathrm{Div}\,\varphi\rangle_{\mathrm{L}^{2,q}(\partial\Omega)} = -\langle \mathrm{Rot}\,\gamma_t\Phi, \varphi\rangle_{\mathrm{L}^{2,q+1}(\partial\Omega)} \\ &= -\langle \gamma_t \mathrm{rot}\Phi, \varphi\rangle_{\mathrm{L}^{2,q+1}(\partial\Omega)} = -\langle \Gamma_t\mathrm{rot}\Phi, \varphi\rangle_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} \quad . \end{aligned}$$

Approximating $E \in \mathbf{R}^q(\Omega)$ with $\Phi \in \mathrm{C}^{\infty,q}(\overline{\Omega})$ by (2.15) we get that $\mathrm{Rot}\,\Gamma_t E$ exists in $\mathbf{H}^{-3/2,q+1}(\partial\Omega)$ and $\mathrm{Rot}\,\Gamma_t E = \Gamma_t\,\mathrm{rot}\,E$. Hence $\mathrm{Rot}\,\Gamma_t E \in \mathbf{H}^{-1/2,q+1}(\partial\Omega)$ since $\mathrm{rot}\,E \in {}_0\mathbf{R}^{q+1}(\Omega) \subset \mathbf{R}^{q+1}(\Omega)$, i.e. $\Gamma_t E \in \mathcal{R}^q(\partial\Omega)$. This proves that the mapping $\Gamma_t : \mathbf{R}^q(\Omega) \to \mathcal{R}^q(\partial\Omega)$ is well defined, and **(ii)**. Clearly we have

$$\|\,\mathrm{Rot}\,\Gamma_t E\|_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} = \|\Gamma_t\,\mathrm{rot}\,E\|_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} \leq c \cdot \|E\|_{\mathbf{R}^q(\Omega)} \qquad ,$$

since $\mathrm{rot} : \mathbf{R}^q(\Omega) \to \mathbf{R}^{q+1}(\Omega)$ is continuous. Thus **(iii)** is proved. ∎

Defining the normal trace acting on $\mathbf{D}^q(\Omega)$ by

$$\Gamma_n := (-1)^{(q-1)N} \circledast \Gamma_t *$$

we achieve (using the star operator)

Theorem 2.15 *For each $H \in \mathbf{D}^q(\Omega)$ the normal trace $\Gamma_n H$ is an element of $\mathcal{D}^{q-1}(\partial\Omega)$. Moreover, the normal trace Γ_n has the following properties:*

(i) $$\bigwedge_{\substack{H \in \mathbf{D}^q(\Omega), \\ \Psi \in \mathbf{H}^{1,q-1}(\Omega)}} \langle \Gamma_n H, \gamma_t\Psi\rangle_{\mathbf{H}^{-1/2,q-1}(\partial\Omega)} = \langle \mathrm{div}H, \Psi\rangle_{\mathrm{L}^{2,q-1}(\Omega)} + \langle H, \mathrm{rot}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}$$

(ii) $$\bigwedge_{H \in \mathbf{D}^q(\Omega)} \mathrm{Div}\,\Gamma_n H = -\Gamma_n \mathrm{div} H$$

(iii) *The mapping $\Gamma_n : \mathbf{D}^q(\Omega) \to \mathcal{D}^{q-1}(\partial\Omega)$ is continuous.*

Our traces possess natural properties. So we have for all $E \in \mathbf{R}^q(\Omega)$

$$\Gamma_t E = 0 \qquad \Longleftrightarrow \qquad E \in \overset{\circ}{\mathbf{R}}{}^q(\Omega) \tag{2.42}$$

and for all $E \in \mathbf{D}^q(\Omega)$

$$\Gamma_n E = 0 \qquad \Longleftrightarrow \qquad E \in \overset{\circ}{\mathbf{D}}{}^q(\Omega) \qquad . \tag{2.43}$$

Furthermore, (2.41) and (2.38) yield

$$\check{\gamma}_n \varphi \in \overset{\circ}{\mathbf{R}}{}^{q+1}(\Omega) \qquad , \qquad \check{\gamma}_t \varphi \in \overset{\circ}{\mathbf{D}}{}^q(\Omega) \tag{2.44}$$

for all $\varphi \in \mathbf{H}^{1/2,q}(\partial\Omega)$.

Now we will construct two extension operators.

Theorem 2.16 *Let $\varepsilon, \nu \in \mathbb{A}^{0,q}(\Omega)$. Then there exist two linear and continuous extension operators*

$$\begin{aligned} \check{\Gamma}_t : \mathcal{R}^q(\partial\Omega) &\longrightarrow \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega) \cap {}_\nu\mathcal{H}^q(\Omega)^{\perp_\varepsilon} \qquad , \\ \check{\Gamma}_n : \mathcal{D}^{q-1}(\partial\Omega) &\longrightarrow \mathbf{D}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{R}^q(\Omega) \cap {}_\nu\tilde{\mathcal{H}}^q(\Omega)^{\perp_\varepsilon} \end{aligned}$$

satisfying $\Gamma_t\check{\Gamma}_t = \mathrm{id}$ and $\Gamma_n\check{\Gamma}_n = \mathrm{id}$.

Remark 2.17

(i) *$\check{\Gamma}_t$ even maps to $\mathbf{R}^q(\Omega) \cap \varepsilon^{-1} \operatorname{div} \operatorname{rot} \big(\overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{H}^{2,q}(\Omega)\big)$.*

(ii) *$\check{\Gamma}_n$ even maps to $\mathbf{D}^q(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \operatorname{div} \big(\overset{\circ}{\mathbf{D}}{}^q(\Omega) \cap \mathbf{H}^{2,q}(\Omega)\big)$.*

(iii) *Because of the missing boundary condition neither $\check{\Gamma}_t$ nor $\check{\Gamma}_n$ maps to $\mathbf{H}^{1,q}(\Omega)$. But this is obvious, since the existence of the left inverse Γ_t resp. Γ_n would imply*

$$\mathcal{R}^q(\partial\Omega) \subset \mathbf{H}^{1/2,q}(\partial\Omega) \qquad \textit{resp.} \qquad \mathcal{D}^{q-1}(\partial\Omega) \subset \mathbf{H}^{1/2,q-1}(\partial\Omega) \qquad .$$

(iv) *$E := \check{\Gamma}_t\lambda \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$ is the unique solution of the boundary value problem*

$$\begin{aligned} \operatorname{div} \varepsilon E &= 0 \qquad , & \operatorname{div} \operatorname{rot} E &= 0 \qquad , \\ \Gamma_t E &= \lambda \qquad , & E &\in {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp_\varepsilon} \qquad . \end{aligned}$$

(v) *$H := \check{\Gamma}_n\lambda \in \mathbf{D}^q(\Omega) \cap \varepsilon^{-1}\mathbf{R}^q(\Omega)$ is the unique solution of the boundary value problem*

$$\begin{aligned} \operatorname{rot} \varepsilon H &= 0 \qquad , & \operatorname{rot} \operatorname{div} H &= 0 \qquad , \\ \Gamma_n H &= \lambda \qquad , & H &\in {}_\varepsilon\tilde{\mathcal{H}}^q(\Omega)^{\perp_\varepsilon} \qquad . \end{aligned}$$

Here ${}_\varepsilon\tilde{\mathcal{H}}^q(\Omega) := {}_0\overset{\circ}{\mathbf{D}}{}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{R}^q(\Omega)$ denotes the space of 'harmonic Neumann forms' and we have ${}_\varepsilon\tilde{\mathcal{H}}^q(\Omega) = *_{*\varepsilon*}\mathcal{H}^{q'}(\Omega)$ as well as $\dim {}_\varepsilon\tilde{\mathcal{H}}^q(\Omega) = d^{q'}$ with $q' := N - q$. Clearly we put again $\tilde{\mathcal{H}}^q(\Omega) := {}_{\mathrm{id}}\tilde{\mathcal{H}}^q(\Omega)$.

Proof: Let $\lambda \in \mathcal{R}^q(\partial\Omega)$. We have to find some $E = \check{\Gamma}_t\lambda \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$ with $\Gamma_t E = \lambda$. We look at

$$\begin{aligned} \mathrm{Y}^q(\Omega) &:= \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{D}^q(\Omega) \qquad , \\ \mathrm{Y}^q_{\mathrm{rot}}(\Omega) &:= \mathrm{Y}^q \cap \mathrm{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega) = \mathrm{rot}\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega) \cap \mathbf{D}^q(\Omega) \qquad , \\ \mathrm{Y}^q_{\mathrm{div}}(\Omega) &:= \mathrm{Y}^q \cap \mathrm{div}\mathbf{D}^{q+1}(\Omega) = \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathrm{div}\mathbf{D}^{q+1}(\Omega) \end{aligned}$$

supplied with $\langle \cdot\,, \cdot \rangle_{\mathbf{R}^q(\Omega)\cap\mathbf{D}^q(\Omega)}$. By Lemma 2.4 and (2.18), (2.19)

$$\mathrm{Y}^q(\Omega) = \mathrm{Y}^q_{\mathrm{rot}}(\Omega) \oplus \mathrm{Y}^q_{\mathrm{div}}(\Omega) \oplus \mathcal{H}^q(\Omega)$$

is an orthogonal decomposition. Due to Theorem 2.8 all spaces are subspaces of $\mathbf{H}^{1,q}(\Omega)$.

We consider the following problem: Find some $F \in \mathrm{Y}^{q+2}_{\mathrm{rot}}(\Omega)$ satisfying

$$\langle \mathrm{div}F, \mathrm{div}\Phi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \langle \mathrm{Rot}\,\lambda, \gamma_n\Phi \rangle_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} \tag{2.45}$$

for all $\Phi \in \mathrm{Y}^{q+2}_{\mathrm{rot}}(\Omega)$.

Because of (2.17) the continuous bilinear form on the left hand side is strongly coercive in $\mathrm{Y}^{q+2}_{\mathrm{rot}}(\Omega)$ and using Theorem 2.8 the right hand side is an antilinear continuous functional on $\mathrm{Y}^{q+2}_{\mathrm{rot}}(\Omega)$. Hence the Lax-Milgram theorem yields a unique solution F with

$$\|F\|_{\mathbf{D}^{q+2}(\Omega)} \leq c \cdot \|\,\mathrm{Rot}\,\lambda\|_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} \qquad . \tag{2.46}$$

Analogously we solve a second problem: Find some $H \in \mathrm{Y}^{q+1}_{\mathrm{rot}}(\Omega)$ with

$$\begin{aligned} &\langle \varepsilon^{-1}\mathrm{div}H, \mathrm{div}\Phi \rangle_{\mathrm{L}^{2,q}(\Omega)} \\ &= \langle \mathrm{div}F, \Phi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \phi_\lambda, \Phi \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \lambda, \gamma_n\Phi \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} \end{aligned} \tag{2.47}$$

for all $\Phi \in \mathrm{Y}^{q+1}_{\mathrm{rot}}(\Omega)$, where

$$\phi_\lambda := \sum_{\ell=1}^{d^{q+1}} \langle \lambda, \gamma_n h_\ell \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} \cdot h_\ell$$

for some $\langle \cdot\,, \cdot \rangle_{\mathrm{L}^{2,q+1}(\Omega)}$-orthonormal basis $\{h_1, \ldots, h_{d^{q+1}}\}$ of $\mathcal{H}^{q+1}(\Omega)$.

Then clearly

$$\|H\|_{\mathbf{D}^{q+1}(\Omega)} \leq c \cdot \big(\|\mathrm{div}F\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\lambda\|_{\mathbf{H}^{-1/2,q}(\partial\Omega)}\big) \tag{2.48}$$

holds and combining (2.46) and (2.48) we have

$$\|F\|_{\mathbf{D}^{q+2}(\Omega)} + \|H\|_{\mathbf{D}^{q+1}(\Omega)} \leq c \cdot \|\lambda\|_{\mathcal{R}^q(\partial\Omega)} \qquad . \tag{2.49}$$

For all h_ℓ the right hand side of (2.47) vanishes. Thus (2.47) also holds for all forms $\Phi \in \mathrm{Y}^{q+1}_{\mathrm{rot}}(\Omega) \oplus \mathcal{H}^{q+1}(\Omega)$. Let $\Phi \in \mathrm{Y}^{q+1}_{\mathrm{div}}(\Omega)$. By (2.18), (2.19) and Lemma 2.4 we may assume $\Phi = \mathrm{div}\Psi$ with $\Psi \in \mathbf{D}^{q+2}(\Omega) \cap {}_0\overset{\circ}{\mathbf{R}}{}^{q+2}(\Omega) \cap \mathcal{H}^{q+2}(\Omega)^{\perp} = \mathrm{Y}^{q+2}_{\mathrm{rot}}(\Omega)$. Since $\Phi \in \mathbf{H}^{1,q+1}(\Omega)$ we obtain by Theorem 2.8 $\Psi \in \mathbf{H}^{2,q+2}(\Omega)$. Here we needed the C^3-requirements on the boundary $\partial\,\Omega$. Using (2.45)

$$\begin{aligned} &\langle \mathrm{div}F, \Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \phi_\lambda, \Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \lambda, \gamma_n\Phi\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} \\ &= \langle \mathrm{div}F, \mathrm{div}\Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \lambda, \gamma_n \mathrm{div}\Psi\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} \\ &= \langle \mathrm{Rot}\,\lambda, \gamma_n\Psi\rangle_{\mathbf{H}^{-1/2,q+1}(\partial\Omega)} + \langle \lambda, \mathrm{Div}\,\gamma_n\Psi\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} = 0 \end{aligned}$$

shows that (2.47) is even valid for $\Phi \in \mathrm{Y}^{q+1}_{\mathrm{div}}(\Omega)$ and hence for all $\Phi \in \mathrm{Y}^{q+1}(\Omega)$. Putting $E := -\varepsilon^{-1}\mathrm{div}H$ we obtain $E \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$ by (2.47) since of course $\overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega) \subset \mathrm{Y}^{q+1}$ and $\mathrm{rot}E = \mathrm{div}F + \phi_\lambda$. Moreover,

$$\check{\gamma}_n\mathbf{H}^{1/2,q}(\partial\Omega) \subset \mathbf{H}^{1,q+1}(\Omega) \cap \overset{\circ}{\mathbf{R}}{}^{q+1}(\Omega) \subset \mathrm{Y}^{q+1}(\Omega)$$

holds and thus $\Gamma_t E = \lambda$ follows again by (2.47). Finally, by (2.49) our tangential extension operator is continuous.

Defining

$$\begin{array}{rcccc} \check{\Gamma}_n & : & \mathcal{D}^{q-1}(\partial\Omega) & \longrightarrow & \mathbf{D}^q(\Omega) \\ & & \lambda & \longmapsto & (-1)^{q(N-q)} * \check{\Gamma}_t \circledast \lambda \end{array}$$

(with $\pm * \varepsilon *$ instead of ε) yields $\Gamma_n\check{\Gamma}_n\lambda = (-1)^{(q-1)N} \circledast \Gamma_t\check{\Gamma}_t \circledast \lambda = \lambda$ as well as $\check{\Gamma}_n\lambda \in \mathbf{D}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{R}^q(\Omega)$. Clearly $\check{\Gamma}_n$ is continuous as well. ■

To finish this section we present a generalization of Theorem 2.8, a regularity theorem handling inhomogeneous boundary data:

Theorem 2.18 *Let $m \in \mathbb{N}_0$, Ω be a bounded $(\mathrm{C}^{m+2} \cap \mathrm{C}^3)$-region and $\varepsilon \in \mathbb{A}^{m+1,q}(\overline{\Omega})$. Furthermore, let $E \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$ with*

$$\mathrm{rot}E \in \mathbf{H}^{m,q+1}(\Omega) \quad , \quad \mathrm{div}\varepsilon E \in \mathbf{H}^{m,q-1}(\Omega) \quad , \quad \Gamma_t E \in \mathbf{H}^{m+1/2,q}(\partial\Omega) \quad .$$

Then $E \in \mathbf{H}^{m+1,q}(\Omega)$ and there exists a positive constant c independent of E, such that

$$\begin{aligned} &\|E\|_{\mathbf{H}^{m+1,q}(\Omega)} \\ &\leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}(\Omega)} + \|\mathrm{rot}E\|_{\mathbf{H}^{m,q+1}(\Omega)} + \|\mathrm{div}\varepsilon E\|_{\mathbf{H}^{m,q-1}(\Omega)} + \|\Gamma_t E\|_{\mathbf{H}^{m+1/2,q}(\partial\Omega)}\right) \quad . \end{aligned}$$

Proof: Let $\check{E} := \check{\gamma}_t\Gamma_t E \in \mathbf{H}^{m+1,q}(\Omega)$. Then $\hat{E} := E - \check{E} \in \overset{\circ}{\mathbf{R}}{}^q(\Omega)$ satisfies the assumptions of Theorem 2.8. Thus we get $\hat{E} \in \mathbf{H}^{m+1,q}(\Omega)$ and using the continuity of $\check{\gamma}_t$ the asserted estimate as well. ■

Remark 2.19 *Clearly using the star operator and some transformation $E \rightsquigarrow \varepsilon E$ the assumption $\Gamma_n\varepsilon E \in \mathbf{H}^{m+1/2,q-1}(\partial\Omega)$ instead of $\Gamma_t E \in \mathbf{H}^{m+1/2,q}(\partial\Omega)$ yields a corresponding theorem. Moreover, these regularity results hold for spaces of the form $\varepsilon^{-1}\mathbf{R}^q(\Omega) \cap \mathbf{D}^q(\Omega)$ as well.*

2.4 Static solution theory

Let Ω be a C^3-region, $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$ an admissible transformation and d^q continuous linear functionals Φ_ε^ℓ as in Theorem 2.6 be given. We consider the following problem:

Find some q-form $E \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$ satisfying

$$\begin{aligned} \mathrm{rot} E &= G \quad , \\ \mathrm{div} \varepsilon E &= F \quad , \\ \Gamma_t E &= \lambda \quad , \\ \Phi_\varepsilon^\ell(E) &= \alpha_\ell \quad , \quad \ell = 1, \dots, d^q \quad . \end{aligned} \tag{2.50}$$

Noting $\mathcal{H}^{q+1}(\Omega) \subset \mathbf{H}^{1,q+1}(\Omega)$ we get

Theorem 2.20 *The conditions $G \in {}_0\mathbf{R}^{q+1}(\Omega)$, $F \in {}_0\mathbf{D}^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^\perp$, $\lambda \in \mathcal{R}^q(\partial\Omega)$, $\alpha \in \mathbb{C}^{d^q}$ and*

$$\mathrm{Rot}\,\lambda = \Gamma_t G \qquad \wedge \qquad \bigwedge_{h \in \mathcal{H}^{q+1}(\Omega)} \langle G, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \langle \lambda, \gamma_n h \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)}$$

are necessary and sufficient for the solvability of (2.50)*. The solution is unique and depends continuously on the data, i.e. there exists a positive constant c independent of E or the data, such that*

$$\|E\|_{\mathbf{R}^q(\Omega)\cap\varepsilon^{-1}\mathbf{D}^q(\Omega)} \le c \cdot \left(\|F\|_{\mathrm{L}^{2,q-1}(\Omega)} + \|G\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\lambda\|_{\mathcal{R}^q(\partial\Omega)} + |\alpha| \right)$$

holds.

Proof: The necessity of the conditions is easily checked. By Theorem 2.16 we obtain $\check{E} := \check{\Gamma}_t \lambda \in \mathbf{R}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$. The ansatz $E := \check{E} + \tilde{E}$ with $\tilde{E} \in \mathring{\mathbf{R}}^q(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q(\Omega)$ leads with (2.42) to the system

$$\begin{aligned} \mathrm{rot} \tilde{E} &= G - \mathrm{rot}\,\check{E} =: \tilde{G} \in {}_0\mathbf{R}^{q+1}(\Omega) \quad , \\ \mathrm{div} \varepsilon \tilde{E} &= F \in {}_0\mathbf{D}^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^\perp \quad , \\ \Phi_\varepsilon^\ell(\tilde{E}) &= \alpha_\ell - \Phi_\varepsilon^\ell(\check{E}) =: \tilde{\alpha}_\ell \quad , \quad \ell = 1, \dots, d^q \quad , \end{aligned}$$

which is uniquely solved by $\tilde{E} := \mathcal{M}\mathrm{ax}_\varepsilon^{-1}(F, \tilde{G}, \tilde{\alpha}_\ell)$ with $\mathcal{M}\mathrm{ax}_\varepsilon$ from Theorem 2.6, if $(F, \tilde{G}, \tilde{\alpha}_\ell) \in W^q(\Omega)$. Hence it remains to show

$$\tilde{G} \in \mathring{\mathbf{R}}^{q+1}(\Omega) \cap \mathcal{H}^{q+1}(\Omega)^\perp \quad .$$

From $\Gamma_t \tilde{G} = \Gamma_t G - \mathrm{Rot}\,\lambda = 0$ we see that $\tilde{G}$ satisfies the homogeneous (electric) boundary condition. To check the orthogonality on the Dirichlet forms we pick some h from $\mathcal{H}^{q+1}(\Omega) \subset \mathbf{H}^{1,q+1}(\Omega)$ (by Theorem 2.8) and compute

$$\langle \tilde{G}, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \langle G, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \underbrace{\Gamma_t \check{E}}_{=\lambda}, \gamma_n h \rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} = 0$$

using Theorem 2.14 (i). This concludes the proof. ■

Finally we shortly turn to the dual problem using the Hodge star operator. Let $d^{q'}$ continuous linear functionals Ψ^ℓ_ε on $\mathbf{D}^q(\Omega) \cap \varepsilon^{-1}\mathbf{R}^q(\Omega)$ with

$$ {}_\varepsilon\tilde{\mathcal{H}}^q(\Omega) \cap \bigcap_{\ell=1}^{d^{q'}} N(\Psi^\ell_\varepsilon) = \{0\} $$

be given. We formulate the dual problem:

Find for given data F, G, λ, α a q-form $H \in \mathbf{D}^q(\Omega) \cap \varepsilon^{-1}\mathbf{R}^q(\Omega)$ satisfying

$$ \begin{aligned} \mathrm{div} H &= F \quad , \\ \mathrm{rot}\varepsilon H &= G \quad , \\ \Gamma_n H &= \lambda \quad , \\ \Psi^\ell_\varepsilon(H) &= \alpha_\ell \quad , \quad \ell = 1, \ldots, d^{q'} \quad . \end{aligned} \tag{2.51} $$

Corollary 2.21 *The conditions $G \in {}_0\mathbf{R}^{q+1}(\Omega) \cap \tilde{\mathcal{H}}^{q+1}(\Omega)^\perp$, $F \in {}_0\mathbf{D}^{q-1}(\Omega)$, $\lambda \in \mathcal{D}^{q-1}(\partial\Omega)$, $\alpha \in \mathbb{C}^{d^{q'}}$ and*

$$ \mathrm{Div}\,\lambda = -\Gamma_n F \qquad \wedge \qquad \bigwedge_{h \in \tilde{\mathcal{H}}^{q-1}(\Omega)} \langle F, h \rangle_{\mathrm{L}^{2,q-1}(\Omega)} = \langle \lambda, \gamma_t h \rangle_{\mathbf{H}^{-1/2,q-1}(\partial\Omega)} $$

are necessary and sufficient for the solvability of (2.51)*. The solution is unique and depends continuously on the data, i.e. there exists a positive constant c independent of H or the data, such that*

$$ \|H\|_{\mathbf{D}^q(\Omega)\cap\varepsilon^{-1}\mathbf{R}^q(\Omega)} \leq c \cdot \Big(\|F\|_{\mathrm{L}^{2,q-1}(\Omega)} + \|G\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\lambda\|_{\mathcal{D}^{q-1}(\partial\Omega)} + |\alpha| \Big) $$

holds.

Proof: Applying the last theorem to the data $\pm * F$, $\pm * G$, $\pm * \lambda$, α, the transformation $\pm * \varepsilon *$ and the linear functionals $\Phi^\ell_\varepsilon := \Psi^\ell_\varepsilon(* \,\cdot\,)$ we obtain our solution by $H := *E$. ■

3 Exterior domains

In this section we will consider an exterior domain $\Omega \subset \mathbb{R}^N$, i.e. $\mathbb{R}^N \setminus \Omega$ is compact, as a special Riemannian manifold of dimension $3 \leq N \in \mathbb{N}$. To this end we need some preliminaries:

3.1 Notations and preliminaries

We fix a radius r_0 and some radii $r_n := 2^n r_0$, $n \in \mathbb{N}$, such that $\mathbb{R}^N \setminus \Omega$ is a compact subset of U_{r_0}. For later purpose we choose a cut-off function $\boldsymbol{\eta}$, such that

$$\boldsymbol{\eta} \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R}) \quad , \quad \operatorname{supp} \boldsymbol{\eta} \subset [1,\infty) \quad , \quad \boldsymbol{\eta}|_{[2,\infty)} = 1 \quad , \tag{3.1}$$

and define two other cut-off functions by

$$\hat{\eta}(t) := \boldsymbol{\eta}\big(1 + \frac{t - r_1}{r_2 - r_1}\big) \tag{3.2}$$

and

$$\eta := \hat{\eta} \circ r \quad . \tag{3.3}$$

Setting $A_r := \mathbb{R}^N \setminus K_r$ and $Z_{r,\tilde{r}} := A_r \cap U_{\tilde{r}}$ we note $\operatorname{supp} \nabla\eta \subset \overline{Z_{r_1,r_2}}$.

The definitions of spaces from section 2 carry over to exterior domains as long as the boundedness of Ω is not necessary. Using the weight function

$$\rho := (1 + r^2)^{1/2}$$

we introduce for $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$ the weighted Sobolev spaces

$$\begin{aligned} \mathrm{H}^m_s(\Omega) &:= \big\{ u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) : \rho^{s+|\alpha|}\,\partial^\alpha u \in \mathrm{L}^2(\Omega) \text{ for all } |\alpha| \le m \big\} \quad , \\ \subset \mathbf{H}^m_s(\Omega) &:= \big\{ u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) : \rho^{s}\,\partial^\alpha u \in \mathrm{L}^2(\Omega) \text{ for all } |\alpha| \le m \big\} \quad . \end{aligned}$$

To distinguish between these different polynomially weighted Sobolev spaces of exterior domains we are forced to use roman and bold roman letters simultaneously. Equipped with their natural norms

$$\begin{aligned} \|\cdot\|_{\mathrm{H}^m_s(\Omega)} &:= \Big(\sum_{|\alpha|\le m} \|\rho^{s+|\alpha|}\,\partial^\alpha \cdot\|^2_{\mathrm{L}^2(\Omega)} \Big)^{1/2} = \Big(\sum_{|\alpha|\le m} \|\,\partial^\alpha \cdot\|^2_{\mathrm{H}^0_{s+|\alpha|}(\Omega)} \Big)^{1/2} \quad , \\ \|\cdot\|_{\mathbf{H}^m_s(\Omega)} &:= \Big(\sum_{|\alpha|\le m} \|\rho^{s}\,\partial^\alpha \cdot\|^2_{\mathrm{L}^2(\Omega)} \Big)^{1/2} = \Big(\sum_{|\alpha|\le m} \|\,\partial^\alpha \cdot\|^2_{\mathrm{H}^0_{s}(\Omega)} \Big)^{1/2} \end{aligned}$$

these are Hilbert spaces. In the special cases $m = 0$ or $s = 0$ we also write

$$\mathrm{H}^m(\Omega) := \mathrm{H}^m_0(\Omega) \quad , \quad \mathbf{H}^m(\Omega) = \mathbf{H}^m_0(\Omega) \quad ,$$
$$\mathrm{L}^2_s(\Omega) := \mathrm{H}^0_s(\Omega) = \mathbf{H}^0_s(\Omega) \quad , \quad \mathrm{L}^2(\Omega) = \mathrm{H}^0_0(\Omega) = \mathbf{H}^0_0(\Omega) \quad .$$

Now we have a global chart (Ω, id) and naturally Ω becomes a N-dimensional smooth Riemannian manifold with Cartesian coordinates $\{x_1, \ldots, x_N\}$. As in section 2 with componentwise partial derivatives $\partial^\alpha u = (\partial^\alpha u_I)\,\mathrm{d}x^I$, if $u = u_I\,\mathrm{d}x^I$, we introduce for $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$ the Sobolev spaces $\mathrm{H}^{m,q}_s(\Omega)$ resp. $\mathbf{H}^{m,q}_s(\Omega)$ of q-forms and denote the natural (componentwise) norms as in the scalar case by $\|\cdot\|_{\mathrm{H}^{m,q}_s(\Omega)}$ resp. $\|\cdot\|_{\mathbf{H}^{m,q}_s(\Omega)}$. Again in the special cases $m = 0$ or $s = 0$ we write

$$\mathrm{H}^{m,q}(\Omega) := \mathrm{H}^{m,q}_0(\Omega) \quad , \quad \mathbf{H}^{m,q}(\Omega) = \mathbf{H}^{m,q}_0(\Omega) \quad ,$$
$$\mathrm{L}^{2,q}_s(\Omega) := \mathrm{H}^{0,q}_s(\Omega) = \mathbf{H}^{0,q}_s(\Omega) \quad , \quad \mathrm{L}^{2,q}(\Omega) = \mathrm{H}^{0,q}_0(\Omega) = \mathbf{H}^{0,q}_0(\Omega) \quad .$$

Especially for $m = s = 0$ and $f = f_I\,\mathrm{d}x^I, g = g_I\,\mathrm{d}x^I \in \mathrm{L}^{2,q}(\Omega)$ we have the scalar product

$$\langle f, g\rangle_{\mathrm{L}^{2,q}(\Omega)} = \int_\Omega f \wedge *\overline{g} = \int_\Omega *\langle f, g\rangle_q = \int_\Omega \langle f, g\rangle_q \, d\lambda = \int_\Omega f_I \overline{g}_I \, d\lambda \quad .$$

Furthermore, for $s \in \mathbb{R}$ we need some special weighted spaces suited for Maxwell's equations:

$$\begin{aligned} \mathrm{R}^q_s(\Omega) &:= \big\{E \in \mathrm{L}^{2,q}_s(\Omega) : \operatorname{rot} E \in \mathrm{L}^{2,q+1}_{s+1}(\Omega)\big\} \quad , \\ \subset \mathbf{R}^q_s(\Omega) &:= \big\{E \in \mathrm{L}^{2,q}_s(\Omega) : \operatorname{rot} E \in \mathrm{L}^{2,q+1}_{s}(\Omega)\big\} \quad , \\ \mathrm{D}^q_s(\Omega) &:= \big\{H \in \mathrm{L}^{2,q}_s(\Omega) : \operatorname{div} H \in \mathrm{L}^{2,q-1}_{s+1}(\Omega)\big\} \quad , \\ \subset \mathbf{D}^q_s(\Omega) &:= \big\{H \in \mathrm{L}^{2,q}_s(\Omega) : \operatorname{div} H \in \mathrm{L}^{2,q-1}_{s}(\Omega)\big\} \end{aligned}$$

Equipped with their natural graph norms these are all Hilbert spaces. To generalize the homogeneous boundary condition we introduce $\mathring{\mathrm{R}}^q_s(\Omega)$ resp. $\mathring{\mathbf{R}}^q_s(\Omega)$ as the closure of $\mathring{\mathrm{C}}^{\infty,q}(\Omega)$ in the corresponding graph norm $\|\cdot\|_{\mathrm{R}^q_s(\Omega)}$ resp. $\|\cdot\|_{\mathbf{R}^q_s(\Omega)}$. The spaces $\mathbf{R}^q_s(\Omega)$, $\mathbf{D}^q_s(\Omega)$ and even $\mathring{\mathbf{R}}^q_s(\Omega)$ are invariant under multiplication with bounded smooth functions. As in the last section a subscript 0 at the lower left corner indicates vanishing rotation resp. divergence, e.g.

$$\begin{aligned} {}_0\mathring{\mathrm{R}}^q_s(\Omega) &:= \big\{E \in \mathring{\mathrm{R}}^q_s(\Omega) : \operatorname{rot} E = 0\big\} = {}_0\mathring{\mathbf{R}}^q_s(\Omega) \quad , \\ {}_0\mathbf{D}^q_s(\Omega) &:= \big\{H \in \mathbf{D}^q_s(\Omega) : \operatorname{div} H = 0\big\} = {}_0\mathrm{D}^q_s(\Omega) \quad , \end{aligned}$$

and in the special case $s = 0$ we neglect the weight index, e.g.

$${}_0\mathrm{D}^q(\Omega) := {}_0\mathrm{D}^q_0(\Omega) \quad , \quad \mathring{\mathbf{R}}^q(\Omega) := \mathring{\mathbf{R}}^q_0(\Omega) \quad .$$

By the star operator we have

$$ {}_{(0)}\overset{(\circ)}{\mathrm{D}}{}^{N-q}_{(s)}(\Omega) = *{}_{(0)}\overset{(\circ)}{\mathrm{R}}{}^{q}_{(s)}(\Omega) \qquad , \qquad {}_{(0)}\overset{(\circ)}{\mathbf{D}}{}^{N-q}_{(s)}(\Omega) = *{}_{(0)}\overset{(\circ)}{\mathbf{R}}{}^{q}_{(s)}(\Omega) \qquad , $$

where $\overset{\circ}{\mathrm{D}}{}^q_s(\Omega)$ and $\overset{\circ}{\mathbf{D}}{}^q_s(\Omega)$ are defined analogously to the corresponding spaces of rotations. Finally we need the local spaces

$$ \begin{aligned} \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) &:= \big\{ E \in \mathrm{A}^q(\Omega) : E \in \mathrm{L}^{2,q}(\Xi) \text{ for all } \Xi \Subset \Omega \big\} \qquad , \\ \mathrm{L}^{2,q}_{\mathrm{loc}}(\overline{\Omega}) &:= \big\{ E \in \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) : E \in \mathrm{L}^{2,q}(\Omega \cap U_\varrho) \text{ for all } \varrho > r_0 \big\} \qquad , \\ \mathrm{R}^{q}_{\mathrm{loc}}(\Omega) &:= \big\{ E \in \mathrm{L}^{2,q}_{\mathrm{loc}}(\Omega) : \operatorname{rot} E \in \mathrm{L}^{2,q+1}_{\mathrm{loc}}(\Omega) \big\} \qquad , \\ \mathrm{R}^{q}_{\mathrm{loc}}(\overline{\Omega}) &:= \big\{ E \in \mathrm{R}^{q}_{\mathrm{loc}}(\Omega) : E \in \mathbf{R}^{q}(\Omega \cap U_\varrho) \text{ for all } \varrho > r_0 \big\} \qquad , \\ \overset{\circ}{\mathrm{R}}{}^{q}_{\mathrm{loc}}(\overline{\Omega}) &:= \big\{ E \in \mathrm{R}^{q}_{\mathrm{loc}}(\overline{\Omega}) : \varphi E \in \overset{\circ}{\mathbf{R}}{}^{q}(\Omega) \text{ for all } \varphi \in \overset{\circ}{\mathrm{C}}{}^{\infty}(\mathbb{R}^N) \big\} \qquad . \end{aligned} $$

In this sense we also may define $\mathrm{H}^{m,q}_{\mathrm{loc}}(\Omega)$, $\mathrm{H}^{m,q}_{\mathrm{loc}}(\overline{\Omega})$ and $\mathrm{D}^{q}_{\mathrm{loc}}(\overline{\Omega})$, $\overset{\circ}{\mathrm{D}}{}^{q}_{\mathrm{loc}}(\overline{\Omega})$. If we consider the whole space, i.e. $\Omega = \mathbb{R}^N$, we omit the dependence on the domain and write for example

$$ {}_0\mathrm{R}^q_s := {}_0\mathrm{R}^q_s(\mathbb{R}^N) \qquad , \qquad \mathbf{H}^{m,q}_s := \mathbf{H}^{m,q}_s(\mathbb{R}^N) \qquad . $$

Exchanging the weight subscript by vox, e.g. ${}_0\mathrm{R}^q_{\mathrm{vox}}(\Omega)$, we indicate, that such functions or forms have bounded support.

Finally in this case of an exterior domain we need some additional decay properties of our transformations. Let $\tau \geq 0$. A transformation ν belongs to $\mathbb{A}^{0,q}_\tau(\Omega)$, if and only if $\nu \in \mathbb{A}^{0,q}(\Omega)$, i.e. ν is admissible, and

$$ \nu = \mathrm{id} + \hat{\nu} \qquad \text{with} \qquad \hat{\nu} = \mathcal{O}(r^{-\tau}) \qquad \text{as} \qquad r \to \infty $$

holds. We call τ the '*order of decay*' of the perturbation $\hat{\nu}$ (or simply of ν). Furthermore, for $\ell \in \mathbb{N}$ we define $\nu \in \mathbb{A}^{\ell,q}_\tau(\Omega)$ resp. $\nu \in \mathbb{A}^{\ell,q}_\tau(\overline{\Omega})$, if and only if $\nu \in \mathbb{A}^{\ell,q}(\Omega)$ resp. $\nu \in \mathbb{A}^{\ell,q}(\overline{\Omega})$ and the transformation ν fulfills the asymptotics

$$ \partial^\alpha \nu = \partial^\alpha \hat{\nu} = \mathcal{O}(r^{-\tau}) \qquad \text{as} \qquad r \to \infty $$

for all $1 \leq |\alpha| \leq \ell$. For $\tau = 0$ this only means boundedness and hence we have $\mathbb{A}^{\ell,q}_0(\Omega) = \mathbb{A}^{\ell,q}(\Omega)$ resp. $\mathbb{A}^{\ell,q}_0(\overline{\Omega}) = \mathbb{A}^{\ell,q}(\overline{\Omega})$.

Similarly to the bounded domain case we need a special property of our boundary $\partial\Omega$:

Definition 3.1 *Ω possesses the 'Maxwell's local compactness property' (*MLCP*), if and only if the embeddings*

$$ \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \mathbf{D}^q(\Omega) \hookrightarrow \mathrm{L}^{2,q}_{\mathrm{loc}}(\overline{\Omega}) $$

are compact for all q.

Remark 3.2 *The following assertions are equivalent:*

(i) Ω *possesses the* MLCP.

(ii) $\Omega \cap U_\varrho$ *possesses the* MCP *for all* $\varrho \geq r_0$.

(iii) *The embeddings*

$$\overset{\circ}{\mathbf{R}}{}^q_s(\Omega) \cap \mathbf{D}^q_s(\Omega) \hookrightarrow \mathrm{L}^{2,q}_t(\Omega)$$

are compact for all $t, s \in \mathbb{R}$ *with* $t < s$ *and all* q .

(iv) *For all* $t, s \in \mathbb{R}$ *with* $t < s$ *, all* q *and all* $\varepsilon_q \in \mathbb{A}^{0,q}(\Omega)$ *the embeddings*

$$\overset{\circ}{\mathbf{R}}{}^q_s(\Omega) \cap \varepsilon_q^{-1}\mathbf{D}^q_s(\Omega) \hookrightarrow \mathrm{L}^{2,q}_t(\Omega)$$

are compact.

Let $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$ and $t \in \mathbb{R}$. We introduce the *'(weighted harmonic) Dirichlet forms'*

$$_\varepsilon\mathcal{H}^q_t(\Omega) := {}_0\overset{\circ}{\mathrm{R}}{}^q_t(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}^q_t(\Omega) \tag{3.4}$$

and in the special case $\varepsilon = \mathrm{id}$ we denote them by $\mathcal{H}^q_t(\Omega)$. If $t = 0$, we also write $_\varepsilon\mathcal{H}^q(\Omega) := {}_\varepsilon\mathcal{H}^q_0(\Omega)$. Moreover, we define the dimension of the Dirichlet forms by

$$d^q_t := \dim {}_\varepsilon\mathcal{H}^q_t(\Omega) \qquad , \qquad d^q := d^q_0 \qquad .$$

The same arguments as in the bounded domain case show, that the $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\Omega)}$-orthogonal decompositions presented in Lemma 2.4 still hold true in unbounded domains. We have

$$\begin{aligned}
\mathrm{L}^{2,q}(\Omega) &= \overline{\mathrm{rot}\,\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_0\mathrm{D}^q(\Omega) = {}_0\overset{\circ}{\mathrm{R}}{}^q(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\mathrm{div}\,\mathbf{D}^{q+1}(\Omega)} \\
&= \varepsilon^{-1}\overline{\mathrm{rot}\,\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon {}_0\mathrm{D}^q(\Omega) = \varepsilon^{-1}{}_0\overset{\circ}{\mathrm{R}}{}^q(\Omega) \oplus_\varepsilon \overline{\mathrm{div}\,\mathbf{D}^{q+1}(\Omega)} \\
&= \overline{\mathrm{rot}\,\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon {}_\varepsilon\mathcal{H}^q(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\mathrm{div}\,\mathbf{D}^{q+1}(\Omega)} \\
&= \varepsilon^{-1}\overline{\mathrm{rot}\,\overset{\circ}{\mathbf{R}}{}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_{\varepsilon^{-1}}\mathcal{H}^q(\Omega) \oplus_\varepsilon \overline{\mathrm{div}\,\mathbf{D}^{q+1}(\Omega)} \qquad ,
\end{aligned} \tag{3.5}$$

where all closures are taken in $\mathrm{L}^{2,q}(\Omega)$.

As in the bounded domain case one easily sees that the dimension of the space of Dirichlet forms $_\varepsilon\mathcal{H}^q(\Omega)$ does not depend on ε . From [12] and [14] we even obtain $\dim {}_\varepsilon\mathcal{H}^q(\Omega) = \dim \mathcal{H}^q(\Omega) = \dim \mathcal{H}^q_{-1}(\Omega) = \beta_{N-q} < \infty$, if Ω possesses the MLCP. For the sake of completeness we also define the *'(weighted harmonic) Neumann forms'*

$$_\varepsilon\tilde{\mathcal{H}}^q_t(\Omega) := {}_0\overset{\circ}{\mathrm{D}}{}^q_t(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{R}^q_t(\Omega) \qquad .$$

3.2 Regularity

Before we discuss regularity results in our exterior domain we consider the whole space case $\Omega = \mathbb{R}^N$. In this special case we are able to characterize the following Sobolev- resp. rotation- and divergence-spaces with the aid of the Fourier transform on q-forms $\mathcal{F}$ using the formulas (2.25)-(2.29):

$$\mathbf{H}^{m,q} = \{E \in \mathrm{L}^{2,q} : \mathcal{F}(E) \in \mathrm{L}^{2,q}_m\} \qquad , \qquad m \in \mathbb{N} \tag{3.6}$$

$$\mathbf{R}^q = \{E \in \mathrm{L}^{2,q} : R\mathcal{F}(E) \in \mathrm{L}^{2,q+1}\} \tag{3.7}$$

$$\mathbf{D}^q = \{E \in \mathrm{L}^{2,q} : T\mathcal{F}(E) \in \mathrm{L}^{2,q-1}\} \tag{3.8}$$

In this sense we also may define $\mathbf{H}^{s,q}$, if $s \in \mathbb{R}$. First we prove

Lemma 3.3 *Let $\varepsilon \in \mathbb{A}^{1,q}$. Then $\mathbf{R}^q \cap \varepsilon^{-1}\mathbf{D}^q = \mathbf{H}^{1,q}$ holds with equivalent norms depending on ε.*

Remark 3.4 *This lemma and a cutting technique easily yield a first inner regularity result.*

Proof: Partial integration and $\Delta = \operatorname{rot}\operatorname{div} + \operatorname{div}\operatorname{rot}$ yield

$$\bigwedge_{\Phi \in \overset{\circ}{\mathrm{C}}{}^{\infty,q}} \qquad \sum_{n=1}^{N} \|\,\partial_n \Phi\|^2_{\mathrm{L}^{2,q}} = \|\operatorname{rot}\Phi\|^2_{\mathrm{L}^{2,q+1}} + \|\operatorname{div}\Phi\|^2_{\mathrm{L}^{2,q-1}} \qquad . \tag{3.9}$$

A combination of this identity and (3.6)-(3.8) as well as (2.22) implies

$$\mathbf{R}^q \cap \mathbf{D}^q = \mathbf{H}^{1,q} \tag{3.10}$$

with equal norms, since $\overset{\circ}{\mathrm{C}}{}^{\infty,q}$ is dense in $\mathbf{H}^{1,q}$.

Now let $E \in \mathbf{R}^q \cap \varepsilon^{-1}\mathbf{D}^q$. By (3.5) and $\big[$[12], Lemma 7$\big]$ we decompose the q-form $E = \operatorname{rot}\Phi + \Psi$ according to

$$\mathrm{L}^{2,q} = \overline{\operatorname{rot}\mathbf{R}^{q-1}} \oplus {}_0\mathrm{D}^q = \operatorname{rot}\left(\mathrm{R}^{q-1}_{-1} \cap {}_0\mathrm{D}^{q-1}_{-1}\right) \oplus {}_0\mathrm{D}^q$$

observing $\operatorname{rot}\Psi = \operatorname{rot}E$ and $\operatorname{div}\Psi = 0$. By (3.10) we obtain $\Psi \in \mathbf{H}^{1,q}$ and the estimate $\|\Psi\|_{\mathbf{H}^{1,q}} \leq c \cdot \|E\|_{\mathbf{R}^q}$ with some constant $c > 0$. Hence $\varepsilon\Psi \in \mathbf{H}^{1,q}$ and Φ solves the elliptic system

$$\operatorname{div}\varepsilon\operatorname{rot}\Phi = \operatorname{div}\varepsilon E - \operatorname{div}\varepsilon\Psi =: F \in \mathrm{L}^{2,q-1} \qquad , \qquad \operatorname{div}\Phi = 0 \qquad ,$$

where $\|F\|_{\mathrm{L}^{2,q}} \leq c \cdot \|E\|_{\mathbf{R}^q \cap \varepsilon^{-1}\mathbf{D}^q}$. Using the operators $\tau_{h,i}$ and $\delta_{h,i}$, $i = 1, \ldots, N$, $h > 0$, defined on $\mathbb{R}^N$ from the proof of Theorem 2.8 as well as $\|\tau^*_{h,i}\phi\|_{\mathrm{L}^{2,q}} = \|\phi\|_{\mathrm{L}^{2,q}}$ and the estimates $\|\delta^*_{h,i}\phi\|_{\mathrm{L}^{2,q}} \leq \|\,\partial_i\,\phi\|_{\mathrm{L}^{2,q}}$, $\|\operatorname{rot}\phi\|_{\mathrm{L}^{2,q+1}} \leq \sum_{n=1}^{N} \|\,\partial_n\,\phi\|_{\mathrm{L}^{2,q}}$ we get

$$\langle \varepsilon\delta^*_{h,i}\operatorname{rot}\Phi, \operatorname{rot}\phi\rangle_{\mathrm{L}^{2,q}} = \langle \operatorname{div}\varepsilon\operatorname{rot}\Phi, \delta^*_{-h,i}\phi\rangle_{\mathrm{L}^{2,q-1}} + \left\langle \operatorname{rot}\Phi, (\delta_{-h,i}\varepsilon)\tau^*_{-h,i}\operatorname{rot}\phi\right\rangle_{\mathrm{L}^{2,q}}$$

and thus by (3.9) uniformly in ϕ and h

$$\begin{aligned}\big|\langle\varepsilon\delta^*_{h,i}\operatorname{rot}\Phi,\operatorname{rot}\phi\rangle_{\mathrm{L}^{2,q}}\big| &\leq c\cdot\|E\|_{\mathbf{R}^q\cap\varepsilon^{-1}\mathbf{D}^q}\cdot\sum_{n=1}^N\|\,\partial_n\,\phi\|_{\mathrm{L}^{2,q-1}}\\ &\leq c\cdot\|E\|_{\mathbf{R}^q\cap\varepsilon^{-1}\mathbf{D}^q}\cdot\big(\|\operatorname{rot}\phi\|_{\mathrm{L}^{2,q}}+\|\operatorname{div}\phi\|_{\mathrm{L}^{2,q-2}}\big)\end{aligned}$$

for all $\phi\in\overset{\circ}{\mathrm{C}}{}^{\infty,q-1}$. By this estimate and since $\overset{\circ}{\mathrm{C}}{}^{\infty,q-1}$ is dense in $\mathrm{R}^{q-1}_{-1}\cap\mathrm{D}^{q-1}_{-1}$ we obtain

$$\|\delta^*_{h,i}\operatorname{rot}\Phi\|_{\mathrm{L}^{2,q}}\leq c\cdot\|E\|_{\mathbf{R}^q\cap\varepsilon^{-1}\mathbf{D}^q}\qquad,$$

where the constant $c>0$ is independent of h. Therefore $\operatorname{rot}\Phi\in\mathbf{H}^{1,q}$ and the estimates $\|\,\partial_i\operatorname{rot}\Phi\|_{\mathrm{L}^{2,q}}\leq c\cdot\|E\|_{\mathbf{R}^q\cap\varepsilon^{-1}\mathbf{D}^q}$, $i=1,\dots,N$, hold, which completes the proof. ∎

Corollary 3.5 *Let $s\in\mathbb{R}$ and $\varepsilon\in\mathbb{A}^{1,q}$.*

(i) *Then $\mathbf{R}^q_s\cap\varepsilon^{-1}\mathbf{D}^q_s=\mathbf{H}^{1,q}_s$ holds with equivalent norms depending on ε.*

(ii) *If additionally $\varepsilon=\mathrm{id}+\hat\varepsilon\in\mathbb{A}^{1,q}_\tau$ with $\tau>0$ and*

$$\partial_n\,\hat\varepsilon=\mathcal{O}(r^{-1})\qquad\textit{as}\qquad r\to\infty\qquad,\qquad n=1,\dots,N\qquad,$$

then also $\mathrm{R}^q_s\cap\varepsilon^{-1}\mathrm{D}^q_s=\mathrm{H}^{1,q}_s$ holds with equivalent norms depending on ε.

Proof: Let $E\in\mathbf{R}^q_s\cap\varepsilon^{-1}\mathbf{D}^q_s$. We have $\rho^sE\in\mathrm{L}^{2,q}$ and by (2.24)

$$\begin{aligned}\operatorname{rot}(\rho^sE)&=\rho^s\operatorname{rot}E+s\rho^{s-2}RE\in\mathrm{L}^{2,q+1}\qquad,\\ \operatorname{div}(\rho^s\varepsilon E)&=\rho^s\operatorname{div}\varepsilon E+s\rho^{s-2}T\varepsilon E\in\mathrm{L}^{2,q-1}\qquad.\end{aligned}$$

Thus using Lemma 3.3 $\rho^sE\in\mathbf{R}^q\cap\varepsilon^{-1}\mathbf{D}^q=\mathbf{H}^{1,q}$ follows and

$$\partial_n(\rho^sE)=\rho^s\,\partial_n\,E+s\rho^{s-2}\mathcal{X}_nE\in\mathrm{L}^{2,q}$$

yields **(i)**.

Looking at $E\in\mathrm{R}^q_s\cap\varepsilon^{-1}\mathrm{D}^q_s\subset\mathbf{R}^q_s\cap\varepsilon^{-1}\mathbf{D}^q_s$ we obtain $E\in\mathbf{H}^{1,q}_s$ by **(i)**. Therefore it only remains to show $\partial_n\,E\in\mathrm{L}^{2,q}_{s+1}$, $n=1,\dots,N$. Choosing the cut-off function $\varphi_t:=\mathbf{1}-\boldsymbol{\eta}(t^{-1}r)$ we calculate with (3.9) or (3.10) uniformly in $t\in\mathbb{R}_+$

$$
\begin{aligned}
&\big\| \partial_n(\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q}_{s+1}} \\
&\leq c \cdot \Big(\big\| \partial_n(\underbrace{\rho^{s+1}\varphi_t \cdot E}_{\in \mathrm{H}^{1,q}_{\mathrm{vox}} \subset \mathbf{H}^{1,q}})\big\|_{\mathrm{L}^{2,q}} + \big\|(s+1)\rho^{s-1}\mathcal{X}_n\varphi_t \cdot E\big\|_{\mathrm{L}^{2,q}}\Big) \\
&\leq c \cdot \Big(\big\| \operatorname{rot}(\rho^{s+1}\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q+1}} + \big\| \operatorname{div}(\rho^{s+1}\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q-1}} + \|\varphi_t \cdot E\|_{\mathrm{L}^{2,q}_s}\Big) \\
&\leq c \cdot \Big(\|\varphi_t \cdot E\|_{\mathrm{R}^q_s \cap \varepsilon^{-1}\mathrm{D}^q_s} + \big\| \operatorname{div}(\varphi_t \cdot \hat{\varepsilon} E)\big\|_{\mathrm{L}^{2,q-1}_{s+1}}\Big) \\
&\leq c \cdot \Big(\|\varphi_t \cdot E\|_{\mathrm{R}^q_s \cap \varepsilon^{-1}\mathrm{D}^q_s} + \sum_{m=1}^{N} \big\| \partial_m(\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q}_{s+1-\tau}}\Big) \qquad .
\end{aligned}
$$

Since $\tau > 0$ and decomposing $\mathbb{R}^N = \overline{U_\vartheta} \cup A_\vartheta$ we get for all $\vartheta \in \mathbb{R}_+$

$$
\big\| \partial_m(\varphi_t \cdot E)\big\|^2_{\mathrm{L}^{2,q}_{s+1-\tau}} \leq c_\vartheta \cdot \big\| \partial_m(\varphi_t \cdot E)\big\|^2_{\mathrm{L}^{2,q}_s} + (1+\vartheta^2)^{-\tau} \cdot \big\| \partial_m(\varphi_t \cdot E)\big\|^2_{\mathrm{L}^{2,q}_{s+1}}
$$

with some constant $c_\vartheta > 0$ depending on ϑ and s, τ. A combination of the latter two estimates yields for some sufficient large ϑ and with **(i)**

$$
\begin{aligned}
&\sum_{n=1}^{N} \big\| \partial_n(\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q}_{s+1}} \\
&\leq c \cdot \Big(\|\varphi_t \cdot E\|_{\mathbf{H}^{1,q}_s} + \big\| \operatorname{rot}(\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q+1}_{s+1}} + \big\| \operatorname{div}(\varphi_t \cdot \varepsilon E)\big\|_{\mathrm{L}^{2,q-1}_{s+1}}\Big) \\
&\leq c \cdot \big(\|E\|_{\mathrm{R}^q_s \cap \varepsilon^{-1}\mathrm{D}^q_s} + \|t^{-1}r^{-1}RE\|_{\mathrm{L}^{2,q+1}_{s+1}(Z_{t,2t})} + \|t^{-1}r^{-1}T\varepsilon E\|_{\mathrm{L}^{2,q-1}_{s+1}(Z_{t,2t})}\big) \qquad .
\end{aligned}
$$

Using $t^{-1} \leq 2r^{-1}$ in $Z_{t,2t}$ we finally obtain the estimate

$$
\sum_{n=1}^{N} \| \partial_n E\|_{\mathrm{L}^{2,q}_{s+1}(U_t)} \leq \sum_{n=1}^{N} \big\| \partial_n(\varphi_t \cdot E)\big\|_{\mathrm{L}^{2,q}_{s+1}} \leq c \cdot \|E\|_{\mathrm{R}^q_s \cap \varepsilon^{-1}\mathrm{D}^q_s} \qquad ,
$$

which holds uniformly in t. Thus letting $t \to \infty$ the monotone convergence theorem implies $E \in \mathrm{H}^{1,q}_s$ and the desired estimate, i.e. **(ii)** is proved. ■

Now we can formulate our first main regularity result in this section:

Theorem 3.6 *Let $\ell \in \mathbb{N}_0$, $s \in \mathbb{R}$ and $\varepsilon \in \mathbb{A}^{\ell+1,q}$ as well as $E \in \mathrm{L}^{2,q}_s$.*

(i) *Then* $\operatorname{rot} E \in \mathbf{H}^{\ell,q+1}_s$, $\operatorname{div} \varepsilon E \in \mathbf{H}^{\ell,q-1}_s$ *is equivalent to $E \in \mathbf{H}^{\ell+1,q}_s$ and there exists a positive constant c, such that*

$$\|E\|_{\mathbf{H}^{\ell+1,q}_s} \leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}_s} + \|\operatorname{rot} E\|_{\mathbf{H}^{\ell,q+1}_s} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{\ell,q-1}_s} \big)$$

holds uniformly in E.

(ii) *If in addition $\varepsilon = \operatorname{id} + \hat{\varepsilon} \in \mathbb{A}^{\ell+1,q}_\tau$ with $\tau > 0$ and for all $1 \leq |\alpha| \leq \ell + 1$*

$$\partial^\alpha \hat{\varepsilon} = \mathcal{O}(r^{-|\alpha|}) \qquad \text{as} \qquad r \to \infty \qquad ,$$

then $\operatorname{rot} E \in \mathrm{H}^{\ell,q+1}_{s+1}$, $\operatorname{div} \varepsilon E \in \mathrm{H}^{\ell,q-1}_{s+1}$ *is equivalent to $E \in \mathrm{H}^{\ell+1,q}_s$ and with some positive constant c the estimate*

$$\|E\|_{\mathrm{H}^{\ell+1,q}_s} \leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}_s} + \|\operatorname{rot} E\|_{\mathrm{H}^{\ell,q+1}_{s+1}} + \|\operatorname{div} \varepsilon E\|_{\mathrm{H}^{\ell,q-1}_{s+1}} \big)$$

holds uniformly in E.

Remark 3.7 *Clearly from this theorem we obtain easily a second inner regularity result by a cutting technique.*

Proof: Corollary 3.5 proves the assertions for $\ell = 0$.
To show **(i)** by induction we assume

$$\varepsilon \in \mathbb{A}^{\ell+1,q} \qquad , \qquad \operatorname{rot} E \in \mathbf{H}^{\ell,q+1}_s \qquad , \qquad \operatorname{div} \varepsilon E \in \mathbf{H}^{\ell,q-1}_s \qquad .$$

The assertion for $\ell - 1$ yields $E \in \mathbf{H}^{\ell,q}_s$ and the corresponding estimate. Then for $n = 1, \dots, N$ we get $\partial_n E \in \mathrm{L}^{2,q}_s$, $\operatorname{rot} \partial_n E \in \mathbf{H}^{\ell-1,q+1}_s$ and

$$\operatorname{div}(\varepsilon\, \partial_n E) = \partial_n \operatorname{div} \varepsilon E - \operatorname{div} \big((\partial_n \varepsilon) E \big) \in \mathbf{H}^{\ell-1,q-1}_s \qquad .$$

Using the assumption for $\ell - 1$ a second time we obtain $\partial_n E \in \mathbf{H}^{\ell,q}_s$ and

$$\|\partial_n E\|_{\mathbf{H}^{\ell,q}_s} \leq c \cdot \Big(\|\partial_n E\|_{\mathrm{L}^{2,q}_s} + \|\operatorname{rot} \partial_n E\|_{\mathbf{H}^{\ell-1,q+1}_s} + \big\|\operatorname{div}(\varepsilon\, \partial_n E)\big\|_{\mathbf{H}^{\ell-1,q-1}_s} \Big)$$

for $n = 1, \dots, N$. Hence $E \in \mathbf{H}^{\ell+1,q}_s$ and

$$\begin{aligned} \|E\|_{\mathbf{H}^{\ell+1,q}_s} &\leq c \cdot \big(\|E\|_{\mathbf{H}^{\ell,q}_s} + \sum_{n=1}^N \|\partial_n E\|_{\mathbf{H}^{\ell,q}_s} \big) \\ &\leq c \cdot \big(\|E\|_{\mathbf{H}^{\ell,q}_s} + \|\operatorname{rot} E\|_{\mathbf{H}^{\ell,q+1}_s} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{\ell,q-1}_s} \big) \qquad . \end{aligned}$$

Similarly we prove **(ii)** paying attention to the fact that the weights in the $\| \cdot \|_{\mathrm{H}^{\ell,q}_s}$-norms grow with the number of derivatives and that this effect is compensated by the decay properties of $\hat{\varepsilon}$ and its derivatives. ■

Using the results from the last theorem we are able to show easily weighted inner regularity in exterior domains with a cutting technique:

Corollary 3.8 *Let $\ell \in \mathbb{N}_0$, $s \in \mathbb{R}$, $\varepsilon \in \mathbb{A}^{\ell+1,q}(\Omega)$ and $E \in \mathrm{L}^{2,q}_s(\Omega)$ as well as $\Xi \subset \mathbb{R}^N$ be another exterior domain, such that $\Xi \subset \Omega$ and $\operatorname{dist}(\Xi, \partial\,\Omega) > 0$ (dist: distance function).*

(i) *Then $\operatorname{rot} E \in \mathbf{H}^{\ell,q+1}_s(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathbf{H}^{\ell,q-1}_s(\Omega)$ imply $E \in \mathbf{H}^{\ell+1,q}_s(\Xi)$ and there exists a positive constant c, such that*

$$\|E\|_{\mathbf{H}^{\ell+1,q}_s(\Xi)} \leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathbf{H}^{\ell,q+1}_s(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{\ell,q-1}_s(\Omega)} \right)$$

holds uniformly in E.

(ii) *If even $\varepsilon = \mathrm{id} + \hat{\varepsilon} \in \mathbb{A}^{\ell+1,q}_\tau(\Omega)$ with $\tau > 0$ and for all $1 \leq |\alpha| \leq \ell + 1$*

$$\partial^\alpha \hat{\varepsilon} = \mathcal{O}(r^{-|\alpha|}) \qquad \text{as} \qquad r \to \infty \qquad ,$$

then $\operatorname{rot} E \in \mathrm{H}^{\ell,q+1}_{s+1}(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathrm{H}^{\ell,q-1}_{s+1}(\Omega)$ imply $E \in \mathrm{H}^{\ell+1,q}_s(\Xi)$ and there exists some constant $c > 0$, such that the estimate

$$\|E\|_{\mathrm{H}^{\ell+1,q}_s(\Xi)} \leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathrm{H}^{\ell,q+1}_{s+1}(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathrm{H}^{\ell,q-1}_{s+1}(\Omega)} \right)$$

holds uniformly in E.

Proof: With the aid of a cut-off function φ with $\operatorname{supp} \varphi \subset \Omega$ and $\varphi|_\Xi = 1$ the form $\varphi \cdot E$ fulfills the assumptions of Theorem 3.6. This yields $\varphi \cdot E \in \mathbf{H}^{\ell+1,q}_s$ resp. $\varphi \cdot E \in \mathrm{H}^{\ell+1,q}_s$, i.e.

$$E \in \mathbf{H}^{\ell+1,q}_s(\Xi) \qquad \text{resp.} \qquad E \in \mathrm{H}^{\ell+1,q}_s(\Xi) \qquad ,$$

and the corresponding estimates can be shown by induction. ∎

Finally we combine the boundary regularity from Theorem 2.8 and the exterior domain regularity:

Theorem 3.9 *Let $\ell \in \mathbb{N}_0$, $s \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ be an exterior domain with a $\mathrm{C}^{\ell+2}$-boundary, i.e. $\Omega \cap U_{r_0}$ is a $\mathrm{C}^{\ell+2}$-region. Furthermore, let $\varepsilon \in \mathbb{A}^{\ell+1,q}(\overline{\Omega})$ and*

$$E \in \left(\mathring{\mathbf{R}}^q_s(\Omega) \cap \varepsilon^{-1} \mathbf{D}^q_s(\Omega) \right) \cup \left(\mathbf{R}^q_s(\Omega) \cap \varepsilon^{-1} \mathring{\mathbf{D}}^q_s(\Omega) \right) \qquad .$$

(i) *Then $\operatorname{rot} E \in \mathbf{H}^{\ell,q+1}_s(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathbf{H}^{\ell,q-1}_s(\Omega)$ imply $E \in \mathbf{H}^{\ell+1,q}_s(\Omega)$ and with some constant $c > 0$*

$$\|E\|_{\mathbf{H}^{\ell+1,q}_s(\Omega)} \leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathbf{H}^{\ell,q+1}_s(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{\ell,q-1}_s(\Omega)} \right)$$

holds uniformly in E.

(ii) *If additionally $\varepsilon = \mathrm{id} + \hat{\varepsilon} \in \mathbb{A}^{\ell+1,q}_\tau(\Omega)$ with $\tau > 0$ and for all $1 \leq |\alpha| \leq \ell + 1$*

$$\partial^\alpha \hat{\varepsilon} = \mathcal{O}(r^{-|\alpha|}) \qquad \text{as} \qquad r \to \infty \qquad ,$$

then $\operatorname{rot} E \in \mathrm{H}^{\ell,q+1}_{s+1}(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathrm{H}^{\ell,q-1}_{s+1}(\Omega)$ imply $E \in \mathrm{H}^{\ell+1,q}_s(\Omega)$ and there exists some positive constant c, such that the estimate

$$\|E\|_{\mathrm{H}^{\ell+1,q}_s(\Omega)} \leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathrm{H}^{\ell,q+1}_{s+1}(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathrm{H}^{\ell,q-1}_{s+1}(\Omega)} \right)$$

holds uniformly in E.

Proof: Let us discuss the case $E \in \overset{\circ}{\mathbf{R}}{}^q_s(\Omega)$. Applying the latter corollary we get $\eta E \in \mathbf{H}^{\ell+1,q}_s$ resp. $\eta E \in \mathrm{H}^{\ell+1,q}_s$. Moreover, with $(1-\eta)E \in \overset{\circ}{\mathbf{R}}{}^q_s(\Omega \cap U_{r_3})$ Theorem 2.8 yields $(1-\eta)E \in \mathbf{H}^{\ell+1,q}(\Omega \cap U_{r_3})$ by induction. Extending $(1-\eta)E$ by zero leads to $(1-\eta)E \in \mathrm{H}^{\ell+1,q}_{\mathrm{vox}}(\Omega)$, which completes the proof. ∎

Remark 3.10 *Using the star operator all these regularity results also hold for all kind of spaces like*

$$\varepsilon^{-1}\mathbf{R}^q_s \cap \mathbf{D}^q_s \qquad \textit{resp.} \qquad \varepsilon^{-1}\mathrm{R}^q_s \cap \mathrm{D}^q_s \qquad .$$

3.3 Trace and extension theorems

We will provide trace and extension theorems on rotation- and divergence spaces of exterior domains using the results corresponding to the adequate spaces of bounded domains known from section 2.

Let Ω have a C^3-boundary and $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$. Our aim is to construct a linear and in some sense 'continuous' tangential trace operator

$$\Gamma_t \ : \ \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega}) \ \longrightarrow \ \mathcal{R}^q(\partial\Omega)$$

with some corresponding linear and continuous tangential extension operator

$$\check{\Gamma}_t \ : \ \mathcal{R}^q(\partial\Omega) \ \longrightarrow \ \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega)$$

satisfying $\Gamma_t\check{\Gamma}_t = \mathrm{id}$ on $\mathcal{R}^q(\partial\Omega)$.

We need some preliminaries. Let $\Omega_{\mathrm{b}} := \Omega \cap U_{r_3}$ as well as $S := S_{r_3}$. Then of course $\partial\,\Omega_{\mathrm{b}} = \partial\,\Omega \dot{\cup} S$ holds and from section 2.3 for $m \in \mathbb{N}$ we have the linear and continuous traces

$$\begin{aligned}
\gamma^{\mathrm{b}}_t &: \mathbf{H}^{m,q}(\Omega_{\mathrm{b}}) \to \mathbf{H}^{m-1/2,q}(\partial\,\Omega_{\mathrm{b}}) \qquad ,\\
\gamma^{\mathrm{b}}_n &: \mathbf{H}^{m,q}(\Omega_{\mathrm{b}}) \to \mathbf{H}^{m-1/2,q-1}(\partial\,\Omega_{\mathrm{b}}) \qquad ,\\
\Gamma^{\mathrm{b}}_t &: \mathbf{R}^{q}(\Omega_{\mathrm{b}}) \to \mathcal{R}^{q}(\partial\,\Omega_{\mathrm{b}}) \qquad ,\\
\Gamma^{\mathrm{b}}_n &: \mathbf{D}^{q}(\Omega_{\mathrm{b}}) \to \mathcal{D}^{q-1}(\partial\,\Omega_{\mathrm{b}})
\end{aligned}$$

together with their corresponding linear and continuous extensions

$$\begin{aligned}
\check{\gamma}^{\mathrm{b}}_t &: \mathbf{H}^{m-1/2,q}(\partial\,\Omega_{\mathrm{b}}) \to \mathbf{H}^{m,q}(\Omega_{\mathrm{b}}) \cap \overset{\circ}{\mathbf{D}}{}^q(\Omega_{\mathrm{b}}) \qquad ,\\
\check{\gamma}^{\mathrm{b}}_n &: \mathbf{H}^{m-1/2,q-1}(\partial\,\Omega_{\mathrm{b}}) \to \mathbf{H}^{m,q}(\Omega_{\mathrm{b}}) \cap \overset{\circ}{\mathbf{R}}{}^q(\Omega_{\mathrm{b}}) \qquad ,\\
\check{\Gamma}^{\mathrm{b}}_t &: \mathcal{R}^{q}(\partial\,\Omega_{\mathrm{b}}) \to \mathbf{R}^{q}(\Omega_{\mathrm{b}}) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega_{\mathrm{b}}) \qquad ,\\
\check{\Gamma}^{\mathrm{b}}_n &: \mathcal{D}^{q-1}(\partial\,\Omega_{\mathrm{b}}) \to \mathbf{D}^{q}(\Omega_{\mathrm{b}}) \cap \varepsilon^{-1}{}_0\mathbf{R}^q(\Omega_{\mathrm{b}}) \qquad .
\end{aligned}$$

First we introduce the tangential trace

$$\begin{array}{rcccl} \gamma_t & : & \mathrm{H}^{m,q}_{\mathrm{loc}}(\overline{\Omega}) & \longrightarrow & \mathbf{H}^{m-1/2,q}(\partial\,\Omega) \\ & & \Phi & \longmapsto & \gamma_t^{\mathrm{b}}\Phi\big|_{\partial\,\Omega} \end{array} \quad , \tag{3.11}$$

which is well defined, i.e. independent of the special choice of Ω_{b}, since $\mathrm{C}^{\infty,q}(\overline{\Omega}_{\mathrm{b}})$ is dense in $\mathbf{H}^{m,q}(\Omega_{\mathrm{b}})$ and $\gamma_t^{\mathrm{b}}\Phi = \gamma_t^{\mathrm{b}}\Psi$ for all $\Phi, \Psi \in \mathrm{C}^{\infty,q}(\overline{\Omega}_{\mathrm{b}})$ with $\Phi = \Psi$ 'near $\partial\,\Omega$'. Analogously we define the normal trace $\gamma_n : \mathrm{H}^{m,q}_{\mathrm{loc}}(\overline{\Omega}) \to \mathbf{H}^{m-1/2,q-1}(\partial\,\Omega)$.

For $\lambda \in \mathbf{H}^{m-1/2,q}(\partial\,\Omega)$ we define $\tilde{\lambda} \in \mathbf{H}^{m-1/2,q}(\partial\,\Omega_{\mathrm{b}})$ by

$$\tilde{\lambda} := \begin{cases} \lambda & \text{on } \partial\,\Omega \\ 0 & \text{on } S \end{cases}$$

and present a tangential extension operator

$$\begin{array}{rcccl} \check{\gamma}_t & : & \mathbf{H}^{m-1/2,q}(\partial\,\Omega) & \longrightarrow & \mathrm{H}^{m,q}_{\mathrm{vox}}(\Omega) \cap \overset{\circ}{\mathrm{D}}{}^q_{\mathrm{vox}}(\Omega) \\ & & \lambda & \longmapsto & (1-\eta)\check{\gamma}_t^{\mathrm{b}}\tilde{\lambda} \end{array} \quad . \tag{3.12}$$

Clearly $\check{\gamma}_t$ depends on the special choice of Ω_{b} and $\mathrm{supp}(\check{\gamma}_t\lambda) \subset \overline{\Omega \cap U_{r_2}}$ holds. Approximating $\check{\gamma}_t^{\mathrm{b}}\tilde{\lambda}$ with $\mathrm{C}^{\infty,q}(\overline{\Omega}_{\mathrm{b}})$-forms we calculate

$$\gamma_t\big((1-\eta)\check{\gamma}_t^{\mathrm{b}}\tilde{\lambda}\big) = \gamma_t^{\mathrm{b}}\big((1-\eta)\check{\gamma}_t^{\mathrm{b}}\tilde{\lambda}\big)\Big|_{\partial\,\Omega} = \gamma_t^{\mathrm{b}}\check{\gamma}_t^{\mathrm{b}}\tilde{\lambda}\Big|_{\partial\,\Omega} = \tilde{\lambda}\Big|_{\partial\,\Omega} = \lambda$$

and thus $\gamma_t\check{\gamma}_t = \mathrm{id}$ on $\mathbf{H}^{m-1/2,q}(\partial\,\Omega)$ holds. In the same way we construct a normal extension operator

$$\begin{array}{rcccl} \check{\gamma}_n & : & \mathbf{H}^{m-1/2,q-1}(\partial\,\Omega) & \longrightarrow & \mathrm{H}^{m,q}_{\mathrm{vox}}(\Omega) \cap \overset{\circ}{\mathrm{R}}{}^q_{\mathrm{vox}}(\Omega) \\ & & \lambda & \longmapsto & (1-\eta)\check{\gamma}_n^{\mathrm{b}}\tilde{\lambda} \end{array} \quad , \tag{3.13}$$

which satisfies $\gamma_n\check{\gamma}_n = \mathrm{id}$. Clearly by our constructions the operators γ_t, γ_n and $\check{\gamma}_t$, $\check{\gamma}_n$ are linear and continuous.

Looking once more at (2.39) this equation even holds true in our exterior domain Ω for pairs $\Phi \in \mathrm{H}^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ and $\Psi \in \mathrm{H}^{1,q+1}_{\mathrm{vox}}(\Omega)$. Especially for $\varphi \in \mathbf{H}^{1/2,q}(\partial\,\Omega)$ we have $\check{\gamma}_n\varphi \in \mathrm{H}^{1,q+1}_{\mathrm{vox}}(\Omega)$ and thus for all $E \in \mathrm{H}^{1,q}_{\mathrm{loc}}(\overline{\Omega})$

$$\langle \mathrm{rot}\, E, \check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \mathrm{div}\,\check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q}(\Omega)} = \langle \gamma_t E, \varphi\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)} \quad .$$

Again this suggests to define a tangential trace

$$\Gamma_t E \in \mathbf{H}^{-1/2,q}(\partial\,\Omega)$$

of a q-form $E \in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})$ by

$$\Gamma_t E(\varphi) = \langle \Gamma_t E, \varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} := \langle \mathrm{rot}\, E, \check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \mathrm{div}\,\check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q}(\Omega)}$$

for all $\varphi \in \mathbf{H}^{1/2,q}(\partial\,\Omega)$. Clearly again for $E \in \mathrm{H}^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ we have

$$\Gamma_t E = \langle \gamma_t E, \,\cdot\,\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}$$

and in this case we identify $\Gamma_t E$ with $\gamma_t E \in \mathbf{H}^{1/2,q}(\partial\,\Omega)$. Furthermore, Γ_t has the familiar properties, which can be proved in the same way as in the case of bounded domains.

Theorem 3.11 *For each $E \in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})$ the tangential trace $\Gamma_t E$ is an element of $\mathcal{R}^q(\partial\Omega)$ and Γ_t possesses the following properties:*

(i) $$\bigwedge_{\substack{E\in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})\,,\\ \Psi\in \mathrm{H}^{1,q+1}_{\mathrm{vox}}(\Omega)}} \langle \Gamma_t E, \gamma_n \Psi\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)} = \langle \mathrm{rot} E, \Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle E, \mathrm{div}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}$$

(ii) $$\bigwedge_{E\in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})} \mathrm{Rot}\, \Gamma_t E = \Gamma_t \mathrm{rot} E$$

(iii) *The mapping $\Gamma_t : \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega}) \to \mathcal{R}^q(\partial\Omega)$ is continuous, i.e. there exists some positive constant c, such that*

$$\|\Gamma_t E\|_{\mathcal{R}^q(\partial\Omega)} \le c \cdot \|E\|_{\mathbf{R}^q(\Omega_{\mathrm{b}})}$$

holds uniformly in $E \in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})$.

(iv) $$E \in \overset{\circ}{\mathrm{R}}{}^q_{\mathrm{loc}}(\overline{\Omega}) \qquad \Longleftrightarrow \qquad E \in \mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega}) \quad \wedge \quad \Gamma_t E = 0$$

Defining the normal trace acting on $\mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega})$ by

$$\Gamma_n := (-1)^{(q-1)N} \circledast \Gamma_t * \tag{3.14}$$

we get

Theorem 3.12 *For each $H \in \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega})$ the normal trace $\Gamma_n H$ is an element of $\mathcal{D}^{q-1}(\partial\Omega)$ and Γ_n has the following properties:*

(i) $$\bigwedge_{\substack{H\in \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega})\,,\\ \Psi\in \mathrm{H}^{1,q-1}_{\mathrm{vox}}(\Omega)}} \langle \Gamma_n H, \gamma_t \Psi\rangle_{\mathbf{H}^{-1/2,q-1}(\partial\Omega)} = \langle \mathrm{div} H, \Psi\rangle_{\mathrm{L}^{2,q-1}(\Omega)} + \langle H, \mathrm{rot}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}$$

(ii) $$\bigwedge_{H\in \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega})} \mathrm{Div}\, \Gamma_n H = -\Gamma_n \mathrm{div} H$$

(iii) *The mapping $\Gamma_n : \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega}) \to \mathcal{D}^{q-1}(\partial\Omega)$ is continuous, i.e. there exists some positive constant c, such that*

$$\|\Gamma_n H\|_{\mathcal{D}^{q-1}(\partial\Omega)} \le c \cdot \|H\|_{\mathbf{D}^q(\Omega_{\mathrm{b}})}$$

holds uniformly in $H \in \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega})$.

(iv) $$H \in \overset{\circ}{\mathrm{D}}{}^q_{\mathrm{loc}}(\overline{\Omega}) \qquad \Longleftrightarrow \qquad H \in \mathrm{D}^q_{\mathrm{loc}}(\overline{\Omega}) \quad \wedge \quad \Gamma_n H = 0$$

Now we show that there exist the corresponding extension operators.

Theorem 3.13 *Let $\varepsilon \in \mathbb{A}^{0,q}(\Omega)$. Then there exist two linear and continuous extension operators*

$$\begin{aligned} \check{\Gamma}_t &: \mathcal{R}^q(\partial\Omega) \to \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega) \qquad , \\ \check{\Gamma}_n &: \mathcal{D}^{q-1}(\partial\Omega) \to \mathrm{D}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{R}^q_{\mathrm{vox}}(\Omega) \end{aligned}$$

satisfying $\Gamma_t\check{\Gamma}_t = \mathrm{id}$ and $\Gamma_n\check{\Gamma}_n = \mathrm{id}$. Moreover, $\check{\Gamma}_t$ and $\check{\Gamma}_n$ map on forms, which have got their supports in $\overline{\Omega \cap U_{r_2}}$.

Proof: For $\lambda \in \mathbf{H}^{-1/2,q}(\partial\,\Omega)$ we define $\tilde{\lambda} \in \mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})$ by

$$\langle\tilde{\lambda}, \varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})} := \langle\lambda, \varphi|_{\partial\,\Omega}\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)}$$

for all $\varphi \in \mathbf{H}^{1/2,q}(\partial\,\Omega_{\mathrm{b}})$. Let $\lambda \in \mathcal{R}^q(\partial\Omega)$. Then we define

$$\check{\Gamma}_t\lambda := (1-\eta)\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda} \in \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega)$$

and note $\operatorname{supp}\check{\Gamma}_t\lambda \subset \overline{\Omega \cap U_{r_2}}$. $\check{\Gamma}_t$ is well defined since $\tilde{\lambda} \in \mathcal{R}^q(\partial\,\Omega_{\mathrm{b}})$ holds, which may be proved picking some $\varphi \in \mathbf{H}^{3/2,q+1}(\partial\,\Omega_{\mathrm{b}}) \subset \mathbf{H}^{3/2,q+1}(\partial\,\Omega)$ and computing

$$\begin{aligned} \langle\tilde{\lambda}, \operatorname{Div}\varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})} &= \langle\lambda, \operatorname{Div}\varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} \\ &= -\langle\operatorname{Rot}\lambda, \varphi\rangle_{\mathbf{H}^{-1/2,q+1}(\partial\,\Omega)} = -\langle\widetilde{\operatorname{Rot}\lambda}, \varphi\rangle_{\mathbf{H}^{-1/2,q+1}(\partial\,\Omega_{\mathrm{b}})} \qquad . \end{aligned}$$

To prove the continuity of $\check{\Gamma}_t$ we estimate

$$\|\check{\Gamma}_t\lambda\|_{\mathbf{R}^q(\Omega)\cap\varepsilon^{-1}\mathbf{D}^q(\Omega)} \leq c \cdot \|\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}\|_{\mathbf{R}^q(\Omega_{\mathrm{b}})} \leq c \cdot \|\tilde{\lambda}\|_{\mathcal{R}^q(\partial\,\Omega_{\mathrm{b}})} \leq c \cdot \|\lambda\|_{\mathcal{R}^q(\partial\,\Omega)} \quad , \tag{3.15}$$

where we used the continuity of $\check{\Gamma}^{\mathrm{b}}_t$ and $\|\tilde{\lambda}\|_{\mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})} \leq \|\lambda\|_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)}$ as well as $\operatorname{Rot}\tilde{\lambda} = \widetilde{\operatorname{Rot}\lambda}$.

It remains to show $\Gamma_t\check{\Gamma}_t = \mathrm{id}$. Thus let $\lambda \in \mathcal{R}^q(\partial\Omega)$ and $\varphi \in \mathbf{H}^{1/2,q}(\partial\,\Omega)$. Using $\operatorname{supp}\check{\gamma}_n\varphi \subset \overline{\Omega}_{\mathrm{b}}$, (2.24), Theorem 2.14 (i) and (2.23) we calculate

$$\begin{aligned} &\langle\Gamma_t\check{\Gamma}_t\lambda, \varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} \\ &= \langle\operatorname{rot}\check{\Gamma}_t\lambda, \check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle\check{\Gamma}_t\lambda, \operatorname{div}\check{\gamma}_n\varphi\rangle_{\mathrm{L}^{2,q}(\Omega)} \\ &= \big\langle \operatorname{rot}\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}, (1-\eta)\check{\gamma}_n\varphi\big\rangle_{\mathrm{L}^{2,q+1}(\Omega_{\mathrm{b}})} + \Big\langle\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}, \operatorname{div}\big((1-\eta)\check{\gamma}_n\varphi\big)\Big\rangle_{\mathrm{L}^{2,q}(\Omega_{\mathrm{b}})} \\ &\qquad - \big\langle\hat{\eta}'(r)r^{-1}R\,\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}, \check{\gamma}_n\varphi\big\rangle_{\mathrm{L}^{2,q+1}(\Omega_{\mathrm{b}})} + \big\langle\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}, \hat{\eta}'(r)r^{-1}T\,\check{\gamma}_n\varphi\big\rangle_{\mathrm{L}^{2,q}(\Omega_{\mathrm{b}})} \\ &= \big\langle\Gamma^{\mathrm{b}}_t\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda}, \gamma^{\mathrm{b}}_n(1-\eta)\check{\gamma}_n\varphi\big\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})} \\ &\qquad + \int_{\Omega_{\mathrm{b}}} \hat{\eta}'(r)r^{-1}\big(-R\,\check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda} \wedge *\check{\gamma}_n\varphi + \check{\Gamma}^{\mathrm{b}}_t\tilde{\lambda} \wedge *T\,\check{\gamma}_n\varphi\big) \\ &= \big\langle\tilde{\lambda}, \gamma^{\mathrm{b}}_n(1-\eta)\check{\gamma}_n\varphi\big\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega_{\mathrm{b}})} = \Big\langle\lambda, \big(\gamma^{\mathrm{b}}_n(1-\eta)\check{\gamma}_n\varphi\big)\big|_{\partial\,\Omega}\Big\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} \\ &= \big\langle\lambda, \gamma_n(1-\eta)\check{\gamma}_n\varphi\big\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} = \langle\lambda, \varphi\rangle_{\mathbf{H}^{-1/2,q}(\partial\,\Omega)} \qquad . \end{aligned}$$

The assertions upon $\check{\Gamma}_n := (-1)^{q(N-q)} * \check{\Gamma}_t \circledast$ follow analogously or by the star operator. ■

Sometimes it might be useful to work with solenoidal or irrotational extensions. With a slightly stronger assumption on ε we get

Theorem 3.14 *Let $\varepsilon \in \mathbb{A}^{0,q}(\Omega) \cap \mathbb{A}^{1,q}(Z_{r_1,r_2})$ and $\nu \in \mathbb{A}^{0,q}(\Omega)$. Then there exist two linear and continuous extension operators*

$$\begin{aligned} {}_0\check{\Gamma}_t : \mathcal{R}^q(\partial\Omega) &\to \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}^q_{\mathrm{vox}}(\Omega) \cap {}_\nu\mathcal{H}^q(\Omega)^{\perp_\varepsilon} \qquad , \\ {}_0\check{\Gamma}_n : \mathcal{D}^{q-1}(\partial\Omega) &\to \mathrm{D}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap {}_\nu\tilde{\mathcal{H}}^q(\Omega)^{\perp_\varepsilon} \end{aligned}$$

satisfying $\Gamma_t {}_0\check{\Gamma}_t = \mathrm{id}$ and $\Gamma_n {}_0\check{\Gamma}_n = \mathrm{id}$.

Remark 3.15 *${}_0\check{\Gamma}_t$ and ${}_0\check{\Gamma}_n$ map on forms, which have got their supports in $\overline{\Omega \cap U_{r_2}}$. Moreover, ${}_0\check{\Gamma}_t$ even maps to*

$$\mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1} \operatorname{div} \big(\mathring{\mathrm{R}}^{q+1}_{\mathrm{vox}}(\Omega) \cap \mathrm{H}^{1,q+1}_{\mathrm{vox}}(\Omega) \big)$$

and ${}_0\check{\Gamma}_n$ to

$$\mathrm{D}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \big(\mathring{\mathrm{D}}^{q-1}_{\mathrm{vox}}(\Omega) \cap \mathrm{H}^{1,q-1}_{\mathrm{vox}}(\Omega) \big) \qquad .$$

Proof: Let $\lambda \in \mathcal{R}^q(\partial\Omega)$ and $\tilde{\lambda} \in \mathcal{R}^q(\partial\,\Omega_{\mathrm{b}})$ as in the proof of Theorem 3.13. The idea is to get the extension as a divergence of some compactly supported form. To do this we look again at the proof of Theorem 2.16. There we have

$$\check{\Gamma}^{\mathrm{b}}_t \tilde{\lambda} = \varepsilon^{-1} \operatorname{div} H \in \mathbf{R}^q(\Omega_{\mathrm{b}})$$

with some $H \in \operatorname{rot}\big(\mathring{\mathbf{R}}^q(\Omega_{\mathrm{b}}) \cap \mathbf{H}^{2,q}(\Omega_{\mathrm{b}})\big) \subset {}_0\mathring{\mathbf{R}}^{q+1}(\Omega_{\mathrm{b}}) \cap \mathbf{H}^{1,q+1}(\Omega_{\mathrm{b}}) \cap \mathcal{H}^{q+1}(\Omega_{\mathrm{b}})^\perp$ satisfying $\|H\|_{\mathbf{H}^{1,q+1}(\Omega_{\mathrm{b}})} \leq c \cdot \|\operatorname{div} H\|_{\mathrm{L}^{2,q}(\Omega_{\mathrm{b}})} \leq c \cdot \|\check{\Gamma}^{\mathrm{b}}_t \tilde{\lambda}\|_{\mathrm{L}^{2,q}(\Omega_{\mathrm{b}})}$ by Theorem 2.8 and (2.17). Putting

$$E := {}_0\check{\Gamma}_t \lambda := \varepsilon^{-1} \operatorname{div} \big((1-\eta) H \big) \in \varepsilon^{-1}{}_0\mathrm{D}^q_{\mathrm{vox}}(\Omega) \cap {}_\nu\mathcal{H}^q(\Omega)^{\perp_\varepsilon}$$

and computing

$$E = (1-\eta)\check{\Gamma}^{\mathrm{b}}_t \tilde{\lambda} - \varepsilon^{-1} \hat{\eta}'(r) r^{-1} T H = \check{\Gamma}_t \lambda - \varepsilon^{-1} \hat{\eta}'(r) r^{-1} T H$$

we get $E \in \mathrm{R}^q_{\mathrm{vox}}(\Omega)$ and $\Gamma_t E = \Gamma_t \check{\Gamma}_t \lambda = \lambda$ since the second term of the sum belongs to $\mathring{\mathrm{H}}^{1,q}_{\mathrm{vox}}(\Omega) \subset \mathring{\mathrm{R}}^q_{\mathrm{vox}}(\Omega)$. The continuity of ${}_0\check{\Gamma}_t$ follows by

$$\|E\|_{\mathbf{R}^q(\Omega)} \leq c \cdot \big(\|\check{\Gamma}^{\mathrm{b}}_t \tilde{\lambda}\|_{\mathbf{R}^q(\Omega_{\mathrm{b}})} + \|H\|_{\mathbf{H}^{1,q+1}(\Omega_{\mathrm{b}})} \big) \leq c \cdot \|\check{\Gamma}^{\mathrm{b}}_t \tilde{\lambda}\|_{\mathbf{R}^q(\Omega_{\mathrm{b}})} \leq c \cdot \|\lambda\|_{\mathcal{R}^q(\partial\,\Omega)}$$

using (3.15). Clearly the normal extension defined by ${}_0\check{\Gamma}_n := (-1)^{q(N-q)} * {}_0\check{\Gamma}_t \circledast$ and acting on $\mathcal{D}^{q-1}(\partial\Omega)$ possesses the corresponding properties. ■

Finally we can prove the analogue to Theorem 2.18:

Theorem 3.16 *Let $\ell \in \mathbb{N}_0$, $s \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ be an exterior domain with a $(\mathrm{C}^{\ell+2} \cap \mathrm{C}^3)$-boundary, i.e. $\Omega \cap U_{r_0}$ is a $(\mathrm{C}^{\ell+2} \cap \mathrm{C}^3)$-region. Furthermore, let $\varepsilon \in \mathbb{A}^{\ell+1,q}(\overline{\Omega})$ as well as*

$$E \in \mathbf{R}^q_s(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_s(\Omega) \qquad , \qquad \Gamma_t E \in \mathbf{H}^{\ell+1/2,q}(\partial\Omega) \qquad .$$

(i) *Then $\operatorname{rot} E \in \mathbf{H}^{\ell,q+1}_s(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathbf{H}^{\ell,q-1}_s(\Omega)$ imply $E \in \mathbf{H}^{\ell+1,q}_s(\Omega)$ and with some constant $c > 0$*

$$\begin{aligned}&\|E\|_{\mathbf{H}^{\ell+1,q}_s(\Omega)}\\ &\leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathbf{H}^{\ell,q+1}_s(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathbf{H}^{\ell,q-1}_s(\Omega)} + \|\Gamma_t E\|_{\mathbf{H}^{\ell+1/2,q}(\partial\Omega)}\big)\end{aligned}$$

holds uniformly in E.

(ii) *If additionally $\varepsilon = \mathrm{id} + \hat{\varepsilon} \in \mathbb{A}^{\ell+1,q}_\tau(\Omega)$ with $\tau > 0$ and for all $1 \leq |\alpha| \leq \ell + 1$*

$$\partial^\alpha \hat{\varepsilon} = \mathcal{O}(r^{-|\alpha|}) \qquad \text{as} \qquad r \to \infty \qquad ,$$

then $\operatorname{rot} E \in \mathrm{H}^{\ell,q+1}_{s+1}(\Omega)$ and $\operatorname{div} \varepsilon E \in \mathrm{H}^{\ell,q-1}_{s+1}(\Omega)$ imply $E \in \mathrm{H}^{\ell+1,q}_s(\Omega)$ and there exists some positive constant c, such that the estimate

$$\begin{aligned}&\|E\|_{\mathrm{H}^{\ell+1,q}_s(\Omega)}\\ &\leq c \cdot \big(\|E\|_{\mathrm{L}^{2,q}_s(\Omega)} + \|\operatorname{rot} E\|_{\mathrm{H}^{\ell,q+1}_{s+1}(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathrm{H}^{\ell,q-1}_{s+1}(\Omega)} + \|\Gamma_t E\|_{\mathbf{H}^{\ell+1/2,q}(\partial\Omega)}\big)\end{aligned}$$

holds uniformly in E.

Proof: Let $\check{E} := \check{\gamma}_t \Gamma_t E \in \mathrm{H}^{\ell+1,q}_{\mathrm{vox}}(\Omega)$. Then $\hat{E} := E - \check{E} \in \overset{\circ}{\mathbf{R}}{}^q_s(\Omega) \cap \varepsilon^{-1}\mathbf{D}^q_s(\Omega)$ satisfies the assumptions of Theorem 3.9. Thus we get $\hat{E} \in \mathbf{H}^{\ell+1,q}_s(\Omega)$ resp. $\mathrm{H}^{\ell+1,q}_s(\Omega)$ and using the continuity of $\check{\gamma}_t$ the asserted estimate as well. ∎

Remark 3.17 *Clearly using the star operator and some transformation $E \rightsquigarrow \varepsilon E$ the assumption $\Gamma_n \varepsilon E \in \mathbf{H}^{\ell+1/2,q-1}(\partial\Omega)$ instead of $\Gamma_t E \in \mathbf{H}^{\ell+1/2,q}(\partial\Omega)$ yields the corresponding theorem. Moreover, these regularity results hold for spaces of the form $\varepsilon^{-1}\mathbf{R}^q_s(\Omega) \cap \mathbf{D}^q_s(\Omega)$ as well.*

3.4 Static solution theory

In this last section we generally assume that our exterior domain Ω has got the MLCP and

$$\varepsilon = \mathrm{id} + \hat{\varepsilon} \in \mathbb{A}^{0,q}_\tau(\Omega) \cap \mathrm{C}^{1,q}(A_{r_0}) \qquad \text{with order of decay} \qquad \tau > 0$$

and the additional property

$$\partial_n \hat{\varepsilon} = \mathcal{O}(r^{-1-\tau}) \qquad \text{as} \qquad r \to \infty \qquad , \qquad n = 1, \dots, N \qquad .$$

First we generalize the electro-magneto static results from [12] to inhomogeneous, anisotropic media, i.e. we replace id by ε. Having done this we will present a static solution theory using our trace and extension theorems, which deals with inhomogeneous boundary conditions.

We need a fundamental estimate:

Lemma 3.18 *There exists some constant $c > 0$ and some compact set $K \subset \mathbb{R}^N$, such that*

$$\|E\|_{\mathrm{L}^{2,q}_{-1}(\Omega)} \leq c \cdot \left(\|\operatorname{rot} E\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2,q-1}(\Omega)} + \|E\|_{\mathrm{L}^{2,q}(\Omega \cap K)}\right)$$

holds true for all $E \in \mathrm{R}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega)$.

Proof: By a usual cutting technique w. l. o. g. we may restrict our considerations to the special case $\Omega = \mathbb{R}^N$ and $\varepsilon \in \mathbb{A}^{1,q}_\tau$ with the asymptotics $\partial_n \hat{\varepsilon} = \mathcal{O}(r^{-1-\tau})$ as $r \to \infty$ for $n = 1, \ldots, N$. Picking some $E \in \mathrm{R}^q_{-1} \cap \varepsilon^{-1}\mathrm{D}^q_{-1}$ by Theorem 3.6 (ii) we get $E \in \mathrm{H}^{1,q}_{-t}$ for all $t \geq 1$ and the estimate (with c depending on t but not on E)

$$\|E\|_{\mathrm{H}^{1,q}_{-t}} \leq c \cdot \left(\|E\|_{\mathrm{L}^{2,q}_{-t}} + \|\operatorname{rot} E\|_{\mathrm{L}^{2,q+1}_{1-t}} + \|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2,q-1}_{1-t}}\right) \qquad . \tag{3.16}$$

From $\big[$[12], Lemma 5$\big]$ we receive a compact set K, such that

$$\|E\|_{\mathrm{L}^{2,q}_{-1}} \leq c \cdot \left(\|\operatorname{rot} E\|_{\mathrm{L}^{2,q+1}} + \|\operatorname{div} E\|_{\mathrm{L}^{2,q-1}} + \|E\|_{\mathrm{L}^{2,q}(K)}\right) \qquad .$$

Then (3.16) (for $t = 1$) and the latter estimate yield

$$\|E\|_{\mathrm{H}^{1,q}_{-1}} \leq c \cdot \left(\|\operatorname{rot} E\|_{\mathrm{L}^{2,q+1}} + \|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2,q-1}} + \|E\|_{\mathrm{L}^{2,q}(K)} + \|E\|_{\mathrm{H}^{1,q}_{-1-\tau}}\right) \qquad .$$

Using (3.16) (for $t = 1 + \tau$) again the term $\|E\|_{\mathrm{H}^{1,q}_{-1-\tau}}$ may be replaced by $\|E\|_{\mathrm{L}^{2,q}_{-1-\tau}}$. Since $\tau > 0$ this one can be swallowed by the left hand side, which maybe produces some other compact set $\tilde{K} \supset K$. ■

We note that we did not need the MLCP for the proof of this lemma. But this lemma and the MLCP yield directly by an indirect argument

Corollary 3.19 *Let $\nu \in \mathbb{A}^{0,q}(\Omega)$. ${}_\varepsilon\mathcal{H}^q_{-1}(\Omega)$ is finite dimensional and there exists some positive constant c, such that*

$$\|E\|_{\mathrm{L}^{2,q}_{-1}(\Omega)} \leq c \cdot \left(\|\operatorname{rot} E\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2,q-1}(\Omega)}\right)$$

holds for all $E \in \mathring{\mathrm{R}}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-1}(\Omega)^{\perp_{-1,\nu}}$. Here we denote by $\perp_{-1,\nu}$ the orthogonality w. r. t. the $\langle \nu\rho^{-1} \cdot, \rho^{-1} \cdot \rangle_\Omega$-scalar product.

Now we are able to prove

Lemma 3.20 *Let $\nu \in \mathbb{A}^{0,q}(\Omega)$. With closures taken in $\mathrm{L}^2(\Omega)$ we have*

(i)
$$\begin{aligned} \overline{\operatorname{rot} \mathring{\mathbf{R}}^q(\Omega)} &= \overline{\operatorname{rot} \mathring{\mathrm{R}}^q_{\mathrm{vox}}(\Omega)} = \operatorname{rot} \mathring{\mathrm{R}}^q_{-1}(\Omega) \\ &= \operatorname{rot} \left(\mathring{\mathrm{R}}^q_{-1}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}^q_{-1}(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-1}(\Omega)^{\perp_{-1,\nu}}\right) \qquad , \end{aligned}$$

(ii)
$$\begin{aligned} \overline{\operatorname{div} \mathbf{D}^q(\Omega)} &= \overline{\operatorname{div} \mathrm{D}^q_{\mathrm{vox}}(\Omega)} = \operatorname{div} \mathrm{D}^q_{-1}(\Omega) \\ &= \operatorname{div} \left(\mathrm{D}^q_{-1}(\Omega) \cap \varepsilon^{-1}{}_0\mathring{\mathrm{R}}^q_{-1}(\Omega) \cap {}_{\varepsilon^{-1}}\mathcal{H}^q_{-1}(\Omega)^{\perp_{-1,\nu}}\right) \qquad . \end{aligned}$$

Proof: The proof is analogous to the one of $\big[$[12], Lemma 7$\big]$. Nevertheless, let us briefly indicate how to prove **(i)**. The other assertion follows similarly. To this end let $G \in \overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^q(\Omega)}$ and $(E_n)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathbf{R}}{}^q(\Omega)$ be some sequence with $\operatorname{rot} E_n \xrightarrow{n\to\infty} G$ in $\mathrm{L}^{2,q+1}(\Omega)$. Using (3.5) w. l. o. g. $E_n \in \overset{\circ}{\mathbf{R}}{}^q(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}^q(\Omega)$ holds. Moreover, by the projection theorem applied in $\mathrm{L}^{2,q}_{-1}(\Omega)$ we may assume

$$E_n \in \overset{\circ}{\mathrm{R}}{}^q_{-1}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}^q_{-1}(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-1}(\Omega)^{\perp_{-1,\nu}} \qquad .$$

By Corollary 3.19 $(E_n)_{n\in\mathbb{N}}$ is a $\mathrm{L}^{2,q}_{-1}(\Omega)$-Cauchy sequence and the limit $E \in \mathrm{L}^{2,q}_{-1}(\Omega)$ even is an element of $\overset{\circ}{\mathrm{R}}{}^q_{-1}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}^q_{-1}(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-1}(\Omega)^{\perp_{-1,\nu}}$, which completes the proof. ■

As in the bounded domain case we introduce the range

$$W^q(\Omega) := \overline{\operatorname{div}\mathbf{D}^q(\Omega)} \times \overline{\operatorname{rot}\overset{\circ}{\mathbf{R}}{}^q(\Omega)} \times \mathbb{C}^{d^q_{-1}} \qquad .$$

An immediate and easy conclusion of Lemma 3.20 is our first main result of this section:

Theorem 3.21 *Let d^q_{-1} continuous linear functionals Φ^ℓ_ε on $\mathrm{R}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega)$ with*

$${}_\varepsilon\mathcal{H}^q_{-1}(\Omega) \cap \bigcap_{\ell=1}^{d^q_{-1}} N(\Phi^\ell_\varepsilon) = \{0\}$$

be given. Then with $\Phi_\varepsilon := (\Phi^1_\varepsilon\,\cdot\,, \dots, \Phi^{d^q_{-1}}_\varepsilon\,\cdot\,)$

$$\begin{array}{rccc} \mathcal{M}\mathrm{ax}_\varepsilon & : & \overset{\circ}{\mathrm{R}}{}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega) & \longrightarrow & W^q(\Omega) \\ & & E & \longmapsto & \big(\operatorname{div}\varepsilon E, \operatorname{rot} E, \Phi_\varepsilon(E)\big) \end{array}$$

is a topological isomorphism.

Remark 3.22

(i) *For any* $\nu \in \mathbb{A}^{0,q}(\Omega)$ *we can choose* $\Phi_\varepsilon^\ell := \langle \nu\rho^{-1}\cdot\,, \rho^{-1}h_\ell\rangle_{\mathrm{L}^{2,q}(\Omega)}$ *with an arbitrary basis* $\{h_1, \ldots, h_{d_{-1}^q}\}$ *of* ${}_\varepsilon\mathcal{H}_{-1}^q(\Omega)$.

(ii) *Let* $(\tilde{\nu}, \hat{\nu}) \in \mathbb{A}^{0,q-1}(\Omega) \times \mathbb{A}^{0,q+1}(\Omega)$. *By* (3.5) *we obtain*

$$W^q(\Omega) = \big({}_0\mathrm{D}^{q-1}(\Omega) \,\cap\, {}_{\tilde{\nu}}\mathcal{H}^{q-1}(\Omega)^\perp\big) \times \big({}_0\overset{\circ}{\mathrm{R}}{}^{q+1}(\Omega) \,\cap\, {}_{\hat{\nu}}\mathcal{H}^{q+1}(\Omega)^{\perp_{\hat{\nu}}}\big) \times \mathbb{C}^{d_{-1}^q} \qquad .$$

(iii) *If we replace* ε *by* ε^{-1} *and consider* ${}_\varepsilon\mathcal{M}\mathrm{ax} = \mathcal{M}\mathrm{ax}_{\varepsilon^{-1}}\,\varepsilon$, *then*

$$\begin{array}{rcccc} {}_\varepsilon\mathcal{M}\mathrm{ax} & : & \varepsilon^{-1}\overset{\circ}{\mathrm{R}}{}_{-1}^q(\Omega) \cap \mathrm{D}_{-1}^q(\Omega) & \longrightarrow & W^q(\Omega) \\ & & E & \longmapsto & \big(\operatorname{div} E, \operatorname{rot}\varepsilon E, \Phi_{\varepsilon^{-1}}(\varepsilon E)\big) \end{array}$$

is a topological isomorphism as well.

(iv) *Clearly we have the corresponding dual results using the star operator.*

Finally we present an electro-magneto static solution theory, which handles inhomogeneous boundary data. To this end we additionally assume that Ω has got a C^3-boundary. Using the functionals Φ_ε^ℓ from Theorem 3.21 we consider the following problem:

Find for some given data G, F, λ, α a q-form $E \in \mathrm{R}_{-1}^q(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{-1}^q(\Omega)$ satisfying

$$\begin{aligned} \operatorname{rot} E &= G \quad , \\ \operatorname{div} \varepsilon E &= F \quad , \\ \Gamma_t E &= \lambda \quad , \\ \Phi_\varepsilon^\ell(E) &= \alpha_\ell \quad , \qquad \ell = 1, \ldots, d_{-1}^q \quad . \end{aligned} \tag{3.17}$$

We obtain the second main result of this section:

Theorem 3.23 *The conditions* $G \in {}_0\mathrm{R}^{q+1}(\Omega)$, $F \in {}_0\mathrm{D}^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^\perp$, $\lambda \in \mathcal{R}^q(\partial\Omega)$, $\alpha \in \mathbb{C}^{d_{-1}^q}$ *and*

$$\operatorname{Rot}\lambda = \Gamma_t G \qquad , \qquad \bigwedge_{h \in \mathcal{H}^{q+1}(\Omega)} \langle G, h\rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \langle \lambda, \gamma_n h\rangle_{\mathbf{H}^{-1/2,q}(\partial\Omega)}$$

are necessary and sufficient for the solvability of (3.17). *The solution is unique and depends continuously on the data, i.e. there exists a positive constant* c *independent of* E *or the data, such that*

$$\|E\|_{\mathrm{R}_{-1}^q(\Omega)\cap\varepsilon^{-1}\mathrm{D}_{-1}^q(\Omega)} \le c \cdot \big(\|F\|_{\mathrm{L}^{2,q-1}(\Omega)} + \|G\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\lambda\|_{\mathcal{R}^q(\partial\Omega)} + |\alpha|\big)$$

holds.

Proof: The proof is similar to the one of Theorem 2.20. By Theorem 3.13 we get for the extension $\check{E} := \check{\Gamma}_t\lambda \in \mathrm{R}^q_{\mathrm{vox}}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{\mathrm{vox}}(\Omega)$. Then the ansatz $E := \check{E} + \tilde{E}$ with $\tilde{E} \in \overset{\circ}{\mathrm{R}}{}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q_{-1}(\Omega)$ leads us with Theorem 3.11 (iv) to the system

$$\begin{aligned} \mathrm{rot}\tilde{E} &= G - \mathrm{rot}\,\check{E} =: \tilde{G} \in {}_0\mathrm{R}^{q+1}(\Omega) \quad , \\ \mathrm{div}\varepsilon\tilde{E} &= F - \mathrm{div}\,\varepsilon\check{E} =: \tilde{F} \in {}_0\mathrm{D}^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^{\perp} \quad , \\ \Phi^\ell_\varepsilon(\tilde{E}) &= \alpha_\ell - \Phi^\ell_\varepsilon(\check{E}) =: \tilde{\alpha}_\ell \quad , \quad \ell = 1, \dots, d^q_{-1} \quad , \end{aligned}$$

which is uniquely solved by $\tilde{E} := \mathcal{M}\mathrm{ax}^{-1}_\varepsilon(\tilde{F}, \tilde{G}, \tilde{\alpha}_\ell)$ with $\mathcal{M}\mathrm{ax}_\varepsilon$ from Theorem 3.21, if $(\tilde{F}, \tilde{G}, \tilde{\alpha}_\ell) \in W^q(\Omega)$. Thus using Remark 3.22 (ii) it only remains to show that $\tilde{G}$ belongs to $\overset{\circ}{\mathrm{R}}{}^{q+1}(\Omega) \cap \mathcal{H}^{q+1}(\Omega)^{\perp}$. As in the bounded domain case $\tilde{G}$ satisfies the homogeneous (electrical) boundary condition. To check the orthogonality on the Dirichlet forms we pick some $h \in \mathcal{H}^{q+1}(\Omega) \subset \mathrm{H}^{1,q+1}(\Omega)$ (by Theorem 3.9 (ii)) and some cut-off function $\xi \in \overset{\circ}{\mathrm{C}}{}^\infty$ with $\xi|_{\Omega_\mathrm{b}} = 1$, e.g. $\xi := 1 - \boldsymbol{\eta}(1 + \frac{t - r_3}{r_4 - r_3})$ and calculate

$$\begin{aligned} \langle \tilde{G}, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} &= \langle G, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \mathrm{rot}\,\check{E}, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \check{E}, \mathrm{div}\,h \rangle_{\mathrm{L}^{2,q}(\Omega)} \\ &= \langle G, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \mathrm{rot}\,\check{E}, \xi h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \check{E}, \mathrm{div}\,\xi h \rangle_{\mathrm{L}^{2,q}(\Omega)} \\ &= \langle G, h \rangle_{\mathrm{L}^{2,q+1}(\Omega)} - \langle \lambda, \underbrace{\gamma_n \xi h}_{= \gamma_n h} \rangle_{\mathrm{H}^{-1/2,q}(\partial\Omega)} = 0 \end{aligned}$$

using Theorem 3.11 (i) since $\xi h \in \mathrm{H}^{1,q+1}_{\mathrm{vox}}(\Omega)$. ■

We finish this paper by shortly turning to the dual problem using the Hodge star operator. To this end we define ${}_\varepsilon\tilde{\mathcal{H}}^q_t(\Omega) := {}_0\overset{\circ}{\mathrm{D}}{}^q_t(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{R}^q_t(\Omega)$ with $t \in \mathbb{R}$, the space of '*(weighted harmonic) Neumann fields*'. Again we denote $\tilde{\mathcal{H}}^q_t(\Omega) := {}_{\mathrm{id}}\tilde{\mathcal{H}}^q_t(\Omega)$ and ${}_\mu\tilde{\mathcal{H}}^q(\Omega) := {}_\mu\tilde{\mathcal{H}}^q_0(\Omega)$. Then we have ${}_\varepsilon\tilde{\mathcal{H}}^q_t(\Omega) = *_{*\varepsilon*}\mathcal{H}^{q'}_t(\Omega)$ and hence the dimension of ${}_\varepsilon\tilde{\mathcal{H}}^q_t(\Omega)$ equals $d^{q'}_t$ (with $q' = N - q$). Furthermore, let $d^{q'}_{-1}$ continuous linear functionals Ψ^ℓ_ε on $\mathrm{D}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{R}^q_{-1}(\Omega)$ with

$${}_\varepsilon\tilde{\mathcal{H}}^q_{-1}(\Omega) \cap \bigcap_{\ell=1}^{d^{q'}_{-1}} N(\Psi^\ell_\varepsilon) = \{0\}$$

be given. We formulate the dual problem:

Find for given data F, G, λ, α a q-form $H \in \mathrm{D}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{R}^q_{-1}(\Omega)$ satisfying

$$\begin{aligned} \mathrm{div}H &= F \quad , \\ \mathrm{rot}\varepsilon H &= G \quad , \\ \Gamma_n H &= \lambda \quad , \\ \Psi^\ell_\varepsilon(H) &= \alpha_\ell \quad , \quad \ell = 1, \dots, d^{q'}_{-1} \quad . \end{aligned} \tag{3.18}$$

Analogously to Corollary 2.21 we obtain

Corollary 3.24 *The conditions* $G \in {}_0\mathrm{R}^{q+1}(\Omega) \cap \tilde{\mathcal{H}}^{q+1}(\Omega)^{\perp}$, $F \in {}_0\mathrm{D}^{q-1}(\Omega)$, $\lambda \in \mathcal{D}^{q-1}(\partial\Omega)$, $\alpha \in \mathbb{C}^{d^{q'}_{-1}}$ *and*

$$\operatorname{Div} \lambda = -\Gamma_n F \qquad , \qquad \bigwedge_{h \in \tilde{\mathcal{H}}^{q-1}(\Omega)} \langle F, h \rangle_{\mathrm{L}^{2,q-1}(\Omega)} = \langle \lambda, \gamma_t h \rangle_{\mathbf{H}^{-1/2,q-1}(\partial\Omega)}$$

are necessary and sufficient for the solvability of (3.18)*. The solution is unique and depends continuously on the data, i.e. there exists a positive constant* c *independent of* H *or the data, such that*

$$\|H\|_{\mathrm{D}^q_{-1}(\Omega) \cap \varepsilon^{-1}\mathrm{R}^q_{-1}(\Omega)} \leq c \cdot \left(\|F\|_{\mathrm{L}^{2,q-1}(\Omega)} + \|G\|_{\mathrm{L}^{2,q+1}(\Omega)} + \|\lambda\|_{\mathcal{D}^{q-1}(\partial\Omega)} + |\alpha| \right)$$

holds.

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