Lectures on quasiconformal and quasisymmetric mappings

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Quasiconformal mappings appeared perhaps for the first time in 1928 in a work by Grötzsch under the name "most nearly conformal mappings". He essentially considered the problem of mapping a planar square to a planar (non-square) rectangle by a diffeomorphism that sends the vertices of the square to the corner points of the rectangle. Even though these two domains are conformally equivalent, the given boundary condition cannot be realized by any conformal mapping. For a conformal mapping f, the ratio $|f'(z)|^2/J_f(z)$, is identically one by the Cauchy-Riemann equations, and one was then lead to try to minimize the maximum of this quantity under diffeomorphisms with the given boundary condition. Similar questions were subsequently considered by Teichmüller in the 1930's. The term "quasifonformal mapping" was coined by Ahlfors in 1935. He relaxed the regularity assumption and considered homeomorphisms in the local Sobolev class $W^{1,2}$ for which

$$|Df(z)|^2 \leq KJ_f(z)$$

almost everywhere for some constant $K \geq 1$. The restriction $K \geq 1$ comes from simple linear algebra: for each $n \times n$-matrix A,

$$\det A \leq |A|^n,$$

where $\det A$ refers to the determinant of A and

$$|A| = \sup_{|h| \leq 1} |Ah| = \sup_{|h| = 1} |Ah|$$

is the operator norm of the linear transformation associated with A.

In 1938, Morrey proved a powerful existence theorem, called the measurable Riemann mapping theorem. This essentially states that, in the plane, quasiconformal mappings with any prescribed ratio $|Df(z)|^2/J_f(z) \in L^\infty$ and any given direction for the the maximal directional derivative can be found. Other important developers of the theory include Lavrantiev and Bojarski. Planar quasiconformal mappings have since then been applied to many entirely different problems. Let us simply here list the following: Kleinian

groups, Nevanlinna theory, surface topology, complex dynamics, partial differential equations, inverse problems and conductivity.

Higher dimensional quasiconformal mappings were already introduced by Lavrantiev in 1938. The theory began to flourish around 1960 when important works by Loewner, Gehring, and Väisälä appeared. Other significant contributors include Callender, Shabat, and Reshetnyak. Subsequently, these mappings were introduced also in non-Euclidean settings by Mostow, who proved his celebrated rigidity theorem in 1968. Another celebrated result is the reverse Hölder inequality of Gehring's from 1972. In higher dimensions, the theory of and techniques introduced to study quasiconformal mappings have been successfully applied in differential geometry, topology, harmonic analysis, partial differential equations, and non-linear elasticity, among other fields.

The purpose of these notes is to give an introduction to the theory. The selected approach has been influenced by recent advances in the metric setting, but the framework is mostly that of a Euclidean space. The concept of quasisymmetry will be crucial in our approach. We have tried to make the notes as self-contained as possible. The reader is nevertheless assumed to know the basics of the Lebesgue integration theory and L^p-spaces. The topics covered reflect the personal taste of the author. Naturally many important aspects must have been left untouched. For further reading, we recommend the classic monograph "Lectures on n-dimensional quasiconformal mappings" by Väisälä [30] and the monograph [4].

These notes are based on courses given at the University of Jyväskylä in 1997, 2004 and 2008 and at the University of Michigan in 2002. The current notes are the outcome of several iterations. We wish to thank all the people who have provided us with lists of typos. In our experience, most of the material can be covered in a one semester, graduate level topics course. Regarding the sources for the presented material, we wish to highlight [8], [14] and [30]. There are rather few historical comments in what follows, and the inclusions or omissions of references are essentially random.

1   The metric definition

We begin by introducing the so-called metric definition of quasiconformality.

To this end, let $(X, |\cdot|), (Y, |\cdot|)$ be metric spaces and $f : X \rightarrow Y$ homeomorphism. Let $x \in X$ and $r > 0$. Define

$$L_f(x, r) := \sup\{|f(x) - f(y)| : |x - y| \leq r\},$$

$$l_f(x, r) := \inf\{|f(x) - f(y)| : |x - y| \geq r\},$$
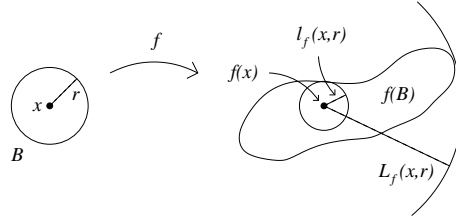
and
$$H_f(x, r) := \frac{L_f(x, r)}{l_f(x, r)}.$$

Figure 1: The definition of $L_f(x, r)$ and $l_f(x, r)$

A homeomorphism f is quasiconformal if there exists $H < \infty$ such that

$$H_f(x) := \limsup_{r \to 0} H_f(x, r) \leq H$$

for all $x \in X$. We then say that f is (metrically) H-quasiconformal.

Here is a list of examples of quasiconformal mappings in the Euclidean setting.

1.1 Examples.

1) Each conformal f is quasiconformal.

2) The planar mapping $f(x, y) = (x, 2y)$ is quasiconformal.

3) The "radial streching" $f(x) = x|x|^{\varepsilon-1}$, $\varepsilon > 0$, is quasiconformal in all dimensions.

4) There is quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(S^1(0, 1))$ is the von Koch snowflake curve.

5) Each diffeomorphism $f : \Omega \to \Omega'$ is quasiconformal in every subdomain $G \subset\subset \Omega$.

Let us begin by considering 1) in the plane. Write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ for a conformal mapping f, where u, v are real functions. Then f is analytic and the Jacobian determinant J_f of f is strictly positive.

By the Cauchy-Riemann equations we have that

$$u_x = v_y, \ u_y = -v_x.$$

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Thus
$$Df(x, y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}.$$
We conclude that $J_f(x, y) = (u_x)^2 + (u_y)^2 = |\nabla u|^2 = |\nabla v|^2$ and that $\nabla u \cdot \nabla v = 0$. Moreover, also the two columns of $Df(x, y)$ are perpendicular and both of length $|\nabla u|$. Thus, given a vector h, we have that
$$|Df(x, y)h| = |\nabla u||h|.$$
By the (complex) differentiability of f we conclude that
$$\limsup_{r \to 0} H_f(x, r) = 1$$
everywhere. Notice also that
$$|Df(x, y)|^2 = J_f(x, y)$$
everywhere, where $|A| = \sup_{|h| \leq 1} |Ah|$ is the usual operator norm. Since $f'(x + iy) = u_x(x + iy) - iu_y(x + iy)$ for the complex derivative f', we also have that $|Df(x, y)| = |f'(x + iy)|$, where the latter term is the modulus of the complex derivative and the former again the operator norm.

For 2) one easily checks that f is indeed quasiconformal, with
$$\limsup_{r \to 0} H_f(x, r) = 2$$
everywhere.

The radial mapping described in 3) requires already some effort, see Chapter 10 below. We will also discuss the mapping referred to in 4) in more detail in Chapter 10.

Regarding 5), notice that the Jacobian $J_f(x)$ of f is locally bounded away from zero and that $|Df(x)| = \sup_{|h| \leq 1} |Df(x)h|$ is locally bounded. Thus, given $G \subset\subset \Omega$, we have that
$$|Df(x)|^n \leq KJ_f(x)$$
for some constant K and all $x \in G$. This implies that
$$|Df(x)| \leq K' \min_{|h|=1} |Df(x)h|$$
with some constant K' in G (in fact, we may take $K' = |K|^{n-1}$). The quasiconformality then follows with $H = K'$ using the differentiability of f.

The metric definition is easy to state but it is hard to deduce properties of quasiconformal mappings directly from it. For example, it is not clear from the definition if quasiconformal mappings form a group. The problem is that the definition is an infinitesimal one. In the next chapter we show that it implies a global estimate which is easier to work with.

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2 From local to global

In this chapter we prove the following global estimate and introduce the machinery needed for its proof.

2.1 Theorem. Let $f : \Omega \to \Omega'$ be H-quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$ are domains, $n \geq 2$. Then $H_f(x, r) \leq H'(H, n)$ whenever $B(x, 7r) \subset \Omega$.

To help to understand the fundamental ideas of the proof, let us begin with a simpler setting.

2.1 Special case

Suppose that $\Omega = \Omega' = \mathbb{R}^2$ and assume that f be a diffeomorphism.

Assume that f is orientation preserving. Let $x \in \Omega$. Then f is differentiable at x with $J_f(x) > 0$. It follows that

$$\max_{|e|=1} |Df(x)e| \leq H \min_{|e|=1} |Df(x)e|$$

and

$$|Df(x)|^2 \leq H J_f(x),$$

see Subsection 11.2 in the appendix. Let us show that $H_f(x_0, r) \leq H'$. We may assume that $x_0 = 0 = f(x_0)$. Denote $L := L_f(0, r)$ and $l := l_f(0, r)$. Define

$$v(y) = \begin{cases} 1 & \text{if } |y| \leq l \\ 0 & \text{if } |y| \geq L \\ \frac{\log \frac{L}{|y|}}{\log \frac{L}{l}} & \text{if } l \leq |y| \leq L \end{cases}.$$

Then

$$|\nabla v(y)| = \begin{cases} 0 & \text{if } |y| < l \\ 0 & \text{if } |y| > L \\ \frac{1}{|y| \log \frac{L}{l}} & \text{if } l < |y| < L \end{cases}.$$

Now

$$\int_{\mathbb{R}^2} |\nabla v(y)|^2 \, dy = \left(\frac{1}{\log \frac{L}{l}} \right)^2 \int_{l \leq |y| \leq L} \frac{dy}{|y|^2}$$

$$= \left(\frac{1}{\log \frac{L}{l}} \right)^2 \int_l^L \int_0^{2\pi} \frac{1}{r^2} \, r \, d\varphi \, dr$$

$$= \left(\frac{1}{\log \frac{L}{l}} \right)^2 (\log L - \log l) 2\pi$$

$$= \frac{2\pi}{\log \frac{L}{l}}.$$

Let $u(x) = v(f(x))$. Then (see Subsection 11.2 in the appendix)

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \leq \int_{\mathbb{R}^2} |\nabla v(f(x))|^2 |Df(x)|^2 \, dx$$

$$\leq H \int_{\mathbb{R}^2} |\nabla v(f(x))|^2 |J_f(x)| \, dx$$

$$= H \int_{\mathbb{R}^2} |\nabla v(y)|^2 \, dy$$

$$= \frac{2\pi H}{\log \frac{L}{l}}.$$

Now, $u = 1$ on $f^{-1}(\overline{B}(0,l))$ and $u = 0$ on $f^{-1}(\mathbb{R}^2 \setminus B(0,L))$. Let w_0, z_0 be such that $|w_0| = |z_0| = r$, $w_0 \in f^{-1}(\mathbb{R}^2 \setminus B(0,L))$ and $z_0 \in f^{-1}(\overline{B}(0,l))$. Set

$$w = \begin{cases} \frac{w_0}{2} & \text{if } |w_0 - z_0| \geq \sqrt{2}r \\ \frac{w_0 + z_0}{|w_0 + z_0|} r & \text{if } |w_0 - z_0| < \sqrt{2}r \end{cases}.$$

Then, for $\frac{\pi r}{4} < t < r$, $S^1(w,t)$ intersects both $f^{-1}(\mathbb{R}^2 \setminus B(0,L))$ and $f^{-1}(\overline{B}(0,l))$. Now, since u oscillates from 0 to 1 on $S^1(w,t)$, we have

$$1 \leq \int_{S^1(w,t)} |\nabla u| \overset{\text{Hölder}}{\leq} (2\pi t)^{1/2} \left(\int_{S^1(w,t)} |\nabla u|^2 \right)^{1/2}$$

for each $\frac{\pi r}{4} < t < r$. Thus

$$\int_{B(0,2r)} |\nabla u|^2 \geq \int_{\frac{\pi r}{4}}^r \left(\int_{S^1(w,t)} |\nabla u|^2 \right) dt \geq \int_{\frac{\pi r}{4}}^r \frac{1}{2\pi t} = C \qquad (1)$$

where C is independent of r. Hence

$$\frac{L}{l} \leq \exp(CH).$$

This gives us the desired global control.

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2.2 Relaxing the regularity assumption

We continue with the planar setting. We begin by disposing of the use of the chain rule.

Let us define

$$\rho(x) = \begin{cases} \frac{|Df(x)|}{|f(x)|}\frac{1}{\log\frac{L}{l}} & \text{on } f^{-1}(B(0,L)\setminus\overline{B}(0,l)) =: A \\ 0 & \text{elsewhere} \end{cases}.$$

Then

$$\int_{\mathbb{R}^2}\rho^2 \le \frac{2\pi H}{\log\frac{L}{l}}.$$

If γ is a subarc of $S^1(w,t)$ which connects $f^{-1}(\mathbb{R}^2\setminus B(0,L))$ to $f^{-1}(\overline{B}(0,l))$, then $f\circ\gamma$ connects $\mathbb{R}^2\setminus B(0,L)$ to $B(0,l)$, and so

$$\int_{S^1(w,t)}\rho\,ds \ge \int_{f\circ\gamma}\frac{ds}{|y|\log\frac{L}{l}} \ge 1,$$

where w and $\frac{\pi r}{4} < t < r$ are as above. Reasoning as in (1), using polar coordinates, we conclude that

$$\int_{B(0,2r)}\rho^2 \ge C > 0.$$

We no longer require the chain rule, but it still looks like we need f to be differentiable. To relax this assumption, let us try to discretize the definition of our function ρ. Recall that we wish to bound L/l from above. We may thus assume that $L \ge 2l$.

Suppose $A = f^{-1}(\overline{B}(0,L)\setminus B(0,l)) \subset \bigcup B_j$, where B_j's are balls. Set

$$\rho(x) = \left(\log\frac{L}{l}\right)^{-1}\sum\frac{\text{diam}(f(B_j))}{\text{diam}(B_j)}\frac{1}{\text{dist}(0,f(B_j))}\chi_{2B_j}(x).$$

Then

$$\int_{S^1(w,t)}\rho\,ds = \left(\log\frac{L}{l}\right)^{-1}\sum\frac{\text{diam}(f(B_j))}{\text{diam}(B_j)}\frac{1}{\text{dist}(0,f(B_j))}\int_{S^1(w,t)}\chi_{2B_j}\,ds.$$

If the B_j's are small, then

$$\int_{S^1(w,t)}\chi_{2B_j}\,ds \ge \frac{\text{diam}(B_j)}{2}$$

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whenever $B_j \cap S^1(w,t) \neq \emptyset$. Hence

$$\int_{S^1(w,t)} \rho \, ds \geq \left(\log \frac{L}{l}\right)^{-1} \frac{1}{2} \sum_{B_j \cap S^1(w,t) \neq \emptyset} \frac{\text{diam}(f(B_j))}{\text{dist}(0, f(B_j))}.$$

Assume that the sets $f(B_j)$ are so small that each $f(B_j)$ intersects at most two annuli $A_i = B(0, 2^i l) \setminus B(0, 2^{i-1} l)$. Write $\lfloor t \rfloor$ for the integer part of a real number t. Then

$$
\begin{aligned}
\sum_{B_j \cap S^1(w,t) \neq \emptyset} \frac{\text{diam}(f(B_j))}{\text{dist}(0, f(B_j))} &\geq \frac{1}{4} \sum_{i=1}^{\lfloor \log_2 \frac{L}{l} \rfloor} \sum_{B_j \cap S^1(w,t) \neq \emptyset, f(B_j) \cap A_i \neq \emptyset} \frac{\text{diam}(f(B_j))}{\text{dist}(0, f(B_j))} \\
&\geq \frac{1}{4} \sum_{i=1}^{\lfloor \log_2 \frac{L}{l} \rfloor} \sum_{B_j \cap S^1(w,t) \neq \emptyset, f(B_j) \cap A_i \neq \emptyset} \frac{\text{diam}(f(B_j))}{2^i l} \\
&\geq \frac{1}{4} \sum_{i=1}^{\lfloor \log_2 \frac{L}{l} \rfloor} \frac{2^{i-1} l}{2^i l} \\
&\geq \frac{1}{8} \log_2 \frac{L}{l},
\end{aligned}
$$

and so

$$\int_{S^1(w,t)} \rho \, ds \geq C > 0$$

whenever $\frac{\pi r}{4} < t < r$. As before, this gives

$$\int_{B(0,2r)} \rho^2 \geq C > 0. \tag{2}$$

When we try to estimate $\|\rho\|_{L^2(B(0,2r))}$ from above, we are faced with the integral

$$\left(\log \frac{L}{l}\right)^{-2} \int_{B(0,2r)} \left(\sum_1^k \frac{\text{diam}(f(B_j))}{\text{diam}(B_j)} \frac{1}{\text{dist}(f(B_j), 0)} \chi_{2B_j}(x)\right)^2 dx. \tag{3}$$

Problems:

1) How to select balls B_j so that we can find an effective estimate on our integral? This requires control on the overlap of the balls B_j.

2) How to get rid of the annoying 2 in χ_{2B_j}?

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3) Even if we can handle 1) and 2), how can we handle dimensions $n \geq 3$? Notice here that the proof of (2) strongly used the fact that we are in the plane.

We next introduce the technology that will allow us to handle the above problems.

2.3 Covering theorems

We will later use covering theorems to select the above balls B_j. We begin with a covering lemma that holds in all metric spaces whose closed balls are compact.

2.2 Theorem. (Vitali) Let $\mathcal{B}$ be a collection of closed balls in $\mathbb{R}^n$ such that

$$\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty.$$

Then there are $B_1, B_2, \ldots$ (possibly a finite sequence) from this collection such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup 5 B_j.$$

For a proof we refer the reader to [20]. Let us anyhow briefly explain the idea in a simple case. Suppose that the family $\mathcal{B}$ consists of balls $B(x, r_x)$, where $x \in A$ and A is bounded. Let $M = \sup_{x \in A} r_x$. Choose a ball $B_1 = B(x, r_x)$ so that $r_x > 3M/4$. Continue by considering points in $A \setminus 3B_1$, and repeating the first step (now letting $M_1 = \sup_{y \in A \setminus 3B_1} r_y$) and after that continue by induction.

In the Euclidean setting, a subcollection often can be chosen so that we only have uniformly bounded overlap for the cover.

2.3 Theorem. (Besicovitch) Let $\mathcal{B}$ be a collection of closed balls in $\mathbb{R}^n$ such that the set A consisting of the centers is bounded. Then there is a countable (possibly finite) subcollection $B_1, B_2, \ldots$ such that

$$\chi_A(x) \leq \sum \chi_{B_j}(x) \leq C(n)$$

for all x.

The selection of the balls B_j eventually will be made using the Besicovitch covering theorem. In more general settings, say, in the Heisenberg group, Besicovitch fails. The reason it holds in the Euclidean setting, is basically the following fact:

Suppose that we are given $B(x_1, r_1)$ and $B(x_2, r_2)$ so that $0 \in B(x_1, r_1) \cap B(x_2, r_2)$, $x_1 \notin B(x_2, r_2)$ and $x_2 \notin B(x_1, r_1)$. Then the angle between the vectors x_1 and x_2 is at least 60 degrees.

For a proof of the Besicovitch covering theorem, we again refer to [20].

2.4 The maximal function

We will need maximal functions to dispose of the constant 2 in the term χ_{2B_j} in (3). Maximal functions turn out to be important for other things as well.

Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. The non-centered maximal function of u is

$$\mathrm{M}u(x) = \sup_{x \in B(y,r)} \fint_{B(y,r)} |u|.$$

Here and in what follows,

$$\fint_A v = \frac{1}{|A|} \int_A v$$

when A is measurable with $0 < |A| < \infty$, and $|A|$ refers to the Lebesgue measure of A.

2.4 Remarks.

1) According to the Lebesgue differentiation theorem (cf. Remarks 4.3),

$$\mathrm{M}u(x) \geq |u(x)|$$

   almost everywhere. This fact is not be needed in this section.

2) There are many other maximal functions. For example the restricted, centered maximal function

$$\mathrm{M}^C_\delta u(x) = \sup_{0 < r < \delta} \fint_{B(x,r)} |u|.$$

3) We always have $\mathrm{M}^C_\infty u(x) \leq \mathrm{M}u(x) \leq 2^n \mathrm{M}^C_\infty u(x)$.

4) Notice that $\{\mathrm{M}u > t\}$ is open for each $t \geq 0$ and, consequently, $\mathrm{M}u$ is measurable. Indeed, if $x \in \{\mathrm{M}u > t\}$, then it immediately follows from the definition that $B(y, r) \subset \{\mathrm{M}u > t\}$, for some $B(y, r)$ containing x.

2.5 Theorem.

1) If $u \in L^1$ and $t > 0$, then $|\{\mathrm{M}u > t\}| \leq \frac{5^n}{t} \int_{\{\mathrm{M}u > t\}} |u| \leq \frac{5^n}{t} \|u\|_1$.

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2) If $u \in L^p$, $p > 1$, then $\int (Mu)^p \le C(p,n) \int |u|^p$.

Proof. 1) We may assume that $M := \int_{\{Mu>t\}} |u| < \infty$. For each $x \in \{Mu > t\}$ there is a ball B such that $x \in B$ and

$$\fint_B |u| > t.$$

Then

$$|B| < t^{-1} \int_B |u|$$

and thus

$$\mathrm{diam}(B) < C(n)t^{-1}||u||_1.$$

If $y \in B$, then $Mu(y) > t$ and thus $B \subset \{Mu > t\}$. So

$$|B| < \frac{1}{t} \int_B |u| \le \frac{1}{t} \int_{\{Mu>t\}\cap B} |u|.$$

By the Vitali covering theorem we find pairwise disjoint balls $B_1, B_2, \ldots$ as above so that $\{Mu > t\} \subset \bigcup 5B_j$. Then

$$|\{Mu > t\}| \le \sum |5B_j| = 5^n \sum |B_j| \le \frac{5^n}{t} \sum \int_{B_j} |u| \le \frac{5^n}{t} \int_{\{Mu>t\}} |u|.$$

2) Recall the Cavalieri principle:

$$\int |v|^p = p \int \int_0^{|v(x)|} t^{p-1} \, dt \, dx$$
$$= p \int \int_0^\infty t^{p-1} \chi_{\{|v|>t\}} \, dt \, dx$$
$$= p \int_0^\infty t^{p-1} |\{|v| > t\}| \, dt.$$

Fix $t > 0$. Define $g(x) = |u(x)| \, \chi_{\{|u(x)|>\frac{t}{2}\}}(x)$. Then $|u(x)| \le g(x) + \frac{t}{2}$ and so $Mu(x) \le Mg(x) + \frac{t}{2}$. Thus $\{x : Mu(x) > t\} \subset \{x : Mg(x) > \frac{t}{2}\}$. Now, the Cavalieri principle, part 1) of our theorem and the Fubini theorem yield the

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estimate

$$\int (\mathrm{M}u(x))^p = p \int_0^\infty t^{p-1} |\{\mathrm{M}u(x) > t\}| \, dt$$

$$\leq p \int_0^\infty t^{p-1} |\{\mathrm{M}g(x) > \frac{t}{2}\}| \, dt$$

$$\leq p \int_0^\infty t^{p-1} \frac{2 \cdot 5^n}{t} ||g||_1$$

$$\leq p \int_0^\infty t^{p-1} \frac{2 \cdot 5^n}{t} \int_{\{|u(x)| > \frac{t}{2}\}|} |u| \, dx \, dt$$

$$\leq 2 \cdot 5^n p \int_0^\infty t^{p-2} \int_{\mathbb{R}^n} \chi_{\{|u(x)| > \frac{t}{2}\}} |u| \, dx \, dt$$

$$= 2 \cdot 5^n p \int_{\mathbb{R}^n} |u(x)| \int_0^{2|u(x)|} t^{p-2} \, dt \, dx$$

$$= \frac{2^p 5^n p}{p-1} \int |u|^p.$$

$\square$

2.6 Remark.

1) Let us single out, for future reference, the estimate

$$|\{\mathrm{M}u(x) > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\{|u(x)| > \frac{t}{2}\}} |u| \, dx$$

from the above proof.

2) Suppose that $u \in L^p(\Omega)$, $p > 1$. Applying Theorem 2.5 to the zero extension of u we conclude that $\int_\Omega (Mu)^p \leq C(p,n) \int_\Omega |u|^p$. Similarly, the inequality in part 1) of this remark can be restricted to Ω when $u \in L^1(\Omega)$.

The case $p = 1$ was not left out by accident from the previous theorem.

2.7 Example. If $u(x) = \chi_{B(0,1)}(x)$, then $\mathrm{M}u \notin L^1(\mathbb{R}^n)$. In fact, $Mu \notin L^1(\mathbb{R}^n)$ unless u is the zero function.

The following lemma from [7] will allow us to handle problem 2) stated after formula 3.

2.8 Lemma. (Bojarski) Fix $1 \le p < \infty$. Let $B_1, B_2, \ldots$ be balls in $\mathbb{R}^n$, $a_j \ge 0$ and $\lambda > 1$. Then

$$\| \sum a_j \chi_{\lambda B_j} \|_p \le C(\lambda, p, n) \| \sum a_j \chi_{B_j} \|_p.$$

Proof. The case $p = 1$ is clear. Let $p > 1$. Then, by the $L^p - L^{p/(p-1)}$-duality (see Subsection 11.3 in the appendix),

$$\| \sum a_j \chi_{\lambda B_j} \|_p = \sup_{\|\varphi\|_{\frac{p}{p-1}} \le 1} \left| \int \sum a_j \chi_{\lambda B_j} \varphi \right|.$$

Now, using monotone convergence and Theorem 2.5 we estimate

$$
\begin{aligned}
\left| \int \sum a_j \chi_{\lambda B_j} \varphi \right| &\le \sum a_j \int_{\lambda B_j} |\varphi| \\
&\le \sum a_j |\lambda B_j| \fint_{\lambda B_j} |\varphi| \\
&\le \sum a_j \lambda^n \int_{B_j} \mathrm{M}\varphi \\
&= \lambda^n \int \sum a_j \chi_{B_j} \mathrm{M}\varphi \\
&\le \lambda^n \| \sum a_j \chi_{B_j} \|_p \| \mathrm{M}\varphi \|_{\frac{p}{p-1}} \\
&\le \lambda^n C(p, n) \| \sum a_j \chi_{B_j} \|_p \| \varphi \|_{\frac{p}{p-1}}.
\end{aligned}
$$

The claim follows. $\qquad\square$

2.5   Upper gradients and Poincaré inequalities

In this section we give a substitute for (1). We will later show that it allows us to prove an analog of (2) in all dimensions.

A Borel function $g \ge 0$ is an upper gradient of u in U, if

$$|u(x) - u(y)| \le \int_{\gamma_{x,y}} g \, ds \tag{4}$$

whenever $x \ne y \in U$ and $\gamma_{x,y}$ is a rectifiable curve that joins x to y in U. Here we agree that inequality (4) holds, whatever an expression we have on the left hand side, if the given line integral is infinite, and that both $u(x)$ and $u(y)$ are finite if the integral in question converges.

The rectifiability of $\gamma : [a, b] \to \mathbb{R}^n$ above means that, for some $M < \infty$,

$$\sum_{j=1}^{k-1} |\gamma(t_{j+1}) - \gamma(t_j)| \leq M$$

whenever $a = t_1 < t_2 < \cdots < t_k = b$ and $k \geq 2$. The supremum of such sums over all $k \geq 2$ and all partitions is then the length of γ. Recall that each rectifiable curve $\gamma : [a, b] \to \mathbb{R}^n$ of length l can be parametrized by $\gamma_0 : [0, l] \to \mathbb{R}^n$ so that $|\gamma_0'(t)| = 1$ for a.e. t and γ_0 is 1-Lipschitz, i.e. $|\gamma_0(t) - \gamma_0(s)| \leq |t - s|$ for all $t, s \in [0, l]$. Then

$$\int_\gamma g \, ds := \int_{[0,l]} g(\gamma_0(t)) \, dt.$$

For all this see [30].

2.9 Examples.

1) $u \in C^1, g = |\nabla u|$. This is simply the fundamental theorem of calculus for the absolutely (even Lipschitz) continuous function $u \circ \gamma_0$ of a single variable:

$$u(\gamma(l)) - u(\gamma(0)) = \int_{[0,l]} <\nabla u(\gamma_0(t)), \gamma_0'(t)> \, dt. \qquad (5)$$

2) u Lipschitz, g the pointwise Lipschitz "constant"

$$\mathrm{Lip}u(x) = \limsup_{r \to 0} \sup_{|x-y| \leq r} \frac{|u(x) - u(y)|}{r}.$$

Notice that $(u \circ \gamma_0)'(t) \leq \mathrm{Lip}u(t)$ for almost every t.

3) u anything, $g \equiv \infty$. In this case, the right hand side of (4) is always infinite.

Integration of (4), the Fubini theorem and spherical coordinates give us the important Poincaré inequality.

2.10 Theorem. (Poincaré inequality) Let $u \in L^1(B(x_0, r)) \subset \mathbb{R}^n$, $n \geq 2$, and let $g \in L^p(B(x_0, r))$, $1 \leq p < \infty$, be an upper gradient of u in $B(x_0, r)$. Then

$$\fint_{B(x_0,r)} |u - u_B| \leq C(n)r \left(\fint_{B(x_0,r)} g^p \right)^{1/p}.$$

Here $u_B := \fint_{B(x_0,r)} u$.

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Proof. Let $x \in B = B(x_0, r)$. Then

$$\int_B |u(x) - u(y)| \, dy \leq \int_B \int_0^1 g(x + t(y-x))|y-x| \, dt \, dy$$

$$= \int_0^1 \int_B g(x + t(y-x))|y-x| \, dy \, dt$$

$$\leq \int_0^1 \int_{B \cap B(x,2tr)} g(z) \left(\frac{|z-x|}{t} \right) t^{-n} \, dz \, dt$$

$$\leq 2r \int_0^1 \int_{B \cap B(x,2tr)} g(z) t^{-n} \, dz \, dt$$

$$\leq 2r \int_B g(z) \int_{\frac{|z-x|}{2r}}^1 t^{-n} \, dt \, dz$$

$$\leq C_n r^n \int_B \frac{g(z)}{|z-x|^{n-1}} \, dz.$$

Integrating with respect to x we obtain the estimate

$$\int_B \int_B |u(x) - u(y)| \, dy \, dx \leq C_n r^n \int_B \int_B \frac{g(y)}{|y-x|^{n-1}} \, dy \, dx$$

$$= C_n r^n \int_B g(y) \int_B \frac{1}{|y-x|^{n-1}} \, dx \, dy$$

$$\leq C_n' r^{n+1} \int_B g.$$

Now

$$\fint_B |u(x) - u_B| \, dx = \fint_B \left| \fint_B u(x) - u(y) \, dy \right| dx \leq \fint_B \fint_B |u(x) - u(y)| \, dy \, dx.$$

Combining the above estimates, we obtain the desired inequality for $p = 1$. The general case follows by Hölder's inequality. $\qquad\square$

2.11 Remarks.

1) The Poincaré inequality also holds when $n = 1$ and the proof is easier: when $x < y$ and $x, y \in I$, where I is a bounded interval, we have that

$$|u(y) - u(x)| \leq \int_x^y g(t) \, dt \leq \int_I g(t) \, dt$$

by the upper gradient inequality. Integrating this estimate over I with respect both of the variables, we obtain the Poincaré inequality by repeating the last steps of the proof of Theorem 2.10.

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2) It is easy to modify the proof of Theorem 2.10 so as to verify

$$\left(\fint_{B(x,r)} |u - u_B|^p\right)^{1/p} \leq C(n,p)r \left(\fint_{B(x,r)} g^p\right)^{1/p}.$$

This is the usual form of the Poincaré inequality.

3) It is harder to prove that

$$\left(\fint_{B(x,r)} |u - u_B|^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}} \leq C(n,p)r \left(\fint_{B(x,r)} g^p\right)^{1/p}$$

when $1 \leq p < n$. This inequality is called the Sobolev-Poincaré inequality.

4) If $u \in L^1(B(x_0, r))$ has an upper gradient $g \in L^\infty(B(x_0, r))$, then it easily follows that u has a representative $\tilde{u}$ (i.e. $\tilde{u} = u$ almost everywhere) that is $||g||_{L^\infty}$-Lipschitz. By the last step of the proof of Theorem 2.10 we then conclude that the Poincaré inequality also holds for $p = \infty$.

We are now ready to prove a substitute for (1).

2.12 Theorem. Let u be continuous in $B(x_0, 3r)$, $g \geq 0$ an upper gradient of u in $B(x_0, 3r)$ and assume that $u \leq 0$ on E, $u \geq 1$ on F where $E, F \subset B(x_0, r)$ are continua with $\min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \geq \delta_0 r > 0$. Then

$$\int_{B(x_0,3r)} g^n \geq \delta(\delta_0, n) > 0.$$

Proof. Let $a = \fint_{B(x_0,r)} u$. Assume that $a \leq 1/2$. Let $x \in F$ and write $r_i = 2^{-i}r$, $i \geq -1$, $B_i = B(x, r_i)$. Then

$$u(x) = \lim_{i \to \infty} u_{B_i} = \lim_{i \to \infty} \fint_{B_i} u.$$

Now

$$\frac{1}{2} \leq |u(x) - u_{B(x_0,r)}| \leq \sum_{i \geq 0} |u_{B_i} - u_{B_{i+1}}| + |u_{B_0} - u_{B(x_0,r)}|.$$

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Also, $B(x_0, r) \subset B(x, 2r)$ and thus a simple estimate and the Poincaré inequality yield

$$
\begin{aligned}
|u_{B_0} - u_{B(x_0,r)}| &\le |u_{B(x,r)} - u_{B(x,2r)}| + |u_{B(x_0,r)} - u_{B(x,2r)}| \\
&\le 2 \cdot 2^n \fint_{B(x,2r)} |u - u_{B(x,2r)}| \\
&\le C(n) 2r \left(\fint_{B(x,2r)} g^n \right)^{1/n} \\
&\le C(n)(2r)^{1/n} \left((2r)^{-1} \int_{B(x,2r)} g^n \right)^{1/n}.
\end{aligned}
$$

Similarly

$$
|u_{B_i} - u_{B_{i+1}}| \le C(n) r_i^{1/n} \left(r_i^{-1} \int_{B_i} g^n \right)^{1/n}.
$$

Thus

$$
\begin{aligned}
\frac{1}{2} &\le \sum_{i=-1}^{\infty} C(n) r_i^{1/n} \left(r_i^{-1} \int_{B_i} g^n \right)^{1/n} \\
&\le C(n) r^{1/n} \sup_{0 < t \le 2r} \left(t^{-1} \int_{B(x,t)} g^n \right)^{1/n}.
\end{aligned}
$$

Thus, for each $x \in F$, there is a ball $B(x, t_x)$ so that $t_x \le 2r$ and

$$
t_x \le C(n) r \int_{B(x,t_x)} g^n.
$$

By Vitali we find pairwise disjoint balls $B_1, B_2, \ldots$ as above such that $F \subset \bigcup 5B_k$. Then

$$
\mathrm{diam}(F) \le \sum \mathrm{diam}(5B_k) \le C(n) r \sum \int_{B_k} g^n \le C(n) r \int_{B(x_0,3r)} g^n.
$$

If $a > 1/2$, then we use E instead of F above. $\qquad\square$

2.13 Remark. By choosing the balls B_i more cleverly, one can show that $B(x_0, 3r)$ may be replaced with $B(x_0, r)$.

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2.6   Proof of Theorem 2.1.

We prove the estimate for $B(x_0, r)$. We may assume that $x_0 = 0 = f(x_0)$. Recalling that we wish to bound $H_f(x_0, r) = \frac{L_f(x_0, r)}{l_f(x_0, r)}$ from above, we may further assume that $L \geq 2l$, where $L := L_f(x_0, r)$ and $l := l_f(x_0, r)$. Let

$$A := f^{-1}((\overline{B}(0, L) \setminus B(0, l)) \cap \Omega') \cap \overline{B}(0, 6r).$$

For each $x \in A$, pick $0 < r_x < r/30$ such that

$$H(x, r_x) < 2H \quad \text{and} \quad \text{diam}(f(B(x, r_x))) < l/4.$$

By the Besicovitch covering theorem we find a subcollection $\{B_j\}_j = \{B(x_j, r_j)\}_j$ of $\{\overline{B}(x, r_x)\}_x$ so that

$$\chi_A(x) \leq \sum \chi_{B_j}(x) \leq C(n)$$

for all x. Because f is a homeomorphism, also

$$\sum \chi_{f(B_j)}(x) \leq C(n).$$

Pick $r_j < r_{x_j} < 2r_j$ so that

$$\text{diam}(f(B(x_j, r_{x_j}))) \leq 2 \, \text{diam}(f(B_j)).$$

Because A is compact, already a finite number of the balls $\hat{B}_j = B(x_j, r_{x_j})$ cover A, say $\hat{B}_1, \ldots, \hat{B}_k$. Define

$$\rho(x) = \left(\log \frac{L}{l} \right)^{-1} \sum_1^k \frac{\text{diam}(f(\hat{B}_j))}{\text{diam}(\hat{B}_j)} \frac{1}{\text{dist}(0, f(\hat{B}_j))} \chi_{2\hat{B}_j}(x).$$

Then

$$\rho(x) \leq 8 \left(\log \frac{L}{l} \right)^{-1} \sum_1^k \frac{\text{diam}(f(B_j))}{\text{diam}(B_j)} \frac{1}{\text{dist}(0, f(B_j))} \chi_{4B_j}(x).$$

By Lemma 2.8

$$\int \rho^n \, dx \leq C(n) \left(\log \frac{L}{l} \right)^{-n} \int \left(\sum_1^k \frac{\text{diam}(f(B_j))}{\text{diam}(B_j)} \frac{1}{\text{dist}(0, f(B_j))} \chi_{B_j}(x) \right)^n dx$$

$$\leq C(n) \left(\log \frac{L}{l} \right)^{-n} \sum_1^k \left(\frac{\text{diam}(f(B_j))}{\text{dist}(0, f(B_j))} \right)^n$$

$$\leq C(n, H) \left(\log \frac{L}{l} \right)^{-n} \sum_1^k \frac{|f(B_j)|}{\text{dist}(0, f(B_j))^n}.$$

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Denote $A_i = \{x : 2^{i-1}l \leq |x| \leq 2^i l\}$ and $i_0 = \lfloor \log \frac{L}{l} / \log 2 \rfloor + 1$. Then

$$\sum_1^k \frac{|f(B_j)|}{\text{dist}(0, f(B_j))^n} \leq \sum_1^{i_0} \sum_{f(B_j) \cap A_i \neq \emptyset} \frac{|f(B_j)|}{\text{dist}(0, f(B_j))^n} \leq \sum_1^{i_0} C(n) \frac{|B(0, 2^{i+1}l)|}{(2^{(i-2)}l)^n},$$

and so

$$\int \rho^n \, dx \leq C(n, H) \left(\log \frac{L}{l} \right)^{1-n}. \tag{6}$$

Notice that $f^{-1}(\mathbb{R}^n \setminus B(0, L))$ contains a continuum F that joins $S^{n-1}(0, r)$ to $S^{n-1}(0, 2r)$ in $\overline{B}(0, 2r)$. Because $f(B(0, r))$ is open, it is easy to check that $B(0, l) \subset f(B(0, r))$. Define $E = f^{-1}(\overline{B}(0, l))$. Then E is a continuum, $\text{diam}(E) \geq r$, $\text{diam}(F) \geq r$, and $E, F \subset \overline{B}(0, 2r)$. If γ is a rectifiable curve that joins E to F, then $f \circ \gamma$ joins $\overline{B}(0, l)$ to $\mathbb{R}^n \setminus B(0, L)$. Reasoning as in 2.2 we see that

$$\int_\gamma \rho \, ds \geq \varepsilon_0 > 0$$

where ε_0 does not depend on f, r or γ. Define

$$u(x) = \frac{1}{\varepsilon_0} \inf \int_{\gamma_x} \rho \, ds,$$

where infimum is taken over all rectifiable curves that join x to F. Then $u = 0$ in F and $u \geq 1$ in E. Remember from the definition of ρ that ρ is bounded. Let $u(y) > u(x)$. Then

$$|u(y) - u(x)| \leq \int_{\gamma_{x,y}} \frac{\rho}{\varepsilon_0} \, ds$$

for all rectifiable curves $\gamma_{x,y}$ connecting x to y. Thus $\frac{\rho}{\varepsilon_0}$ is an upper gradient of u. Note that u is Lipschitz because

$$|u(x) - u(y)| \leq \sup_{z \in B(x, 2|x-y|)} \frac{\rho(z)}{\varepsilon_0} |x - y|.$$

By Theorem 2.12 we conclude that

$$\int_{B(0,6r)} \rho^n \, dx \geq \varepsilon_0^n \delta > 0. \tag{7}$$

A bound on L/l follows combining (6) and (7), as desired.

2.14 Remarks.

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1) The assumption that $n \geq 2$ was needed to ensure that the exponent $1 - n$ in (6) is negative. Thus the proof does not extend to the case $n = 1$. This is no accident. The simple quasiconformal mapping $f(x) = x + \exp(x)$ of a single variable shows that the claim of Theorem 2.1 fails for $n = 1$.

2) We only needed that

$$\liminf_{r \to 0} H_f(x, r) \leq H$$

for all $x \in \Omega$ for our homeomorphism in the proof of Theorem 2.1. Thus the quasiconformality assumption can be relaxed to this condition.

3) It is now natural to inquire if the uniform boundedness of the $\limsup$ or $\liminf$ of $H_f(x, r)$ is really necessary. To this end, let $E \subset [0, 1]$ be the $\frac{1}{3}$-Cantor set. Then the Cantor function $\xi : [0, 1] \to [0, 1]$ maps E to a set of positive length. Let $\Omega =]0, 1[\times \mathbb{R}$ and define $f(x, y) = (x + \xi(x), y)$. Then

$$\limsup_{r \to 0} H_f(x, r) = 1$$

outside $E \times \mathbb{R}$ and $f : \Omega \to f(\Omega)$ is a homeomorphism that takes a set of zero area to a set of positive area. We will soon prove that a quasiconformal mapping cannot do this (one can also show directly using the properties of the Cantor function that f cannot be quasiconformal). We can replace the $\frac{1}{3}$-Cantor set in this example with any Cantor set, even of Hausdorff dimension zero. Consequenty, uniform boundedness of $H_f(x)$ outside a set of dimension one when $n = 2$ does not suffice for the uniform boundedness of $H_f(x, r)$. In higher dimensions, one replaces $\mathbb{R}$ above by $\mathbb{R}^{n-1}$ to see that the analog of dimension one is then $n - 1$.

On the other hand, if

$$\liminf_{r \to 0} H_f(x, r) \leq H$$

outside a set of σ-finite $(n - 1)$-measure, one can prove that f is quasiconformal. This is rather easily seen from our previous arguments when $n = 2$: Let $\tilde{E}$ be the exceptional set of σ-finite length. Instead of picking small balls centered at each $x \in A$, do this for $A \setminus \tilde{E}$. Define ρ as before. Then still

$$\int \rho^2 \leq C(H) \left(\log \frac{L}{l} \right)^{-1}.$$

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What about the lower bound? Let us refer to our previous argument in 2.2. It could well happen that our balls do not cover the subarc of $S^1(w,t)$. However, one can prove that, for almost every $t > 0$, the set $\tilde{E} \cap S^1(w,t)$ is countable. Then the balls we selected cover the subarc of $S^1(w,t)$ up to a countable set for almost every $t > 0$. Hence the images of the balls cover $f(S^1(w,t))$ up to a countable set. Thus

$$\int_{S^1(w,t)} \rho \, ds \geq \varepsilon_0 > 0$$

for almost every $t > 0$. The general setting is similar in spirit to that in the plane.

4) In the above proof, H' depends on H, n. It is not known if the claim could hold with some H' that does not depend on the dimension n. This is an interesting open problem. One cannot in general take $H' = H$ even when f is a conformal mapping of the unit disk onto a simply connected planar domain.

5) Theorem 2.1 extends to a rather abstract setting. Let X, Y be Ahlfors Q-regular [1], $Q > 1$, suppose that closed balls are compact, and the Poincaré inequality with exponent $p = Q$ holds for both X and Y. If $f : X \to Y$ is quasiconformal, then

$$H_f(x,r) \leq H'$$

for all $x \in X$, $r > 0$. In fact, even

$$\liminf_{r \to 0} H_f(x,r) \leq H$$

for all x suffices. These results can be proved by suitably modifying the argument that we used above, see [5]. The real difficulty is in circumventing the Besicovitch covering theorem. The size of exceptional sets is not yet entirely understood in this general setting, see [18].

6) A metric space X is called linearly locally connected (LLC), if there is a constant C so that

   i) each pair of points in any ball B can be joined by a continuum in CB, and

---

[1] A metric measure space X is Ahlfors Q-regular, if there is a constant C so that

$$C^{-1}r^Q \leq \mu(B(x,r)) \leq Cr^Q$$

for all x, r for some Borel measure μ.

ii) each pair of points outside any ball B can be joined by a continuum in $X \setminus C^{-1}B$.

The spaces in 5) are LLC. This connectivity condition is used to find substitutes for the sets E and F in the proof of Theorem 2.1.

3 Quasisymmetric mappings

By Theorem 2.1 we know that quasiconformality implies the uniform local boundedness of $H_f(x, r)$. We introduce the equivalent concept of quasisymmetry that turns out to be very useful.

Let X and Y be metric spaces and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism $f : X \to Y$ is η-quasisymmetric (η-qs), if

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \eta\left(\frac{|a - x|}{|b - x|}\right)$$

for all $a \neq x \neq b$.

3.1 Remark. If f is η-quasisymmetric, then

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)} \leq \eta(1).$$

So, quasisymmetric mappings are quasiconformal.

We next prove that quasiconformal mappings are locally quasisymmetric.

3.2 Theorem. Let $f : B(x_0, 3r_0) \to \Omega' \subset \mathbb{R}^n$ be a homeomorphism such that $H_f(x, r) \leq H$ for all $x \in B(x_0, r_0)$ and $0 < r < 2r_0$. Then $f_{|B(x_0,r_0)}$ is η-quasisymmetric, where η depends only on n and H.

Proof. Let $a \neq x \neq b$ be points in $B(x_0, r_0)$ and let $t = |a - x|/|b - x|$.
<u>Case 1: $t > 1$</u>. Write

$$a_j = x + j|b - x|\frac{a - x}{|a - x|}$$

for $j = 0, 1, \ldots, k$, where $k = \lfloor t \rfloor$. Then

$$|f(a_j) - f(a_{j-1})| \leq H|f(a_{j-1}) - f(a_{j-2})|,$$

for $j \geq 2$, and so

$$|f(a_j) - f(a_{j-1})| \leq H^{j-1}|f(a_1) - f(x)| \leq H^j|f(b) - f(x)|.$$

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Since $|f(a) - f(a_k)| \le H|f(a_k) - f(a_{k-1})|$, we obtain

$$|f(a) - f(x)| \le |f(a) - f(a_k)| + \sum_{j=1}^{k} |f(a_j) - f(a_{j-1})|$$
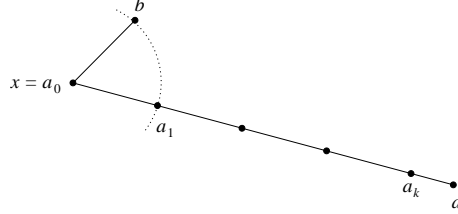$$\le (k+1)H^{k+1}|f(b) - f(x)|$$
$$\le (t+1)H^{t+1}|f(b) - f(x)|.$$

Figure 2: Case 1

Case 2: $t < 1/9$. Denote $b_j = x + 3^{-j}(b - x)$, for $j \ge 0$, and

$$B_j = B((b_j + b_{j-1})/2, 3^{-j}|b - x|),$$

for $j \ge 1$. Let $j \le k = \lfloor \log_3(1/t) \rfloor$. Then $|a - x| \le |b_j - x|$ and so

$$|f(a) - f(x)| \le H|f(b_j) - f(x)| \le H^2|f(b_j) - f(b_{j-1})| \le H^2 \operatorname{diam}(f(B_j)).$$

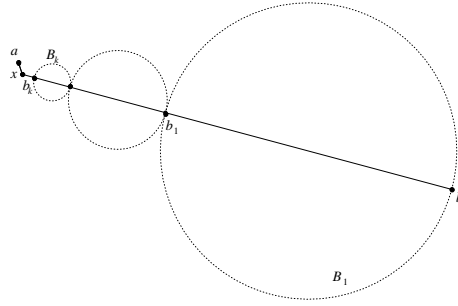This implies that

$$|f(a) - f(x)|^n \le C(H, n)|f(B_j)|.$$

Figure 3: Case 2

Since the balls B_j are pairwise disjoint and

$$f(B_j) \subset f(B(x, |b - x|)) \subset B(f(x), H|f(b) - f(x)|),$$

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we obtain

$$k|f(a) - f(x)|^n \leq C(H,n) \sum_{j=1}^{k} |f(B_j)|$$
$$\leq C(H,n)|B(f(x), H|f(b) - f(x)|)|$$
$$\leq C'(H,n)|f(b) - f(x)|^n.$$

Thus

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq C''(H,n)(\log(1/t))^{-1/n}.$$

Case 3: $1/9 \leq t \leq 1$. Clearly

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq H.$$

Select a homeomorphism $\eta : [0, \infty[\to [0, \infty[$ that is greater than or equal to the above bounds. $\square$

3.3 Remarks.

1) The proof goes through if $f : X \to Y$, X is LLC and both X and Y are Q-regular.

2) In fact, one can choose C and s depending on n and H so that the restriction of f to $B(x_0, r)$ is $\tilde{\eta}$-quasisymmetric with $\tilde{\eta}(t) = C \max\{t^s, t^{1/s}\}$. This requires a bit more work.

3.4 Corollary. Let $\Omega, \Omega' \subset \mathbb{R}^n$, where $n \geq 2$. Suppose that $f : \Omega \to \Omega'$ is quasiconformal and let $0 < \lambda < 1$. Then there is an $\eta = \eta(n, H, \lambda)$ so that the restriction of f to $B(x, \lambda d(x, \partial\Omega))$ is η-quasisymmetric whenever $x \in \Omega$.

Proof. By Theorem 2.1, the assumptions of Theorem 3.2 are satisfied for balls $B(x, d(x, \partial\Omega)/15)$. If $1/15 < \lambda < 1$, one then iterates the quasisymmetry estimate for the case $\lambda = 1/15$ so as to obtain quasisymmetry in $B(x, \lambda d(x, \partial\Omega))$ (with a new control function η that also depends on λ). $\square$

It is easy to check, from the definition, that quasisymmetric mappings form a group. The following proposition follows directly from the definition.

3.5 Proposition. Let $f : A_1 \to A_2$ be η_1-quasisymmetric and let $g : A_2 \to A_3$ be η_2-quasisymmetric. Then $f^{-1} : A_2 \to A_1$ is $\hat{\eta}$-quasisymmetric, where $\hat{\eta}(0) = 0$ and

$$\hat{\eta}(t) = \frac{1}{\eta_1^{-1}(\frac{1}{t})},$$

for $t > 0$, and $g \circ f : A_1 \to A_3$ is $\tilde{\eta}$-quasisymmetric, where $\tilde{\eta}(t) = \eta_2(\eta_1(t))$.

As a consequence of Corollary 3.4 and Proposition 3.5 we now conclude that quasiconformal mappings also form a group. This cannot be easily proven from the definition.

3.6 Theorem. Let $f : \Omega_1 \to \Omega_2$ be H_1-quasiconformal and let $g : \Omega_2 \to \Omega_3$ be H_2-quasiconformal. Then f^{-1} is $H(H_1, n)$-quasiconformal and $g \circ f$ is $H(H_1, H_2, n)$-quasiconformal.

Proof. By Corollary 3.4 there is $\eta = \eta(n, H)$ so that the restriction of f to any ball $B = B(x, d(x, \partial\Omega_1)/2)$ is η-quasisymmetric. Then $f^{-1} : f(B) \to B$ is $\hat{\eta}$-quasisymmetric by Proposition 3.5. Given $y \in \Omega_2$, choose $x = f^{-1}(y)$, let $B = B(x, d(x, \partial\Omega)/2)$, notice that $B(y, r) \subset f(B)$ for $r < l_f(x, d(x, \partial\Omega_1)/2)$, and apply Remark 3.1 to f^{-1}.

The quasiconformality of the composition follows by a similar argument. □

3.7 Remark. Let $\Omega \subset \mathbb{R}^2$ be bounded and simply connected. Let $f : B^2(0, 1) \to \Omega$ be quasiconformal. Then the following are equivalent:

1) f is quasisymmetric.

2) Ω is LLC.

3) There is a quasiconformal mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ so that $g_{|B^2(0,1)} = f$.

The fact that 1) implies 2) is easy to prove. By Corollary 3.4, 3) yields 1). The remaining implications are harder. To see that 2) implies 1), one reasons as in the proof of Theorem 2.1 using Remark 2.13 and a suitable case study. The fact that 1) implies 3) can be shown relying on techniques from Chapter 10 below.

4 Gehring's lemma and regularity of quasiconformal mappings

We will prove that quasiconformal mappings are differentiable almost everywhere, preserve the null sets for Lebesgue measure, and belong to the Sobolev class $W^{1,p}_{\text{loc}}$ for some $p = p(n, H) > n$. This amounts to absolute continuity of the component functions of f on almost all lines parallel to the coordinate axes (in the domain in question) and local p-integrability of the classical partial derivatives.

4.1 The volume derivative

It will be important for us to pull back the Lebesgue measure under our quasiconformal mapping.

4.1 Proposition. Let $f : \Omega \to \Omega'$ be a homeomorphism. Then

$$\mu'_f(x) = \lim_{r \to 0} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}$$

exists almost everywhere in Ω, belongs to $L^1_{\text{loc}}(\Omega)$ and

$$\int_E \mu'_f(x)\, dx \leq |f(E)|$$

for each Borel set $E \subset \Omega$, with equality whenever $|A| = 0$ implies $|f(A)| = 0$.

This is a direct consequence of the following Radon-Nikodym theorem when one chooses $\mu(A) = |f(A)|$ and $\lambda(A) = |A|$.

4.2 Theorem. (Radon-Nikodym) Let μ and λ be Radon measures on $\Omega \subset \mathbb{R}^n$. Then

$$D(\mu, \lambda, x) := \lim_{r \to 0} \frac{\mu(\overline{B}(x,r))}{\lambda(\overline{B}(x,r))}$$

exists λ-a.e., is locally integrable with respect to λ, and

$$\int_E D(\mu, \lambda, x)\, d\lambda(x) \leq \mu(E)$$

for each Borel set E with equality if an only if μ is absolutely continuous with respect to λ.

Recall that a measure μ is Radon, if $\mu(K) < \infty$ for compact sets, Borel sets are measurable,

$$\mu(U) = \sup\{\mu(K) : K \subset U \text{ compact}\}$$

for open U, and
$$\mu(A) = \inf\{\mu(U) : A \subset U \text{ open}\}$$

for arbitrary A.

We refer the reader to [20] for a proof of the Radon-Nikodym theorem. It is a rather direct application of a covering theorem that we have not discussed.

Let us however briefly explain how a weaker version of Proposition 4.1 can be justified using the covering theorems from 2.3. Instead of μ'_f, let us consider

$$u(x) = \limsup_{r \to \infty} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|},$$

and let us assume that we already know the Borel measurability of u. Let $E \subset \Omega$ be a Borel set. We may assume that $E \subset\subset \Omega$. Given $k \in \mathbb{Z}$, write $E_k = \{x \in E : 2^{k-1} < u(x) \le 2^k\}$, and set $E^0 = \{x \in E : u(x) = 0\}$, $E_\infty = \{x \in E : u(x) = \infty\}$.

Consider first E_∞. Let $0 < r < d(E, \partial\Omega)$ and fix $M \ge 1$. For each $x \in E_\infty$, we find $0 < r_x < r$ so that

$$|\overline{B}(x,r)| \le M|f(\overline{B}(x,r_x))|.$$

By the Vitali covering theorem, we find pairwise disjoint balls $\overline{B}_1, \overline{B}_2, \cdots$ as above and so that $E_\infty \subset \cup_j 5\overline{B}_j$. Thus

$$|E_\infty| \le 5^n \sum |\overline{B}_j| \le 5^n M^{-1} |\cup_j f(\overline{B}_j)|.$$

There exists a compact set $F \subset \Omega$, independent of M, so that $\cup_j \overline{B}_j \subset F$. Thus $|\cup_j f(\overline{B}_j)| \le |f(F)| < \infty$. By letting M tend to infinity, we conclude that $|E_\infty| = 0$.

Fix then $\varepsilon > 0$ and let $k \in \mathbb{Z}$. Pick an open set U_k so that $f(E_k) \subset U_k$ and $|U_k| < |f(E_k)| + \varepsilon$. For each $x \in E_k$, pick $0 < r_x < d(E, \partial\Omega)$ so that

$$2^{k-1}|\overline{B}(x,r_x)| \le |f(\overline{B}(x,r_x))|$$

and $f(\overline{B}(x,r_x)) \subset U_k$. Using the Vitali covering theorem as above, we conclude that

$$2^{k-1}5^{-n}|E_k| \le |U_k| < |f(E_k)| + \varepsilon,$$

and letting $\varepsilon \to 0$, we infer that

$$2^{k-1}5^{-n}|E_k| \le |f(E_k)|. \tag{8}$$

Regarding the opposite inequality, we choose an open set U_k containing E_k so that $|U_k| < |E_k| + \varepsilon$. Given $x \in E_k$ pick then r_x so that

$$2^k|\overline{B}(x, r_x)| \ge |f(\overline{B}(x, r_x))|$$

and $\overline{B}(x, r_x) \subset U_k$. By the Besicovitch covering theorem, we find balls $\overline{B}_1, \overline{B}_2, \cdots$ as above and so that

$$\chi_{E_k}(x) \le \sum_j \chi_{\overline{B}_j} \le C_n \chi_{U_k}.$$

Summing over j and letting $\varepsilon \to 0$, we conclude that

$$|f(E_k)| \le 2^k C_n |E_k| \tag{9}$$

Summing over k in (8) and (9), and noticing that $\int_{E^0} u = 0$, we arrive at

$$C_n^{-1}|f(E \setminus E_\infty)| \le \int_E u(x) \le C_n|f(E)|, \tag{10}$$

where C_n depends only on n. Recalling that $|E_\infty| = 0$, we may replace $E \setminus E_\infty$ with E, provided f maps sets of measure zero to sets of measure zero.

One can establish (10) with $C_n = 1$ by substituting a suitable more refined covering theorem [20] for the Besicovitch and Vitali covering theorems above. The almost everywhere existence of the limit in the definition of μ_f' also follows from suitable versions of (8) and (9). The measurability of μ_f' is rutine.

4.3 Remarks.

1) (Lebesgue's differentiation theorem) Let $u \in L^1_{\mathrm{loc}}$. Then

$$\lim_{r \to 0} \fint_{B(x,r)} u(y)\, dy = u(x)$$

for almost every x.

Proof. By considering the positive and negative parts of u separately, we may assume that $u \ge 0$. Define $\mu(E) = \int_E u$ for Lebesgue measurable $E \subset \mathbb{R}^n$. Then μ is a Radon measure and the Radon-Nikodym theorem gives

$$\int_E \lim_{r \to 0} \fint_{B(x,r)} u(y)\, dy\, dx = \int_E u\, dx.$$

Thus the claim follows.

2) The Lebesgue differentiation theorem can be improved to: If $u \in L^p_{\mathrm{loc}}$, $p \geq 1$, then
$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|^p \, dy = 0$$
for almost every x. This follows by applying the Lebesgue differentation theorem to the functions $u_q(y) = |u(y) - q|^p$, $q \in \mathbb{Q}$.

3) Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. From 1), with $u = \chi_E$, we see that
$$\lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1$$
for almost every $x \in E$.

4) The use of balls centered at x in the Lebesgue differentation theorem is not essential. Indeed, consider the collection $\mathcal{Q}$ consisting of all cubes $Q \subset \mathbb{R}^n$. If $u \in L^1_{\mathrm{loc}}$, then, for almost every x,
$$\lim_{j \to \infty} \fint_{Q_j} u(y) \, dy = u(x)$$
whenever $Q_j \in \mathcal{Q}$ satisfy $\cap_j Q_j = \{x\}$. This can be proved, for example, by first noticing that the claim is trivial if u is continuous, approximating a general locally integrable function by continuous ones, and by controlling the error terms via the weak boundedness (as in part 1) of Theorem 2.5) of the maximal operator [26].

4.2   The maximal streching

Set
$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}.$$

4.4 Lemma. Let $f : \Omega \to \Omega'$ be a homeomorphism. The function L_f is Borel measurable and
$$\mu'_f(x) \leq L_f(x)^n \leq H_f(x)^n \mu'_f(x)$$
for almost every $x \in \Omega$. In particular, $L_f \in L^n_{\mathrm{loc}}(\Omega)$ when f is quasiconformal.

Proof.   The Borel measurability of L_f follows from the fact that, given a compact subset E of Ω,
$$\{x \in E : \ L_f(x) < t\} = \bigcup A_i,$$

where the sets

$$A_i = \left\{ x \in E : \frac{|f(x+h) - f(x)|}{|h|} \leq t - \frac{1}{i} \text{ for all } 0 < |h| < d(E, \partial\Omega)/i \right\}$$

are closed by continuity of f. Let $x \in \Omega$, $0 < r < d(x, \partial\Omega)$. Then

$$\frac{|f(\overline{B}(x,r))|}{|B(x,r)|} \leq \left(\frac{L_f(x,r)}{r} \right)^n.$$

Now

$$\left(\frac{L_f(x,r)}{r} \right)^n \leq \left(\frac{L_f(x,r)}{l_f(x,r)} \right)^n \left(\frac{l_f(x,r)}{r} \right)^n \leq \left(\frac{L_f(x,r)}{l_f(x,r)} \right)^n \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}.$$

Hence the claim follows by letting r tend to zero. $\qquad\square$

Notice that, at a point x, where $Df(x)$ exists, $|Df(x)|$ is controlled in terms of $L_f(x)$. However, integrability of L_f does not a priori guarantee absolute continuity of f on almost all lines parallel to the coordinate axes. Indeed, $L_f(x) = 1$ almost everywhere for the homeomorphism f from part 3) of Remarks 2.14, but f is not absolutely continuous on any line parallel to the x-axis. We are thus lead to modify the definition of L_f.

For a homeomorphism $f : \Omega \to \Omega'$ and $\varepsilon > 0$ define

$$L_f^\varepsilon(x) = \sup_{r \leq \varepsilon} \frac{L_f(x,r)}{r}.$$

Then L_f^ε is Borel measurable. Note that $\varepsilon \mapsto L_f^\varepsilon(x)$ is increasing and that $L_f^\varepsilon(x) \to L_f(x)$, as $\varepsilon \to 0$.

It is easy to check that, in dimension one, local integrability of L_f^ε guarantees the absolute continuity of f. The following result is a generalization of this fact.

4.5 Lemma. Let $f : \Omega \to \Omega'$ be a homeomorphism, and let $\varepsilon > 0$. Then

$$|f(x) - f(y)| \leq \int_\gamma 2L_f^\varepsilon \, ds$$

for all rectifiable curves connecting x to y in Ω. In particular, $2L_f^\varepsilon$ is an upper gradient of the component functions f_i of f in Ω, and of the function

$$u(x) = |f(x) - f(x_0)|,$$

whenever $x_0 \in \Omega$ is fixed.

Proof. Fix $x, y \in \Omega$ and let $\gamma = \gamma_0 : [0, l] \to \Omega$, be a rectifiable curve joining x to y. Assume first that $d := \operatorname{diam}(\gamma([0, l])) < \varepsilon$. Let $z \in \gamma([0, l])$. Then $\gamma([0, l]) \subset B(z, d)$ and so

$$|f(x) - f(y)| \leq \operatorname{diam}(f(\gamma([0, l]))) \leq 2L_f(z, d).$$

Hence

$$|f(x) - f(y)| \leq \int_{[0,l]} \frac{2L_f(\gamma(s), d)}{l} \, ds \leq \int_{[0,l]} \frac{2L_f(\gamma(s), d)}{d} \, ds \leq \int_\gamma 2L_f^\varepsilon \, ds.$$

If $d \geq \varepsilon$, choose $0 = t_1 < \cdots < t_k = l$ such that $\operatorname{diam}(\gamma([t_i, t_{i+1}])) < \varepsilon$, for $1 \leq i < k$, and use the triangle inequality.

The rest of the claim follows from the facts that

$$|f_i(x) - f_i(y)| \leq |f(x) - f(y)|,$$

for $1 \leq i \leq n$, and

$$|u(x) - u(y)| = \left| |f(x) - f(x_0)| - |f(y) - f(x_0)| \right| \leq |f(x) - f(y)|.$$

$\square$

The above proof did not employ the fact that f is a homeomorphism. In fact, the conclusion holds for each continuous $f : \Omega \to \mathbb{R}^k$, $k \geq 1$.

We next show that L_f^ε is locally p-integrable for all $p < n$, provided f is quasisymmetric.

4.6 Lemma. Let f be η-quasisymmetric in $2B$, where $B = B(x_0, r_0) \subset \mathbb{R}^n$, and let $0 < \varepsilon < \operatorname{diam}(B)/100$. Then

$$|\{x \in B : L_f^\varepsilon(x) > t\}| \leq [5\eta(1)\eta(2)/t]^n |f(B)|$$

for $t > 0$.

Proof. If $L_f^\varepsilon(x) > t$, then there exists $0 < r_x \leq \varepsilon$ such that

$$\frac{L_f(x, r_x)}{r_x} > t.$$

Write $E_t = \{x \in B : L_f^\varepsilon(x) > t\}$. By the Vitali covering theorem we find pairwise disjoint closed balls $B_1 = \overline{B}(x_1, r_1), B_2 = \overline{B}(x_2, r_2), \ldots$ as above so

that $E_t \subset \bigcup 5\overline{B}_j$. Thus

$$\begin{aligned}
|E_t| &\leq 5^n \sum |B_j| \leq 5^n |B(0,1)| t^{-n} \sum L_f(x_j, r_j)^n \\
&\leq |B(0,1)| \, [5\eta(1)/t]^n \sum l_f(x_j, r_j)^n \\
&\leq [5\eta(1)/t]^n \sum |f(B(x_j, r_j))| \\
&\leq [5\eta(1)/t]^n \, |f(2B)|.
\end{aligned}$$

By quasisymmetry,

$$\begin{aligned}
|f(2B)| &\leq |B(0,1)| L_f(x_0, 2r_0)^n \\
&\leq |B(0,1)| l_f(x_0, r_0)^n \eta(2)^n \\
&\leq |f(B)| \eta(2)^n.
\end{aligned}$$

$\square$

4.7 Lemma. Let f be η-quasisymmetric in $2B$, where $B = B(x_0, r_0) \subset \mathbb{R}^n$, and let $0 < \varepsilon < \operatorname{diam}(B)/100$. Then

$$\fint_B (L_f^\varepsilon)^p \leq C(n, \eta, p) \left(\frac{|f(B)|}{|B|} \right)^{p/n}$$

for $1 \leq p < n$.

Proof. Applying the Cavalieri formula and the previous lemma we see that

$$\begin{aligned}
\int_B (L_f^\varepsilon)^p &= p \int_0^\infty t^{p-1} |\{x \in B : L_f^\varepsilon(x) > t\}| \, dt \\
&= p \left[\int_0^{t_0} + \int_{t_0}^\infty \right] \\
&\leq p \int_0^{t_0} t^{p-1} |B| \, dt + C(n, \eta, p) |f(B)| \int_{t_0}^\infty t^{p-n-1} \, dt \\
&= |B| t_0^p + C(n, \eta, p) |f(B)| t_0^{p-n}.
\end{aligned}$$

Solve for t_0 so that the two terms are equal. $\square$

4.8 Corollary. Let $f : \Omega \to \Omega'$ be quasisymmetric (or quasiconformal), where $\Omega, \Omega' \subset \mathbb{R}^n$ are domains, $n \geq 2$. Then $f \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$: $|f| \in L_{\text{loc}}^n(\Omega)$, the component functions are absolutely continuous on almost all lines parallel to the coordinate axes in Ω, and the classical partial derivatives of the coordinate functions belong to $L_{\text{loc}}^n(\Omega)$.

Recall that absolute continuity of a function $u : \Omega \to \mathbb{R}$ on almost all lines parallel to the coordinate axes in Ω requires that, for $(n-1)-$ almost every $(x_2, \cdots, x_n)$, $u(t, x_2, \cdots, x_n)$ is absolutely continuous on each compact line segment in the x_1-direction in Ω, as a function of t, and analogously when x_1 above is replaced by x_j, $j = 2, \cdots, n$.

Proof. Fix a cube Q with $\overline{Q} \subset \Omega$, and pick $0 < \varepsilon < 1$ so that $L_f^\varepsilon \in L^1(Q)$, see Lemma 4.7. Fix a coordinate direction, say x_1. Fiber Q by line segments parallel to the x_1-axis. Denote $J(x_2, \ldots, x_n) = \{y \in Q : y_2 = x_2, \ldots, y_n = x_n\}$. By the Fubini theorem $L_f^\varepsilon \in L^1(J(x_2, \ldots, x_n))$ for $(n-1)$-almost every $(x_2, \ldots, x_n)$. Let $J = J(x_2, \ldots, x_n)$ be such a line segment. By Lemma 4.5 we have, for $1 \le j \le n$,

$$|f_j(t_1, x_2, \ldots, x_n) - f_j(t_2, x_2, \ldots, x_n)| \le \int_{J(t_1, t_2)} 2 L_f^\varepsilon \, ds,$$

where $J(t_1, t_2) = \{x \in J : t_1 \le x_1 \le t_2\}$. Since Lebesgue integral is absolutely continuous with respect to Lebesgue measure, it follows that f_j is absolutely continuous on J and that $\partial_1 f_j(x)$ exists at almost every $x \in J$. Furthermore,

$$|\partial_1 f_j(x)| \le L_f(x),$$

for such points. The above clearly shows that f_j is absolutely continuous on almost all lines parallel to the coordinate axes in Ω. Next, from Lemma 4.4 we know that $L_f \in L^n_{\text{loc}}(\Omega)$. Because a quasiconformal mapping is locally quasisymmetric by Corollary 3.4, the claim follows. $\square$

4.9 Remark. The previous results do not allow us to conclude that a quasisymmetric mapping of the real line onto itself is absolutely continuous. Indeed, in the proof of Lemma 4.7 we only obtain the p-integrability of L_f^ε for $p < 1$ and thus Lemma 4.5 gives no estimate on the oscillation of f. This does not mean any weakness in our technique because one can give examples of quasisymmetric mappings $f : \mathbb{R} \to \mathbb{R}$ that fail to be absolutely continuous.

Next we will show that $L_f \in L^p_{\text{loc}}(\Omega)$ for some $p = p(n, H) > n$.

4.3   Gehring's lemma

The following result is the starting point for the higher integrability of L_f.

4.10 Lemma. (Reverse Hölder Inequality) Let f be η-quasisymmetric on $2B \subset \mathbb{R}^n$. Then

$$\left(\fint_B L_f^n\right)^{1/n} \leq C(n,\eta)\fint_B L_f.$$

Proof. There is nothing to be proved when $n = 1$. Thus assume that $n \geq 2$. Let $\varepsilon > 0$ be small. Suppose $B = B(x_0, r_0)$. Define $u(x) = |f(x) - f(x_0)|$. Then, by Lemma 4.5, $2L_f^\varepsilon$ is an upper gradient of u and thus, by the Poincaré inequality,

$$\fint_B |u - u_B| \leq C(n)r_0\fint_B L_f^\varepsilon.$$

Since L_f^ε is locally integrable, the monotone convergence theorem implies that

$$\fint_B |u - u_B| \leq C(n)r_0\fint_B L_f. \tag{11}$$

Now

$$u_B = \fint_B |f(x) - f(x_0)| \geq \frac{1}{|B|}\int_{B\backslash\frac{1}{2}B} |f(x) - f(x_0)| \geq \frac{2^{-n}}{\eta(2)}L_f(x_0, r_0),$$

and there is a $\delta = \delta(n,\eta) > 0$ such that

$$u(x) = |f(x) - f(x_0)| \leq \frac{2^{-n}}{2\eta(2)}L_f(x_0, r_0),$$

whenever $x \in \delta B$. Thus

$$\fint_B |u - u_B| \geq \frac{1}{|B|}\int_{\delta B} (u_B - u) \geq C(n,\eta)L_f(x_0, r_0).$$

This, combined with (11), gives

$$L_f(x_0, r_0) \leq C(n,\eta)r_0\fint_B L_f. \tag{12}$$

So, by Lemma 4.4 and Proposition 4.1,

$$\left(\fint_B L_f^n\right)^{1/n} \leq C(n,\eta)\left(\fint_B \mu_f'\right)^{1/n} \leq C(n,\eta)\frac{|f(B)|^{1/n}}{r_0}$$
$$\leq C(n,\eta)\frac{L_f(x_0, r_0)}{r_0} \leq C(n,\eta)\fint_B L_f.$$

$\square$

4.11 Remarks.

1) Applying Hölder's inequality to the right hand side of (12) we obtain the estimate
$$L_f(x_0, r_0) \leq C(n, \eta) r_0 \left(\fint_B L_f^p \right)^{1/p}$$
for $p \geq 1$. In particular, with $p = n$, we have
$$L_f(x_0, r_0) \leq C(n, \eta) \left(\int_B L_f^n \right)^{1/n}.$$

2) If X is Q-regular and if we have a p-Poincaré inequality for some $p < Q$, then the proof of Lemma 4.10 gives
$$\left(\fint_B L_f^Q \right)^{1/Q} \leq C(\text{data}) \left(\fint_B L_f^p \right)^{1/p}$$
when f is η-quasisymmetric.

3) The reverse Hölder inequality also holds with balls replaced by cubes (assuming that f is quasisymmetric on $\sqrt{n}Q$). Indeed, let $B = B(x_0, r_0) \subset Q$, where the edge length of Q is $\operatorname{diam}(B)$. Then $Q \subset \sqrt{n}B$. By 1),
$$L_f(x_0, r_0) \leq C r_0 \fint_B L_f,$$
and by quasisymmetry
$$\operatorname{diam}(f(Q)) \leq 2\eta(\sqrt{n}) L_f(x_0, r_0).$$
Following the proof of Lemma 4.10 we see that
$$\left(\fint_Q L_f^n \right)^{1/n} \leq C \frac{\operatorname{diam}(f(Q))}{r_0},$$
and we conclude that
$$\left(\fint_Q L_f^n \right)^{1/n} \leq C \fint_B L_f \leq C \fint_Q L_f.$$

One can also verify the reverse Hölder inequality directly for cubes, without using the Poincaré inequality. Let us sketch this in dimension

two. Suppose that f is η-quasisymmetric on Q. Assume for notational simplicity that $Q = [-1, 1]^2$. By quasisymmetry,

$$\operatorname{diam}(f(Q)) \leq 2\eta(\sqrt{2})|f(1,t) - f(0,0)| \leq 2\eta(1)\eta(\sqrt{2})|f(1,t) - f(-1,t)|$$

for each $-1 \leq t \leq 1$. As at the end of the proof of Lemma 4.10, we have that

$$\left(\fint_Q L_f^2 \right)^{1/2} \leq C(n, \eta) \operatorname{diam}(f(Q)).$$

The claim follows by noticing (see the proof of Corollary 4.8)

$$|f(1,t) - f(-1,t)| \leq \int_{J_t} 2L_f \, ds$$

for almost every $-1 \leq t \leq 1$, where J_t is the line segment between the points $(1,t)$ and $(-1,t)$, and then integrating with respect to t.

As the first consequence of the reverse Hölder inequality we show that quasiconformal mappings preserve the class of sets of measure zero.

4.12 Corollary. Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$. Then $|f(E)| = 0$, if and only if $|E| = 0$. In particular,

$$|f(E)| = \int_E \mu'_f \, dx,$$

for Borel (and all Lebesgue measurable) sets E, and f maps Lebesgue measurable sets to Lebesgue measurable sets. Moreover, $\mu'_f(x) > 0$ almost everywhere.

Proof. Let $|E| = 0$. We may assume that E is bounded and $\overline{E} \subset \Omega$. Pick open $U \supset E$ so that $U \subset\subset \Omega$. Then $L_f \in L^n(U)$ by Lemma 4.4. Given $\varepsilon > 0$, we further find an open set V with $E \subset V \subset U$ and $|V| < \varepsilon$. For each $x \in E$, pick a ball $\overline{B}(x, r_x)$ so that $\overline{B}(x, 15r_x) \subset V$. By the Vitali covering theorem, we find such balls $\overline{B}_1, \overline{B}_2, \ldots$ so that $\overline{B}_i \cap \overline{B}_j = \emptyset$ when $i \neq j$ and $E \subset \cup 5\overline{B}_j$. Then $f(E) \subset f(\cup 5\overline{B}_j)$ and

$$|f(\cup 5\overline{B}_j)| \leq \sum |f(5\overline{B}_j)| \leq \eta(5)C(n) \sum L_f(x_j, r_j)^n$$

and so, by part 1) of Remark 4.11,

$$|f(\cup 5\overline{B}_j)| \leq C(n, \eta) \sum \int_{B_j} L_f^n = C(n, \eta) \int_{\cup B_j} L_f^n \leq C(n, \eta) \int_V L_f^n.$$

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Letting $\varepsilon \to 0$, we conclude that $|f(E)| = 0$. The "only if" part follows from the fact that f^{-1} is also quasiconformal. By the Radon-Nikodym theorem,

$$|f(E)| = \int_E \mu'_f \, dx \tag{13}$$

for all Borel sets E. Let $E \subset \Omega$ be Lebesgue measurable. Pick a Borel set $F \supset E$ so that $|F \setminus E| = 0$. Then $f(F)$ is a Borel set, $f(E) \subset f(F)$ and $|f(F) \setminus f(E)| = |f(F \setminus E)| = 0$. It follows that $f(E)$ is Lebesgue measurable and that (13) holds also for E. Suppose finally that $\mu'_f(x) = 0$ in E with $|E| > 0$. Then

$$|f(E)| = \int_E \mu'_f \, dx = 0,$$

which contradicts the fact that $|f(E)| = 0$ if and only if $|E| = 0$. $\qquad \square$

We continue with a powerful tool from harmonic analysis, the Calderón-Zygmund decomposition, and some consequences of this decomposition.

The dyadic decomposition of a cube Q_0 consists of open cubes $Q \subset Q_0$ with faces parallel to the faces of Q_0 and of edge length $l(Q) = 2^{-i}l(Q_0)$, where $i = 1, 2, \ldots$ refer to the generation in the construction. The cubes in each generation cover Q_0 up to a set of measure zero and the closures of the cubes in a fixed generation cover Q_0; there are 2^{in} cubes of edge length $2^{-i}l(Q_0)$ in the ith generation and the cubes corresponding to the same generation are pairwise disjoint. For almost every $x \in Q_0$, there is a (unique) decreasing sequence $Q_0 \supset Q_1 \supset \ldots$ of cubes in the dyadic decomposition so that $\{x\} = \bigcap Q_i$. In what follows, Q, Q_0, Q_x etc. are cubes.

4.13 Theorem. (Calderón-Zygmund decomposition) Let $Q_0 \subset \mathbb{R}^n$, $u \in L^1(Q_0)$, and suppose that

$$t \geq \fint_{Q_0} u \geq 0.$$

Then there is a subcollection $\{Q_j\}$ from the dyadic decomposition of Q_0 so that $Q_i \cap Q_j = \emptyset$ when $i \neq j$,

$$t < \fint_{Q_j} u \leq 2^n t$$

for each j, and $u(x) \leq t$ for almost every $x \in Q_0 \setminus \bigcup Q_j$.

Proof. For almost every $x \in Q_0$ there is a decreasing sequence $\{Q_j\}$ of dyadic cubes so that $\{x\} = \bigcap Q_j$. By the Lebesgue differentiation theorem

(see part 3) of Remarks 4.3)

$$\lim_{j \to \infty} \fint_{Q_j} u = u(x)$$

for almost every such x. Let $u(x) > t$ and assume that the above holds for x with the sequence $\{Q_j\}$. Then there must be maximal $Q_x := Q_{j(x)}$ so that

$$\fint_{Q_x} u > t.$$

For this cube we have

$$t < \fint_{Q_x} u \le 2^n \fint_{Q_{j(x)-1}} u \le 2^n t.$$

We can pick such a cube Q_x for almost every x with $u(x) > t$. It is then easy to choose the desired subcollection from the cubes Q_x. $\qquad\square$

The dyadic maximal function of a measurable function u (with respect to a cube Q_0) is defined by

$$M_{Q_0} u(x) = \sup_{x \in \overline{Q} \subset Q_0} \fint_Q |u|,$$

where the supremum is taken over all cubes Q that belong to the dyadic decomposition of Q_0 and whose closures contain x.

4.14 Remark. As for the usual maximal function, we have the weak type estimate

$$|\{x \in Q_0 : \ M_{Q_0} u(x) > t\}| \le \frac{2 \cdot 5^n}{t} \int_{\{x \in Q_0 : |u(x)| > \frac{t}{2}\}} |u|$$

for the dyadic maximal function. Moreover,

$$\int_{Q_0} (M_{Q_0} u)^p \le C(p, n) \int_{Q_0} |u|^p$$

for $p > 1$. The proof of the weak type estimate is actually easier than for the usual maximal operator because no covering theorem is needed.

The following simple consequence of the Calderón-Zygmund decomposition is essentially the converse of the weak type estimate for the dyadic maximal function.

4.15 Lemma. Let $u \in L^1(Q_0)$ and suppose $t \geq \fint_{Q_0} |u|$. Then

$$\int_{\{x \in Q_0 : |u(x)| > t\}} |u| \leq 2^n t |\{x \in Q_0 : M_{Q_0} u(x) > t\}|.$$

Proof. By the Calderón-Zygmund decomposition we find pairwise disjoint cubes $Q_1, Q_2, \ldots$ so that

$$t < \fint_{Q_j} |u| \leq 2^n t$$

for all j, and $|u(x)| \leq t$ almost everywhere in $Q_0 \setminus \bigcup Q_j$. Then

$$\int_{\{x \in Q_0 : |u(x)| > t\}} |u| \leq \sum \int_{Q_j} |u|$$
$$\leq \sum 2^n t |Q_j|$$
$$\leq 2^n t |\{x \in Q_0 : M_{Q_0} u(x) > t\}|,$$

because

$$M_{Q_0} u(x) \geq \fint_{Q_j} |u| > t$$

for each $x \in Q_j$. □

We are now ready to prove an important result. For historical reasons, it is only called a lemma (Gehring's lemma). I learned the truncation trick employed in the proof below from Xiao Zhong.

4.16 Lemma. (Gehring's lemma, 1973) Let $u \in L^q(Q_0)$, $1 < q < \infty$ and suppose that

$$\left(\fint_Q |u|^q \right)^{1/q} \leq C \fint_Q |u| \tag{14}$$

for all dyadic subcubes $Q \subset Q_0$. Then there is $s = s(q, n, C) > q$ so that

$$\left(\fint_{Q_0} |u|^s \right)^{1/s} \leq 2^{1/s} C \fint_{Q_0} |u|. \tag{15}$$

In particular, $u \in L^s(Q_0)$.

Proof. We begin by noticing that

$$M_{Q_0}(|u|^q)(x) \leq C^q M_{Q_0} u(x) \tag{16}$$

for each $x \in Q$. Let then $t \geq t_0 := \fint_{Q_0} |u|^q$. Combining Lemma 4.15, (16) and the weak type estimate from Remark 4.14, we conclude that

$$\int_{\{x \in Q_0 : |u(x)|^q > t\}} |u|^q \leq 2^n t |\{x \in Q_0 : M_{Q_0}(|u|^q)(x) > t\}|$$

$$\leq 2^n t |\{x \in Q_0 : M_{Q_0} u(x) > C^{-q} t^{1/q}\}|$$

$$\leq 2^{n+1} 5^n t^{1-\frac{1}{q}} \int_{\{x \in Q_0 : |u(x)| > \frac{1}{2} C^{-q} t^{1/q}\}} |u|,$$

provided $t \geq t_0$. Consequently, for these values of t,

$$\int_{\{x \in Q_0 : |u(x)|^q > t\}} |u|^q \leq C_n t^{1-\frac{1}{q}} \int_{\{x \in Q_0 : |u(x)| > \delta t^{1/q}\}} |u|, \tag{17}$$

where C_n depends only on n and $\delta = 2^{-1} C^{-q}$. Multiplying both sides of (17) by t^{p-2} and integrating over the interval $[t_0, j]$, where $j > t_0$ is fixed, results in

$$\int_{t_0}^{j} t^{p-2} \int_{\{x \in Q_0 : |u(x)|^q > t\}} |u|^q \, dx dt \leq \int_{t_0}^{j} t^{p-1-1/q} \int_{\{x \in Q_0 : |u(x)| > \delta t^{1/q}\}} |u| \, dx dt. \tag{18}$$

Write $b(j, s, u(x)) = \min\{j, s|u(x)|^q\}$ when $s > 0$. Notice that

$$b(j, s, u(x)) \leq s b(j, 1, u(x))$$

when $s \geq 1$. By the Fubini theorem,

$$\int_{t_0}^{j} t^{p-1-1/q} \int_{\{x \in Q_0 : |u(x)| > \delta t^{1/q}\}} |u| \, dx dt$$

$$= \int_{Q_0} |u| \int_{t_0}^{b(j, \delta^{-q}, u(x))} t^{p-1-1/q} \, dt dx$$

$$\leq q(pq-1)^{-1} \int_{Q_0} b(j, \delta^{-q}, u(x))^{p-1/q} |u| \, dx$$

$$\leq q(pq-1)^{-1} \delta^{1-pq} \int_{Q_0} b(j, 1, u(x))^{p-1/q} |u| \, dx$$

$$\leq q(pq-1)^{-1} \delta^{1-pq} \int_{Q_0} b(j, 1, u(x))^{p-1} |u|^q.$$

Similarly,

$$\int_{t_0}^{j} t^{p-2} \int_{\{x \in Q_0 : |u(x)|^q > t\}} |u|^q \, dx dt = \int_{Q_0} |u|^q \int_{t_0}^{b(j, 1, u(x))} t^{p-2} \, dt dx$$

$$= (p-1)^{-1} \int_{Q_0} \left(b(j, 1, u(x))^{p-1} - t_0^{p-1} \right) |u|^q \, dx.$$

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Combining the above estimates for the left and right hand sides of (18) we conclude that

$$\fint_{Q_0} \min\{j, |u(x)|^q\}^{p-1} |u(x)|^q \le C' (\fint_{Q_0} |u|^q)^p \le C' C^{pq} (\fint_{Q_0} |u|)^{pq}$$

where $C' = ((p-1)^{-1} - q(pq-1)^{-1} \delta^{1-pq})^{-1}$, provided $C' > 0$. We used (14) at the last step. Choosing $p > 1$ so that $C' = 2$ allows us to conclude the claim via the monotone convergence theorem. $\qquad\qquad\square$

Given a domain $\Omega \subset \mathbb{R}^n$ and $1 \le p \le \infty$, we let $W^{1,p}(\Omega)$ denote the collection of all functions $u \in L^p(\Omega)$ that are absolutely continuous on almost all lines parallel to the coordinate axes in Ω and whose classical partial derivatives belong to $L^p(\Omega)$. Then $W^{1,p}(\Omega, \mathbb{R}^n)$ refers to mappings $f : \Omega \to \mathbb{R}^n$ whose each component function f_j, $j = 1, \cdots, n$, belongs to $W^{1,p}(\Omega)$. The definitions of $W^{1,p}_{\mathrm{loc}}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ should then be obvious.

4.17 Corollary. Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \ge 2$. There is $p = p(n, H) > n$ and a constant $C = C(n, p, H)$ so that

1) $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and

$$\left(\fint_Q L_f^p \right)^{1/p} \le C \left(\fint_Q L_f^n \right)^{1/n}$$

whenever $2Q \subset \Omega$.

2) If $2Q \subset \Omega$ and $E \subset Q$ is measurable, then

$$\frac{|f(E)|}{|f(Q)|} \le C \left(\frac{|E|}{|Q|} \right)^{1-n/p}.$$

Proof. 1) By Remark 4.11 we have

$$\left(\fint_Q L_f^n \right)^{1/n} \le C \fint_Q L_f$$

whever $2Q \subset \Omega$. The Sobolev regularity and the asserted inequality follow from Gehring's lemma because f is absolutely continuous on almost all lines parallel to the coordinate axes and

$$|\partial_j f_i(x)| \le L_f(x)$$

for almost every x, see Corollary 4.8 and its proof.

2) By Corollary 4.12, Lemma 4.4, Hölder's inequality, Proposition 4.1, and part 1) we see that

$$
\begin{aligned}
|f(E)| = \int_E \mu'_f &\leq C \int_E L_f^n \\
&\leq C \left(\int_E L_f^p \right)^{n/p} |E|^{1-n/p} \\
&\leq C \left(\fint_Q L_f^p \right)^{n/p} |E|^{1-n/p} |Q|^{n/p} \\
&\leq C \fint_Q \underbrace{L_f^n}_{\leq C\mu'_f} |E|^{1-n/p} |Q|^{n/p} \\
&\leq C |f(Q)| |E|^{1-n/p} |Q|^{n/p-1}.
\end{aligned}
$$

$\square$

4.4 A_p-weights

We will briefly point out the connection between A_p-weights and reverse Hölder inequalities. The results of this section will not be needed later on. We refer the reader to [26] for proofs of the facts presented in this section.

Let $w \in L^1_{\mathrm{loc}}$, $w > 0$ almost everywhere. If $-\infty < s < t < \infty$ and $|E| > 0$, then

$$
\left(\fint_E w^s \right)^{1/s} \leq \left(\fint_E w^t \right)^{1/t}.
$$

So, when $p > 1$, we have that

$$
\left(\fint_B w^{1/(1-p)} \right)^{1-p} \leq \left(\fint_B w^p \right)^{1/p}
$$

for each ball B. We say that w is an A_p-weight (belongs to the Muckenhoupt A_p-class), if for all balls

$$
\left(\fint_B w^p \right)^{1/p} \leq C_{p,w} \left(\fint_B w^{1/(1-p)} \right)^{1-p},
$$

when $1 < p < \infty$, and

$$
\fint_B w \leq C_{1,w} \mathrm{essinf}_B w,
$$

when $p = 1$. Clearly $A_1 \subset A_p \subset A_q$ when $1 \leq p \leq q$. We finally set $A_\infty = \bigcup_{p>1} A_p$.

One of the connections between A_p-classes and reverse Hölder inequalities is given by the following result.

4.18 Fact. Let $w \in L^1_{\text{loc}}$, $w > 0$ almost everywhere. Then $w \in A_\infty$ if and only if there exist $q > 1$ and C such that

$$\left(\fint_B w^q \right)^{1/q} \leq C \fint_B w$$

for all balls B.

4.19 Corollary. Let $n \geq 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal, then $\mu_f \in A_\infty$.

Given $w \in A_p$, $p > 1$, one can use the above reverse Hölder inequality to prove that $w \in A_q$ for some $q < p$ that depends on $n, p, C_{p,w}$.

A_p-weights are of their own interest. One of their important properties is that they work well with maximal functions.

4.20 Fact. Let $1 < p < \infty$. The inequality

$$\int (Mu)^p w \leq C \int |u|^p w$$

holds for each Lebesgue measurable u if and only if $w \in A_p$.

One can further characterize A_∞ by the following condition. There are constants C and δ so that

$$\frac{\int_E w}{\int_Q w} \leq C \left(\frac{|E|}{|Q|} \right)^\delta \tag{19}$$

for each cube Q and each measurable $E \subset Q$. Given a domain G, let us write $A_\infty(G)$ for the collection of all w for which (19) holds with uniform constants for each cube $Q \subset G$ with $\text{diam}(Q) \leq d(Q, \partial G)$. Then, in dimensions $n \geq 2$, a homeomorphism $f : \Omega \to \Omega'$ is quasiconformal if and only if, for each subdomain $G \subset \Omega$, $w \circ f^{-1} \in A_\infty(f(G))$ for each $w \in A_\infty(G)$ and $w \circ f \in A_\infty(G)$ for each $w \in A_\infty(f(G))$ with uniform bounds in both cases. For this see [25].

4.5 Differentiability almost everywhere

We begin with an almost everywhere differentiability result that goes back to Cesari and Calderón. Recall that $u \in W_{\text{loc}}^{1,p}(\Omega)$ means that u is locally p-integrable, absolutely continuous on almost all lines parallel to the coordinate axes in Ω and that the classical partial derivatives are locally p-integrable.

4.21 Theorem. Let $p > n$ and let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be continuous. Then u is differentiable almost everywhere.

This result is optimal in the sense that there exist continuous functions in $W_{\text{loc}}^{1,n}$ that are nowhere differentiable.

We need a few technical results for the proof of this theorem.

4.22 Lemma. Let $u \in W_{\text{loc}}^{1,1}(\Omega)$ and $\Omega_0 \subset\subset \Omega$. Given $0 < r < d(\Omega_0, \partial\Omega)$, set

$$u_r(x) = \fint_{B(x,r)} u(y)\, dy$$

for $x \in \Omega_0$. Then $u_r \in C^1(\Omega_0)$ and

$$\nabla u_r(x) = \fint_{B(x,r)} \nabla u(y)\, dy.$$

Proof. Fix $0 < r < d(\Omega_0, \partial\Omega)$, $x \in \Omega_0$ and $1 \le j \le n$. Let $0 < |t| < d(\Omega_0, \partial\Omega) - r$. By the absolute continuity of u on almost all lines parallel to the x_j-axis in Ω,

$$u(y + te_j) - u(y) = \int_{[0,t]} \partial_j u(y + se_j)\, ds$$

for almost all $y \in B(x, r)$. Integrating this estimate and invoking the Fubini theorem we infer that

$$\frac{u_r(x + te_j) - u_r(x)}{t} = \fint_{B(x,r)} \frac{u(y + te_j) - u(y)}{t}\, dy$$

$$= \fint_{B(x,r)} \fint_{[0,t]} \partial_j u(y + se_j)\, ds\, dy$$

$$= \fint_{[0,t]} \fint_{B(x,r)} \partial_j u(y + se_j)\, dy\, ds.$$

$$= \fint_{[0,t]} \underbrace{\fint_{B(x+se_j,r)} \partial_j u(y)\, dy}_{=:f(s)}\, ds.$$

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Since $\partial_j u \in L^1(\Omega)$, it follows that f is continuous. Hence

$$\partial_j u_r(x) = \lim_{t \to 0} \fint_{[0,t]} f(s) \, ds = f(0) = \fint_{B(x,r)} \partial_j u(y) \, dy.$$

$\square$

4.23 Lemma. Suppose that $v \in L^p(\lambda B)$, $1 \le p < \infty$, where $\lambda > 1$. Given $0 < \varepsilon < d(B, \lambda B^c)$, set

$$v_\varepsilon(x) = \fint_{B(x,\varepsilon)} v(y) \, dy$$

for $x \in B$. Then $v_\varepsilon \to v$ in $L^p(B)$.

Proof. Let $w \in L^p(\lambda B)$. Let $0 \le \psi_\varepsilon \in L^\infty$ be such that $\int \psi_\varepsilon = 1$ and $\mathrm{spt}\,\psi_\varepsilon \subset \overline{B}(0,\varepsilon)$. Extend w as zero to $\mathbb{R}^n \setminus \lambda B$. Then

$$w_{\psi_\varepsilon} := \int_{\mathbb{R}^n} \psi_\varepsilon(y) w(x - y) \, dy$$

is bounded on B:

$$|w_{\psi_\varepsilon}(x)| \le \|\psi_\varepsilon\|_{L^\infty} \int_{\lambda B} |w|.$$

Choose now

$$\psi_\varepsilon(y) = \frac{1}{|B(0,\varepsilon)|} \chi_{B(0,\varepsilon)}(y)$$

and write $w_\varepsilon = w_{\psi_e}$. By the Hölder inequality,

$$|w_\varepsilon| = \int_{\mathbb{R}^n} \psi_\varepsilon(y)^{1/p} |w(x-y)| \psi_\varepsilon(y)^{(p-1)/p} \le \left(\int_{\mathbb{R}^n} \psi_\varepsilon(y) |w(x-y)|^p \, dy \right)^{1/p}$$

and so

$$\int_B |w_\varepsilon|^p \le \int_{\lambda B} \int_{\mathbb{R}^n} \psi_\varepsilon(y) |w(x-y)|^p \, dy \, dx$$

$$= \int_{\mathbb{R}^n} \psi_\varepsilon(y) \int_{\lambda B} |w(x-y)|^p \, dx \, dy$$

$$\le \int_{\lambda B} |w|^p.$$

If w is continuous on λB, then

$$\|w - w_\varepsilon\|_{L^p(B)} \to 0,$$

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as $\varepsilon \to 0$. Let $\delta > 0$. Recall that continuous functions are dense in $L^p(\lambda B)$, see Subsection 11.3 in the appendix. Choose a continuous w such that

$$\|v - w\|_{L^p(\lambda B)} < \delta,$$

and take $\varepsilon > 0$ so small that

$$\|w - w_\varepsilon\|_{L^p(B)} < \delta.$$

Then

$$\|v - v_\varepsilon\|_{L^p(B)} \leq \|v - w\|_{L^p(B)} + \|w - w_\varepsilon\|_{L^p(B)} + \|\underbrace{w_\varepsilon - v_\varepsilon}_{=(w-v)_\varepsilon}\|_{L^p(B)} < 3\delta.$$

Thus $v_\varepsilon \to v$ in $L^p(B)$. $\qquad\qquad\square$

4.24 Corollary. If $u \in W^{1,1}(B)$, then

$$\int_B |u - u_B|\, dx \leq C \operatorname{diam}(B) \int_B |\nabla u|\, dx.$$

Proof. Let $0 < \delta < 1$. Then, for $0 < r < \delta/2$, u_r is well defined and C^1 in $(1-\delta)B$. Thus, by the usual Poincaré inequality,

$$\int_{(1-\delta)B} |u_r(x) - (u_r)_{(1-\delta)B}|\, dx \leq C(1-\delta)\operatorname{diam}(B) \int_{(1-\delta)B} |\nabla u_r(x)|\, dx.$$

By letting $r \to 0$ we see that this inequality holds for u (v_r tends to v in L^1 when $v \in L^1$ and $r \to 0$ by Lemma 4.23). The claim follows by letting $\delta \to 0$; notice that $u_{(1-\delta)B} \to u_B$. $\qquad\square$

4.25 Corollary. Let $u \in W^{1,p}(5B)$ and let $p > n$. Then

$$|u(x) - u(y)| \leq C(n, p)|x - y|^{1-n/p} \left(\int_{B(x, 2|x-y|)} |\nabla u|^p \right)^{1/p}$$

for all Lebesgue points $x, y \in B$ of u.

Proof. Let $x, y \in B$ be Lebesgue points of u. Define $B_i = B(x, 2^{-i}|x - y|)$ for $i \geq 0$. Then, by Corollary 4.24 and the Hölder inequality,

$$|u(x) - u_{B_0}| \leq \sum_{i=1}^{\infty} |u_{B_{i-1}} - u_{B_i}|$$

$$\leq 2^n \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|$$

$$\leq C(n, p) \sum_{i=0}^{\infty} 2^{-i}|x - y| \left(\fint_{B_i} |\nabla u|^p \right)^{1/p}$$

$$\leq C(n, p) \sum_{i=0}^{\infty} (2^{-i}|x - y|)^{1-n/p} \left(\int_{B_i} |\nabla u|^p \right)^{1/p}$$

$$\leq C(n, p)|x - y|^{1-n/p} \left(\int_{B(x,|x-y|)} |\nabla u|^p \right)^{1/p}.$$

Similarly,

$$|u(y) - u_{B(y,|x-y|)}| \leq C(n, p)|x - y|^{1-n/p} \left(\int_{B(y,|x-y|)} |\nabla u|^p \right)^{1/p}.$$

Moreover, denoting $B_x = B(x, |x - y|)$, $B_y = B(y, |x - y|)$ and $\Delta = B_x \cap B_y$, we have

$$|u_{B_x} - u_{B_y}| \leq |u_{B_x} - u_\Delta| + |u_\Delta - u_{B_y}|$$

$$\leq \fint_\Delta |u - u_{B_x}| + \fint_\Delta |u - u_{B_y}|$$

$$\leq C(n) \left(\fint_{B_x} |u - u_{B_x}| + \fint_{B_y} |u - u_{B_y}| \right)$$

$$\leq C(n, p)|x - y|^{1-n/p} \left(\fint_{B(x,2|x-y|)} |\nabla u|^p \right)^{1/p}.$$

The claim follows by the triangle inequality. $\qquad\square$

4.26 Remarks.

1) If $u \in W^{1,p}_{\text{loc}}(\Omega)$, $p > n$, then

$$\tilde{u}(x) = \limsup_{r \to 0} \fint_{B(x,r)} u$$

is continuous and satisfies the modulus of continuity given in the corollary. This easily follows from the previous corollary. Notice that, by the Lebesgue differentiation theorem, $\tilde{u} = u$ almost everywhere. We call $\tilde{u}$ the continuous representative of u. A function $u \in W_{\text{loc}}^{1,n}(\Omega)$ does not need to have a continuous representative when $n > 1$. An example of this is $u(x) = \log\log|x|^{-1}$, $|x| < e^{-1}$.

2) The continuous representative $\tilde{u}$ belongs to $W_{\text{loc}}^{1,p}(\Omega)$: By the continuity of $\tilde{u}$ and the fact that $\tilde{u} = u$ almost everywhere, we have that

$$\tilde{u}(x) = \lim_{r\to 0}(\tilde{u})_r(x) = \lim_{r\to 0}u_r(x)$$

for all x. By Lemma 4.22, $\partial_j(u_r)(x) = (\partial_j u)_r(x)$ for all x and $1 \le j \le n$. Fix a cube $Q \subset\subset \Omega$ and $1 \le j \le n$. Since $(\partial_j u)_r \to \partial_j u$ in $L^1(Q)$, it follows that $\int_J (\partial_j u)_r \to \int_J \partial_j u$ for almost every line segment $J \subset Q$ parallel to the x_j-axis. Let J be such a line segment with endpoints x and y. Then

$$\tilde{u}(x) - \tilde{u}(y) = \lim_{r\to 0}(u_r(x) - u_r(y)) = \lim_{r\to 0}\int_J \partial_j(u_r) = \lim_{r\to 0}\int_J (\partial_j u)_r = \int_J \partial_j u.$$

It follows that $\tilde{u}$ is absolutely continuous on almost all lines in Ω and that $\partial_j\tilde{u} = \partial_j u$ almost everywhere, as desired.

We are now ready to prove Theorem 4.21.

Proof. Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be continuous. Then, by part 2) of Remark 4.3, at almost every x_0, $\nabla u(x_0)$ exists and

$$\lim_{r\to 0}\fint_{B(x_0,r)}|\nabla u(x) - \nabla u(x_0)|^p\,dx = 0.$$

Fix such an x_0 and define

$$w(x) = u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0).$$

Then $w \in W_{\text{loc}}^{1,p}(\Omega)$ and $\nabla w(x) = \nabla u(x) - \nabla u(x_0)$ whenever $\nabla u(x)$ exists. By Corollary 4.25,

$$|w(x) - w(x_0)| \le C(n,p)|x - x_0|\left(\fint_{B(x_0,5|x-x_0|)}|\nabla u(y) - \nabla u(x_0)|^p\,dy\right)^{1/p}.$$

Thus

$$\lim_{x\to x_0}\frac{|u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|}{|x - x_0|} = \lim_{x\to x_0}\frac{|w(x) - w(x_0)|}{|x - x_0|} = 0.$$

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$\square$

4.27 Remark. By Theorem 4.21, Lipschitz functions are differentiable almost everywhere. This immediately implies that each Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is almost everywhere differentiable.

Given a domain $\Omega \subset \mathbb{R}^n$, recall that $W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^n)$ denotes the collection of mappings $f : \Omega \to \mathbb{R}^n$ whose each component function f_j, $j = 1, \cdots, n$, belongs to $W_{\mathrm{loc}}^{1,p}(\Omega)$.

4.28 Corollary. Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, are domains. Then f belongs to $W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^n)$ for some $p > n$ and, for almost every $x \in \Omega$, f is differentiable at x with $J_f(x) \neq 0$ and satisfies
$$|Df(x)|^n \leq H_f(x)^{n-1}|J_f(x)|.$$

Proof. By Corollary 4.17 and Theorem 4.21 applied to the coordinate functions of f, f belongs to $W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^n)$, for some $p > n$, and is differentiable almost everywhere.

Suppose that f is differentiable at x_0 and that $J_f(x_0) = \det Df(x_0) = 0$. Then
$$|f(B(x_0, r))| \leq (|Df(x_0)| + \varepsilon(r))^{n-1}\, r^{n-1}\varepsilon(r)r,$$
where $\varepsilon(r) \to 0$, as $r \to 0$. Thus
$$\mu_f'(x_0) = \lim_{r \to 0} \frac{|f(\overline{B}(x_0, r))|}{|B(x_0, r)|} = 0.$$

Because $\mu_f' > 0$ almost everywhere by Corollary 4.12 and f is differentiable almost everywhere, $J_f \neq 0$ almost everywhere.

Suppose that f is differentiable at x_0 with $J_f(x_0) = \det Df(x_0) \neq 0$. Then
$$|Df(x_0)| \leq H_f(x_0) \min_{|h|=1} |Df(x_0)h|.$$

Because
$$|J_f(x_0)| \geq \left(\min_{|h|=1} |Df(x_0)h| \right)^{n-1} |Df(x_0)|,$$
see Subsection 11.2 we conclude that
$$|Df(x_0)|^n \leq H_f(x_0)^{n-1}|J_f(x_0)|.$$

$\square$

4.29 Remarks.

1) The exponent $n - 1$ for H in Corollary 4.28 is optimal. This is seen by considering the quasiconformal mapping $f(x) = Ax$, where A is a diagonal matrix whose diagonal entrees are all 1 expect for a single entry which is, say, 2.

2) If $f : \Omega \to \Omega'$, both domains in $\mathbb{R}^n$, is a homeomorphism and differentiable at $x, y \in \Omega$, then either $J_f(x) \geq 0$ and $J_f(y) \geq 0$ or $J_f(x) \leq 0$ and $J_f(y) \leq 0$. This can be proved using the so-called topological degree, which we have not introduced. Combining this with Corollary 4.28 allows us to conclude that, given a quasiconformal mapping f, defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, either $J_f(x) > 0$ almost everywhere in Ω or $J_f(x) < 0$ almost everywhere in Ω.

3) If $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a domain, is a homeomorphism and $p > n - 1$ ($p \geq 1$ in the plane), then f is differentiable almost everywhere, see [23]. If $p = n-1$ and $n \geq 3$, then f need not be differentiable anywhere. The positive results are non-trivial. For the counterexample, one picks a continuous function $u \in W^{1,n-1}_{\text{loc}}(\mathbb{R}^{n-1})$ of $n - 1$ variables that fails to be differentiable anywhere and defines

$$f(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1}, x_n + u(x_1, \cdots, x_{n-1})).$$

4) If $p < n - 1$, it is not known if the Jacobian of a homeomorphism $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ can change its sign. For $p > n - 1$, the Jacobian determinant cannot change its sign by 1) and 2) and this is expected to also hold when $p = n - 1$.

Added: Hencl and Malý, Jacobians of Sobolev homeomorphisms, to appear in Calc. Var. have very recently shown that one can relax the assumption $p > n - 1$ to $p > p_n$, where p_n is the integer part of $n/2$, especially $p_3 = 1$. The case $1 < p \leq p_n$ remains open when $n > 3$.

5 The analytic definition

In this chapter we give an analytic definition for quasiconformality by establishing the following characterization of quasiconformality.

5.1 Theorem. Suppose that $\Omega, \Omega' \subset \mathbb{R}^n$ are domains, $n \geq 2$. Let $f : \Omega \to \Omega'$ be a homeomorphism. Then the following are equivalent:

1) f is quasiconformal.

2) There exists η such that $f|_B$ is η-quasisymmetric for each ball B with $2B \subset \Omega$.

3) $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and there is K such that

$$|Df(x)|^n \leq K|J_f(x)|$$

almost everywhere in Ω.

5.2 Remark. It follows that either $J_f > 0$ almost everywhere in Ω or that $J_f < 0$ almost everywhere in Ω, see Corollary 4.28 and Remarks 4.29.

We already saw in Chapter 3 that 1) and 2) are equivalent and Corollary 4.28 shows that 1) implies 3). In order to deduce 1) from 3) we introduce some preliminary results.

Recall the notation

$$\mu'_f(x) = \lim_{r \to 0} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}$$

that we used for homeomorphisms. One of our aims is to show that J_f is locally integrable for a homeomorphism that is locally in the Sobolev class $W^{1,1}_{\text{loc}}$. This will be done by relating J_f to μ'_f. It is rather easy to do this at the points of differentiability of our homeomorphism. The problem is that, in dimensions $n \geq 3$, our regularity assumption $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ (see Remarks 4.29) does not by itself guarantee differentiability even at a single point. In order to overcome this, we will use Lipschitz "approximations" to f, but the prize we have to pay is that these Lipschitz mappings need not be injective. Given a continuous mapping $f : \Omega \to \mathbb{R}^n$, we write

$$\mu'_f(x) = \limsup_{r \to 0} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}.$$

Because $f(\overline{B}(x,r))$ is compact and so measurable, $\mu'_f(x)$ is indeed defined. We cannot however apply the Radon-Nikodym theorem as we did in the connection with Proposition 4.1: $\mu(A) = |f(A)|$ does not necessarily define a measure when f fails to be injective. We will be able to get around this problem.

5.3 Lemma. Let $f : \Omega \to \mathbb{R}^n$ be continuous and assume that $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$. Then

$$|J_f(x)| \leq \mu'_f(x)$$

almost everywhere in Ω.

The proof of this result will be based on a sequence of lemmas.

5.4 Lemma. Let $f : \Omega \to \mathbb{R}^n$ be continuous and assume that f is differentiable at $x_0 \in \Omega$. Then
$$|J_f(x_0)| = \mu_f'(x_0).$$

Proof. We already saw in the proof of Corollary 4.28 that if $J_f(x_0) = 0$ and f is differentiable at x_0, then $\mu_f'(x_0) = 0$. Suppose that $J_f(x_0) \neq 0$. We may assume that $x_0 = 0 = f(x_0)$. Because $J_f(0) \neq 0$, the inverse matrix $(Df(0))^{-1}$ exists. Define $g(x) = (Df(0))^{-1}f(x)$. Then g is differentiable at 0, $Dg(0) = I$, and moreover,
$$|f(B(0, r))| = |Df(0)\, g(B(0, r))| = |J_f(0)||g(B(0, r))|.$$

Thus it suffices to show that
$$\lim_{r \to 0} \frac{|g(B(0, r))|}{|B(0, r)|} = 1.$$

Because g is differentiable at 0 and $Dg(0) = I$,
$$|g(x) - x| \le \varepsilon(|x|)|x|, \tag{20}$$

where $\varepsilon(|x|) \to 0$ as $|x| \to 0$. It follows that
$$\frac{|g(B(0, r))|}{|B(0, r)|} \le \frac{|B(0, r + \varepsilon(r)r)|}{|B(0, r)|} = (1 + \varepsilon(r))^n \longrightarrow 1, \quad \text{as } r \to 0,$$

so especially
$$\limsup_{r \to 0} \frac{|g(B(0, r))|}{|B(0, r)|} \le 1.$$

For the opposite inequality we use the fact that
$$B\big(0, (1 - \varepsilon)r\big) \subset g\big(B(0, r)\big) \tag{21}$$

for given $\varepsilon > 0$ whenever $0 < r < r_\varepsilon$. This follows from Lemma 11.10 in the appendix, since now $|g(x) - x| \le \varepsilon$ for $|x| < r_\varepsilon$ by inequality (20). Thus by (21) we obtain for $r < r_\varepsilon$ that
$$\frac{|g(B(0, r))|}{|B(0, r)|} \ge \frac{|B(0, (1 - \varepsilon)r)|}{|B(0, r)|} = (1 - \varepsilon)^n \longrightarrow 1, \quad \text{as } \varepsilon \to 0,$$

so
$$\liminf_{r \to 0} \frac{|g(B(0, r))|}{|B(0, r)|} \ge 1.$$

This proves the lemma. $\qquad\qquad\square$

5.5 Lemma. (McShane extension) Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be L-Lipschitz, that is
$$|f(x) - f(y)| \leq L|x - y|$$
for all $x, y \in A$. Then there exists a $(\sqrt{m}L)$-Lipschitz $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\tilde{f}|_A = f$.

Proof. Let $m = 1$. Define
$$\tilde{f}(x) = \inf_{a \in A}\{f(a) + L|x - a|\}.$$

Then $\tilde{f}(x) = f(x)$ when $x \in A$: Since f is L-Lipschitz on A,
$$f(x) \leq f(a) + L|x - a| \quad \text{when } x, a \in A,$$
and so $\tilde{f}(x) \geq f(x)$. Also, clearly $\tilde{f}(x) \leq f(x)$.

Given $x, y \in \mathbb{R}^n$, we have that
$$\tilde{f}(x) = \inf_{a \in A}\{f(a) + \underbrace{L|x - a|}_{\leq L(|y-a|+|y-x|)} \}$$
$$\leq L|y - x| + \tilde{f}(y).$$

Because this also holds with x replaced by y, we conclude that $\tilde{f}$ is L-Lipschitz.

Let us then consider the case $m \geq 2$. For given $f = (f_1, \ldots, f_m)$ define $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m)$ as in the previous case. Now
$$|\tilde{f}(x) - \tilde{f}(y)|^2 = \sum_1^m |\tilde{f}_i(x) - \tilde{f}_i(y)|^2 \leq mL^2|x - y|^2,$$

and the claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.6 Remark. By choosing a suitable extension different from the McShane extension, one could require above $\tilde{f}$ to be L-Lipschitz. This can be done using the so-called Kirszbaum extension.

5.7 Lemma. Let $u \in W^{1,1}(3B)$ and $\varepsilon > 0$. Then there is a set $A_\varepsilon \subset B$ so that $|B \setminus A_\varepsilon| < \varepsilon$ and $u|_{A_\varepsilon}$ is Lipschitz.

Proof. Write $B = B(x_0, r_0)$. Let $x, y \in B$ be Lebesgue points of u. Choose $B_j = B(x, 2^{-j}|x - y|)$ for $j \geq 0$ and $B_j = B(y, 2^{j+1}|x - y|)$ for $j < 0$. Then by the Poincaré inequality (as in the proof of Theorem 2.12),

$$
\begin{aligned}
|u(x) - u(y)| &\leq \sum_{-\infty}^{\infty} |u_{B_j} - u_{B_{j+1}}| \leq \sum_{-\infty}^{\infty} C_n \fint_{B_j} |u - u_{B_j}| \\
&\leq C_n \sum_{-\infty}^{\infty} r_j \fint_{B_j} |\nabla u| \\
&\leq C_n |x - y| \big(\mathrm{M}_{3r_0} |\nabla u(x)| + \mathrm{M}_{3r_0} |\nabla u(y)| \big) \\
&\leq 2C_n |x - y| \lambda
\end{aligned}
$$

when both x and y belong to the set $\{z \in B : \mathrm{M}_{3r_0}|\nabla u(z)| \leq \lambda\}$. Thus we have $C_n \lambda$-Lipschitz continuity outside the set

$$
\mathrm{Bad}_\lambda = \{z \in B : \mathrm{M}_{3r_0}|\nabla u(z)| > \lambda\} \cup \{z \in B : z \text{ non-Lebesgue point of } u\}.
$$

By Remark 2.6,

$$
|\mathrm{Bad}_\lambda| \leq \frac{5^n 2}{\lambda} \underbrace{\int_{\{|\nabla u(z)| > \frac{\lambda}{2}\} \cap 3B} |\nabla u|}_{\substack{\longrightarrow \\ \lambda \to \infty}} \, 0 = o\left(\tfrac{1}{\lambda}\right)
$$

and the claim follows. $\qquad\square$

5.8 Remark. The above proof shows that u is $C_n \lambda$-Lipschitz in $B \setminus \mathrm{Bad}_\lambda$, where $|\mathrm{Bad}_\lambda| = o\left(\tfrac{1}{\lambda}\right)$. Use the McShane extension theorem to extend the restriction of u to this set as $C_n \lambda$-Lipschitz function u_λ to all of B. Then

$$
\int_B |\nabla u - \nabla u_\lambda| \leq \int_{\mathrm{Bad}_\lambda} |\nabla u| + |\nabla u_\lambda| \leq \int_{\mathrm{Bad}_\lambda} |\nabla u| + C_n \lambda o\left(\tfrac{1}{\lambda}\right) \xrightarrow[\lambda \to \infty]{} 0
$$

because

$$
\nabla u_\lambda(x) = \nabla u(x) \tag{22}
$$

at almost every point x of $G_\lambda = B \setminus \mathrm{Bad}_\lambda$.
Reason: If $E \subset \Omega$ is measurable, $\partial_i v$ and $\partial_i w$ exist almost everywhere in E and $v = w$ on E, then $\partial_i v = \partial_i w$ almost everywhere in E: Simply notice that almost every point x of E is of linear density one in the x_i-direction.

One can do even better. Consider the set

$$
\mathrm{Bad}'_\lambda = \{x \in B : \mathrm{M}_{3r_0} u(x) \geq \lambda\}.
$$

Then $|\mathrm{Bad}'_\lambda| = o\left(\frac{1}{\lambda}\right)$. So, when λ is large, the distance from any point in Bad'_λ to $B \setminus \mathrm{Bad}'_\lambda$ is at most one. Thus the McShane extension u_λ of u from $B \setminus (\mathrm{Bad}_\lambda \cup \mathrm{Bad}'_\lambda)$ is $C_n \lambda$-Lipschitz and bounded in absolute value by $2C_n \lambda$ on B. It follows that

$$\int_B |u - u_\lambda| + |\nabla u - \nabla u_\lambda| \underset{\lambda \to \infty}{\longrightarrow} 0.$$

The final estimate of the preceding remark yields the following corollary:

5.9 Corollary. If $u \in W^{1,1}(3B)$, then there is a sequence $(\varphi_j)_1^\infty$ of Lipschitz functions such that

$$|\{x \in B : \varphi_j(x) \neq u(x)\}| \to 0$$

and

$$\int_B |u - \varphi_j| + |\nabla u - \nabla \varphi_j| \to 0$$

as $j \to \infty$.

5.10 Remarks.

1) One can get rid of the constant 3 above (see Figure 4)

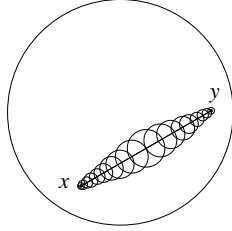
Figure 4: Remark 5.10 (1).

2) The same argument as above gives the corollary for $W^{1,p}$ and with

$$\int_B |u - \varphi_j|^p + |\nabla u - \nabla \varphi_j|^p \to 0 \quad \text{as } j \to \infty.$$

Proof of Lemma 5.3. Assume that $f : \Omega \to \mathbb{R}^n$ is continuous and $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Let $B \subset \Omega$ be a ball with $3B \subset \Omega$. It suffices to prove that

$$|J_f(x)| \leq \mu'_f(x) \quad \text{for a.e. } x \in B.$$

Let $\varepsilon > 0$. Pick a Lipschitz mapping $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that for the set $\mathcal{B} = \{x \in B : \tilde{f}(x) \neq f(x)\}$ we have $|\mathcal{B}| < \varepsilon$, see Corollary 5.9. Because $\tilde{f}$ is Lipschitz, it is differentiable almoste everywhere in $B \setminus \mathcal{B}$; see Remark 4.27. By Lemma 5.4, $|J_{\tilde{f}}(x)| = \mu'_{\tilde{f}}(x)$ at the points of differentiability. By the reasoning in Remark 5.8, see (22), $J_{\tilde{f}}(x) = J_f(x)$ almost everywhere in $B \setminus \mathcal{B}$. So it suffices to prove that $\mu'_{\tilde{f}}(x) \leq \mu'_f(x)$ almost everywhere in $G = B \setminus \mathcal{B}$. Let $x \in G$. Then

$$\frac{|\tilde{f}(B(x,r))|}{|B(x,r)|} \leq \frac{|\tilde{f}(B(x,r) \cap G)| + |\tilde{f}(B(x,r) \cap \mathcal{B})|}{|B(x,r)|}$$
$$\leq \frac{|f(B(x,r))|}{|B(x,r)|} + \frac{L^n |B(x,r) \cap \mathcal{B}|}{|B(x,r)|},$$

and the claim follows because the last term tends to zero for almost every $x \in G$ by Remarks 4.3 3). $\qquad\square$

5.11 Corollary. Let $f : \Omega \to \Omega'$ be a homeomorphism with $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$. If

$$|Df(x)|^n \leq K|J_f(x)|$$

almost everywhere in Ω for some $1 \leq K < \infty$, then $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$.

Proof. By Lemma 5.3, $|J_f(x)| \leq \mu'_f(x)$ almost everywhere in Ω. The claim follows because $\mu'_f \in L^1_{\text{loc}}(\Omega)$, by Proposition 4.1. $\qquad\square$

5.12 Lemma. Let $f : \Omega \to \Omega'$ be a homeomorphism, $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, and let $u : \Omega' \to [0, \infty)$ be Borel measurable. Then

$$\int_\Omega u(f(x))|J_f(x)| \leq \int_{\Omega'} u.$$

Proof. Let $a > 1$ and set $G_j = \{y \in \Omega' : a^j < u(y) \leq a^{j+1}\}$ for $j \in \mathbb{Z}$. Then $\Omega' \setminus \bigcup G_j = \{y \in \Omega' : u(y) = 0\}$. Thus, by Proposition 4.1 and Lemma

5.3,

$$\int_\Omega u(f(x))|J_f(x)| = \int_{f^{-1}(\cup G_j)} u(f(x))|J_f(x)|$$

$$= \sum \int_{f^{-1}(G_j)} u(f(x))|J_f(x)|$$

$$\leq \sum \int_{f^{-1}(G_j)} a^{j+1} \mu'_f(x)\, dx$$

$$\leq \sum a^{j+1}|G_j| \leq a \sum \int_{G_j} u\, dy = a \int_{\Omega'} u.$$

Let $a \to 1$ to complete the proof. $\qquad\square$

5.13 Lemma. Let $f : \Omega \to \Omega'$ be a homeomorphism, $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $|Df(x)|^n \leq K|J_f(x)|$ almost everywhere in Ω. If u is C^1 on Ω', then $u \circ f \in W^{1,n}_{\text{loc}}(\Omega)$ and

$$\int_\Omega |\nabla(u \circ f)|^n \leq K \int_{\Omega'} |\nabla u|^n\,.$$

Proof. Clearly $u \circ f$ is absolutely continuous on almost all lines parallel to the coordinate axes in Ω because f is and u is locally Lipschitz. Because $u \circ f$ is locally bounded, it thus suffices to show the local n-integrability of $|\nabla(u \circ f)|$ and the asserted inequality. Let $\tilde{f}$ be as in the proof of Lemma 5.3. Then $\tilde{f}$ is differentiable almost everywhere and $Df(x) = D\tilde{f}(x)$ almost everywhere in G (see Remark 5.8). Thus, using the usual chain rule and Proposition 11.1, we see that

$$|\nabla(u \circ f)(x)|^n = |\nabla u(f(x))Df(x)|^n$$
$$\leq |Df(x)|^n|\nabla u(f(x))|^n \leq K|J_f(x)||\nabla u(f(x))|^n$$

almost everywhere in G. It follows that this inequality holds almost everywhere in Ω. Use Lemma 5.12 to complete the proof. $\qquad\square$

Proof of 3) $\Rightarrow$ 1) in Theorem 5.1. Let $B = B^n(x_0, r_0) \subset 2B \subset \Omega$, and define l, L as in the proof of Theorem 2.1, see Figure 5. We may again assume

that $L \geq 2l$. By Corollary 5.11, $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$. Define

$$
u(y) = \begin{cases} 1 & \text{if } |y - f(x_0)| \leq l \\ 0 & \text{if } |y - f(x_0)| \geq L \\ \dfrac{\log \dfrac{1}{|y - f(x_0)|} - \log \dfrac{1}{L}}{\log \dfrac{L}{l}} & \text{if } l \leq |y - f(x_0)| \leq L \,, \end{cases}
$$

and set $u_\varepsilon(y) = \fint_{B(y,\varepsilon)} u(z)dz$ for $\varepsilon > 0$. Then u_ε is C^1 by Lemma 4.22 and
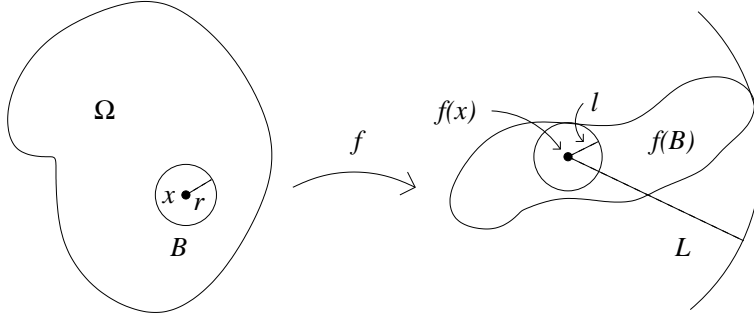

Figure 5: $f\big(B(x,r)\big)$

thus, by Lemma 5.13, $u_\varepsilon \circ f \in W^{1,n}_{\text{loc}}(\Omega)$ and

$$
\int_\Omega |\nabla(u_\varepsilon \circ f)|^n \leq K \int_{\Omega'} |\nabla u_\varepsilon|^n \,. \tag{23}
$$

Next

$$
\int_{\Omega'} |\nabla u_\varepsilon|^n \to \int_{\Omega'} |\nabla u|^n = \omega_{n-1} \left(\log \tfrac{L}{l}\right)^{1-n} \tag{24}
$$

when $\varepsilon \to 0$ by Lemma 4.22 and dominated convergence (also by Lemma 4.22 and Lemma 4.23). Here ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere.

Notice that $f^{-1}(B^n(f(x_0), l))$ is a connected set containing x_0 and its closure intersects $S^{n-1}(x_0, r)$. Furthermore, $f^{-1}(\mathbb{R}^n \setminus \overline{B}(f(x_0), L))$ has an open component G whose closure intersects both $S^{n-1}(x_0, r)$ and $S^{n-1}(x_0, \frac{3r}{2})$. We may then select continua $E \subset f^{-1}(B^n(f(x_0), l))$ and $F \subset G$, both of diameter at least $r_0/4$ so that $u_\varepsilon = 1$ on E and $u_\varepsilon = 0$ on F for all sufficiently small ε. Thus

$$
\int_{\Omega_1} |\nabla(u_\varepsilon \circ f)|^n \geq \delta_n > 0 \tag{25}
$$

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for all sufficiently small $\varepsilon > 0$ because of the size of the 0- and 1-sets of u_ε and the fact that $u \circ f \in W^{1,1}(2B)$; notice that the proof of Theorem 2.12 only assumed a Poincaré inequality, which holds in our setting by Corollary 4.24.

A bound on L/l and so also quasiconformality of f follow by combining (23), (24) and (25). □

5.14 Remarks. 1) Regarding the relationship between the constants H and K in parts 1) and 3) of Theorem 5.1, we have the estimates $K \leq H^{n-1}$ and $H \leq \exp(C_n K^{1/(n-1)})$.

The first of these is sharp and contained in Corollary 4.28 and the second follows from the proof of Theorem 5.1 above. The second estimate can be improved to $H_f(x) \leq K$ almost everywhere, but, for example, for the simple planar quasiconformal mapping defined by $f(x, y) = (x, 2y)$ in the upper closed half plane and by $f(x, y) = (x, y/2)$ in the lower half plane, one has $K = 2$ and $H = 4$. On the other hand, one can construct examples (in the plane [19]) that show the sharpness of the given global bound on H.

2) Notice that the analytic definition requires the pointwise inequality at almost every point. One could then expect that the metric definition could also be slightly relaxed. This is indeed the case in the sense that a homeomorphism $f : \Omega \to \Omega'$ satisfies

$$f \in W_{\mathrm{loc}}^{1,n}(\Omega, \mathbb{R}^n) \quad \text{and} \quad |Df(x)| \leq K \min_{|h|=1} |Df(x)h|$$

almost everywhere if and only if $\liminf_{r \to 0} H_f(x, r) < \infty$ outside a set of σ-finite $(n-1)$-measure and $\liminf_{r \to 0} H_f(x, r) \leq K$ almost everywhere. Above, $\limsup$ instead of $\liminf$ naturally works as well.

6 K-quasiconformal mappings

Let us call from now on a homeomorphism $f : \Omega \to \Omega'$ with $f \in W_{\mathrm{loc}}^{1,1}(\Omega, \mathbb{R}^n)$ and

$$|Df(x)|^n \leq K|J_f(x)| \quad \text{a.e. in } \Omega$$

K-*quasiconformal* (K-qc) according to the analytic definition. We will typically abuse the notation and only talk about K-quasiconformal mappings below. Above, $\Omega, \Omega' \subset \mathbb{R}^n$ are domains and we assume that $n \geq 2$. Notice that each conformal f is 1-qc.

6.1 Remark. If f is K-qc, then

(i) f is differentiable almost everywhere,

(ii) $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some $p = p(n, K) > n$,

(iii) either $J_f(x) > 0$ a.e. in Ω or $J_f(x) < 0$ a.e. in Ω,

(iv) f is locally Hölder continuous,

(v) $|f(E)| = \int_E |J_f|$ whenever $E \subset \Omega$ is measurable,

(vi) $\dfrac{|f(E)|}{|f(Q)|} \leq C \left(\dfrac{|E|}{|Q|}\right)^{\alpha}$ whenever $E \subset Q \subset 2Q \subset \Omega$, where

$$C = C(n, K), \ 0 < \alpha = \alpha(n, K).$$

All this follows by combining our previous results.

By Theorem 3.6 we know that quasiconformal mappings form a group. It turns out that the analytic definition allows us to give sharp estimates on the associated constants of quasiconformality.

6.2 Theorem. Let $f_1 : \Omega_1 \to \Omega_2$ be K_1-qc and $f_2 : \Omega_2 \to \Omega_3$ be K_2-qc. Then $f_2 \circ f_1 : \Omega_1 \to \Omega_3$ is $K_1 K_2$-qc.

Proof. We already know that $f_2 \circ f_1$ is quasiconformal because the three different definitions (in Theorem 5.1) give the same class of mappings. Thus $f_2 \circ f_1 \in W^{1,1}_{\text{loc}}(\Omega_1, \mathbb{R}^n)$ (and even $W^{1,p}_{\text{loc}}(\Omega_1, \mathbb{R}^n)$ for some $p > n$). Now f_2 is differentiable almost everywhere in Ω_2, f_1 is differentiable almost everywhere in Ω_1, and because f_1 cannot map a set of positive measure to a set of measure zero, we conclude that

$$D(f_2 \circ f_1)(x) = Df_2(f_1(x))Df_1(x)$$

for almost every $x \in \Omega_1$. In particular, for such a point x,

$$|D(f_2 \circ f_1)(x)|^n = |Df_2(f_1(x))Df_1(x)|^n.$$

For almost every $x \in \Omega_1$,

$$|Df_1(x)|^n \leq K_1 |J_{f_1}(x)|,$$

and for almost every $y = f_1(x) \in \Omega_2$,

$$|Df_2(f_1(x))|^n \leq K_2 |J_{f_2}(f_1(x))|.$$

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Because f_1 can not map a set of positive measure to a set of measure zero, both inequalities hold for almost every $x \in \Omega_1$. Thus

$$|D(f_2 \circ f_1)(x)|^n \leq K_2 K_1 |J_{f_2}(f_1(x))||J_{f_1}(x)| = K_2 K_1 |J_{f_2 \circ f_1}(x)|$$

for almost every $x \in \Omega_1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.3 Theorem. Let $f : \Omega \to \Omega'$ be K-qc. Then $f^{-1} : \Omega' \to \Omega$ is K^{n-1}-qc.

6.4 Remark. The constants $K_1 K_2$ and K^{n-1} in Theorem 6.2 and Theorem 6.3 are sharp. To see this, simply consider the linear quasiconformal mappings f_1, f_2, f associated to the diagonal matrices A_1, A_2 and A where the first diagonal entry of A_1 is $K_1^{1/(n-1)}$, of A_2 is $K_2^{1/(n-1)}$ and all the rest are 1, and the $n-1$ first diagonal entries of A are all K and the last one is 1.

For the proof of Theorem 6.3 we need some elementary linear algebra:

6.5 Proposition. If $\det A \neq 0$ and $|A|^n \leq K |\det A|$, then

$$|A^{-1}|^n \leq K^{n-1}|\det A^{-1}|.$$

Proof. By Proposition 11.2 in the appendix, we find two orthonormal bases so that the matrix of A with respect to these bases is diagonal. Notice that the associated changes of bases preserve lenghts. Thus the operator norms of A and A^{-1} and the determinants of A, A^{-1} can be readily read of from this diagonal representation D of A (see Lemma 11.4 in the appendix). We may assume that

$$D = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$. Then

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1/\lambda_n \end{bmatrix}.$$

Because $|\lambda_1|^n \leq K|\lambda_1 \ldots \lambda_n|$, we have that $|\lambda_j| \leq K|\lambda_n|$ for each j. Thus

$$|D^{-1}|^n = \frac{1}{|\lambda_n|^n} = \frac{1}{|\lambda_n|}\left(\frac{1}{|\lambda_n|}\right)^{n-1} \leq \frac{K^{n-1}}{|\lambda_1 \ldots \lambda_n|} = K^{n-1}|\det A^{-1}|.$$

$\square$

Proof of Theorem 6.3.   We already know that f^{-1} is quasiconformal and so $f^{-1} \in W^{1,n}_{loc}(\Omega', \mathbb{R}^n)$. Also f preserves the null sets for the Lebesgue measure and, at almost every x, f is differentiable with $J_f(x) \neq 0$. In particular, for almost every $x \in \Omega$

$$I = D(f^{-1} \circ f)(x) = Df^{-1}(f(x))Df(x).$$

So

$$Df^{-1}(f(x)) = \left[Df(x)\right]^{-1}$$

for almost every $x \in \Omega$ and so also for almost every $y = f(x) \in \Omega'$. Because f is K-qc, we have $|Df(x)|^n \leq K|\det Df(x)|$, and consequently Proposition 6.5 gives the claim. $\qquad\square$

6.6 Remark. Combining Corollary 4.25, almost everywhere differentiability of qc mappings and Corollary 4.17 we see that each K-qc mapping is locally Hölder-continuous:

$$|f(x) - f(y)| \overset{p>n}{\leq} C|x-y|^{1-n/p} \left(\int_B |Df|^p\right)^{1/p}$$

$$= C\,|x-y|^{1-n/p}\,r^{n/p} \left(\fint_B |Df|^p\right)^{1/p}$$

$$\overset{\text{Gehring}}{\leq} \tilde{C}|x-y|^{1-n/p}\,r^{n/p}\fint_B |Df|,$$

where $p = p(n, K) > n$. Thus f is Hölder continuous with some exponent that depends on K, n. It is then natural to ask for the best possible Hölder exponent.

6.7 Theorem. Let $f : \Omega \to \Omega'$ be K-qc. If $7B \subset \Omega$, then

$$\frac{|f(x) - f(y)|}{\operatorname{diam} f(B)} \leq C(n, K) \left(\frac{|x-y|}{\operatorname{diam} B}\right)^{C_1/K},$$

where $C_1 = C_1(n)$, whenever $x, y \in B$.

Proof. Let $g : \Omega_1 \to \Omega_2$ be K-qc, let $y_0 \in \Omega_2$ and let $y \in \Omega_2$ satisfy $|y - y_0| < d(y_0, \partial\Omega_2)/3$. Write $r = d(y_0, \partial\Omega_2)/2$. We define

$$
v(z) = \begin{cases}
1, & \text{if } |z - y_0| \leq |y - y_0| \\
0, & \text{if } |z - y_0| \geq r \\
\dfrac{\log \dfrac{1}{|z - y_0|} - \log \dfrac{1}{r}}{\log \dfrac{r}{|y - y_0|}}, & \text{if } |y - y_0| \leq |z - y_0| \leq r.
\end{cases}
$$

Write $u = v \circ g$ and extend u as zero to the exterior of Ω_1. Then, as in the proof of Theorem 5.1,

$$
\int_{\Omega_1} |\nabla u|^n \leq K \int_{\Omega_2} |\nabla v|^n \leq K\,\omega_{n-1} \left(\log \left(\frac{r}{|y - y_0|} \right) \right)^{1-n}, \qquad (26)
$$

where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere. Suppose that we could show that

$$
\int_{\Omega_1} |\nabla u|^n \geq \omega_{n-1} \left(\log \left(\frac{C(n) L_{g^{-1}}(y_0, r)}{|g^{-1}(y) - g^{-1}(y_0)|} \right) \right)^{1-n}. \qquad (27)
$$

Then, combining (26) and (27) and the fact that the support of u is compactly contained in Ω, we would conclude that

$$
|g^{-1}(y) - g^{-1}(y_0)| \leq C(n) L_{g^{-1}}(y_0, r) \left(\frac{|y - y_0|}{r} \right)^{K^{-1/(n-1)}}.
$$

Applying this to the K^{n-1}-qc (see Theorem 6.3) mapping $g = f^{-1} : \Omega' \to \Omega$, the claim would easily follow with $C_1 = 1$. It is not easy to establish (27), but it is not hard to prove the lower bound with some constant C_n, which is sufficient for the claim of our theorem:
Write $L = L_{g^{-1}}(y_0, r)$, $x_0 = g^{-1}(y_0)$ and $s = |g^{-1}(y) - g^{-1}(y_0)|$. If

$$
\fint_{B(x_0, 3s)} u \leq \tfrac{2}{3},
$$

then

$$
\int_{B(x_0, 3s)} |\nabla u|^n \geq \delta(n) > 0
$$

by the proof of the corresponding earlier estimate (Theorem 2.12); notice that $u = 1$ on the compact, connected set $g^{-1}(\overline{B}(y_0, |y - y_0|))$ of diameter at

least s. Notice further that $u(x) = 0$ on $\mathbb{R}^n \setminus B(y_0, L)$. Pick $w \in S^{n-1}(y_0, 2L)$. Then

$$\fint_{B(w,L)} u = 0.$$

Now, we may assume that

$$\fint_{B(x_0, 3s)} u \geq \tfrac{2}{3} \quad \text{and} \quad \fint_{B(w,L)} u = 0,$$
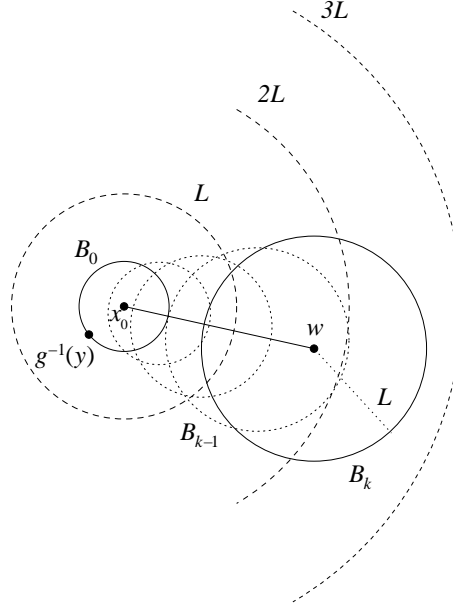
and thus (see Figure 6)

Figure 6: Choice of B_j's in the proof of 6.7

$$\tfrac{1}{3} \leq \sum_{j=1}^{k} |u_{B_j} - u_{B_{j-1}}| \leq \sum_{j=0}^{k} C r_j \left(\fint_{B_j} |\nabla u|^n \right)^{1/n}$$

$$\leq \sum_{j=0}^{k} \tilde{C} \left(\int_{B_j} |\nabla u|^n \right)^{1/n}$$

$$\leq \tilde{C} \underbrace{(k+1)}_{\leq c \log \frac{cL}{s}}{}^{(n-1)/n} \left(\int_{\bigcup B_j} |\nabla u|^n \right)^{1/n}.$$

This gives the desired lower bound for $\int_{\Omega_1} |\nabla u|^n$. $\qquad\square$

6.8 Remarks. 1) Given a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and compact sets $E, F \subset \overline{\Omega}$ with $E \cap F = \emptyset$, set

$$\mathrm{cap}_n(E, F; \Omega) = \inf_{u \in A(E,F;\Omega)} \int_\Omega |\nabla u|^n \,,$$

where

$$A(E, F; \Omega) = \left\{ u \in C(\Omega \cup E \cup F) \cap W_{\mathrm{loc}}^{1,n}(\Omega) : u \geq 1 \text{ in } E \text{ and } u \leq 0 \text{ in } F \right\}.$$

This is called *the conformal capacity* (variational n-capacity, n-capacity) *of E and F with respect to Ω.* As a part of the proof of Theorem 5.1 we essentially showed the fact that

$$\mathrm{cap}_n\left(f^{-1}(E), f^{-1}(F); \Omega\right) \leq K \, \mathrm{cap}_n(E, F; \Omega')$$

whenever $E, F \subset \Omega'$ are compact and $f : \Omega \to \Omega'$ is K-qc.

The basic estimates are:

(i) If $E \subset E'$, $F \subset F'$ and $\Omega \subset \Omega'$, then

$$\mathrm{cap}_n(E, F; \Omega) \leq \mathrm{cap}_n(E', F'; \Omega').$$

(ii) If $\overline{B}(x, r) \subset B(x, R) \subset \Omega$, then

$$\mathrm{cap}_n\left(\overline{B}(x, r), S^{n-1}(x, R); \Omega\right) = \mathrm{cap}_n\left(\overline{B}(x, r), S^{n-1}(x, R); B(x, R)\right)$$
$$\leq \frac{\omega_{n-1}}{\left(\log \frac{R}{r}\right)^{n-1}}\,.$$

In fact, the inequality can also be reversed:

If $u \in A(\overline{B}(x, r), S^{n-1}(x, R); B(x, R))$ is C^1, then the fundamental theorem of calculus and Hölder's inequality give

$$1 \leq \int_r^R |\nabla u(tw)| \, dt$$
$$\leq \int_r^R |\nabla u(tw)| t^{\frac{n-1}{n} - \frac{n-1}{n}} \, dt$$
$$\leq \left(\int_r^R \frac{dt}{t} \right)^{\frac{n-1}{n}} \left(\int_r^R |\nabla u(tw)|^n t^{n-1} \, dt \right)^{1/n}$$

for every $w \in S^{n-1}(0, 1)$. The desired inequality follows by raising both sides of this inequality to power n and integrating over $S^{n-1}(0, 1)$ with respect to w. Approximation then gives the same for general test functions.

(iii) If $E, F \subset B(x, r)$ are continua with

$$\frac{\min\{\operatorname{diam} E, \operatorname{diam} F\}}{r} \geq \delta_1 > 0,$$

then

$$\operatorname{cap}_n(E, F; B(x, r)) \geq \delta(\delta_1, n) > 0.$$

(iv) If Ω is bounded and $E \subset \Omega$ is a continuum, then

$$\operatorname{cap}_n(E, \partial\Omega; \Omega) \geq \frac{\omega_{n-1}}{\left(\log \frac{C(n)\operatorname{diam}\Omega}{\operatorname{diam} E}\right)^{n-1}}$$

This is not trivial; one uses symmetrization [11].

One in fact also has the estimates [11]

$$\operatorname{cap}_n(E, F; \mathbb{R}^n) \geq \frac{\omega_{n-1}}{(\log(C(n)(1+t)))^{n-1}}$$

given continua $E, F \subset \mathbb{R}^n$, where $t = \frac{d(E,F)}{\min\{\operatorname{diam}(E),\operatorname{diam}(F)\}}$, and

$$\operatorname{cap}_n(E, F; \mathbb{R}^n_+) \geq \operatorname{cap}_n(E, F; \mathbb{R}^n)/2$$

when $E, F \subset \mathbb{R}^n_+$.

2) If we use (iv) in the proof of the Hölder-continuity estimate, we see that one can take $C_1 = 1$, so that the Hölder exponent is $\alpha = 1/K$. This is sharp:

$$f(x) = x|x|^{-(1-1/K)} \quad \text{is } K\text{-qc.}$$

3) The Hölder exponent we found is thus $1/K$ (in terms of "H", $1/H$ in dimension two). By Corollary 4.25, $f \in W^{1,p}_{\text{loc}}$ is locally Hölder-continuous with exponent $1 - \frac{n}{p}$. To obtain the Hölder exponent $1/K$ via Corollary 4.25, one would need f to be in the Sobolev class W^{1,p_K}_{loc} with

$$p_K = \frac{nK}{K-1}.$$

The radial mapping

$$f(x) = x|x|^{-(1-1/K)}$$

belongs to $W^{1,p}_{\text{loc}}$ exactly when p is strictly less than this p_K.

6.9 Conjecture. Let $\Omega, \Omega' \subset \mathbb{R}^n$ be domains, where $n \geq 2$. If $f : \Omega \to \Omega'$ is K-quasiconformal, then $f \in W^{1,p}_{\text{loc}}$ for all $p < p_K$.

This holds when $n = 2$ by results by Astala [2]. In higher dimensions, the conjecture would follow if a certain conjecture in calculus of variations gets proved [16].

7 Sobolev spaces and convergence of quasiconformal mappings

We will show that quasiconformality is stable under locally uniform convergence in the following sense.

7.1 Theorem. Let $f_j : \Omega \to \Omega_j$ be K-qc for each $j \geq 1$, and suppose that $f_j \to f : \Omega \to \Omega'$ locally uniformly. If f is a homeomorphism, then f is K-qc.

7.2 Remarks. 1) In the plane, one obtains the following conclusion in terms of the metric definition. Suppose that $f_j : \Omega \to \Omega_j$ are quasiconformal in terms of the metric definition with $H_{f_j}(x) = \limsup_{r \to 0} H_{f_j}(x, r) \leq H$ almost everywhere in Ω for each j. If the sequence $(f_j)_j$ converges locally uniformly to a homeomorphism $f : \Omega \to \Omega'$, then f is quasiconformal with $H_f(x) \leq H$ almost everywhere in Ω.

To see this notice first that each f_j is H-qc by Corollary 4.28. Thus Theorem 7.1 shows that f is H-qc. By Corollary 4.28 we know that f is differentiable at almost every x with $J_f(x) \neq 0$. Fix such an x. As in the proof of Proposition 6.5, we may assume that $Df(x)$ is diagonal with diagonal entries λ_1, λ_2 satisfying $|\lambda_1| \geq |\lambda_2| > 0$. Then

$$\lambda_1^2 \leq H|\lambda_1 \lambda_2|$$

and it follows that $|\lambda_1| \leq H|\lambda_2|$. This implies that $H_f(x) \leq H$, as desired.

2) Let $n \geq 3$. There is a sequence of qc mappings $f_j : \mathbb{R}^n \to \mathbb{R}^n$ so that $f_j \to f$ locally uniformly, the (metric) H-dilatations of f_j are all almost everywhere bounded by some $H_0 > 1$ and the H-dilatation of f is not essentially bounded by H_0. Such examples have been found by Iwaniec [15].

3) The assumption that the limit function be a homeomorphism is not superfluous. Indeed, the sequence $(f_j)_j$ of 1-quasiconformal mappings defined by setting $f_j(x) = x/j$ converges locally uniformly to the constant function $f(x) \equiv 0$.

4) One can characterize the class of K-quasiconformal mappings by a completeness property related to Theorem 7.1. We will return to this in Chapter 8.

In order to prove Theorem 7.1 we need a better understanding of the Sobolev spaces than what immediately follows from the definition that we have used this far. We begin by stating a characterization for the membership in the Sobolev class and by sketching its proof.

7.3 Theorem. (Definitions of Sobolev spaces.) Let $u \in L^p(\Omega)$, $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$. Then the following are equivalent:

1) (**ACL**) There is $\tilde{u} \in W^{1,p}(\Omega)$ with $\tilde{u} = u$ almost everywhere.

2) (**H**) There is a sequence $(\varphi_j)_j \subset C^1(\Omega)$ so that $\varphi_j \to u$ in $L^p(\Omega)$ and $(\nabla \varphi_j)_j$ is Cauchy in $L^p(\Omega)$.

3) (**W**) For each $1 \leq j \leq n$ there is $v_j \in L^p(\Omega)$ so that

$$\int_\Omega u \, \partial_j \varphi = - \int_\Omega v_j \varphi$$

for each $\varphi \in C_0^\infty(\Omega)$.

4) There is $\tilde{u}$ and $g \in L^p(\Omega)$ so that $\tilde{u} = u$ almost everywhere in Ω and g is an upper gradient of $\tilde{u}$ in Ω.

Proof. (sketch)
2) $\Rightarrow$ 1): Passing to a subsequence, we may assume that $(\varphi_j(x))_j$ converges for almost every x. We define

$$\tilde{u}(x) = \lim_{j \to \infty} \varphi_j(x)$$

whenever the limit exists, and set, say, $\tilde{u}(x) = 0$ for the remaining $x \in \Omega$. Then $\tilde{u}(x) = u(x)$ almost everywhere in Ω. By the fundamental theorem of calculus applied to the functions φ_j and the Hölder inequality, one obtains absolute continuity in Ω on the lines for which both

$$\int_I |\nabla u - \nabla \varphi_j|^p \underset{j \to \infty}{\longrightarrow} 0$$

for each compact subinterval I in Ω and $\lim_{j \to \infty} \varphi_j(x)$ exists for some $x \in I$. By the Fubini theorem, this holds for almost all lines parallel to the coordinate axes. It also easily follows that the classical partial derivatives of $\tilde{u}$ exist almost everywhere in Ω and that they are obtained as limits of the partial derivatives of the approximating functions.

1) $\Rightarrow$ 2): We already proved in Chapter 5 that u can be approximated in this manner by Lipschitz functions, provided $\Omega = \mathbb{R}^n$. In this case, the claim follows by taking averages, see Lemma 4.22. For the general case, one uses a partition of unity: $0 \leq \psi_i \leq 1$, $\psi_i \in C_0^\infty(\Omega)$ such that $\sum_1^\infty \psi_i = 1$ in Ω and the supports have bounded overlap. Considering $u\psi_i$, the statement easily follows.

1) $\Rightarrow$ 3): Integrate by parts, v_j is the classical partial derivative.

3) $\Rightarrow$ 2): We use the (smooth) convolution approximation: Let

$$\psi_1(x) = \begin{cases} 0, & |x| \geq 1 \\ C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \end{cases}$$

where C is chosen so that $\int_{\mathbb{R}^n} \psi_1 \, dx = 1$. Define

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n} \psi_1 \left(\frac{x}{\varepsilon}\right).$$

If $v \in L^p_{\mathrm{loc}}$, set

$$v^\varepsilon(x) = (\psi_\varepsilon * v)(x) = \int \psi_\varepsilon(x-y)v(y) \, dy,$$

when $B(x,\varepsilon) \subset\subset \Omega$. If $v \in L^p(\mathbb{R}^n)$, then $v^\varepsilon \to v$ in $L^p(\mathbb{R}^n)$, see the proof of Lemma 4.23. Also $v^\varepsilon(x) \to v(x)$ when x is a Lebesgue point of u.

Fix $x \in \Omega$ and $\varepsilon > 0$ small compared to $d(x, \partial\Omega)$. Now

$$\frac{u^\varepsilon(x+he_i) - u^\varepsilon(x)}{h}$$

$$= \frac{1}{\varepsilon^n} \int_\Omega \underbrace{\frac{1}{h}\left[\psi_1\left(\frac{x+he_i-y}{\varepsilon}\right) - \psi_1\left(\frac{x-y}{\varepsilon}\right)\right]}_{\xrightarrow[h\to 0]{} \frac{1}{\varepsilon}\frac{\partial\psi_1}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right) = \varepsilon^n \frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)} u(y) \, dy$$

$$\xrightarrow[h\to 0]{} \int_\Omega \frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)\, u(y) \, dy$$

by the dominated convergence theorem:

$$\int_G \left|\frac{1}{h}\left[\cdots\right]u(y)\right| \, dy \leq \frac{1}{\varepsilon}\int_G \|\nabla\psi_1\|_\infty \, |u| \, dy.$$

Thus

$$\exists \, \frac{\partial u^\varepsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)\, u(y) \, dy$$

and because ψ_ε is smooth, we see that u^ε is C^1. Moreover, when $u \in W^{1,p}$,

$$\frac{\partial u^\varepsilon}{\partial x_i}(x) = \int \frac{\partial\psi_\varepsilon(x-y)}{\partial x_i}u(y) \, dy$$

$$= -\int \frac{\partial\psi_\varepsilon(x-y)}{\partial y_i}u(y) \, dy$$

$$= \int \psi_\varepsilon(x-y)\, v_i(y) \, dy.$$

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If $v_i \in L^p(\mathbb{R}^n)$, then this convolution sequence converges to v_i in $L^p(\mathbb{R}^n)$. When u is given, use a partition of unity to reduce the setting to that of $\mathbb{R}^n$.

2) $\Rightarrow$ **4)**: Recall that we have already shown that 2) implies 1). Pick a Cauchy sequence $(\varphi_j)_j$ of C^1-functions in the norm $\|\varphi\|_{L^p(\Omega)} + \|\nabla\varphi\|_{L^p(\Omega)}$ so that $\varphi_j \to u$ and $\nabla\varphi_j \to \nabla u$ in $L^p(\Omega)$. Then a subsequence of (φ_j) converges to u almost everywhere and we define $\tilde{u}$ as the pointwise limit of such a fixed subsequence. Write E for the set where this subsequence does not converge. We set $\tilde{u}(x) = 0$ when $x \in E$. We may assume that

$$\|\nabla u - \nabla\varphi_j\|_{L^p(\Omega)} \leq 2^{-j}.$$

Let γ be a rectifiable curve. If

$$\lim_{j\to\infty} \int_\gamma |\nabla u - \nabla\varphi_j|^p = 0, \quad \text{then} \quad \lim_{j\to\infty} \int_\gamma |\nabla u - \nabla\varphi_j| = 0,$$

and if further the sequence $(\varphi_j(x))_j$ converges for some $x \in \gamma$, then the upper gradient inequality holds for the pair $\tilde{u}$, $|\nabla u|$ along γ and along any subcurve of γ (see (5); in fact $(\varphi_j(y))_j$ then converges for all $y \in \gamma$). Consider then a rectifiable curve γ so that

$$\int_\gamma |\nabla u - \nabla\varphi_j| \nrightarrow 0.$$

when $j \to \infty$. Then there is $\delta > 0$ so that

$$\int_\gamma |\nabla u - \nabla\varphi_j| \geq \delta \tag{28}$$

for infinitely many j. Now

$$\int_\gamma \sum_j |\nabla u - \nabla\varphi_j|\, ds = \infty$$

and

$$\int_{\mathbb{R}^n} \left(\sum |\nabla u - \nabla\varphi_j| \right)^p \leq 1.$$

Define

$$g = h(x) + |\nabla u(x)| + \sum |\nabla u - \nabla\varphi_j|,$$

where we set $h(x)$ to be infinite if $x \in E$ and $h(x) = 0$ when $x \notin E$. We may assume that g is a Borel function. It now easily follows that g is an upper gradient of $\tilde{u}$.

4) $\Rightarrow$ 1): This is immediate from the definitions. $\qquad\square$

We now easily obtain the important weak compactness property of $W^{1,p}(\Omega)$, $p > 1$. Recall that $v_{j_k} \rightharpoonup v$ in $L^p(\Omega)$ refers to weak convergence, see Subsection 11.3 in the appendix.

7.4 Corollary. Let $(u_j)_j$ be bounded in $W^{1,p}(\Omega)$, $1 < p < \infty$. Then there is $u \in W^{1,p}(\Omega)$ so that $u_{j_k} \rightharpoonup u$ in $L^p(\Omega)$ and $\nabla u_{j_k} \rightharpoonup \nabla u$ in $L^p(\Omega)$ for a subsequence $(u_{j_k})_k$.

Proof. Both $(u_j)_j$ and $(\nabla u_j)_j$ are bounded in $L^p(\Omega)$. Thus there exist u and $v = (v_1, \ldots, v_n)$ in $L^p(\Omega)$ so that

$$u_{j_k} \rightharpoonup u \quad \text{and} \quad \nabla u_{j_k} \rightharpoonup v \quad \text{in } L^p(\Omega),$$

see Subsection 11.3 in the appendix. Now

$$
\begin{array}{ccc}
\displaystyle\int_\Omega \partial_i\,\varphi u_{j_k} & = & -\displaystyle\int_\Omega \varphi\,\partial_i u_{j_k} \\
\downarrow & & \downarrow \\
\displaystyle\int_\Omega \partial_i\varphi\,u & = & -\displaystyle\int_\Omega \varphi v_i
\end{array}
$$

by the weak convergence, when $\varphi \in C_0^1(\Omega)$. Thus $u \in W^{1,p}(\Omega)$ and $\nabla u = (v_1, \ldots, v_n)$. $\qquad\square$

7.5 Remark. Corollary 7.4 does not extend to the case $p = 1$. For example, when $\Omega = B^2(0,1)$ and $u_j(x) = \min\{1, \max\{0, jx_2\}\}$, we have that $u_j \rightharpoonup u = \chi_{B_+^2(0,1)}$, where $B_+^2(0,1) = B^2(0,1) \cap \{(x_1, x_2) : x_2 > 0\}$. Thus the only potential weak limit of a subsequence of $(u_j)_j$ is u. Moreover, our sequence $(u_j)_j$ is bounded in $W^{1,1}(\Omega)$ and $u \notin W^{1,1}(\Omega)$.

Proof of Theorem 7.1. Fix $B \subset 2B \subset\subset \Omega$. Then $f_j|_{\frac{3}{2}B}$ is η-quasisymmetric with η independent of j (Corollary 3.4 and Theorem 5.1). It follows from the uniform convergence of the mappings f_j that f is η-quasisymmetric on $\frac{3}{2}B$. Because B was arbitrary, we conclude from Theorem 5.1 that f is K_1-qc in Ω for some K_1. It remains to be proven that we may choose $K_1 = K$.

Let $B = B(x, r)$ be as above. For each $\varepsilon > 0$ there is j_ε such that

$$f(B(x, r - \varepsilon)) \subset f_j(B) \subset f(B(x, r + \varepsilon)) \tag{29}$$

for $j \geq j_\varepsilon$. Indeed, it suffices to check that

$$B(x, r - \varepsilon) \subset f^{-1}(f_j(B)) \subset B(x, r + \varepsilon).$$

The second inclusion follows using the uniform convergence of our sequence and the uniform continuity of f^{-1} on $f(\frac{3}{2}B)$. Regarding the first inclusion, notice that, given $\tilde{\varepsilon}$ there is $j_{\tilde{\varepsilon}}$ so that

$$|f^{-1} \circ f_j(y) - y| \leq \tilde{\varepsilon}$$

for all y with $|x - y| = r$ when $j \geq j_{\tilde{\varepsilon}}$. Thus the desired inclusion follows by applying Lemma 11.10 to

$$h(z) = \frac{1}{r} \left(f^{-1} \circ f_j(rz + x) - f^{-1} \circ f_j(x) \right).$$

Because f is quasiconformal and $|\partial B| = 0$, we conclude from Corollary 4.12 that $|f(\partial B)| = 0$, Thus, it follows from Remark 6.1 and (29) that

$$\int_B |J_{f_j}| = |f_j(B)| \overset{j \to \infty}{\longrightarrow} |f(B)| = \int_B |J_f|.$$

Now

$$\int_B |Df_j|^n \leq K \int_B |J_{f_j}| \leq M$$

for some finite M because $|f_j(B)| \to |f(B)| < \infty$. Moreover, there is $M' < \infty$ so that $|f_j(x)| \leq M'$ for $x \in \overline{B}$ for all j. Thus the sequence (f_j) is bounded in $W^{1,n}(B, \mathbb{R}^n)$ and so a subsequence converges to some $g \in W^{1,n}(B, \mathbb{R}^n)$ weakly, i.e.

$$f_{j_k} \rightharpoonup g, \ Df_{j_k} \rightharpoonup Dg \quad \text{in } L^n(\Omega).$$

Because $f_j \to f$ uniformly on B we conclude that $g = f$. Thus

$$\int_B |Df|^n = \int_B |Dg|^n \leq \liminf_{k \to \infty} \int_B |Df_{j_k}|^n$$

$$\leq K \liminf_{k \to \infty} \int_B |J_{f_k}| = K \int_B |J_f|.$$

Let $x \in \Omega$ be a Lebesgue point both for $|Df(x)|^n$ and $|J_f(x)|$. Then

$$|Df(x)|^n = \lim_{r \to 0} \fint_{B(x,r)} |Df|^n \leq K \lim_{r \to 0} \fint_{B(x,r)} |J_f| = K|J_f(x)|,$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.6 Corollary. Let $f_j : B \to f_j(B) \subset \mathbb{R}^n$ be K-qc. Assume that the sequence $(f_j)_j$ is bounded in $W^{1,1}(B; \mathbb{R}^n)$. Then a subsequence converges locally uniformly to a continuous mapping $f \in W^{1,n}(\Omega; \mathbb{R}^n)$. If f is a homemorphism, then f is K-qc.

Proof. Fix $\tilde{B} \subset 2\tilde{B} \subset B$. By Remark 6.6

$$|f_j(x) - f_j(y)| \leq C|x - y|^{1 - \frac{n}{p}} |B|^{\frac{1}{p} - 1} \int_B |Df_j|$$

whenever $x, y \in \tilde{B} \subset 2\tilde{B} \subset B$, and so our sequence is equicontinuous on $\tilde{B}$. Also, each f_j is η-quasisymmetric in $\tilde{B}$ with some η independent of j, which together with the estimate

$$\int_B |f_j| \leq M < \infty$$

implies that $|f_j| \leq M'$ on $\tilde{B}$. Invoking the Arzela-Ascoli theorem we may apply Theorem 7.1 to conclude the claim. $\qquad\square$

8 On 1-quasiconformal mappings

As mentioned earlier, each conformal mapping is 1-qc. Thus there are plenty of 1-qc mappings in the plane. However, the structure of global 1-qc mappings is simple in all dimensions $n \geq 2$. This also holds for 1-quasiconformal mappings according to the metric definition, see part 1) of Remarks 5.14.

8.1 Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be 1-qc, $n \geq 2$. Then there is a constant $M > 0$ so that
$$|f(x) - f(y)| = M|x - y|$$
for all $x, y \in \mathbb{R}^n$.

Theorem 8.1 does not extend to the case $n = 1$ (for the metric definition) as is seen by considering the 1-quasiconformal mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. We postpone the proof of Theorem 8.1 for a while and continue with a version of the Liouville theorem according to which there are very few 1-qc mappings in dimensions $n \geq 3$. This result is due to Gehring.

8.2 Theorem. Let $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 3$, be domains and $f : \Omega \to \Omega'$ be 1-qc. Then f is the restriction of a Möbius transformation to Ω.

Recall that a Möbius transformation is a finite composition of reflections with respect to spheres and hyperplanes.

The proof of Theorem 8.2 will be based on the usual Liouville theorem which assumes *a priori* regularity of the mappings in question.

8.3 Theorem. (Liouville) Let $\Omega, \Omega' \subset \mathbb{R}^n$ be domains, $n \geq 3$, and $f : \Omega \to \Omega'$ be 1-qc, $f \in C^3(\Omega)$ and $J_f > 0$ in Ω. Then f is the restriction of a Möbius transformation to Ω.

We omit the proof and refer the reader to [17] for a proof.

Proof of Theorem 8.2. We may assume that $J_f(x) \geq 0$ almost everywhere in Ω, see Remark 5.2. Notice that f is locally Lipschitz and so is f^{-1} (both are Hölder continuous with exponent 1 by part 2) of Remarks 6.8. Thus

$$\frac{|x - y|}{C} \leq |f(x) - f(y)| \leq C|x - y|$$

when x is fixed and y is sufficiently close to x; C may depend on x but it is locally bounded. Consequently J_f is bounded away from zero locally (almost everywhere). Because f is 1-qc, we have that $|Df(x)|^n = J_f(x)$ almost everywhere with $J_f(x) > 0$. Fix such an x. We conclude from basic linear algebra (see Proposition 11.3 and Proposition 11.4 in the appendix) that

$$|Df(x)h| = J_f(x)^{1/n}|h|$$

for each $h \in \mathbb{R}^n$. Thus

$$\operatorname{ad} Df(x) = J_f(x)^{1-2/n} Df(x)^t$$

by Proposition 11.5. Let e_j be one of the coordinate vectors. Then the previous equation shows that

$$\operatorname{ad} Df(x)e_j = J_f(x)^{1-2/n} \nabla f_j(x).$$

Notice further that $|\nabla f_j(x)| = |Df(x)^t e_j| = J_f(x)^{1/n}$. We thus conclude from Proposition 11.8 that f_j is n-harmonic in Ω, and thus C^1 by Proposition 11.6. Because $|\nabla f_j(x)|$ is (locally) bounded away from zero, it follows from Proposition 11.7 that f is C^∞-smooth. The claim thus follows from Theorem 8.3.

8.4 Lemma. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism so that

$$f\big(S^{n-1}(x, r)\big) = S^{n-1}\big(f(x), R_{x,r}\big)$$

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for all $x \in \mathbb{R}^n$, $r > 0$. Then there is $M > 0$ so that

$$|f(x) - f(y)| = M|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Proof. Let us first observe that lines get mapped to lines: If z is the midpoint
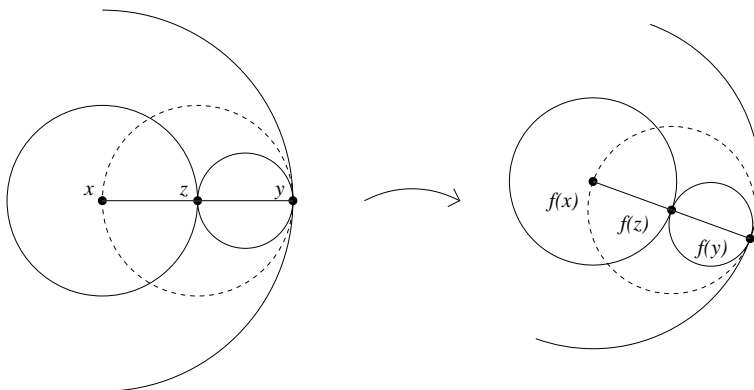

Figure 7: Line segment is mapped to a line segment

of $[x, y]$, then $f(z)$ lies on $[f(x), f(y)]$ and furthermore

$$|f(x) - f(z)| = |f(z) - f(y)|,$$

as we can see from Figure 7. By iterating, we see that for a given line L there is M_L so that $|f(x) - f(y)| = M_L|x - y|$ whenever $x, y \in L$.
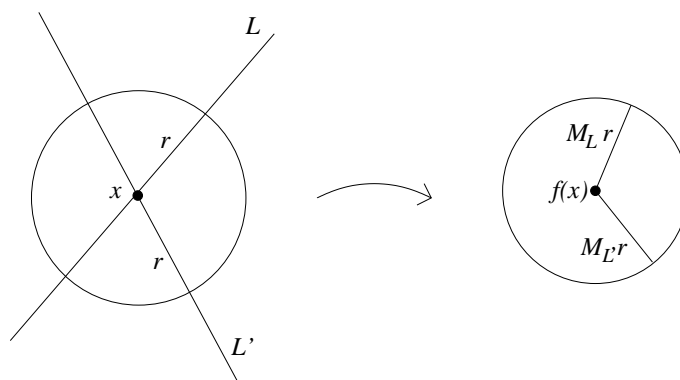
Figure 8: when $L \cap L' \neq \emptyset$

Let then L and L' be lines. Suppose first that $L \cap L' \neq \emptyset$. If $L = L'$, then $M_L = M_{L'}$. Otherwise the setting looks like in Figure 8 and thus $M_L = M_{L'}$.

If $L \cap L' = \emptyset$, pick L'' so that $L \cap L'' \neq \emptyset$ and $L' \cap L'' \neq \emptyset$. $\qquad\square$

Proof of Theorem 8.1. It suffices to show that

$$f\big(S^{n-1}(x,r)\big) = S^{n-1}\big(f(x), R_{x,r}\big) \tag{30}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ (Lemma 8.4 will then give the claim). Fix x and r. By using translations, rotations and dilations, we may assume that $x = 0$, $r = 1$, $f(e_1) = e_1$ and $B(0,1) \subset f(B(0,1))$.

Set

$$W = \big\{ f : \mathbb{R}^n \to \mathbb{R}^n : f \text{ is 1-qc }, f(0) = 0, f(e_1) = e_1 \text{ and } B(0,1) \subset f(B(0,1)) \big\}$$

Define $a = \sup_{f \in W} |f(B(0,1))|$. Then $a < \infty$ because each such f is η-qs with a fixed η and so $f(B(0,1)) \subset B(0, \eta(1))$.

We will show that $a = |B(0,1)|$. Clearly $a \geq |B(0,1)|$. Suppose $a > |B(0,1)|$ and pick a sequence $(f_j)_j$ of mappings in W so that $|f_j(B(0,1))| \to a$. Then (f_j) is bounded in $W^{1,n}(2B)$. Indeed

$$f_j(B(0,2)) \subset B(0, \eta(1)\eta(2)),$$

and so

$$\int_{2B} |Df|^n \leq \int_{2B} |J_f| \leq c_0.$$

Thus, by Corollary 7.6, $f_{j_k} \to g$ uniformly in $B(0, 3/2)$ for some mapping g and some subsequence $(f_{j_k})_k$. Because $f_{j_k}(0) = 0$ and $f_{j_k}(e_1) = e_1$ and each f_{j_k} is η-quasisymmetric, it follows from the uniform convergence that g is a homeomorphism. Invoking Corollary 7.6 again, we conclude that g is 1-qc. As in the proof of Theorem 7.1, we see that $B(0,1) \subset g(B(0,1))$ (and that $|g(B(0,1))| = a$). Thus $g \in W$. Notice that $g(B(0,1)) \setminus B(0,1)$ contains some non-trivial open set U because $|g(B(0,1))| = a > 1$. Clearly $|g(U)| > 0$.

Consider $h = g \circ g$. Now $h \in W$ and

$$|h(B(0,1))| = |g\big(g(B(0,1))\big)| \geq |g(B(0,1)) \cup g(U)| \geq a + |g(U)| > a,$$

which contradicts the definition of a.

We have proven that $a = |B(0,1)|$. Returning to our fixed mapping f, this shows that $|f(B(0,1))| = |B(0,1)|$. By assumption, $B(0,1) \subset f(B(0,1))$, and we conclude that $f(B(0,1)) = B(0,1)$. It follows that $f(B(x,r)) = B(f(x), R_{x,r})$ for all x, r. This implies (30). $\qquad\square$

8.5 Remark. The proof of Theorem 8.1 was based on a compactness argument. In fact, compactness can be used to characterize quasiconformality in the following sense.

We call a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ similarity if there is a constant $\lambda > 0$ so that $|T(x) - T(y)| = \lambda |x - y|$ for all $x, y \in \mathbb{R}^n$. Next, we say that a family $\mathcal{F}$ of homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ is complete with respect to similarities if, for each $f \in \mathcal{F}$ and all similarities T, S, the composite mapping $g = T \circ f \circ S$ also belongs to $\mathcal{F}$. We call a homemorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ normalized if $f(0) = 0$ and $f(e_1) = e_1$, where e_1 is the unit vector in the x_1-direction. Then the family $\mathcal{F}$ is said to satisfy the compactness condition if every infinite set of normalized mappings in $\mathcal{F}$ contains a subsequence which converges locally uniformly to a homeomorphism.

We have the following result: Let a family $\mathcal{F}$ of homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, be complete with respect to similarities. Then $\mathcal{F}$ satisfies the compactness condition if and only if there is $1 \leq K < \infty$ so that each $f \in \mathcal{F}$ is K-qc.

The above statement is not hard to prove using the results and ideas gathered this far. The compactness condition for K-qc mappings follows using Corollary 7.6 and the normalization once we recall that each of the mappings f is η-quasisymmetric with a fixed η. For the converse, one first proves that there is $H < \infty$ so that $H_f(x, r) \leq H$ for each $f \in \mathcal{F}$, all $x \in \mathbb{R}^n$ and every $r > 0$ and then applies the equivalence of the metric and analytic definitions.

Here is a sketch of a proof of the estimate on $H_f(x, r)$. By the compactness property it easily follows that there is $H < \infty$ so that $|f(x)| \leq H$ for each normalized $f \in \mathcal{F}$ and all $x \in S^{n-1}(0, 1)$. Given $f \in \mathcal{F}$, x, and $r > 0$, pick $y \in S^{n-1}(x, r)$ that realizes $l_f(x, r)$. Map e_1 to y and 0 to x using a similarity S, $f(x)$ to 0 and $f(y)$ to e_1 using a similarity T, and apply the above bound to $g = T \circ f \circ S$.

9 Mapping theorems

We begin by discussing the planar setting. It is convenient to use complex notation: we identify $\mathbb{R}^2$ with $\mathbb{C}$ and write a point $z \in \mathbb{C}$ as $z = x + iy$, where x, y are real. Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{C})$ be continuous, where $\Omega \subset C$ is a domain. Writing $f(z) = u(z) + iv(z)$ with u, v real-valued, we notice that both u and v have, at almost every z, partial derivatives u_x, u_y, v_x, v_y with respect to x, y. Then

$$\partial_x f(z) = u_x(z) + iv_x(z),$$
$$\partial_y f(z) = u_y(z) + iv_y(z).$$

We will employ the derivatives ∂f, $\overline{\partial} f$ defined by

$$\partial f(z) = \frac{1}{2}(\partial_x f(z) - i\partial_y(f)),$$

$$\overline{\partial} f(z) = \frac{1}{2}(\partial_x f(z) + i\partial_y(f)).$$

Recalling the Cauchy-Riemann equations

$$u_x = v_y, u_y = -v_x,$$

we notice that $\overline{\partial} f(z) = 0$ if f is analytic. In fact, for a continuous $f \in W^{1,1}_{loc}(\Omega; \mathbb{C})$, $\overline{\partial} f(z) = 0$ almost everywhere only when f is analytic.

Let us further denote by $\partial_\alpha f(z)$ the derivative of f in the direction $e^{i\alpha}$ (if it happens to exist). In the real notation, this is simply $Df(x,y)(\cos\alpha, \sin\alpha)$ if f is differentiable at the point (x, y) and it is easy to check that, in our complex notation,

$$\partial_\alpha f(z) = \partial f(z)e^{i\alpha} + \overline{\partial} f(z)e^{-i\alpha}. \tag{31}$$

In fact, one has for each $h \in \mathbb{C}$

$$Df(z)h = \partial f(z)h + \overline{\partial} f(z)\overline{h},$$

where $\overline{h}$ is the complex conjugate of h (for $h = x + iy$, $\overline{h} = x - iy$). Now $\partial_\alpha f(z)$ has maximal length when the two vectors in the sum (31) point to the same direction, i.e. when

$$\alpha + \arg \partial f(z) = -\alpha + \arg \overline{\partial} f(z)$$

(modulo 2π), and minimal length when these two vectors point to opposite directions. Here $\arg w$ denotes the argument of a complex number w. Thus the maximal directional derivative has the value

$$|\partial f(z)| + |\overline{\partial} f(z)|$$

and corresponds to the choice

$$\alpha = \frac{1}{2}(\arg \overline{\partial} f(z) - \arg \partial f(z))$$

and one has the minimal value

$$||\partial f(z)| - |\overline{\partial} f(z)||$$

corresponding to

$$\alpha = \frac{\pi}{2} + \frac{1}{2}(\arg \overline{\partial} f(z) - \arg \partial f(z)).$$

Moreover,

$$|J_f(z)| = |(|\partial f(z)| + |\overline{\partial} f(z)|)(|\partial f(z)| - |\overline{\partial} f(z)|)|$$
$$= ||\partial f(z)|^2 - |\overline{\partial} f(z)|^2|.$$

9.1 Theorem. Let $\mu : \mathbb{C} \to \mathbb{C}$ satisfy $||\mu||_{L^\infty} < 1$. Then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ so that

$$\overline{\partial} f(z) = \mu(z) \partial f(z)$$

almost everywhere.

This is a very strong existence theorem. Notice that $J_f(z) \neq 0$ almost everywhere because f is quasiconformal. Thus the discussion before Theorem 9.1 shows that

$$\frac{|Df(z)|^2}{|J_f(z)|} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

almost everywhere. Moreover, for almost every z,

$$H_f(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

and the differential $Df(z)$ maps disks $B(z, r)$ centered at z to ellipses with major axes of the length

$$2|Df(z)|r = 2|\partial f(z)|r(1 + |\mu(z)|)$$

and minor axes of the length

$$2|\partial f(z)|r(1 - |\mu(z)|).$$

The orientation of these ellipses is not determined by $\mu(z)$. However, consider the collection of all ellipses E with center x so that the ratio of the major and the minor axis is $H_f(z)$ and the angle determined by the minor axis and the real line is

$$\alpha = \frac{1}{2} \arg \mu(z).$$

Then the differential $Df(z)$ maps these ellipses to discs centered at $f(z)$.

We will omit the proof of Theorem 9.1 and refer the reader to [4] for the proof and further extensions of this existence theorem.

Let us recall the Riemann mapping theorem, see [23] for a proof.

9.2 Theorem. (Riemann Mapping Theorem) Each simply connected domain $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disk.

It follows that, given simply connect, proper subdomains Ω, Ω' of the plane, there is a conformal mapping $f : \Omega \to \Omega'$. We continue with a quasiconformal version of this statement.

9.3 Theorem. (Measurable Riemann Mapping Theorem) Let $\Omega, \Omega' \subsetneq \mathbb{C}$ be simply connected subdomains and suppose that $\mu : \Omega \to \mathbb{C}$ satisfies $\|\mu\|_{L^\infty} < 1$. Then there is a quasiconformal mapping $f : \Omega \to \Omega'$ so that

$$\overline{\partial} f(z) = \mu(z) \partial f(z) \quad \text{a.e. in } \Omega.$$

In fact, f is $\dfrac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$-qc.

Proof. Given Ω, Ω' and μ, we extend μ as zero to the rest of $\mathbb{C}$. Then Theorem 9.1 gives us a quasiconformal mapping as asserted, except for the requirement that $f(\Omega) = \Omega'$. In any case, $f(\Omega)$ is a simply connected proper subdomain of $\mathbb{C}$, and thus the usual Riemann mapping theorem provides us with a conformal mapping $g : f(\Omega) \to \Omega'$. Setting $\tilde{f} = g \circ f$, it is easy to check using the "chain rules"

$$\partial(g \circ h) = \partial g(h) \partial h + \overline{\partial} g(h) \partial \overline{h},$$

$$\overline{\partial}(g \circ h) = \partial g(h) \overline{\partial} h + \overline{\partial} g(h) \overline{\partial h},$$

that $\tilde{f}$ has all the required properties. $\qquad\square$

We deduce from Theorem 8.2 that there is no Riemann mapping theorem in higher dimensions.

9.4 Corollary. Let $f : B^n \to f(B^n) \subset \mathbb{R}^n$ be 1-qc, $n \geq 3$. Then $f(B^n)$ is a ball or a half space.

One could still hope for a "quasiconformal Riemann mapping theorem" for $n \geq 3$. Unfortunately, this hope is futile:

9.5 Example. Let $\Omega \subset \mathbb{R}^3$ be as in Figure 9. Then there is no quasiconformal mapping $f : B^3(0, 1) \to \Omega$.
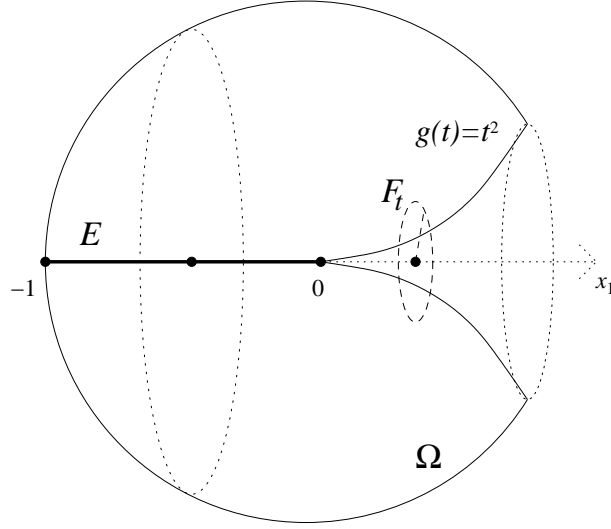
Figure 9: Domain Ω of the example 9.5

Reason : Suppose there is a quasiconformal mapping $f : B^3(0,1) \rightarrow \Omega$. Pick a circle F_t of radius $2t^2$ around the cusp at the level $x_1 = t$ and let $E = [-1,0]$ on x_1-axis. Then (see Figure 10)

$$\mathrm{cap}_n(E, F_t\,;\Omega) \leq \mathrm{cap}_n\left(\overline{B}\big((t,0,0), 2t^2\big)\,,\; S^2\big((t,0,0)\,,t\big)\,;\Omega\right)$$

$$= \frac{\omega_2}{\left(\log \frac{t}{2t^2}\right)^2} \; \longrightarrow 0 \quad \text{when } t \rightarrow 0\,.$$
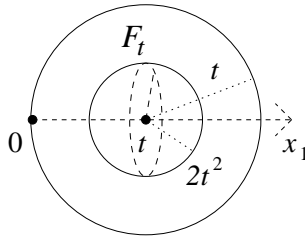
Figure 10: $\overline{B}\big((t,0,0)\,,2t^2\big)$ and $S^2\big((t,0,0)\,,t\big)$.

Because f is K-qc for some K, it follows that

$$\mathrm{cap}_3\left(f^{-1}(E), f^{-1}(F_t); B^3(0,1)\right) \leq K\,\mathrm{cap}_3(E, F_t\,;\Omega) \; \longrightarrow 0 \quad \text{when } t \rightarrow 0\,.$$

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But, on the other hand,

$$\frac{\min\left\{\operatorname{diam} f^{-1}(E), \operatorname{diam} f^{-1}(F_t)\right\}}{d\left(f^{-1}(E), f^{-1}(F_t)\right)} \geq 10^{-6}$$

for all t, and thus

$$\operatorname{cap}_3\left(f^{-1}(E), f^{-1}(F_t); B^3(0,1)\right) \geq \delta(3, 10^{-6}) > 0.$$

To be precise, we have cheated a bit above. Indeed, E intersects the boundary of Ω and thus it is not clear if $f^{-1}(E)$ is compact (nor even if f^{-1} has an extension to the points $-1, 0$). It is easy to fix this by replacing E with $E_j \subset E$ which is the segment $[-1 + 1/j, -1/j]$ with j sufficiently large. Notice that $f^{-1}(y)$ necessarily tends to the boundary of $B^3(0,1)$ when y tends to $\partial\Omega$.

By the above example, not every topologically nice $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is quasiconformally equivalent to the unit ball. One does not in fact know any general geometric criteria for this equivalence. The following result due to Gehring gives a sufficient condition for quasiconformal equivalence. For a proof see [30].

9.6 Theorem. If $\partial\Omega$ is diffeomorphic to $S^{n-1}(0,1)$, then there is a quasiconformal mapping $f : B^n(0,1) \to \Omega$.

Based on Corollary 9.4 it is natural to ask if domains in $\mathbb{R}^n$, $n \geq 3$, that are K-qc equivalent to the unit ball for a suitably small K are more regular than one a priori expects. This turns out to be true in the sense that they are even quasisymmetrically equivalent to the unit ball.

9.7 Theorem. Let $n \geq 3$. There exists $K_0 = K_0(n) > 1$ such that if $f : B^n \to f(B^n) \subset \mathbb{R}^n$ is K-qc, $1 \leq K < K_0$ and bounded, then f is quasisymmetric. In particular, f extends to a homeomorphism $\tilde{f} : \overline{B}^n \to \overline{f(B^n)}$.

This theorem is from [3], [27]. The proof heavily relies on results due to Reshetnyak [23] that essentially give an asymptotic version of Theorem 8.2 when the distortion K tends to 1.

9.8 Remark. There are still plenty of quasiconformal mappings. For example, there is a quasiconformal mapping $f : B^n \to f(B^n) \subset \mathbb{R}^n$ so that $|\partial\Omega| = \infty$. See [31] for this.

10   Examples of quasiconformal mappings

10.1 Example. (Basic mappings)

1) *Linear transformations:* If $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then f is quasiconformal.

2) *Radial stretchings:* Let $f(x) = x|x|^{a-1} = \frac{x}{|x|}|x|^a$, where $0 < a < \infty$. Then f is K-qc, where

$$K = \begin{cases} a^{n-1} & \text{if } a \geq 1 \\ a^{-1} & \text{if } 0 < a < 1 \,. \end{cases}$$

In the planar setting, it is easy to establish this estimate on K by using complex notation. Indeed, let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z|z|^{a-1} = z^{(1+a)/2} \, \overline{z}^{(a-1)/2}$. Then

$$\overline{\partial} f(z) = \tfrac{1}{2}(a-1)z^{\frac{1}{2}(1+a)}\overline{z}^{\frac{1}{2}(a-3)}$$

$$\partial f(z) = \tfrac{1}{2}(a+1)z^{\frac{1}{2}(a-1)}\overline{z}^{\frac{1}{2}(a-1)} \,,$$

so

$$\mu(z) = \frac{\overline{\partial} f}{\partial f} = \frac{a-1}{a+1}\frac{z}{\overline{z}} \,.$$

Thus $|\mu(z)| = |a-1|/(a+1)$, and the desired estimate follows by the discussion in the beginning of Chapter 9.

The higher dimensional setting requires a bit more thinking. We leave this to the reader with the following hints. First of all, f maps balls centered at the origin to balls centered at the origin. Let $x \neq 0$. The matrix $Df(x)$ is diagonal when x lies on the x_1-axis and the required estimate then easily follows. Also, the image of $B(x, r)$ in this case is approximatively determined by the image ellipsoid of $B(x, r)$ under the linear transformation corresponding to $Df(x)$. Next, given $x \neq 0$, the image of $B(x, r)$ under f is, modulo a rotation, the image of $B(z, r)$, where z lies on the x_1-axis and satisfies $|z| = |x|$, and f is differentiable at x. Combining this with the approximation from above gives the claim.

3) *Folding maps:* Let (r, φ, z) be the cylindrical coordinates of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$; this means that $r > 0$, $0 \leq \varphi < 2\pi$, $z \in \mathbb{R}^{n-2}$, and

$$x_1 = r\cos\varphi \,, \quad x_2 = r\sin\varphi \quad \text{and } z = (x_3, \ldots, x_n) \,.$$

Let $0 < \alpha, \beta \leq 2\pi$, and let $\Omega_\alpha = \{(r, \varphi, z) : \ 0 < \varphi < \alpha\}$, $\Omega_\beta = \{(r, \varphi, z) : 0 < \varphi < \beta\}$. Then the mapping $f : \Omega_\alpha \to \Omega_\beta$, $(r, \varphi, z) \to (r, (\beta/\alpha)\varphi, z)$ is K-qc, where

$$K = \begin{cases} (\beta/\alpha)^{n-1} & \text{for } \alpha \leq \beta \\ \alpha/\beta & \text{for } \alpha > \beta \,. \end{cases}$$

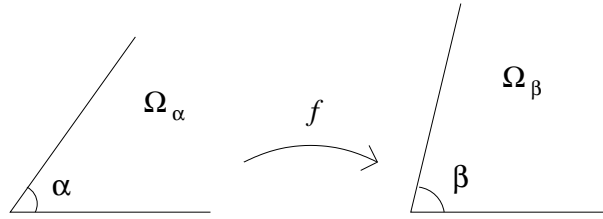The estimate on K is obtained using the diagonal representation of $Df(x)$ obtained using suitable orthonormal coordinates.

Figure 11: Folding map $f : \Omega_\alpha \to \Omega_\beta$

4) *Cone map:* Let (R, φ, θ) be the spherical coordinates of $(x_1, x_2, x_3) \in \mathbb{R}^3$; this means that $R > 0$, $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$, and

$$x_1 = R \sin \theta \cos \varphi \quad x_2 = R \sin \theta \sin \varphi \quad \text{and} \quad x_3 = R \cos \theta \,.$$

For $0 < \alpha \le \pi$ the domain $C_\alpha = \{(R, \varphi, \theta) : 0 \le \theta < \alpha\}$ is called a cone of angle α. The mapping $f : C_\alpha \to C_\beta$, $(R, \varphi, \theta) \to (R, \varphi, \beta\theta/\alpha)$ (see Fig. 12 for the special case where $\beta = \pi/2$), is K-qc for $0 < \alpha \le \beta < \pi$, where

$$K = \frac{\beta^2 \sin \alpha}{\alpha^2 \sin \beta} \,.$$

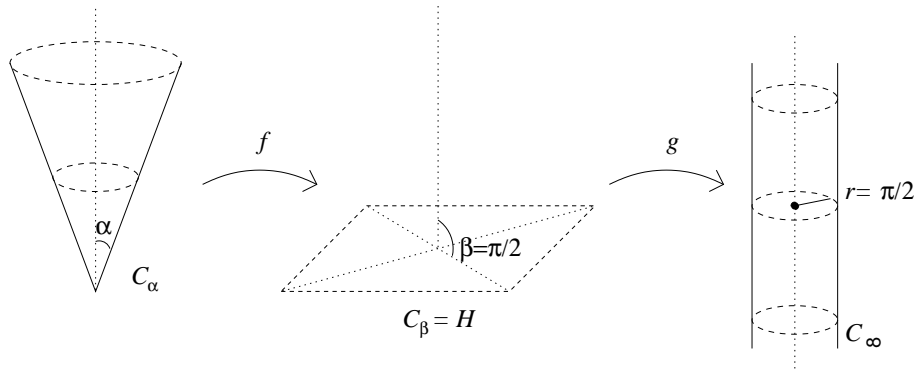For $\beta = \pi$ the quasiconformality fails. Use similar coordinates as for 3) to verify the claim.

Figure 12: Maps $f : C_\alpha \to C_{\pi/2}$ and $g : H \to C_\infty$.

5) *Cone to an infinite cylinder:* Let H be the half-space determined by $H = C_{\pi/2}$. Let C_∞ be the infinite cylinder $C_\infty = \{(r, \varphi, x_3) : r \le \pi/2\}$

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(in cylindrical coordinates). Then $g : H \to C_\infty$, which maps the point $(R, \varphi, \theta) \in H$ (spherical) to $(r = \theta, \varphi, x_3 = \log R) \in C_\infty$ (cylindrical), is $\pi^2/4$-qc; see Figure 12. Especially, for each cone C_α of angle $0 < \alpha < \pi$, there is a quasiconformal mapping $h : C_\alpha \to C_\infty$.

10.2 Example. ("Dust to dust") Given $n \geq 2$ and $0 < \lambda < n$, $0 < \lambda' < n$, there is a K-qc map $f : \mathbb{R}^n \to \mathbb{R}^n$ and Cantor sets E, E' of Hausdorff dimensions λ, λ', respectively, so that $f(E) = E'$. Here K depends on n, λ, λ'.

Reason : Let $I = [0, 1]^n \subset \mathbb{R}^n$ and I_i, $i = 1, \ldots, 2^n$ be the dyadic subcubes of I with side length $\frac{1}{2}$. Fix $0 < s < \frac{1}{2}$ and for each $i = 1, \ldots 2^n$ pick a similarity mapping $g_i : I \mapsto I_i : x \mapsto sx + a_i$, where $a_i \in I_i$ is chosen so that the centers of I_i and $Q_i = g_i(I)$ coincide. Let

$$F_j = \bigcup_{1 \leq i_1, i_2 \ldots, i_j \leq 2^n} g_{i_1} \circ g_{i_2} \circ \ldots \circ g_{i_j}(I). \tag{32}$$

It is easy to see that $F_1 \supset F_2 \supset \ldots$. Moreover, the cubes $g_{i_1} \circ \ldots \circ g_{i_j}(I)$ and $g_{i'_1} \circ \ldots \circ g_{i'_j}(I)$ are disjoint if $i_k \neq i'_k$ for some $1 \leq k \leq j$. We define a Cantor set C_s^n by setting

$$C_s^n = \bigcap_{j=1}^{\infty} F_j. \tag{33}$$

Then the Hausdorff dimension of C_s^n is $n\frac{\log \frac{1}{2}}{\log s}$, see [20].

Fix $0 < s < \frac{1}{2}$ and $0 < s' < \frac{1}{2}$ and the corresponding Cantor constructions as above. It is easy to see that there exists a K-quasiconformal $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ so that $f_1(x) = x$ outside I, and $f_1(x) = g'_i \circ g_i^{-1}(x)$ if $x \in Q_i$, where K depends basically only on the ratio $\frac{\frac{1}{2}-s'}{\frac{1}{2}-s}$. For example, define $\psi : [-\frac{1}{4}, \frac{1}{4}]^n \to [-\frac{1}{4}, \frac{1}{4}]^n$ by

$$\psi(x) = \begin{cases} \frac{s'}{s}x, & \text{when } 0 \leq q||x||_{\max} \leq \frac{s}{2} \text{ and} \\ x\left(\frac{\frac{1}{2}-s'}{\frac{1}{2}-s} + \frac{s'-s}{4||x||_{\max}(\frac{1}{2}-s)}\right) & \text{when } \frac{s}{2} \leq ||x||_{max} \leq \frac{1}{4}, \end{cases}$$

and finally set for $x \in I_i$ that

$$f_1(x) = \psi(x - b_i) + b_i, \tag{34}$$

where b_i denotes the center of I_i (which also is the center of Q_i). On I^c, we define f_1 to be identity. It is an easy exercise to check that f_1 satisfies the desired properties (see Figure 13 in the two-dimensional case).
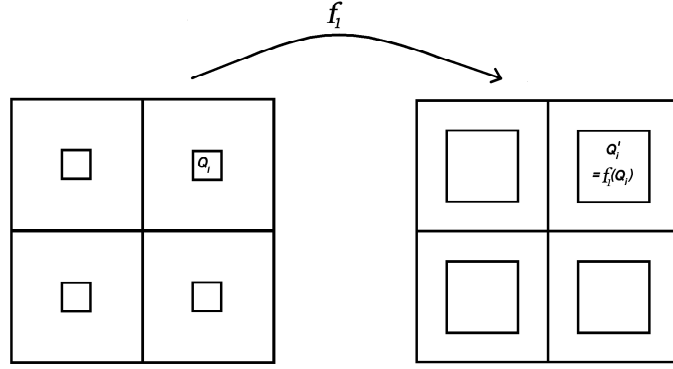
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Figure 13: The initial map f_1

We define a sequence of functions f_j inductively: assuming that f_j is defined, we define the mapping f_{j+1} by setting $f_{j+1}(x) = f_j(x)$ outside F_j and

$$f_{j+1}(x) = g'_{i_1} \circ f_j \circ g_{i_1}^{-1}, \text{ if } x \in g_{i_1} \circ \cdots \circ g_{i_j}(I) \tag{35}$$

when $x \in F_j$. It is easy to check that f_j is a homemorphism that maps F_j onto F'_j. Moreover, because each g_i and g_j^{-1} is 1-qc, each f_j is K-qc with the constant K corresponding to the construction of f_1 above.

It is immediate from the construction that the sequence $(f_j)_j$ of K-qc maps converges uniformly to a homemorphism f that maps C_n^s onto $C_n^{s'}$. From Theorem 7.1 we deduce that f is K-qc.

10.3 Example. (Reflection) Let $f : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a K-qc map that maps bounded sets to bounded sets. Then f is quasisymmetric and thus f extends to a (quasisymmetric) homeomorphism $\tilde{f} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$. Define

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x_n > 0 \\ \tilde{f}(x) & \text{if } x_n = 0 \\ \overline{f(\overline{x})} & \text{if } x_n < 0 \,, \end{cases}$$

where $\overline{x} = (x_1, x_2, \ldots, -x_n)$. Then $\hat{f}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is K-qc.

Reason : Repeat the argument we used to prove that the analytic definition implies the metric definition (Theorem 5.1) to see that f is quasisymmetric (see Figure 14). For the K-quasiconformality of $\hat{f}$ it suffices to check that $\hat{f} \in W_{\mathrm{loc}}^{1,n}$. For each bounded $G \subset \mathbb{R}^n$ we have

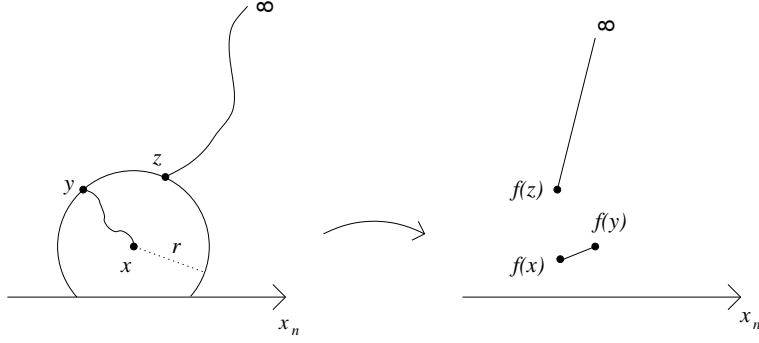$$\int_{G_+} |Df|^n \le K \int_{G_+} |J_f| < \infty,$$

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Figure 14: $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

because f maps bounded sets to bounded sets. Similarly, $\int_{G_-} |D\hat{f}|^n < \infty$.

Thus we only need to check that

$$\int \partial_i f_j \varphi = - \int f_j \partial_i \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n).$$

This is trivial when $i = 1, \ldots, n-1$; almost every line parallel to the first $n-1$ coordinate axes lies either in the upper half space or in the lower one. For $i = n$, integrate by parts along lines up to boundary in both sides; the boundary term showing up gets cancelled because f is continuous.

10.4 Example. (Lifting) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasisymmetric, $n \geq 1$. Then there is a quasiconformal mapping $\hat{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ so that $\hat{f}|_{\mathbb{R}^n} = f$.

Reason : For $n = 1$ define

$$\hat{f}(x, y) = \left(\tfrac{1}{2} \int_0^1 f(x + ty) + f(x - ty)\, dt, \int_0^1 f(x + ty) - f(x - ty)\, dt \right)$$

for $y > 0$ and use reflection (Example 10.3). This is the Beurling-Ahlfors extension.

The high dimensional case is hard essentially because of topological difficulties. The setting $n = 2$ is due to Ahlfors [1], $n = 3$ to Carleson [9] and $n \geq 4$ to Tukia and Väisälä [29]. Notice that, in dimensions $n \geq 2$, we could simply assume that f be quasiconformal. For $n = 1$ one really needs to assume quasisymmetry because there exist quasiconformal mappings of the real line that fail to be quasisymmetric.

10.5 Example. ("Generalized lifting") Let $f : \mathbb{R} \to f(\mathbb{R}) \subset \mathbb{R}^2$ be quasisymmetric. Then there is quasiconformal mapping $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ so that $\hat{f}|_{\mathbb{R}} = f$.

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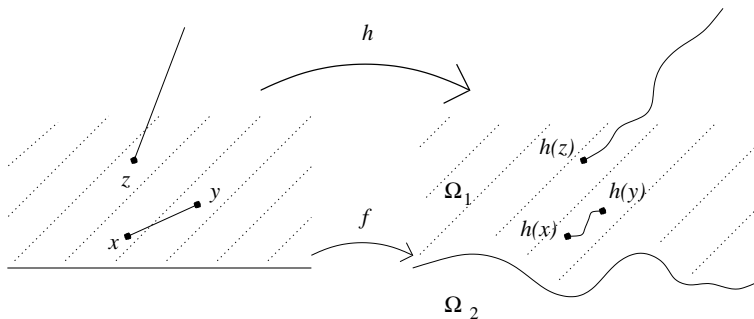Figure 15: Conformal $h : \mathbb{R}^2_+ \to \Omega_1$

Reason : (See Figure 15.) One can show using the fact that $f : \mathbb{R} \to \partial\Omega_1$ is quasisymmetric that Ω_1 is LLC. Then the Riemann mapping theorem gives us a conformal mapping $h : \mathbb{R}^2_+ \to \Omega_1$ and g is quasisymmetric by the usual arguments (we may assume that h maps bounded sets to bounded sets; see the proof of Theorem 5.1). We can then extend h to a quasisymmetric mapping $\tilde{h} : \overline{\mathbb{R}^2_+} \to \overline{\Omega}_1$. Now $\tilde{h}^{-1} \circ f : \mathbb{R} \to \mathbb{R}$ is also quasisymmetric. By lifting, there is a quasiconformal mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ so that $g|_{\mathbb{R}} = \tilde{h}^{-1} \circ f$. Then $f_1 = \tilde{h} \circ g : \overline{\mathbb{R}^2_+} \to \overline{\Omega}_1$ is quasisymmetric and $f_1|_{\mathbb{R}} = f$. Repeat the same procedure to obtain a quasisymmetric mapping $f_2 : \overline{\mathbb{R}^2_-} \to \overline{\Omega}_2$ so that $f_2|_{\mathbb{R}} = f$, and define $\hat{f}$ in pieces.

10.6 Remark. There are quasisymmetric mappings $f : \mathbb{R}^n \to f(\mathbb{R}^n) \subset \mathbb{R}^{n+1}$ that do not extend to a homeomorphism $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, when $n \geq 2$.

10.7 Definition. A Jordan curve $\gamma \subset \hat{\mathbb{C}}$ is a quasicircle if there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ so that $\gamma = f(S^1)$ or $\gamma \setminus \{\infty\} = f(\mathbb{R})$.

Above, $\hat{\mathbb{C}}$ refers to the Riemann sphere (the one-point compactification of $\mathbb{C}$.) One can check that each quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ extends to a homeomorphism $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$; this extension is also quasiconformal on the Riemann sphere.
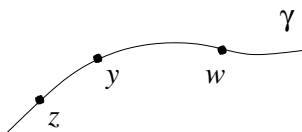
10.8 Remark. The following are equivalent:

(1) γ is a quasicircle

(2) one of the components of $\mathbb{C} \setminus \gamma$ is LLC

(3) both components are LLC

(4) If $z, w, y \in \gamma$ and y is "between" z and w, then

$$|z - y| + |w - y| \leq C|z - w|$$

with $C > 0$ independent of z, w and y.

10.9 Example. (The snowflake mapping) Take piecewise linear mappings $f_k : [0, 1] \to \mathbb{C}$ as in Figure 16. Then extend the construction to entire $\mathbb{R}$ as
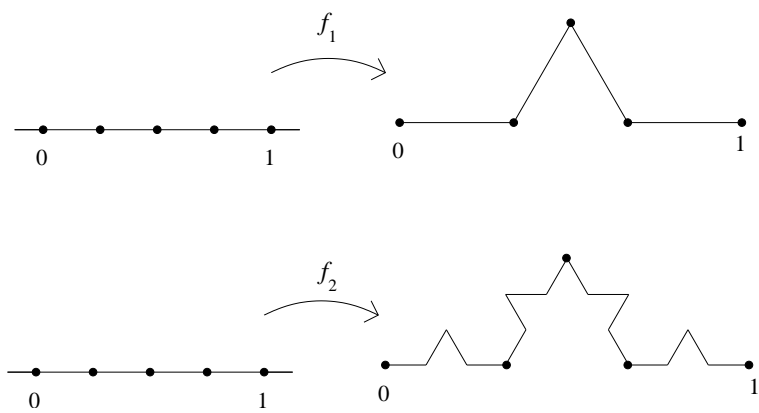

Figure 16: First iterations of the snowflake map

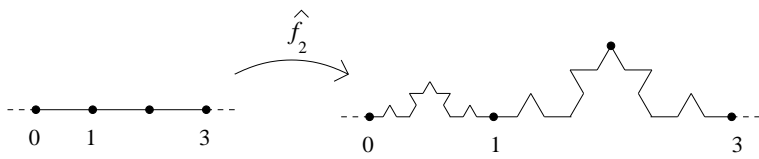in Figure 17 to obtain piecewise linear mappings $\hat{f}_k : \mathbb{R} \to \mathbb{C}$. The mappings

Figure 17: Extension of f_2 to R

$\hat{f}_k$ are uniformly quasisymmetric. By Arzela-Ascoli, we obtain a quasisymmetric mapping $f : \mathbb{R} \to \gamma \subset \mathbb{C}$, where γ is a version of the von Koch snowflake curve. The mapping f satisfies the estimate

$$\frac{1}{C} |x - y|^{\frac{\log 3}{\log 4}} \leq |f(x) - f(y)| \leq C |x - y|^{\frac{\log 3}{\log 4}}$$

for $x, y \in [0, 1]$. Eventually, take a quasiconformal extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$ of f.

10.10 Remarks. 1) One can change the construction so that, for a given $\frac{1}{2} < \alpha \leq 1$, there is f_α so that

$$\frac{1}{C}\, |x - y|^\alpha \leq |f(x) - f(y)| \leq C\, |x - y|^\alpha$$

for $x, y \in \mathbb{R}$.

2) In higher dimensions, similar constructions have been made by David and Toro for α close to 1 [10].

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11 Appendix

11.1 Conformal mappings of a square onto a rectangle

Let us explain why one cannot map a square conformally to a rectangle which is not a square, so that the vertices get mapped to the vertices. Notice that this statement is a bit ambiguous. Indeed, the conformal mapping is a priori only defined in the open square and thus the meaning of vertices being mapped to vertices is not clear. Because of the simple geometry of both of these domains, one can easily verify that any conformal mapping necessarily extends to a homeomorphism of the closed square onto the closed rectangle. Thus, we are claiming that there is no homeomorphism between

a closed square and and a closed, non-square rectangle which is conformal in the open square and maps the sides of the square to the sides of the rectangle.

Let us call the square Q and the rectangle R and the mapping f. By translating and scaling, we may assume without loss of generality that $Q =]0, 1[\times]0, 1[$ and that $R =]0, 1[\times]0, L[$ for some $L > 0$. We may further assume that the vertical sides of Q get mapped to the vertical sides of R. Consider the line segment $I_y = \{(t, y) : 0 \leq t \leq 1\}$ for $0 < y < 1$. Because $f(I_y)$ joins the vertical sides of R, we conclude that

$$\int_{I_y} |Df(x, y)| \, dx \geq L.$$

By Hölder's inequality we deduce that

$$L^2 \leq \int_{I_y} |Df(x, y)|^2 \, dx.$$

Integrating with respect to y and using the inequality

$$|Df(x, y)|^2 \leq J_f(x, y)$$

that follows from the Cauchy-Riemann equations (cf. Section 1) we arrive at

$$L^2 \leq \int_Q |Df(x, y)|^2 \, dx \, dy \leq \int_Q J_f(x, y) \, dx \, dy \leq |R|,$$

where $|R|$ is the area of R. Since $|R| = L$, we conclude that $L \leq 1$. The opposite inequality follows by reversing the roles of Q and R in the above argument.

11.2   Some linear algebra

Let A be a $n \times n$-matrix. The operator norm of A is defined by

$$|A| = \sup_{|h|=1} |Ah|,$$

where $|h|, |Ah|$ are the euclidean lengths of the given vectors. Sometimes one also uses the norm

$$\|A\|_{HS} = \sqrt{\sum_{i,j} a_{ij}^2}$$

called the Hilbert-Schmidt norm. These two norms are clearly equivalent.

11.1 Proposition. For each $h \in \mathbb{R}^n$, we have that

$$|hA| \leq |A||h|.$$

To see that this estimate holds, notice first that

$$|hA| = |A^t h^t|.$$

It thus suffices to show that $|A^t| \leq |A|$. To this end, choose a unit vector $h \in \mathbb{R}^n$ so that $|A^t h| = |A|$. Now

$$|A^t|^2 = < A^t h, A^t h > = < AA^t h, h > \leq |A||A^t h||h| \leq |A||A^t|,$$

and the claim follows.

Notice that we acually proved above that $|A^t| \leq |A|$. Recalling that $(A^t)^t = A$, we arrive at the equality

$$|A^t| = |A|. \tag{36}$$

We continue with a result according to which linear mappings can always be represented by diagonal matrices.

11.2 Proposition. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Then there are orthonormal bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ so that the matrix of L with respect to these bases is diagonal.

Proof. Pick v_1, with $|v_1| = 1$, so that $|Lv_1| = \sup_{|h|=1} |Lh|$. If $|Lv_1| = 0$, then the zero matrix will do (use then the standard bases). We will take $w_1 = \frac{Lv_1}{|Lv_1|}$. Assume for simplicity that $|Lv_1| = 1$.
Claim: If $v \perp v_1$, then $Lv \perp Lv_1$.

Suppose not. We can assume that $|v| = 1$ and that $\langle Lv, Lv_1 \rangle > 0$. Then $Lv = cLv_1 + w$ for some $c > 0$, where $w \perp Lv_1$. Thus $|L(v_1 + \varepsilon v)| \geq 1 + c\varepsilon$ for $\varepsilon > 0$. Now

$$\frac{|L(v_1 + \varepsilon v)|}{|v_1 + \varepsilon v|} \geq \frac{1 + c\varepsilon}{\sqrt{1 + \varepsilon^2}} = \frac{1 + 2c\varepsilon + c^2 \varepsilon^2}{1 + \varepsilon^2} > 1,$$

when $\varepsilon > 0$ is small, which contradicts the fact $1 = |Lv_1| = \sup_{|h|=1} |Lh|$. This proves the above claim.

Then pick v_2 with $|v_2| = 1$ and so that

$$|Lv_2| = \sup_{|h|=1, h \perp v_1} |Lh|.$$

Repeat the argument as in the first step. After n steps, we have found the required basis. $\qquad \square$

We deduce the following familiar property of linear transformations.

11.3 Proposition. If $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and one-to-one, then L maps balls to ellipsoids.

Proof. By the linearity of L, it suffices to show that $L(\overline{B}^n(0,1))$ is an ellipsoid. Relying on the preceding proposition, we may assume that the matrix corresponding to L is diagonal. Since L is one-to-one, the diagonal entries of this matrix are non-zero. The claim follows. $\square$

Let us recall the standard fact that, under a linear transformation, the measure of the image of a set E is obtained by multiplying the measure of E by the absolute value of the determinant of the matrix representing the linear transformation.

11.4 Proposition. We have

$$|AE| = |\det A||E|$$

for each measurable set E.

Notice that our claim is trivial when A is diagonal. Thus the previous proposition essentially gives our claim. The only problem is that one would need the fact that the determinant does not depend on the choice of the orthonormal bases involved. A rigorous elementary proof of our claim can be found in [24].

To each $n \times n$- matrix A we associate the adjunct matrix $\operatorname{ad} A$, defined by setting

$$(\operatorname{ad} A)_{ji} = \det A'_{ij},$$

where $(\operatorname{ad} A)_{ji}$ refers to the entry of $\operatorname{ad} A$ at row i and column j and A'_{ij} is the matrix obtained from A by replacing the entry at row i and column j with 1 and all the other entries in the corresponding row and column by 0. If A is invertible, then $\operatorname{ad} A = A^{-1} \det A$, and, more generally,

$$A \operatorname{ad} A = I \det A, \tag{37}$$

where I is the identity matrix.

11.5 Proposition. Let $\lambda > 0$ and suppose that A satisfies

$$|Ah| = \lambda |h|$$

for all $h \in \mathbb{R}^n$. Then

$$\operatorname{ad} A = (\det A)^{1-2/n} A^t.$$

Proof. Clearly $|AE| = \lambda^n|E|$ for each measurable set. Thus Proposition 11.4 shows that
$$\lambda = (\det A)^{1/n}. \tag{38}$$
Write $B = \frac{1}{\lambda}A$. Then $|Bh| = |h|$ for all $h \in \mathbb{R}^n$, and so $|B| = 1$. From (36) we conlude that also $|B^t| = 1$.

Fix h with $|h| = 1$. Then

$$1 = <Bh, Bh> = <B^tBh, h>,$$

and because
$$|B^tBh| \le |B^t||B||h| = 1,$$
we conclude that $B^tBh = h$. It follows that $B^{-1} = B^t$.

Now
$$A^t = \lambda B^t = \lambda B^{-1} = \lambda^2 A^{-1}. \tag{39}$$

Combining (39) with (37) and (38) we conclude that

$$\text{ad } A = A^{-1} \det A = (\det A)^{1-2/n} A^t,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

11.3 L^p-spaces

Recall that $L^p(\Omega)$, $1 \le p < \infty$, consists of (equivalence classes) of measurable functions u with
$$\int_\Omega |u|^p < \infty.$$
We write
$$||u||_{L^p} = ||u||_p := \left(\int_\Omega |u|^p\right)^{1/p}.$$

Furthermore, $L^\infty(\Omega)$ consists of those measurable functions on Ω that are essentially bounded. Then $||u||_{L^\infty} = ||u||_\infty$ is the essential supremum of $|u|$ over Ω. If $1 < p < \infty$, we set $p' = p/(p-1)$, and we define $1' = \infty$. With this notation, we have the Minkowski

$$||u + v||_p \le ||u||_p + ||v||_p$$

and Hölder

$$||uv||_1 \le ||u||_p||v||_{p'}$$

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inequalities.

One often needs the following spherical coordinates. Given a Borel function $u \in L^1(B^n(0,1))$ we have that

$$\int_{B(0,1)} u = \int_{S^{n-1}(0,1)} \int_0^1 u(tw)t^{n-1} \, dtdw.$$

We say that a sequence $(u_i)_i$ converges to u in $L^p(\Omega)$ if all these functions belong to $L^p(\Omega)$ and if $||u - u_i||_p \to 0$ when $i \to \infty$. We then write $u_i \to u$ in $L^p(\Omega)$. If $u_i \to u$ in $L^p(\Omega)$, then there is a subsequence $(u_{i_i})_k$ of $(u_i)_i$ which converges to u pointwise almost everywhere. For $1 \leq p < \infty$, continuous functions are dense in $L^p(\Omega)$: given $u \in L^p(\Omega)$ one can find continuous u_i with $u_i \to u$ both in $L^p(\Omega)$ and almost everywhere. This can easily seen by first approximating u by simple functions, then approximating the associated measurable sets by compact sets and finally approximating the characteristic functions of the compact sets by continuous functions.

The dual of $L^p(\Omega)$ is $L^{p/(p-1)}(\Omega)$ when $1 < p < \infty$. Then

$$||u||_p = \sup_{||\varphi||_{\frac{p}{p-1}} = 1} ||u\varphi||_1.$$

One of the inequalities easily follows by Hölder's inequality and the other by choosing φ to be a suitable constant multiple of $|u|^{p-1}$.

We also need the following weak compactness property: if $(u_j)_j$ is a bounded sequence in $L^p(\Omega)$, $1 < p < \infty$, then there is a subsequence $(u_{j_k})_k$ and a function $u \in L^p(\Omega)$ so that

$$\lim_{k \to \infty} \int_\Omega u_{j_k}\varphi = \int_\Omega u\varphi$$

for each $\varphi \in L^{p/(p-1)}(\Omega)$. We then write

$$u_{u_k} \rightharpoonup u.$$

This notation should in principle include the exponent p, but the exponent in question is typically only indicated when its value is not obvious. This function u, called the weak limit, is unique and satisfies

$$||u||_p \leq \liminf_{k \to \infty} ||u_{j_k}||_p.$$

The existence of the weak limit u follows from the fact that $L^p(\Omega)$, $1 < p < \infty$, is reflexive. Furthermore, the norm estimate on u is a consequence of a general result according to which a norm is lower semicontinuous with

respect to the associated weak convergence. In general, weak convergence is defined by considering bounded linear mappings $T : X \to \mathbb{R}$; in the case of $L^p(\Omega)$, $1 < p < \infty$, they can be identified with elements of $L^{p/(p-1)}(\Omega)$. If $v_j = (v_1^j, \cdots, v_n^j) \in L^p(\Omega)$, then

$$v_j \rightharpoonup u$$

means that

$$v_i^j \to u_i$$

for each $1 \le i \le n$.

When we apply the above to a sequence $A_j(x)$ of $n \times n$-matrix functions, we conclude that the boundedness in $L^p(\Omega)$, $1 < p < \infty$ of the sequence $(|A_j(x)|)_j$ guarantees the existence of an $n \times n$-matrix function $A(x) \in L^p(\Omega)$ so that the rows (or columns) of a subsequence of $(|A_j(x)|)_j$ converge weakly to the corresponding rows (or columns) of $A(x)$. Notice that boundedness above is independent of the initial norm (like the operator or Hilbert-Schmidt one).Then $||A||_p \le C_n \liminf_{k \to infty} ||A_{j_k}||_p$. In fact, one can show that

$$||A||_p \le \liminf_{k \to infty} ||A_{j_k}||_p;$$

the L^p-norms generated by the operator or Hilbert-Schmidt norms are equivalent and so the associated concepts of weak convergence coincide.

11.4 Regularity of p-harmonic functions

Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that a continuous function $u \in W_{loc}^{1,p}(\Omega)$ is p-harmonic, $1 < p < \infty$, if

$$\int_\Omega < |\nabla u|^{p-2} \nabla u(x), \nabla \varphi > dx = 0$$

for each $\varphi \in C_0^\infty(\Omega)$.

11.6 Proposition. Each function u, p-harmonic function in Ω, is (locally) $C^{1,\alpha}$-smooth, where $\alpha = \alpha(p, n)$.

Notice that when $p = 2$, our p-harmonic function is in fact harmonic and then C^∞-smooth. In general, for $p \ne 2$, this is not true. The difficulty lies in the fact that our equation is the degenerate. This is in fact the only obstacle as the the next proposition asserts.

11.7 Proposition. Let u be p-harmonic in Ω and C^1 with $|\nabla u(x)| > 0$ (locally). Then u is C^∞-smooth.

Proofs for the regularity results above can be found for example in the paper "Regularity of the derivatives of solutions to certain degenerate elliptic equations" by J.L.Lewis in Indiana Math. J. 32 (1983), pp. 849–858.

In the planar setting, the coordinate functions of a conformal mapping are harmonic. In higher dimensions, they turn out to be n-harmonic. This is based on the following result.

11.8 Proposition. Let $f \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$ and $\varphi \in C_0^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain. Let e_j be one of the coordinate vectors. Then

$$\int_\Omega < \operatorname{ad} Df(x)e_j, \nabla\varphi(x) > dx = 0.$$

If f is C^2-smooth, the claim follows from a direct computation using the definition of $\operatorname{ad} Df(x)$ and the fact that, for a C^2-smooth function u, $\partial_j\partial_j u(x) = \partial_i\partial_j u(x)$. To relax the regularity assumption to $f \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$, approximate f by smooth mappings and observe that the entries of $\operatorname{ad} Df(x)$ ane $(n-1)$-fold products of the partial derivatives of the coordinate mappings of f.

11.5 Fixed point theorem and related results

The following result is the Brouwer fixed point theorem.

11.9 Theorem. If $G : \overline{B}(0,1) \to \overline{B}(0,1)$ is continuous, then there is at least one fixed point $x \in \overline{B}(0,1)$ (i.e. $G(x) = x$).

Notice that, in dimension one, the claim easily follows from the mean value theorem. The higher dimensional version can be rather easily reduced to the "Hairly Ball Theorem" according to to which an even dimensional sphere does not admit any continuous field of non-zero tangent vectors. This reduction and a surprisingly simple analytic proof of this classical topological result can be found in the paper "Analytic proofs of the 'hairy ball theorem' and the Brouwer fixed point theorem," by John Milnor in the American Mathematical Montly, Vol. 85, No. 7, pp. 521–524.

We employ the fixed point theorem to prove the following result that could also be established using degree theory.

11.10 Lemma. Let $h : \overline{B}^n(0,1) \to \mathbb{R}^n$ be a continuous mapping satisfying

$$|h(x) - x| \leq \varepsilon \quad \text{when } |x| = 1.$$

Then $B(0, 1-\varepsilon) \subset h\big(B(0,1)\big)$.

Proof. Assume that there is $x_0 \in B(0, 1-\varepsilon) \setminus h(B(0,1))$ and define

$$F(x) = \begin{cases} h(2x) & \text{when } |x| < \frac{1}{2} \\ (2|x| - 1)\frac{x}{|x|} + 2(1 - |x|) h\left(\frac{x}{|x|}\right) & \text{when } \frac{1}{2} \leq |x| \leq 1 . \end{cases}$$

Then F is continuous, $F\big(B(0, 1/2)\big) = h\big(B(0,1)\big)$ and $F(x) = x$ if $|x| = 1$.
Also, for $\frac{1}{2} \leq |x| \leq 1$ we see that $F(x) \in \left[\frac{x}{|x|}, \frac{h(x)}{|h(x)|}\right]$, and so $|F(x)| > 1-\varepsilon$ (see
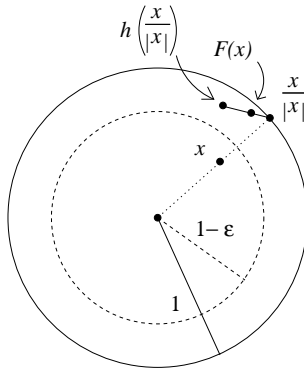Figure 18). Thus $x_0 \notin F\big(B(0,1)\big)$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism

Figure 18: $F(x)$ when $\frac{1}{2} \leq |x| \leq 1$.

so that $g(x) = x$ if $|x| = 1$ and $g(x_0) = 0$ and define for $x \in B(0,1)$

$$G(x) = -\frac{g\big(F(x)\big)}{\big|g\big(F(x)\big)\big|} .$$

Then $G : \overline{B}(0,1) \to \{x : |x| = 1\}$ is continuous, and if $|x| = 1$, then
$G(x) = -x$. This means that G does not have a fixed point, which contra-
dicts the previous Brouwer's fixed point theorem. □

The techniques from algebraic topology that are usually used to prove
the Brouwer fixed point theorem also yield related results. One of them is
"invariance of domain" which is a stronger version of the previous lemma.

11.11 Theorem. Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^n$ be continuous
and one-to-one. Then $f(\Omega)$ is a domain and $f : \Omega \to f(\Omega)$ is a homeomor-
phism.

Notice that the claim is trivial when $n = 1$. Indeed, then f is either
strictly increasing or stricly decreasing.