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RESCALING PRINCIPLE FOR ISOLATED ESSENTIAL SINGULARITIES OF QUASIREGULAR MAPPINGS

YUSUKE OKUYAMA AND PEKKA PANKKA

Abstract. We establish a rescaling theorem for isolated essential singularities of quasiregular mappings. As a consequence we show that the class of closed manifolds receiving a quasiregular mapping from a punctured unit ball with an essential singularity at the origin is exactly the class of closed quasiregularly elliptic manifolds, that is, closed manifolds receiving a non-constant quasiregular mapping from a Euclidean space.

1. Introduction

A continuous mapping \( f : M \to N \) between oriented Riemannian \( n \)-manifolds is \( K \)-quasiregular if \( f \) belongs to the Sobolev space \( W^{1,n}_{\text{loc}}(M,N) \) and satisfies the distortion inequality

\[
\|Df\|^n \leq KJ_f \quad \text{a.e.,}
\]

where \( \|Df\| \) is the operator norm and \( J_f \) is the Jacobian determinant of the differential \( Df \) of \( f \).

The main result of this paper is the following rescaling theorem. We denote the open unit ball about the origin in \( \mathbb{R}^n \) by \( B^n \). We say that a quasiregular mapping \( f \) from \( \mathbb{B}^n \setminus \{0\} \) to a closed and oriented Riemannian \( n \)-manifold \( N \) has an essential singularity at the origin if the limit \( \lim_{x \to 0} f(x) \) does not exist in \( N \).

**Theorem 1.** Let \( N \) be a closed and oriented Riemannian \( n \)-manifold, \( n \geq 2 \), and let \( f : \mathbb{B}^n \setminus \{0\} \to N \) be a \( K \)-quasiregular mapping with an essential singularity at the origin, \( K \geq 1 \). Then there exist a non-constant \( K \)-quasiregular mapping \( g : X \to N \), where \( X \) is either \( \mathbb{R}^n \) or \( \mathbb{R}^n \setminus \{0\} \), and sequences \( (x_k) \) and \( (\rho_k) \) in \( \mathbb{B}^n \) and \( (0, \infty) \), respectively, such that \( \lim_{k \to \infty} x_k = 0 \), \( \lim_{k \to \infty} \rho_k = 0 \) and

\[
\lim_{k \to \infty} f(x_k + \rho_kv) = g(v)
\]

locally uniformly on \( X \).

Theorem 1 bears a close resemblance to Miniowitz’s Zalcman lemma for quasiregular mappings; see Miniowitz [10] and Zalcman [16]. It seems, however, that this version for isolated essential singularities has gone unnoticed.

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in the quasiregular literature although the heuristic idea behind this rescaling principle is well known in the classical holomorphic case \((n = 2\) and \(K = 1\)); see, e.g., Bergweiler [1] and Minda [9].

Theorem 1 readily yields the following characterization of closed and oriented Riemannian manifolds receiving a quasiregular mapping with an isolated essential singularity.

**Theorem 2.** Let \(N\) be a closed and oriented Riemannian \(n\)-manifold, \(n \geq 2\). If there exists a \(K\)-quasiregular mapping \(f : \mathbb{B}^n \setminus \{0\} \to N\) having an essential singularity at the origin, \(K \geq 1\), then there exists a non-constant \(K'\)-quasiregular mapping \(g : \mathbb{R}^n \to N\) satisfying \(g(\mathbb{R}^n) \subset f(\mathbb{B}^n \setminus \{0\})\).

Conversely, if there exists a non-constant \(K\)-quasiregular mapping \(g : \mathbb{R}^n \to N, K \geq 1\), then there exists a \(K'\)-quasiregular mapping \(f : \mathbb{B}^n \setminus \{0\} \to N\) having an essential singularity at the origin such that \(f(\mathbb{B}^n \setminus \{0\}) \subset g(\mathbb{R}^n)\).

Here \(K' = K'(n, K) \geq 1\) depends only on \(n\) and \(K\), and \(K'(2, K) = K\).

Having Theorem 2 at our disposal, we readily obtain “big” versions of Varopoulos’s theorem [15, pp. 146-147] and the Bonk–Heinonen theorem [2, Theorem 1.1], which respectively give a bound of the fundamental group and the de Rham cohomology ring of a closed quasiregularly elliptic manifold. Recall that a connected and oriented Riemannian \(n\)-manifold \(N, n \geq 2\), is called quasiregularly elliptic if there exists a non-constant quasiregular mapping from \(\mathbb{R}^n\) to \(N\).

**Corollary 1.** Let \(N\) be a closed, connected, and oriented Riemannian \(n\)-manifold, \(n \geq 2\), with a \(K\)-quasiregular mapping \(\mathbb{B}^n \setminus \{0\} \to N\) having an essential singularity at the origin, \(K \geq 1\). Then the fundamental group \(\pi_1(N)\) of \(N\) has polynomial growth of order at most \(n\), and the de Rham cohomology ring \(H^*(N)\) of \(N\) satisfies

\[
\dim H^*(N) := \sum_{k=0}^n \dim H^k(N) \leq C,
\]

where \(C = C(n, K) > 0\) depends only on \(n\) and \(K\).

Although the former half of Corollary 1, the big Varopoulos theorem, is well-known to the experts, we have been unable to find it in the literature. For a direct proof of the big Bonk–Heinonen theorem, i.e., the bound (1.1), see [12].

We would also like to note that together with the Holopainen–Rickman Picard theorem for quasiregularly elliptic manifolds [6], we obtain a big Picard type theorem for quasiregular mappings into closed manifolds; see also [5].

**Corollary 2.** Let \(N\) be a closed, oriented, and connected Riemannian \(n\)-manifold, \(n \geq 2\), and \(f : \mathbb{B}^n \setminus \{0\} \to N\) be a \(K\)-quasiregular mapping with an essential singularity at the origin, \(K \geq 1\). Then for every \(x \in N\), except for at most \(q - 1\) points, it holds that \(#f^{-1}(x) = \infty\), where \(q = q(n, K) \in \mathbb{N}\) depends only on \(n\) and \(K\).

We conclude this introduction with an application of Theorem 1 to the Ahlfors five islands theorem; see, e.g., Bergweiler [1] or Nevanlinna [11, XII §7, §8] for a detailed discussion.
Let $f$ be a quasimeromorphic function on a domain $U$ in the 2-sphere $\mathbb{S}^2$, i.e., a quasiregular mapping from $U$ to $\mathbb{S}^2$. We say that $f$ has a simple island $\Omega$ over a Jordan domain $D'$ in $\mathbb{S}^2$ if $\Omega$ is a relatively compact subdomain in $U$ and is mapped univalently onto $D'$ by $f$. The Ahlfors five islands theorem states that given five Jordan domains in $\mathbb{S}^2$ with pairwise disjoint closures, any non-constant quasimeromorphic function on $\mathbb{R}^2$ has a simple island over one of these Jordan domains.

**Corollary 3.** Let $f$ be a quasimeromorphic function on $\mathbb{R}^2 \setminus \{0\}$ having an essential singularity at the origin. Then given five Jordan domains $D_1, \ldots, D_5$ in $\mathbb{S}^2$ with pairwise disjoint closures, $f$ has a simple island over one of $D_1, \ldots, D_5$.

**Proof.** Applying Theorem 1 to $f$, we obtain sequences $(x_k)$ and $(\rho_k)$, and a non-constant quasimeromorphic function $g$ on $\mathbb{R}^2 \setminus \{0\}$, where $f_k$ is the mapping $v \mapsto f(x_k + \rho_k v)$ and $g$ is the locally uniform limit of $(f_k)$, as in Theorem 1. We may fix Jordan domains $D'_1, \ldots, D'_5$ in $\mathbb{S}^2$ satisfying $D'_j \subset D'_j$ for every $j \in \{1, 2, 3, 4, 5\}$ and having pairwise disjoint closures. By the Ahlfors five islands theorem, the quasimeromorphic function $g \circ \exp$ on $\mathbb{R}^2$ has a simple island $\Omega'$ over one of these Jordan domains, say $D'_j$. Hence $g$ has a simple island $\Omega'$ over $D'_j$. By Rouché’s theorem, for every $k \in \mathbb{N}$ large enough, $f_k$ has a simple island $\Omega_k \subset \Omega'$ over $D_j \subset D'_j$.

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2. Preliminaries

Let $\mathbb{B}^n(x, r)$ be the open ball in $\mathbb{R}^n$ about $x \in \mathbb{R}^n$ of radius $r > 0$. Set $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ for each $r > 0$ and set $\mathbb{B}^n := \mathbb{B}^n(1)$. The corresponding closed balls are denoted by $\overline{\mathbb{B}}^n(x, r)$, $\mathbb{B}^n(r)$, and $\overline{\mathbb{B}}^n$, respectively.

Let $M$ be an oriented Riemannian $n$-manifold, $n \geq 2$. We denote by $|x - y|$ the distance between $x$ and $y$ in $M$, and by $B(x, r)$ the Riemannian ball $\{y \in M : |x - y| < r\}$ about $x \in M$ of radius $r > 0$ in $M$. Similarly, we denote by $B(x, r)$ the corresponding closed ball about $x \in M$ of radius $r > 0$.

By [14, III.1.11], every $K$-quasiregular mapping from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is locally $\alpha$-Hölder continuous with $\alpha = (1/K)^{1/(n-1)}$. We refer to [14] and [7] for the Euclidean theory of quasiregular mappings.

For every $x \in M$, there exist $r > 0$ and a 2-bilipschitz chart $B(x, r) \to \mathbb{R}^n$. Thus every $K$-quasiregular mapping from an open set $U \subset \mathbb{R}^n$ to $M$ is locally $\beta$-Hölder continuous with $\beta = \beta(n, K)$ depending only on $n$ and $K$.

The local Hölder continuity plays a key role in the proof of the following manifold version ([7, Theorem 19.9.3]) of Miniowitz’s Zalcman lemma [10, §4]. Recall that a family $\mathcal{F}$ of $K$-quasiregular mappings from a domain $\Omega$ in $\mathbb{R}^n$ to $M$ is normal on an open subset $U$ in $\Omega$ if every infinite sequence $(f_k)$ in $\mathcal{F}$ contains a subsequence which converges locally uniformly on $U$, and is normal at a point $a \in \Omega$ if $\mathcal{F}$ is normal on some open neighborhood of $a$. 

...
Let \( \Omega \) be a domain in \( \mathbb{R}^n \), and \( N \) be a closed, connected, and oriented Riemannian \( n \)-manifold, \( n \geq 2 \). Let \( \mathcal{F} \) be a family of \( K \)-quasiregular mappings from \( \Omega \) to \( N, K \geq 1 \). If \( \mathcal{F} \) is not normal at \( a \in \Omega \), then there exist sequences \( (x_j) \), \( (\rho_j) \), and \( (f_j) \) in \( \Omega \), \( (0, \infty) \), and \( \mathcal{F} \), respectively, and a non-constant \( K \)-quasiregular mapping \( g : \mathbb{R}^n \to N \) such that \( \lim_{j \to \infty} x_j = a \), \( \lim_{j \to \infty} \rho_j = 0 \) and
\[
\lim_{j \to \infty} f_j(x_j + \rho_j v) = g(v)
\]
locally uniformly on \( \mathbb{R}^n \).

3. Proof of Theorem 1

We begin the proof of Theorem 1 by showing a manifold version of a classical lemma on isolated essential singularities due to Lehto and Virtanen [8]; see also Heinonen and Rossi [4, Theorem 2.3] and Gauld and Martin [3].

Lemma 3.1. Let \( M \) be a closed and oriented Riemannian \( n \)-manifold, \( n \geq 2 \), and \( f : \mathbb{B}^n \setminus \{0\} \to M \) be a quasiregular mapping with an essential singularity at the origin. Then

\[
\limsup_{r \to 0} \text{diam}(f(\partial \mathbb{B}^n(r))) > 0.
\]

Proof. The proof follows the argument of Heinonen and Rossi in [4]. Set \( A(r', r) := \mathbb{B}^n(r) \setminus \mathbb{B}^n(r') \) for each \( r, r' > 0, r > r' \).

Since the origin is an isolated essential singularity of \( f \) and \( M \) is compact, there exist sequences \( \{z_k\} \) and \( \{w_j\} \) in \( \mathbb{B}^n \setminus \{0\} \) such that \( \lim_{k \to \infty} z_k = \lim_{j \to \infty} w_j = 0 \) and that both limits
\[
a := \lim_{k \to \infty} f(z_k) \quad \text{and} \quad b := \lim_{j \to \infty} f(w_j)
\]
each exist in \( M \) and are distinct.

Using the exponential map \( \exp_a : T_a M \to M\), we find \( R \in (0, |b - a|/4) \) and a \( 2 \)-bilipschitz map \( \varphi : B(a, 4R) \to \mathbb{B}^n(4R) \) satisfying \( \varphi(a) = 0 \). Note that \( b \notin B(a, 4R) \) by our choice of \( R \).

Suppose that (3.1) does not hold. Then there is \( r_0 > 0 \) such that, for every \( r \in (0, r_0) \), diam \( f(\partial \mathbb{B}^n(r)) < R/8 \).

We fix \( k_1 \in \mathbb{N} \) so large that \( |z_{k_1}| < r_0 \) and that \( |f(z_{k_1}) - a| \leq R/8 \). Let \( r_1 := |z_{k_1}| \). Since \( \text{diam}(f(\partial \mathbb{B}^n(r_1))) < R/8 \), we have
\[
\text{diam}(f(\partial \mathbb{B}^n(r_1))) < \varphi^{-1}(\mathbb{B}^n(R/4)) \subset B(a, R).
\]

Since \( f(w_j) \to b \notin \bar{B}(a, 4R) \) as \( j \to \infty \), the continuity of \( f \) implies the existence of the maximal element, say \( r_2 \in (0, r_1) \), of the subset
\[
\{r \in (0, r_1) : f(\partial \mathbb{B}^n(r)) \notin B(a, 2R)\}.
\]

By maximality of \( r_2 \), \( f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R) \neq \emptyset \). Note that \( f(A(r_2, r_1)) \subset B(a, 2R) \) and \( b \notin f(A(r_2, r_1)) \).

Let \( c \in f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R) \). Then \( \text{diam}(f(\partial \mathbb{B}^n(r_2))) < R/8 \) and
\[
f(\partial \mathbb{B}^n(r_2)) \subset \varphi^{-1}(\mathbb{B}^n(\varphi(c), R/4)) \subset B(c, R/2).
\]

We join \( \partial \mathbb{B}^n(r_1) \) and \( \partial \mathbb{B}^n(r_2) \) by a line segment \( \ell \), which is contained, except for the end points, in the ring domain \( A(r_2, r_1) \). Then the path \( f(\ell) \)
in $M$ joins $f(\partial \mathbb{B}^n(r_1))$ and $f(\partial \mathbb{B}^n(r_2))$, and by (3.2), (3.3) and the choice of $r_1$ and $r_2$, we may fix $y_0 \in \ell$ such that

$$f(y_0) \in B(a, 2R) \setminus (B(a, R) \cup B(c, R/2)).$$

Since both $\mathbb{B}^n(4R) \setminus (\mathbb{B}^n(R/4) \cup \mathbb{B}^n(\varphi(c), R/4))$ and $\varphi^{-1}(\partial \mathbb{B}^n(3R))$ are connected, also $M \setminus (\varphi^{-1}(\mathbb{B}^n(R/4)) \cup \varphi^{-1}(\mathbb{B}^n(\varphi(c), R/4)))$ is connected.

Thus $f(y_0)$ can be joined with $b \notin B(a, 4R)$ by a path $\beta: [0, 1] \to M$ such that $\beta([0, 1]) \cap f(\partial(A(r_2, r_1))) = \emptyset$.

Let $\alpha: I_0 \to \mathbb{B}^n \setminus \{0\}$, where $I_0 = [0, 1]$ or $[0, t_0)$ for some $t_0 \in (0, 1]$, be a maximal lift of $\beta$ under $f$ starting at $y_0 = \alpha(0) \in A(r_2, r_1)$. If $I_0 = [0, 1]$, then $f(\alpha(1)) = b \notin f(A(r_2, r_1))$, so $\alpha(1) \notin A(r_2, r_1)$. If $I_0 \neq [0, 1]$, then $\lim \text{dist}(\alpha(t), \partial \mathbb{B}^n \cup \{0\}) \to 0$ as $t \to t_0$. In both cases, $\beta(I_0) \cap f(\partial A(r_2, r_1)) \neq \emptyset$. This is a contradiction and (3.1) holds. \hfill \Box

Proof of Theorem 1. Define a function $Q_f: \mathbb{B}^n(2/3) \setminus \{0\} \to [0, \infty)$ by

$$Q_f(x) := \sup_{y, y' \in \mathbb{B}^n(x, |x|/2), y \neq y'} \frac{|f(y) - f(y')|}{|y - y'|^\beta},$$

where $\beta = \beta(n, K)$ is as in Section 2. Put

$$M_f := \limsup_{x \to 0} Q_f(x)|x|^{\beta}.$$

Suppose first that

$$M_f = \infty,$$

or equivalently, that there exists a sequence $(y_k)$ in $\mathbb{B}^n \setminus \{0\}$ satisfying $y_k \to 0$ and $Q_f(y_k)|y_k|^{\beta} \to \infty$ as $k \to \infty$.

Fix $\delta \in (0, 1)$ small enough, and, for each $k \in \mathbb{N}$, define a mapping $g_k: \mathbb{B}^n(1 + \delta) \to N$ by

$$g_k(z) := f(y_k + \frac{|y_k|}{2}z).$$

By (3.4), there exist sequences $(z_k)$ and $(w_k)$ in $\mathbb{B}^n$ satisfying

$$\limsup_{k \to \infty} \frac{|g_k(z_k) - g_k(w_k)|}{|z_k - w_k|^{\beta}} \geq \limsup_{k \to \infty} \frac{1}{2} Q_f(y_k) \left( \frac{|y_k|}{2} \right)^{\beta} = \infty.$$

Hence the family $\{g_k: k \in \mathbb{N}\}$ is not normal on $\mathbb{B}^n(1 + \delta)$. Indeed, otherwise, there exists a locally uniform limit point of $\{g_k: k \in \mathbb{N}\}$, which is $K$-quasiregular on $\mathbb{B}^n(1 + \delta)$ but not $\beta$-Hölder continuous on $\mathbb{B}^n$. This is impossible.

The non-normality of $\{g_k: k \in \mathbb{N}\}$ on $\mathbb{B}^n(1 + \delta)$ is equivalent to the non-normality of $\{g_k: k \in \mathbb{N}\}$ at some $a \in \mathbb{B}^n(1 + \delta)$. By Theorem 2.1, there exist sequences $(z_j)$, $(\rho_j)$ and $(k_j)$ in $\mathbb{B}^n(1 + \delta)$, $(0, \infty)$, and $\mathbb{N}$, respectively, and a non-constant $K$-quasiregular mapping $g: \mathbb{R}^n \to N$ such that $\lim_{j \to \infty} z_j = a$, $\lim_{j \to \infty} \rho_j = 0$, $\lim_{j \to \infty} k_j = \infty$, and

$$\lim_{j \to \infty} g_{k_j}(z_j + \rho_j v) = g(v)$$

locally uniformly on $\mathbb{R}^n$. Observing that

$$f \left( (|y_k|/2)z_j + ((|y_k|/2)\rho_j) v \right) = g_{k_j}(z_j + \rho_j v),$$
\[
\lim_{j \to \infty} (y_k + (|y_k|/2)z_j) = 0, \text{ and } \lim_{j \to \infty} (|y_k|/2)\rho_j = 0, \text{ completes the proof in this case.}
\]

Suppose next that
\[
M_f < \infty.
\]
Let \( \{g_k : \mathbb{B}^n(e^k) \setminus \{0\} \to N; k \in \mathbb{N}\} \) be the family of \( K \)-quasiregular mappings defined as
\[
g_k(v) := f(e^{-k}v).
\]
By (3.5), we have for every \( v \in \mathbb{R}^n \setminus \{0\}, \)
\[
\limsup_{k \to \infty} \sup_{w \in \mathbb{B}^n(e^k) \setminus \mathbb{B}^n(e^k)/2, w \neq v} \frac{|g_k(w) - g_k(v)|}{|w - v|^\beta} \leq M_f|v|^{-\beta} < \infty,
\]
so the family \( \{g_k : k \in \mathbb{N}\} \) is locally equicontinuous on \( \mathbb{R}^n \setminus \{0\}. \)

By Lemma 3.1, there exists a sequence \( (r_j) \) in \((0, \infty)\) tending to 0 as \( j \to \infty \) such that \( \lim_{j \to \infty} \text{diam} f(\partial \mathbb{B}(r_j)) > 0. \) Fix a sequence \( (k_j) \) in \( \mathbb{N} \) such that for every \( j \in \mathbb{N}, e^{-k_j-1} < r_j \leq e^{-k_j}. \) Then for every \( j \in \mathbb{N}, \)
\[
g_k_j(\mathbb{B}^n(0) \setminus \mathbb{B}^n(e^{-j})) = f(\mathbb{B}^n(e^{-k_j}) \setminus \mathbb{B}^n(e^{-k_j-1})) \supset f(\partial \mathbb{B}(r_j)). \]
In particular,
\[
\liminf_{j \to \infty} \text{diam}(g_{k_j}(\mathbb{B}^n \setminus \mathbb{B}^n(e^{-j}))) > 0.
\]
By passing to a further subsequence of \( (g_{k_j}) \) if necessary, we may assume, by the Arzelà–Ascoli theorem, that \( (g_{k_j}) \) converges locally uniformly on \( \mathbb{R}^n \setminus \{0\} \) to a mapping \( g : \mathbb{R}^n \setminus \{0\} \to N. \) Since \( (g_{k_j}) \) is a sequence of \( K \)-quasiregular mappings, \( g \) is \( K \)-quasiregular, and by (3.6), non-constant. This completes the proof.

\[\square\]

**Example 3.2.** To see that both cases in the proof of Theorem 1 actually occur, we give two examples, which are similar to Examples 23 and 24 in [13].

For \( M_f < \infty, \) we may take the conformal mapping \( f : \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times \mathbb{S}^1, x \mapsto (x/|x|, e^{-1\log |x|}). \) Then \( f \) is the composition \( \psi \circ h \) of a conformal homeomorphism \( h : \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times \mathbb{R}, x \mapsto (x/|x|, -\log |x|) \), and a locally isometric covering map \( \psi : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n-1} \times \mathbb{S}^1. \) Since \( |h(sx) - h(st)| \leq |\log s - \log t| + |x - y| \) for all \( s, t \in (0, 1) \) and all \( x, y \in \mathbb{S}^{n-1}, \) we easily observe that \( M_f < \infty. \)

For \( M_f = \infty, \) we construct \( f : \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^n \) using the winding map \( h : \mathbb{S}^n \to \mathbb{S}^n, \)
\[
(x_1, \ldots, x_{n-2}, re^{i\theta}) \mapsto (x_1, \ldots, x_{n-2}, re^{i\theta}),
\]
which is a quasiregular endomorphism on \( \mathbb{S}^n; \) we identify \( \mathbb{R}^{n+1} \) with \( \mathbb{R}^{n-1} \times \mathbb{C}. \)

Let \( \sigma : \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n \) be the stereographic projection, and \( S \) the lower hemisphere \( \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_{n+1} \leq 0\} \) of \( \mathbb{S}^n. \) We note that \( h|\partial S \) is the identity.

Let \( (r_k) \) be a sequence tending to 0 in \((0, 1/2)\) and put \( x_k := r_k e_1 \) and \( B_k := \mathbb{B}^n(x_k, r_k/2) \) for each \( k \in \mathbb{N}. \) We may assume that balls \( B_k, k \in \mathbb{N}, \) are mutually disjoint. For each \( k \in \mathbb{N}, \) put \( x'_k := \sigma^{-1}(x_k) \) and \( B'_k := \sigma^{-1}(B_k) \) and let \( \rho_k \) be the the Möbius transformation on \( \mathbb{S}^n \) defined as
\[
\rho_k(y) = \begin{cases} 
\sigma^{-1} \circ \alpha_k \circ \sigma(y), & y \neq e_n + 1 \\
e_n + 1, & y = e_n + 1
\end{cases}
\]

where \(\alpha_k\) is the affine transformation \(x \mapsto r_k^{-1}(x-x_k)\) on \(\mathbb{R}^n\). Then \(\rho_k(B'_k) = S\) and \(\rho_k(x_k) = -e_n + 1\), so the mapping \(f : \mathbb{B}^n \setminus \{0\} \to S^n\) defined by

\[
f(x) = \begin{cases}
\sigma^{-1}(x), & x \in (\mathbb{B}^n \setminus \{0\}) \setminus \bigcup_{k=1}^\infty B_k \\
(p_k^{-1} \circ h \circ \rho_k) \circ \sigma^{-1}(x), & x \in B_k
\end{cases}
\]

is quasiregular with the same distortion constant \(K\) as \(h\). Recall the definition of \(\beta = \beta(n, K)\) in Section 2. We observe that, for every \(k \in \mathbb{N}\), there exists a unique \(y_k \in B_k\) satisfying \(f(y_k) = e_1\) and \(|x_k - y_k| \leq Cn^2\), where \(C > 0\) is independent of \(k\). Since \(f(x_k) = e_n + 1\) for every \(k \in \mathbb{N}\), by a direct computation, there is \(C' > 0\) such that for every \(k \in \mathbb{N}\), \(Q_f(x_k)|x_k|^{\beta} \geq C' n^{-\beta}\), so \(f\) satisfies \(M_f = \infty\).

4. Proof of Theorem 2

Let \(Z_n : \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}, n > 2\), be the Zorich mapping (see [17] or [14, I.3.3] for the construction of \(Z_n\)), which is \(K_n\)-quasiregular for some \(K_n \geq 1\) and an analog of the exponential function \(Z_2 : \mathbb{C} \to \mathbb{C} \setminus \{0\}\). The mapping \(Z_2\) is \(K_2\)-quasiregular for \(K_2 = 1\). Set \(K' = K \cdot K_n \geq 1\) for each \(n \geq 2\) and each \(K \geq 1\).

Let \(N\) be a closed, connected, and oriented Riemannian \(n\)-manifold.

Suppose there exists a \(K\)-quasiregular mapping \(f : \mathbb{B}^n \setminus \{0\} \to N\) with an essential singularity at the origin. By Theorem 1 and a manifold version of Hurwitz’s theorem (cf. the proof of [10, Lemma 2]), there exists a non-constant \(K\)-quasiregular mapping \(g : X \to N\), where \(X\) is either \(\mathbb{R}^n\) or \(\mathbb{R}^n \setminus \{0\}\), satisfying \(g(X) \subset f(\mathbb{B}^n \setminus \{0\})\). If \(X = \mathbb{R}^n\), then the mapping \(g\) has the desired properties. If \(X = \mathbb{R}^n \setminus \{0\}\), then the mapping \(g \circ Z_n : \mathbb{R}^n \to N\) has the desired properties.

Suppose now that \(g : \mathbb{R}^n \to N\) is a non-constant \(K\)-quasiregular mapping. Let \(\iota\) be an orientation preserving conformal involution of \(\mathbb{R}^n \setminus \{0\}\) satisfying \(\iota(\mathbb{B}^n \setminus \{0\}) = \mathbb{R}^n \setminus \overline{\mathbb{B}^n}\). If \(g\) has an essential singularity at the infinity, \(f : \mathbb{B}^n \setminus \{0\} \to N, x \mapsto g \circ \iota(x)\), has an essential singularity at the origin. If \(g\) has a removable singularity at the infinity, \(g\) extends to a quasiregular mapping \(S^n \to N\). Then the mapping \(f : \mathbb{B}^n \setminus \{0\} \to N, x \mapsto g \circ Z_n(\iota(x))\), has the desired properties.

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