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Logarithmic mean inequality for generalized trigonometric and hyperbolic functions

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Abstract. In this paper we study the convexity and concavity properties of generalized trigonometric and hyperbolic functions in case of Logarithmic mean.

1 Introduction

Recently, the study of the generalized trigonometric and generalized hyperbolic functions has got huge attention of numerous authors, and has appeared the huge number of papers involving the equalities and inequalities and basis properties of these function, e.g. see [7, 8, 9, 6, 10, 13, 14, 18, 23] and the references therein. These generalized trigonometric and generalized hyperbolic functions p-functions depending on the parameter $p > 1$ were introduced by Lindqvist [19] in 1995. These functions coincides with the usual functions for $p = 2$. Thereafter Takesheu took one further step and generalized these function for two parameters $p, q > 1$, so-called (p, q)-functions. In [8], some convexity and concavity properties of p-functions were studied. Thereafter those results were extended in [5] for two parameters in the sense of Power mean inequality. In this paper we study the convexity and concavity property of p-function with

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respect Logarithmic mean. Before we formulate our main result we will define generalized trigonometric and hyperbolic functions customarily.

The eigenfunction $\sin_p$ of the so-called one-dimensional p-Laplacian problem [12]

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u,\ u(0)=u(1)=0, \quad p>1,$$

is the inverse function of $F:(0,1)\to\left(0,\frac{\pi_p}{2}\right)$, defined as

$$F(x) = \mathrm{arcsin}_p(x) = \int_0^x (1-t^p)^{-\frac{1}{p}}\,dt,$$

where

$$\pi_p = 2\mathrm{arcsin}_p(1) = \frac{2}{p}\int_0^1 (1-s)^{-1/p} s^{1/p-1}\,ds = \frac{2}{p}\,B\left(1-\frac{1}{p},\frac{1}{p}\right) = \frac{2\pi}{p\,\sin\left(\frac{\pi}{p}\right)},$$

here $B(.,.)$ denotes the classical beta function.

The function arcsin_p is called the generalized inverse sine function, and coincides with usual inverse sine function for $p=2$. Similarly, the other generalized inverse trigonometric and hyperbolic functions arccos_p: $(0,1)\to(0,\pi_p/2)$, arctan_p: $(0,1)\to(0,b_p)$, $\mathrm{arcsinh}_p$: $(0,1)\to(0,c_p)$, $\mathrm{arctanh}_p$: $(0,1)\to(0,\infty)$, where

$$b_p = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right)-\psi\left(\frac{1}{2p}\right)\right) = 2^{-\frac{1}{p}} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right),$$

$$c_p = \left(\frac{1}{2}\right)^{\frac{1}{p}} F\left(1,\frac{1}{p};1+\frac{1}{p},\frac{1}{2}\right),$$

are defined as follows

$$\mathrm{arccos}_p(x) = \int_0^{(1-x^p)^{\frac{1}{p}}} (1-t^p)^{-\frac{1}{p}}\,dt, \quad \mathrm{arctan}_p(x) = \int_0^x (1+t^p)^{-1}\,dt,$$

$$\mathrm{arcsinh}_p(x) = \int_0^x (1+t^p)^{-\frac{1}{p}}\,dt, \quad \mathrm{arctanh}_p(x) = \int_0^x (1-t^p)^{-1}\,dt,$$

where $F(a,b;c;z)$ is *Gaussian hypergeometric function* [1].

The generalized cosine function is defined by

$$\frac{d}{dx}\sin_p(x) = \cos_p(x), \quad x\in[0,\pi_p/2].$$

It follows from the definition that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p},$$

and

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1, \quad x \in \mathbb{R}. \tag{1}$$

Clearly we get

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}.$$

The generalized tangent function $\tan_p$ is defined by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)},$$

and applying (1) we get

$$\frac{d}{dx}\tan_p(x) = 1 + \tan_p(x)^p.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_p(x)$ is defined by

$$\operatorname{arcsinh}_p(x) = \int_0^x (1 + t^p)^{-1/p}\,dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_p(x) = \frac{d}{dx}\sinh_p(x), \quad \tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)},$$

respectively. It follows from the definitions that

$$|\cosh_p(x)|^p - |\sinh_p(x)|^p = 1. \tag{2}$$

From above definition and (2) we get the following derivative formulas,

$$\frac{d}{dx}\cosh_p(x) = \cosh_p(x)^{2-p}\sinh_p(x)^{p-1}, \quad \frac{d}{dx}\tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

Note that these generalized trigonometric and hyperbolic functions coincide with usual functions for $p = 2$.

For two distinct positive real numbers x and y, the Arithmetic mean, Geometric mean, Logarithmic mean, Harmonic mean and the Power mean of order $p \in \mathbb{R}$ are respectively defined by

$$A(x, y) = \frac{x + y}{2}, \quad G(x, y) = \sqrt{xy},$$

$$L(x,y)=\frac{x-y}{\log(x)-\log(y)},\quad x\neq y,$$

$$H(x,y)=\frac{1}{A(1/x,1/y)},$$

and

$$M_t=\begin{cases}\left(\dfrac{x^t+y^t}{2}\right)^{1/t}, & t\neq 0,\\ \sqrt{xy}, & t=0.\end{cases}$$

Let $f: I\to(0,\infty)$ be continuous, where I is a sub-interval of $(0,\infty)$. Let M and N be the means defined above, the we call that the function f is MN-convex (concave) if

$$f(M(x,y))\leq(\geq)N(f(x),f(y))\quad\text{for}\quad\text{all } x,y\in I.$$

Recently, Generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [2]. We recall one of their results as follows

Lemma 1 [2, Theorem 2.4] *Let* I *be an open sub-interval of* $(0,\infty)$ *and let* $f: I\to(0,\infty)$ *be differentiable. Then* f *is* HH*-convex (concave) on* I *if and only if* $x^2f'(x)/f(x)^2$ *is increasing (decreasing).*

In [4], Baricz studied that if the functions f is differentiable, then it is (a,b)-convex (concave) on I if and only if $x^{1-a}f'(x)/f(x)^{1-b}$ is increasing (decreasing).

It is important to mention that $(1,1)$-convexity means the AA-convexity, $(1,0)$-convexity means the AG-convexity, and $(0,0)$-convexity means GG-convexity.

Motivated by the results given in [2, 4], we contribute to the topic by giving the following result.

Theorem 1 *Let* $f: I\to(0,\infty)$ *be a continuous and* $I\subseteq(0,\infty)$*, then*

1. $L(f(x),f(y))\geq(\leq)f(L(x,y))$,
2. $L(f(x),f(y))\geq(\leq)f(A(x,y))$,

if f *is increasing and* log*-convex (concave).*

Theorem 2 *For* $x,y\in(0,\pi_p/2)$*, the following inequalities*

1. $L(\sin_p(x),\sin_p(y))\leq\sin_p(L(x,y)),\quad p>1,$

2. $L(\cos_p(x), \cos_p(y)) \leq \cos_p(L(x, y)), \quad p \geq 2.$

Theorem 3 *For* $p > 1$*, we have*

1. $L(1/\sin_p(x), 1/\sin_p(y)) \geq 1/\sin_p(A(x, y)), \quad x, y \in (0, \pi_p/2),$
2. $L(1/\cos_p(x), 1/\cos_p(y)) \geq 1/\cos_p(L(x, y)), \quad x, y \in (0, \pi_p/2),$
3. $L(\tanh_p(x), \tanh_p(y)) \leq \tanh_p(A(x, y)), \quad x, y \in (0, \infty),$
4. $L(\operatorname{arcsinh}_p(x), \operatorname{arcsinh}_p(y)) \leq \operatorname{arcsinh}_p(A(x, y)), \quad x, y \in (0, 1),$
5. $L(\arctan_p(x), \arctan_p(y)) \leq \arctan_p(A(x, y)), \quad x, y \in (0, 1).$

2 Preliminaries and Proofs

We give the following lemmas which will be used in the proof of our main result.

Lemma 2 [22] *Let* $f, g : [a, b] \to R$ *be integrable functions, both increasing or both decreasing. Furthermore, let* $p : [a, b] \to R$ *be a positive, integrable function. Then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \tag{3}$$

If one of the functions f *or* g *is non-increasing and the other non-decreasing, then the inequality in (3) is reversed.*

Lemma 3 [17] *If* $f(x)$ *is continuous and convex function on* $[a, b]$*, and* $\varphi(x)$ *is continuous on* $[a, b]$*, then*

$$f\left(\frac{1}{b-a}\int_a^b \varphi(x)dx\right) \leq \frac{1}{b-a}\int_a^b f(\varphi(x))\, dx. \tag{4}$$

If function $f(x)$ *is continuous and concave on* $[a, b]$*, then the inequality in* (4) *reverses.*

Lemma 4 [3] *For two distinct positive real numbers* a, b*, we have* $L < A$*.*

Lemma 5 *For* $p > 1$*, the function* $\sin_p(x)$ *is* HH*-concave on* $(0, \pi_p/2)$*.*

Proof. Let $f(x) = f_1(x)f_2(x), x \in (0, \pi_p/2)$, where $f_1(x) = 1/\sin(x)$ and $f_2(x) = x^2 \cos_p(x)/\sin_p(x)$. Clearly, f_1 is decreasing, so it is enough to prove that f_2 is decreasing, then the proof follows from Lemma 1. We get

$$\begin{aligned} f_2'(x) &= \frac{\sin_p(x)(\cos_p(x) - x\cos_p(x)^{2-p}\sin_p(x)^{p-1}) - x\cos_p(x)^2}{\sin_p(x)^2} \\ &= \frac{\cos_p(x)^2((1 - x\tan_p(x)^{p-1})\tan_p(x) - x)}{\sin_p(x)^2} = f_3(x)\frac{\cos_p(x)^2}{\sin_p(x)^2}, \end{aligned}$$

where $f_3(x) = \tan_p(x) - x\tan_p(x)^p - 1$. Again, one has

$$f_3'(x) = p\tan_p(x)^{p-1}(1 + \tan_p(x)^p)x < 0.$$

Thus, f_3 is decreasing and $g(x) < g(0) = 0$. This implies that $f_2' < 0$, hence f_2 is strictly decreasing, the product of two decreasing functions is decreasing. This implies the proof. □

Proof of Theorem 1. We get

$$L(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} 1dt}{\int_{f(y)}^{f(x)} \frac{1}{t}dt} = \frac{\int_y^x f'(u)du}{\int_y^x \frac{f'(u)}{f(u)}du}. \tag{5}$$

It is assumed that the function $f(x)$ is increasing and $\log f$ is convex, this implies that $\frac{f'(x)}{f(x)}$ is increasing. Letting $p(x) = 1, f(x) = f(u)$ and $g(x) = f'(u)/f(u)$ in Lemma 2, we get

$$\int_y^x 1du \int_y^x f'(u)du \geq \int_y^x \frac{f'(u)}{f(u)}du \int_y^x f(u)du.$$

This is equivalent to

$$L(f(x), f(y)) = \frac{\int_y^x f'(u)du}{\int_y^x \frac{f'(u)}{f(u)}du} \geq \frac{\int_y^x f(u)du}{\int_y^x 1du}.$$

By Lemmas 3 and 4, and keeping in mind that log-convexity of f implies the convexity of f, we get

$$L(f(x), f(y)) \geq f\left(\frac{\int_y^x udu}{x - y}\right) = f\left(\frac{x + y}{2}\right) \geq f\left(L(x, y)\right).$$

The proof of converse follows similarly. If we repeat the lines of proof of part (1), and use the concavity of the function, and Lemmas 3 & 4 then we arrive at the proof of part (2).

Proof of Theorem 2. It is easy to see that the function $\sin_p(x)$ is increasing and log-concave. So the proof of part (1) follows easily from Theorem 1. We also offer another proof as follows:

It can be observed easily that

$$L\left(\sin_p(x), \sin_p(y)\right) = \frac{\int_y^x \cos_p(u)du}{\int_{\sin_p(y)}^{\sin_p(x)} \frac{1}{t}dt} = \frac{\int_y^x \cos_p u du}{\int_y^x \frac{\cos_p u}{\sin_p(u)}du},$$

and

$$\sin_p\left(L(x,y)\right) = \sin_p\left(\frac{x-y}{\log\frac{x}{y}}\right) = \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \frac{1}{u}du}\right).$$

Clearly, $\cos_p(u)$ and $\sin_p(1/u)$, utilizing Chebyshev inequality, we have

$$\int_y^x \cos_p(u)du \int_y^x \sin_p(1/u)du \leq \int_y^x 1du \int_y^x \cos_p u \sin_p \frac{1}{u}du.$$

So, we get

$$\int_y^x \cos_p u du \int_y^x \sin_p(1/u)du < \int_y^x 1du \int_y^x \frac{\cos_p(u)}{\sin_p(u)}du.$$

Where we apply simple inequality $\sin_p\left(\frac{1}{u}\right) < \frac{1}{\sin_p(u)}$. In order to prove inequality (1), we only prove

$$\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du} \leq \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du}\right).$$

Consider a partition T of the interval $[y,x]$ into n equal length sub-interval by means of points $y = x_0 < x_1 < \cdots < x_n = x$ and $\Delta x_i = \frac{x-y}{n}$. Picking an arbitrary point $\xi_i \in [x_{i-1}, x_i]$ and using Lemma 1.2, we have

$$\frac{n}{\sum\limits_{i=1}^{n} \sin_p \frac{1}{\xi_i}} \leq \sin_p\left(\frac{n}{\sum\limits_{i=1}^{n} \frac{1}{\xi_i}}\right)$$

$\Leftrightarrow$

$$\frac{x-y}{\lim\limits_{n\to\infty}\left(\frac{x-y}{n}\sum\limits_{i=1}^{n} \sin_p \frac{1}{\xi_i}\right)} \leq \sin_p\left(\frac{x-y}{\lim\limits_{n\to\infty}\left(\frac{x-y}{n}\sum\limits_{i=1}^{n} \frac{1}{\xi_i}\right)}\right)$$

$\Leftrightarrow$
$$\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du} \leq \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du}\right).$$
This completes the proof.

For (2), clearly $\cos_p(x)$ is decreasing and $\tan_p(x)^{p-1}$ is increasing. One has
$$(\cos_p(x))'' = \cos_p(x)\tan_p(x)^{p-2}\left(1-p+(2-p)\tan_p(x)^p\right) < 0,$$
this implies that $\cos_p(x)$ is concave on $(0, \pi_p/2)$.

Using Tchebyshef inequality, we have
$$\int_y^x 1du \int_y^x \cos_p(u)\tan_p(u)^{p-1}du \leq \int_y^x \cos_p(u)du \int_y^x \tan_p(u)^{p-1}du,$$
which is equivalent to
$$\frac{\int_y^x \cos_p(u)\tan_p(u)^{p-1}du}{\int_y^x \tan_p(u)^{p-1}du} \leq \frac{\int_y^x \cos_p(u)du}{\int_y^x 1du}. \tag{6}$$
Substituting $t = \cos_p(u)$ in (6), we get
$$L(\cos_p(x), \cos_p(y)) = \frac{\int_{\cos_p(y)}^{\cos_p(x)} 1dt}{\int_{\cos_p(y)}^{\cos_p(x)} \frac{1}{t}dt} = \frac{\int_y^x \cos_p(u)\tan_p(u)^{p-1}du}{\int_y^x \tan_p(u)^{p-1}du} \leq \frac{\int_y^x \cos_p(u)du}{\int_y^x 1du}.$$
Using Lemma 3 and concavity of $\cos_p(x)$, we obtain
$$L(\cos_p(x), \cos_p y) \leq \cos_p\left(\frac{\int_y^x udu}{x-y}\right) = \cos_p\left(\frac{x+y}{2}\right) \leq \cos_p\left(L(x,y)\right).$$

Proof of Theorem 3. Let $g_1(x) = 1/\cos_p(x)$, $x \in (0, \pi_p/2)$ and $g_2(x) = \tanh_p(x)$, $x > 0$. We get
$$(\log(g_1(x)))'' = (p-1)\tan_p(x)^{p-2}(1+\tan_p(x)^p) > 0,$$
and
$$(\log(g_2(x)))'' = \frac{1-\tanh_p(x)^p}{\tanh_p(x)^2}((1-p)\tanh_p(x)^p - 1) < 0.$$
This implies that g_1 and g_2 are log-convex, clearly both functions are increasing, and log-convexity implies the convexity, so g_1 and g_2 are convex functions. Now the proof follows easily from Theorem 1. The rest of proof follows similarly.

Corollary 1 *For* $p > 1$*, we have*

1. $L(\tan_p(x), \tan_p(y)) \geq \tan_p(L(x, y)), \quad x, y \in (s_p, \pi_p/2)$, *where* s_p *is the unique root of the equation* $\tan_p(x) = 1/(p-1)^{1/p}$,
2. $L(\mathrm{arctanh}_p(x), \mathrm{arctanh}_p(y)) \geq \mathrm{arctanh}_p(L(x, y)), \quad x, y \in (r_p, 1)$, *where* r_p *is the unique root of the equation* $x^{p-1}\mathrm{arctanh}_p(y) = 1/p$.

Proof. Write $f_1(x) = \tan_p(x)$. We get

$$\left(\frac{f_1'(x)}{f(x)}\right)' = \left(\frac{1 + \tan_p^p(x)}{\tan_p(x)}\right)' = \frac{1 + \tan_p^p(x)}{\tan_p^2(x)}\left[(p-1)\tan_p^p(x) - 1\right] > 0$$

on $\left(s_p, \frac{\pi_p}{2}\right)$. This implies that f_1 is log-convex, clearly f_1 is increasing, and the proof follows easily from Theorem 1. The proof of part (2) follows similarly. ☐

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References

[1] M. Abramowitz, I. Stegun, eds., *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards, Dover, New York, 1965.

[2] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, Genenalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007), 1294–1308.

[3] H. Alzer, S.-L Qiu, Inequalities for means in two variables, *Arch. Math.* **80** (2003), 201–205.

[4] Á. Baricz, Geometrically concave univariate distributions, *J. Math. Anal. Appl.* **363** (1) (2010), 182–196.

[5] Á. Baricz, B. A. Bhayo, R. Klén, Convexity properties of generalized trigonometric and hyperbolic functions, *Aequat. Math.* DOI 10.1007/s00010-013-0222-x.

[6] Á. Baricz, B. A. Bhayo, M. Vuorinen, Turán type inequalities for generalized inverse trigonometric functions, available online at `http://arxiv.org/abs/1209.1696`.

[7] B. A. Bhayo, Power mean inequality of generalized trigonometric functions, *Mat. Vesnik*, (to appear) `http://mv.mi.sanu.ac.rs/Papers/MV2013_033.pdf`.

[8] B. A. Bhayo, M. Vuorinen, On generalized trigonometric functions with two parameters, *J. Approx. Theory*, **164** (10) (2012),1415–1426.

[9] B. A. Bhayo, M. Vuorinen, Inequalities for eigenfunctions of the p-Laplacian, *Issues of Analysis* **2** (20), No 1, (2013), `http://arxiv.org/abs/1101.3911`

[10] P. J. Bushell, D. E. Edmunds, Remarks on generalised trigonometric functions, *Rocky Mountain J. Math.*, **42** (1) (2012), 25–57.

[11] B. C. Carlson, Some inequalities for hypergeometric functions, *Proc. Amer. Math. Soc.*, **17** (1), (1966), 32–39.

[12] P. Drábek, R. Manásevich, On the closed solution to some $\mathrm{p}-$Laplacian nonhomogeneous eigenvalue problems, *Diff. and Int. Eqns.*, **12** (1999), 723–740.

[13] D. E. Edmunds, P. Gurka, J. Lang, Properties of generalized trigonometric functions, *J. Approx. Theory*, **164** (2012) 47–56, doi:10.1016/j.jat.2011.09.004.

[14] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, F. Qi,Convexity of the generalized sine function and the generalized hyperbolic sine function, *J. Approx. Theory*, **174** (2013), 1–9.

[15] D. B. Karp, E. G. Prilepkina, Parameter convexity and concavity of generalized trigonometric functions, `arXiv:1402.3357[math.CA]`

[16] R. Klén, M. Visuri, M. Vuorinen, On Jordan type inequalities for hyperbolic functions, *J. Ineq. Appl.*, vol. 2010, pp. 14.

[17] J.-C. Kuang, Applied inequalities (Second edition), Shan Dong Science and Technology Press. Jinan, 2002.

[18] R. Klén, M. Vuorinen, X.-H. Zhang, Inequalities for the generalized trigonometric and hyperbolic functions, *J. Math. Anal. Appl.*, **409** (1) (2014), 521-29.

[19] P. Lindqvist, Some remarkable sine and cosine functions, *Ricerche di Matematica*, Vol. XLIV (1995), 269–290.

[20] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, USA, 1970.

[21] E. Neuman, J. Sándor, Optimal inequalities for hyperbolic and trigonometric functions, *Bull. Math. Anal. Appl*, **3**(3), (2011), 177–181. `http://www.emis.de/journals/BMAA/repository/docs/BMAA3_3_20.pdf`.

[22] F. Qi, Z. Huang, Inequalities of the complete elliptic integrals, *Tamkang J. Math*, **29** (3) (1998), 165–169.

[23] S. Takeuchi, Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian, *J. Math. Anal. Appl.* **385** (2012) 24–35.

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