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# GROMOV HYPERBOLICITY AND QUASIHYPHERBOLIC GEODESICS

## (GROMOV HYPERBOLICITÉ ET QUASI-HYPHERBOLIQUE GÉODÉSIIQUES)

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ABSTRACT. We characterize Gromov hyperbolicity of the quasihyperbolic metric space  $(\Omega, k)$  by geometric properties of the Ahlfors regular length metric measure space  $(\Omega, d, \mu)$ . The characterizing properties are called the Gehring–Hayman condition and the ball–separation condition.

RÉSUMÉ. Nous caractérisons l’hyperbolicité au sens de Gromov de l’espace quasi-hyperbolique  $(\Omega, k)$  par des propriétés géométriques (dites Gehring–Hayman condition et ball–separation condition) de l’espace métrique mesuré Ahlfors–régulière  $(\Omega, d, \mu)$ .

### 1. INTRODUCTION

Given a proper subdomain  $\Omega$  of Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , equipped with Euclidean distance, one defines the *quasihyperbolic metric*  $k$  in  $\Omega$  as the path metric generated by the density

$$\rho(z) = \frac{1}{d(z)},$$

where  $d(z) = \text{dist}(z, \partial\Omega)$ . Precisely, one sets

$$k(x, y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \rho(z) \, ds,$$

where the infimum is taken over all rectifiable curves  $\gamma_{xy}$  that join  $x$  and  $y$  in  $\Omega$  and the integral is the usual line integral. Then  $\Omega$  equipped with  $k$  is a geodesic metric space: there is a curve  $\gamma_{xy}$  whose length in the above sense equals  $k(x, y)$ . Let us denote by  $[x, y]$  any such geodesic; these geodesics are not necessarily unique as can be easily seen, for example for  $\Omega = \mathbb{R}^n \setminus \{0\}$ . The quasihyperbolic metric  $k$  was introduced in [GP] and [GO] where the basic properties of it were established.

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If for all triples of geodesics  $[x, y]$ ,  $[y, z]$ ,  $[z, x]$  in  $\Omega$  every point in  $[x, y]$  is within  $k$ -distance  $\delta$  from  $[y, z] \cup [z, x]$  then the space  $(\Omega, k)$  is called  $\delta$ -hyperbolic. Roughly speaking this means that *geodesic triangles* in  $\Omega$  are  $\delta$ -thin. Moreover, we say that  $(\Omega, k)$  is *Gromov hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta$ . The following theorem from [BB] that extends results from [BHK] gives a complete characterization of Gromov hyperbolicity of  $(\Omega, k)$ .

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a proper subdomain. Then  $(\Omega, k)$  is Gromov hyperbolic if and only if  $\Omega$  satisfies both a Gehring–Hayman condition and a ball separation condition.*

Above, the Gehring–Hayman condition means that there is a constant  $C_{\text{gh}} \geq 1$  such that for each pair of points  $x, y$  in  $\Omega$  and for each quasihyperbolic geodesic  $[x, y]$  it holds that

$$\text{length}([x, y]) \leq C_{\text{gh}} \text{length}(\gamma_{xy}),$$

where  $\gamma_{xy}$  is any other curve joining  $x$  to  $y$  in  $\Omega$ . In other words, it says that quasihyperbolic geodesics are essentially the shortest curves in  $\Omega$ .

The other condition, a ball separation condition, requires the existence of a constant  $C_{\text{bs}} \geq 1$  such that for each pair of points  $x$  and  $y$ , each quasihyperbolic geodesic  $[x, y]$ , every  $z \in [x, y]$ , and every curve  $\gamma_{xy}$  joining  $x$  to  $y$  it holds that

$$B(z, C_{\text{bs}}d(z)) \cap \gamma_{xy} \neq \emptyset.$$

Here the ball is taken with respect to the inner metric of  $\Omega$ .

Notice that the three conditions in Theorem 1.1, Gromov hyperbolicity and the Gehring–Hayman and the ball separation conditions, are only based on metric concepts. It is then natural to ask for an extension of this characterization to an abstract metric setting. Such an extension was given in [BB] that relies on an analytic assumption that essentially requires the space in question to support a suitable Poincaré inequality. This very same condition, expressed in terms of moduli of curve families [HeiK], is already in force in [BHK].

The purpose of this paper is to show that Poincaré inequalities are not critical for geometric characterizations of Gromov hyperbolicity of a non-complete metric space equipped with the quasihyperbolic metric. Our main result reads as follows.

**Theorem 1.2.** *Let  $Q > 1$  and let  $(X, d, \mu)$  be a  $Q$ -regular metric measure space with  $(X, d)$  a locally compact and annularly quasiconvex length space. Let  $\Omega$  be a bounded and proper subdomain of  $X$ , and let  $d_\Omega$  be the inner metric on  $\Omega$  associated to  $d$ . Then  $(\Omega, k)$  is Gromov hyperbolic if and only if  $(\Omega, d_\Omega)$  satisfies both a Gehring–Hayman condition and a ball separation condition.*

The main point in Theorem 1.2 is the necessity of the Gehring-Hayman and ball separation conditions; their sufficiency is already given in [BB, Theorem 2.4 and Theorem 6.1].

Above, annular quasiconvexity means that there is a constant  $\lambda \geq 1$  so that for any  $x \in X$  and all  $0 < r' < r$  each pair of points  $y, z$  in  $B(x, r) \setminus B(x, r')$  can be joined with a path  $\gamma_{yz}$  in  $B(x, \lambda r) \setminus B(x, r'/\lambda)$  such that  $\text{length}(\gamma_{yz}) \leq \lambda d(y, z)$ ,  $Q$ -regularity requires the existence of a constant  $C_q$  so that

$$r^Q/C_q \leq \mu(B(x, r)) \leq C_q r^Q$$

for all  $r > 0$  and all  $x \in X$ , and the other concepts are defined analogously to the Euclidean setting described in the beginning of our introduction. See Section 2 for the precise definitions. In fact, the assumptions of Theorem 1.2 can be somewhat relaxed, see Section 5.

This paper is organized as follows. Section 2 contains necessary definitions. In Section 3 we give preliminaries relating the quasihyperbolic metric and Whitney balls. Section 4 is devoted to the proof of our main technical estimate, and Section 5 contains the proof of our main result and some generalizations.

## 2. DEFINITIONS

Let  $(X, d)$  be a metric space. A *curve* is a continuous map  $\gamma: [a, b] \rightarrow X$  from an interval  $[a, b] \subset \mathbb{R}$  to  $X$ . We also denote the image set  $\gamma([a, b])$  of  $\gamma$  by  $\gamma$ . The *length*  $\ell_d(\gamma)$  of  $\gamma$  with respect to the metric  $d$  is defined as

$$\ell_d(\gamma) = \sup \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions  $a = t_0 < t_1 < \dots < t_m = b$  of the interval  $[a, b]$ . If  $\ell_d(\gamma) < \infty$ , then  $\gamma$  is said to be a *rectifiable curve*. When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c, d]}),$$

where the supremum is taken over all compact subintervals  $[c, d]$ .

When every pair of points in  $(X, d)$  can be joined with a rectifiable curve, the space  $(X, d)$  is called *rectifiably connected*. If  $\ell_d(\gamma_{xy}) = d(x, y)$  for some curve  $\gamma_{xy}$  joining points  $x, y \in X$ , then  $\gamma_{xy}$  is said to be a *geodesic*. If every pair of points in  $(X, d)$  can be joined with a geodesic, then  $(X, d)$  is called a *geodesic space*. Moreover, a *geodesic ray* in  $X$  is an isometric image in  $(X, d)$  of the interval  $[0, \infty)$ . Furthermore, for a rectifiable curve  $\gamma$  we define the *arc length*  $s: [a, b] \rightarrow [0, \infty)$  along  $\gamma$  by

$$s(t) = \ell_d(\gamma|_{[a, t]}).$$

Let  $(X, d)$  be a geodesic metric space and let  $\delta \geq 0$ . Denote by  $[x, y]$  any geodesic joining two points  $x$  and  $y$  in  $X$ . If for all triples of

geodesics  $[x, y]$ ,  $[y, z]$ ,  $[z, x]$  in  $X$  every point in  $[x, y]$  is within distance  $\delta$  from  $[y, z] \cup [z, x]$ , the space  $(X, d)$  is called  $\delta$ -hyperbolic. In other words, *geodesic triangles* in  $X$  are  $\delta$ -thin. Moreover, we say that a space is *Gromov hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta$ . All Gromov hyperbolic spaces in this paper are assumed to be unbounded.

Next, let  $(X, d)$  be a locally compact, rectifiably connected and non-complete metric space, and denote by  $\overline{X}_d$  its metric completion. Then the *boundary*  $\partial_d X := \overline{X}_d \setminus X$  is nonempty. We write

$$d(z) := \text{dist}_d(z, \partial_d X) = \inf\{d(z, x) : x \in \partial_d X\}$$

for  $z \in X$ .

Given a real number  $D \geq 1$ , a curve  $\gamma: [a, b] \rightarrow X$  is called a  $D$ -quasiconvex curve if

$$\ell_d(\gamma) \leq Dd(\gamma(a), \gamma(b)).$$

If  $\gamma$  also satisfies the *cigar condition*

$$\min\{\ell_d(\gamma|_{[a,t]}), \ell_d(\gamma|_{[t,b]})\} \leq Dd(\gamma(t))$$

for every  $t \in [a, b]$ , the curve is called a  $D$ -uniform curve. A metric space  $(X, d)$  is called a  $D$ -quasiconvex space or  $D$ -uniform space if every pair of points in it can be joined with a  $D$ -quasiconvex curve or a  $D$ -uniform curve respectively.

Let  $\rho: X \rightarrow (0, \infty)$  be a continuous function. For each rectifiable curve  $\gamma: [a, b] \rightarrow X$  we define the  $\rho$ -length  $\ell_\rho(\gamma)$  of  $\gamma$  by

$$\ell_\rho(\gamma) = \int_\gamma \rho ds = \int_a^b \rho(\gamma(t)) ds(t).$$

Because  $(X, d)$  is rectifiably connected, the density  $\rho$  determines a metric  $d_\rho$ , called the  $\rho$ -metric, defined by

$$d_\rho(x, y) = \inf_{\gamma_{xy}} \ell_\rho(\gamma_{xy}),$$

where the infimum is taken over all rectifiable curves  $\gamma_{xy}$  joining  $x, y \in X$ . If  $\rho \equiv 1$ , then  $\ell_\rho(\gamma) = \ell_d(\gamma)$  is the length of the curve  $\gamma$  with respect to the metric  $d$ , and the metric  $d_\rho = \ell_d$  is the *inner metric associated with  $d$* . Generally, if the distance between every pair of points in the metric space is the infimum of the lengths of all curves joining the points, then the metric space is called a *length space*.

If we choose

$$\rho(z) = \frac{1}{d(z)},$$

we obtain the *quasihyperbolic metric* in  $X$ . In this special case, we denote the metric  $d_\rho$  by  $k$  and the quasihyperbolic length of the curve  $\gamma$  by  $\ell_k(\gamma)$ . Moreover,  $[x, y]_k$  refers to a  $k$ -geodesic (i.e., quasihyperbolic geodesic) joining points  $x$  and  $y$  in  $X$ . Because we are dealing with many different metrics, the usual metric notations will have an additional subscript that refers to the metric in use. For ease of notation, terms

which refer to the metric  $d_\rho$  will have an additional subscript  $\rho$  instead of  $d_\rho$ .

We say that  $(X, d)$  satisfies a *ball separation condition* if there is a constant  $C_{\text{bs}} \geq 1$  such that for each pair of points  $x, y \in X$ , for every  $k$ -geodesic  $[x, y]_k \subset X$ , for every  $z \in [x, y]_k$ , and for every curve  $\gamma_{xy}$  joining points  $x$  and  $y$ , it holds that

$$(BS) \quad B_d(z, C_{\text{bs}}d(z)) \cap \gamma_{xy} \neq \emptyset.$$

Thus the condition says that the ball  $B_d(z, C_{\text{bs}}d(z))$  either includes at least one of the endpoints of the  $k$ -geodesic or it separates the endpoints. This condition was introduced in [BHK, §7]. We also say that  $(X, d)$  satisfies the *Gehring–Hayman condition* if there is a constant  $C_{\text{gh}} \geq 1$  such that for every  $k$ -geodesic  $[x, y]_k$  it holds that

$$(GH) \quad \ell_d([x, y]_k) \leq C_{\text{gh}}\ell_d(\gamma_{xy}),$$

where  $\gamma_{xy}$  is any other curve joining  $x$  to  $y$  in  $X$ .

Following [BB], we say that  $(X, d)$  is *minimally nice* if  $(X, d)$  is a locally compact, rectifiably connected and non-complete metric space, and the identity map from  $(X, d)$  to  $(X, \ell_d)$  is continuous. If  $(X, d)$  is minimally nice, then the identity map from  $(X, d)$  to  $(X, k)$  is a homeomorphism, and  $(X, k)$  is complete (see [BHK, Theorem 2.8]); in particular,  $(X, k)$  is proper (i.e. closed balls are compact) and geodesic (recall the Hopf–Rinow theorem).

Furthermore, we define a proper, geodesic space  $(X, k)$  to be  *$K$ -roughly starlike*,  $K > 0$ , with respect to a base point  $w \in X$  if for every point  $x \in X$  there exists some geodesic ray emanating from  $w$  whose distance to  $x$  is at most  $K$ .

Let  $\mu$  be a Borel regular measure on  $(X, d)$  with dense support. We call  $\rho$  a *conformal density* provided it satisfies both a *Harnack inequality*  $\text{HI}(A)$  for some constant  $A \geq 1$  :

$$\text{HI}(A) \quad \frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A \quad \text{for all } x, y \in B_d(z, \frac{1}{2}d(z)) \text{ and all } z \in X,$$

and a *volume growth condition*  $\text{VG}(B)$  for some constant  $B > 0$  :

$$\text{VG}(B) \quad \mu_\rho(B_\rho(z, r)) \leq Br^Q \quad \text{for all } z \in X \text{ and } r > 0.$$

Here  $\mu_\rho$  is the Borel measure on  $X$  defined by

$$\mu_\rho(E) = \int_E \rho^Q d\mu \quad \text{for a Borel set } E \subset X,$$

and  $Q$  is a positive real number. Generally  $Q$  will be the Hausdorff dimension of our space  $(X, d)$ . There is nothing special about the constant  $\frac{1}{2}$  in condition  $\text{HI}(A)$ : we may replace it by any constant  $0 < c \leq \frac{1}{2}$ , actually by any constant  $c \in (0, 1)$ . Suppose that we have fixed  $0 < c \leq \frac{1}{2}$ . Then each ball  $B_d(z, cd(z))$  is called a *Whitney type ball*.

In general, we say that  $(X, d, \mu)$  is  $Q$ -upper regular for some  $Q > 0$  if there is a constant  $C_u \geq 1$  such that

$$(2.1) \quad \mu(B_d(z, r)) \leq C_u r^Q$$

for each  $z \in X$  and every  $r > 0$ . We also say that  $(X, d, \mu)$  is  $Q$ -regular on Whitney type balls for some  $Q > 0$  if there are constants  $C_w \geq 1$  and  $0 < \epsilon \leq 1$  such that

$$(2.2) \quad C_w^{-1} r^Q \leq \mu(B_d(z, r)) \leq C_w r^Q$$

for each  $z \in X$  and every  $r \leq \epsilon d(z)/2$ .

### 3. THE METRIC MEASURE SPACE $(X, d_\epsilon, \mu_\epsilon)$

Let  $(X, d, \mu)$  be a minimally nice metric measure space so that the measure  $\mu$  is Borel regular and  $(X, k)$  is Gromov hyperbolic. Let  $w \in X$  be a base point. We define two deformations by setting

$$\rho_\epsilon(z) = \exp\{-\epsilon k(w, z)\} \quad \text{and} \quad \sigma_\epsilon(z) = \frac{\rho_\epsilon(z)}{d(z)}$$

for a given  $\epsilon > 0$ . This generates a metric space  $(X, d_\epsilon)$  with

$$d_\epsilon(x, y) = \inf_{\gamma_{xy}} \ell_\epsilon(\gamma_{xy}) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \rho_\epsilon(z) ds_k = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \sigma_\epsilon(z) ds,$$

where the infimum is taken over all curves  $\gamma_{xy}$  joining points  $x$  and  $y$  in  $X$ , and  $ds_k = ds/d(z)$  denotes the quasihyperbolic arc length element. For ease of notation we write  $d_\epsilon$  instead of  $d_{\sigma_\epsilon}$ . We also refer to the metric  $d_\epsilon$  via an additional subscript  $\epsilon$ ; for instance  $\overline{X}_\epsilon$  denotes the metric completion of  $(X, d_\epsilon)$ . If  $Q > 0$  is fixed we also attach the Borel measure  $d\mu_\epsilon(z) = \sigma_\epsilon(z)^Q d\mu(z)$  to  $(X, d_\epsilon)$ .

Because  $(X, k)$  is geodesic, for a given  $x \in X$  and geodesic  $[w, x]_k$  we have that

$$d_\epsilon(w, x) \leq \int_{[w, x]_k} \rho_\epsilon ds_k \leq \int_0^\infty e^{-\epsilon t} dt = \frac{1}{\epsilon}.$$

Thus  $(X, d_\epsilon)$  is always bounded. Moreover, by the triangle inequality the density  $\rho_\epsilon$  satisfies a Harnack type inequality:

$$(3.1) \quad \exp\{-\epsilon k(x, y)\} \leq \frac{\rho_\epsilon(x)}{\rho_\epsilon(y)} \leq \exp\{\epsilon k(x, y)\}$$

for all  $x, y \in X$  and all  $\epsilon > 0$ . We also obtain that the density  $\sigma_\epsilon$  satisfies the Harnack inequality HI( $A$ ) with the constant  $A = 3 \exp\{2\epsilon\}$ :

$$\begin{aligned} \frac{1}{3} \exp\{-2\epsilon\} &\leq \frac{1}{3} \exp\{-\epsilon k(x, y)\} \leq \frac{\sigma_\epsilon(x)}{\sigma_\epsilon(y)} \\ &\leq 3 \exp\{\epsilon k(x, y)\} \leq 3 \exp\{2\epsilon\} \end{aligned}$$

for all  $x, y \in B_d(z, \frac{1}{2}d(z))$  and for every  $z \in X$ .

Bonk, Heinonen and Koskela proved in [BHK, §4 and Theorem 5.1] that there is  $\epsilon_0 > 0$  depending on  $\delta$  such that the metric space  $(X, d_\epsilon)$

is  $D_\epsilon$ -uniform for every  $0 < \epsilon \leq \epsilon_0$ , where  $k$ -geodesics serve as  $D_\epsilon$ -uniform curves with  $D_\epsilon = D(\delta, \epsilon, \epsilon_0) \geq 1$ . Especially, we have a version of the *Gehring-Hayman condition*: there is a constant  $D_\epsilon \geq 1$  such that when  $\epsilon \leq \epsilon_0$

$$(3.2) \quad \ell_\epsilon([x, y]_k) \leq D_\epsilon \ell_\epsilon(\gamma_{xy})$$

for each  $k$ -geodesic  $[x, y]_k$  in  $X$  and for each curve  $\gamma_{xy}$  joining  $x$  to  $y$  in  $X$ . Furthermore, if  $(X, k)$  is  $K$ -roughly starlike with respect to the base point  $w$ , then by [BHK, Lemma 4.17] we have that

$$(3.3) \quad \begin{aligned} \frac{1}{\epsilon e} \sigma_\epsilon(x) d(x) &= \frac{1}{\epsilon e} \rho_\epsilon(x) \leq d_\epsilon(x) \\ &\leq \frac{2 \exp\{\epsilon K\} - 1}{\epsilon} \rho_\epsilon(x) \\ &= \frac{2 \exp\{\epsilon K\} - 1}{\epsilon} \sigma_\epsilon(x) d(x) \end{aligned}$$

for all  $\epsilon > 0$  and every  $x \in X$ . Thus there exists  $c = c(\delta, K) \in (0, 1)$  such that

$$(3.4) \quad c\epsilon k(x, y) \leq k_\epsilon(x, y) \leq e\epsilon k(x, y)$$

for all  $x, y \in X$  and every  $0 < \epsilon \leq \epsilon_0$ , where  $k_\epsilon$  is the quasihyperbolic metric derived from  $d_\epsilon$ .

Moreover, we obtain from [BHK, Theorem 6.39] that Whitney type balls in  $(X, d_\epsilon)$  are also Whitney type balls in  $(X, d)$ . To be more specific, let

$$(3.5) \quad C_0 = \max\left\{3 \exp\{2\epsilon_0\}, \epsilon_0 e, \frac{2 \exp\{\epsilon_0 K\} - 1}{\epsilon_0}\right\}.$$

Then

$$(WB1) \quad B_\epsilon(z, \epsilon d_\epsilon(z)) \subset B_d(z, \epsilon C_0^2 d(z))$$

whenever  $z \in X$  and  $0 < \epsilon \leq \min\{\epsilon_0, \frac{1}{2C_0^2}\}$ . Furthermore, if  $(X, d)$  is a  $D$ -quasiconvex space then Whitney type balls in  $(X, d)$  are also Whitney type balls in  $(X, d_\epsilon)$ :

$$(WB2) \quad B_d(z, \epsilon d(z)) \subset B_\epsilon(z, \epsilon DC_0^2 d_\epsilon(z))$$

whenever  $z \in X$  and  $0 < \epsilon \leq \min\{\epsilon_0, \frac{1}{8D}\}$ .

Moreover, if  $(X, d, \mu)$  is  $Q$ -regular on Whitney type balls and  $Q$ -upper regular then  $(X, d_\epsilon, \mu_\epsilon)$  is  $Q$ -regular on Whitney type balls when  $\epsilon \leq \min\{\epsilon_0, \frac{1}{2C_0^2}\}$ . Indeed, let  $z \in X$  and let  $r \leq \epsilon d_\epsilon(z) \leq \frac{1}{2} d_\epsilon(z)$ . From (WB1) we obtain that

$$(3.6) \quad B_\epsilon(z, r) \subset B_\epsilon(z, \epsilon d_\epsilon(z)) \subset B_d(z, \epsilon C_0^2 d(z)) \subset B_d(z, d(z)/2),$$

and by  $\text{HI}(A)$  it follows that  $B_\epsilon(z, r) \subset B_d\left(z, \frac{A}{\sigma_\epsilon(z)}r\right)$ . Now, because  $\mu$  is  $Q$ -upper regular, with  $\text{HI}(A)$  it follows that

$$\begin{aligned}
 \mu_\epsilon(B_\epsilon(z, r)) &= \int_{B_\epsilon(z, r)} (\sigma_\epsilon(u))^Q d\mu(u) \\
 &\leq (A\sigma_\epsilon(z))^Q \int_{B_\epsilon(z, r)} d\mu(u) \\
 &\leq (A\sigma_\epsilon(z))^Q \mu\left(B_d\left(z, \frac{A}{\sigma_\epsilon(z)}r\right)\right) \\
 &\leq A^{2Q} C_u r^Q.
 \end{aligned}
 \tag{3.7}$$

The lower bound follows similarly: when  $(X, d)$  is  $D$ -quasiconvex, by  $\text{HI}(A)$  we have that  $B_d\left(z, \frac{1}{AD\sigma_\epsilon(z)}r\right) \subset B_\epsilon(z, r)$ . Thus  $\text{HI}(A)$  together with (2.2) yields

$$\begin{aligned}
 \mu_\epsilon(B_\epsilon(z, r)) &= \int_{B_\epsilon(z, r)} (\sigma_\epsilon(u))^Q d\mu(u) \\
 &\geq \left(\frac{\sigma_\epsilon(z)}{A}\right)^Q \int_{B_\epsilon(z, r)} d\mu(u) \\
 &\geq \left(\frac{\sigma_\epsilon(z)}{A}\right)^Q \mu\left(B_d\left(z, \frac{1}{AD\sigma_\epsilon(z)}r\right)\right) \\
 &\geq \left(\frac{1}{A}\right)^{2Q} \frac{1}{D^Q C_w} r^Q.
 \end{aligned}
 \tag{3.8}$$

Let  $Q > 1$  and let  $(X, d, \mu)$  be a minimally nice  $D$ -quasiconvex and  $Q$ -upper regular space so that the measure  $\mu$  is  $Q$ -regular on Whitney type balls and  $(X, k)$  is a  $K$ -roughly starlike Gromov hyperbolic space. Let the constants  $\epsilon_0$  and  $C_0$  be as in the paragraph containing (3.2) and (3.5). We may define a Whitney covering of the space  $(X, d_\epsilon, \mu_\epsilon)$  when  $\epsilon \leq \min\{\epsilon_0, \frac{1}{8D}, \frac{1}{2C_0^2}\}$ , see e.g. [CW, Theorem III.1.3], [MaSe, Lemma 2.9], [HKT, Lemma 7] and [KL, §3].

Let  $r(z) = \epsilon d_\epsilon(z)/50$ . From the family  $\{B_\epsilon(z, r(z))\}_{z \in X}$  of balls we select a maximal (countable) subfamily  $\{B_\epsilon(z_i, r(z_i)/5)\}_{i \in I}$  of pairwise disjoint balls. We write  $\mathcal{B} = \{B_i\}_{i \in I}$ , where  $B_i = B_\epsilon(z_i, r_i)$  and  $r_i = r(z_i)$ . We call the family  $\mathcal{B}$  a *Whitney covering* of  $(X, d_\epsilon)$ . We list the basic properties of the Whitney covering in Lemma 3.1. As in [KL, Lemma 3.2], the property (iv) is a consequence of the  $Q$ -regularity condition of  $\mu_\epsilon$  on Whitney type balls, and the property (v) follows from the proof of Lemma 3.2 in [KL] via uniformity and the  $Q$ -regularity condition of  $\mu_\epsilon$  on Whitney type balls.

**Lemma 3.1.** *There is  $N \in \mathbb{N}$  such that*

- (i) *the balls  $B_\epsilon(z_i, r_i/5)$  are pairwise disjoint,*

(ii)  $X = \bigcup_{i \in I} B_\epsilon(z_i, r_i)$ ,

(iii)  $5r_i \leq d_\epsilon(z_i)/10$ ,

(iv)  $\sum_{i=1}^\infty \chi_{B_\epsilon(z_i, 5r_i)}(x) \leq N$  for all  $x \in X$ .

Furthermore, suppose  $d_\epsilon(x, y) \geq d_\epsilon(x)/2$  and let  $\gamma_{xy}$  be a  $D_\epsilon$ -uniform curve joining  $x$  to  $y$ . There exists a constant  $C_\epsilon > 0$ , that depends quantitatively on  $\epsilon$  and the hypotheses, such that

(v)  $C_\epsilon^{-1} N_\epsilon(x, y) \leq \ell_{k_\epsilon}(\gamma_{xy}) \leq C_\epsilon N_\epsilon(x, y)$ ,

where  $N_\epsilon(x, y)$  is the number of balls  $B \in \mathcal{B}$  intersecting  $\gamma_{xy}$ .

Fix a ball  $B_0$  from the Whitney covering  $\mathcal{B}$  and let  $z_0$  be its center. We define the *shadow*  $S(B)$  of a ball  $B \in \mathcal{B}$  by

$$S(B) = \{x \in X : 5B \cap [z_0, x]_k \neq \emptyset\}.$$

For  $n \in \mathbb{N}$  we set

$$\mathcal{B}_n = \{B_i \in \mathcal{B} : n \leq k_\epsilon(z_0, z_i) < n + 1\}.$$

Similarly as in [KL, Lemma 3.3 and Lemma 3.4] we may prove, when  $\epsilon \leq \min\{\epsilon_0, \frac{1}{8D}, \frac{1}{2C_0^2}\}$ , that there is a constant  $C_0 > 0$  that depends quantitatively on  $\epsilon$  and the hypotheses, such that

$$(3.9) \quad \sum_{B \in \mathcal{B}_n} \chi_{S(B)}(x) \leq C_0.$$

#### 4. THE MAIN LEMMA

Next we prove a lemma which is the central tool for proving our main theorem. From now on we assume that  $Q > 1$ ,  $(X, d, \mu)$  is a minimally nice  $Q$ -upper regular  $D$ -quasiconvex metric measure space such that the measure  $\mu$  is  $Q$ -regular on Whitney type balls and  $(X, k)$  is a  $K$ -roughly starlike Gromov hyperbolic space. We also assume that  $(X, d_\epsilon, \mu_\epsilon)$  is a deformation of  $(X, d, \mu)$  as described above, where  $w \in X$  is a base point,  $\epsilon_0 > 0$  is as in the paragraph containing (3.2),  $C_0 > 1$  as in (3.5) and  $0 < \epsilon \leq \min\{\epsilon_0, \frac{1}{8D}, \frac{1}{2C_0^2}\}$ .

**Lemma 4.1.** *Let  $u \in X$  be a point and  $\gamma \subset X$  be a curve such that*

$$\text{dist}_\epsilon(u, \gamma) \leq \min\{C_1 d_\epsilon(u), C_2 \text{diam}_\epsilon(\gamma)\}$$

*for some  $C_1, C_2 > 0$ . Then there exists a constant  $M \geq 1$  that depends quantitatively on  $\epsilon$  and the constants in our hypotheses so that*

$$\text{dist}_d(u, \gamma) \leq Md(u).$$

Before giving a detailed proof for the above lemma, let us briefly discuss potential approaches. One would like to prove the claim via a suitable modulus of curve families argument. Indeed, the assumptions on  $\gamma$  seem to indicate that the modulus of the family of curves joining  $\gamma$  to  $B_\epsilon(u, d(u)/2)$  should be bounded from below. On the other

hand,  $(X, d)$  is “conformally equivalent” to  $(X, d_\epsilon)$  and the converse to the claim for large  $M$  should then contradict this lower bound. The obstacle here is that there need not be such a lower bound for the modulus. What we will do is consider a suitable Whitney decomposition, attach to it a “discrete test function for the modulus”, integrate over  $\gamma$  against a suitable measure, and eventually use the upper volume growth condition to bound  $M$ .

*Proof.* Suppose that for a fixed  $M > 2$  there is  $u \in X$  and a curve  $\gamma$  such that  $\text{dist}_\epsilon(u, \gamma) \leq \min\{C_1 d_\epsilon(u), C_2 \text{diam}_\epsilon(\gamma)\}$  but  $\text{dist}_d(u, \gamma) > Md(u)$ . We will show that such an  $M$  has an upper bound in terms of our data. Towards this end, let us fix  $u$  and  $\gamma$  as above. By replacing  $\gamma$  with a suitable subcurve of  $\gamma$ , we may assume without loss of generality that

$$\gamma \subset B_\epsilon(u, 2Cd_\epsilon(u)),$$

where  $C = \max\{C_1, C_1/C_2\}$ .

Let  $\mathcal{B}$  be a Whitney covering of  $(X, d_\epsilon, \mu_\epsilon)$  as in Section 3. We choose  $B_\epsilon(u, r(u))$  as the fixed ball  $B_0 \in \mathcal{B}$ . Let  $\hat{u} \in \partial_d X$  be such that  $d(u) = d(u, \hat{u})$ . Let  $y \in \gamma$  and let  $[u, y]_k$  be a  $k$ -geodesic joining  $u$  to  $y$ . Moreover, let  $I_y, J_y \subset \mathbb{N}$  be index sets defined by setting

$$I_y = \{i \in \mathbb{N} : B_i \in \mathcal{B}, [u, y]_k \cap 5B_i \cap B_d(u, Md(u)) \neq \emptyset\}$$

and

$$J_y = \{j \in \mathbb{N} : B_j \in \mathcal{B}, [u, y]_k \cap B_j \cap B_d(u, Md(u)) \neq \emptyset\}.$$

Then  $\emptyset \neq J_y \subset I_y$ . Let  $\tilde{y} \in [u, y]_k$  be the first point in  $[u, y]_k$  as the path is traversed from  $u$  to  $y$  such that  $\tilde{y} \notin B_d(u, Md(u))$ . Thus  $[u, \tilde{y}]_k$  is a subcurve of  $[u, y]_k$  in  $\overline{B_d(u, Md(u))}$ .

We may assume that  $[u, \tilde{y}]_k$  is not entirely contained in any  $5B_i$ ,  $i \in I_y$ . Indeed, if  $[u, \tilde{y}]_k \subset 5B_i$ , then (WB1) guarantees that  $d(u, \tilde{y})$  is no more than a constant multiple of  $d(u)$ . Thus a bound for  $M$  would immediately follow.

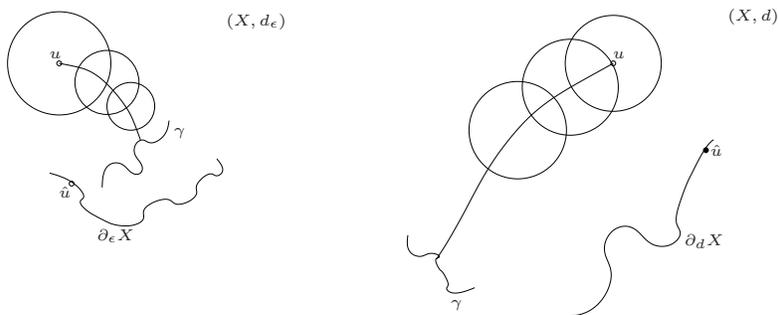


FIGURE 1. Figure of the setting of Lemma 4.1

Subclaim. We have that

$$(4.1) \quad \frac{P}{\log(M-1)} \sum_{i \in I_y} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} \geq 1,$$

where  $P \geq 1$  is a constant that depends only on  $\epsilon$  and on the constants in the hypotheses.

Towards this end, let  $[u_i, y_i]_k \subset [u, \tilde{y}]_k$ ,  $i \in I_y$ , be a subcurve of  $[u, \tilde{y}]_k$  such that  $u_i$  is the first point and  $y_i$  is the last point where  $[u, \tilde{y}]_k$  intersects  $5\overline{B}_i$ . Moreover, for  $j \in J_y$ , by the reduction before Subclaim, we may choose a subcurve  $\alpha_j \subset [u_j, y_j]_k \cap 5\overline{B}_j$  so that  $\ell_\epsilon(\alpha_j) \geq \text{diam}_\epsilon(B_j)$ .

We define  $g: X \rightarrow (0, \infty)$  by setting

$$g(x) = \sum_{B \in \mathcal{B}} \frac{\text{diam}_d(B)}{\log(M-1) \text{dist}_d(B, \hat{u}) \text{diam}_\epsilon(B)} \chi_{5B}(x).$$

Because  $k$ -geodesics are  $D_\epsilon$ -uniform curves in  $(X, d_\epsilon)$ , we obtain that

$$(4.2) \quad \begin{aligned} \int_{[u, \tilde{y}]_k} g ds_\epsilon &\leq \frac{1}{\log(M-1)} \sum_{i \in I_y} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u}) \text{diam}_\epsilon(B_i)} \ell_\epsilon([u_i, y_i]_k) \\ &\leq \frac{5D_\epsilon}{\log(M-1)} \sum_{i \in I_y} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})}. \end{aligned}$$

Here  $ds_\epsilon = \sigma_\epsilon(z) ds$  is the arlength element in the metric  $d_\epsilon$ .

Moreover, by Lemma 3.1 (iv) and (WB2) we obtain that

$$(4.3) \quad \begin{aligned} \int_{[u, \tilde{y}]_k} g ds_\epsilon &\geq \frac{1}{N \log(M-1)} \sum_{j \in J_y} \frac{\text{diam}_d(B_j)}{\text{dist}_d(B_j, \hat{u}) \text{diam}_\epsilon(B_j)} \ell_\epsilon(\alpha_j) \\ &\geq \frac{1}{N \log(M-1)} \sum_{j \in J_y} \frac{\text{diam}_d(B_j)}{\text{dist}_d(B_j, \hat{u})} \\ &\geq \frac{\epsilon}{25C^2 D N \log(M-1)} \sum_{j \in J_y} \frac{d(z_j)}{\text{dist}_d(B_j, \hat{u})}. \end{aligned}$$

Let  $j \in J_y$ . Because  $\text{diam}_{k_\epsilon}(B_j) \leq \frac{2\epsilon}{50-\epsilon}$ , using (WB1) and (3.4), we obtain the estimate

$$(4.4) \quad \begin{aligned} \int_{[u, y]_k \cap B_j} ds &= \int_{[u, y]_k \cap B_j} \frac{d(z_j)}{d(z_j)} ds \\ &\leq \frac{50 + \epsilon C^2}{50} d(z_j) \text{diam}_k(B_j) \\ &\leq \frac{50 + \epsilon C^2}{50} d(z_j) \frac{1}{c\epsilon} \text{diam}_{k_\epsilon}(B_j) \\ &\leq \frac{50 + \epsilon C^2}{25c} \frac{1}{50 - \epsilon} d(z_j). \end{aligned}$$

Combining this with (4.3) and (4.2) we have that

$$\begin{aligned}
& \frac{5D_\epsilon}{\log(M-1)} \sum_{i \in I_y} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} \\
& \geq \frac{c\epsilon}{C^2 DN \log(M-1)} \frac{50-\epsilon}{50+\epsilon C^2} \sum_{j \in J_y} \int_{[u,y]_k \cap B_j} \frac{1}{\text{dist}_d(B_j, \hat{u})} ds \\
(4.5) \quad & \geq \frac{c\epsilon}{C^2 DN \log(M-1)} \frac{50-\epsilon}{50+\epsilon C^2} \int_{[u,\tilde{y}]_k} \frac{1}{d(z, \hat{u})} ds \\
& \geq \frac{c\epsilon}{C^2 DN \log(M-1)} \frac{50-\epsilon}{50+\epsilon C^2} \log\left(\frac{d(\tilde{y}, \hat{u})}{d(u, \hat{u})}\right) \\
& \geq \frac{c\epsilon}{C^2 DN} \frac{50-\epsilon}{50+\epsilon C^2},
\end{aligned}$$

and (4.1) follows.

What we would like to do next is to integrate (4.1) over  $y \in \gamma$ . We know that  $\gamma$  is a curve of reasonably large diameter in the metric  $d_\epsilon$ , but the length of  $\gamma$  could well be infinite, or  $\gamma$  could wiggle. Frostman's lemma (cf. [Ma] and [KL, Theorem 4.1]) provides us with a suitable substitute for the  $d_\epsilon$ -length measure on  $\gamma$ : there is a Radon measure  $\nu$  on  $\gamma \subset (X, d_\epsilon)$  so that

$$\begin{aligned}
(4.6) \quad & \nu(E) \leq \text{diam}_\epsilon(E) \quad \text{for every } E \subset \gamma, \text{ and} \\
& \nu(\gamma) \geq \frac{1}{30} \frac{\text{diam}_\epsilon(\gamma)}{2}.
\end{aligned}$$

Towards the use of the Fubini theorem, recall the definition of the shadow  $S(B)$  of a Whitney ball, that  $B_\epsilon(u, r(u))$  is our fixed ball and the definition of  $\mathcal{B}_n$  from the end of the previous section.

Using (4.1), Fubini's theorem and Hölder's inequality we obtain that

$$\begin{aligned}
(4.7) \quad & \nu(\gamma) = \int_\gamma d\nu \leq \frac{P}{\log(M-1)} \int_\gamma \sum_{i \in I_y} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} d\nu \\
& \leq \frac{P}{\log(M-1)} \sum_{n=0}^{\infty} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} \nu(S(B_i) \cap \gamma) \\
& \leq \frac{P}{\log(M-1)} \left( \sum_{n=0}^{\infty} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} \left( \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} \right)^Q \right)^{\frac{1}{Q}} \\
& \quad \left( \sum_{n=0}^{\infty} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} (\nu(S(B_i) \cap \gamma))^{\frac{Q}{Q-1}} \right)^{\frac{Q-1}{Q}}.
\end{aligned}$$

We will complete the proof of our claim by carefully estimating the terms in (4.7). The key points here are that the upper volume growth

condition allows us to estimate the first double sum from above by a multiple of  $\log M$  and that the geometry of  $(X, d_\epsilon)$  and the properties of our Frostman measure allow us to cancel  $\nu(\gamma)$  by the indicated power of the second double sum.

Let us begin by estimating the first double sum in (4.7). Let

$$A_n = \overline{B}_d(\hat{u}, 2^n d(u)) \setminus B_d(\hat{u}, 2^{n-1} d(u)).$$

Pick an integer  $m$  with  $2^{m-1} < M + 1 \leq 2^m$ . Write  $\tilde{B} = B_d(z, \frac{\epsilon C_0^2}{50} d(z))$  for  $B = B_\epsilon(z, r(z)) \in \mathcal{B}$ . Given  $B \in \mathcal{B}$ , (WB1) ensures that  $B \subset \tilde{B}$ . Moreover, if  $B \cap A_n \neq \emptyset$  for some  $n \in \mathbb{Z}$ , then  $\tilde{B} \subset A_{n-1} \cup A_n \cup A_{n+1}$ . Thus, by the  $Q$ -upper regularity condition and the  $Q$ -regularity condition on Whitney type balls in  $(X, d)$  we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} \left( \frac{\text{diam}_d(B_i)}{\text{dist}_d(B_i, \hat{u})} \right)^Q &\leq \sum_{n=0}^m \sum_{\substack{B \in \mathcal{B} \\ B \cap A_n \neq \emptyset}} \left( \frac{\text{diam}_d(B)}{\text{dist}_d(B, \hat{u})} \right)^Q \\
&\leq \sum_{n=0}^m \sum_{\substack{B \in \mathcal{B} \\ B \cap A_n \neq \emptyset}} \frac{2^Q C_w \mu(\tilde{B})}{(\text{dist}_d(B, \hat{u}))^Q} \\
(4.8) \quad &\leq 2^Q C_w^3 (5DC_0^4)^Q \sum_{n=0}^m \frac{\mu(B_d(\hat{u}, 2^{n+1} d(u)) \cap X)}{(2^{n-2} d(u))^Q} \\
&\leq 2^{4Q} C_w^3 C (5DC_0^4)^Q (m+1) \\
&\leq 2^{4(Q+1)} C_w^3 C (5DC_0^4)^Q \log M.
\end{aligned}$$

Above, in moving from the second line to the third, we used the pairwise disjointness of the balls  $\frac{1}{5}B$ , (WB2) and the  $Q$ -regularity of  $\mu$  on Whitney type balls.

Let us then estimate the second double sum in (4.7). Let  $B_i \in \mathcal{B}$ , where  $i \in I_y$  with  $y \in \gamma$ , and let  $z_i \in B_i$  be its center. Because  $k$ -geodesics are  $D_\epsilon$ -uniform curves and  $\text{dist}_\epsilon(u, \gamma) \leq \min\{C_1 d_\epsilon(u), C_2 \text{diam}_\epsilon(\gamma)\}$ , we conclude that

$$\begin{aligned}
d_\epsilon(z_i) &\leq d_\epsilon(u) + D_\epsilon \text{dist}_\epsilon(u, \gamma) + \frac{\epsilon}{10} d_\epsilon(z_i) \\
&\leq (1 + D_\epsilon C_1) d_\epsilon(u) + \frac{\epsilon}{10} d_\epsilon(z_i),
\end{aligned}$$

and thus

$$(4.9) \quad d_\epsilon(z_i) \leq \frac{10}{10 - \epsilon} (1 + D_\epsilon C_1) d_\epsilon(u).$$

We also know that  $B_\epsilon(u, \epsilon d_\epsilon(u)) \subset B_d(u, \frac{1}{2}d(u)) \subset B_d(u, Md(u))$  and  $\gamma \cap B_d(u, Md(u)) = \emptyset$ . Especially  $\gamma \cap B_\epsilon(u, \epsilon d_\epsilon(u)) = \emptyset$  and hence

$$\begin{aligned} d_\epsilon(z_i) &\leq d_\epsilon(u) + D_\epsilon \operatorname{dist}_\epsilon(u, \gamma) + \frac{\epsilon}{10} d_\epsilon(z_i) \\ &\leq \frac{1}{\epsilon} \operatorname{dist}_\epsilon(u, \gamma) + D_\epsilon \operatorname{dist}_\epsilon(u, \gamma) + \frac{\epsilon}{10} d_\epsilon(z_i). \end{aligned}$$

Thus

$$(4.10) \quad d_\epsilon(z_i) \leq \frac{10}{10 - \epsilon} \frac{1 + \epsilon D_\epsilon}{\epsilon} C_2 \operatorname{diam}_\epsilon(\gamma).$$

If  $B_i \in \mathcal{B}_n$ , since  $(X, d_\epsilon)$  is a  $D_\epsilon$ -uniform space, by [BHK, Lemma 2.13] we have that

$$(4.11) \quad n \leq k_\epsilon(u, z_i) \leq 4D_\epsilon^2 \log\left(1 + \frac{d_\epsilon(u, z_i)}{\min\{d_\epsilon(u), d_\epsilon(z_i)\}}\right).$$

Moreover, since  $k$ -geodesics are  $D_\epsilon$ -uniform curves in  $(X, d_\epsilon)$ , (4.10) implies that

$$\begin{aligned} d_\epsilon(u, z_i) &\leq D_\epsilon \operatorname{dist}_\epsilon(u, \gamma) + \frac{\epsilon}{10} d_\epsilon(z_i) \\ &\leq \left(D_\epsilon + \frac{1 + \epsilon D_\epsilon}{10 - \epsilon}\right) C_2 \operatorname{diam}_\epsilon(\gamma), \end{aligned}$$

and thus (4.9), (4.11) and this give us when  $n \geq 4D_\epsilon^2$  that

$$\begin{aligned} (4.12) \quad d_\epsilon(z_i) &\leq \frac{10}{10 - \epsilon} (1 + D_\epsilon C_1) \min\{d_\epsilon(u), d_\epsilon(z_i)\} \\ &\leq \frac{10}{10 - \epsilon} (1 + D_\epsilon C_1) \frac{d_\epsilon(u, z_i)}{\exp\{\frac{n}{4D_\epsilon^2}\} - 1} \\ &\leq 2^{\frac{-n}{4D_\epsilon^2}} \frac{2(10 + 10D_\epsilon C_1)(10D_\epsilon + 1)}{(10 - \epsilon)^2} C_2 \operatorname{diam}_\epsilon(\gamma). \end{aligned}$$

Because  $k$ -geodesics are  $D_\epsilon$ -uniform curves in  $(X, d_\epsilon)$  and  $\gamma \subset B_\epsilon(u, 2Cd_\epsilon(u))$ , it easily follows that there is a constant  $C_s$  only depending on  $D_\epsilon$  and  $C$  so that

$$(4.13) \quad \operatorname{diam}_\epsilon(S(B_i) \cap \gamma) \leq C_s \operatorname{diam}_\epsilon(B_i)$$

for each  $y \in \gamma$  and every  $i \in I_y$ . Now, inequalities (3.9), (4.6), (4.13), (4.10) and (4.12) yield

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} (\nu(S(B_i) \cap \gamma))^{\frac{Q}{Q-1}} &\leq \sum_{n=0}^{\infty} \max_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} (\nu(S(B_i) \cap \gamma))^{\frac{1}{Q-1}} \sum_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} \nu(S(B_i) \cap \gamma) \\
 &\leq C_0 \nu(\gamma) \sum_{n=0}^{\infty} \left( \max_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} (\text{diam}_{\epsilon}(S(B_i) \cap \gamma)) \right)^{\frac{1}{Q-1}} \\
 (4.14) \quad &\leq C_0 \left( \frac{\epsilon C_s}{25} \right)^{\frac{1}{Q-1}} \nu(\gamma) \sum_{n=0}^{\infty} \left( \max_{\substack{B_i \in \mathcal{B}_n \\ i \in I_y, y \in \gamma}} (d_{\epsilon}(z_i)) \right)^{\frac{1}{Q-1}} \\
 &\leq C' \nu(\gamma) (\text{diam}_{\epsilon}(\gamma))^{\frac{1}{Q-1}},
 \end{aligned}$$

where  $C' > 0$  is a constant depending on  $\epsilon$  and the hypotheses.

Combining (4.8) and (4.14) with (4.7) we obtain

$$\nu(\gamma) \leq C'' \frac{(\log M)^{\frac{1}{Q}}}{\log(M-1)} \nu(\gamma)^{\frac{Q-1}{Q}} (\text{diam}_{\epsilon}(\gamma))^{\frac{1}{Q}}.$$

Inserting (4.6) we conclude that

$$1 \leq 60(C'')^Q \frac{\log(M)}{(\log(M-1))^Q}.$$

This gives the desired upper bound on  $M$  and the claim follows.  $\square$

## 5. PROOF OF THEOREM 1.2

We begin by proving the following theorem.

**Theorem 5.1.** *Let  $Q > 1$  and let  $(X, d, \mu)$  be a minimally nice  $Q$ -upper regular  $D$ -quasiconvex space such that the measure  $\mu$  is  $Q$ -regular on Whitney type balls. Suppose that  $(X, k)$  is a  $K$ -roughly starlike Gromov hyperbolic space. Then  $(X, d)$  satisfies both the Gehring–Hayman condition and the ball separation condition.*

*Proof.* Let us first prove that  $(X, d)$  satisfies the Gehring–Hayman condition. Because  $(X, k)$  is Gromov hyperbolic and  $K$ -roughly starlike,  $(X, d_{\epsilon}, \mu_{\epsilon})$  is uniform for a deformation as in Section 3 with respect to a base point  $w \in X$ , where we choose  $\epsilon \leq \min\{\epsilon_0, \frac{1}{8D}, \frac{1}{2C_0^2}\}$ , where  $\epsilon_0 > 0$  is as in the paragraph containing (3.2), with  $C_0 > 1$  as in (3.5). We know from (3.7) and (3.8) that the measure  $\mu_{\epsilon}$  is  $Q$ -regular on Whitney type balls. We will consider  $(X, d, \mu)$  as a conformal deformation of  $(X, d_{\epsilon}, \mu_{\epsilon})$  so that we deduce the Gehring–Hayman inequality from results in [KL].

Towards this end define  $\tilde{\rho}: (X, d_{\epsilon}) \rightarrow (0, \infty)$  by setting

$$\tilde{\rho}(z) = \frac{d(z)}{\rho_{\epsilon}(z)} = (\sigma_{\epsilon}(z))^{-1}.$$

First, let us prove that  $\tilde{\rho}$  satisfies the Harnack inequality  $\text{HI}(A)$  with some constant  $A \geq 1$ . Let  $z \in X$  and  $x \in B_\epsilon(z, \epsilon d_\epsilon(z))$ . By inequality (WB1) and the triangle inequality we obtain that

$$(5.1) \quad \tilde{\rho}(x) = \frac{d(x)}{\exp\{-\epsilon k_d(w, x)\}} \leq \frac{(1 + \epsilon C_0^2)d(z)}{\exp\{-\epsilon(k_d(w, z) + 1)\}} \\ \leq \exp\{\epsilon\}(1 + \epsilon C_0^2)\tilde{\rho}(z),$$

and

$$(5.2) \quad \tilde{\rho}(x) = \frac{d(x)}{\exp\{-\epsilon k_d(w, x)\}} \geq \frac{(1 - \epsilon C_0^2)d(z)}{\exp\{-\epsilon(k_d(w, z) - 1)\}} \\ \geq \frac{1 - \epsilon C_0^2}{\exp\{\epsilon\}}\tilde{\rho}(z).$$

Thus, for  $A = \max\{\exp\{2\epsilon\}(1 + \epsilon C_0^2)^2, \frac{\exp\{2\epsilon\}}{(1 - \epsilon C_0^2)^2}\}$ , the density  $\tilde{\rho}$  satisfies

$$(5.3) \quad A^{-1} \leq \frac{\tilde{\rho}(x)}{\tilde{\rho}(y)} \leq A$$

for all  $x, y \in B_\epsilon(z, \epsilon d_\epsilon(z))$  and each  $z \in X$ .

The density  $\tilde{\rho}$  also satisfies the volume growth condition  $\text{VG}(B)$  with the constant  $B = C_u D^Q$ . Indeed, observe that

$$(5.4) \quad d_{\tilde{\rho}}(x, y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \tilde{\rho}(z) ds_\epsilon(z) \\ = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} (\sigma_\epsilon(z))^{-1} \sigma_\epsilon(z) ds(z) \\ = \inf_{\gamma_{xy}} \ell_d(\gamma_{xy}),$$

where the infimum is taken over all curves  $\gamma_{xy}$  joining points  $x$  and  $y$ . Since  $(X, d)$  is  $D$ -quasiconvex, it follows that  $(X, d_{\tilde{\rho}})$  is bi-Lipschitz equivalent to  $(X, d)$  and furthermore, we have that  $B_{\tilde{\rho}}(z, r) \subset B_d(z, Dr)$  for all  $z \in X$  and  $r > 0$ . Thus from the  $Q$ -upper regularity condition (2.1) it follows that

$$(5.5) \quad \mu_{\tilde{\rho}}(B_{\tilde{\rho}}(z, r)) = \int_{B_{\tilde{\rho}}(z, r)} \tilde{\rho}(x)^Q d\mu_\epsilon(x) \\ = \int_{B_{\tilde{\rho}}(z, r)} (\sigma_\epsilon(x))^{-Q} (\sigma_\epsilon(x))^Q d\mu(x) \\ = \mu(B_{\tilde{\rho}}(z, r)) \\ \leq \mu(B_d(z, Dr)) \\ \leq C_u D^Q r^Q$$

for every  $z \in X$  and  $r > 0$ .

Hence,  $\tilde{\rho}$  is a conformal density. The corresponding deformation of the metric space  $(X, d_\epsilon)$  with  $\tilde{\rho}$  results in an inner metric space  $(X, \ell_d)$  which is bi-Lipschitz equivalent to the original metric space  $(X, d)$ .

We aim to apply [KL, Theorem 1.1] which gives a Gehring-Hayman condition for conformal deformations of certain uniform spaces. As stated in [KL] this theorem applies in our setting to quasihyperbolic geodesics with respect to the metric  $d_\epsilon$  but not directly to the geodesics  $[x, y]_k$ . However, the proof in [KL] gives the estimate

$$\ell_d(\beta_{xy}) \leq C_{\text{gh}} \ell_d(\gamma_{xy})$$

for each  $D_\epsilon$ -uniform curve  $\beta_{xy}$ , with  $C_{\text{gh}}$  depending only on  $D_\epsilon$  and the data associated to our conformal deformation and  $(X, d_\epsilon, \mu_\epsilon)$ . Recalling that each quasihyperbolic geodesic  $[x, y]_k$  is a  $D_\epsilon$ -uniform curve in  $(X, d_\epsilon)$ , the Gehring-Hayman condition follows.

Let us now prove that  $(X, d)$  satisfies the ball separation condition. Let  $x, y \in X$ ,  $u \in [x, y]_k$  and  $\gamma_{xy} \subset X$  be a curve joining  $x$  and  $y$ . Let  $(X, d_\epsilon, \mu_\epsilon)$  be the deformation of  $(X, d, \mu)$  as before. We may assume that  $\ell_\epsilon([x, u]_k) \leq \ell_\epsilon([u, y]_k)$ . Because  $[x, y]_k$  is a  $D_\epsilon$ -uniform curve in  $(X, d_\epsilon)$ , we have that

$$\text{dist}_\epsilon(u, \gamma_{xy}) \leq \ell_\epsilon([x, u]_k) \leq D_\epsilon d_\epsilon(u),$$

and

$$\text{dist}_\epsilon(u, \gamma_{xy}) \leq \ell_\epsilon([x, u]_k) \leq D_\epsilon \text{diam}_\epsilon(\gamma_{xy}).$$

Thus assumptions of Lemma 4.1 hold, and hence there is a constant  $C_{\text{bs}} \geq 1$  depending on  $\epsilon$  and the hypotheses such that

$$\gamma \cap B_d(u, C_{\text{bs}} d(u)) \neq \emptyset.$$

□

Balogh and Buckley proved in [BB, Theorem 2.4 and Theorem 6.1] that, for a minimally nice length space  $(X, d)$  that satisfies both the Gehring-Hayman condition and the ball separation condition, the associated space  $(X, k)$  is Gromov hyperbolic. Therefore we have the following corollary to Theorem 5.1.

**Corollary 5.2.** *Let  $Q > 1$  and let  $(X, d, \mu)$  be a minimally nice  $Q$ -upper regular length space such that the measure  $\mu$  is  $Q$ -regular on Whitney type balls. Suppose that  $(X, k)$  is  $K$ -roughly starlike. Then the quasihyperbolic space  $(X, k)$  is Gromov hyperbolic if and only if  $(X, d)$  satisfies both the Gehring-Hayman condition and the ball separation condition.*

Now we are able to deduce Theorem 1.2.

*Proof of Theorem 1.2.* From [BB, Theorem 3.1] it follows that  $(\Omega, k)$  is  $K$ -roughly starlike, because  $(X, d)$  is annularly quasiconvex and  $\Omega \subset X$  is a bounded and proper subdomain. Hence the claim follows from Corollary 5.2. □

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