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## **Variational Principle and Bifurcations in Stability Analysis of Panels**

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# Variational Principle and Bifurcations in Stability Analysis of Panels\*

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## Abstract

In this paper, the stability of a simply supported axially moving elastic panel is considered. A complex variable technique and bifurcation theory are applied. As a result, variational equations and a variational principle are derived. Analysis of the variational principle allows the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point is analyzed and presented.

It is shown analytically that the eigenvalue curves in the  $(\omega, V_0)$  plane cross both the  $\omega$  and  $V_0$  axes perpendicularly. It is also shown that near each bifurcation point, the dependence  $\omega(V_0)$  for each mode approximately follows the shape of a square root near the origin.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

## 1 Introduction

The aim of our studies has been to develop mathematical models representing the behaviour of the paper making process. Previously (see e.g., Banichuk et al. [2013b,a, 2011a,b]), we have considered many approaches for modelling moving materials and their stability. Conclusions that have been drawn can be applied for example, the processing of paper or steel, fabric, rubber or some other continuous material, and looping systems such as band saws and timing belts.

Typically systems of axially moving web have been modelled as travelling flexible strings, membranes, beams, and plates. Classical articles in this field are, for example, Mote [1972], Archibald and Emslie [1958], Simpson [1973], Wang et al.

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[2005], Parker [1998] Kong and Parker [2004], Miranker [1960] Chonan [1986], Wickert and Mote [1990], Bhat et al. [1982], Perkins [1990], Wickert [1994] and Parker [1999].

In the case of beams interacting with external media, one can read e.g. the article by Chang and Moretti [1991], and the articles by Banichuk and Neittaanmäki [2008a,b,c]. The study has been extended in Banichuk et al. [2010] for a two-dimensional model of the web, considered as a moving plate under homogeneous tension but without external media. The most straightforward and efficient way to study stability is to use linear stability analysis. In a recent article by Hatami et al. [2009], the free vibration of a moving orthotropic rectangular plate was studied at sub- and supercritical speeds, and its flutter and divergence instabilities at supercritical speeds. The study is limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving material, many necessary fundamentals can be found in the book by Marynowski [2008b]. An extensive literature review about dynamics of axially moving continua can be found in Marynowski and Kapitaniak [2014] or in the book by Banichuk et al. [2014]. However, in this article the effect of surrounding media have been excluded.

The dynamical properties of moving plates have been studied by Shen et al. [1995] and by Shin et al. [2005], and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. [2007a,b]. Critical regimes and other problems of stability analysis have been studied e.g. by Wang [2003] and Sygulski [2007]. Moreover, in the articles Marynowski [2002, 2004, 2008a] the dynamical aspects of the axially moving web are discussed extensively. In Yang and Chen [2005], the authors considered transverse vibrations of the axially accelerating viscoelastic beam, and in Pellicano and Vestroni [2000], dynamic behavior of a simply supported beam subjected to an axial transport of mass was studied. An extensive literature review related to areas presented in this paper, can be found for example in Ghayesh et al. [2013]. Note also some approaches to bifurcation problems and estimation of critical parameters presented by Nečas et al. [1987] and Neittaanmäki and Ruotsalainen [1985].

The focus of this article is the stability of a simply supported axially moving elastic panel. We have used a complex variable technique and bifurcation theory. Our main task has been the derivation of variational equations and a variational principle. Moreover, we have performed an analysis of the variational principle, which allows the study of qualitative properties of the bifurcation points. Furthermore, asymptotic behaviour around an arbitrary bifurcation point is analyzed and presented. As a result, we show analytically that the eigenvalue curves in the  $(\omega, V_0)$  plane cross both the  $\omega$  and  $V_0$  axes perpendicularly. It is also shown that near each bifurcation point, the dependence  $\omega(V_0)$  for each mode approximately follows the shape of a square root near the origin. Gained results complement existing numerical studies on the stability of axially moving materials, and especially materials with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

## 2 Basic relations and complex functions

Consider the problem of free harmonic vibrations of an elastic panel, moving axially at a constant velocity  $V_0$ . In a stationary orthogonal coordinate system, the transverse vibrations are characterized by the function  $w = w(x, t)$ , which is determined by the following partial differential equation:

$$w_{,tt} + 2V_0 w_{,xt} + (V_0^2 - C^2) w_{,xx} + \frac{D}{\rho S} w_{,xxxx} = 0, \quad 0 < x < \ell, \quad C = \sqrt{\frac{T_0}{\rho S}}. \quad (1)$$

At the ends of the considered interval  $x \in [0, \ell]$ , we have the simply supported boundary conditions

$$w(0, t) = w(\ell, t) = Dw_{,xx}(0, t) = Dw_{,xx}(\ell, t) = 0. \quad (2)$$

Here  $x$  is the axial space coordinate,  $t$  time,  $T_0$  tension,  $\rho$  material density,  $S$  the area of the panel cross section,  $\ell$  the length of the free span, and  $D$  the bending rigidity of the panel.

For harmonic vibrations at frequency  $\omega$ , the transverse displacement can be represented in the form

$$w(x, t) = e^{i\omega t} u(x), \quad i = \sqrt{-1}, \quad (3)$$

where  $u(x)$  is an amplitude function that satisfies the following boundary value problem:

$$u_{,xxxx} + (V_0^2 - C^2) u_{,xx} + 2i\omega V_0 u_{,x} - \omega^2 u = 0, \quad (4)$$

$$u(0) = u(\ell) = u_{,xx}(0) = u_{,xx}(\ell) = 0, \quad (5)$$

which is written in dimensionless variables

$$x = \ell \tilde{x}, \quad \frac{\rho S \omega^2 \ell^4}{D} = \tilde{\omega}^2, \quad \frac{\rho S \ell^2}{D} V_0^2 = \tilde{V}_0^2, \quad \frac{\rho S \ell^2}{D} C^2 = \tilde{C}^2. \quad (6)$$

In what follows, the tilde will be omitted.

The amplitude function  $u(x)$  determined from the boundary value problem (4)–(5) is a complex-valued function, i.e.

$$u(x) = u^1(x) + iu^2(x), \quad \hat{u}(x) = u^1(x) - iu^2(x), \quad (7)$$

where  $u^1(x)$  and  $u^2(x)$  are real-valued functions and  $\hat{u}(x)$  is the complex conjugate of  $u(x)$ .

In the following we present a variational formulation of the spectral problem (4)–(5). This formulation allows us to make important conclusions about the frequencies of free vibrations of moving elastic systems without knowing the rigorous solution of the spectral boundary value problem. To derive the variational formulation of (4)–(5), we multiply the differential equation by the complex conjugate (adjoint) amplitude function  $\hat{u}(x)$  and integrate the result on the interval  $(0, 1)$ . We will also take into account the boundary conditions

$$\begin{aligned} u^1(0) = u^2(0) = u^1(1) = u^2(1) = 0, \\ u^1_{,xx}(0) = u^2_{,xx}(0) = u^1_{,xx}(1) = u^2_{,xx}(1) = 0, \end{aligned} \quad (8)$$

which follow from the boundary conditions (5). We obtain the functional equation

$$a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0 , \quad (9)$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are integral functional depending on the problem (4)–(5). The functional  $a$  is given by

$$a = \int_0^1 u\hat{u} \, dx = \int_0^1 ((u^1)^2 + (u^2)^2) \, dx > 0 . \quad (10)$$

Using the boundary conditions (8), we can write the functional  $b$  as

$$\int_0^1 u_{,x}\hat{u} \, dx = i \int_0^1 ((u^2)_{,x}u^1 - (u^1)_{,x}u^2) \, dx = ib , \quad (11)$$

where  $b$  is real-valued. The functionals  $c$  and  $d$  are obtained by integrating by parts (once in the case of  $c$  and twice for  $d$ ), and taking into account the corresponding boundary conditions in (8). We have

$$c = - \int_0^1 u_{,xx}\hat{u} \, dx = \int_0^1 ([(u^1)_{,x}]^2 + [(u^2)_{,x}]^2) \, dx > 0 , \quad (12)$$

$$d = \int_0^1 u_{,xxxx}\hat{u} \, dx = \int_0^1 ([(u^1)_{,xx}]^2 + [(u^2)_{,xx}]^2) \, dx > 0 . \quad (13)$$

### 3 Variational analysis and variational principle in complex variables

Let us write the variation of the functional equation (9). To do this, we take into account the variations of the considered functionals,

$$\begin{aligned} \delta a &= \int_0^1 (\hat{u}\delta u + u\delta\hat{u}) \, dx , \\ i\delta b &= \int_0^1 (\hat{u}\delta u_{,x} + u_{,x}\delta\hat{u}) \, dx , \\ \delta c &= \int_0^1 (u_{,x}\delta\hat{u}_{,x} + \hat{u}_{,x}\delta u_{,x}) \, dx , \\ \delta d &= \int_0^1 (u_{,xx}\delta\hat{u}_{,xx} + \hat{u}_{,xx}\delta u_{,xx}) \, dx , \end{aligned} \quad (14)$$

and perform standard transformations in (9), replacing  $u$ ,  $\hat{u}$  and  $\omega$  with  $u + \delta u$ ,  $\hat{u} + \delta\hat{u}$  and  $\omega + \delta\omega$ , respectively. We will have the variation

$$\begin{aligned} 2(a\omega + bV_0)\delta\omega + \int_0^1 [-\omega^2 u + 2i\omega V_0 u_{,x} + (V_0^2 - C^2)u_{,xx} + u_{,xxxx}] \delta\hat{u} \, dx \\ + \int_0^1 [-\omega^2 \hat{u} - 2i\omega V_0 \hat{u}_{,x} + (V_0^2 - C^2)\hat{u}_{,xx} + \hat{u}_{,xxxx}] \delta u \, dx = 0 . \end{aligned} \quad (15)$$

For  $u(x)$  and  $\hat{u}(x)$ , which are solutions of the spectral boundary value problem (4)–(5) and its complex conjugate, the integral expressions in (15) are identically zero. Taking this into account, we are left with

$$2(a\omega + bV_0) \delta\omega = 0 . \quad (16)$$

Thus, if  $a\omega + bV_0 \neq 0$  for the spectral problem (4)–(5), then the frequency variation for free vibrations  $\delta\omega$  is zero. That is,

$$a\omega + bV_0 \neq 0 , \quad \delta\omega = 0 . \quad (17)$$

Solving (9) for  $\omega$ , we arrive at the variational representation for harmonic vibrations, corresponding to each of the two solution branches of equation (9):

$$\omega_{\pm}(V_0) = \frac{1}{a} \left( -bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad} \right) \rightarrow \underset{u(x), \hat{u}(x)}{\text{extr}} . \quad (18)$$

## 4 Analysis of extremum conditions and bifurcation analysis

From (16), the other possibility is

$$a\omega + bV_0 = 0 \quad (19)$$

and  $\delta\omega$  free. To perform analysis for this case, we consider equation (9) as an implicit function  $F(\omega, V_0)$ :

$$F(\omega, V_0) = 0 , \quad F(\omega, V_0) = a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d . \quad (20)$$

Again, we can solve (9) for  $\omega$ , obtaining the following two solution branches:

$$\omega_{\pm}(V_0) = \frac{1}{a} \left( -bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad} \right) \quad (21)$$

Let  $(\omega^*, V_0^*)$  denote the bifurcation point, i.e. the values of  $\omega$  and  $V_0$  at which the solution of (20) branches. At the bifurcation point, the conditions of the implicit function theorem must be violated, i.e. we will have

$$F(\omega, V_0) = 0 , \quad \frac{\partial F(\omega, V_0)}{\partial \omega} = 0 . \quad (22)$$

Using (20), these conditions become

$$a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0 , \quad a\omega + bV_0 = 0 . \quad (23)$$

As a result, we find the following representation for bifurcation values of the frequency and panel velocity:

$$\omega^* = -\frac{b}{a}V_0^* , \quad (ac - b^2)(V_0^*)^2 = acC^2 + ad . \quad (24)$$

Alternatively, these values can be obtained from the condition  $\omega_+(V_0) = \omega_-(V_0)$  and the representation (21) for  $\omega_{\pm}(V_0)$ . Note also that if some solutions have  $b = 0$ , the corresponding bifurcation points are distributed along the  $V_0$  axis in the  $(V_0, \omega)$  plane, i.e.

$$\omega^* = 0, \quad (V_0^*)^2 = C^2 + \frac{d}{c} \quad (\text{for solutions with } b = 0). \quad (25)$$

Let us differentiate  $\omega(V_0)$  with respect to the parameter  $V_0$ . To do this, in (20) we replace  $V_0, u$  and  $\omega$  with  $V_0 + \delta V_0, u + \delta u$  and  $\omega + \delta \omega$ , respectively. Using the standard transformations (as was done in (16)) we obtain

$$2(a\omega + bV_0) \delta \omega + 2(b\omega + cV_0) \delta V_0 = 0. \quad (26)$$

Consequently,

$$\frac{d\omega}{dV_0} = -\frac{b\omega + cV_0}{a\omega + bV_0}. \quad (27)$$

In particular, it follows from (27) that for all bifurcation points  $(\omega^*, V_0^*)$  we have the limit

$$\lim_{V_0 \rightarrow V_0^*} \frac{d\omega_{\pm}(V_0)}{dV_0} = \pm \infty. \quad (28)$$

In the case  $V_0 = 0$ , we have  $b = 0$ , and find that

$$\frac{d\omega_{\pm}(V_0=0)}{dV_0} = 0. \quad (29)$$

It follows from (28)–(29) that the curves  $\omega_{\pm}(V_0)$  cross the  $\omega$  and  $V_0$  axes at right angles; see Figure 1.

## 5 Nonlinear analysis of asymptotic behaviour of the frequencies in the vicinity of bifurcation points

Let  $(\omega_1^*, V_{01}^*), (\omega_2^*, V_{02}^*), \dots$  be solutions of the system of nonlinear equations (20). Consider the behaviour of the functions  $\omega_i(V_0)$  ( $i = 1, 2, \dots$ ), determined in a small neighbourhood of the bifurcation point  $(\omega_k^*, V_{0k}^*)$ , in implicit form, by the equation  $F(\omega, V_0) = 0$ . For brevity, we will omit the indices of the functions  $\omega_i(V_0)$  and the bifurcation points  $(\omega_k^*, V_{0k}^*)$ .

To study the behaviour of the function  $F(\omega, V_0)$ , we expand it in series around the bifurcation point  $(\omega^*, V_0^*)$ . We have

$$F(\omega, V_0) = F(\omega^*, V_0^*) + \frac{\partial F(\omega^*, V_0^*)}{\partial \omega} (\omega - \omega^*) + \frac{\partial F(\omega^*, V_0^*)}{\partial V_0} (V_0 - V_0^*) + \frac{1}{2} \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} (\omega - \omega^*)^2 + \dots \quad (30)$$



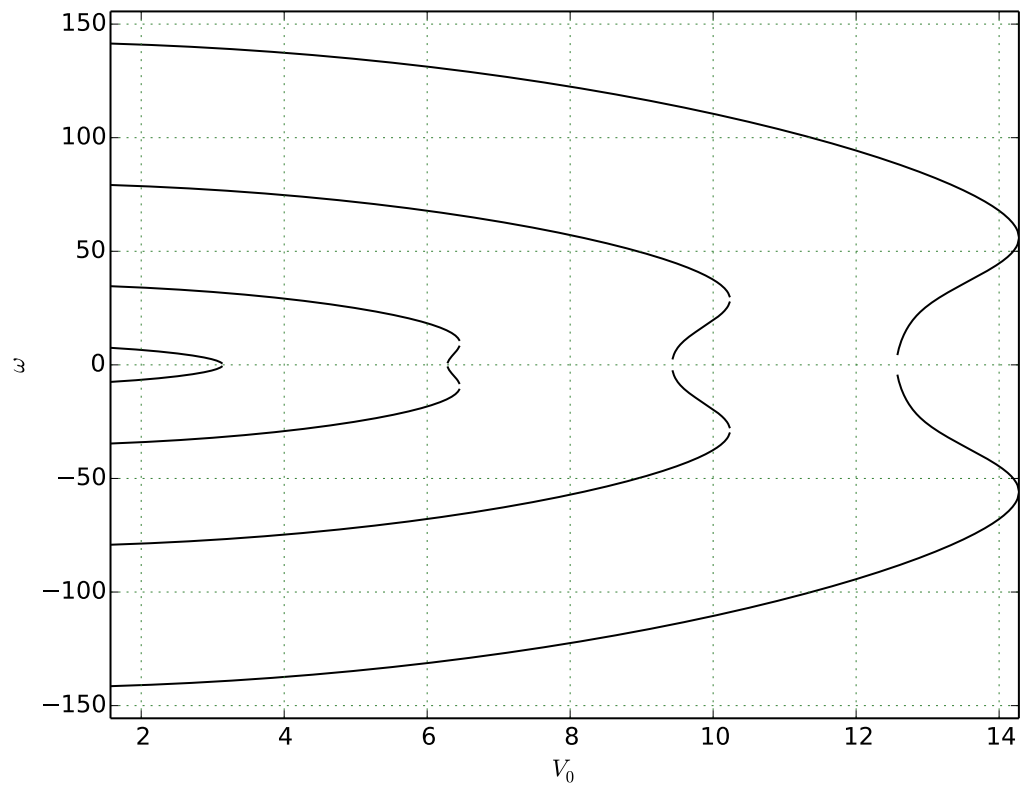


Figure 1: Behaviour of the natural frequencies  $\omega$  as a function of the panel axial velocity  $V_0$ . Numerical solution using finite elements.

Taking into account that at each bifurcation point  $(\omega^*, V_0^*)$ , relation (22) holds, we have that the first two terms in (30) vanish, obtaining

$$F(\omega, V_0) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{1}{2} \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}(\omega - \omega^*)^2 + \dots \quad (31)$$

Observe that all terms that have been omitted in (30) have a higher order of smallness. The expression (31) thus contains all leading-order terms, and describes completely general behaviour of  $F(\omega, V_0)$  in a small neighbourhood of a given bifurcation point  $(\omega^*, V_0^*)$ . This is the general case; the special cases where one or both of  $\partial F(\omega^*, V_0^*)/\partial V_0$  and  $\partial^2 F(\omega^*, V_0^*)/\partial \omega^2$  are zero must be studied separately.

Without loss of generality, we may represent the function  $\omega = \omega(V_0)$  in the small neighbourhood of the bifurcation point  $(\omega^*, V_0^*)$  as a power series:

$$\omega(V_0) = \omega^* + \alpha_1(V_0 - V_0^*)^{\varepsilon_1} + \alpha_2(V_0 - V_0^*)^{\varepsilon_2} + \dots, \quad \text{where } 0 < \varepsilon_1 < \varepsilon_2 < \dots \quad (32)$$

The values of the constants  $\alpha_1, \alpha_2, \dots$  and  $\varepsilon_1, \varepsilon_2, \dots$  are determined with the help of the condition  $F(\omega, V_0) = 0$ . After substitution of (32) into (31), the equation  $F(\omega, V_0) = 0$  reduces to the corresponding equation

$$\Psi(V - V_0^*) = 0, \quad (33)$$

where  $\Psi$  is a function of one variable.

In order for (33) to hold, the coefficient of each power of  $(V - V_0^*)$  in the expression of  $\Psi$  must be equal to zero. This requirement allows us to determine the values of  $\alpha_1, \alpha_2, \dots$  and  $\varepsilon_1, \varepsilon_2, \dots$  in the power series (32). In the following, for simplicity we consider only the determination of  $\alpha_1$  and  $\varepsilon_1$ , i.e. we approximate  $\omega(V_0)$  as

$$\omega(V_0) \approx \omega^* + \alpha_1(V_0 - V_0^*)^{\varepsilon_1}. \quad (34)$$

After substitution of (34) into (31), we obtain

$$\Psi(V_0 - V_0^*) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{\alpha_1^2}{2} \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}(V_0 - V_0^*)^{2\varepsilon_1} + \dots \equiv 0. \quad (35)$$

The expression (35) contains the leading-order terms; all omitted terms are of a higher order of smallness.

We will analyze the case where

$$\frac{\partial F(\omega^*, V_0^*)}{\partial V_0} \neq 0, \quad \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} \neq 0. \quad (36)$$

A separate analysis is needed if one or both values in (36) are zero.

Consider now the cases  $2\varepsilon_1 < 1$ ,  $2\varepsilon_1 = 1$  and  $2\varepsilon_1 > 1$ , which together cover all possibilities for  $\varepsilon_1$ . If  $2\varepsilon_1 < 1$ , then the first term in (35) is of a higher order of smallness with respect to the second term, and consequently in order for (35) to hold,  $\partial^2 F(\omega^*, V_0^*)/\partial \omega^2$  must be zero, which contradicts the second condition in (36). Similarly, if  $2\varepsilon_1 > 1$ , then in order for (35) to hold,  $\partial F(\omega^*, V_0^*)/\partial V_0$  must be zero,

which contradicts the first condition in (36). As a result, the only possible value is  $2\varepsilon_1 = 1$ , which transforms (35) into

$$\Psi(V_0 - V_0^*) = \left[ \frac{\partial F(\omega^*, V_0^*)}{\partial V_0} + \frac{\alpha_1^2}{2} \frac{\partial F(\omega^*, V_0^*)}{\partial \omega^2} \right] (V_0 - V_0^*) + \dots \equiv 0. \quad (37)$$

The value of  $\alpha_1$  is found from the condition that the coefficient of  $(V - V_0^*)$  is zero. We have

$$\alpha_1^2 = -2 \frac{\partial F(\omega^*, V_0^*) / \partial V_0}{\partial^2 F(\omega^*, V_0^*) / \partial \omega^2}. \quad (38)$$

Because the functionals (10)–(13) are all real-valued, and thus  $F(\omega, V_0)$  is real-valued, it follows from (38) that  $\alpha_1^2$  is real-valued, and thus  $\alpha_1$  is either purely real or purely imaginary.

Thus we find the asymptotic dependence  $\omega(V_0)$  in the small neighbourhood of the bifurcation point as

$$\omega(V_0) \approx \omega^* \pm \alpha_1 \sqrt{V_0 - V_0^*}, \quad |V_0 - V_0^*| \ll 1. \quad (39)$$

From (39) it follows that in the small neighbourhood around each bifurcation point  $(\omega^*, V_0^*)$ , the frequency of harmonic vibrations  $\omega$  obtains complex values. If the coefficient  $\alpha_1$  is real, then the frequency becomes complex for  $V_0 < V_0^*$ ; otherwise ( $\alpha_1$  imaginary) the frequency becomes complex for  $V_0 > V_0^*$ .

The appearance of complex frequencies and their complex conjugates means that according to the model considered, the displacement will grow exponentially, which corresponds to instability in the Lyapunov sense. Thus, the considered elastic system exhibits elastic instability at the bifurcation points, and from a mathematical point of view, the bifurcation points correspond to static (divergence, buckling,  $\omega^* = 0$ ) and dynamic (flutter,  $\omega^* \neq 0$ ) kinds of instability in the Bolotin classification. Both kinds of instabilities are caught by the present analysis, because in both cases ( $\omega^* = 0$  and  $\omega^* \neq 0$ ) we have instability in the Lyapunov sense.

## 6 Example

As an example, consider the harmonic vibrations of an elastic panel (plate undergoing cylindrical deformation) moving in the axial direction at a constant velocity  $V_0$ , and with zero axial tension ( $C = 0$ ). In this case, some of the bifurcation points lie on the  $V_0$  axis ( $\omega = 0$ ), corresponding to static instabilities (divergence). For this set of points, the bifurcation values of the velocities are

$$V_{0k}^* = k\pi, \quad k = 1, 2, 3, \dots \quad (\text{static instabilities})$$

and the dependences  $\omega_k(V_0)$  in the small neighbourhood of the points  $(0, V_{0k}^*)$  are given by

$$\omega_k(V_0) \approx \pm \alpha_{1k} \sqrt{V_0 - k\pi} + \dots, \quad |V_0 - k\pi| \ll 1, \quad k = 1, 2, \dots \quad (40)$$

where we have used  $\varepsilon_1 = 1/2$ . Taking into account that  $\alpha_{11}^2 < 0$ , the first branch  $\omega_1(V_0)$  is complex for  $V_0 > \pi$ , and consequently we will have instability for  $V_0 = V_{01}^* = \pi$ . It can be shown that the values  $\alpha_{1k}^2$  are positive for all  $k \geq 2$ , and consequently each branch  $\omega_k(V_0)$  takes complex values at  $V_0 < k\pi$ .

The results of asymptotic analysis of  $\omega_k(V_0)$  agree with the numerical solution presented in Figure 1, which was obtained by solving the spectral boundary value problem (4)–(5) as an eigenvalue problem for  $(\omega, u(x))$  using finite elements of the Hermite type. For this picture, we present the bifurcation values for critical points outside the axis  $V_0(\omega=0)$ , denoted by two lower indices:

$$\begin{aligned} (V_{021}^* = 6.45, \omega_{21}^* = \pm 10.58) \\ (V_{031}^* = 10.23, \omega_{31}^* = \pm 32.01) \end{aligned}$$

## 7 Conclusion

In this paper, the stability of an axially moving elastic panel was considered. The panel was travelling at constant velocity between a system of rollers. Small transverse elastic displacements of the panel were described by a fourth-order differential equation that included the centrifugal and Coriolis effects (induced by the axial motion), axial tension, and bending resistance. The same formulation directly applies also to the small out-of-plane elastic displacements of an axially travelling beam.

To study the stability of the system, a complex variable technique and bifurcation theory were applied. As a result, variational equations and a variational principle were derived. Analysis of the variational principle allowed the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point was analyzed and presented. The bifurcation points were found by determining conditions where the conditions of the implicit function theorem (which concerns the uniqueness of a local explicit representation of an implicit function) are violated.

It was shown analytically that the eigenvalue curves in the  $(\omega, V_0)$  plane cross both the  $\omega$  and  $V_0$  axes perpendicularly. It was also shown that near each bifurcation point, the dependence  $\omega_k(V_0)$ , for each mode  $k$ , approximately follows the shape of a square root function (considered near the origin). From this analysis it was also seen that, as expected for this class of systems, the eigenvalues appear in conjugate pairs.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

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## References

- F. R. Archibald and A. G. Emslie. The vibration of a string having a uniform motion along its length. *ASME Journal of Applied Mechanics*, 25:347–348, 1958.
- N. Banichuk and P. Neittaanmäki. On the transverse vibrations analysis of traveling membranes and plates submerged into ideal fluid. Technical Report B2/2008, University of Jyväskylä, 2008a. Reports of the Department of Mathematical Information Technology. B: Scientific Computing.
- N. Banichuk and P. Neittaanmäki. On application of the Galerkin method and perturbation techniques for vibration analysis of a moving web submerged into ideal fluid. Technical Report B3/2008, University of Jyväskylä, 2008b. Reports of the Department of Mathematical Information Technology. B: Scientific Computing.
- N. Banichuk and P. Neittaanmäki. Instability analysis for traveling paper web interacting with surrounding air. Technical Report B4/2008, University of Jyväskylä, 2008c. Reports of the Department of Mathematical Information Technology. B: Scientific Computing.
- N. Banichuk, J. Jeronen, P. Neittaanmäki, and T. Tuovinen. On the instability of an axially moving elastic plate. *International Journal of Solids and Structures*, 47(1):91–99, 2010. URL <http://dx.doi.org/10.1016/j.ijsolstr.2009.09.020>.
- N. Banichuk, J. Jeronen, M. Kurki, P. Neittaanmäki, T. Saksa, and T. Tuovinen. On the limit velocity and buckling phenomena of axially moving orthotropic membranes and plates. *International Journal of Solids and Structures*, 48(13):2015–2025, 2011a. ISSN 0020-7683. URL <http://dx.doi.org/10.1016/j.ijsolstr.2011.03.010>.
- N. Banichuk, J. Jeronen, P. Neittaanmäki, and T. Tuovinen. Dynamic behaviour of an axially moving plate undergoing small cylindrical deformation submerged in axially flowing ideal fluid. *Journal of Fluids and Structures*, 27(7):986–1005, 2011b. ISSN 0889-9746. URL <http://dx.doi.org/10.1016/j.jfluidstructs.2011.07.004>.
- N. Banichuk, J. Jeronen, P. Neittaanmäki, T. Saksa, and T. Tuovinen. Theoretical study on travelling web dynamics and instability under non-homogeneous tension. *International Journal of Mechanical Sciences*, 66C:132–140, 2013a. URL <http://dx.doi.org/10.1016/j.ijmecsci.2012.10.014>.

- N. Banichuk, M. Kurki, P. Neittaanmäki, T. Saksa, M. Tirronen, and T. Tuovinen. Optimization and analysis of processes with moving materials subjected to fatigue fracture and instability. *Mechanics Based Design of Structures and Machines: An International Journal*, 41(2):146–167, 2013b. URL <http://dx.doi.org/10.1080/15397734.2012.708630>.
- N. Banichuk, J. Jeronen, P. Neittaanmäki, T. Saksa, and T. Tuovinen. *Mechanics of moving materials*, volume 207 of *Solid mechanics and its applications*. Springer, 2014. ISBN: 978-3-319-01744-0 (print), 978-3-319-01745-7 (electronic).
- R. B. Bhat, G. D. Xistris, and T. S. Sankar. Dynamic behavior of a moving belt supported on elastic foundation. *Journal of Mechanical Design*, 104(1):143–147, 1982. URL <http://dx.doi.org/10.1115/1.3256304>.
- Y. B. Chang and P. M. Moretti. Interaction of fluttering webs with surrounding air. *TAPPI Journal*, 74(3):231–236, 1991.
- S. Chonan. Steady state response of an axially moving strip subjected to a stationary lateral load. *Journal of Sound and Vibration*, 107:155–165, 1986. URL [http://dx.doi.org/10.1016/0022-460X\(86\)90290-7](http://dx.doi.org/10.1016/0022-460X(86)90290-7).
- M. H. Ghayesh, M. Amabili, and M. P. Païdoussis. Nonlinear dynamics of axially moving plates. *Journal of Sound and Vibration*, 332(2):391–406, 2013. ISSN 0022-460X. URL <http://dx.doi.org/10.1016/j.jsv.2012.08.013>.
- S. Hatami, M. Azhari, M. M. Saadatpour, and P. Memarzadeh. Exact free vibration of webs moving axially at high speed. In *AMATH'09: Proceedings of the 15th American Conference on Applied Mathematics*, pages 134–139, Stevens Point, Wisconsin, USA, 2009. World Scientific and Engineering Academy and Society (WSEAS). ISBN 978-960-474-071-0. Houston, USA.
- L. Kong and R. G. Parker. Approximate eigensolutions of axially moving beams with small flexural stiffness. *Journal of Sound and Vibration*, 276:459–469, 2004. URL <http://dx.doi.org/10.1016/j.jsv.2003.11.027>.
- A. Kulachenko, P. Gradin, and H. Koivurova. Modelling the dynamical behaviour of a paper web. Part I. *Computers & Structures*, 85:131–147, 2007a. URL <http://dx.doi.org/10.1016/j.compstruc.2006.09.006>.
- A. Kulachenko, P. Gradin, and H. Koivurova. Modelling the dynamical behaviour of a paper web. Part II. *Computers & Structures*, 85:148–157, 2007b. URL <http://dx.doi.org/10.1016/j.compstruc.2006.09.007>.
- K. Marynowski. Non-linear dynamic analysis of an axially moving viscoelastic beam. *Journal of Theoretical and Applied Mechanics*, 2(40):465–482, 2002.
- K. Marynowski. Non-linear vibrations of an axially moving viscoelastic web with time-dependent tension. *Chaos, Solitons & Fractals*, 21(2):481 – 490, 2004. ISSN 0960-0779. DOI: 10.1016/j.chaos.2003.12.020.

- K. Marynowski. Non-linear vibrations of the axially moving paper web. *Journal of Theoretical and Applied Mechanics*, 46(3):565 – 580, 2008a.
- K. Marynowski. *Dynamics of the Axially Moving Orthotropic Web*, volume 38 of *Lecture Notes in Applied and Computational Mechanics*. Springer–Verlag, Germany, 2008b. URL <http://dx.doi.org/10.1007/978-3-540-78989-5>. ISBN 978-3-540-78988-8 (print), 978-3-540-78989-5 (online).
- K. Marynowski and T. Kapitaniak. Dynamics of axially moving continua. *International Journal of Mechanical Sciences*, 81:26–41, apr 2014. ISSN 0020-7403. URL <http://dx.doi.org/10.1016/j.ijmecsci.2014.01.017>.
- W. L. Miranker. The wave equation in a medium in motion. *IBM Journal of Research and Development*, 4:36–42, 1960. URL <http://dx.doi.org/10.1147/rd.41.0036>.
- C. D. Mote. Dynamic stability of axially moving materials. *Shock and Vibration Digest*, 4(4):2–11, 1972.
- P. Neittaanmäki and K. Ruotsalainen. On the numerical solution of the bifurcation problem for the sine-Gordon equation. *Arab Journal of Mathematics*, 6(1 and 2), 1985.
- J. Nečas, A. Lehtonen, and P. Neittaanmäki. On the construction of Lusternik–Schnirelmann critical values with application to bifurcation problems. *Applicable Analysis*, 25(4):253–268, 1987. URL <http://dx.doi.org/10.1080/00036818708839689>.
- R. G. Parker. On the eigenvalues and critical speed stability of gyroscopic continua. *ASME Journal of Applied Mechanics*, 65:134–140, 1998. URL <http://dx.doi.org/10.1115/1.2789016>.
- R. G. Parker. Supercritical speed stability of the trivial equilibrium of an axially-moving string on an elastic foundation. *Journal of Sound and Vibration*, 221(2): 205–219, 1999. URL <http://dx.doi.org/10.1006/jsvi.1998.1936>.
- F. Pellicano and F. Vestroni. Nonlinear dynamics and bifurcations of an axially moving beam. *Journal of Vibration and Acoustics*, 122(1):21–30, 2000.
- N. C. Perkins. Linear dynamics of a translating string on an elastic foundation. *Journal of Vibration and Acoustics*, 112(1):2–7, 1990. URL <http://dx.doi.org/10.1115/1.2930094>.
- J. Y. Shen, L. Sharpe, and W. M. McGinley. Identification of dynamic properties of plate-like structures by using a continuum model. *Mechanics Research Communications*, 22(1):67–78, Jan. 1995. URL [http://dx.doi.org/10.1016/0093-6413\(94\)00043-D](http://dx.doi.org/10.1016/0093-6413(94)00043-D).

- C. Shin, J. Chung, and W. Kim. Dynamic characteristics of the out-of-plane vibration for an axially moving membrane. *Journal of Sound and Vibration*, 286(4-5):1019–1031, Sept. 2005. URL <http://dx.doi.org/10.1016/j.jsv.2005.01.013>.
- A. Simpson. Transverse modes and frequencies of beams translating between fixed end supports. *Journal of Mechanical Engineering Science*, 15:159–164, 1973. URL [http://dx.doi.org/10.1243/JMES\\_JOUR\\_1973\\_015\\_031\\_02](http://dx.doi.org/10.1243/JMES_JOUR_1973_015_031_02).
- R. Sygulski. Stability of membrane in low subsonic flow. *International Journal of Non-Linear Mechanics*, 42(1):196–202, Jan. 2007. URL <http://dx.doi.org/10.1016/j.ijnonlinmec.2006.11.012>.
- X. Wang. Instability analysis of some fluid-structure interaction problems. *Computers & Fluids*, 32(1):121–138, Jan. 2003. URL [http://dx.doi.org/10.1016/S0045-7930\(01\)00103-7](http://dx.doi.org/10.1016/S0045-7930(01)00103-7).
- Y. Wang, L. Huang, and X. Liu. Eigenvalue and stability analysis for transverse vibrations of axially moving strings based on Hamiltonian dynamics. *Acta Mechanica Sinica*, 21:485–494, 2005. URL <http://dx.doi.org/10.1007/s10409-005-0066-2>.
- J. A. Wickert. Response solutions for the vibration of a traveling string on an elastic foundation. *Journal of Vibration and Acoustics*, 116(1):137–139, 1994. URL <http://dx.doi.org/10.1115/1.2930389>.
- J. A. Wickert and C. D. Mote. Classical vibration analysis of axially moving continua. *ASME Journal of Applied Mechanics*, 57:738–744, 1990. URL <http://dx.doi.org/10.1115/1.2897085>.
- X.-D. Yang and L.-Q. Chen. Bifurcation and chaos of an axially accelerating viscoelastic beam. *Chaos, Solitons & Fractals*, 23(1):249–258, 2005.