Variational Principle and Bifurcations in Stability Analysis of Panels

Nikolay Banichuk  Alexander Barsuk
Tero Tuovinen  Juha Jeronen
Variational Principle and Bifurcations in Stability Analysis of Panels

Nikolay Banichuk Alexander Barsuk Tero Tuovinen
Juha Jeronen

Abstract

In this paper, the stability of a simply supported axially moving elastic panel is considered. A complex variable technique and bifurcation theory are applied. As a result, variational equations and a variational principle are derived. Analysis of the variational principle allows the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point is analyzed and presented.

It is shown analytically that the eigenvalue curves in the \((\omega, V_0)\) plane cross both the \(\omega\) and \(V_0\) axes perpendicularly. It is also shown that near each bifurcation point, the dependence \(\omega(V_0)\) for each mode approximately follows the shape of a square root near the origin.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

1 Introduction

The aim of our studies has been to develop mathematical models representing the behaviour of the paper making process. Previously (see e.g., Banichuk et al. [2013b,a, 2011a,b]), we have considered many approaches for modelling moving materials and their stability. Conclusions that have been drawn can be applied for example, the processing of paper or steel, fabric, rubber or some other continuous material, and looping systems such as band saws and timing belts.

Typically systems of axially moving web have been modelled as travelling flexible strings, membranes, beams, and plates. Classical articles in this field are, for example, Mote [1972], Archibald and Emslie [1958], Simpson [1973], Wang et al.

*This research was supported by RFBR (grant 14-08-00016-a), RAS Program 12, Program of Support of Leading Scientific Schools (grant 2954.2014.1), and the Finnish Cultural Foundation.
In the case of beams interacting with external media, one can read e.g. the article by Chang and Moretti [1991], and the articles by Banichuk and Neittaanmäki [2008a,b,c]. The study has been extended in Banichuk et al. [2010] for a two-dimensional model of the web, considered as a moving plate under homogeneous tension but without external media. The most straightforward and efficient way to study stability is to use linear stability analysis. In a recent article by Hatami et al. [2009], the free vibration of a moving orthotropic rectangular plate was studied at sub- and supercritical speeds, and its flutter and divergence instabilities at supercritical speeds. The study is limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving material, many necessary fundamentals can be found in the book by Marynowski [2008b]. An extensive literature review about dynamics of axially moving continua can be found in Marynowski and Kapitaniak [2014] or in the book by Banichuk et al. [2014]. However, in this article the effect of surrounding media have been excluded.

The dynamical properties of moving plates have been studied by Shen et al. [1995] and by Shin et al. [2005], and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. [2007a,b]. Critical regimes and other problems of stability analysis have been studied e.g. by Wang [2003] and Sygulski [2007]. Moreover, in the articles Marynowski [2002, 2004, 2008a] the dynamical aspects of the axially moving web are discussed extensively. In Yang and Chen [2005], the authors considered transverse vibrations of the axially accelerating viscoelastic beam, and in Pellicano and Vestrioni [2000], dynamic behavior of a simply supported beam subjected to an axial transport of mass was studied. An extensive literature review related to areas presented in this paper, can be found for example in Ghayesh et al. [2013]. Note also some approaches to bifurcation problems and estimation of critical parameters presented by Nečas et al. [1987] and Neittaanmäki and Ruotsalainen [1985].

The focus of this article is the stability of a simply supported axially moving elastic panel. We have used a complex variable technique and bifurcation theory. Our main task has been the derivation of variational equations and a variational principle. Moreover, we have performed an analysis of the variational principle, which allows the study of qualitative properties of the bifurcation points. Furthermore, asymptotic behaviour around an arbitrary bifurcation point is analyzed and presented. As a result, we show analytically that the eigenvalue curves in the \((\omega, V_0)\) plane cross both the \(\omega\) and \(V_0\) axes perpendicularly. It is also shown that near each bifurcation point, the dependence \(\omega(V_0)\) for each mode approximately follows the shape of a square root near the origin. Gained results complement existing numerical studies on the stability of axially moving materials, and especially materials with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.
2 Basic relations and complex functions

Consider the problem of free harmonic vibrations of an elastic panel, moving axially at a constant velocity $V_0$. In a stationary orthogonal coordinate system, the transverse vibrations are characterized by the function $w = w(x, t)$, which is determined by the following partial differential equation:

$$w_{tt} + 2V_0 w_{xt} + (V_0^2 - C^2)w_{xx} + \frac{D}{\rho S} w_{xxxx} = 0, \quad 0 < x < \ell, \quad C = \sqrt{\frac{T_0}{\rho S}}. \quad (1)$$

At the ends of the considered interval $x \in [0, \ell]$, we have the simply supported boundary conditions

$$w(0, t) = w(\ell, t) = Dw_{xx}(0, t) = Dw_{xx}(\ell, t) = 0. \quad (2)$$

Here $x$ is the axial space coordinate, $t$ time, $T_0$ tension, $\rho$ material density, $S$ the area of the panel cross section, $\ell$ the length of the free span, and $D$ the bending rigidity of the panel.

For harmonic vibrations at frequency $\omega$, the transverse displacement can be represented in the form

$$w(x, t) = e^{i\omega t} u(x), \quad i = \sqrt{-1}, \quad (3)$$

where $u(x)$ is an amplitude function that satisfies the following boundary value problem:

$$u_{xxxx} + (V_0^2 - C^2)u_{xx} + 2i\omega V_0 u_x - \omega^2 u = 0, \quad (4)$$

$$u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0, \quad (5)$$

which is written in dimensionless variables

$$\tilde{x} = \ell \tilde{x}, \quad \rho \sqrt{\frac{\omega^2 \ell^4}{D}} = \tilde{\omega}, \quad \rho \sqrt{\frac{\ell^2}{S}} V_0^2 = \tilde{V}_0^2, \quad \rho \sqrt{\frac{\ell^2}{S}} C^2 = \tilde{C}^2. \quad (6)$$

In what follows, the tilde will be omitted.

The amplitude function $u(x)$ determined from the boundary value problem (4)–(5) is a complex-valued function, i.e.

$$u(x) = u^1(x) + i u^2(x), \quad \hat{u}(x) = u^1(x) - i u^2(x), \quad (7)$$

where $u^1(x)$ and $u^2(x)$ are real-valued functions and $\hat{u}(x)$ is the complex conjugate of $u(x)$.

In the following we present a variational formulation of the spectral problem (4)–(5). This formulation allows us to make important conclusions about the frequencies of free vibrations of moving elastic systems without knowing the rigorous solution of the spectral boundary value problem. To derive the variational formulation of (4)–(5), we multiply the differential equation by the complex conjugate (adjoint) amplitude function $\hat{u}(x)$ and integrate the result on the interval $(0, 1)$. We will also take into account the boundary conditions

$$u^1(0) = u^2(0) = u^1(1) = u^2(1) = 0, \quad (8)$$

$$u_{xx}^1(0) = u_{xx}^2(0) = u_{xx}^1(1) = u_{xx}^2(1) = 0,$$
which follow from the boundary conditions (5). We obtain the functional equation

$$a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0,$$

(9)

where $a$, $b$, $c$ and $d$ are integral functional depending on the problem (4)–(5). The functional $a$ is given by

$$a = \int_0^1 \hat{u} \hat{u} \, dx = \int_0^1 ((u^1)^2 + (u^2)^2) \, dx > 0.$$  

(10)

Using the boundary conditions (8), we can write the functional $b$ as

$$\int_0^1 u_x \hat{u} \, dx = i \int_0^1 ((u^2)_x u^1 - (u^1)_x u^2) \, dx = ib,$$

(11)

where $b$ is real-valued. The functionals $c$ and $d$ are obtained by integrating by parts (once in the case of $c$ and twice for $d$), and taking into account the corresponding boundary conditions in (8). We have

$$c = - \int_0^1 u_{xx} \hat{u} \, dx = \int_0^1 \left( (u^1)_{xx}^2 + (u^2)_{xx}^2 \right) \, dx > 0,$$

(12)

$$d = \int_0^1 u_{xxxx} \hat{u} \, dx = \int_0^1 \left( (u^1)_{xxxx}^2 + (u^2)_{xxxx}^2 \right) \, dx > 0.$$  

(13)

3 Variational analysis and variational principle in complex variables

Let us write the variation of the functional equation (9). To do this, we take into account the variations of the considered functionals,

$$\delta a = \int_0^1 (\hat{u} \delta u + u \delta \hat{u}) \, dx,$$

$$i \delta b = \int_0^1 (\hat{u} \delta u_x + u_x \delta \hat{u}) \, dx,$$

$$\delta c = \int_0^1 (u_x \delta \hat{u}_x + \hat{u}_x \delta u_x) \, dx,$$

$$\delta d = \int_0^1 (u_{xx} \delta \hat{u}_{xx} + \hat{u}_{xx} \delta u_{xx}) \, dx,$$

(14)

and perform standard transformations in (9), replacing $u$, $\hat{u}$ and $\omega$ with $u + \delta u$, $\hat{u} + \delta \hat{u}$ and $\omega + \delta \omega$, respectively. We will have the variation

$$2(a\omega + bV_0) \delta \omega + \int_0^1 \left[ -\omega^2 u + 2i\omega V_0 u_x + (V_0^2 - C^2)u_{xx} + u_{xxxx} \right] \delta \hat{u} \, dx$$

$$+ \int_0^1 \left[ -\omega^2 \hat{u} - 2i\omega V_0 \hat{u}_x + (V_0^2 - C^2)\hat{u}_{xx} + \hat{u}_{xxxx} \right] \delta u \, dx = 0.$$  

(15)
For \(u(x)\) and \(\hat{u}(x)\), which are solutions of the spectral boundary value problem (4)–(5) and its complex conjugate, the integral expressions in (15) are identically zero. Taking this into account, we are left with

\[
2(a\omega + bV_0)\delta\omega = 0 .
\]

Thus, if \(a\omega + bV_0 \neq 0\) for the spectral problem (4)–(5), then the frequency variation for free vibrations \(\delta\omega\) is zero. That is,

\[
a\omega + bV_0 \neq 0 , \quad \delta\omega = 0 .
\]

Solving (9) for \(\omega\), we arrive at the variational representation for harmonic vibrations, corresponding to each of the two solution branches of equation (9):

\[
\omega_{\pm}(V_0) = \frac{1}{a} \left(-bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad}\right) \rightarrow \text{extr}_{u(x),\hat{u}(x)} .
\]

4 Analysis of extremum conditions and bifurcation analysis

From (16), the other possibility is

\[
a\omega + bV_0 = 0
\]

and \(\delta\omega\) free. To perform analysis for this case, we consider equation (9) as an implicit function \(F(\omega,V_0)\):

\[
F(\omega,V_0) = 0 , \quad F(\omega,V_0) = a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d .
\]

Again, we can solve (9) for \(\omega\), obtaining the following two solution branches:

\[
\omega_{\pm}(V_0) = \frac{1}{a} \left(-bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad}\right)
\]

Let \((\omega^*, V_0^*)\) denote the bifurcation point, i.e. the values of \(\omega\) and \(V_0\) at which the solution of (20) branches. At the bifurcation point, the conditions of the implicit function theorem must be violated, i.e. we will have

\[
F(\omega,V_0) = 0 , \quad \frac{\partial F(\omega,V_0)}{\partial\omega} = 0 .
\]

Using (20), these conditions become

\[
a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0 , \quad a\omega + bV_0 = 0 .
\]

As a result, we find the following representation for bifurcation values of the frequency and panel velocity:

\[
\omega^* = -\frac{b}{a}V_0^* , \quad (ac - b^2)(V_0^*)^2 = acC^2 + ad .
\]
Alternatively, these values can be obtained from the condition \( \omega_+(V_0) = \omega_-(V_0) \) and the representation (21) for \( \omega_\pm(V_0) \). Note also that if some solutions have \( b = 0 \), the corresponding bifurcation points are distributed along the \( V_0 \) axis in the \((V_0, \omega)\) plane, i.e.

\[
\omega^* = 0, \quad (V_0^*)^2 = C^2 + \frac{d}{c} \quad \text{for solutions with } b = 0.
\]

Let us differentiate \( \omega(V_0) \) with respect to the parameter \( V_0 \). To do this, in (20) we replace \( V_0, u \) and \( \omega \) with \( V_0 + \delta V_0, u + \delta u \) and \( \omega + \delta \omega \), respectively. Using the standard transformations (as was done in (16)) we obtain

\[
2(a\omega + bV_0) \delta \omega + 2(b\omega + cV_0) \delta V_0 = 0. \tag{26}
\]

Consequently,

\[
\frac{d\omega}{dV_0} = \frac{b\omega + cV_0}{a\omega + bV_0}. \tag{27}
\]

In particular, it follows from (27) that for all bifurcation points \((\omega^*, V_0^*)\) we have the limit

\[
\lim_{V_0 \to V_0^*} \frac{d\omega_\pm(V_0)}{dV_0} = \pm\infty. \tag{28}
\]

In the case \( V_0 = 0 \), we have \( b = 0 \), and find that

\[
\frac{d\omega_\pm(V_0=0)}{dV_0} = 0. \tag{29}
\]

It follows from (28)–(29) that the curves \( \omega_\pm(V_0) \) cross the \( \omega \) and \( V_0 \) axes at right angles; see Figure 1.

## 5 Nonlinear analysis of asymptotic behaviour of the frequencies in the vicinity of bifurcation points

Let \((\omega_1^*, V_{01}^*), (\omega_2^*, V_{02}^*), \ldots\) be solutions of the system of nonlinear equations (20). Consider the behaviour of the functions \( \omega_i(V_0) \) \((i = 1, 2, \ldots)\), determined in a small neighbourhood of the bifurcation point \((\omega_k^*, V_{0k}^*)\), in implicit form, by the equation \( F(\omega, V_0) = 0 \). For brevity, we will omit the indices of the functions \( \omega_i(V_0) \) and the bifurcation points \((\omega_k^*, V_{0k}^*)\).

To study the behaviour of the function \( F(\omega, V_0) \), we expand it in series around the bifurcation point \((\omega^*, V_{0}^*)\). We have

\[
F(\omega, V_0) = F(\omega^*, V_{0}^*) + \frac{\partial F(\omega^*, V_{0}^*)}{\partial \omega}(\omega - \omega^*) + \frac{\partial F(\omega^*, V_{0}^*)}{\partial V_0}(V_0 - V_{0}^*) + \frac{1}{2} \frac{\partial^2 F(\omega^*, V_{0}^*)}{\partial \omega^2}(\omega - \omega^*)^2 + \ldots \tag{30}
\]
Figure 1: Behaviour of the natural frequencies $\omega$ as a function of the panel axial velocity $V_0$. Numerical solution using finite elements.
Taking into account that at each bifurcation point \((\omega^*, V_0^*)\), relation (22) holds, we have that the first two terms in (30) vanish, obtaining
\[
F(\omega, V_0) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{1}{2} \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}(\omega - \omega^*)^2 + \ldots
\] (31)

Observe that all terms that have been omitted in (30) have a higher order of smallness. The expression (31) thus contains all leading-order terms, and describes completely general behaviour of \(F(\omega, V_0)\) in a small neighbourhood of a given bifurcation point \((\omega^*, V_0^*)\). This is the general case; the special cases where one or both of \(\frac{\partial F(\omega^*, V_0^*)}{\partial V_0}\) and \(\frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}\) are zero must be studied separately.

Without loss of generality, we may represent the function \(\omega = \omega(V_0)\) in the small neighbourhood of the bifurcation point \((\omega^*, V_0^*)\) as a power series:
\[
\omega(V_0) = \omega^* + \alpha_1(V_0 - V_0^*)^\varepsilon_1 + \alpha_2(V_0 - V_0^*)^\varepsilon_2 + \ldots , \quad \text{where } 0 < \varepsilon_1 < \varepsilon_2 < \ldots
\] (32)
The values of the constants \(\alpha_1, \alpha_2, \ldots\) and \(\varepsilon_1, \varepsilon_2, \ldots\) are determined with the help of the condition \(F(\omega, V_0) = 0\). After substitution of (32) into (31), the equation \(F(\omega, V_0) = 0\) reduces to the corresponding equation
\[
\Psi(V - V_0^*) = 0 ,
\] (33)
where \(\Psi\) is a function of one variable.

In order for (33) to hold, the coefficient of each power of \((V - V_0^*)\) in the expression of \(\Psi\) must be equal to zero. This requirement allows us to determine the values of \(\alpha_1, \alpha_2, \ldots\) and \(\varepsilon_1, \varepsilon_2, \ldots\) in the power series (32). In the following, for simplicity we consider only the determination of \(\alpha_1\) and \(\varepsilon_1\), i.e. we approximate \(\omega(V_0)\) as
\[
\omega(V_0) \approx \omega^* + \alpha_1(V_0 - V_0^*)^\varepsilon_1.
\] (34)
After substitution of (34) into (31), we obtain
\[
\Psi(V_0 - V_0^*) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{\alpha_1^2}{2} \frac{\partial F(\omega^*, V_0^*)}{\partial \omega^2}(V_0 - V_0^*)^{2\varepsilon_1} + \ldots \equiv 0 .
\] (35)
The expression (35) contains the leading-order terms; all omitted terms are of a higher order of smallness.

We will analyze the case where
\[
\frac{\partial F(\omega^*, V_0^*)}{\partial V_0} \neq 0 , \quad \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} \neq 0 .
\] (36)
A separate analysis is needed if one or both values in (36) are zero.

Consider now the cases \(2\varepsilon_1 < 1, 2\varepsilon_1 = 1\) and \(2\varepsilon_1 > 1\), which together cover all possibilities for \(\varepsilon_1\). If \(2\varepsilon_1 < 1\), then the first term in (35) is of a higher order of smallness with respect to the second term, and consequently in order for (35) to hold, \(\frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}\) must be zero, which contradicts the second condition in (36). Similarly, if \(2\varepsilon_1 > 1\), then in order for (35) to hold, \(\frac{\partial F(\omega^*, V_0^*)}{\partial V_0^2}\) must be zero,
which contradicts the first condition in (36). As a result, the only possible value is $2\varepsilon_1 = 1$, which transforms (35) into

$$\Psi(V_0 - V_0^*) = \left[ \frac{\partial F(\omega^*, V_0^*)}{\partial V_0} + \frac{\alpha_1^2}{2} \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} \right] (V_0 - V_0^*) + \cdots \equiv 0 . \quad (37)$$

The value of $\alpha_1$ is found from the condition that the coefficient of $(V - V_0^*)$ is zero. We have

$$\alpha_1^2 = -2 \frac{\partial F(\omega^*, V_0^*)}{\partial V_0} / \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} . \quad (38)$$

Because the functionals (10)–(13) are all real-valued, and thus $F(\omega, V_0)$ is real-valued, it follows from (38) that $\alpha_1^2$ is real-valued, and thus $\alpha_1$ is either purely real or purely imaginary.

Thus we find the asymptotic dependence $\omega(V_0)$ in the small neighbourhood of the bifurcation point as

$$\omega(V_0) \approx \omega^* \pm \alpha_1 \sqrt{V_0 - V_0^*} , \quad |V_0 - V_0^*| \ll 1 . \quad (39)$$

From (39) it follows that in the small neighbourhood around each bifurcation point $(\omega^*, V_0^*)$, the frequency of harmonic vibrations $\omega$ obtains complex values. If the coefficient $\alpha_1$ is real, then the frequency becomes complex for $V_0 < V_0^*$; otherwise ($\alpha_1$ imaginary) the frequency becomes complex for $V_0 > V_0^*$.

The appearance of complex frequencies and their complex conjugates means that according to the model considered, the displacement will grow exponentially, which corresponds to instability in the Lyapunov sense. Thus, the considered elastic system exhibits elastic instability at the bifurcation points, and from a mathematical point of view, the bifurcation points correspond to static (divergence, buckling, $\omega^* = 0$) and dynamic (flutter, $\omega^* \neq 0$) kinds of instability in the Bolotin classification. Both kinds of instabilities are caught by the present analysis, because in both cases ($\omega^* = 0$ and $\omega^* \neq 0$) we have instability in the Lyapunov sense.

6 Example

As an example, consider the harmonic vibrations of an elastic panel (plate undergoing cylindrical deformation) moving in the axial direction at a constant velocity $V_0$, and with zero axial tension ($C = 0$). In this case, some of the bifurcation points lie on the $V_0$ axis ($\omega = 0$), corresponding to static instabilities (divergence). For this set of points, the bifurcation values of the velocities are

$$V_{0k}^* = k\pi , \quad k = 1, 2, 3, \ldots \quad \text{(static instabilities)}$$

and the dependences $\omega_k(V_0)$ in the small neighbourhood of the points $(0, V_{0k}^*)$ are given by

$$\omega_k(V_0) \approx \pm \alpha_{1k} \sqrt{V_0 - k\pi} \ldots , \quad |V_0 - k\pi| \ll 1 , \quad k = 1, 2, \ldots \quad (40)$$
where we have used $\varepsilon_1 = 1/2$. Taking into account that $\alpha_{11}^2 < 0$, the first branch $\omega_1(V_0)$ is complex for $V_0 > \pi$, and consequently we will have instability for $V_0 = V_{01}^* = \pi$. It can be shown that the values $\alpha_{1k}^2$ are positive for all $k \geq 2$, and consequently each branch $\omega_k(V_0)$ takes complex values at $V_0 < k\pi$.

The results of asymptotic analysis of $\omega_k(V_0)$ agree with the numerical solution presented in Figure 1, which was obtained by solving the spectral boundary value problem (4)–(5) as an eigenvalue problem for $(\omega, u(x))$ using finite elements of the Hermite type. For this picture, we present the bifurcation values for critical points outside the axis $V_0(\omega = 0)$, denoted by two lower indices:

$$(V_{021}^* = 6.45, \omega_{21}^* = \pm 10.58)$$
$$(V_{031}^* = 10.23, \omega_{31}^* = \pm 32.01)$$

7 Conclusion

In this paper, the stability of an axially moving elastic panel was considered. The panel was travelling at constant velocity between a system of rollers. Small transverse elastic displacements of the panel were described by a fourth-order differential equation that included the centrifugal and Coriolis effects (induced by the axial motion), axial tension, and bending resistance. The same formulation directly applies also to the small out-of-plane elastic displacements of an axially travelling beam.

To study the stability of the system, a complex variable technique and bifurcation theory were applied. As a result, variational equations and a variational principle were derived. Analysis of the variational principle allowed the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point was analyzed and presented. The bifurcation points were found by determining conditions where the conditions of the implicit function theorem (which concerns the uniqueness of a local explicit representation of an implicit function) are violated.

It was shown analytically that the eigenvalue curves in the $(\omega, V_0)$ plane cross both the $\omega$ and $V_0$ axes perpendicularly. It was also shown that near each bifurcation point, the dependence $\omega_k(V_0)$, for each mode $k$, approximately follows the shape of a square root function (considered near the origin). From this analysis it was also seen that, as expected for this class of systems, the eigenvalues appear in conjugate pairs.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.
Acknowledgements

This research was supported by RFBR (grant 14-08-00016-a), RAS Program 12, Program of Support of Leading Scientific Schools (grant 2954.2014.1), and the Finnish Cultural Foundation.

References


