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LOCAL DIMENSIONS OF MEASURES ON INFINITELY GENERATED SELF-AFFINE SETS

EINO ROSSI

Abstract. We show the existence of the local dimension of an invariant probability measure on an infinitely generated self-affine set, for almost all translations. This implies that an ergodic probability measure is exactly dimensional. Furthermore the local dimension equals the minimum of the local Lyapunov dimension and the dimension of the space.

1. Introduction

The upper and lower local dimensions of a locally finite Borel measure $\mu$, denoted by $\dim_{\text{loc}}(\mu, x)$ and $\dim_{\text{loc}}(\mu, x)$ respectively, are the lim sup and lim inf of the ratio

$$\frac{\log \mu(B(x, r))}{\log r},$$

as $r \to 0$. When they agree, we say that the local dimension, denoted by $\dim_{\text{loc}}(\mu, x)$, exists and equals to this common value. If the local dimension is constant almost everywhere, we say that $\mu$ is exactly dimensional. The local dimension does not only give information about the geometry of the measure, but also about the support of the measure. For example, if the upper local dimension of $\mu$ is smaller than $t$ for all $x \in A$, then the packing dimension of $A$ is at most $t$, see e.g. [6, Proposition 2.3(d)].

Our main interest is to study the local dimensions of the canonical projection $\pi \mu$ of an invariant Borel probability measure $\mu$ onto a self-affine set. In 2009, Feng and Hu [10] showed that the local dimension of $\pi \mu$ exists almost everywhere if the underlying iterated function system, IFS for short, is conformal. They also showed that the local dimension exists if the mappings of the IFS satisfy $f_i(x) = A_i x + a_i$ and the matrices $A_i$ commute. When $\mu$ is ergodic, these results give that $\mu$ is exactly dimensional. The general affine case remained open. In 2011, Falconer and Miao [8] calculated the local dimension in a specific affine case. They showed that $\pi \mu$ is exactly dimensional for Lebesgue almost all translation vectors $a \in \mathbb{R}^d$, where $\kappa$ is the number of mappings in the IFS, assuming that $\mu$ is a Bernoulli measure and that $\sup_i ||A_i|| < \frac{1}{2}$, see [8, Theorem 6.1]. By taking a minor change in the proof of [11, Theorem 4.3] we can have the same result for any ergodic measure. This was noted in a very restrictive case by Barral and Feng in [9, Theorem 2.6] and giving the general proof was one of our motivations at the beginning of this work. In [1], the published version of [9], it is mentioned that this generalization is also known by the authors of [11]. However the proof is not written out. All the works mentioned here assume that the IFS is finitely generated.

Our main result, Theorem 1.2, generalizes the results mentioned above. We show that even in the infinitely generated case, the local dimension of an invariant Borel measure...
exists, assuming again that \( \sup_i |A_i| < \frac{1}{2} \). As a corollary we get that ergodic measures are exactly dimensional. We also remark how to obtain estimates for the local dimensions that hold for all translations, with only assuming that the mappings \( A_i \) are contractive. Finally, we make remarks on the connections of our results to the dimensions of the limit set.

Let us now introduce some notation. Let \( I \) be a finite or countable set. We define \( I^* = \bigcup_{n=1}^{\infty} I^n \). If \( I \) is finite, we say that \( I^* \) is finitely generated and otherwise \( I^* \) is infinitely generated. When \( i \in I^* \), we denote by \( ij \) the symbol obtained by juxtaposing \( i \) and \( j \). Furthermore, for \( i \in I^* \), we set \( [i] = \{ij : j \in I^*\} \) and call this set a cylinder of \( i \). When \( i = (i_1, i_2, \ldots) \) we denote \( |i|_n = (i_1, \ldots, i_n) \). On the symbol space \( I^* \) we consider the left shift \( \sigma \), defined by \( \sigma(i_1, i_2, i_3, \ldots) = (i_2, i_3, \ldots) \) and study Borel measures that are invariant with respect to this shift, that is \( \mu(B) = \mu(\sigma^{-1}B) \) for all borel sets \( B \). An invariant measure is called ergodic, if for all Borel sets \( B \) with \( B = \sigma^{-1}B \), we have \( \mu(B) = 0 \) or \( \mu(B) = 1 \). We denote the set of invariant and ergodic Borel probability measures on \( I^* \) by \( \mathcal{M}_{\sigma}(I^*) \) and \( \mathcal{E}_{\sigma}(I^*) \) respectively. Throughout the paper, \( \mu \) denotes a Borel probability measure. By \( \pi_{\mu} \), we mean the push-forward measure \( \mu \circ \sigma^{-1} \).

For each \( i \in I \), we fix an invertible \( d \times d \) matrix \( A_i \) and a translation vector \( a_i \in Q \), where \( Q = [-\frac{1}{2}, \frac{1}{2}]^d \). Due to Kolmogorov extension theorem \( Q^N \) supports a natural probability measure \( m = (\mathcal{L}^d|Q)^N \). We assume that \( \sup_{i \in I} |A_i| = \sigma < 1 \) and consider the IFS \( \{f_i\}_{i \in I} \), where \( f_i(x) = A_ix + a_i \), and the canonical projection \( \pi_{\sigma} : I^N \to \mathbb{R}^d \) defined by \( \{\pi_{\sigma}(i)\} = \bigcap_{n \in \mathbb{N}} f_{i_n}(B(0, R)) \), where \( f_{i_n} = f_{i_1} \circ \cdots \circ f_{i_n} \) and \( R \) is so large that \( f_i(B(0, R)) \subseteq B(0, R) \) for all \( i \in I \). We call \( F_{\sigma} = \bigcup_{i \in I^*} \pi_{\sigma}(i) \) the limit set of this IFS. It is not restrictive to assume that each \( a_i \) is in the cube \( Q \), since this is just a matter of scaling the limit set. This only excludes the case where \( \sup_{i \in I} |a_i| = \infty \).

The singular values \( |A_i|_n = \alpha_1(\hat{i}|_n) \cdots \alpha_d(\hat{i}|_n) > 0 \) of \( A_i|_n = A_i \cdots A_i \) are the lengths of the principal semiaxis of the ellipsoid \( A_i|_n(B(0,1)) \). For \( 0 \leq s < d \), the singular value function is defined as
\[
\phi^s(\hat{i}|_n) = \alpha_1(\hat{i}|_n) \cdots \alpha_k(\hat{i}|_n) \alpha_{k+1}(\hat{i}|_n)^{s-k},
\]
where \( k \) is the integer part of \( s \). When \( s \geq d \), we set \( \phi^s(\hat{i}|_n) = (\alpha_1(\hat{i}|_n) \cdots \alpha_d(\hat{i}|_n))^{s/d} \).

For convenience, fix partitions \( \mathcal{P}_n = \{[i]\}_{i \in I^n} \), and set \( H_{\mu}(\mathcal{P}_n) = -\sum_{i \in I^n} \mu([i]) \log \mu([i]) \). Entropy and energy of \( \mu \in \mathcal{M}_{\sigma}(I^N) \), defined by
\[
h_{\mu} = -\lim_{n \to \infty} \frac{1}{n} H(\mathcal{P}_n) \quad \text{and} \quad \Lambda_{\mu}(s) = \lim_{n \to \infty} \frac{1}{n} \int_{I^n} \log \phi^s(\hat{i}|_n) d\mu,
\]
respectively, are the basic tools in the study of ergodic measures in the field of iterated function systems. In order to work with invariant measures, we need to localize these concepts. By Theorems [18, Theorem 7 in section 2] and [22, Theorem 10.1], for \( \mu \in \mathcal{M}_{\sigma}(I^N) \), there exist \( L^1(\mu) \) functions \( h_{\mu}(\hat{i}) \) and \( \Lambda_{\mu}(s, \hat{i}) \) so that
\[
\begin{align*}
(1.1) \quad h_{\mu}(\hat{i}) &= -\lim_{n \to \infty} \frac{1}{n} \log \mu([\hat{i}|_n]) \quad \text{and} \quad \Lambda_{\mu}(s, \hat{i}) = \lim_{n \to \infty} \frac{1}{n} \log \phi^s(\hat{i}|_n),
\end{align*}
\]
for \( \mu \) almost all \( \hat{i} \in I^N \) and
\[
\begin{align*}
(1.2) \quad \int_{I^n} h_{\mu}(\hat{i}) d\mu &= h_{\mu} \quad \text{and} \quad \int_{I^n} \Lambda_{\mu}(s, \hat{i}) d\mu &= \Lambda_{\mu}(s).
\end{align*}
\]
Furthermore, for \( \mu \in \mathcal{E}_{\sigma}(I^N) \), we have \( h_{\mu}(\hat{i}) = h_{\mu} \) and \( \Lambda_{\mu}(s, \hat{i}) = \Lambda_{\mu}(s) \) for \( \mu \) almost all \( \hat{i} \in I^N \). We call \( h_{\mu}(\hat{i}) \) the local entropy of \( \mu \) at \( \hat{i} \) and \( \Lambda_{\mu}(s, \hat{i}) \) the local energy of \( \mu \) at \( \hat{i} \).

In order to use [18, Theorem 7 in section 2], we need to assume that \( H(\mathcal{P}_n) < \infty \) at some
level \( n \). Since \( h_{\mu} = \lim_{n \to \infty} \frac{1}{n} H(P_n) = \inf_{n \in \mathbb{N}} \frac{1}{n} H(P_n) \) we can equivalently assume that \( h_{\mu} < \infty \).

We define the measure-theoretical pressure function \( P_\mu(\cdot, i) : [0, \infty] \to \mathbb{R} \) by
\[
P_\mu(s, i) = -\lim_{n \to \infty} \frac{1}{n} \log \frac{\mu[i|n]}{\phi^n(i|n)},
\]
when \( i \) is so that both equations in (1.1) hold. If \( h_{\mu} < \infty \) the limit exists for \( \mu \) almost all \( i \in I^N \). When \( h_{\mu}(i) < \infty \) or \( \Lambda_\mu(s, i) > -\infty \), then \( P_\mu(s, i) \) is just \( h_{\mu}(i) + \Lambda_\mu(s, i) \).

It is not yet said, that there exists \( i \in I^N \), so that \( \lim_{n \to \infty} \frac{1}{n} \log \phi^n(i|n) \) exists for all \( s \). Fortunately, this happens for \( \mu \) almost all \( i \in I^N \). By repetitive use of the second equation in (1.1), we get that for \( \mu \) almost all \( i \in I^N \), the limit \( \lim_{n \to \infty} \frac{1}{n} \log a(i|n) \) exists for all \( 1 \leq l \leq d \). We call these values the Lyapunov exponents of \( \mu \) at \( i \) and denote them by \( \lambda_l(i, \mu) \). For \( s < d \), it now easily follows that
\[
(1.3) \quad \Lambda_\mu(s, i) = \lambda_1(i, \mu) + \cdots + \lambda_k(i, \mu) + (s-k)\lambda_{k+1}(i, \mu),
\]
where \( k \) is the integer part of \( s \), with the interpretation that \( 0 \cdot (-\infty) = 0 \). If \( s \geq d \), we get \( \Lambda_\mu(s, i) = \frac{s}{d}(\lambda_1(i, \mu) + \cdots + \lambda_d(i, \mu)) \). From this we see, that \( \Lambda_\mu(\cdot, i) \) is strictly decreasing function with \( \Lambda_\mu(0, i) = 0 \). Also we see that \( \Lambda_\mu(\cdot, i) \) has at most one point of discontinuity and at this point it is continuous from left. The point of discontinuity equals \( \min\{k : \lambda_{k+1}(i, \mu) = -\infty\} \).

With the assumption \( h_{\mu} < \infty \), we have that for \( \mu \) almost all \( i \in I^N \), the equations in (1.1) hold for all \( s \). Also, the first equation in (1.2) gives that \( h_{\mu}(i) < \infty \) for \( \mu \) almost all \( i \in I^N \). In this light, we give the following definition.

**Definition 1.1.** Let \( \mu \in \mathcal{M}_\sigma(I^N) \) and \( h_{\mu} < \infty \). When \( i \) is so that \( h_{\mu}(i) < \infty \) and both equations in (1.1) hold, the local Lyapunov dimension of \( \mu \) at \( i \), denoted by \( \text{dim}_{LY}(\mu, i) \), is defined to be the infimum of the numbers \( s \), for which \( P_\mu(s, i) < 0 \).

We remark that, for ergodic \( \mu \), the above functions \( h_{\mu}(i), \lambda_l(i, \mu), \Lambda_\mu(s, i), P_\mu(s, i) \) and \( \text{dim}_{LY}(\mu, i) \) are constants for \( \mu \) almost all \( i \). In such case, we use the notations \( h_{\mu}, \lambda_l(\mu), \Lambda_\mu(s), P_\mu(s) \) and \( \text{dim}_{LY}(\mu) \) to emphasize the independence of \( i \). We are now ready to state our main result.

**Theorem 1.2.** Assume that \( \mu \in \mathcal{M}_\sigma(I^N) \), \( h_{\mu} < \infty \), \( \sup_{i \in I} ||A_i|| < \frac{1}{7} \) and that there exists \( s \in [0, \infty) \) so that \( 0 > P_\mu(s, i) > -\infty \). Then \( \text{dim}_{loc}(\pi_{a\mu}, \pi_{a}(i)) = \min\{d, \text{dim}_{LY}(\mu, i)\} \) for \( \mu \) almost all \( i \in I^N \) and \( m \) almost all \( a \in Q^N \).

We only need the assumption \( 0 > P_\mu(s, i) > -\infty \) in the proof of the upper bound to ensure that \( \lambda_{k+1}(i, \mu) > -\infty \), where \( k \) is the integer part of \( \text{dim}_{LY}(\mu, i) \).

2. LOCAL DIMENSIONS OF INVARIANT MEASURES

In this section we prove Theorem 1.2. The proof is divided into upper and lower estimates, namely to Theorems 2.1 and 2.2. We remark that Theorem 2.1 was proven in [11, Proposition 4.4] for an ergodic measure on a finitely generated affine IFS.

**Theorem 2.1.** Assume that \( \mu \in \mathcal{M}_\sigma(I^N) \), \( h_{\mu}(i) < \infty \) and \( \sup_{i \in I} \alpha_1(i) < \frac{1}{7} \). Then we have that \( \text{dim}_{loc}(\pi_{a\mu}, \pi_a(i)) \geq \min\{d, \text{dim}_{LY}(\mu, i)\} \) for \( \mu \) almost all \( i \in I^N \) and \( m \) almost all \( a \in Q^N \).

**Proof.** Assume first that \( \text{dim}_{LY}(\mu, i) \leq d \). For arbitrary \( \varepsilon > 0 \), we choose \( \gamma(i) = \text{dim}_{LY}(\mu, i) - 2\varepsilon \) and \( \theta(i) = \text{dim}_{LY}(\mu, i) - \varepsilon \). Since \( P_\mu(\cdot, i) \) is strictly decreasing, we
find $\varepsilon' > 0$, so that $\Lambda_\mu(\theta(1), 1) \geq -h_\mu(1) + 2\varepsilon'$. By Egoroff’s theorem, for each $\delta > 0$ there is a measurable set $H_\delta \subset I^N$ and integer $N_\delta$, such that $\mu(I^N \setminus H_\delta) < \delta$ and

$$
\frac{1}{n} \log \mu[1_n] - \varepsilon' \leq \Lambda_\mu(\theta(1), 1) - \varepsilon' \leq \frac{1}{n} \log \phi^{(1)}(1_n)
$$

for all $n \geq N_\delta$ and $i \in H_\delta$. Therefore we find a constant $c' > 0$, independent of $i$, so that

$$
\mu[1_n] \leq c' \phi^{(1)}(1_n)
$$

for all $n \in \mathbb{N}$ and $i \in H_\delta$. Next we consider the integral

$$
\int_{\mathbb{Q}^N} \frac{dm(a)}{|\pi_a(1) - \pi_a(j)|^{\gamma(1)}} = \int_{\mathbb{Q}^N} \int_{\mathbb{Q}^N} \frac{dL^d(a_1)}{|\pi_a(1) - \pi_a(j)|^{\gamma(1)}} dm(a')
$$

where $a = (a_1, a') \in \mathbb{Q}^N$. We can make the change of variable in the inner integral as in [3, Lemma 3.1]. By using this Lemma with Fubini’s theorem, and then inequality (2.1) and the properties of the singular value function, we get

$$
\int_{\mathbb{Q}^N} \int_{\mathbb{R}^n} \frac{d\mu(j)d\mu(1)dm(a)}{|\pi_a(1) - \pi_a(j)|^{\gamma(1)}} \leq c \int_{H_\delta} \int_{\mathbb{Q}^N} (\phi^{(1)}(1 \land j))^{-1} d\mu(j)d\mu(1)
$$

$$
\leq c \int_{H_\delta} \sum_{n=1}^\infty (\phi^{(1)}(1_n))^{-1} \mu[1_n]d\mu(1)
$$

$$
\leq cc' \int_{H_\delta} \sum_{n=1}^\infty (\phi^{(1)}(1_n))^{-1} \phi^{(1)}(1_n)d\mu(1)
$$

$$
\leq cc' \int_{H_\delta} \sum_{n=1}^\infty 2^{n(\gamma(1) - \theta(1))} d\mu(1)
$$

$$
\leq cc' \int_{H_\delta} \sum_{n=0}^\infty 2^{-ne} < \infty,
$$

where $1 \land j = 1_{\min\{k-1; i_k \neq j_k\}}$ and $c$ is the constant from [3, Lemma 3.1], independent of $i$ and $j$. Originally, the bound of the norms of the linear maps in [3, Lemma 3.1] is $\frac{1}{\beta}$, but by [21, Proposition 3.1], $\frac{1}{2}$ suffices. Now we have that

$$
\int_{H_\delta} \int_{\mathbb{R}^n} \frac{d\mu(j)d\mu(1)}{|\pi_a(1) - \pi_a(j)|^{\gamma(1)}} < \infty
$$

for $m$ almost all $a \in \mathbb{Q}^N$. Next we fix $a$ so that (2.2) holds. We deduce that the integral

$$
\int_{\mathbb{R}^n} |\pi_a(1) - \pi_a(j)|^{-\gamma(1)} d\mu(j)
$$

is finite for $\mu$ almost all $i \in H_\delta$ and so we find constants $M(i)$ for $\mu$ almost every $i \in H_\delta$, so that $\int_{\mathbb{R}^n} |\pi_a(1) - \pi_a(j)|^{-\gamma(1)} d\mu(j) < M(i)$. This implies that $\pi_a M(B(\pi_a(1), r)) \leq r^{\gamma(1)} M(i)$ for all $r > 0$ and for $\mu$ almost all $i \in H_\delta$.

We have obtained that $\dim_{loc}(\pi_a M, \pi_a(1)) \geq \gamma(1)$ for $\mu$ almost all $i \in H_\delta$ and $m$ almost all $a \in \mathbb{Q}^N$. Since $\delta$ was arbitrary, this also holds for $\mu$ almost all $i \in I^N$.

If $\dim_{LY}(\mu, i) > d$, then we get the proof by choosing $\theta(1) = d$ and $\gamma(1) = d - \varepsilon$.

**Theorem 2.2.** If $\mu \in \mathcal{M}_r(I^N)$, $h_\mu < \infty$ and $\Lambda_\mu(s, i) > -\infty$ for some $s > \dim_{LY}(\mu, i)$, then $\dim_{loc}(\pi_a M, \pi_a(1)) \leq \min\{d, \dim_{LY}(\mu, i)\}$ for $\mu$ almost all $i \in I^N$ and for all $a \in \mathbb{Q}^N$.

**Proof.** As mentioned in the introduction, we follow the lines of the proof of [11, Theorem 4.3].

We may assume that $\dim_{LY}(\mu, i) < d$. Fix an integer $k$, so that $k \leq \dim_{LY}(\mu, i) < k+1$. We have that $\pi_a[1_n] \in f_{1_n}(B(0, R))$ for some $R \in \mathbb{N}$. The ellipsoid $f_{1_n}(B(0, R))$ can be
covered by a rectangular box, call it $B(\mathbf{i}|n)$, with side-lengths $2R\alpha_1(\mathbf{i}|n), \ldots , 2R\alpha_d(\mathbf{i}|n)$. We can cover $B(\mathbf{i}|n)$ with $N(\mathbf{i}|n)$ non-overlapping “half-open” boxes with side-lengths

$$\alpha_k+1(\mathbf{i}|n), \ldots , \alpha_k+1(\mathbf{i}|n), \alpha_k+2(\mathbf{i}|n), \ldots , \alpha_d(\mathbf{i}|n),$$

where $N(\mathbf{i}|n) \leq (2R)^d \alpha_1(\mathbf{i}|n) \cdots \alpha_d(\mathbf{i}|n) \alpha_k+1(\mathbf{i}|n)^{-k}$. Let $P_n(\mathbf{i})$ be the box that contains $\pi_n(\mathbf{i})$, and let $Q_n(\mathbf{i}) := [\mathbf{i}|n] \cap \pi_n^{-1}(P_n(\mathbf{i}))$. In other words, $Q_n(\mathbf{i})$ is the part of the cylinder $[\mathbf{i}|n]$ that gets projected into $P_n(\mathbf{i})$. For fixed $j$ we define

$$A^j_n := \{ \mathbf{i} \in \mathcal{I}^N : \mu(Q_n(\mathbf{i})) \geq 2^{-nj} \frac{\mu(\mathbf{i}|n)}{N(\mathbf{i}|n)} \}$$

for all $n \in \mathbb{N}$. Now we have

$$\mu(\mathcal{I}^N \setminus A^j_n) = \mu(\bigcup_{\mathbf{i} \in \mathcal{I}^n} Q_n(\mathbf{i}) \setminus A^j_n) = \sum_{Q_n(\mathbf{i}) \notin A^j_n} \mu(Q_n(\mathbf{i})) \leq \sum_{Q_n(\mathbf{i}) \notin A^j_n} N(\mathbf{i}) 2^{-nj} \frac{\mu(\mathbf{i}|n)}{N(\mathbf{i}|n)} = 2^{-nj}.$$

Thus for the set $A^j := \bigcup_{N \in \mathbb{N}} \bigcap_{n=N}^\infty A^j_n$ we have

$$\mu(A^j) = \lim_{N \to \infty} \mu(\bigcap_{n=N}^\infty A^j_n) = 1 - \lim_{N \to \infty} \mu \left( \bigcup_{n=N}^\infty (\mathcal{I}^N \setminus A^j_n) \right) \geq 1 - \lim_{N \to \infty} \sum_{n=N}^\infty 2^{-nj} = 1.$$

By definition, for all $\mathbf{i} \in A^j$, we find $M(\mathbf{i}) \in \mathbb{N}$ such that the inequality

$$(2.3) \quad \mu(Q_n(\mathbf{i})) \geq 2^{-nj} \frac{\mu(\mathbf{i}|n)}{N(\mathbf{i}|n)}$$

holds for all $n \geq M(\mathbf{i})$.

We assumed that $\Lambda_{\mu}(s, \mathbf{i}) > -\infty$ for some $s > \dim_{LY}(\mu, \mathbf{i})$. This implies that $\lambda_l(\mu) > -\infty$ for all $1 \leq l \leq k+1$. For $\gamma(\mathbf{i}) > \dim_{LY}(\mu, \mathbf{i})$, we have

$$\lim_{n \to \infty} \frac{\log \mu(\mathbf{i}|n)}{\log \alpha_{k+1}(\mathbf{i}|n-1)} = \lim_{n \to \infty} \frac{1}{n} \left( \log \mu(\mathbf{i}|n) - \sum_{l=1}^k \log \alpha_l(\mathbf{i}|n) - \log \alpha_{k+1}(\mathbf{i}|n)^{-k} \right) \leq h_\mu(\mathbf{i}) + \lambda_1(\mu, \mathbf{i}) + \cdots + \lambda_k(\mu, \mathbf{i}) + k < \gamma(\mathbf{i})$$

for $\mu$ almost all $\mathbf{i} \in \mathcal{I}^N$. The first inequality follows by the definition of $N(\mathbf{i}|n)$ and the fact that $h_\mu(\mathbf{i})$ and $\lambda_l(\mu, \mathbf{i})$ are finite for $1 \leq l \leq k+1$ and $0 > \lambda_{k+1}(\mu, \mathbf{i})$. The second inequality follows by (1.3), since $P_n(\gamma(\mathbf{i}), \mathbf{i}) < 0$. In the calculation, we have omitted the constant $(2R)^d$ from $N(\mathbf{i}|n)$, since it has no effect on the result.

Let $r_l$ be any sequence of positive numbers converging to zero. For each $l$, we find an integer $n_l$, so that $\sqrt[2^l]{\alpha_{k+1}(\mathbf{i}|n_l)} \leq r_l < \sqrt[2^l]{\alpha_{k+1}(\mathbf{i}|n_l-1)}$. To avoid complicated notation, we only write $n$ instead of $n_l$. We have $\mathbf{i} \in Q_n(\mathbf{i})$ and $\pi_n Q_n(\mathbf{i}) \subset P_n(\mathbf{i})$ and the greatest side-length of $P_n(\mathbf{i})$ is $\alpha_{k+1}(\mathbf{i}|n)$. Therefore we have $\pi_n Q_n(\mathbf{i}) \subset B(\pi_n(\mathbf{i}), \sqrt[2^l]{\alpha_{k+1}(\mathbf{i}|n)})$. 
Using this and (2.3) and (2.4), we get that
\[
\limsup_{i \to \infty} \frac{\log \pi_a B(\pi_a(i), r_i)}{\log r_i} \leq \limsup_{n \to \infty} \frac{\log \pi_a B(\pi_a(i), \sqrt{\alpha_{k+1}(i_1|n)})}{\log \alpha_{k+1}(i_1|n)} \\
\leq \limsup_{n \to \infty} \frac{\log \mu Q_n(i)}{\log \alpha_{k+1}(i_1|n-1)} \\
\leq \limsup_{n \to \infty} \left( \frac{\log 2^{-n/j}}{\log \alpha_{k+1}(i_1|n-1)} + \frac{\log \mu [i_1|n] - \log N(i_1|n)}{\log \alpha_{k+1}(i_1|n-1)} \right) \\
\leq j^{-1} \frac{\log 2}{\log \alpha} + \gamma(i),
\]
where \( \overline{\alpha} = \sup_{i \in I} \alpha_1(i) < 1 \). Since \( j \) and the sequence \( r_i \) were arbitrary and \( \mu(A^j) = 1 \) for all \( j \in \mathbb{N} \), we have obtained \( \overline{\dim}_{\text{loc}}(\pi_a \mu, \pi_a(i)) \leq \gamma(i) \) for \( \mu \) almost all \( i \in I^\mathbb{N} \). \( \square \)

**Remark 2.3.** It is natural to ask, what can be said of the local dimensions, when one only assumes \( \sup_{i \in I} \alpha_1(i) \leq \overline{\alpha} < 1 \), and what results can be obtained for all translations \( a \). Observe that Theorem 2.2 already applies to this case. By using an essentially identical proof to the proof of [10, Theorem 2.6], one can get the following estimates.

Assume that \( \mu \in \mathcal{M}_d(I^\mathbb{N}), h_\mu < \infty \) and \( \log \alpha_d(i_1) \in L^1(\mu) \). Then we have for \( \mu \) almost all \( i \in I^\mathbb{N} \) and for all \( a \in \mathbb{Q}^\mathbb{N} \) that
\[
(2.5) \quad \frac{h_\mu^a(i)}{-E_\mu(\log \alpha_d(i_1))} \leq \overline{\dim}_{\text{loc}}(\pi \mu, \pi(i)) \leq \overline{\dim}_{\text{loc}}(\pi \mu, \pi(i)) \leq \frac{h_\mu^a(i)}{-E_\mu(\log \alpha_1(i_1))},
\]
where \( h_\mu^a(i) \) is the local projection entropy defined as in [10, Definition 2.1], \( m \) is so that \( H(P_m) < \infty \), and \( I \) is the \( \sigma \)-algebra of \( \sigma \) invariant sets. For the definition of the conditional expectation \( E_\mu \), see [18]. If the index set \( I \) is finite then (2.5) is strictly included in [10, Theorem 2.6].

The assumptions \( h_\mu < \infty \) and \( \log \alpha_d(i_1) \in L^1(\mu) \) are needed in the ergodic theorems that are used in the proof and the number \( m \) can be chosen to be the least integer for which \( H(P_m) < \infty \). In the finitely generated case these assumptions are of course satisfied and \( m = 1 \). We also note that the proof of [10, Theorem 6.2], which a more general version of [10, Theorem 2.6], deals with a direct product of two IFS and the conditional measures used there are not needed to obtain (2.5).

In most cases, the upper bound in equation (2.5) is not as good as the result of Theorem 2.2. However, in the exceptional case, where Theorem 2.1 does not hold, the upper estimate in (2.5) might give a better estimate since \( h_\mu^a \leq h_\mu \), see [10, Proposition 4.1].

### 3. Pressure function and dimensions of the limit set

In order to determine the Hausdorff dimension of the limit set \( F_\mu \), one often considers the pressure function defined by
\[
P(s) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in I^n} \phi^s(i).
\]
In the finitely generated setting it is known that if \( \max_{i \in I} ||A_i|| < \frac{1}{2} \), then the Hausdorff dimension of \( F_\mu \) equals to the zero of the pressure for \( \mathcal{L}^d_A \) almost all \( a \in \mathbb{R}^d \), where \( \kappa \) denotes cardinality of the index set \( I \), see [3]. In [13, Theorem B], Käenmäki and Reeve generalize this result for an infinitely generated affine IFS, with the extra assumption of quasi-multiplicativity, see [13, (2.1)] for the definition. Since their results on the Hausdorff
Suppose that the singular value function $\dim$ and not for all $i$.

We do not know whether a similar approximation holds for $\dim$ and not for all $i$. Mauldin and Urbanski have given an example of an infinitely generated
self similar set $F$ satisfying the open set condition, for which $\dim H F < \dim P F$, see [16, Example 5.2]. On the other hand, for all finite subsystems it holds that $\dim H \pi_a(J^N) = \dim P \pi_a(J^N)$, see [4]. Therefore the dimension approximation property does not hold for this, or similar examples. Note also that $\dim_B F_a = \dim_P F_a$ for infinitely generated self-affine sets $F_a$ by [16, Theorem 3.1]. The following theorem gives an estimate for the relation between Hausdorff and packing dimensions of infinitely generated self affine sets. For $x \in F_a$, we set the notation $L_n(x) = \{f_i(x) : i \in I^n\}$.

**Theorem 3.3.** Let $\{f_i\}_{i \in I}$ be an infinitely generated affine IFS. Then we have that

$$\sup_{x \in F_a} \{\dim_H F_a, \dim_B L_n(x)\} \leq \dim_P F_a \leq \sup_{x \in F_a} \{s_0, \dim_B L_n(x)\},$$

where $s_0 = \inf \{s : \lim_{n \to \infty} \frac{1}{n} \sum_{i \in I^n} \alpha_1(i)^s = 0\}$.

**Proof.** We have $\dim_P F_a = \dim_B F_a$ by [16, Theorem 3.1] and so the first inequality is trivial. The proof of the last inequality is essentially the same as the proof of [17, Lemma 2.8], since $\|f_i'\| = \alpha_1(i)$.

Note that if $s_0 \leq 1$, then $s_0 = \inf \{s : P(s) < 0\} = \dim_H F_a = \dim_P F_a$ for $m$ almost all $a \in \mathbb{Q}^N$ by [13, Theorem B].

4. Examples and final remarks

Here we give some examples on the entropies and pressures of measures. In example 4.1 we show that the measure-theoretical pressure function can be non-zero everywhere and in example 4.2 we show that the pressure function can be non-zero everywhere, as mentioned earlier. In the examples, we make use of Bernoulli measures: Fix reals $0 \leq p_i \leq 1$ so that $\sum_{i=1}^{\infty} p_i = 1$. The unique measure satisfying $\mu[I_n] = p_1 p_2 \cdots p_n$ is called a Bernoulli measure. It is well known that Bernoulli measures are ergodic. It is also easy to see that the entropy of a Bernoulli measure can be infinite.

**Example 4.1.** $(P_t(s) \neq 0$ everywhere$)$ Let $\mu$ be a Bernoulli measure with $\mu[i] = c(i+1)^{-2}$, where $c = \left(\frac{\pi^2}{6} - 1\right)^{-1}$. Let

$$A_i = \begin{bmatrix} 2\mu[i] & 0 \\ 0 & c^{4^{-i}} \end{bmatrix}.$$

We can now calculate

$$h_\mu = -\sum_{i=2}^{\infty} c i^{-2} \log c i^{-2} = \log c + 2c \sum_{i=2}^{\infty} i^{-2} \log i < \infty$$

and thus $\mu$ is a probability measure with finite entropy. Also, by induction we see that $\lambda_1(\mu) = \sum_{i=1}^{\infty} \mu[i] \log 2 \mu[i] = \log 2 - h_\mu$ and

$$\lambda_2(\mu) = c \sum_{i=1}^{\infty} (i + 1)^{-2} \log c 4^{-i} = -\infty.$$

Thus $\mu$ is a Bernoulli measure with finite entropy and $P_t(\mu)$ $\geq \log 2$ for all $t \leq 1$ and $P_t(\mu) = -\infty$ for all $t > 1$. Since $\sup_{i \in I} \alpha_1(i) = 2\mu[1] = \frac{N}{2} \leq \frac{4}{3}$, Theorem 2.1 gives that $\dim_{\text{loc}}(\pi_a \mu, \pi_a(\mu)) \geq 1$ for $\mu$ almost all $i \in I^N$ and $m$ almost all $a \in \mathbb{Q}^N$. 
Example 4.2. \((P(s) \neq 0 \text{ everywhere})\) Let \(c_i = i^{-\frac{1}{2}}, d_i = i^{-1}\) and \(A_i = \begin{bmatrix} c_i & 0 \\ 0 & d_i \end{bmatrix}\) for all \(i \in \mathbb{N}\). Now \(A_i = \begin{bmatrix} c_i & 0 \\ 0 & d_i \end{bmatrix}\) for all \(i \in I^n\), where \(c_1 = c_1 \cdots c_n\) and \(d_1 = d_1 \cdots d_n\). Therefore, for all \(t = 1 + s\), we have

\[
\phi^t(i) = \frac{1}{i_1^t} \cdots \frac{1}{i_2^t} \cdots \frac{1}{i_n^t},
\]

which implies

\[
\sum_{i \in I^n} \phi^t(i) = \left( \sum_{i \in \mathbb{N}} \frac{1}{i^{t+s}} \right)^n \quad \text{and} \quad P(t) = \log \sum_{i \in \mathbb{N}} \frac{1}{i^{t+s}}.
\]

Choose \(I = \{ i | (\log i)^2 : i \geq n_0 \}\). Now we have that

\[
P(t) = \log \sum_{i \in I} \frac{1}{i^{t+s}} = \log \sum_{i=n_0}^{\infty} \frac{1}{i (\log i)^2} < 0
\]

for \(n_0\) large enough. For all \(t < \frac{3}{2}\) we get \(P(t) = \infty\), since \(\log i \leq i^\delta\) for large \(i\) when \(\delta > 0\).

We end with final remarks on the assumptions and results of this paper.

Remarks 4.3. (1) Considering the proof of Theorem 2.2, suppose that \(\lambda_{k+1}(\mu, i) = -\infty\). We face difficulties at (2.4) since \(\log a(d, i) \leq \sum_{i \in \mathbb{N}} \frac{1}{i^{t+s}}\); and so the assumption \(\log a(d, i) \in L^1\) in equation (2.5) is necessary.

(2) Since we assumed that the limit set \(F\) is bounded it is reasonable to also assume that \(\alpha_d(i) \to 0\) as \(i \to \infty\). Therefore we could have \(-\sum_{i \in \mathbb{N}} \mu[i] \log a_d(i) = \infty\) and so the assumption \(\log a_d(i) \in L^1\) is necessary.

(3) Considering the finitely generated case, suppose that \(#I = \kappa\) and that \(s_0\) is the zero of the pressure function. Käenmäki proved the existence of an equilibrium measure \(\mu_{s_0}\) in [12]. For this measure, \(\dim_{LY}(\mu_{s_0})\) equals to \(s_0\). By Theorem 1.2, we get that \(\dim_{LY}(F) \geq s_0\) for Lebesgue almost all \(a \in \mathbb{R}^d\). This shows that we can not remove the assumption \(\sup_{i \in I} \|a_i\| = \frac{1}{\rho}\) from Theorem 1.2. For examples where \(\dim_{LY}(F) < s_0\), see [2, 19, 20]. Also there are examples showing that for particular \(a\), Theorem 1.2 cannot hold, see e.g. [5, Example 9.11]. The size of the set of these exceptional translation has been studied by Falconer and Miao in [7].

(4) Supposing that \(b^\kappa_{\mu}(i) < \infty\), we may slightly modify the definition of the Lyapunov dimension, namely by setting

\[
\dim_{LY}^f(\mu, i) = \inf \{ s : b^\kappa_{\mu}(i) - \Lambda_\mu(s, i) < 0 \}.
\]

Perhaps we could have \(\dim_{loc}(\sigma, \mu, i) = \min \{ d, \dim_{LY}^f(\mu, i) \} \) for \(\mu\) almost all \(i \in I^n\) and all \(a \in Q^N\), when \(\mu \in \mathcal{E}(I^n)\), \(b^\kappa_{\mu} < \infty\) and \(\sup_{i \in I} \|A_i\| < 1\).
References


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