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Alexander Zaretskiy

Mathematical Models and Stability Analysis of Three-Phase Synchronous Machines

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ABSTRACT

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This work is devoted to the investigation of stability and oscillations of three-phase synchronous machines with four-pole rotor at various connection (series and parallel) in feed system. Nowadays, they are widely used as generators for power generation in power plants and power systems.

To study these machines new mathematical models are developed under the assumption of a uniformly rotating magnetic field generated by the stator windings. This assumption goes back to classical ideas of N. Tesla and G. Ferraris. The obtained models completely take into account rotor geometry in contrast to the well-known mathematical models of synchronous machines.

Then the conditions of steady-state and global stability for synchronous machines are established. The dynamical stability is considered in the context of the limit load problem. The limit permissible loads on synchronous machines without control are estimated by the second Lyapunov method. In order to increase dynamical stability, the direct torque control is suggested. The sufficient conditions of the existence of circular solutions and at the limit cycles of the second kind for the models of synchronous machines are obtained by the non-local reduction method. The obtained analytical results are an extension of Tricomi's classical results to multidimensional models of synchronous machines. Moreover, numerical modeling of systems, describing the synchronous machines under the load without control, with a proportional control and with a step control, is carried out. The conclusions on more preferred type of connection are made.

Keywords: synchronous machines, four-pole rotor, stability, transient processes, the limit load problem, the non-local reduction method, circular solutions, limit cycles of the second kind

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1 INTRODUCTION AND THE STRUCTURE OF THE WORK

The three-phase synchronous machines are the primary electromechanical energy converters widely used as compensators for reactive power compensation (Thorpe, 1921; Miller, 1982; Eremia and Shahidehpour, 2013), as generators for power generation in power systems (Shenkman, 1998; Tewari, 2003; Rashid, 2010; Emadi et al., 2010; Manwell et al., 2010; Wu et al., 2011), as motors in industrial drives (Stephen, 1958; Humphries, 1988; Bose, 1997; Tewari, 2003; Thumann and Mehta, 2008) and in automatic voltage control (McFarland, 1948; Thumann and Mehta, 2008; Bhattacharya, 2011; Trout, 2011). It was invented first by F. A. Haselwander in 1887 (Boveri, 1992; Hall, 2008). The principle of operation of this synchronous machine was based on an electromagnetic induction discovered by M. Faraday and a rotating magnetic field obtained first with help of stator windings by N. Tesla and G. Ferraris (Tesla, 1888b,a; Ferraris, 1888). At the same time both phenomena are a base of constructing modern electrical machines of alternate current till now (McFarland, 1948; Stephen, 1958; Nasar, 1987; Humphries, 1988; Manwell et al., 2010; Bhattacharya, 2011; Hemami, 2011).

Electrical machines are usually divided into three types: direct current (d.c.) machines, alternating current (a.c.) asynchronous (induction) machines and alternating current (a.c) synchronous machines. *"Of these machine types the d.c. machines are no longer of practical interest as generators because of several drawbacks; they require more maintenance effort, have an unfavourable power to mass ratio and are not suitable for hight voltage windings. Of the a.c. machines, both asynchronous and synchronous types are use"* (Stiebler, 2008). The main difference between an synchronous machine and the induction one is that a speed of the rotor of synchronous machine coincides with a speed of stator magnetic field, being generated by supply voltage.

The theory of synchronous machines was developed during the first half of the 20th century. There were a few of hundreds of engineers and scientists who have published their results in this area (see, e.g., Blondel, 1923; Doherty and Nickle, 1926, 1927; Park, 1928, 1933; Kilgore, 1930; Lyon, 1954; Concordia, 1951). First of all it was related to the problems of the construction of synchronous generators and power systems.

J.K. Maxwell was the first who applied analytical mechanics to analysis of electromechanical systems. In his work (Maxwell, 1954) he established that the electric circuit equations can be write in the form of Lagrange equations. These equations are known as Maxwell equations. Despite the fact that he did not investigate synchronous machines but Maxwell equations had a great influence on development of the theory of electrical machines in the future.

The first mathematical models of synchronous machines were suggested in (Lyon and Edgerton, 1930a; Edgerton and Fourmarier, 1931; Tricomi, 1931, 1933). The fundamental works on mathematical theory of synchronous machines are works of Italian mathematician F. Tricomi (Tricomi, 1931, 1933). He derived the simplest differential equation of a synchronous machine, namely the second order equation, and carried out a global qualitative investigation of this equation. He also proved the existence of nontrivial global bifurcation and obtained the estimations of bifurcation values of parameters. This equation became known as Tricomi's equation.

In the works (Amerio, 1949; Seifert, 1952, 1953, 1959; Hayes, 1953; Belustina, 1954, 1955) Tricomi's equation is investigated in details and more accurate estimations of bifurcation parameter values are obtained. Further results of this equation investigation were essentially theoretical and referred to the phase synchronization theory.

The theory of steady-state operating mode of synchronous machines was developed fairly deeply in the works (Doherty and Nickle, 1926, 1927, 1930; Bohm, 1953). For this purpose the mathematical models such as the vector diagrams (Concordia, 1951; Kimbark, 1956; Puchstein, 1954) and the equivalent circuit models (Pender and Mar, 1922) were used. The main disadvantage of these models are that they do not describe the dynamical processes arising during operation of synchronous machines.

The next important step in the investigation of synchronous machines was the development of mathematical models which describe the transient processes. These models were first suggested by R. Park in (Park, 1928, 1929a,b; Park and Bancer, 1929; Park, 1933). In 1928 A. A. Gorev (Gorev, 1927, 1960, 1985) published general equations of a salient-pole synchronous machine similar to the Park's equations. They were obtained from general equations of electrodynamic system motion. Park-Gorev's equations describe the synchronous machines under transient conditions in stator and rotor windings.

A very valuable contribution to the development of the transient process theory of synchronous machines was made by V. Bush and R.D. Booth (Bush and Booth, 1925), R. E. Doherty and C. A. Nickle (Doherty and Nickle, 1926, 1927, 1930), R. Rüdenberg (Rüdenberg, 1931, 1942, 1975), E. Clarke (Clarke, 1943), F. R. Longley (Longley, 1954), B. Adkins (Adkins, 1957), D. White and H. Woodson (White and Woodson, 1959), R.A. Luter (Luter, 1939), A. Blondel (Blondel, 1923), A.I. Vajnov (Vajnov, 1969), M.P. Kostenko (Kostenko and Piotrovskiĭ, s. a.), G.N. Petrov (Petrov, 1963).

Among the works, it should be marked the fundamental works of G. Kron

(Kron, 1935, 1939, 1942, 1963) on the mathematical theory of electrical machines. He suggested a new mathematical model for the generalized electric machine. It is an idealized two-pole machine with two pairs of windings on the stator and two pairs of windings on the rotor. This model allowed one to reveal the characteristics of electromechanical energy conversion.

In monograph (White and Woodson, 1959) the equations for idealized two-phase electric machine are derived. It was shown that on the basis of these equations almost all used electromechanical converters can be analyzed. However, this model does not take into account any qualitative characteristics of synchronous machines such as the rotor geometry, inductances in damper windings.

Nowadays, different mathematical models of synchronous machines, described by ordinary differential equations (Rodriguez and Medina, 2002, 2003; Wang et al., 2007; Lipo, 2012) or partial differential equations (Lefevre et al., 1989; Silvester and Ferrari, 1996; Toliyat and Kliman, 2010; Krishnan, 2010), are used. The differences between the models depend on the chosen coordinate system and the made simplifying assumptions (Xu et al., 1991; Arrillaga et al., 1995; Srinivas, 2007; Bakshi and Bakshi, 2009c; Kumar, s. a.). In the same time the equations of synchronous machines can be obtained using the Kirchhgoff's and Newton's laws. The motion of the rotor can be described in any of an infinite number of coordinate systems. However, in practice two systems of coordinates (a, b) and (d, q) are in the most extensive use. The first system is the stationary reference frame with the reference axes a and b rigidly connected to the stator. The mathematical models developed in (a, b) coordinate system are called the fixed frame models (Subramaniam and Malik, 1971; Kron, 1938). They are used for investigation of synchronous machine operation under abnormal conditions, since they allow one to take into account the time-varying mutual inductances between the stator and rotor. The second system is the rotating reference frame with the reference axis d and q rigidly connected to the rotor. The mathematical models obtained in (d, q) coordinate system are called the rotating frame models (Lipo, 2012; Fuchs and Masoum, 2011; Smith, 1990). They are used for studying the steady-state operation modes, as well as for estimating the transient processes. R. Park in his work (Park and Bancer, 1929) suggested a transformation of coordinates which associated (d, q) coordinate system with (a, b) coordinate system. Mathematical models of synchronous machines can be also presented in uniformly rotating coordinate systems (Clarke, 1943). The most suitable coordinate system is used for solving particular problem which occurs when the induction motor operates. In this thesis we introduce the rotating system of coordinates rigidly connected to the stator rotating magnetic field. It allows one to simplify the derivation of differential equations of synchronous machines and obtain more accurate mathematical models of these machines.

Mathematical models of synchronous machines described by partial differential equations take into account more completely a magnetic field, temperature distribution, and another particular qualities of synchronous machines, but they turn out to be considerably complicated for investigations. Due to the complexity of such models they can not be analysed by analytical methods. Numerical analysis also does not provide exact results due to errors in computational procedures and finiteness of computational time interval. At the same time analytical analysis of mathematical models of synchronous machines, described by ordinary differential equations, allows one to obtain qualitative behaviour of systems. Therefore, these models are mostly used to describe the synchronous machines.

The engineering and analytical methods for investigating the stability of synchronous machines have developed in parallel with the development of new models. For example, step-by-step method (Hume and Johnson, 1934; Lipo, 2012), the energy criterion of stability, the equal-area criterion (Sarma, 1979; Murty, 2008), the second Lyapunov method (Eremia and Shahidehpour, 2013) were used. Among these methods, the second Lyapunov method is mostly used in dynamical stability analysis of synchronous machines. The actual application of this method to synchronous machines and power systems first appeared in publications of the "Russian school" (see, e.g., Yanko-Trinitskii, 1958; Gorev, 1960; Putilova and Tagirov, 1971; Zaslavskaya et al., 1967). The second Lyapunov method was developed in the monograph A.H. Gelig, G.A. Leonov, V.A. Yakubovich (Gelig et al., 1978), where in addition to typical functions of Lyapunov, the functions involving the information on solutions of Tricomi's equation are used. These Lyapunov-type functions are the essence of the non-local reduction method (Leonov, 1984a,b; Leonov et al., 1992; Yakubovich et al., 2004)

Due to the development of modern computer technology, the numerical methods are widely used at present. A new information about the behavior of trajectories can be obtained. However, in the practice it is insufficient to study numerically one or several solutions of the systems, since some applied problems require finding the estimations of attraction domain of equilibrium states. The limit load problem (Bryant and Johnson, 1935; Sah, 1946; Annett, 1950; Blalock, 1950; Yanko-Trinitskii, 1958; Barbashin and Tabueva, 1969; Caprio, 1986; Chang and Wang, 1992; Miller and Malinowski, 1994; Nasar and Trutt, 1999; Leonov et al., 2001; Das, 2002; Bianchi, 2005; Leonov, 2006a; Wadhwa, 2006; Lawrence, 2010; Glover et al., 2011) is one of these problems and related with synchronous machine stability under sudden changes of load. Numerical solution of the limit load problem for particular values of the parameters is given in works of W.V. Lyon, H.E. Edgerton (Lyon, 1928; Lyon and Edgerton, 1930b), as well as in monograph of D. Stoker (Stoker, 1950). In these works to find the limit load, the equal-area method was used.

The dynamical stability of synchronous machines can be increased by implementing a controller. The controller may influence either on stator and rotor currents or directly on the torque of the rotor. A variable frequency drive (VFD) is frequently used as a controller, which allows one to change amplitude and frequency of current. Two main techniques for the control of synchronous machines are used: field-oriented control (Quang and Dittrich, 2008; De Doncker et al., 2011) and direct torque control (Ozturk, 2008; Jin and Lin, 2011; Alacoque, 2012).

Field-oriented control (FOC) is a control technique, which is based on changing the stator magnetic field (De Doncker et al., 2011) by regulation of amplitude and frequency of the stator supply voltage. K. Hasse and S. F. Blaschke first suggested such control for a.c. motors (Hasse, 1969; Blaschke, 1971, 1973). FOC is divided into direct FOC (feedback vector control) and indirect FOC (feedforward vector control). The first method is less used, since it requires direct computation of flux magnitude and angle feedback signals (Hasse, 1969; Wu, 2006; Quang and Dittrich, 2008). The second method uses an information obtained directly from the sensors (Blaschke, 1971, 1973; Wu, 2006). Mathematical models for realization FOC are usually developed in (d, q) coordinate system. Such models allow one to determine the magnetic fluxes along two axes of the stator and effectively control the synchronous machine.

Direct torque control is based on changing directly the rotor torque through the rotor and stator supply voltage or the additional external devices. This control was first developed by M. Depenbrock (Depenbrock, 1988). Mathematical models for realization DTC are basically determined in (a, b) coordinate system. Direct torque control has many advantages, for example faster torque control, high torque at low speeds and high speed sensitivity. In this thesis two control laws, which can be achieved by DTC technique, are considered.

Despite much research and numerous publications devoted to the study of synchronous machines, some problems still remain unsolved. One of the main problems is the providing stable operation during changing process conditions. In recent years the interest in this problem has increased significantly due to the accident on the Sayano-Shushenskaya hydropower plant (Rostehnadzor, 2009) and blackouts in the U.S. and Europe (Thomas and Hall, 2003; Bialek, 2004; Goodrich, 2005). The main reasons of loss of synchronism, which led to accidents, were the increasing of load torque and voltage collapses. For example, Rostehnadzor made the following conclusion about the accident at Sayano-Shushenskaya hydropower plant: the accident happened due to the multiple additional variable loads on a hydraulic aggregate connected with transition through non-recommended operation domain of a turbine (Rostehnadzor, 2009). The loss of synchronism can occur between one machine and the rest of the system or between groups of machines. Synchronous machines are essential elements in any power system. Because of these reasons the qualitative analysis of transient processes in synchronous machines under sudden changes of load is required.

To study of synchronous machines it is important to develop mathematical models, which adequately describe their behaviour. Incorrect mathematical modeling leads to occurrence of instability zone in corresponding models, which is lacking in real induction machines. For example, increasing the supply voltage proportionally to increasing the torque load gives us stable mathematical model, however, in practice the rotor sometimes starts rotating with acceleration, i.e., synchronous machine is unstable. So ill-posed mathematical models of synchronous machines cause the incorrect conclusions about stability of machines.

The mathematical models of synchronous machines are often described by high-order differential equations with trigonometric nonlinearities. Due to the complexity of these models they practically can not be studied by analytical methods, therefore, numerical methods are also used for investigation of these equations. However, some complicated effects such as semi-stable cycle solutions and hidden oscillations[1] which may occur in electromechanical systems can not be found and studied only by numerical methods. Hence, it is necessary to develop analytical methods for stability analysis of mathematical models.

Thus, the main goals of this thesis are the derivation of adequate mathematical models of synchronous machines under sudden changes of load and their stability analysis with help of analytical and numerical methods.

Structure of the work. This thesis is divided into five main parts. The motivation of this thesis, the literature review of the mathematical models of synchronous machines and methods of their investigation found in the first chapter. Also the main results obtained by the author and the articles which are the basis of this thesis are presented.

In the second chapter the synchronous machines with fore-pole rotor at series and parallel connections in feed system are considered and their operation principle is described. Then the simplifying assumptions which go back to the classical ideas of N. Tesla and G. Ferraris are introduced. Based on these assumptions and laws of classical mechanics and electrodynamics, the mathematical models that completely take into account rotor geometry of synchronous machines are developed. They are described by ordinary differential equations.

The third chapter is devoted to stability analysis of synchronous machines under load conditions. The conditions of steady-state stability and the conditions of global stability for synchronous machines are obtained. The limit load problem is formulated and the estimations of the limit permissible load on synchronous machines are found with help of the second Lyapunov method. By introducing the direct torque control it was shown that the value of the limit permissible load can be increased. Next the sufficient conditions of existence of circular solutions and limit cycles of second kind, which correspond to unstable modes, are established. The analytical results for a direct-torque-controlled synchronous machine are obtained by the non-local reduction method.

In fourth chapter the numerical modeling of considered synchronous machines is carried out by the standard computational tools of MATLAB and a modification of the event-driven method. Moreover, two types of control laws are studied. Numerical results are analyzed and compared with theoretical results.

Conclusions and future research directions are presented and discussed shortly in the fifth chapter.

At the end of this thesis the reader finds Finnish summary, references, three appendices and five included articles. In the first appendix the cylindrical phase space for synchronous machines is introduced. In the second appendix the main

[1] See chaotic hidden attractors in electronic Chua circuits (Leonov et al., 2010; Kuznetsov et al., 2011a,b; Bragin et al., 2011; Leonov et al., 2011b, 2012; Leonov and Kuznetsov, 2012; Kuznetsov et al., 2013; Leonov and Kuznetsov, 2013a; Leonov et al., 2011b,a; Kuznetsov et al., 2010), in drilling systems (Kiseleva et al., 2012, 2014; Leonov et al., 2013), in aircrafts (Leonov et al., 2012a,b; Andrievsky et al., 2012), in two-dimensional polynomial quadratic systems (Kuznetsov et al., 2013; Leonov et al., 2011a; Leonov and Kuznetsov, 2010; Kuznetsov and Leonov, 2008; Leonov et al., 2008; Leonov and Kuznetsov, 2007; Kuznetsov, 2008), in PLL (Leonov and Kuznetsov, 2014), and in Aizerman and Kalman problems (Leonov et al., 2010b,a; Leonov and Kuznetsov, 2011, 2013b,c)

theorems are proved. Listings of programs in MATLAB are represented in the third appendix.

Included articles. This thesis is based on publications (Kondrat'eva, Solov'yova and Zaretsky, 2010; Zaretskiy, 2012; Leonov, Solovyeva and Zaretskiy, 2013; Leonov, Zaretskiy and Solovyeva, 2013; Leonov, Kuznetsov, Kiseleva, Solovyeva and Zaretskiy, 2014) and included articles (PI–PV). In these papers, the statements of problems are due to the supervisors, while the development of mathematical models of synchronous machines, their analytical analysis and computer modeling are due to the author. Four-pole rotor synchronous machines with damper windings and without any control are studied at series connection in PII and at parallel connection in PIII. The limit load problem for these machines is formulated and the estimations of the limit permissible load are obtained by so-called equal-area method. In PIV the direct torque control is introduced for four-pole rotor synchronous machines with damper windings at series connection. In this case the estimations of the limit permissible load are found by the non-local reduction method. The effective sufficient conditions for the existence of circular solutions and the limit cycles of the second kind are also established. In PV similar results are obtained for four-pole rotor synchronous machines without damper windings at different connections in feed system. In PI the non-local reduction method is applied to the solution of the limit load problem for power system of two connected synchronous machines.

The results of this thesis were also reported at the international conferences 5th IFAC International Workshop on Periodic Control Systems (Caen, France – 2013), 7th Vienna International Conference on Mathematical Modelling (Vienna, Austria – 2012), International Conference TRIZfest-2011 (St.Petersburg, Russia – 2011), 4th All-Russian Multi-Conference on Control Problems "MKPU–2011" (Divnomorskoe, Russia – 2011), 7th European Nonlinear Dynamics Conference (Rome, Italy – 2011), XI International Conference "Stability and Oscillations of Nonlinear Control Systems" (Moscow, Russia – 2010, 2012), International Workshop "Mathematical and Numerical Modeling in Science and Technology" (Finland, Jyväskylä – 2010) and at the seminars on the department of Applied Cybernetics (Saint Petersburg State University, Russia 2009 – 2013) and the department of Information Technology (University of Jyväskylä, Finland 2010 – 2013).

2 MATHEMATICAL MODELS OF SYNCHRONOUS MACHINES

The construction of electrical machines has been constantly improved and complicated from the beginning of electrical machinery history. Obviously, the mathematical models of these machines become more complex and difficult for investigations. In this chapter we start with consideration of rather simple electromechanical models, namely four-pole rotor synchronous machines without damper windings. Next, these machines with different connections in feed system are studied (PV). Finally, we introduce the damper windings into construction of the four-pole rotor (PII, PII, PIV). For all considered machines, the mathematical models are developed by the author. The difference between models are explained. Unlike well-known mathematical models of synchronous machines the obtained models completely take into the account geometry of rotors.

2.1 Electromechanical models of salient pole synchronous machines

Like any electrical machine, a synchronous machine consists of a stator and a rotor separated by the air-gap. The stator is a hollow laminated cylinder, carrying windings in slots on its inner surface. The stator winding generates a rotating magnetic field when this winding is connected to the three-phase supply (in the motor case) or produces a three-phase voltage (in the generator case).

Depending on the rotor construction, synchronous machines can be divided into two types: salient pole and non-salient pole (or cylindrical pole) machines (Tewari, 2003; Theraja, 2012). Synchronous machines, except those for very high speed, are always constructed with salient poles (Prasad, 2005). However, the the theoretical description of a salient pole rotor is much more complicated than that of a non-salient rotor due to its asymmetries (Quaschning, 2005). In this thesis we study synchronous machines with the salient pole rotor because of constantly increasing use of these machines in many applications.

A salient pole rotor has four or more salient poles (Fig. 1). Further, for simplicity, we consider a four-pole rotor. It consists of a dc field winding and is fed from a dc source through slip rings and brushes which are mounted on the rotor. The field winding is presented as two orthogonal pairs of parallel coils, each of which is made of several turns of insulated wire. Modern salient pole rotors have an additional winding, known as damper winding. Damper winding is provided to damp the oscillations during transient processes of machine (Sivanagaraju et al., 2009). It consists of bars short-circuited at each end by two rings and is very similar to squirrel-cage winding of an induction motor. The scheme of four-pole rotor with damper winding is shown in Fig. 2.

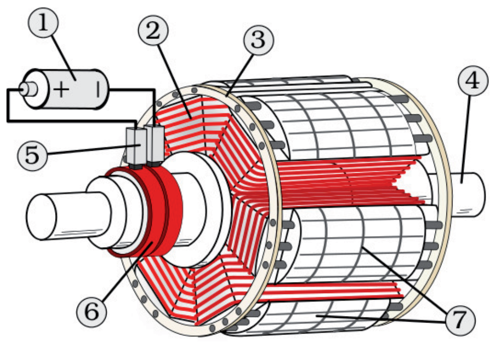


FIGURE 1 The salient pole rotor with damper winding: 1 – source of constant voltage, 2 – field winding, 3 – damper winging, 4 – shaft, 5 – brushes, 6 – rings, 7 – poles

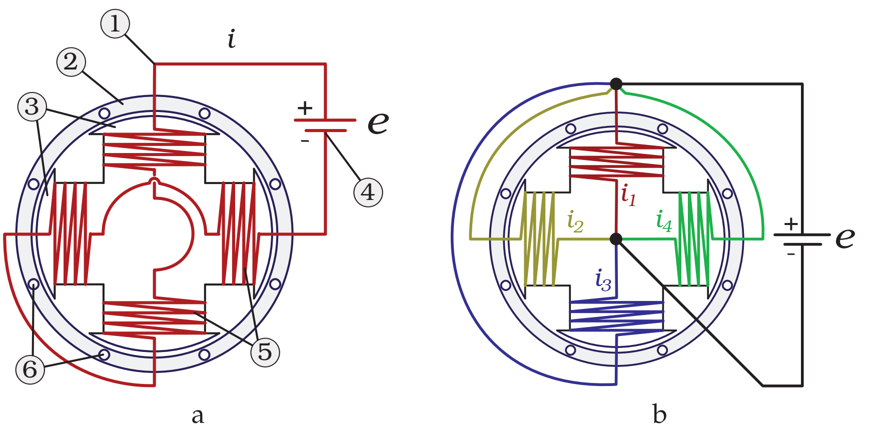


FIGURE 2 Scheme of four-pole rotor with different connections: 1 – field winding, 2 – damper winging, 3 – poles, 4 – source of constant voltage, 5 – coils, 6 – bars. a – series connection; b – parallel connection

In the four-pole rotor of synchronous machines there exists two types of connections in feed system, namely two different ways of connection of field winding to one constant voltage source:

1. series connection, when each coil is connected in series to a constant voltage source (Fig. 2, a);
2. parallel connection, when each coil is connected in parallel to a constant voltage source (Fig. 2, b).

Both types of connections are studied in this thesis.

More detailed description of construction of synchronous machines can be found in (Bakshi and Bakshi, 2009b,a; Bhattacharya and Singh, 2006; Gönen, 2012).

2.2 Modeling assumptions

Synchronous machines obey the reversibility principle, i.e., they can operate as either motor converting electrical energy into mechanical one or generator converting mechanical energy into electrical one. The principle of reversibility allows one to conclude that mathematical models of synchronous machines operating in the generator mode preserve the same structure as synchronous machines operating in the motor mode. In what follows we consider synchronous machines operating as a motor.

The classical derivation of expressions for currents in rotor windings and electromagnetic torque of synchronous motor are based on the following simplifying assumptions (Popescu, 2000; Leonhard, 2001; Skubov and Khodzhaev, 2008; Solovyeva, 2013):

1. It is assumed that the magnetic permeability of stator and rotor steel[1] is equal to infinity. This assumption makes it possible to use the principle of superposition for the determination of magnetic field, generated by stator;
2. one may neglect energy losses in electrical steel, i.e., motor heat losses, magnetic hysteresis losses, and eddy-current losses;
3. the saturation of rotor steel is not taken into account, i.e. the current of any force can run in rotor winding;
4. one may neglect the effects, arising at the ends of rotor winding and in rotor slots, i.e., one may assume that a magnetic field is distributed uniformly along a circumference of rotor.

   Let us make an additional assumption[2]:
5. stator windings are fed from a powerful source of sinusoidal voltage.

Then, following (Adkins, 1957; White and Woodson, 1959; Skubov and Khodzhaev, 2008), by the latter assumption the influence of rotor currents on stator currents may be ignored. Thus, a stator generates a uniformly rotating magnetic field with a constant in magnitude induction. So, it can be assumed that the magnetic

[1] Usually both stator and rotor are made of laminated electrical steel.

[2] Without this assumption it is necessary to consider a stator, what leads to more complicated derivation of equations and more complicated equations themselves, which are difficult for analytical and numerical analyzing.

induction vector B is constant in magnitude and rotates with a constant angular velocity n_1. This assumption goes back to the classical ideas of N. Tesla and G. Ferraris and allows one to consider the dynamics of synchronous motor from the point of view of its rotor dynamics (PV; Leonov, 2006a).

2.3 Mathematical models of four-pole rotor synchronous motors

In order to develop mathematical models of synchronous motors, the operation of these motors is considered. The operation principle of synchronous motors is based on the interaction of the magnetic fields of the stator and the rotor (magnetic locking).

Let us consider the motor starting. When the stator winding is excited by a three-phase ac supply, a rotating magnetic field is produced. The speed at which the magnetic field rotates is called the synchronous speed. At this instant the rotor is stationary. In order to start the synchronous motor it is necessary to rotate the rotor at a speed close to or equal to synchronous speed. For this purpose the rotor is driven with help of some external device or damper winding in the direction of rotating magnetic field. When the rotor achieves the speed close to synchronous speed, the dc supply to the rotor winding is switched on. Now the rotor also produces a rotating magnetic field. At some instant, the magnetic field of the rotor locks with the magnetic field of the stator and the motor operates at synchronous speed. Then the external devise used to rotate rotor is removed. However, the rotor continue to rotate at synchronous speed due to magnetic locking.

Now the rotor rotates at synchronous speed. Let us introduce the uniformly rotating system of coordinates, rigidly connected with the stator magnetic field and consider the motion of four-pole rotor in this coordinate system. Suppose that the stator magnetic field rotates clockwise. Also, assume that the positive direction of the rotation of the rotor coincides with the direction of the rotation of the stator magnetic field.

The stator rotating magnetic field interacts with currents flowing in the field winding. According to Ampere's force law (Theraja and Theraja, 1999), the electromagnetic forces F_k arise, the directions of which are shown in Fig. 3, a.

The value of electromagnetic force, induced in a coil, is determined by Ampere's force law:

$$F = Bl_0 i, \tag{1}$$

where B – induction of the stator magnetic field, l_0 – width of the coil, i – current in the coil.

Let us define electromagnetic torque of a synchronous motor with four-pole rotor in the case of series connection, which is produced by the electromagnetic forces F_k, $k = 1, ..., 8$. The projections of force F_1 and F_2, (Fig. 3, b), acting on one

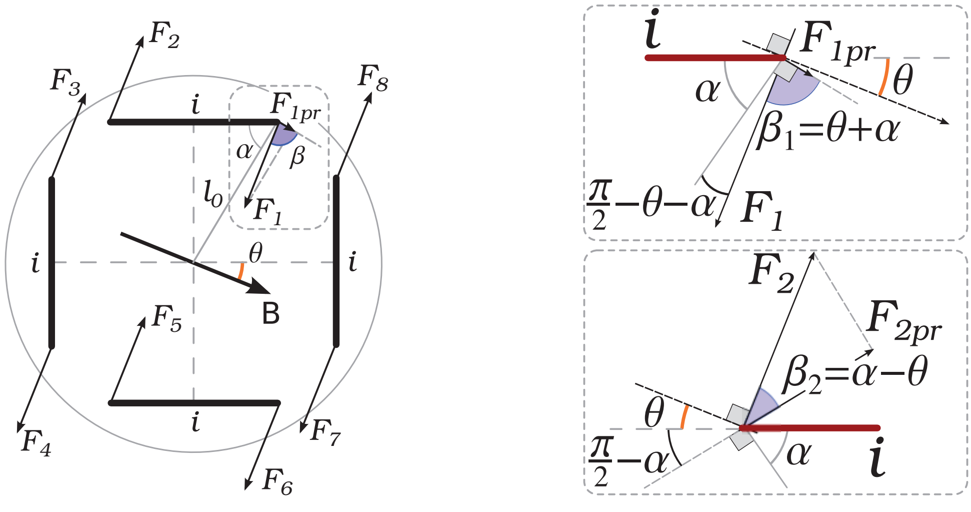


FIGURE 3 Geometry of four-pole rotor at series connection: a – the directions of electromagnetic forces and currents; b – the projection of force F_1.

wind of the first coil, are given by formula

$$F_{1\text{pr}} = F_1 \cos\beta_1 = Bl_0 i\cos(\theta+\alpha),$$

$$F_{2\text{pr}} = F_2 \cos\beta_2 = Bl_0 i\cos(\alpha-\theta),$$

where β_1 – the angle between F_1 and the perpendicular to the radius-vector;. β_2 – the angle between F_2 and the perpendicular to the radius-vector; α – the angle between radius-vector and the plane of the first coil; θ – the angle between the plane perpendicular to the vector of the stator magnetic field and the plane of the first coil.

Taking into account the number of winds in the first coil and a positive direction of the rotor rotation axis, it follows that the produced electromagnetic torque, acting on the first coil, is equal to the following:

$$M_1 = nl\left(F_{1pr} + F_{2pr}\right) = nl_0 lBi\left(\cos(\theta+\alpha) + \cos(\alpha-\theta)\right),$$

$$= n(2l_0 l\cos\alpha)Bi\cos\theta = nSBi\cos\theta,$$

where n – the number of winds in the coil; l – the length of radius-vector; S – the area of one wind of the coil.

Electromagnetic torques, acting on other coils, are similarly determined:

$$M_2 = nSBi\cos\left(\theta+\frac{\pi}{2}\right) = -nSBi\sin\theta,$$

$$M_3 = nSBi\cos\theta,$$

$$M_4 = -nSBi\sin\theta,$$

Thus, the electromagnetic torque of synchronous motor with four-pole rotor at series connection is equal to

$$M_{\rm em} = M_1 + M_2 + M_3 + M_4 = 2nSBi\,(\cos\theta - \sin\theta) =$$

$$= 2\sqrt{2}nSBi\sin\left(\frac{\pi}{4} - \theta\right).$$

In the case of parallel connection the electromagnetic torque of synchronous motor with four-pole rotor is equal to

$$M_{\rm em} = nSB\left[\cos\theta(i_1 + i_3) - \sin\theta(i_2 + i_4)\right],$$

where i_1, i_2, i_3, i_4 – currents in the coils; other parameters have the same meanings as before. We assume that all four coils are identical.

The dynamics of synchronous motor is described by the voltage equations and the torque equation

$$J\ddot{\theta} = M_{\rm em} - M_{\rm l},$$

where θ corresponds to mechanical angle of rotor rotation; J – the moment of inertia of the rotor; $M_{\rm em}$ – electromagnetic torque; $M_{\rm l}$ – load torque, which can include a control law.

Now we find the voltage equations. Consider an electrical circuit of four-pole rotor with series connection, shown in Fig. 4. Note that this electric circuit is equivalent to that, presented in Fig. 3. Define the current in each coil.

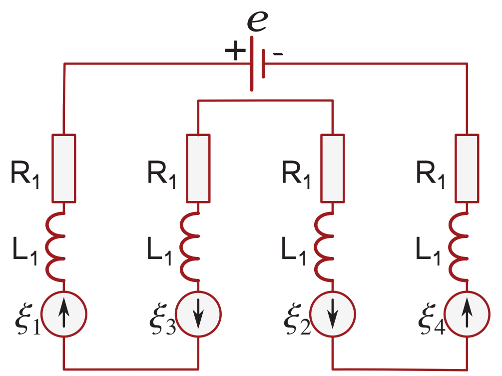


FIGURE 4 Equivalent electrical circuit of four-pole rotor with series connection

Applying Kirchhoff's second law (Theraja and Theraja, 1999) to the closed loop and choosing a positive direction as clockwise to traverse the loop, one arrivals at the following differential equation

$$L\dot{i} + Ri = \xi_1 - \xi_2 + \xi_3 - \xi_4 + e,$$

where R, L – active and inductive resistances of each coil, respectively; ξ_k – electromotive force, induced in k-th coil by rotating magnetic field; e – constant voltage source. The directions of electromotive forces are shown in Fig. 4. According to the law of electromagnetic induction (Theraja and Theraja, 1999), the electromotive force which arises in the first coil moving in magnetic field, is given by formula

$$\xi_1 = l_0 B v_1 \sin\zeta_1 + l_0 B v_2 \sin\zeta_2,$$

where v_1, v_2 are velocities of coil relative to magnetic field, the directions of which are shown in Fig. 5, a; ζ_1, ζ_2 are angles between a vector of velocity and a vector of magnetic field induction. The angles ζ_1 and ζ_2 are defined in Fig. 5, b. Thus, taking into account the number of turns, emf in the first coil is equal to

$$\begin{aligned} \xi_1 &= -nl_0 Bl\omega \left[\sin\left(\frac{\pi}{2}+\alpha+\theta\right) + \sin\left(\frac{\pi}{2}+\alpha-\theta\right)\right] = \\ &= -n(2l_0 l\cos\alpha)B\omega\cos\theta = -nSB\omega\cos\theta, \end{aligned} \tag{2}$$

where $\omega = \dot{\theta}$. Similarly, the expressions for emf in the rest coils are obtained and take the form

$$\begin{aligned} \xi_2 &= -nSB\omega\sin\theta, \\ \xi_3 &= -nSB\omega\cos\theta, \\ \xi_4 &= -nSB\omega\sin\theta. \end{aligned} \tag{3}$$

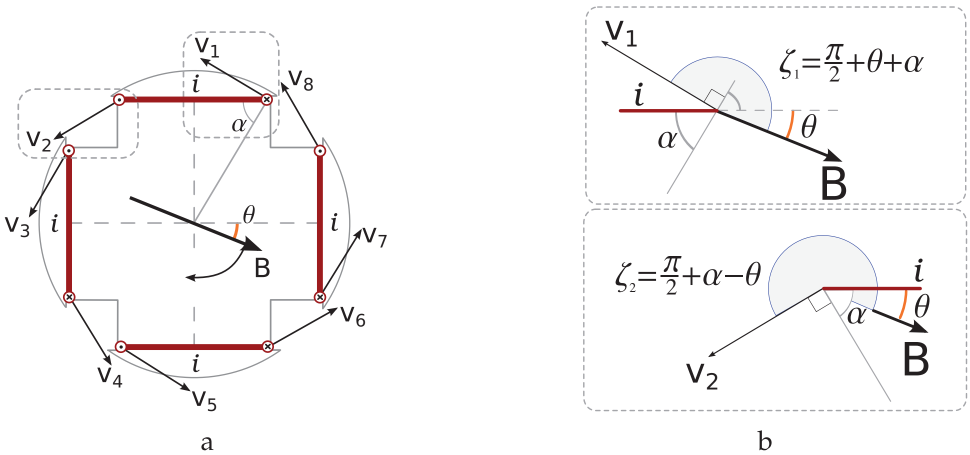


FIGURE 5 Geometry of four-pole rotor at parallel connection: a – the directions of velocity and emf; b – the definitions of angles ζ_1 and ζ_2.

Unlike the electrical circuit of four-pole rotor at series connection, the electrical circuit of four-pole rotor with parallel connection has four closed loops (Fig. 6). Using Kirchhoff's second law for each closed loop and choosing a positive direction as before, one obtains the following differential equation for currents in the case of parallel connection

$$
\begin{aligned}
&L\dot{i}_1 + Ri_1 = \xi_1 + e = -nSB\omega\cos\theta + e,\\
&L\dot{i}_2 + Ri_2 = -\xi_2 + e = nSB\omega\sin\theta + e,\\
&L\dot{i}_3 + Ri_3 = \xi_3 + e = -nSB\omega\cos\theta + e,\\
&L\dot{i}_4 + Ri_4 = -\xi_4 + e = nSB\omega\sin\theta + e.
\end{aligned}
$$

The emf, induced in coils, are determined similarly to series model by equations (2) and (3).

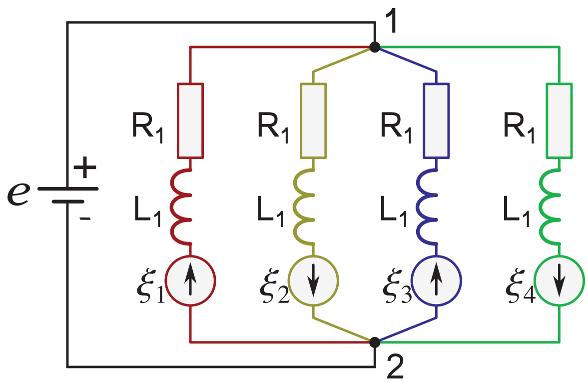


FIGURE 6 Equivalent electrical circuit of four-pole rotor with series connection

Thus, the dynamics of four-pole rotor synchronous motor without damper winding in the case of series connection is described by the following system of differential equations

$$
\begin{aligned}
&\dot{\theta} = \omega,\\
&J\dot{\omega} = 2\sqrt{2}nSBi\sin\left(\frac{\pi}{4} - \theta\right) - M_l,\\
&L\dot{i} + Ri = -2\sqrt{2}nSB\omega\sin\left(\frac{\pi}{4} - \theta\right) + e,
\end{aligned}
\tag{4}
$$

and in the case of parallel connection is described by

$$\begin{aligned}
&\dot{\theta} = \omega, \\
&J\dot{\omega} = nSB\left[\cos\theta(i_1 + i_3) - \sin\theta(i_2 + i_4)\right] - M_l, \\
&L\dot{i}_1 + Ri_1 = -nSB\omega\cos\theta + e, \\
&L\dot{i}_2 + Ri_2 = nSB\omega\sin\theta + e, \\
&L\dot{i}_3 + Ri_3 = -nSB\omega\cos\theta + e, \\
&L\dot{i}_4 + Ri_4 = nSB\omega\sin\theta + e.
\end{aligned} \tag{5}$$

As was mentioned above, most modern salient pole rotors have an damper winding used to start the motor. The damper winding is presented as the squirrel-cage rotor winding of an induction motor. Using results obtained in (Leonov et al., 2013) for the squirrel-cage rotor winding, we get the equations for currents in bars of damper winding of synchronous motor

$$L_1\dot{j}_k + R_1 j_k = -l_0 l_1 B\cos(\theta + \frac{2k\pi}{n_1})\dot{\theta}, \quad k = 1, ..., n_1, \tag{6}$$

and the electromagnetic torque of damper winding

$$M_{\text{em dam}} = l_0 l_1 B \sum_{k=1}^{n_1} \cos(\theta + \frac{2k\pi}{n_1}) j_k.$$

Here n_1 – the numbers of bars; j_k – the current in the k-th bar; R_1 – the bar resistance; L_1 – the bar inductance; l_1 and l_0 – the radius and the length of the squirrel-cage, respectively.

Since the field winding and damper winding are presented as two independent windings, then we obtain that the dynamics of four-pole rotor synchronous motor with damper winding in the case of series connection is described by

$$\begin{aligned}
&\dot{\theta} = \omega, \\
&J\dot{\omega} = 2\sqrt{2}nSBi\sin\left(\frac{\pi}{4} - \theta\right) + l_0 l_1 B \sum_{k=1}^{n_1} \cos(\theta + \frac{2k\pi}{n_1}) j_k - M_l, \\
&L\dot{i} + R\, i = -2\sqrt{2}nSB\omega\sin\left(\frac{\pi}{4} - \theta\right) + e, \\
&L_1\dot{j}_k + R_1 j_k = -l_0 l_1 B\cos(\theta + \frac{2k\pi}{n_1})\dot{\theta}, \quad k = 1, ..., n_1,
\end{aligned} \tag{7}$$

and in the case of parallel connection is described by

$$
\begin{aligned}
&\dot{\theta} = \omega, \\
&J\dot{\omega} = nSB\left[\cos\theta(i_1 + i_3) - \sin\theta(i_2 + i_4)\right] + \\
&\qquad\qquad\qquad + l_0 l_1 B \sum_{k=1}^{n_1} \cos(\theta + \frac{2k\pi}{n_1}) j_k - M_l, \\
&L\dot{i}_1 + Ri_1 = -nSB\omega\cos\theta + e, \\
&L\dot{i}_2 + Ri_2 = nSB\omega\sin\theta + e, \\
&L\dot{i}_3 + Ri_3 = -nSB\omega\cos\theta + e, \\
&L\dot{i}_4 + Ri_4 = nSB\omega\sin\theta + e, \\
&L_1\dot{j}_k + R_1 j_k = -l_0 l_1 B\cos(\theta + \frac{2k\pi}{n_1})\dot{\theta}, \quad k = 1, ..., n_1.
\end{aligned} \tag{8}
$$

Let us transform systems (7) and (8) to a form more convenient for the further study. The nonsingular change of coordinates

$$
\begin{aligned}
\vartheta &= -\theta - \frac{3\pi}{4}, \\
s &= -\omega, \\
x &= i + \frac{e}{R}, \\
\mu &= -\frac{2L_1}{n_1 l_0 l_1 B} \sum_{k=1}^{n_1} i_k \sin\left(\theta + \frac{2k\pi}{n_1}\right), \\
\nu &= -\frac{2L_1}{n_1 l_0 l_1 B} \sum_{k=1}^{n_0} i_k \cos\left(\theta + \frac{2k\pi}{n_1}\right), \\
z_k &= \sum_{k=-\frac{n_1}{4}}^{\frac{n_1}{4}} i_{(k+j) \bmod n_1} + i_k \mathrm{ctg}\frac{\pi}{n_1} \qquad k = 3, ..., n_1.
\end{aligned}
$$

reduces system (7) to the form

$$\begin{aligned}
\dot{\vartheta} &= s, \\
\dot{s} &= ax\sin\vartheta + b\nu - \varphi_1(\vartheta), \\
\dot{x} &= -cx - ds\sin\vartheta, \\
\dot{\mu} &= -c_1\mu + \nu s, \\
\dot{\nu} &= -c_1\nu - \mu s - s, \\
\dot{z}_k &= -c_1 z_k \qquad k = 3, ..., n_1,
\end{aligned} \tag{9}$$

where

$$a = 2\sqrt{2}\frac{nBS}{J}; \quad b = \frac{n_0(S_0B)^2}{J}; \quad c = \frac{R}{L}; \quad c_1 = \frac{R_1}{L_1}; \quad d = 2\sqrt{2}\frac{nBS}{L};$$

$$\gamma_{\max} = 2\sqrt{2}\frac{nBSe}{JR}; \quad \gamma_1 = \frac{M_1}{J};$$

$$\varphi_1(\vartheta) = \gamma_{\max}\sin\vartheta - \gamma_1.$$

Note that the system (9) can be studied without last $n_1 - 2$ differential equations because they do not affect on stability of the system and can be integrated independently on the remaining equations. Therefore, further we consider the system of fifth order differential equations

$$\begin{aligned}
\dot{\vartheta} &= s, \\
\dot{s} &= ax\sin\vartheta + b\nu - \varphi_1(\vartheta), \\
\dot{x} &= -cx - ds\sin\vartheta, \\
\dot{\mu} &= -c_1\mu + \nu s, \\
\dot{\nu} &= -c_1\nu - \mu s - s,
\end{aligned} \tag{10}$$

Using nonsingular change of coordinates

$$\vartheta = \frac{\pi}{4} - \theta,$$

$$s = -\omega,$$

$$x = -\frac{2L}{nSB}\left[-\left(i_1 + i_3 + \frac{2e}{L}\right)\sin\theta - \left(i_1 + i_3 + \frac{2e}{L}\right)\cos\theta\right],$$

$$y = -\frac{2L}{nSB}\left[\left(i_1 + i_3 + \frac{2e}{L}\right)\cos\theta - \left(i_1 + i_3 + \frac{2e}{L}\right)\sin\theta\right],$$

$$\mu = -\frac{2L_1}{n_1 l_0 l_1 B}\sum_{k=1}^{n_1} i_k \sin\left(\theta + \frac{2k\pi}{n_1}\right),$$

$$\nu = -\frac{2L_1}{n_1 l_0 l_1 B}\sum_{k=1}^{n_0} i_k \cos\left(\theta + \frac{2k\pi}{n_1}\right),$$

$$z_1 = i_1 - i_3,$$

$$z_2 = i_2 - i_4,$$

$$z_k = \sum_{k=-\frac{n_1}{4}}^{\frac{n_1}{4}} i_{(k+j) \bmod n_1} + i_k \mathrm{ctg}\frac{\pi}{n_1} \qquad k = 3, ..., n_1.$$

system (8) reduces to the form

$$\begin{aligned}
\dot{\vartheta} &= s,\\
\dot{s} &= ay + b\nu - \varphi_1(\vartheta),\\
\dot{x} &= -cx + ys,\\
\dot{y} &= -cy - xs - s,\\
\dot{\mu} &= -c_1\mu + \nu s,\\
\dot{\nu} &= -c_1\nu - \mu s - s,\\
\dot{z}_1 &= -cz_1,\\
\dot{z}_2 &= -cz_2,\\
\dot{z}_k &= -c_1 z_k \qquad k = 3, ..., n_1,
\end{aligned} \tag{11}$$

where

$$a = \frac{(nBS)^2}{JL}; \quad b = \frac{n_0(S_0B)^2}{JL_0}; \quad c = \frac{R}{L}; \quad c_1 = \frac{R_1}{L_1};$$

$$\gamma_{\max} = 2\sqrt{2}\frac{nBSe}{JR}; \quad \gamma_1 = \frac{M_1}{J};$$

$$\varphi_1(\vartheta) = \gamma_{\max}\sin\vartheta - \gamma_1.$$

Note that the equations with the variables z_k can be integrated independent of the rest of the system and do not affect on its stability. Therefore, it suffices to consider the system of sixth order differential equations

$$\begin{aligned}
\dot{\vartheta} &= s, \\
\dot{s} &= ay + b\nu - \varphi_1(\vartheta), \\
\dot{x} &= -cx + ys, \\
\dot{y} &= -cy - xs - s, \\
\dot{\mu} &= -c_1\mu + \nu s, \\
\dot{\nu} &= -c_1\nu - \mu s - s,
\end{aligned} \tag{12}$$

Thus, investigation of four-pole rotor synchronous motors with damper winding at different types of connections is reduced to study systems (10) and (12).

3 STABILITY AND OSCILLATIONS OF SYNCHRONOUS MOTORS

Stability analysis is one of the most important problems in operation, optimization and control of synchronous machines. By stability we imply that the synchronous machine re-establishes an operating mode after some disturbances. In order to solve the stability analysis problems we use classical approach which is based on analysis of properties of the systems of equations, describing the dynamics of synchronous machines.

This chapter is devoted to analysis of stability and existence of oscillations in synchronous motors with different connections in the feed system. For both cases the conditions that determine the stable operation characteristics of a synchronous motor and the conditions of global stability are established by the author. Next the permissible changes of loads on synchronous motors, under which a transient process is stable, are found. The torque control with linear growth in the slip is suggested. It allows one to improve stability of these motors. At the end the sufficient conditions of existence of circular solutions and limit cycles of second kind, which correspond to unstable modes, are obtained by the author.

3.1 Steady-state stability analysis of synchronous motors

Steady-state stability is a fundamental requirement for normal operation of synchronous machines. By steady-state (static, local) stability we mean the ability of an synchronous motor to maintain an operating mode after its arbitrarily small disturbances. The steady-state stability of modes of synchronous motors is studied with help of classical theorem on stability in the first approximation (Halanay, 1966; Merkin and Afagh, 1997; Menini and Tornambè, 2011). An asymptotically stable equilibrium point corresponds to a operating mode of an synchronous motor. An unstable equilibrium point corresponds to a physically unrealizable mode.

Let us first study the steady-state stability of four-pole rotor synchronous motor with series connection, which is described by system

$$
\begin{aligned}
&\dot{\vartheta} = s, \\
&\dot{s} = ax \sin\vartheta + b\nu - \varphi(\vartheta), \\
&\dot{x} = -cx - ds\sin\vartheta, \\
&\dot{\mu} = -c_1\mu + \nu s, \\
&\dot{\nu} = -c_1\nu - \mu s - s,
\end{aligned}
\tag{10}
$$

The stationary set of system (10) is empty if $|\gamma| > \gamma_{\max}$. Let $|\gamma| \le \gamma_{\max}$, then the stationary set consists of countable number of isolated points

$$
\Lambda = \left\{ \begin{pmatrix} \vartheta_i + 2\pi k \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^5 \;\middle|\; i = \{0,1\}, \quad \forall k \in \mathbb{Z} \right\}.
$$

Here ϑ_0 and ϑ_1 are roots of the equation

$$
\gamma_{\max} \arcsin(\vartheta) = \gamma, \qquad \vartheta \in [0, 2\pi), \tag{13}
$$

and satisfy the following conditions

$$
\begin{aligned}
&\vartheta_0 = \arcsin\left(\frac{\gamma}{\gamma_{\max}}\right), \qquad \varphi(\vartheta_0) = 0, \qquad \varphi'(\vartheta_0) > 0, \\
&\vartheta_1 = \pi - \arcsin\left(\frac{\gamma}{\gamma_{\max}}\right), \qquad \varphi(\vartheta_1) = 0, \qquad \varphi'(\vartheta_1) < 0.
\end{aligned}
\tag{14}
$$

Let us determine which equilibrium points are stable. The characteristic polynomial of the Jacobian matrix of (10) in stationary points is

$$
f(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ -\varphi'(\vartheta_i) & -\lambda & a\sin\vartheta_i & 0 & b \\ 0 & -d\sin\vartheta_i & -c-\lambda & 0 & 0 \\ 0 & 0 & 0 & -c_1-\lambda & 0 \\ 0 & -1 & 0 & 0 & -c_1-\lambda \end{pmatrix} =
$$

$$
\begin{aligned}
= -(c_1+\lambda)\,\big[\lambda^4 + (c+c_1)\lambda^3 + \big(b + cc_1 + ad\sin^2\vartheta_i + \varphi'(\vartheta_i)\big)\,\lambda^2 + \\
+ \Big(bc + adc_1\sin^2\vartheta_i + (c+c_1)\varphi'(\vartheta_i)\Big)\,\lambda + cc_1\varphi'(\vartheta_i)\Big].
\end{aligned}
\tag{15}
$$

The first-order polynomial situated in round brackets of (15) has one negative real root. Hence, stability of the characteristic polynomial is determined by stability of polynomial of the fourth order situated in square brackets of (15). Stability of this polynomial is defined by Gurvic criterion: for the fourth-order polynomial

$$f(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

the necessary and sufficient conditions of stability are the following

$$\begin{gathered} a_3 > 0, \qquad a_2 > 0, \qquad a_1 > 0, \qquad a_0 > 0, \\ a_1(a_2a_3 - a_1) - a_0a_3^2 > 0. \end{gathered} \tag{16}$$

Check these conditions in stationary points:

$$\begin{aligned} & a_3 = c + c_1 > 0, \\ & a_2 = b + cc_1 + ad\sin^2\vartheta_i + \varphi'(\vartheta_i) > 0, \\ & a_1 = bc + adc_1\sin^2\vartheta_i + (c + c_1)\varphi'(\vartheta_i) > 0, \\ & a_0 = cc_1\varphi'(\vartheta_i) > 0, \\ & a_1(a_2a_3 - a_1) - a_0a_3^2 = \left(bc + adc_1\sin^2\vartheta_i\right)\left(bc_1 + adc\sin^2\vartheta_i\right) + \\ & \qquad + (c + c_1)\left(bc_1 + adc\sin^2\vartheta_i\right)\varphi'(\vartheta_i) + \\ & \qquad\qquad + cc_1(c + c_1)\left(bc + adc_1\sin^2\vartheta_i\right) > 0 \end{aligned}$$

It is obvious that the first condition is fulfilled for any stationary point of the system (10). The other conditions of Gurvic criterion are satisfied in the case $\varphi'(\vartheta_i) > 0$ and is not satisfied in the case $\varphi'(\vartheta_i) < 0$. Hence, the equilibrium states $(\vartheta_0 + 2k\pi,\ 0,\ 0,\ 0,\ 0)^T$ are asymptotically stable and correspond to operating modes. The equilibrium states $(\vartheta_1 + 2k\pi,\ 0,\ 0,\ 0,\ 0)^T$ are unstable and correspond to physically unrealizable modes.

Let us study next the steady state stability of synchronous motor with four-pole rotor at parallel connection, which is described by system

$$\begin{aligned} & \dot{\vartheta} = s, \\ & \dot{s} = ay + bv - \varphi(\vartheta), \\ & \dot{x} = -cx + ys, \\ & \dot{y} = -cy - xs - s, \\ & \dot{\mu} = -c_1\mu + \nu s, \\ & \dot{\nu} = -c_1\nu - \mu s - s, \end{aligned} \tag{12}$$

Similarly to the case of serial connection, the stationary set of the system (12) is empty when $|\gamma| > \gamma_{\max}$. If $|\gamma| \leq \gamma_{\max}$, then the stationary set is as follows:

$$\Lambda = \left\{ \begin{pmatrix} \vartheta_i + 2\pi k \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^6 \;\middle|\; i = \{0, 1\}, \quad \forall k \in \mathbb{Z} \right\},$$

where ϑ_0 and ϑ_1 are roots of the equation (13) and satisfy conditions (14). The characteristic polynomial of the Jacobian matrix of system (12) in stationary states is as follows:

$$f_p(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 \\ -\varphi'(\vartheta_i) & -\lambda & 0 & a & 0 & b \\ 0 & 0 & -c-\lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & -c-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_1-\lambda & 0 \\ 0 & -1 & 0 & 0 & 0 & -c_1-\lambda \end{pmatrix} =$$

$$= (c+\lambda)(c_0+\lambda)\Big[\lambda^4 + (c+c_1)\lambda^3 + \big(a+b+cc_1+\varphi'(\vartheta_i)\big)\lambda^2 + \\ + \big(ac_1+bc+(c+c_1)\varphi'(\vartheta_i)\big)\lambda^1 + cc_1\varphi'(\vartheta_i)\Big]. \tag{17}$$

The stability of polynomial $f_p(\lambda)$ is determined by the stability of the fourth order polynomial in square brackets of (17). Using Gurvic criterion for stability analysis of the fourth order polynomial (16), we obtain

$$a_3 = c + c_1 > 0,$$

$$a_2 = a + b + cc_1 + \varphi'(\vartheta_i) > 0,$$

$$a_1 = ac_1 + bc + (c+c_1)\varphi'(\vartheta_i) > 0,$$

$$a_0 = cc_1\varphi'(\vartheta_i) > 0,$$

$$a_1(a_2a_3 - a_1) - a_0a_3^2 = (ac_1+bc)(ac+bc_1) + \\ +(c+c_1)(ac+bc_1)\varphi'(\vartheta_i) + cc_1(c+c_1)(ac_1+bc) > 0$$

Taking into account conditions (14), the characteristic polynomial $f_p(\lambda)$ is stable for $\vartheta_i = \vartheta_0$ and unstable for $\vartheta_i = \vartheta_1$. Hence, the equilibrium points $(\vartheta_0 + 2k\pi,\ 0,\ 0,\ 0,\ 0,\ 0)^T$ are asymptotic stable and the equilibrium points $(\vartheta_1 + 2k\pi,\ 0,\ 0,\ 0,\ 0,\ 0)^T$ are unstable.

The presence of the angular coordinate ϑ in the equations of synchronous motors allows one to introduce the cylindrical phase space (see the main notions and approaches in Appendix 1). In the cylindrical phase space $\mathbb{R}^5/H$, where $H = \{(2k\pi,\ 0,\ 0,\ 0,\ 0)^T \in \mathbb{R}^5 \mid k \in \mathbb{Z}\}$ the stationary set of system (10) is presented by two points (Fig. 7):

$$\begin{pmatrix} \vartheta_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \vartheta_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^5/H.$$

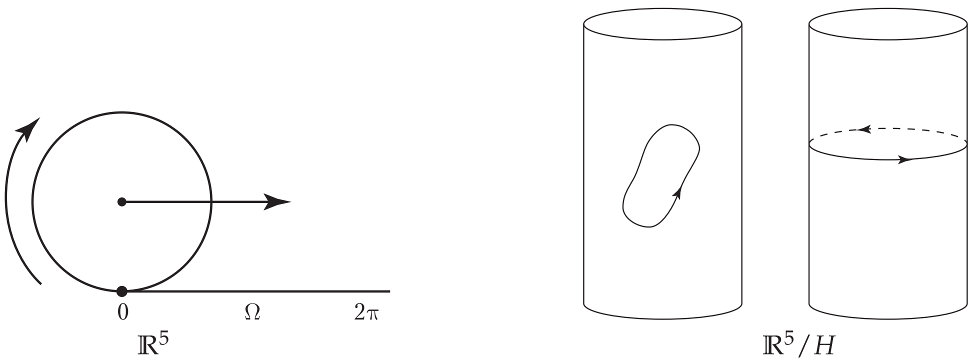


FIGURE 7 Phase space and cylindrical phase space

The stationary set of system (12) also corresponds to two points in the cylindrical phase space $\mathbb{R}^6/H$, where $H = \{(2k\pi,\ 0,\ 0,\ 0,\ 0,\ 0)^T \in \mathbb{R}^6 \mid k \in \mathbb{Z}\}$:

$$\begin{pmatrix} \vartheta_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \vartheta_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^6/H.$$

The use of cylindrical phase space gives us the following advantages:

- multiplicity related to angular coordinates disappears in $\mathbb{R}^n/H$, i.e., all values $\vartheta + 2\pi k$ of the angular coordinate correspond to only one value of a physical model (in our case one position of the rotor);
- in cylindrical phase space $\mathbb{R}^n/H$ the notion of boundedness of solutions can be naturally introduced. The bounded solution in $\mathbb{R}^n/H$ is the bounded solution in the phase space $\mathbb{R}^n$ if we exclude angular coordinates;
- it is convenient to classify the cycle solutions in cylindrical phase space (see Appendix 1).

In the next section the global stability of systems (10) and (12) is proved in the cylindrical phase space.

3.2 Dynamical stability of synchronous machines without load

The term of dynamical stability means that a synchronous machine returns to an operating mode after large disturbances. As well as the synchronous machine is said to be globally stability if the machine returns to an operating mode after any disturbances. In this section we prove the global stability of idle running synchronous motors (synchronous motors under no-load conditions).

The models of synchronous motors developed in section 2.3 can be described by the autonomous system of the form

$$\dot{y} = f(y), \qquad y \in \mathbb{R}^n, \tag{18}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable vector-function satisfying the condition

$$f\left(y + \begin{pmatrix} 2k\pi, \\ \mathbf{0} \end{pmatrix}\right) = f(y), \qquad \forall k \in \mathbb{Z}.$$

We assume that any solution $y(t, y_0)$ of system (18) with initial data $y(0) = y_0$ exists and is defined for all $t \geq 0$.

Now we introduce the definition of global stability for system of differential equation (18).

Definition 1. *(see, e.g., Zinober, 1994; Leonov et al., 1996; Colonius and Kliemann, 2000) System* (18) *is called a gradient-like system if any solution tends to an equilibrium state as $t \to +\infty$.*

If the stationary set in cylindrical phase space $\mathbb{R}^n/H$ consists of only one asymptotically stable equilibrium point and other equilibrium states are unstable in the sense of Lyapunov, then such gradient-like system is said to be *globally stable*.

In the context of theory of electrical machines the term "global stability" is more acceptable than the term "gradient-like system", since here only unique globally stable synchronism is observed physically.

The global stability of synchronous motors under no-load conditions is proved by the following theorem, which is a extension of the well-known Barbashin-Krasovskii theorem () and LaSalle's principle () on the systems with cylindrical phase space $\mathbb{R}^n/H$, where $H = \{(2k\pi,\ \mathbf{0})^T \in \mathbb{R}^n \mid k \in \mathbb{Z}\}$.

Theorem 1. *Suppose that there exists a continuous function $V(y) : \mathbb{R}^n \to \mathbb{R}$ such that the following conditions hold*

1. *$V(y + h) = V(y), \forall y \in \mathbb{R}^n, \forall h \in H$;*
2. *$V(y) + \vartheta^2 \to +\infty$ as $y \to \infty$;*
3. *for any solution $y(t)$ of system* (18) *the function $V(y(t))$ is nonincreasing function;*
4. *if $V(y(t)) \equiv V(y(0))$, then $y(t) \equiv const$.*

Then system (18) *is the gradient-like system.*

The proof of this theorem can be found in (Leonov and Kondrat'eva, 2009).

Theorem 2. *If* $\gamma_l = 0$*, then systems* (10) *and* (12) *are the gradient-like systems, i.e., the idle running synchronous motors are globally stable.*

Proof. We begin with proof for system (10). If $\gamma = 0$, then the stationary set of system (10) consists of isolated points of two types: asymptotically stable points $(2\pi k, 0, 0, 0, 0)$ and unstable points $((2k+1)\pi, 0, 0, 0, 0)$.

Let us show that the function

$$V(\vartheta, s, x, \mu, \nu) = \frac{1}{2}s^2 + \frac{a}{2d}x^2 + \frac{b}{2}\mu^2 + \frac{b}{2}\nu^2 + \int\limits_{\vartheta_1}^{\vartheta} \varphi(\zeta)d\zeta.$$

satisfies all conditions of theorem 1.

The function $V(\vartheta, s, x, \mu, \nu)$ is periodic in ϑ with period 2π, since

$$V(\vartheta + 2\pi k, s, x, \mu, \nu) = V(\vartheta, s, x, \mu, \nu) + \int\limits_{\vartheta}^{\vartheta + 2\pi k} \varphi(\zeta)d\zeta = V(\vartheta, s, x, \mu, \nu).$$

The second condition of theorem 1 for function $V(\vartheta, s, x, \mu, \nu)$ is implied by the relation

$$\int\limits_{\vartheta_1}^{\vartheta} \varphi(\zeta)d\zeta < C, \qquad \forall \vartheta \in \mathbb{R}.$$

On the solutions of system (10) ($\gamma = 0$) the function $V(\vartheta, s, x, \mu, \nu)$ is nonincreasing function:

$$\begin{aligned} \dot{V}(\vartheta, s, x, \mu, \nu) = s(ax\sin\vartheta + b\nu - \varphi(\vartheta)) + \frac{a}{d}x(-cx - ds\sin\vartheta) + \\ + b\mu(-c_1\mu + s\nu) + b\nu(-c_1\nu - s\mu - s) + \\ + s\varphi(\vartheta) = -\frac{ac}{d}x^2 - bc_1\mu^2 - bc_1\nu^2 \leq 0, \end{aligned} \tag{19}$$

Assume that the solution of system (10) with $\gamma = 0$ satisfies the condition

$$V(\vartheta(t), s(t), x(t), \mu(t), \nu(t)) \equiv V(\vartheta(0), s(0), x(0), \mu(0), \nu(0)).$$

Then from (19) and (10) it follows that

$$x(t) \equiv 0, \qquad \mu(t) \equiv 0, \qquad \nu(t) \equiv 0, \qquad s(t) \equiv 0.$$

Thus, $s(t) = \dot{\vartheta}(t) \equiv 0$, and hence $\vartheta(t) \equiv const$, i.e., the fourth condition of 1 is fulfilled.

So system (10) with $\gamma = 0$ is the gradient-like system. In cylindrical phase space $\mathbb{R}^5/H$, where $H = \{(2k\pi,\ 0,\ 0,\ 0,\ 0)^T \in \mathbb{R}^5 \mid k \in \mathbb{Z}\}$ system (10) with $\gamma = 0$ has unique asymptotically stable equilibrium point $(0,\ 0,\ 0,\ 0,\ 0)^T$. Therefore this system is globally stable.

The proof of globally stability of system (12) with $\gamma = 0$ is carried out similarly to the proof for system (10). It is used the cylindrical phase space $\mathbb{R}^6/H$, where $\mathbb{R}^6/H$, where $H = \{(2k\pi,\ 0,\ 0,\ 0,\ 0,\ 0)^T \in \mathbb{R}^6 \mid k \in \mathbb{Z}\}$ and the function

$$V(\vartheta, s, x, y, \mu, \nu) = \frac{1}{2}s^2 + \frac{a}{2}x^2 + \frac{a}{2}y^2 + \frac{b}{2}\mu^2 + \frac{b}{2}\nu^2 + \int\limits_{\vartheta_1}^{\vartheta} \varphi(\zeta)d\zeta.$$

□

The synchronous machines are widely used as compensators which are in fact synchronous motors running without a mechanical load. The synchronous compensator can absorb or generate reactive power, keeping the voltage level constant. The globally stability of idle running synchronous motors guarantees that compensators pull into operating mode at any voltage level (in this case we do not take into account overloads in motor windings).

3.3 Dynamical stability of synchronous machines under constant load

In the previous section it was proved that if a synchronous motor is started without a load, then it pulls in an operating mode after transient process. In other words after start-up the motor operates in synchronism. Now the problem on the maximum permissible load, under which the motor continues to operate, naturally arises. This problem is known in engineering practice as the ultimate (limit) load problem.

Let us describe the ultimate load problem using the example of a rolling mill (Fig. 8). A synchronous motor drives mill rolls. We do not take into account the interaction of connecting mechanisms. The model of the rolling mill in this simplest case can be described by equations of the synchronous motor.

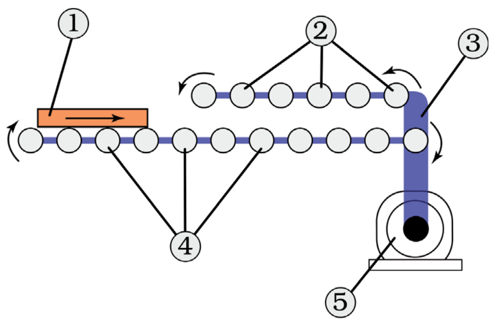


FIGURE 8 Scheme of rolling mill without load: 1– blank, 2 – top rolls, 3 – connecting mechanism, 4 – bottom rolls, 5 – synchronous motor

While uniform metal blank moves only in bottom rolls, we assume that a load on shaft of the synchronous motor is equal to zero and the motor operates in an operating mode. Then at some instant $t = \tau$ the blank enters in those part of rolling mill, where the process of rolling happens (Fig. 9). Due to rotation of top and bottom rolls in the opposite directions the blank moves on, decreasing in thickness. Hence, at time $t = \tau$ the instantaneous load-on arose. The problem is to find loads, under which the synchronous motor pulls in a new operating mode after transient processes.

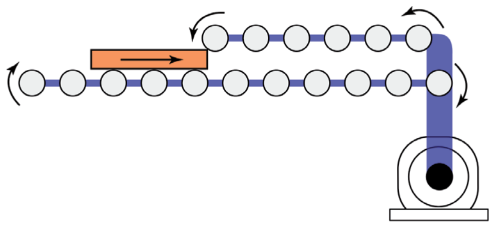


FIGURE 9 Scheme of rolling mill under load

Let us formulate the ultimate load problem mathematically. As previously mentioned the synchronous motors can be described by the following system of differential equations

$$\dot{y} = f(y), \qquad y \in \mathbb{R}^n, \tag{18}$$

Suppose that the synchronous motor without load works in an operating mode which corresponds to the asymptotically stable equilibrium point $y(t) = y_*$ of system (18). For $t > \tau$ the load γ_l is not already zero. Hence, the operating mode of the motor changes. A new operating mode of the motor under load corresponds to the asymptotically stable equilibrium point y_0 of the system (18) with initial data $y(0) = y_*$. A mathematical formulation of the ultimate load problem for synchronous motors is as follows: to find conditions, under which the solution of the system (18) with the initial data $y(0) = y_*$ belongs to the attraction domain of the stationary solution $y(t) = y_0$. The latter means that the following relations hold

$$\lim_{t\to\infty} y(t) = y_0. \tag{20}$$

Thus, the ultimate load problem is closely related to the problem of estimation of attraction domains of stable equilibrium points.

Consider the ultimate load problem for systems (10) and (12), which describe the dynamics of four-pole rotor synchronous motors with damper windings at series connection and at parallel connection, respectively. The problem for system (10) is as follows: to find the conditions under which the solution $\vartheta(t), s(t), x(t), \mu(t), \nu(t)$ of system (10) with zero initial data satisfies the relations

$$\begin{aligned} &\lim_{t\to\infty} \vartheta(t) = \vartheta_0, \quad \lim_{t\to\infty} s(t) = 0, \quad \lim_{t\to\infty} x(t) = 0, \\ &\lim_{t\to\infty} \mu(t) = 0, \quad \lim_{t\to\infty} \nu(t) = 0. \end{aligned} \tag{21}$$

The problem for system (12) is as follows: to find the conditions under which the solution $\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)$ of system (10) with zero initial data satisfies the relations

$$\lim_{t\to\infty} \vartheta(t) = \vartheta_0, \quad \lim_{t\to\infty} s(t) = 0, \quad \lim_{t\to\infty} x(t) = 0,$$
$$\lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} \mu(t) = 0, \quad \lim_{t\to\infty} \nu(t) = 0. \tag{22}$$

The posed problems for systems (10) and (12) are studied by the second method of Lyapunov in (PII, PII). The following results was obtained.

Theorem 3 (PII). *If γ_l satisfies the inequality*

$$\int_{\vartheta_1}^{0} \left(\sin\vartheta - \frac{\gamma_l}{\gamma_{max}} \right) \mathrm{d}\vartheta < 0, \tag{23}$$

then γ_l is a permissible load, i.e., the solution of system (10) *with initial data $\vartheta(0) = s(0) = x(0) = \mu(0) = \nu(0) = 0$ satisfies relations* (21).

Theorem 4 (PIII). *If the following condition is fulfilled*

$$\int_{\vartheta_1}^{0} \left(\sin\vartheta - \frac{\gamma_l}{\gamma_{max}} \right) \mathrm{d}\vartheta < 0, \tag{24}$$

then γ_l is a permissible load, i.e., the solution of system (12) *with initial data $\vartheta(0) = s(0) = x(0) = y(0) = \mu(0) = \nu(0) = 0$ satisfies relations* (22).

Theorems 3 and 4 are a justification of the widely used in engineering practice the equal-area criterion.

The value of maximum permissible load can be increased by torque controllers with control law

$$u(s) = -ks. \tag{25}$$

This control belongs to torque direct control. The systems (10) and (12) with control law (25) take the form

$$\begin{aligned}
\dot{\vartheta} &= s, \\
\dot{s} &= u(s) + ax\sin\vartheta + b\nu - \varphi(\vartheta), \\
\dot{x} &= -cx - ds\sin\vartheta, \\
\dot{\mu} &= -c_1\mu + \nu s, \\
\dot{\nu} &= -c_1\nu - \mu s - s,
\end{aligned} \tag{26}$$

and

$$
\begin{aligned}
&\dot{\vartheta} = s,\\
&\dot{s} = u(s) + ay + bv - \varphi(\vartheta),\\
&\dot{x} = -cx + ys,\\
&\dot{y} = -cy - xs - s,\\
&\dot{\mu} = -c_1\mu + vs,\\
&\dot{v} = -c_1 v - \mu s - s,
\end{aligned} \tag{27}
$$

Since the equilibrium points of new systems (26) and (27) coincide with the the equilibrium points of initial systems (10) and (12), therefore the formulation of the ultimate load problem remains unchanged.

The following theorem gives us the conditions under which the ultimate load on synchronous motors can be increased by suggested control law.

Theorem 5. *Suppose that there exists a number $\lambda \in \mathbb{R}$ such that the following conditions hold*

1. $0 < \lambda < \min\{k, c, c_1\}$;
2. *the solution of the differential equation*

$$F\frac{\mathrm{d}F}{\mathrm{d}\sigma} = -2\sqrt{\lambda(k-\lambda)}\,F - \varphi(\sigma). \tag{28}$$

with initial data

$$F(\vartheta_1) = 0,$$

satisfies the condition

$$F(0) > 0. \tag{29}$$

Then the solution of system (26) *with initial data $\vartheta = s = x = \mu = v = 0$ satisfies the relations* (21) *and the solution of system* (27) *with initial data $\vartheta = s = x = y = \mu = v = 0$ satisfies the relations* (22).

The proof of theorem 5 is based on the modification of the non-local reduction method and is carried out for system (26) in PIV and for system (27) in Appendix 2.

An unstable mode of a synchronous machine operation may lead to breakdown or even failure not only of the machine but also a power system. By this reason it is necessary to determine the conditions under which unstable modes arise. Circular solutions and limit cycles of second kind (see the definitions in Appendix 1) correspond to modes, in which the rotor rotates through an arbitrary large angle. Thus, the presence of these solutions exclude stability of equations of synchronous machines.

The following theorem gives us sufficient conditions under which circular solutions and limit cycles of second kind arise.

Theorem 6 (PIV). *Suppose that there exists a number* $\lambda \in \mathbb{R}$ *such that the following conditions hold*

1. $0 < \lambda < \min\{c, c_1\}$ *and*

$$\lambda - k - \frac{(a+d)^2}{4(c-\lambda)} - \frac{(b+1)^2}{4(c_1-\lambda)} \geq 0;$$

2. *the solution* $F(\sigma)$ *of the equation*

$$F\frac{\mathrm{d}F}{\mathrm{d}\sigma} = -\lambda F - \varphi(\sigma), \tag{30}$$

with initial data $F(\vartheta(0)) = 0$ *satisfies the condition*

$$\inf F(\sigma) > 0 \qquad \forall \sigma > \vartheta(0). \tag{31}$$

Then for any $\varepsilon > 0$ *system* (26) *has a circular solution* $(\vartheta, s, x, \mu, \nu)$ *with initial data* $\vartheta(0), s(0), x(0), \mu(0), \nu(0))$*, satisfying the conditions*

$$s(0) > 0, \qquad |s(0)| + |x(0)| + |\mu(0)| + |\nu(0)| < \varepsilon. \tag{32}$$

Moreover, if $k > 0$*, then system* (26) *has at least one limit cycle of the second kind.*

Theorem 7. *Suppose that there exists a number* $\lambda \in \mathbb{R}$ *such that the following conditions hold*

1. $0 < \lambda < \min\{c, c_1\}$ *and*

$$\lambda - k - \frac{(a+1)^2}{4(c-\lambda)} - \frac{(b+1)^2}{4(c_1-\lambda)} \geq 0; \tag{33}$$

2. *the solution* $F(\sigma)$ *of the equation*

$$F\frac{\mathrm{d}F}{\mathrm{d}\sigma} = -\lambda F - \varphi(\sigma), \tag{34}$$

with initial data $F(\vartheta(0)) = 0$ *satisfies the condition*

$$\inf F(\sigma) > 0 \qquad \forall \sigma > \vartheta(0). \tag{35}$$

Then for any $\varepsilon > 0$ *system* (27) *has a circular solution* $(\vartheta, s, x, y, \mu, \nu)$ *with initial data* $(\vartheta(0), s(0), x(0), y(0), \mu(0), \nu(0))$*, satisfying the conditions*

$$s(0) > 0, \qquad |s(0)| + |x(0)| + |y(0)| + |\mu(0)| + |\nu(0)| < \varepsilon. \tag{36}$$

Moreover, if $k > 0$*, then system* (27) *has at least one limit cycle of the second kind.*

Theorem 7 is proved in Appendix 2.

Numerical analysis of the conditions of the theorems gave us the following results. From the conditions of theorems 3 and 4 the region of permissible loads

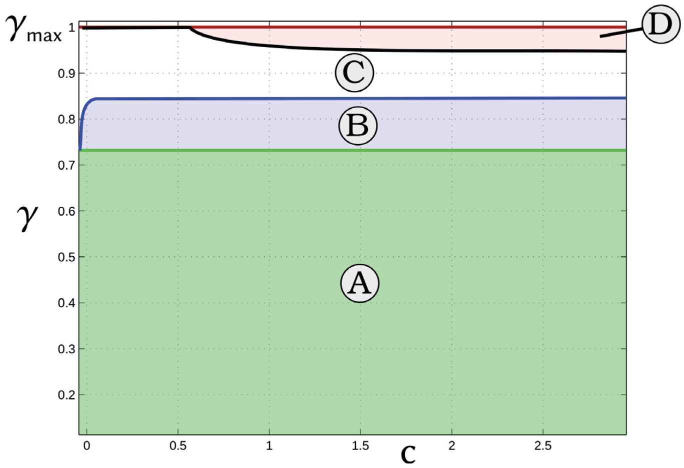


FIGURE 10 A – the region of permissible loads on uncontrolled synchronous machines; B – the region of permissible loads on controlled synchronous machines; C – the region which is not investigated analytically; D – the region of the existence circular solutions and the cycles of the second kind

on uncontrolled synchronous machines (the region A in Fig. 10) is obtained in the parameter space. From the conditions of theorem 5 we get the region of permissible loads on controlled synchronous machines (the regions A and B in Fig. 10). From the conditions of theorems 6 and 7 the region of the existence circular solutions and the cycles of the second kind (the region D in Fig. 10). Thus, the region C in Fig. 10 remains not investigated analytically.

Note that the region of the existence circular solutions and the cycles of the second kind decreases and the region of permissible loads increases as the parameter k increases. For any initial points in the phase space and any permissible parameters of the systems we can find enough big k to make the system stable.

4 NUMERICAL MODELING

In this chapter numerical modeling of differential equations (10), (12), (26) and (27), which describe the dynamics of synchronous machines under load conditions, is carried out. The standard computational tools of MATLAB (Shampine and Reichelt, 1997; Ashino et al., 2000) and a modified algorithm for modeling the systems with discontinuous right hand-sides, presented in (Piiroinen and Kuznetsov, 2008) are used.

We consider the following situation, when an idle synchronous machines operates in synchronism and at some instant the machine is put under load. Moreover, three cases are studied:

1. there are no any controls, i.e., $u(s) \equiv 0$;
2. a proportional control law of the form $u(s) = -ks$ is used (see Fig. 11, a). This control can be achieved by the direct torque control technique;

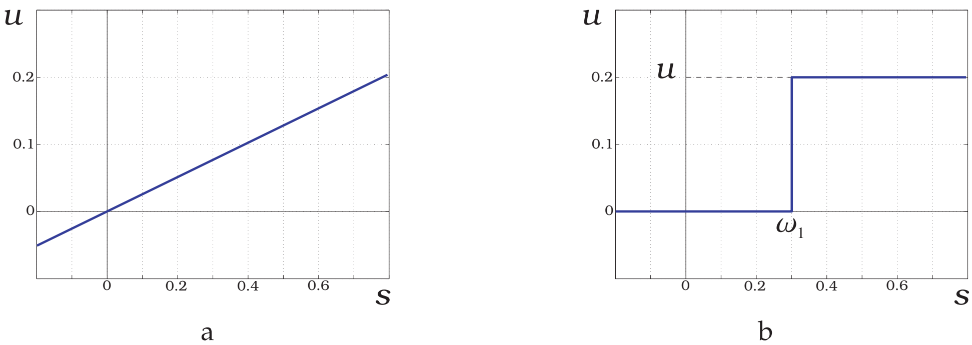


FIGURE 11 a – proportional control law; b – step control law.

3. a step control law of the form

$$u(s) = \begin{cases} 0, & \text{if} \quad s < n_1, \\ -T, & \text{if} \quad s \geq n_1. \end{cases}$$

is used (see Fig. 11, b). This control law describes a process of instantaneous torque change when the particular conditions are achieved (in our case when the given slip is achieved).

For all cases the numerical estimations of limit permissible load on synchronous machines are found. Moreover, the regions of the existence of the second kind limit cycles are plotted. Obtained results are compared with theoretical results.

4.1 The dynamics of uncontrolled synchronous machines

Let us recall that uncontrolled synchronous machines under load at series and parallel connections are described by systems (10) and (12), respectively. It was shown that after start-up the machines without load pull in synchronism, that is, the solutions of the systems tend to zero equilibrium points. It follows that the initial conditions is zero. The parameter γ is varied from 0 up to γ_{max}. If $\gamma > \gamma_{max}$, then the systems don't have equilibrium points, hence, the machines don't have operating modes.

Results of numerical modeling are presented in Fig. 12. For all parameters γ, taken from the regions 1 and 2, the trajectories of systems (10) and (12) tend to asymptotically stable equilibrium points (Figs. 13, 14), i.e., these loads on uncontrolled synchronous machines are permissible. In the same time, the trajectories of systems (10) and (12) for parameters γ, taken from the regions 3, tend to infinity (Figs. 15, 16), i.e., these loads are impermissible. Since in the case of parallel connection the region 2 is more than the region 2 in the series connection case, then uncontrolled synchronous machines at parallel connection are more stable to sudden changes of load than ones at series connection.

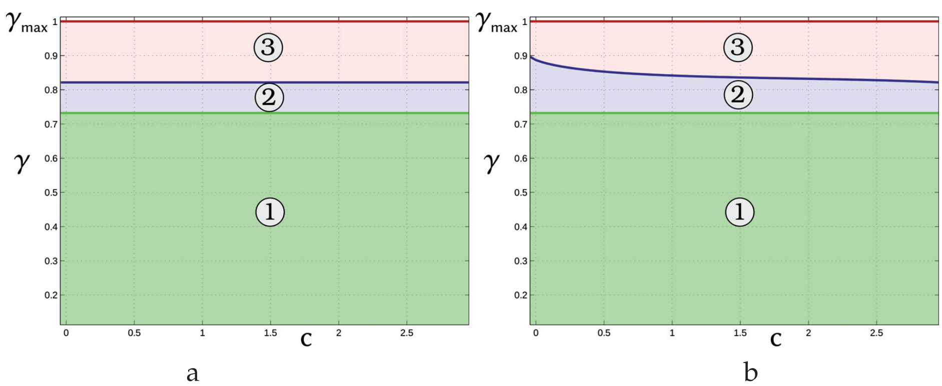
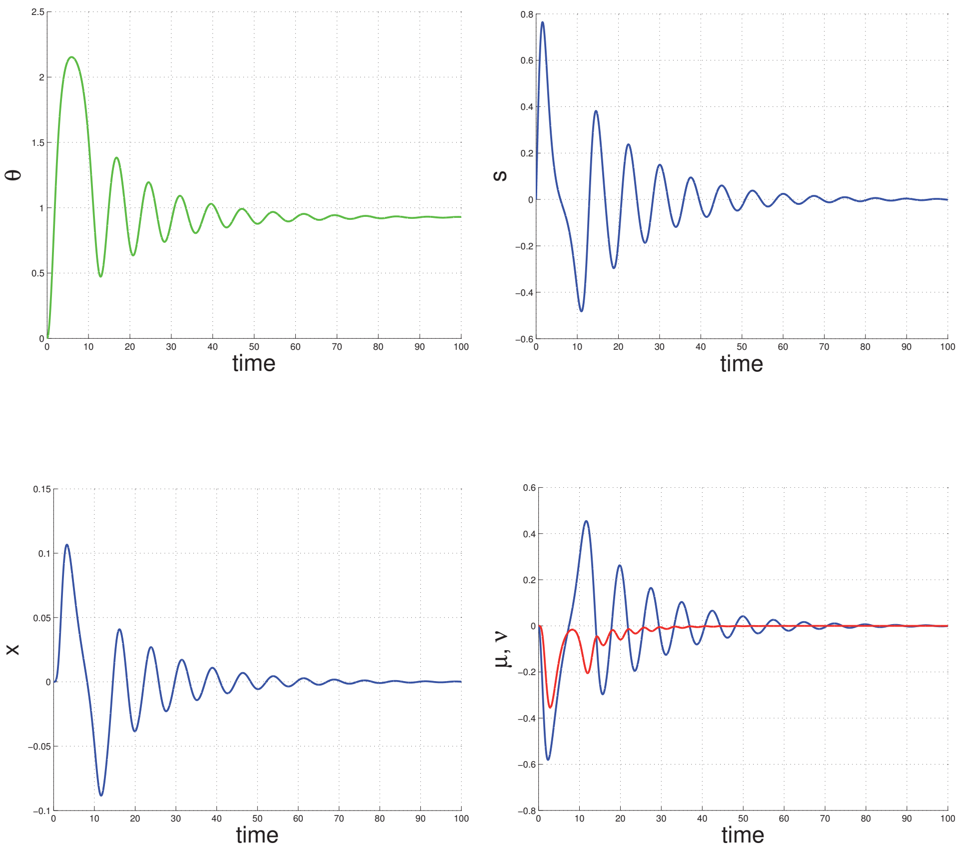


FIGURE 12 Parameter spaces of systems (10) (a) and (12) (b) without control: 1 – permissible loads, obtained by theorems; 2 – permissible loads, obtained numerically; 3 – impermissible loads

time space



phase space

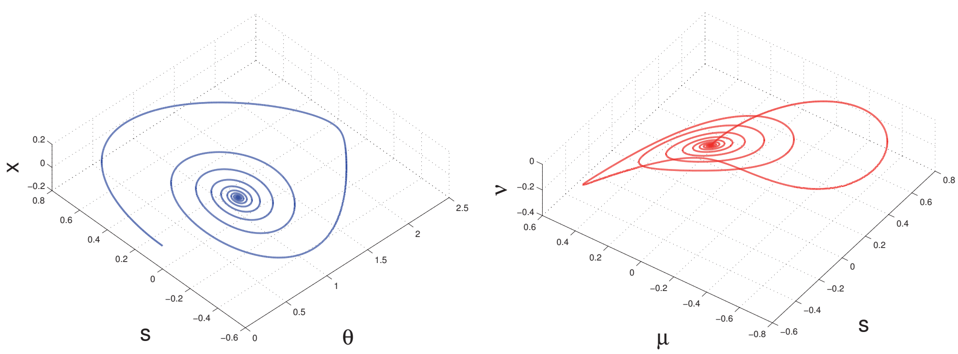
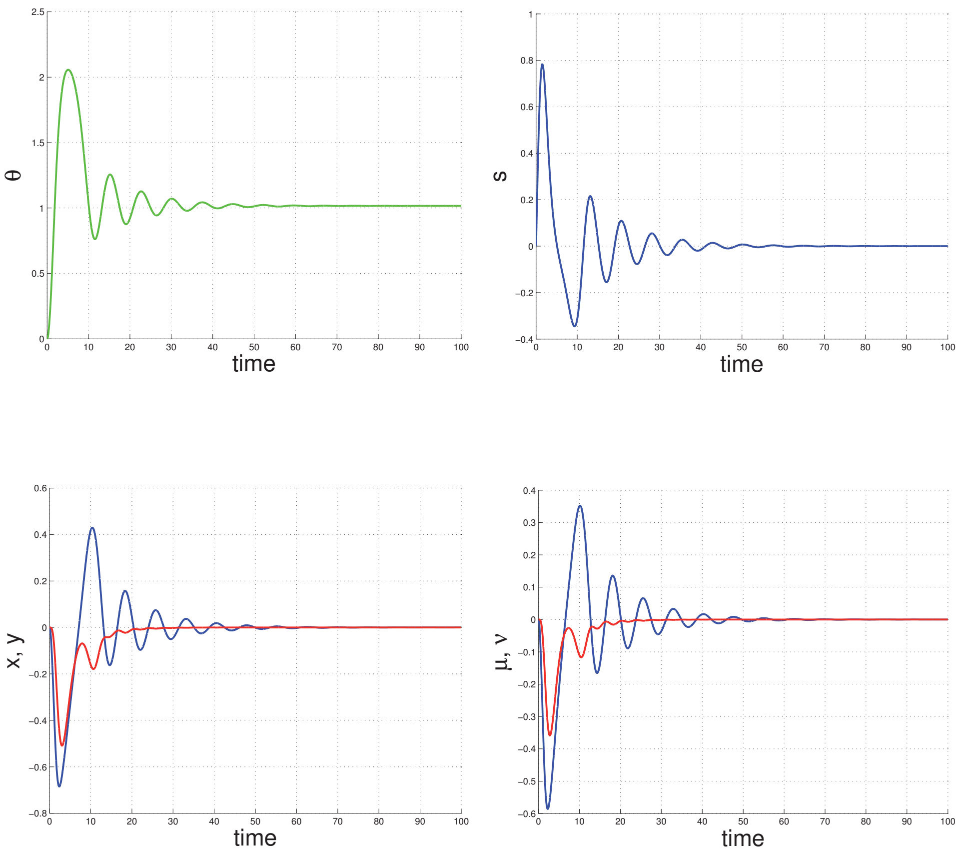


FIGURE 13 The trajectory of system (10) without control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.8$.

time space



phase space

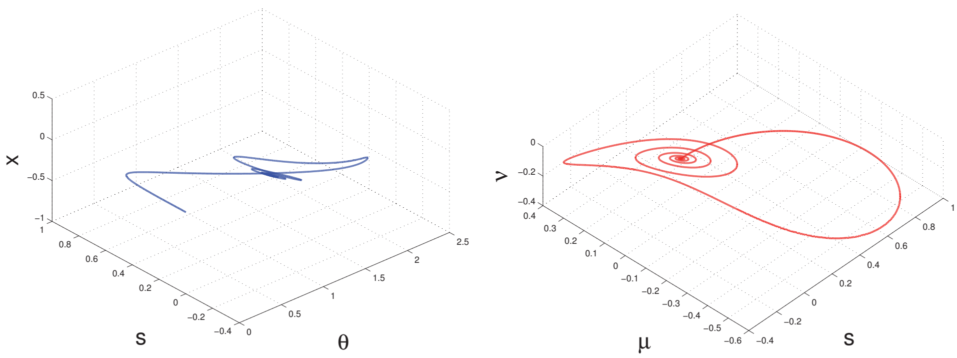


FIGURE 14 The trajectory of system (12) without control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.85$.

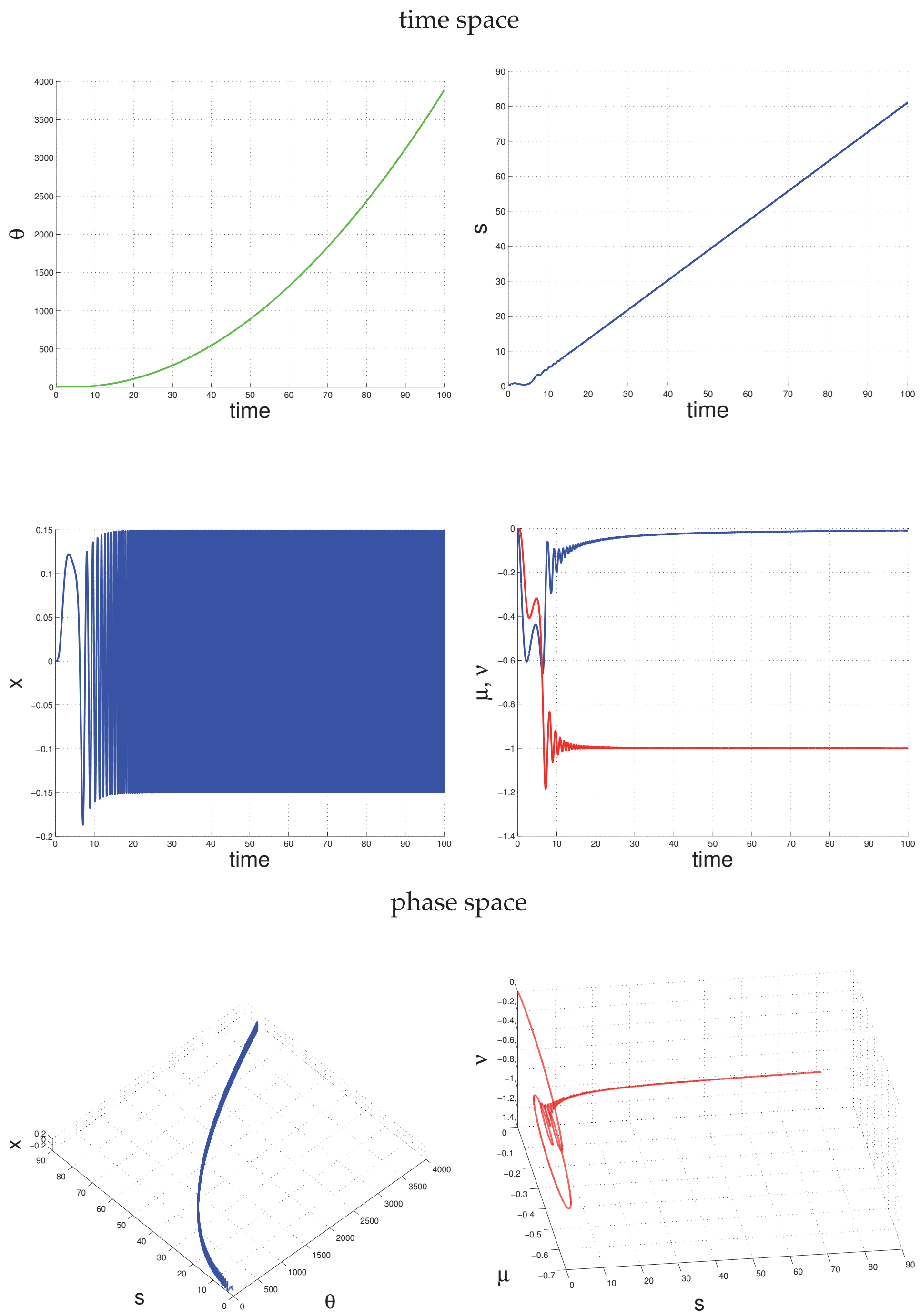


FIGURE 15 The trajectory of system (10) without control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.85$.

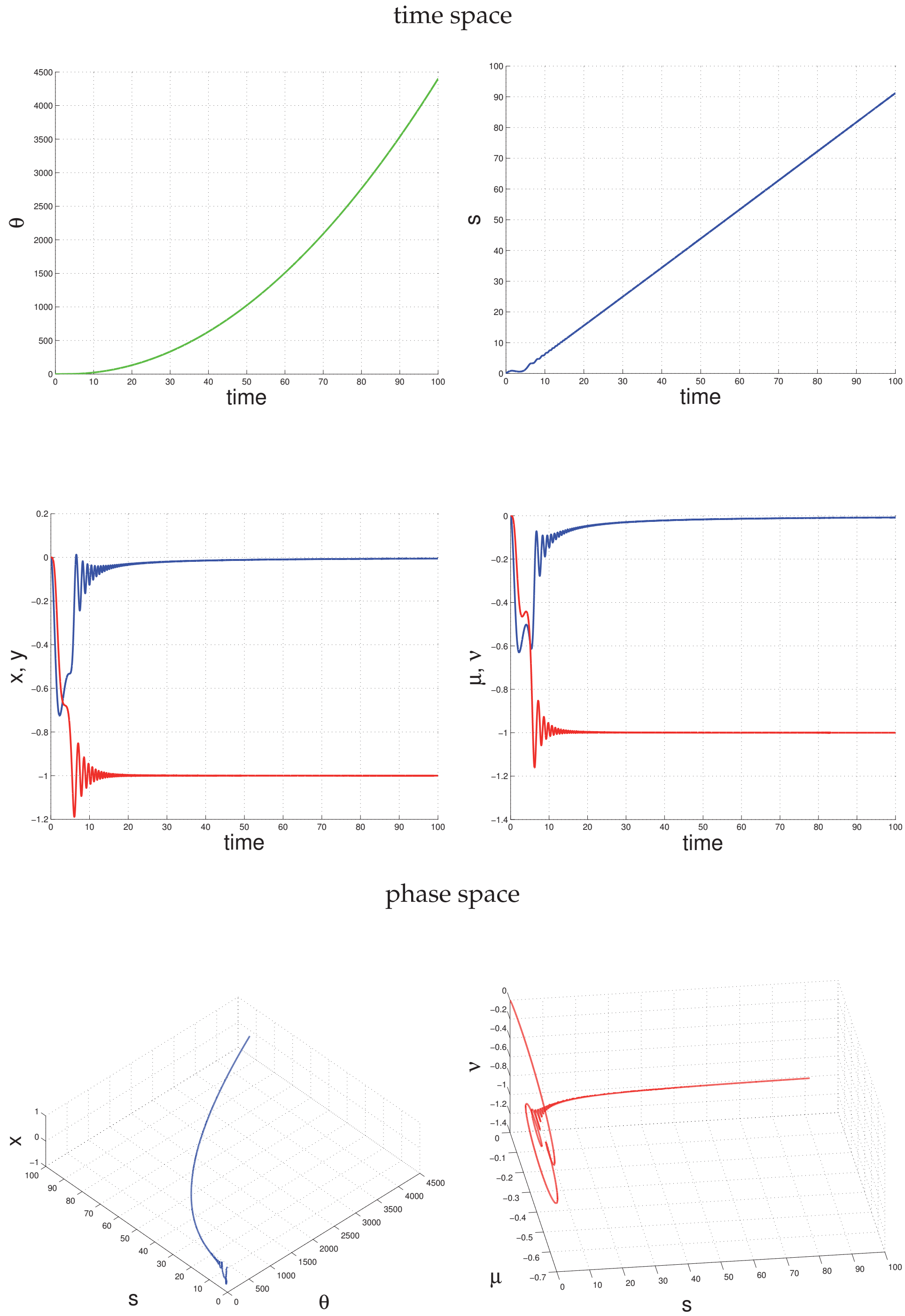


FIGURE 16 The trajectory of system (12) without control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.95$.

4.2 The dynamics of synchronous machines with proportional control

In this section we study the synchronous machines with proportional control under load conditions. Controlled synchronous machines under loads at series and parallel connections are described by systems (26) and (27), respectively. We assume that first synchronous machines operate without control and under no-load conditions, then at some instant simultaneously the load-on occurs and a controller is connected to the machine. Therefore, initial date are chosen zero. They correspond to the operating modes of idle synchronous machines. The parameter γ is varied from 0 up to γ_{max}. It is used the proportional control law $u(s) = 0.1s$.

Numerical results obtained in modeling systems (26) and (27) with proportional control law are shown in Fig. 17. The regions 1 and 2 correspond to permissible loads, since for all parameters γ from these regions the trajectories of systems (10) and (12) tend to asymptotically stable equilibrium points (Figs. 18, 19). The regions 3 correspond to impermissible loads, since the trajectories of systems (10) and (12) with parameters γ from these regions tend to the limit cycles of the second kind, in other words the regions 3 are the region of existence of limit cycles of the second kind (Figs. 20, 21). Thus, the estimations obtained by theorem 5 are improved by numerical modeling. From Fig. 17 it is obvious that in the context of permissible loads the parallel connection for synchronous machines with proportional control is more preferred than series connection.

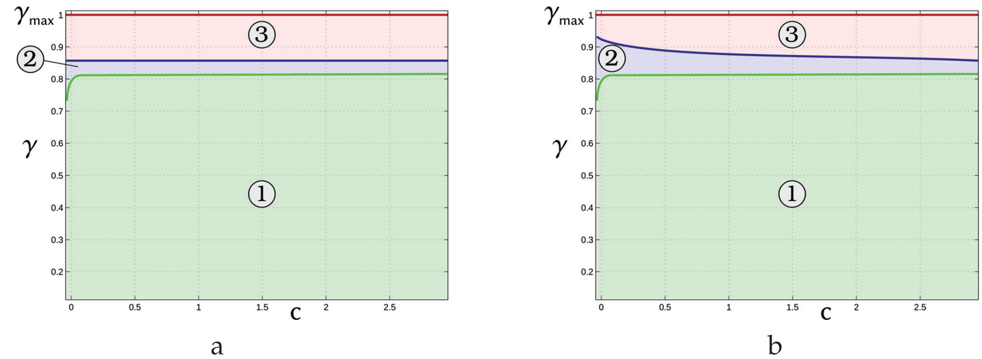
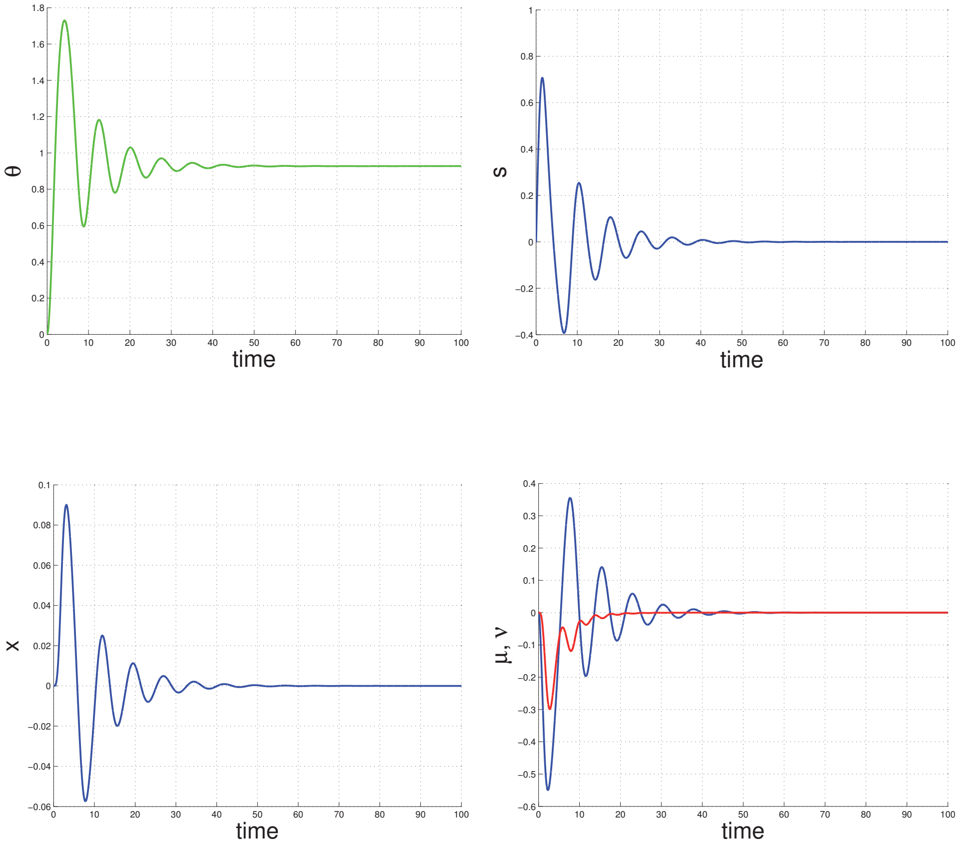


FIGURE 17 Parameter spaces of systems (26) (a) and (27) (b) with proportional control law: 1 – permissible loads, obtained by theorems; 2 – permissible loads, obtained numerically; 3 – impermissible loads

time space



phase space

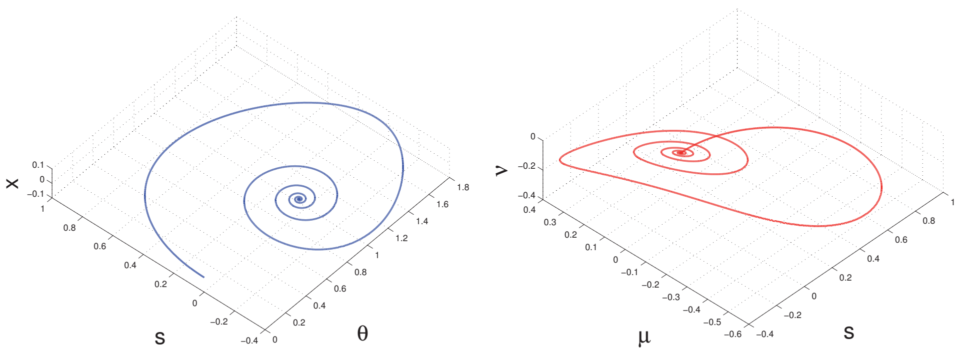


FIGURE 18 The trajectory of system (26) with proportional control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.8$.

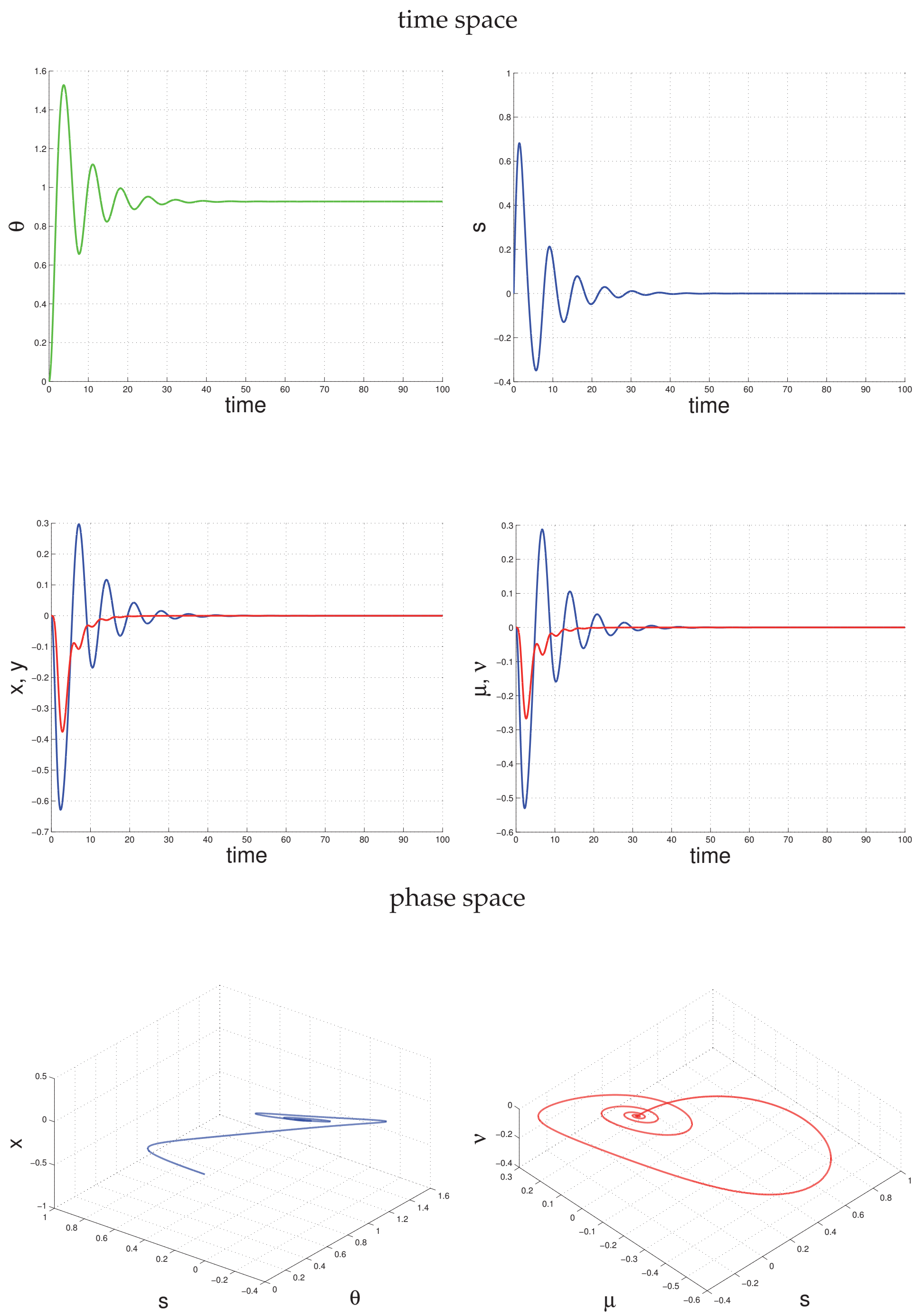


FIGURE 19 The trajectory of system (27) with proportional control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.8$.

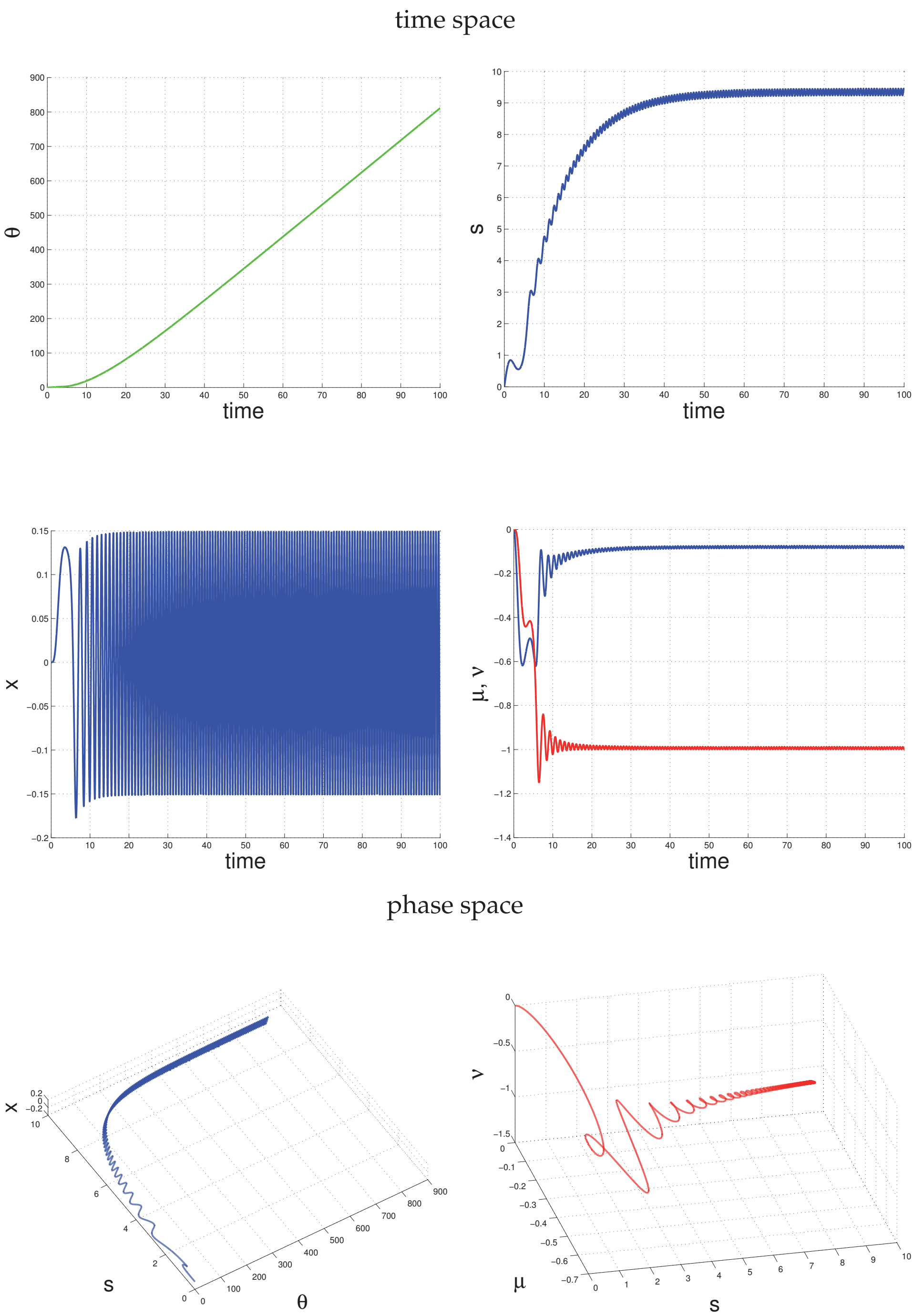


FIGURE 20 The trajectory of system (26) with proportional control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.95$.

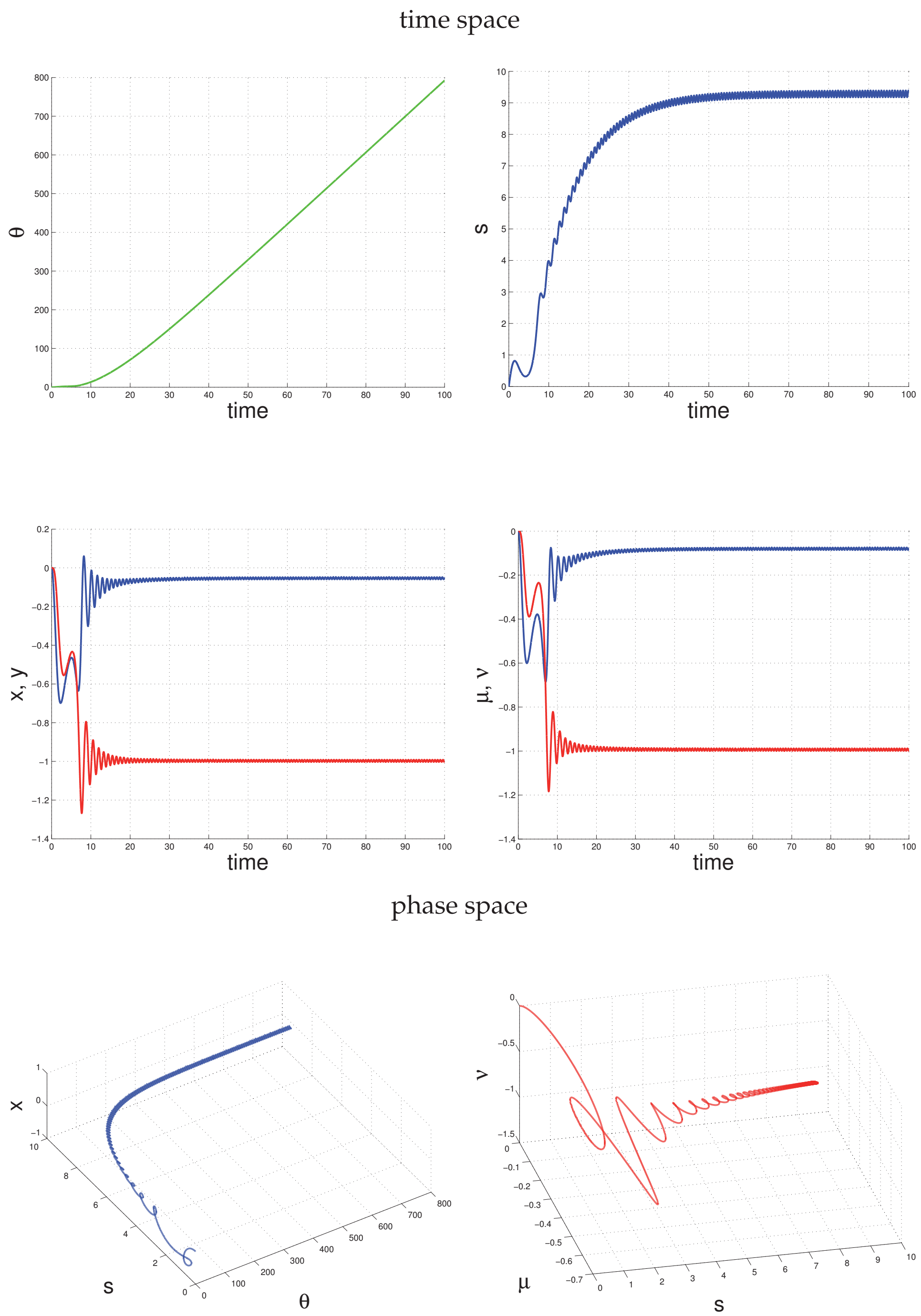


FIGURE 21 The trajectory of system (27) with proportional control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.95$.

4.3 The dynamics of synchronous machines with step control

In this section we consider the synchronous machines with step control under load conditions. Controlled synchronous machines under loads at series and parallel connections are described by systems (26) and (27), respectively. As before we assume that first synchronous machines operate without control and under no-load conditions, then at some instant simultaneously the load-on occurs and a controller is additionally connected to the machine. Initial date are zero. The parameter γ is varied from 0 up to γ_{max}. We use the step control law

$$u(s) = \begin{cases} 0, & \text{if} \quad s < n_1, \\ -0.1, & \text{if} \quad s \geq n_1. \end{cases}$$

Results of numerical modeling are presented in Fig. 22. For parameters γ from the regions 1, the trajectories of systems(26) and (27) with step control tend to asymptotically stable equilibrium points (Figs. 23, 24), i.e., these loads are permissible. In the same time, the trajectories of systems (10) and (12) with step control for parameters γ from the regions 2 tend to infinity (Figs. 25, 26), i.e., these loads are impermissible. As in previous two cases synchronous machines with step control at parallel connection are more stable to sudden changes of load than ones at series connection.

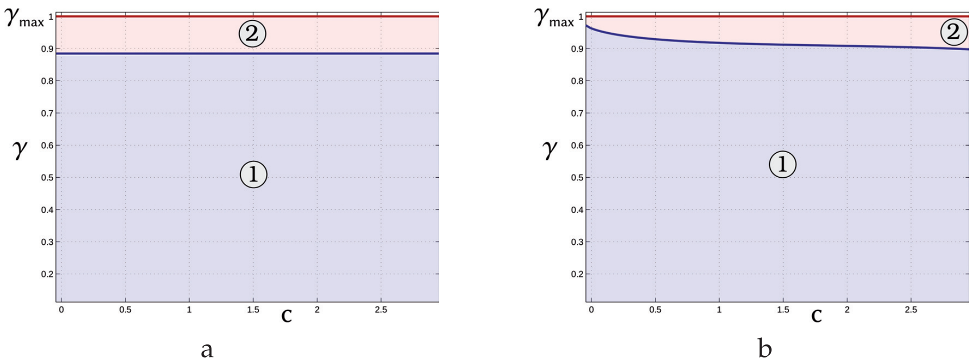
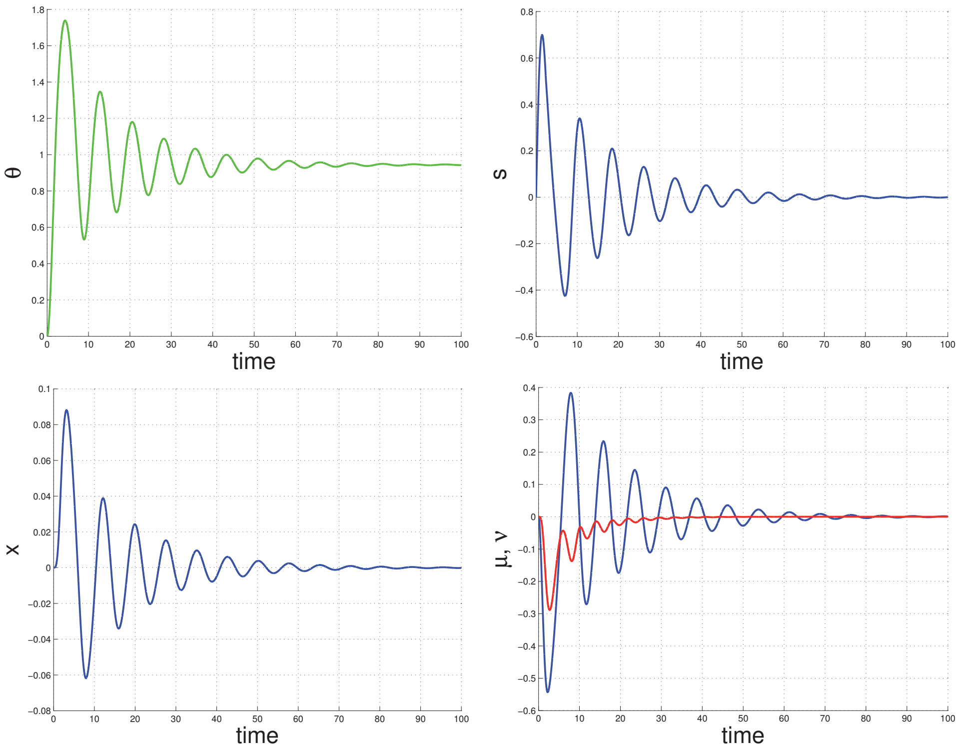


FIGURE 22 Parameter spaces of systems (26) (a) and (27) (b) with step control law: 1 – permissible loads; 2 – impermissible loads

time space



phase space

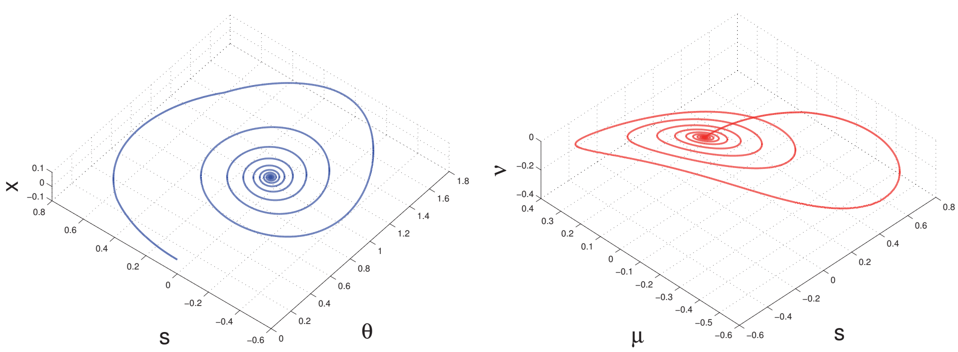
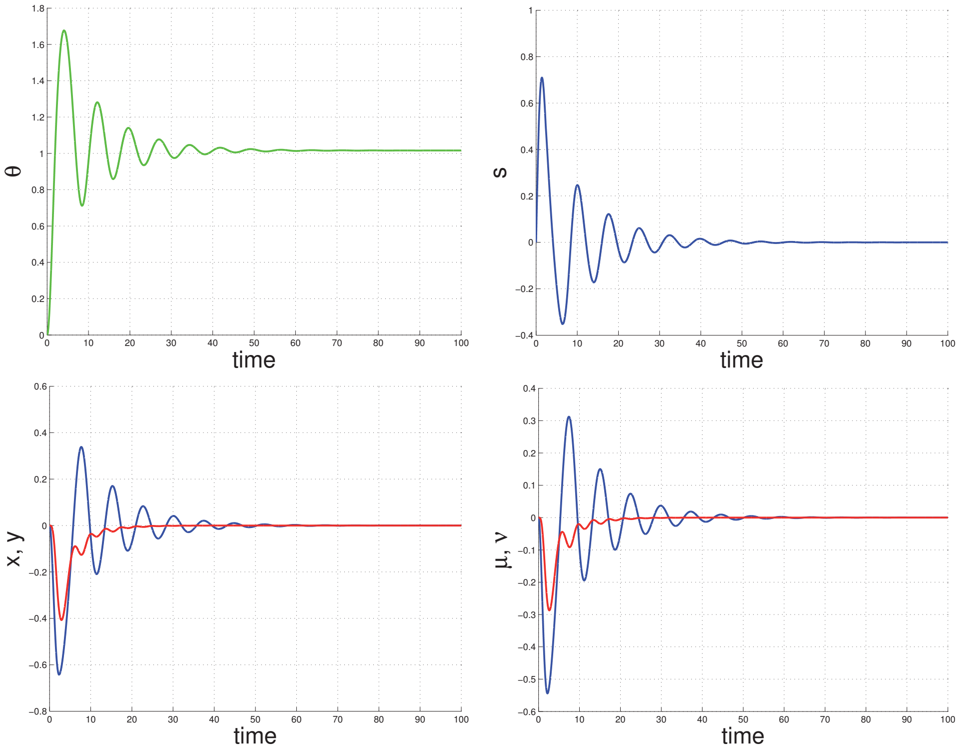


FIGURE 23 The trajectory of system (26) with step control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.81$.

time space



phase space

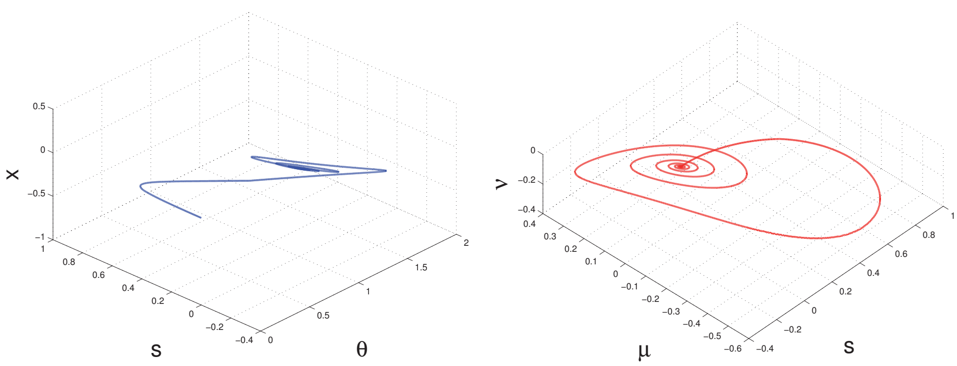


FIGURE 24 The trajectory of system (27) with step control. Permissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.85$.

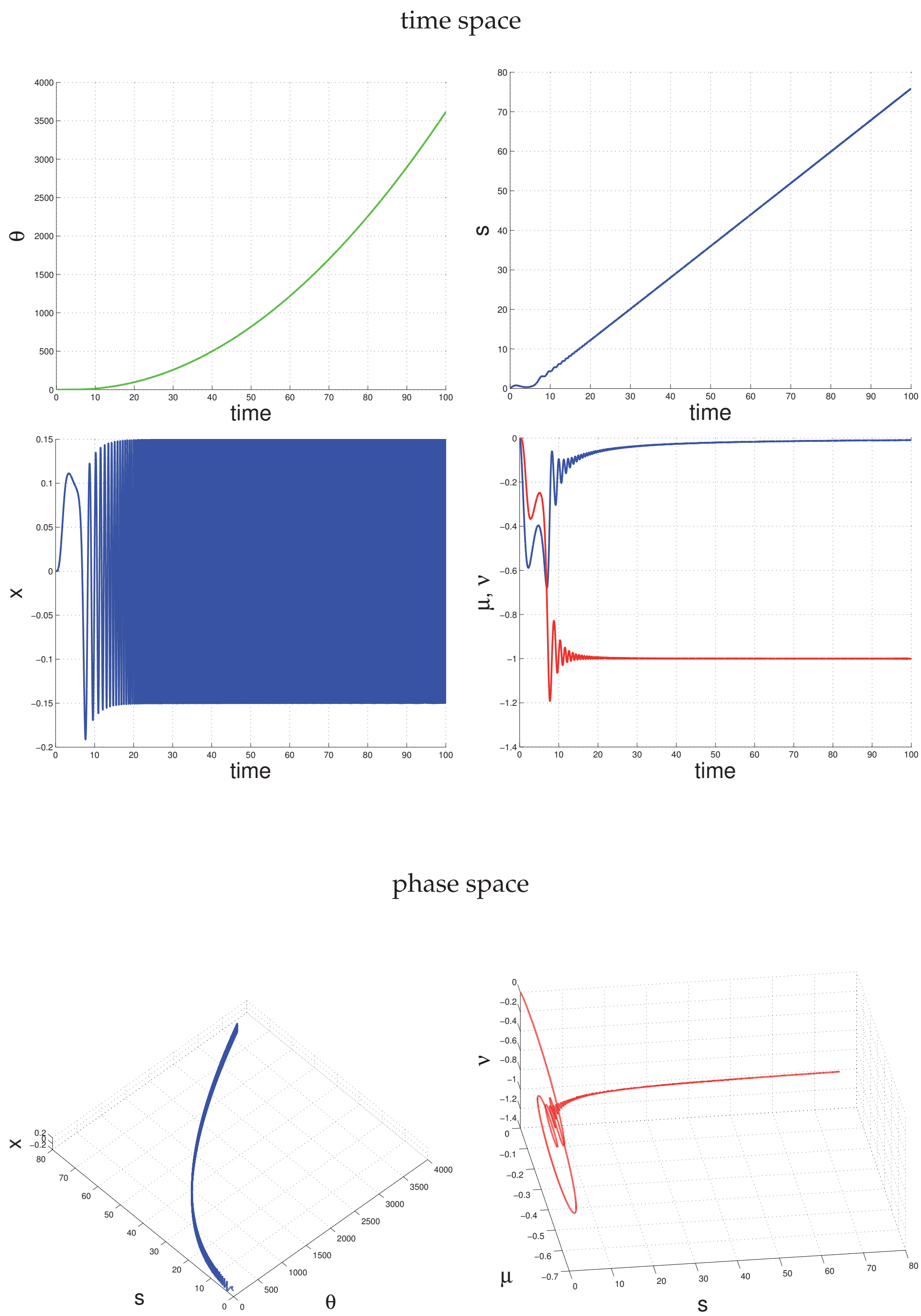


FIGURE 25 The trajectory of system (26) with step control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $d = 0.15$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.9$.

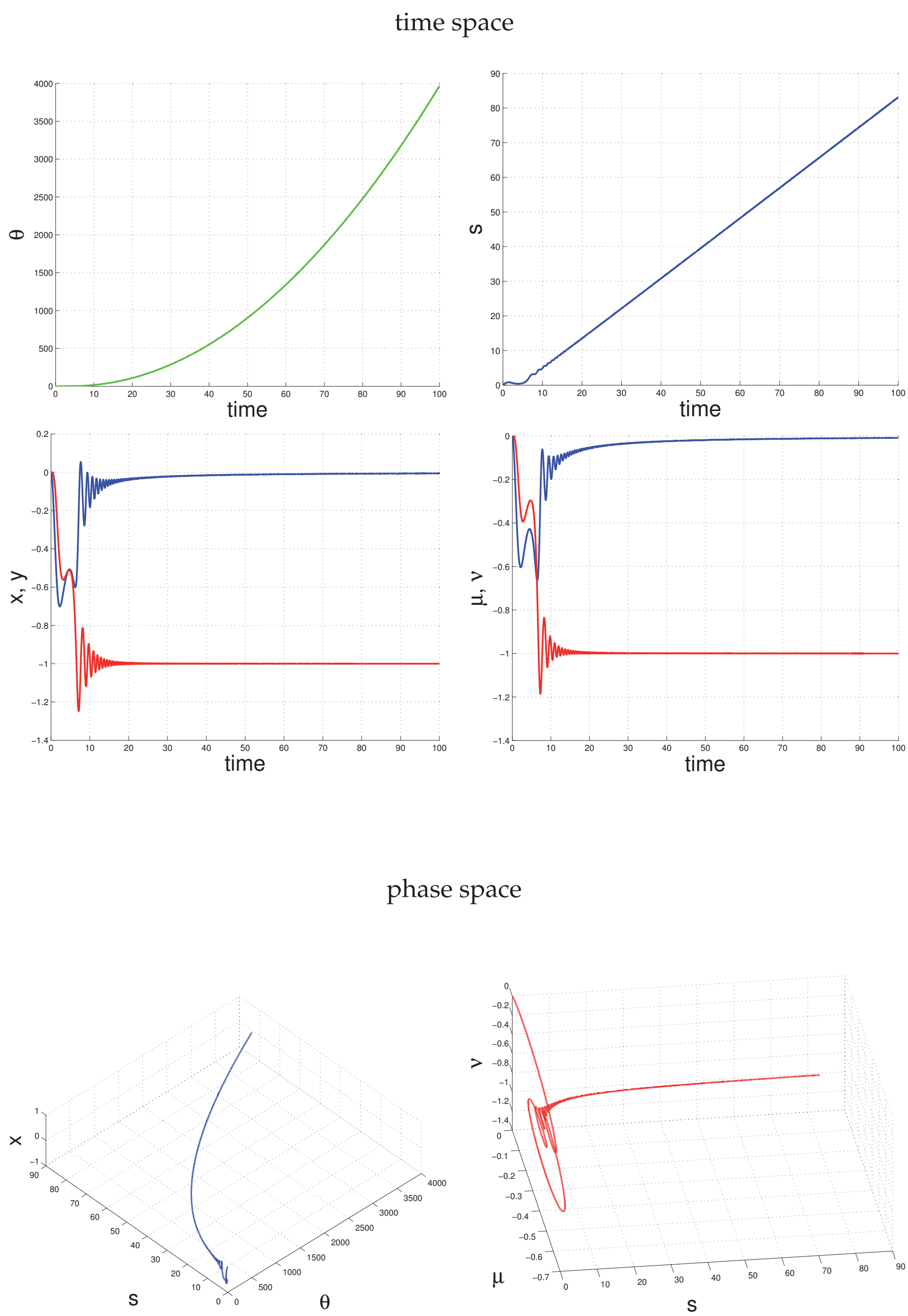


FIGURE 26 The trajectory of system (27) with step control. Impermissible load. Modeling parameters: $a = 0.1$, $b = 0.2$, $c = 0.5$, $c_1 = 0.75$, $\gamma_{max} = 1$, $\gamma = 0.975$.

5 CONCLUSIONS

Two mathematical models of four-pole rotor synchronous motors with damper windings at series and parallel connections were constructed based on laws of classical mechanics and electrodynamics and some simplifying assumptions. The steady-state stability analysis of these machines was carried out. Both models have asymptotically stable equilibrium points. These points correspond to the operating mode of the synchronous machines.

The dynamical stability of the idle synchronous machines is performed. It was shown that they are global stable under no-load conditions. Thus, independently on initial position of the rotors and currents in the rotor windings, the machines pull in synchronism.

The theorems about the estimates of the permissible limit load on uncontrolled and controlled synchronous machines were proved by the equal-area method and the modified non-local reduction method. Analytical results shown that the estimations of the permissible limit load for both models with equal parameters and different connection in exciting systems are equal. However, the numerical modeling gave us greater value of the permissible limit load in comparison with the value obtained by theorem. Moreover, it was shown that synchronous machines at parallel connection are more stable to sudden changes of load than ones at series connection.

In this thesis we consider the case when the synchronous machines are already running under different conditions, i.e., we do not study the start-up. Therefore, according to classical theoretical works (Adkins, 1957; White and Woodson, 1959; Skubov and Khodzhaev, 2008) the basic assumptions behind the modelling is permissible. On practical grounds we need engineering specialists to verify the obtained results. It is the next step of our work.

YHTEENVETO (FINNISH SUMMARY)

Tässä työssä tarkastellaan nelinapaisella roottorilla varustettujen kolmivaihesynkronikoneiden stabiilisuutta ja värähtelyjä syöttöjärjestelmien erilaisissa peräkkäis- ja rinnakkaiskytkentäsovelluksissa. Nykyisin näitä koneita käytetään yleisesti generaattoreina sähköenergian tuottamiseen sähkövoimaloissa ja energiajärjestelmissä.

Näiden koneiden tutkimista varten on kehitetty uusia matemaattisia malleja käyttäen oletusta tasaisesti pyörivästä magneettikentästä, jonka staattorin käämit tuottavat. Tämä oletus on lähtöisin N. Teslan ja G. Ferrarisin klassisista ideoista. Aikaansaaduissa malleissa on huomioitu täysin roottorin ulkomuodot erotuksena tunnetuista asynkronikoneiden matemaattisista malleista.

Työssä on muodostettu ehdot synkronikoneiden lokaalille ja globaalille stabiilisuudelle. Dynaamista stabiilisuutta on tarkasteltu huippukuormitustehtävän yhteydessä. Synkronikoneiden huippukuormitusta ilman säätöä on arvioitu käyttäen toista Ljapunovin menetelmää. Dynaamisen stabiilisuuden tehostamiseksi ehdotetaan välitöntä momentin säätöä (direct torque control). Ei-lokaalin menetelmän avulla on johdettu riittävät ehdot ympyränmuotoisten ratkaisujen ja toisen luokan rajasyklien olemassaololle. Saadut analyyttiset tulokset yleistävät Tricomin klassisisia tuloksia synkronikoneiden moniulotteisiin malleihin. Lisäksi on simuloitu numeerisesti synkronikoneiden eri kuormitukselle ilman säätöä, määräsuhdesäädöllä ja asteittaissäädöllä.

Mallien ja simulointien perusteella voidaan tehdä johtopäätöksiä eri roottorikytkentöjen paremmuudesta eri tilanteissa.

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APPENDIX 1 CYLINDRICAL PHASE SPACE

Systems of differential equations (10), (12), (26), (27), describing the dynamics of synchronous machines, contain angular coordinate which determines position of the rotor relative to the stator. For these equations following (Yakubovich et al., 2004; Leonov and Kondrat'eva, 2009) we introduce a cylindrical phase space. In some cases it turns out to be useful for qualitative investigation of obtained systems.

Let us consider the differential equation

$$\dot{y} = F(y), \quad y \in \mathbb{R}^n, \tag{37}$$

where $F(y)$ is a vector function defined in $\mathbb{R}^n$. Assume that for linearly independent vectors $d_j \in \mathbb{R}^n (j = 1, ..., m)$ the identity

$$F(y + d_j) = F(y), \quad \forall y \in \mathbb{R}^n. \tag{38}$$

is satisfied. The value $d_j^* / |d_j|$ is often called an angular or phase coordinate.

In order to introduce a cylindrical phase space we need some definitions from theory of groups (see, e.g., Robinson, 1996).

Definition 2. *Suppose that G is a nonempty set, $(\cdot)$ is an associative binary operation defined on the set G and the following conditions are fulfilled:*

1. *there exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for any $a \in G$. The element e is called the neutral element; called neutral,*
2. *for any $a \in G$ there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$. The element a^{-1} is called the inverse element for a.*

Then a set G with given the operation $(\cdot)$ on it is called a group.

Definition 3. *A group G is said to be abelian (commutative) if the operation is commutative, i.e.,*

$$a \cdot b = b \cdot a$$

for all $a, b \in G$.

In what follows, an operation given in a group is denoted by $(\cdot)$.

Definition 4. *A subset H of elements of a group G is called a subgroup of the group G if H is a group under the same operation defined in G.*

Definition 5. *A subgroup H of a group G is called a normal subgroup if*

$$a \in H, \quad b \in G, \quad \exists c \in G : b = c^{-1} \cdot a \cdot c \Longrightarrow b \in H.$$

Definition 6. *Let H be a subgroup of a group G and let a be some element of G. The set*

$$a\,H = \{a \cdot h \quad | \quad h \in H\}$$

is called a left coset of H in G generated by a. Similarly, the set

$$H\,a = \{h \cdot a \quad | \quad h \in H\}$$

is called a right coset of H in G generated by a.

It is easy to show that if H is a normal subgroup, its left cosets and right cosets coincide (Robinson, 1996; Leonov and Kondrat'eva, 2009).

Definition 7. *A group formed by the set of all cosets of a normal subgroup H in a group G is called a factor group of the group G by the normal subgroup H and is denoted by G/H.*

The definition of an abelian group implies every subgroup of an abelian group is normal. This property of the abelian group is true, since in such group $a \cdot c = c \cdot a$ for any elements a, b from the group. Hence, a factor group can be constructed by any subgroup of the abelian group.

The vector space $\mathbb{R}^n$ form an abelian group with respect to an operation which is always called "addition" and denoted by $+$. Let us introduce a discrete subgroup of this group

$$H = \left\{ y = kd_\vartheta \;\middle|\; k \in \mathbb{Z} \right\}.$$

Consider the cosets of a subgroup H in a group $\mathbb{R}^n$, defined in the following way

$$[y] = \{y + h | \forall h \in H, y \in \mathbb{R}^n\}.$$

They form a factor group $\mathbb{R}^n/H$.

Let us define the so-called flat metric in the space $\mathbb{R}^n/H$

$$\rho([x],[y]) = \inf_{u \in [x], v \in [y]} |u - v|. \tag{39}$$

Here $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$.

Proposition 1. *(Leonov and Kondrat'eva, 2009) If $y(t)$ is a solution of differential equation (37) defined on the interval (t_1, t_2), then $y(t) + kd_j$ for any integer number k and for d_j satisfying* (38) *is also a solution of equation (37) on the interval (t_1, t_2).*

From proposition 1 it follows that the metric space $\mathbb{R}^n/H$ introduced above is the phase space of system (37). It means that the space $\mathbb{R}^n/H$ can be partitioned into disjoint trajectories $[y(t)]$ of system (37).

The space $\mathbb{R}^n/H$ is called a cylindrical phase space for the systems of the form (37), since it is diffeomorphic to the surface of the cylinder

$$\underbrace{C \times C \times \ldots \times C}_{m} \times \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n-m}$$

with C being a circle.

Now we define the notions of a circular solution and a limit cycle of the second kind which are used in this thesis.

Definition 8. *A solution $y(t)$ of differential equations* (37) *is said to be circular if there exists the numbers $\varepsilon > 0$ and τ such that the following inequality holds*

$$d_j^* \dot{y}(t) \geq \varepsilon \qquad \forall t \geq \tau.$$

Definition 9. *A solution $y(t)$ of differential equations* (37) *is said to be a limit cycle of the second kind if there exists the numbers $T > 0$ and $k \in \mathbb{Z}$, $k \neq 0$ such that the following relation holds*

$$y(t+T) = y(t) + kd_j \qquad \forall t \in \mathbb{R}.$$

The concept of the limit cycle of the second kind is clear in a cylindrical phase space. This cycle represents a trajectory which is closed in the cylindrical phase space, but ceases to be closed if we turn to the phase space $\mathbb{R}^n$. Unlike the usual cycle (the cycle of the first kind), this trajectory in the cylindrical phase space can not homotopically contracted to a point.

APPENDIX 2 PROOF OF THEOREMS

To prove the main theorems we need some definitions and lemma 1.

Definition 10. *A set $M \in \mathbb{R}^n$ is called a positively invariant set of the autonomous system*

$$\dot{y} = f(y), \qquad y \in \mathbb{R}^n \tag{40}$$

if for any point $p \in M$ it follows that

$$y(t, p) \in M \qquad \forall t \geq 0.$$

Definition 11. *System* (40) *is said to be dichotomic if any solution bounded on $[t_0, +\infty)$ tends to stationary set as $t \to +\infty$.*

Lemma 1. *Suppose that the function $u(t)$, continuously differentiable on $[0, +\infty)$, satisfies the following conditions:*

1. *there exists a constant C such that*

$$|\dot{u}(t)| \leq C, \qquad \forall t \geq 0,$$

2. *$u(t) \geq 0, \forall t \geq 0$,*
3. *$\int\limits_0^{+\infty} u(t)dt < +\infty$.*

Then $\lim\limits_{t\to\infty} u(t) = 0$.

The proof of this lemma can be found in (Leonov, 2006b; Leonov and Kondrat'eva, 2009)

In this appendix we consider system (27) with $u(s) = -ks$.

Theorem 8. *Any solution of system (27) bounded for $t \geq 0$ tends to some equilibrium point as $t \to +\infty$, i.e., system* (27) *is dichotomic.*

Proof. Let $\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)$ be a solution of system (27) bounded for $t \geq 0$. Let us consider the function

$$V(\vartheta, s, x, y, \mu, \nu) = \frac{1}{2}s^2 + \frac{a}{2}x^2 + \frac{a}{2}y^2 + \frac{b}{2}\mu^2 + \frac{b}{2}\nu^2 + \int\limits_0^{\vartheta} \varphi(\zeta)d\zeta. \tag{41}$$

For any solution of system (27) the following relation holds

$$\dot{V}(\vartheta, s, x, y, \mu, \nu) = -ks^2 - acx^2 - acy^2 - bc_1\mu^2 - bc_1\nu^2 \leq 0. \tag{42}$$

Hence, the function $V(\vartheta, s, x, y, \mu, \nu)$ on the solutions of system (27) does not increases with respect to T. The boundedness of the solution implies the boundedness of the function V for $t \geq 0$. Thus, there exists a finite limit

$$\lim_{t\to+\infty} V\left(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)\right) = \text{const.} \tag{43}$$

Consider the function

$$u(t) = ks(t)^2 + acx(t)^2 + acy(t)^2 + bc_1\mu(t)^2 + bc_1\nu(t)^2$$

and apply lemma 1 to this function. Obviously, condition 1 of lemma 1 is fulfilled. Taking into account (42), we obtain

$$\int_0^t u(\tau)\mathrm{d}\tau = V(\vartheta(0), s(0), x(0), y(0), \mu(0), \nu(0)) - V(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)).$$

Consequently, from (43) we get

$$\int_0^{+\infty} u(t)dt < \infty,$$

that is, condition 1 of lemma 1 is satisfied. Moreover, due to the boundedness of the solution of system (27), the following relation is valid

$$|\dot{u}| = |-2\left(ks(-ks + ay + b\nu - \varphi(\vartheta)) + acx(-cx + ys) + \right.$$

$$+acy(-cy - xs - s) + bc_1\mu(-c_1\mu + \nu s) +$$

$$\left. +bc_1\nu(-c_1\nu - \mu s - s)\right)| \leq C,$$

that is, condition 3 of lemma 1 is fulfilled. Finally, the application of lemma 1 yield

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty}\left(ks(t)^2 + acx(t)^2 + acy(t)^2 + bc_1\mu(t)^2 + bc_1\nu(t)^2\right) = 0$$

From the last relation it follows that

$$\lim_{t\to+\infty} s(t) = 0,$$

$$\lim_{t\to+\infty} x(t) = 0, \qquad \lim_{t\to+\infty} y(t) = 0, \tag{44}$$

$$\lim_{t\to+\infty} \mu(t) = 0, \qquad \lim_{t\to+\infty} \mu(t) = 0.$$

Hence, taking account equality (43) and expression for the function $V(\vartheta, s, x, y, \mu, \nu)$, we have

$$\lim_{t\to+\infty}\int_0^t \varphi(\vartheta(\tau))\mathrm{d}\tau = \text{const}.$$

Then the periodicity of the function $\varphi(\vartheta)$ implies the existence of some number ϑ_0 such that

$$\lim_{t\to+\infty}\vartheta(t) = \vartheta_0.$$

Thus, the point $\vartheta = \vartheta_0, s = 0, x = 0, y = 0, \mu = 0, \nu = 0$ is equilibrium point of system 27. This completes the proof of the theorem. □

Theorem 5. *Suppose that there exists a number $\lambda \in \mathbb{R}$ such that the following conditions hold*

1. $0 < \lambda < \min\{k, c, c_1\}$;
2. *the solution of the differential equation*

$$F\frac{\mathrm{d}F}{\mathrm{d}\sigma} = -2\sqrt{\lambda(k-\lambda)}\, F - \varphi(\sigma). \tag{28}$$

with initial data

$$F(\vartheta_1) = 0,$$

satisfies the condition

$$F(0) > 0. \tag{29}$$

Then the solution of system (26) *with initial data* $\vartheta = s = x = \mu = \nu = 0$ *satisfies the relations* (21) *and the solution of system* (27) *with initial data* $\vartheta = s = x = y = \mu = \nu = 0$ *satisfies the relations* (22).

Proof. The proof of the theorem for system (26) is given in PIV. Here we prove the theorem for system (27).

Introduce the function

$$V(\vartheta, s, x, y, \mu, \nu) = \frac{1}{2}\left(s^2 + a(x^2+y^2) + b(\mu^2+\nu^2) - F^2(\vartheta)\right).$$

Taking into account (28), for any solution of system (27) the following relation is satisfied

$$\dot{V}(\vartheta, s, x, y, \mu, \nu) + 2\lambda V(\vartheta, s, x, y, \mu, \nu) = s(-ks + ay + b\nu - \varphi(\vartheta)) + ax(-cx + ys) +$$

$$+ay(-cy - xs - s) + b\mu(-c_1\mu + \nu s) + b\nu(-c_1\nu - \mu s - s) - s\left[F\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right](\vartheta) + \lambda s^2 +$$

$$+\lambda a x^2 + \lambda a y^2 + \lambda b \mu^2 + \lambda b \nu^2 - \lambda F^2(\vartheta) = -a(c-\lambda)y^2 - a(c-\lambda)x^2 -$$

$$-b(c_1-\lambda)\mu^2 - b(c_1-\lambda)\nu^2 - (k-\lambda)s^2 + \left(-\varphi(\vartheta) - \left[F\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right](\vartheta)\right)s -$$

$$-\lambda F^2(\vartheta) \le -(k-\lambda)s^2 - 2\sqrt{\lambda(k-\lambda)}\, s\,F(\vartheta) - \lambda F^2(\vartheta) =$$

$$= -(\sqrt{k-\lambda}\, s + \sqrt{\lambda}\, F(\vartheta))^2 \le 0.$$

Hence, on the boundary of the set

$$\Omega_0 = \left\{V(\vartheta, s, x_1, y_1, x_2, y_2) \le 0\right\}$$

the relation

$$\dot{V}(\vartheta, s, x, y, \mu, \nu) \le 0$$

is fulfilled. Thus, Ω_0 is the positively invariant set. Using the 2π-periodicity of the function $\varphi(\vartheta)$, it can be similarly shown the positive invariance of the sets

$$\Omega_k = \left\{s^2 + a(x^2+y^2) + b(\mu^2+\nu^2) - F^2(\vartheta + 2k\pi) \le 0\right\}, \quad \forall k \in \mathbb{Z}.$$

According to condition (29), differential equation (28) has

1. either a solution F such that there exists a point $\vartheta_2 < 0$ for which

$$F(\vartheta_2) = F(\vartheta_1) = 0, \qquad F(\sigma) > 0, \quad \forall \sigma \in (\vartheta_2, \vartheta_1);$$

2. or a solution satisfying the inequality

$$F(\sigma) > 0, \forall \sigma \in (-\infty, \vartheta_1).$$

In the first case the positively invariant set Ω_0 is bounded. In the second case the set $\Omega = \Omega_1 \cap \Omega_0$ is bounded. Obviously, the set Ω is also positively invariant, since it is the intersection of the positively invariant sets.

Let us show that if the conditions of the theorem are satisfied then the sets Ω and Ω_0 contain the initial data $\vartheta = s = x = y = \mu = \nu = 0$ and the equilibrium point $\vartheta = \vartheta_0, s = x_1 = y_1 = x_2 = y_2 = 0$ of system (27). Since $\vartheta_0 \in (0, \vartheta_1)$ and $F(\sigma) > 0$ for all $\sigma \in (0, \vartheta_1)$, then

$$\begin{pmatrix} \vartheta_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \Omega_0, \qquad \begin{pmatrix} \vartheta_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \Omega. \tag{45}$$

From the invariance of equation (28) with respect to the shift by $2\pi k$, $k \in \mathbb{Z}$ and condition (29) it follows that

$$-F(0 + 2k\pi) = -F(0) < 0, \qquad \forall k \in \mathbb{Z}.$$

Thus, we get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \Omega_0, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \Omega. \tag{46}$$

In theorem 8 was proved that system (27) is dychotomic. This fact, the boundedness and positive invariance of the sets Ω and Ω_0 and inclusions (45) and (46) imply (22). □

Theorem 7. *Suppose that there exists a number $\lambda \in \mathbb{R}$ such that the following conditions hold*

1. *$0 < \lambda < \min\{c, c_1\}$ and*

$$\lambda - k - \frac{(a+1)^2}{4(c-\lambda)} - \frac{(b+1)^2}{4(c_1-\lambda)} \geq 0; \tag{33}$$

2. *the solution $F(\sigma)$ of the equation*

$$F\frac{\mathrm{d}F}{\mathrm{d}\sigma} = -\lambda F - \varphi(\sigma), \tag{34}$$

with initial data $F(\vartheta_{init}) = 0$ satisfies the condition

$$\inf F(\sigma) > 0 \qquad \forall \sigma > \vartheta_{init}. \tag{35}$$

Then for any $\varepsilon > 0$ system (27) *has a circular solution $(\vartheta, s, x, y, \mu, \nu)$ with initial data $(\vartheta(0) = \vartheta_{init}, s(0), x(0), y(0), \mu(0), \nu(0))$, satisfying the conditions*

$$s(0) > 0, \qquad |s(0)| + |x(0)| + |y(0)| + |\mu(0)| + |\nu(0)| < \varepsilon. \tag{36}$$

Moreover, if $k > 0$, then system (27) *has at least one limit cycle of the second kind.*

Proof. Introduce the function

$$U(\vartheta, s, x, y, \mu, \nu) = \frac{1}{2}\Big(F^2(\vartheta) - s^2 + x^2 + y^2 + \mu^2 + \nu^2\Big).$$

Note that for $\vartheta = \vartheta_{\text{init}}$ there exists a vector $s_{\text{init}}, x_{\text{init}}, y_{\text{init}}, \mu_{\text{init}}, \nu_{\text{init}}$ satisfying condition (36) for which

$$U(\vartheta_{\text{init}}, s_{\text{init}}, x_{\text{init}}, y_{\text{init}}, \mu_{\text{init}}, \nu_{\text{init}}) < 0.$$

From the continuity $U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t))$ it follows that on some interval $[0, T)$ we have $U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0$. Then

$$0 < -\frac{1}{2}F^2(\vartheta(t)) + \frac{1}{2}s^2(t), \qquad \forall t \in [0, T).$$

This inequality and conditions (35) and (36) imply

$$F(\vartheta(t)) < s(t), \qquad \forall t \in [0, T). \tag{47}$$

Let us show that the function

$$U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) + \int_0^t \Big[2\lambda U(\vartheta(\tau), s(\tau), x(\tau)) + \delta(\tau)\Big] d\tau \tag{48}$$

is a nonincreasing function of t on the interval $[0, T)$, where

$$\delta(t) = \frac{1}{2}\Big(\lambda - k - \frac{(a+1)^2}{4(c-\lambda)} - \frac{(b+1)^2}{4(c_1-\lambda)}\Big)s^2(t).$$

It follows from condition (33) of the theorem that the function $\delta(t)$ is the nonnegative function. Using (33), (47) and F being a solution of equation (34), for any solution of system (27) we obtain

$$\dot{U} + 2\lambda U + \delta = -(c_2 - \lambda)z^2 - \left(\sqrt{c_1 - \lambda}x - \tfrac{a+d}{2\sqrt{c_1-\lambda}}s\right)^2 +$$

$$+\varphi(\vartheta)s + F\frac{dF}{d\vartheta}s + \lambda F^2(\vartheta) \leq \left(F\frac{dF}{d\vartheta} + \lambda F + \varphi(\vartheta)\right)s = 0.$$

The last inequality implies that the function (48) is nonincreasing on the interval $[0, T)$.

Now we show that $U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0$ on the interval $[0, \infty)$.

Let $V(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0$ on the interval $[0, T)$. We prove that there does not exist T such that

$$U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T)) = 0. \tag{49}$$

Suppose by contradiction that $U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T)) = 0$. Then there exists $T_1 < T$ such that

$$\delta(t) > |2\lambda U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T))| \qquad \forall t \in (T_1, T).$$

Since the function (48) does not increase on the interval $[0, T)$, then the following relation holds

$$U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T)) - U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) +$$
$$\int_t^T \Big[2\lambda U(\vartheta(\tau), s(\tau), x(\tau), y(\tau), \mu(\tau), \nu(\tau)) + \delta(\tau)\Big] d\tau \leq 0 \qquad \forall t \in (T_1, T).$$

From the last two inequalities we have

$$U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T)) < U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t))$$

on the interval (T_1, T) and, hence,

$$U(\vartheta(T), s(T), x(T), y(T), \mu(T), \nu(T)) < 0.$$

This contradicts (49). Thus,

$$U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0, \qquad t \geq 0,$$

and therefore,

$$F(\vartheta(t)) < s(t) \qquad \forall t \geq 0. \tag{50}$$

Thus, there exists a solution of system (27) with initial data $\vartheta(0) = \vartheta_{\text{init}}$, $s(0)$, $x(0)$, $y(0)$, $\mu(0)$, $\nu(0)$, which satisfy (36) and $U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0$ for $t \geq 0$. Consequently, for estimates (50) and (35) we can conclude that such solution is circular.

Now we prove the existence of the cycles of the second kind under the condition $k > 0$.

Consider the set

$$\Omega = \left\{ (\vartheta, s, x, y, \mu, \nu) \middle| U(\vartheta, s, x, y, \mu, \nu) < 0, s > 0, \vartheta \geq \vartheta(0) \right\}.$$

Since for any solution of system (27) with initial data from Ω the inequality $U(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) < 0$ is fulfilled for all $t \geq 0$, then it follows the positive invariance of the set Ω. Due to continuous dependence of the solutions of system (27) on the initial data, the closure $\overline{\Omega}$ is also positively invariant.

Let $\delta < \min\{k, c, c_1\}$. Consider the function

$$W(\vartheta, s, x, y, \mu, \nu) = \frac{1}{2}s^2 + \frac{a}{2d}x^2 - w,$$

where $w > \delta^{-1} \max |\varphi(\vartheta)|$. The boundedness the function $\varphi(\vartheta)$ implies $\nu < +\infty$. For any solution of system (27) we have

$$\begin{aligned} &\dot{W}(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) + \\ &\quad + \delta W(\vartheta(t), s(t), x(t), y(t), \mu(t), \nu(t)) \leq 0, \qquad \forall t \geq 0. \end{aligned}$$

Hence, the set

$$\Sigma = \left\{ (\vartheta, s, x, y, \mu, \nu) \quad \middle| \quad W(\vartheta, s, x, y, \mu, \nu) \leq 0 \right\}.$$

is positively invariant.

The continuity of $F(\sigma)$ on $[\vartheta_{\text{init}}, +\infty)$ and the relations $F(\vartheta_{\text{init}}) = 0$, $F(\sigma) > 0$ $\forall \sigma > \vartheta_{\text{init}}$ imply the existence of a number $\vartheta_* > \vartheta_{\text{init}}$ such that

$$F(\vartheta_* + 2\pi) > F(\vartheta_*).$$

It follows from estimate (50) that for any vector

$$u_1 = (\vartheta_1, s_1, x_1, y_1, \mu_1, \nu_1) \in \Sigma_1 = \left\{ (\vartheta, z, x) \in \overline{\Omega} \cup \Sigma \quad \middle| \quad \vartheta = \vartheta_* \right\}$$

there exists a instant $\tau_1 = t(u_1) > 0$, for which the following conditions are satisfied

$$u(\tau_1, u_1) \in \Sigma_2 = \left\{ ((\vartheta, s, x, y, \mu, \nu)) \in \overline{\Omega} \cup \Sigma \quad \middle| \quad \vartheta = \vartheta_* + 2\pi \right\}$$

$$u(t, u_1) \notin \Sigma_2 \qquad \forall t \geq 0, \quad t \neq \tau_1.$$

Here $u(t, u_1)$ denotes the solution of system (27) with initial data $u_1 = (\vartheta_1, s_1, x_1, y_1, \mu_1, \nu_1)$. Let us define a transformation

$$T : \Sigma_1 \to \Sigma_2$$

$$u \longmapsto u(t(u), u), \quad \text{where} \quad u \in \Sigma_1,$$

and an operator

$$Q : \Sigma_2 \to \Sigma_1$$

$$(\vartheta, s, x, y, \mu, \nu) \longmapsto (\vartheta - 2\pi, s, x, y, \mu, \nu).$$

The continuous dependence of the solutions of system (27) on the initial data and the fact that the set Σ_2 is contactless provide the continuity of the transformation T. Consequently, the operator QT is continuous. It is easy to show that the set $\Sigma_1 = \Psi \cup \Sigma$ is convex. Hence, according to the well-known Brouwer's theorem on fixed point (Petrovskii, 1984), there exists a point u_0 with the property $QTu_0 = u_0$. This means that

$$\vartheta(t(u_0), u_0) = \vartheta(0, u_0) + 2\pi, \qquad s(t(u_0), u_0) = s(0, u_0),$$

$$x(t(u_0), u_0) = x(0, u_0), \qquad y(t(u_0), u_0) = y(0, u_0),$$

$$\mu(t(u_0), u_0) = \mu(0, u_0), \qquad \nu(t(u_0), u_0) = \nu(0, u_0),$$

Thus, the solution $u(t, u_0)$ of system (27) is a limit cycle of the second kind.

□

APPENDIX 3 COMPUTER MODELING OF SYSTEMS DESCRIBING SYNCHRONOUS MOTORS UNDER CONSTANT LOADS (MATLAB IMPLEMENTATION)

Program code in Matlab for simulating the behavior of synchronous motors with different torque controls under constant loads.

```
function [] = runSynch(time, n, g0, g1, model, controlName)
global params control;
    control = getContpolParams(controlName);
    gammas = g0:(g1-g0)/n:g1;
    for j = 1:1:(n+1)
        [init, params] = getParams(model, gammas(j));
        [t, z] = modelling( strcat(model, '_model'),     ...
                        strcat('jacobian_', model), ...
                        [0, time], init );
        [tF, pF] = plotter(model, t, z);
        saver(model, tF, pF, gammas(j), 'png');
    end
end

function [control] = getContpolParams(index)
    switch(index)
        case 1
            k = 0;   M = 0;    n_1 = 0;
        case 2
            k = 0.1; M = 0;    n_1 = 0;
        case 3
            k = 0;   M = 0.2;  n_1 = 0.5;
        case 4
            k = 0.1; M = 0.15; n_1 = 0.5;
        otherwise
            disp('Error in runSynch:getContpolParams');
    end
    control = [k, M, n_1];
end

function [init, params] = getParams(model, g)
    a = 0.1; b = 0.2; c = 0.5; d = 0.15;
    c_1 = 0.75; gamma_max = 1;
    switch (model)
        case 'parallel'
            init = [ 0, 0, 0, 0, 0, 0 ];
            params = [a, b, c, c_1, gamma_max];
        case 'serial'
            init = [ 0, 0, 0, 0, 0 ];
            params = [a, b, c, d, c_1, gamma_max];
        otherwise
            disp('Error in runSynch:getParams');
    end
    params = [params, g*params(length(params))];
end


function [tv, zv] = modelling(model, jacobian, tspan, z0)
[state,dir] = findstate(model, jacobian, z0);
options = odeset('RelTol', 1e-5, 'AbsTol', 1e-6, 'MaxStep', 0.01, ...
                 'Events', @fevents);
zv = []; tv = [];
while 1
  [t,z,TE,YE,IE] = feval('ode45', @func, tspan, z0, options, ...
```

```
                        model, jacobian, state, dir);
  z0 = z(end,:); tspan =[t(end),tspan(end)];
  zv = [zv;z];   tv = [tv;t];

  if ~isempty(IE) && (t(end)~=tspan(end))
    for k = 1:length(IE)
      if IE(k) ~= 4
        switch 1
          case state(3)
            switch IE(k)
              case {2,3}
                state([IE(k)-1, 3, 4, 5]) = -state([IE(k)-1, 3, 4, 5]);
                dir([1, IE(k)]) = -[1, dir(IE(k))];
             end
          case state(4)
            switch IE(k)
              case 1,
                state([1, 2]) = -state([1, 2]);
                dir(IE(k))    = -dir(IE(k));
              case {2,3}
                state([4, 5]) = -state([4, 5]);
                dir(IE(k))    = -dir(IE(k));
            end
          case state(5)
            switch IE(k)
              case 1,
                state([1, 2, 3]) = -[1, 1, state(3)];
                dir(IE(k))       = -dir(IE(k));
              case {2,3}
                state([4, 5]) = -state([4, 5]);
                dir(IE(k))    = -dir(IE(k));
            end
          otherwise, disp('Error in runSynch:modelling');
        end
      end
    end
  else
    break;
  end
end
end

function [state, dir] = findstate(model, jacobian, z)
    %% State
    [F1,F2,H,dH] = feval(model, z);
    dHF1  = dH*F1; dHF2 = dH*F2;
    state = - ones(1,5);
    dir   = [-sign(H), -sign(real(dHF1)), -sign(real(dHF2))];
    if     H > 0
        state(1) = -state(1);
    elseif H < 0
        state(2) = -state(2);
    elseif sign(dHF1)*sign(dHF2) < 0
        state(3) = -state(3);
    else
        if sign(dHF1) > 0
            state(1) = -state(1);
        else
            state(2) = -state(2);
        end
    end
    %% Difficalty function
    if sign(dHF1)*sign(dHF2) > 0
        state(4) = -state(4);
    elseif sign(dHF1)*sign(dHF2) < 0
```

```
            state(5) = -state(5);
        else
            if isempty(jacobian)
                state(4) = -state(4);
            else
                [J1,J2,d2H] = feval(jacobian, z);
                if dHF1 == 0
                    HxF1x_F1Hxx = dH*J1 + F1'*d2H;
                    sig = sign(HxF1x_F1Hxx*F1)*sign(dHF2);
                    dir(2) = -sign(HxF1x_F1Hxx*F1);
                elseif dHF2 == 0
                    HxF2x_F2Hxx = dH*J2 + F2'*d2H;
                    sig = sign(HxF2x_F2Hxx*F2)*sign(dHF1);
                    dir(3) = -sign(HxF2x_F2Hxx*F2);
                else
                    disp('Error in runSynch:findstate')
                    sig = 1;
                end

                if sig < 0
                    state(5) = -state(5);
                else
                    state(4) = -state(4);
                end
            end
        end
    end

    function dy = func(t, z, model, jacobian, state, dir);
        [F1,F2,H,dH] = feval(model, z);
        switch 1
            case state(1) % H > 0
                dy = F1;
            case state(2) % H < 0
                dy = F2;
            case state(3) % H == 0
                dHF1 = dH*F1; dHF2 = dH*F2;
                Hu   = -((dHF1 + dHF2)/(dHF2 - dHF1));
                dy   = 0.5*(F1 + F1 + Hu*(F2 - F1)) - H*dH';
            otherwise
                disp('Error in runSynch:func');
        end
    end

    function [value,isterminal,direction] = fevents(t, z, ...
                                                model, jacobian, state, dir)
        [F1, F2, H, dH] = feval(model,z);
        dHF1 = dH*F1; dHF2 = dH*F2;
        value      = [real([H, dHF1, dHF2]), 1];
        isterminal = [1, 1, 1, 0];
        direction  = [dir, 0];
        switch 1
        case {state(1),state(2)}
            direction(1) = -state(1);
        case state(3)
            [J1, J2, d2H] = feval(jacobian,z);
            dHF1p2 = dHF1 + dHF2; dHF2m1 = dHF2 - dHF1;
            F2m1 = F2-F1; F1p2 = F1+F2;
            dHu  = -( ((F1p2')*d2H + dH*(J1+J2))*(dHF2m1)    ...
                     -((F2m1')*d2H + dH*(J2-J1))*(dHF1p2) ) ...
                    /(dHF2m1^2);
            F    = 0.5*F1p2 - 0.5*F2m1*((dHF1p2)/(dHF2m1)) - C*H*dH';
            value      = [1, value([2, 3]), real(dHu*F)];
        otherwise
            disp('Error in runSynch:fevents')
```

```
    end
end

function [tF, pF] = plotter(model, t, z)
    theta = z(:,1); s = z(:,2); x = z(:,3);

    tF = [];    %% Figures in time space
    pF = [];    %% Figures in phase space

    tF = [tF, figure(1)];
        plot(t, 0.*t, '-k');
        hold on; grid on;
            plot(t, s, '-b');
            xlabel('time'); ylabel('s');
        hold off;

    pF = [pF, figure(2)];
        plot3(theta, 0.*theta, 0.*theta, '-k', ...
            0.*s, s, 0.*s, '-k',              ...
            0.*x, 0.*x, x, '-k');
        hold on; grid on;
            plot3(theta, s, x, '-k');
            xlabel('\theta'), ylabel('s'), zlabel('x')
        hold off;
end

function [] = saver(model, tF, pF, gamma, format)
    global control;
    controlStr = strcat( '[', ...
        num2str(control(1)), '|', ...
        num2str(control(2)), '|', ...
        num2str(control(3)), ']');

    time_prefix  = strcat(model, '-time-');
    time_dir = strcat(time_prefix, format);
    mkdir(time_dir);
    for i=1:1:length(tF)
        fName = strcat(time_prefix, ...
            num2str(i), '-', ...
            controlStr, '-', ...
            num2str(gamma));
        saveas(figure(tF(i)), strcat(time_dir,'/',fName, '.eps'), format);
    end

    phase_prefix = strcat(model, '-phase-');
    phase_dir = strcat(phase_prefix, format);
    mkdir(phase_dir);
    for i=1:1:length(pF)
        fName = strcat(phase_prefix, ...
            num2str(i), '-', ...
            controlStr, '-', ...
            num2str(gamma));
        saveas(figure(pF(i)), strcat(phase_dir,'/',fName, '.eps'), format);
    end
end
```

```
function [ f1, f2, j1, j2, H, dH, d2H ] = control( z )
global control; %% Parameters
    k = control(1); M = control(2); n_1 = control(3);
    b = M - k*n_1; s = z(2);
    %% Vector fields  of control
    f1 = zeros(length(z), 1);                       % H > 0
    f2 = zeros(length(z), 1); f2(2) = -(k*s+b); % H < 0
    %% Jacobian of control
    j1 = zeros(length(z));                          % H > 0
```

```
    j2 = zeros(length(z)); j2(2,2) = -k;          % H < 0
    %% Discontinuity surface
    H = n_1 - s;
    if (n_1 == 0)
        H = -1;
    end
    dH  = zeros(1, length(z)); dH(2) = -1;
    d2H = zeros(length(z));
end
```

```
function [F1, F2, H, dH] = serial_model(z)
global params; %% Parameters sets in run_ciynch:getParams()
    a = params(1); b = params(2); c = params(3); d = params(4);
    c_1 = params(5); gamma_max = params(6); gamma = params(7);
%% Variabals and non-linearity
    theta = z(1); s = z(2); x = z(3); mu = z(4); nu = z(5);
    phi   = - gamma_max * sin(theta) + gamma;
%% Vector fields
    F = [ s;                                ... % = \dot{\theta}
          a*x*sin(theta) + b*mu + phi;      ... % = \dot{s}
        - c*x + d*s*sin(theta);             ... % = \dot{x}
        - c_1*mu - nu*s - s;                ... % = \dot{\mu}
        - c_1*nu + mu*s                ];   ... % = \dot{\nu}
% Control. Function defining the discontinuity surface (H = 0)
    [f1, f2, j1, j2, H, dH, d2H] = control(z);
    F1 = F + f1; % H > 0
    F2 = F + f2; % H < 0
```

```
function [J1, J2, d2H] = jacobian_serial(z)
global params; %% Parameters sets in run_ciynch:getParams()
    a = params(1); b = params(2); c = params(3); d = params(4);
    c_1 = params(5); gamma_max = params(6); gamma = params(7);
%% Variabals and non-linearity
    theta = z(1); s = z(2); x = z(3); mu = z(4); nu = z(5);
    [f1, f2, j1, j2, H, dH, d2H] = control(z);
%% Jacobians
    j21 = a*x*cos(theta) - gamma_max*cos(theta);
    j23 = a*sin(theta); j31 = d*s*cos(theta); j32 = d*sin(theta);
    J = [ 0,     1,    0,    0,     0;  ... % theta
        j21,     0,  j23,    b,     0;  ... % s
        j31,   j32,   -c,    0,     0;  ... % x
          0, -nu-1,    0,  c_1,    -s;  ... % mu
          0,    mu,    0,    s,  -c_1]; ... % nu
    J1 = J + j1; %% H > 0
    J2 = J + j2; %% H < 0
```

```
function [F1, F2, H, dH] = parallel_model(z)
global params; %% Parameters sets in run_ciynch:getParams()
    a = params(1); b = params(2); c = params(3);
    c_1 = params(4); gamma_max = params(5); gamma = params(6);
%% Variabals and non-linearity
    theta=z(1); s=z(2); y=z(3); x=z(4); mu=z(5); nu=z(6);
    phi = - gamma_max*sin(theta) + gamma;
%% Vector fields
F = [ s;                       ... % dot theta
      a * y + b * mu + phi;    ... % dot s
    - c * y - x * s - s;       ... % dot y
    - c * x + y * s;           ... % dot x
    - c_1 * mu - nu * s - s;   ... % dot mu
    - c_1 * nu + mu * s    ];      % dot nu
% Control. Function defining the discontinuity surface (H = 0)
    [f1, f2, j1, j2, H, dH, d2H] = control(z);
    F1 = F + f1; % H > 0
    F2 = F + f2; % H < 0
```

```
function [J1,J2,d2H] = jacobian_parallel(z)
global params; %% Parameters sets in runSynch:getParams()
    a = params(1); b = params(2); c = params(3);
    c_1 = params(4); gamma_max = params(5); gamma = params(6);
%% Variabals
    theta=z(1); s=z(2); y=z(3); x=z(4); mu=z(5); nu=z(6);
    d_phi = - gamma_max * cos(theta);
    [f1, f2, j1, j2, H, dH, d2H] = feval(control,z);
%% Jacobians
    J = [0,      1,  0,  0,    0,     0; ... % theta
     d_phi,      0,  0,  a,    b,     0; ... % s
         0,      y, -c,  s,    0,     0; ... % x
         0,   -x-1, -s, -c,    0,     0; ... % y
         0,  -nu-1,  0,  0,-c_1,    -s; ... % mu
         0,     mu,  0,  0,    s, -c_1];... % nu
    J1 = J + j1; %% H > 0
    J2 = J + j2; %% H < 0
```