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Elena Solovyeva

Mathematical Models and Stability Analysis of Induction Motors under Sudden Changes of Load

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ABSTRACT

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This work is devoted to development of mathematical models of induction motors with various rotors and their analysis of stability and oscillations. This subject is of great current interest due to continued increase in the use of an induction motor as the drive for many industrial applications.

The mathematical models of induction motors with squirrel cage, double squirrel cage, wound rotors are developed in a rotating coordinate system rigidly connected with rotating magnetic field. Unlike the well-known mathematical models of induction machines the obtained models completely take into account rotor geometry. Using special nonsingular changes of coordinates the investigation of stability of these systems can be reduced to stability analysis of third and fifth order differential equations. Thus, the derived models are described by rather simple differential equations which allow for the indepth qualitative study of such models.

Further for these models the conditions of local and global stability are established. Dynamic stability of induction motors under various type of load torque is considered. In the case of constant load torque the behaviour of induction motors is defined by ordinary differential equations, in the case of dry friction load torque the behaviour of induction motors is described by differential inclusions. The qualitative analysis of these systems makes it possible to obtain the conditions on permissible changes of induction motor parameters, such as resistance, inductance, torque, under which an induction motor remains in operational mode after a transient process. Moreover, the limit load problem for induction motors is discussed and estimations of the limit load are obtained. The obtained theoretical conclusions are supported by results of numerical experiments. Also with the help of computer modeling, the particular case of a sudden load appearing on an idle induction motor is studied.

Keywords: induction motors, squirrel-cage rotor, double squirrel-cage rotor, wound rotor, stability, transient processes, the limit load problem, the nonlocal reduction method, the event-driven method

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PV G.A. Leonov, E.P. Solovyeva, A.M. Zaretskiy. Method of estimating transients in induction machines . *Vestnik Saint-Petersburg University. Mathematics, Vol. 46, No. 3, pp. 150–168*, 2013.

1 INTRODUCTION AND THE STRUCTURE OF THE WORK

Induction motors are the most widely used electrical motors of alternating current in both household and industrial applications. About 65-70% of all electric energy is consumed by electric motors (Machowski et al., 2011; O'Brien et al., 2012), and over 90% of them are induction motors (Machowski et al., 2011; Sumper and Baggini, 2012). They are essential elements in any power system.

The majority of modern devices using induction motors as drives operate under various loads. For example, a constant load occurs in hoists, lifts, cranes, but drills, boring machines, conveyors operates under dry friction load. A rapid change of load allows one to increase productivity of the motor, but at the same time it may lead to the different undesirable effects such as motor stopping, motor vibrations, damage or failure of the device itself. By this reason the investigation of induction motor operation under sudden changes of load is an actual and practically important problem.

In the study of induction motors it is important to develop mathematical models, which adequately describe their dynamics. In this context the following two utterances are of interest from the viewpoint of mathematical modeling of induction motors. In the book "Feedback control systems: analysis, synthesis, and design" its authors Gille, Pelegrin, and Decaulne write (Gille-Maisani et al., 1959): *"Contrary to what is sometimes thought, the derivation or writing of the system equations is more important than the study of the equations themselves. Indeed, experience shows that the majority of errors arise from inexact equations rather than from faulty solutions. In addition, the equations, once written, can be solved and studied through the use of a computer, but no calculating machine can write the equations of the system under study."* The second utterance of Slemon in his book "Electric machines and drivers" (Slemon, 1992) is: *"Use of a purely mathematical approach without adequate attention to the physical model can frequently lead to serious error. Modeling is an art which will develop as knowledge and experience grow."* Therefore, it is necessary to construct mathematical models of induction motors that are consistent with the real motors.

Quite often the mathematical models of induction motors are described by differential equations of high-order with trigonometric nonlinearities that practically can not be analysed by analytical methods. Therefore, numerical methods are also used for investigation of these equations. However, some complicated effects such as hidden oscillations[1] can not be found only by numerical methods. Hence, it is necessary to develop analytical methods of investigation.

Thus, the main goals of this thesis include the derivation of mathematical models of induction motors adequately describing the dynamics of these motors under sudden changes of load, and investigation of such models with help of analytical and numerical methods.

The brief history of induction motors. The induction motor was invented in the late 19th century. By this time, both phenomena which underlie the principle of operation of induction motors have been already discovered: electromagnetic induction and rotating magnetic field (Sah, 1946; Adkins, 1957; Alger, 1970; Sarma, 1994; Salam, 2005; Gross, 2007).

In 1824 French physicist D.F. Arago discovered phenomenon of so-called rotation magnetism. It has the greatest importance in electrical machines. He showed that a fixed copper disk on the vertical axis begins to rotate, if a permanent magnet rotates above it, and conversely if a copper disc rotates beneath a suspended magnet, then the magnet begins to rotate. This phenomenon was explained on the basis of induced currents by British scientist M. Faraday in 1831. He discovered electromagnetic induction, that is, the production of an electric current in a circuit placed in a varying magnetic field. The discovery of this phenomenon was an important step in the development of electrical machines, gained practical significance and became the foundation of all modern electronics.

However, it took more than a half of a century to understand that Arago's rotation could be used for the construction of induction motors. In 1879 British scientist W. Bailey developed a motor in which he replaced the rotating magnet by a rotating magnetic field, generated by alternative switching of four pole pieces to direct current power supply. Thus, it was the first primitive commutatorless induction motor. A little later, M. Depre (France, 1880—1883), I. Tomson (USA, 1887) and others developed devices, also based on properties of a rotating magnetic field, and achieved some improvements in this direction, but these were only steps towards the future induction motors.

In 1888 Serbian engineer and physicist N. Tesla and Italian physicist G. Ferraris invented a rotating magnetic field (Tesla, 1888a; Ferraris, 1888), produced by an alternating two-phase current in stationary windings of stator. They con-

[1] See chaotic hidden attractors in electronic Chua circuits (Leonov et al., 2010; Kuznetsov et al., 2011a,b; Bragin et al., 2011; Leonov et al., 2011b, 2012; Leonov and Kuznetsov, 2012; Kuznetsov et al., 2013; Leonov and Kuznetsov, 2013a; Leonov et al., 2011b,a; Kuznetsov et al., 2010), in drilling systems (Kiseleva et al., 2012, 2014; Leonov et al., 2013), in aircrafts (Leonov et al., 2012a,b; Andrievsky et al., 2012), in two-dimensional polynomial quadratic systems (Kuznetsov et al., 2013; Leonov et al., 2011a; Leonov and Kuznetsov, 2010; Kuznetsov and Leonov, 2008; Leonov et al., 2008; Leonov and Kuznetsov, 2007; Kuznetsov, 2008), in PLL (Leonov and Kuznetsov, 2014), and in Aizerman and Kalman problems (Leonov et al., 2010b,a; Leonov and Kuznetsov, 2011, 2013b,c)

structed the first alternating current commutatorless induction motors (Tesla, 1888b). Ferraris's motor had the solid copper rotor and concentrated two-phase windings on the stator. Tesla's motor had two-phase concentrated windings on the stator and the same windings on the rotor. However, Tesla's and Ferraris's motors did not receive the wide use. At the same time a rotating magnetic field is a base of constructing electrical machines of alternating current till now (Sheldon and Nostrand, 1902; Dawes, 1922; Bryant and Johnson, 1935; Sah, 1946; Hehre and Harness, 1949; Annett, 1950; Adkins, 1957; Yanko-Trinitskii, 1958; Walsh, 1967; Kashkari, 1969; Alger, 1970; Mablekos, 1980; Rajagopalan, 1987; Sarma, 1994; Beaty and Kirtley, 1998; Drury, 2001; Rajput, 2002; Klempner and Kerszenbaum, 2004; Kothari and Nagrath, 2006; Rajput, 2006; Hughes, 2006; Begamudre, 2007; Gross, 2007; Bakshi and Bakshi, 2009; Araujo, 2012).

In 1889 Russian engineer and inventor M.O. Dolivo-Dobrovolsky invented a three-phase squirrel-cage induction motor and a three-phase transformer. In 1890 he developed a three-phase induction motor with a slip-ring rotor into which resistors could be connected for starting and control. From 1890 the three-phase system received general recognition and marked the beginning of wide application of alternating current. The construction materials, the design and performance of induction motors have been improved for many years. However, fundamental engineering solutions proposed by Dolivo-Dobrovolsky remain mostly unchanged.

Mathematical theory of induction machines and methods of investigations. The history of development of the theory of electrical machines is partly taken from (Adkins, 1957; Kopylov, 1984).

Beginning with the pioneering works of Nikola Tesla and Galileo Ferraris (Tesla, 1888a,b; Ferraris, 1888), much research is devoted to the study of induction machines (see, e.g., (Levine, 1935; Kron, 1951; Adkins, 1957; White and Woodson, 1959; Alger, 1970; Krause, 1986; Ong, 1998; Boldea and Nasar, 2001; Bahram, 2001; Leonhard, 2001; Singh, 2005; Marino et al., 2010; Simion, 2010; Araujo, 2012)). Originally, the theory of induction motors was based on construction of equivalent circuits and their analysis. The following methods were used: the construction of the vector diagram, the symbolic method, the circle diagram method developed in the works of A. Heyland (Heyland, 1894, 1906), B.A. Behrend (Behrend, 1921), K.A. Krug, A. Blondel, G. Osanna, E. Arnold, I. Lakur, O. Bloch, G.N. Petrov, M.P. Kostenko, the method of symmetric components invented by C.L. Fortescue (Fortescue, 1918).

Then the period followed when the theory of induction motors was developed only in the framework of the general theory of electric machines, which was stimulated by the problems of the construction of synchronous generators and power systems. The first works concerned with the mathematical theory of electrical machines appeared in the middle of the 1920s, in the 1930s and 1940s. Among the authors, mention should be made of F. Tricomi (Tricomi, 1931, 1933), R. Park (Park, 1928, 1929, 1933), A.A. Gorev (Gorev, 1927, 1960), G. Kron (Kron, 1935, 1939, 1942, 1963) and G.N. Petrov.

A very valuable contribution to the development of the mathematical theory was made by Park in a set of three papers (Park, 1928, 1929, 1933). These papers develop not only the general two-axes equations of the synchronous machine, but they indicate how the equations can be applied to many important practical problems. The works of P.Park is difficult to understand because they do not contain the original stress equations with periodic coefficients for the real machine and use the artificial system of relative units.

Park's transformation provided the development of Kron's generalized theory, which was first published in a series of papers (Kron, 1935) and later in books (Kron, 1939, 1942, 1963). Kron suggested the model and equations for the generalized (primitive) electric machine.

The generalized electric machine is an idealized two-pole machine with two pairs of windings on the stator and two pairs of windings on the rotor. Despite the infinite variety of constructions of electric machines, any electrical machine with a circular field in the air gap (also the induction motor) can be reduced to the generalized electric machine.

By the end of the 1930s E. Arnold, R. Richter, A. Blondel, L. Dreyfus, M. Vidmar, C.P. Steinmetz, K.A. Krug, K.I. Shenfer, V.A. Tolvinsky and M.P. Kostenko considerably extended and advanced the theory of steady-state operation of electric machines.

The next important step in the development of the mathematical theory of electrical machines was the creation of mathematical models describing the transient processes. Initial studies of transient processes in power systems were conducted at the beginning of the 1920's in the USA and the results of this work were published by V. Bush and R.D. Booth, B.L. Robertson, E. Clark, R. E. Doherty and C. A. Nickle (Doherty and Nickle, 1926, 1927, 1930), R. Rüdenberg (Rüdenberg, 1931, 1942, 1975) and F. R. Longley (Longley, 1954).

A large contribution to the development of the theory of transient processes was made by B. Adkins (Adkins, 1957), A. Blondel, T. Laybl, A.I. Vajnov (Vajnov, 1969), A.A. Gorev (Gorev, 1960, 1985), I.A. Glebov, D.A. Gorodsky, M.P. Kostenko, R.A. Luter (Luter, 1939), G.N. Petrov, D. White, H. Woodson.

In (Adkins, 1957) on the basis of the generalized electric machine the general theory of electrical machines is explained. Many examples of its application for analysis of individual machines are demonstrated. Some questions concerning the operation of electrical machines in automatic control systems and power systems are considered.

In the fundamental work of D. White and H. Woodson (White and Woodson, 1959) the equations for generalized electric machine are derived. On the basis of these equations almost all used electromechanical converters can be analyzed. A good deal of attention is paid to the dynamic modes of operation of electromechanical devices.

By the end of the 1930s there were many practical criteria of steady-state stability. They were not rigorously proved, but contributed to stability analysis of both simple and complex electric systems. These criteria are related to the names of I.M. Markovich, S.A. Sovalov, I.S. Brooke, P.S. Zhdanov, K.A. Smirnov, D.I. Azarev.

In the engineering practice the method of numerical integration is commonly used for stability analysis of general equations of electromechanical systems. By this method, the trajectory of motion as a function of time is obtained. However, this method allows one to calculate the trajectory only on a finite time interval. At the same time amount of computation increases rapidly with increasing the dimension of the system. A.A. Gorev pointed to shortcomings in the methods of numerical integration by considering specific perturbations of the phase variables. He also developed a stability criterion for conservative models that use the kinetic and potential energy of the system. Therefore, these criteria were called energy criterion. The energy criteria of stability are used in the works of R.S. Magnusson (Magnusson, 1947), C. Szendy, B. Bokau, I.A. Gruzdev, M.L.Levinshtein, S. Sovalov, A.A. Andronov for stability analysis of models of power systems.

In the study of the steady-state stability of induction motors the mechanical characteristics obtained from the Kloss formula (Kazmierkowski and Tunia, 1994) are also commonly used in the engineering practice. This approach is consistent with the practice of operation of induction motors. However, it is not rigorous mathematically.

A very effective method of investigation of transient processes in dynamic systems is the second method of A.M. Lyapunov. This method was first used for stability analysis of synchronous motors by A.A. Yanko-Trinitskii in 1958 (Yanko-Trinitskii, 1958).

The dynamic stability problem of electric machines is closely related to the limit (ultimate) load problem (Annett, 1950; Yanko-Trinitskii, 1958; Haque, 1995; Das, 2002; Bianchi, 2005; Leonov and Kondrat'eva, 2009). In the practice of operation of electric machines there are situations, in which a sudden change of outer load or change of voltage occurs. The problem on design of limit load, under which electric machine does not pull out of synchronism, arises.

A typical situation for induction motor is as follows: the motor is started without load, then in transient process it pulls in synchronism and only after that a load-on occurs (for example, when a motor is used in a driver of metal cutter).

The urgency of the limit load problem has increased significantly due to the large number of outages in the modern world.

Numerical solution of the limit load problem for particular values of the parameters is given in works of W.V. Lyon, H.E. Edgerton (Lyon, 1928; Lyon and Edgerton, 1930), as well as in monograph of D. Stoker (Stoker, 1950). To determine the limit load the so called equal-area method is used in the engineering practice.

The limit load problem for electrical machines is related to the problem of finding attraction domains of stable equilibria, which correspond to operating modes of this machine.

Mathematical setting of the limit load problem for electrical machines and the methods of its solution are considered in (Bryant and Johnson, 1935; Sah, 1946; Annett, 1950; Blalock, 1950; Yanko-Trinitskii, 1958; Barbashin and Tabueva, 1969; Caprio, 1986; Chang and Wang, 1992; Miller and Malinowski, 1994; Nasar and

Trutt, 1999; Leonov et al., 2001; Das, 2002; Bianchi, 2005; Leonov, 2006a; Wadhwa, 2006; Lawrence, 2010; Glover et al., 2011), where a mathematical justification of "equal-area method" is given and the estimates of limit loads are obtained. In these works the different mathematical models of electrical machines and the Lyapunov function of the type: "quadratic form plus an integral of nonlinearity" are used.

The complexity of constructing Lyapunov functions for multidimensional models of dynamical systems has led to the necessity to develop the various generalizations of the second Lyapunov method. In the monograph A.H. Gelig, G.A. Leonov, V.A. Yakubovich (Gelig et al., 1978) to investigate the stability of electric motors, in addition to typical functions of Lyapunov, the functions, involving the information on solutions of equation of comparison, namely Tricomi equation, are used. These Lyapunov-type functions constitute the essence of the non-local reduction method (Leonov, 1984a,b).

The non-local reduction method was originally developed to study the global stability and oscillations of differential equations for automatic control systems and synchronous electric machines (Gelig et al., 1978; Leonov et al., 1992; Yakubovich et al., 2004). For alternating current induction machines the elements of this method were introduced in (Leonov, 2004, 2006a; Leonov and Kondrat'eva, 2009), where the limit load problem was considered.

This work is a direct continuation and extension of studies initiated in (Leonov, 2006a; Leonov and Kondrat'eva, 2009; Solovyeva, 2011)

Structure of the work. The work is devoted to the development of differential equations of induction motors and their stability analysis. The following problems are considered: steady-state stability problem, dynamic stability problem, the limit load problem.

This thesis is structured as follows. It consists of five main parts (five chapters), Finnish summary, list of references, three appendices and five included articles. The motivation for this thesis, the construction history of induction motors, the literature review of the development of the mathematical theory of induction motors, in particular, the stability problems and methods of their solutions are presented in chapter 1. Also the main results obtained by the author and the articles that are the basis of this thesis are given shortly.

In chapter 2 the classification, the construction and the operation principle of induction motors are described. Based on laws of classical mechanics and electrodynamics and some simplifying assumptions, the differential equations of induction motors with squirrel-cage, double squirrel-cage, wound rotors are derived by the author. Unlike the well-known mathematical models of induction motors the obtained model completely takes into account rotor geometry (rotor winding configuration). Using special nonsingular transformation of coordinates the investigation of stability of these systems is reduced to stability analysis of third and fifth order differential equations. Thus, the derived mathematical models are described by rather simple differential equations which allow for the indepth qualitative study of such models.

In chapter 3 stability analysis of induction motors is carried out. The conditions of steady-state stability for developed mathematical models are established. Dynamic stability of induction motors under no-load conditions is proved. Dynamic stability of induction motors under different type of load torque (constant and dry friction) is considered. For induction motors with these types of loads, the limit load problem is formulated. Using the second method of Lyapunov, the modification of the non-local reduction method, the estimations of the limit load are obtained by the author. These estimates substantially improve estimates, obtained by the equal-area criterion and used in engineering practice.

In chapter 4 theoretical results are verified by numerical experiments. The systems describing the dynamics of induction motors under different sudden changes of external loads are simulated by the event-driven method. The numerical results are presented and analyzed.

Conclusions and some future research plans are presented and discussed briefly in chapter 5.

In appendix 1 the nonsingularity of coordinates transformations is proved. In appendix 2 the proofs of the main theorems are given. Listings of programs in Matlad are represented in appendix 3.

Included articles. This thesis is based on 10 published articles by the author (PI;PII;PIII;PIV;PV; Zaretsky, Kondrat'eva and Solov'yova, 2010; Leonov, Kondrat'eva, Zaretskiy and Solov'eva, 2011; Solovyeva, 2012; Leonov, Solovyeva and Zaretskiy, 2013; Leonov, Kuznetsov, Kiseleva, Solovyeva and Zaretskiy, 2014). The main results are presented in five included papers. In all the publications the statements of problems are due to the supervisors.

In included articles the following results are obtained by the author. In PI, PII a new mathematical model which describes the dynamics of induction machine with squirrel-cage rotor is proposed. Also the author obtained analytical and numerical estimations of the limit constant load on this machine by the equal-area criterion (PI) and a modification of the non-local reduction method (PII). Similar investigations are carried out for induction machines with double-cage rotor in article PIII. In PIV a new mathematical model of induction machine with wound rotor is developed. The speed control through changing the external resistance in the rotor circuit is considered and estimations of speed control range are obtained. In PV the mathematically rigorous derivation of differential equations for induction motors with squirrel-cage and wound rotors is presented in details. Using the qualitative analysis of these systems the author also obtained the conditions on permissible changes of induction motor parameters, such as resistance, inductance, torque, under which an induction motor remains in an operational mode after a transient process.

The results of this thesis were also reported at the international conferences International Workshop "Mathematical and Numerical Modeling in Science and Technology" (Finland, Jyväskylä – 2010), XI International Conference "Stability and Oscillations of Nonlinear Control Systems" (Moscow, Russia – 2010, 2012), 7th European Nonlinear Dynamics Conference (Rome, Italy – 2011), 4th All-Russian Multi-Conference on Control Problems "MKPU–2011" (Divnomorskoe, Russia – 2011), International Conference TRIZfest-2011 (Saint-Petersburg, Russia

– 2011), 7th Vienna International Conference on Mathematical Modelling (Vienna, Austria – 2012), 5th IFAC International Workshop on Periodic Control Systems (Caen, France – 2013) and at the seminars at the department of Applied Cybernetics (Saint Petersburg State University, Russia 2009 – 2013) and the department of Information Technology (University of Jyväskylä, Finland 2010 – 2013).

2 MODELS OF INDUCTION MOTORS

In the theory of electrical machines the mathematical models, described by ordinary differential equations (White and Woodson, 1959; Leonhard, 2001; Marino et al., 2010; Araujo, 2012; Ahmad, 2010; Chiasson, 2005; Krause et al., 2002; Ong, 1998; Sul, 2011) or partial differential equations (Arkkio, 1987; Marriott and Griner, 1992; Huai et al., 2002; Rachek and Merzouki, 2012), are used. They describe relations between electromagnetic torque and main electrical and mechanical quantities.

Partial differential equations allow one to describe more completely the magnetic field, temperature distribution, and another particular qualities of electrical machines, but they turn out to be considerably complicated for investigations. The complexity of such models excludes analytical analysis and numerical analysis also does not provide exact results due to errors in computational procedures and finiteness of computational time interval. At the same time the use of analytical methods for investigation of mathematical models of electrical machines, described by ordinary differential equations, allows one to obtain qualitative behaviour of system. Therefore these models are mostly used for describing electrical machines.

In the derivation of mathematical models, describing the behavior of induction electrical motors, the rotation of rotor relative to stationary stator is considered. For this purpose different coordinate systems can be used. The most common coordinates are the following: the fixed frame, connected with stator, and the rotating frame, connected with rotor. The fixed frame was used first by N.S. Stanley (Stanley, 1938). The rotating frame was obtained by applying Park's transform (Park, 1929) to induction machines in (Brereton et al., 1957). The equations of induction motors in these coordinates under different simplifying assumptions were derived and studied in the works (Ahmad, 2010; Bose, 2006; Chiasson, 2005; De Doncker et al., 2011; Krause et al., 2002; Marino et al., 2010; Ong, 1998; Sul, 2011; Wach, 2011). However, in this thesis following (Leonov, 2004, 2006a; Leonov and Kondrat'eva, 2009) the coordinate system, rigidly connected with rotating magnetic field, is chosen. It allows one to describe in more details dynamics of rotor. Comparatively low (third and fifth) order of obtained models permits one to use the analytical methods.

The classification of induction motors is given and the construction of induction motors with squirrel-cage, double squirrel-cage and wound rotors is presented in this chapter. The operation principle of induction motors is explained on the example of a motor with double squirrel-cage rotor. For this motor mathematically rigorous derivation of the equations, based on the voltage equations and the torque equation is carried out. The derivation of mathematical models for induction motors with cage and wound rotors is performed in PV. Unlike the well-known mathematical models of induction machines the obtained models completely take into account their rotor geometry (rotor winding configuration).

2.1 Classification and electromechanical models of induction motors

The induction motor belongs to the class of electric rotating machines. The important property of all electrical machines is the principle of reversibility: electrical machine can transform mechanical energy into electrical energy and vice versa (Nasar, 1995; Kothari and Nagrath, 2006). Thus, an induction machine can operate as a motor (in the motor mode), converting electrical energy into mechanical one or as a generator (in the generator mode), converting mechanical energy into electrical one (Voldek, 1974; Ivanov-Smolenskii, 1980). Induction machines are rarely used as generators because their performance characteristics as generators are unsatisfactory for most applications.

Further we consider a classification of electrical motors (similarly for electrical generators). There are many characteristics of electric motors, which create various classifications. Moreover, with the development of new motor types, the classifications of these motors become less rigorous or unsuitable. Following (Bakshi and Bakshi, 2008; Fuchs and Masoum, 2011; Tan and Putra, 2013), the main classification is shown below in Fig. 1.

Electric motors are classified generally by the type of electrical system to which the motor is connected: direct current (DC) motors and alternating current (AC) motors. AC motors are more reliable than DC motors and have less maintenance. Motors with AC supply are subdivided into synchronous and asynchronous ones. The basic difference between an induction machine and a synchronous one is that a speed of the rotor of the induction machine under load does not coincide (is asynchronous) with the speed of magnetic field, being generated by supply voltage.

Induction motors are divided into two main categories: single-phase and three-phase. Single-phase induction motor is fed by single-phase power supply and its stator winding produces a pulsating magnetic field. These motors are used in house-hold applications, like washing machines, fans, coolers, refrigerators, etc. At the same time three-phase induction motor is fed by three-phase power supply and its stator windings produces a rotating magnetic field. Such

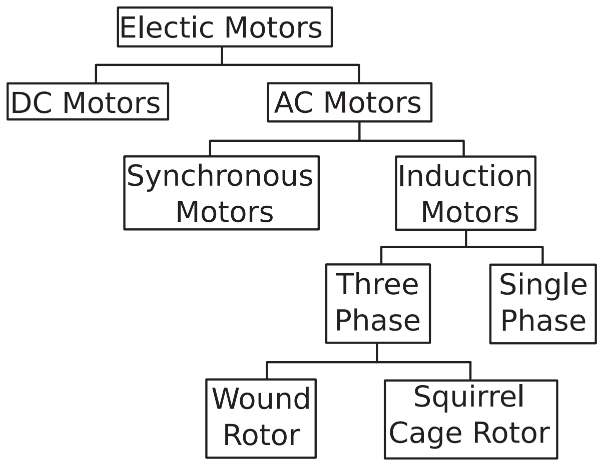


FIGURE 1 Types of electrical motors

motors are used as industrial drives in rolling mills, stamping presses, metal cutting machine tools, conveyors, pumps, drills, etc. The first type of induction motors is not studied in this work because we assume that a magnetic field, generated by stator windings, rotates with a constant angular velocity and the magnetic induction vector is constant in magnitude.

Three-phase induction motors are classified according to the rotor type as follows: squirrel-cage rotor and wound rotor. The both types are studied in the work. In additional, a induction motor with double cage rotor, which is modification of squirrel cage rotor, is studied too.

The squirrel-cage rotor induction motor is the most widely used AC motor because of its simple construction, low cost, reliability in operation and easy maintenance (Voldek, 1974; Ivanov-Smolenskii, 1980). However, it should be noted that the difficulty of speed control, high starting currents and low starting torque are their serious disadvantages. In order to improve the starting properties an induction motor with double squirrel-cage rotor was suggested. Induction motors with wound rotor don't have disadvantages described above. But it is achieved through complication of rotor design, that leads to increasing in the cost and decreasing the reliability. Therefore, wound rotor induction motors are mainly used under heavy starting duty as well as in drivers, where the speed control is required.

Now let us consider the construction of three-phase induction motors. The main constructive elements of induction motors are stationary stator and rotating rotor. The fixed part of the motor, called the stator, is a hollow laminated cylinder (stator core) with axial slots on its inner surface (Fig. 2). The stator core is made of electrical steel laminations. The laminations are insulated from one another. The three-phase winding is placed in the stator core slots. This winding, fed by three-phase supply, is arranged in such a way that it produces a rotating magnetic field.

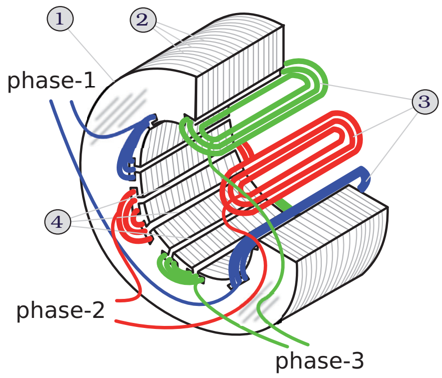


FIGURE 2 Stator of three-phase induction motors: 1 – stator core, 2 – electrical steel laminations, 3 – three-phase winding, 4 – slots

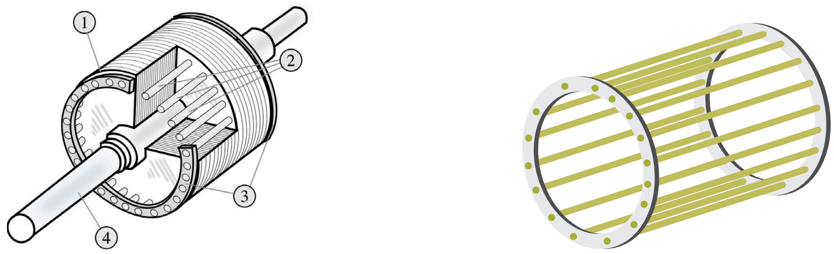
The rotating part of the motor, called the rotor, is placed inside the stator. There is the airgap between stator and rotor. The rotor (Figs. 3, 4, 5) consists of laminated cylinder (rotor core) with axial slots on its outside surface, winding and the shaft. A working gear is connected to a rotor shaft. Thus, by the transformation of electric energy into the mechanical one the induction motor imparts rotational motion to the working gear via shaft. Like the stator core, rotor core is constructed of a stack of electrical steel laminations insulated from one another. The rotor winding is placed in the slots. The slot shape, width of the slot opening and slot depth can be different (for example, rectangular, round, closed, semiclosed, semiopen, open and so on).

Induction motors have the same construction of stator and differ only in rotors (rotor windings). As already mentioned above, there are two types of rotor: squirrel-cage and wound (another name is a slip-ring rotor) ones. The winding of squirrel-cage rotor can be constructed in the form of single squirrel-cage or double squirrel-cage. Respectively, in this case a rotor is called squirrel-cage rotor (for short cage rotor) or double squirrel-cage rotor (for short double cage rotor).

A cage rotor winding consists of bars short-circuited at each end by two rings (Fig. 3, a). The entire structure resembles a squirrel cage (Fig. 3, b), therefore the rotor is called squirrel cage rotor.

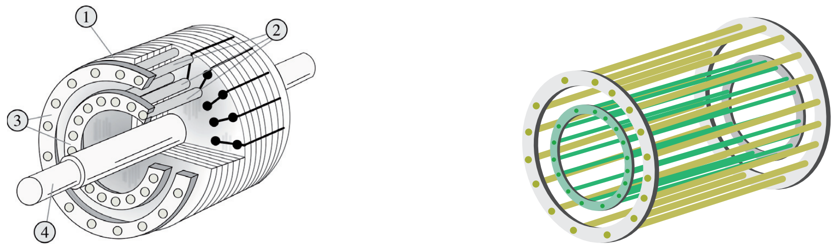
A winding of the double cage rotor (Fig. 4, a) is made of two independent windings – these being two squirrel-cages, one inside the other (Fig. 4, b).

A winding of wound rotor (Fig. 5) consists of coils, each of which is made of several turns of insulated wire. Some coils of one phase, connected in series, form a coil group. In the work we consider the simplest case, when wound rotor winding consists of three coil groups (three-phase winding) and each coil group is made up of one coil. Thus, the wound rotor winding is considered as three identical coils displaced 120 electrical degrees apart (Fig. 5, b). Some ends of coils a, b, c (Fig. 5, b) are connected to the rotor itself at one point o (such a type of connection is called a star). Another ends of coils a', b', c' are connected to slip rings, mounted on rotor shaft and isolated from it and each other. That is why



a b

FIGURE 3 a - Squirrel-cage rotor: 1 – rotor core, 2 – bars, 3 – end rings, 4 – shaft; b - winding of squirrel-cage rotor



a b

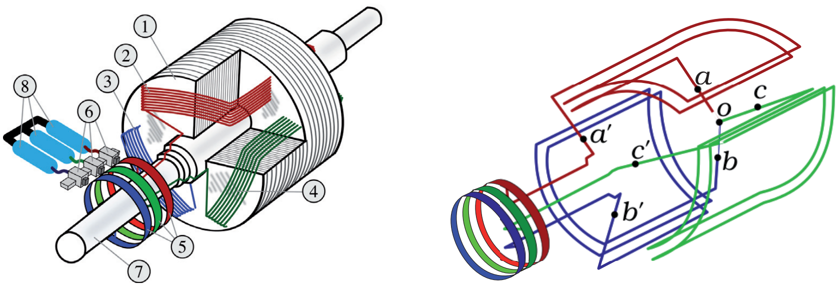
FIGURE 4 a – Double squirrel-cage rotor: 1 – rotor core, 2 – bars, 3 – end rings, 4 – shaft; b - winding of double squirrel-cage rotor

a rotor is called also a slip-ring rotor. The brushes are resting on the slip rings. The brushes, sliding over the surfaces of rotor rings, always make electric contact with them and are connected, thus, with rotor windings. The rotor winding can be either short-circuited, either connected with another external devices through the brushes, for example, with rheostat (Fig. 5, a), inductor, windings of other electrical machines. Such devices are often used for speed control of induction motors with wound rotor.

The design description of induction motors given above is quite enough for development of mathematical models of these induction motors. More detailed description of construction of induction motors can be found in (Vajnov, 1969; Bruskin et al., 1979; Trzynadlowski, 2001; Gaucheron, 2004; Upadhyay, 2008).

2.2 Mathematical models of induction motors with different types of rotors

The derivation of the equations of induction motors is based on laws of classical mechanics and electrodynamics under the following simplifying assumptions (Popescu, 2000; Leonhard, 2001; Skubov and Khodzhaev, 2008; Zaretskiy, 2013):



a b

FIGURE 5 a - Wound rotor with rheostat: 1– rotor core, 2 – first coil with current i_1, 3 – second coil with current i_2, 4 – third coil with current i_3, 5 – slip rings, 6 – brashes, 7 – shaft; 8 – rheostat; b - winding of wound rotor with slip rings

1. it is assumed that magnetic permeability of stator and rotor steel is equal to infinity. This assumption makes it possible to use the principle of superposition for the determination of magnetic field, generated by stator;
2. one may neglect energy losses in electrical steel, i.e., motor heat losses, magnetic hysteresis losses, and eddy-current losses;
3. the saturation of rotor steel is not taken into account, i.e. the current of any force can run in rotor winding;
4. one may neglect the effects, arising at the ends of rotor winding and in rotor slots, i.e., one may assume that a magnetic field is distributed uniformly along a circumference of rotor.

   Let us make an additional assumption[1]:

5. stator windings are fed from a powerful source of sinusoidal voltage.

Then, following (Adkins, 1957; White and Woodson, 1959; Skubov and Khodzhaev, 2008), due to the last assumption the influence of rotor currents on stator currents may be ignored. Thus, a stator generates a uniformly rotating magnetic field with a constant in magnitude induction. So, it can be assumed that the magnetic induction vector B is constant in magnitude and rotates with a constant angular velocity ω_1. This assumption goes back to the classical ideas of N. Tesla and G. Ferraris and allows one to consider the dynamics of induction motor from the point of view of its rotor dynamics (PV; Leonov, 2006a).

Now we focus on derivation of equations of induction motor with double cage rotor. Let us introduce the uniformly rotating coordinates, rigidly connected with the magnetic induction vector B, and consider the motion of double cage rotor in this coordinate system (Fig. 6). Suppose that a magnetic field rotates clockwise. Also, suppose that the positive direction of the rotation axis of the rotor coincides with the direction of the rotation of the magnetic induction vector.

[1] Without this assumption it is necessary to consider a stator, what leads to more complicated derivation of equations and more complicated equations themselves, which are difficult for analytical and numerical analysis.

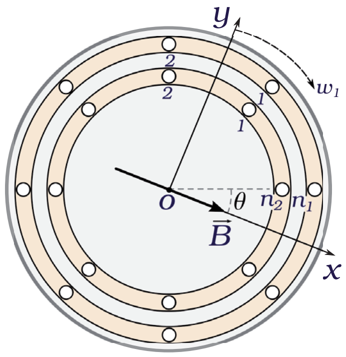


FIGURE 6 The rotating coordinates, rigidly connected with the rotating magnetic field B

The rotating magnetic field crosses bars of rotor winding and, according to the law of electromagnetic induction (Theraja and Theraja, 1999), it induces electromotive force (EMF) in them, the direction of which is shown in Fig. 7, a. EMF, which arises in bars of cages moving in magnetic field, is given by formula

$$\varepsilon = v\, B\, l\, \sin\alpha, \tag{1}$$

where B – induction of magnetic field; v – velocity of bar relative to magnetic field, the directions of which is shown in Fig. 7, a; α – angle between a vector of velocity and a vector of magnetic field induction, l – length of the bar. The direction of EMF in the bar changes, when the sign of $\sin\alpha$ changes. Taking into account the positive direction of the rotor rotation axis, the velocity and the angle α (Fig. 7, b) are calculated as follows:

$$v = -l_1\,\dot{\theta}, \qquad \alpha = \frac{\pi}{2} + \frac{2k\pi}{n_1} + \theta, \qquad k = 1, ..., n_1$$

for the bar with number k of outer cage and

$$v = -l_2\,\dot{\theta}, \qquad \alpha = \frac{\pi}{2} + \frac{2k\pi}{n_2} + \theta, \qquad k = 1, ..., n_2,$$

for the bar with number k of inner cage. Here l_1, l_2 are the lengths of the radiuses of the outer and inner cages, respectively; n_1, n_2 are the numbers of bars in the outer and inner cages, respectively; θ is the angle between a vector of magnetic field induction and the radius-vector directed to the nth bar.

Hence, formula (1) for the EMF of the kth bar of the outer and inner cages correspondingly takes the following form

$$\begin{aligned}
\varepsilon_{1,k} &= -B\, l_1\, l\, \sin(\frac{\pi}{2} + \frac{2k\pi}{n_1} + \theta)\,\dot{\theta} = -B\, l_1\, l\, \cos(\frac{2k\pi}{n_1} + \theta)\,\dot{\theta}, \quad k = 1, ..., n_1, \\
\varepsilon_{2,k} &= -B\, l_2\, l\, \sin(\frac{\pi}{2} + \frac{2k\pi}{n_2} + \theta)\,\dot{\theta} = -B\, l_2\, l\, \cos(\frac{2k\pi}{n_2} + \theta)\,\dot{\theta}, \quad k = 1, ..., n_2.
\end{aligned} \tag{2}$$

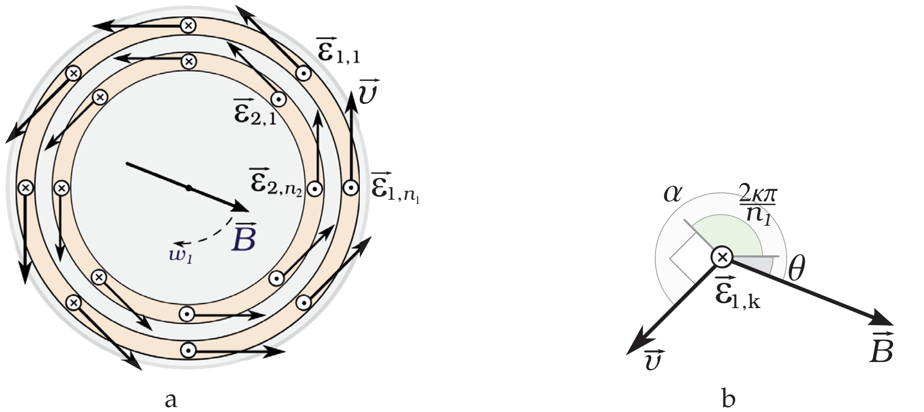


FIGURE 7 Geometry of double cage rotor: a – the directions of velocities and EMF; b – a definition of angle α for the outer cage

Under the action of EMF, an alternating current arises in rotor winding. According to Ampere's force law (Theraja and Theraja, 1999), as a result of the interaction between currents in bars and rotating magnetic field, there arise electromagnetic forces $F_{1,k}$ and $F_{2,k}$, the directions of which are shown in Fig. 8, a. Electromagnetic forces generate electromagnetic torque, and hence the rotor starts rotating with a certain frequency ω_2. The direction of rotor rotation coincides with the direction of magnetic field rotation. If the magnetic field rotates with the speed ω_1 called synchronous, the rotor rotates with the speed ω_2 slightly less than synchronous one. Hence the speed is called asynchronous. The difference between the two speeds, i.e., synchronous speed and rotor speed is called slip speed and is denoted as $s = \omega_1 - \omega_2$.

Define electromagnetic torque of induction motor with double cage rotor, produced by the electromagnetic forces $F_{1,k}$, $k = 1, ..., n_1$, and $F_{2,k}\, k = 1, ..., n_2$. The value of electromagnetic force, induced in the kth bar of outer cage, is determined by Ampere's force law:

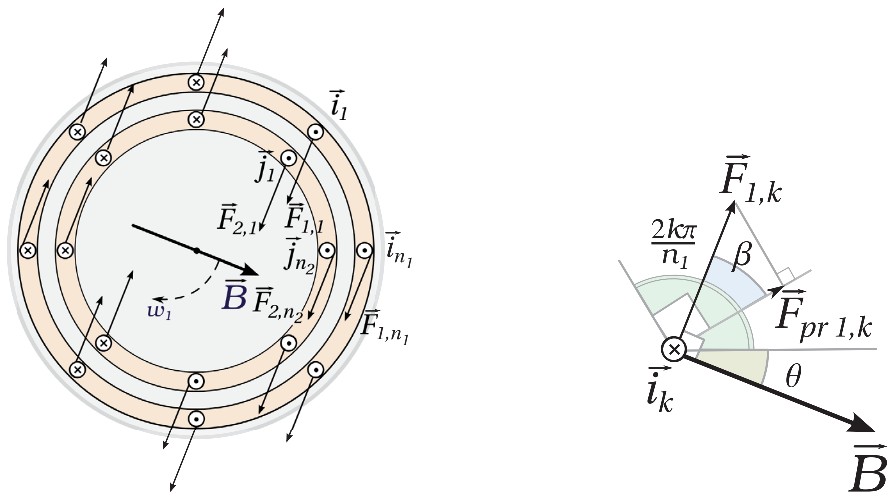
$$F_{1,k} = lBi_k,$$

where l – length of the bar, i_k – current in the kth bar of outer cage. The projections of force $F_{1,k}$ (Fig. 8, b), acting on a bar of outer cage with the current i_k, are given by formula

$$F_{pr\ 1,k} = B\, l\, l_0\, \cos(\beta) i_k = B\, l\, l_0\, \cos(\theta + \frac{2k\pi}{n_1}) i_k. \tag{3}$$

Taking into account the number of bars in outer cage and a positive direction of the rotor rotation axis, it follows that the produced electromagnetic torque, acting on the outer cage, is equal to the following:

$$M_1 = l_1\, lB \sum_{k=1}^{n_1} \cos(\theta + \frac{2k\pi}{n_1}) i_k.$$



a b

FIGURE 8 Geometry of double cage rotor: a – the directions of electromagnetic forces and currents; b – the projection of force $F_{1,k}$

Electromagnetic torques, acting on the inner cage can be similarly determined:

$$M_2 = l_2\, lB \sum_{k=1}^{n_2} \cos(\theta + \frac{2k\pi}{n_2}) j_k,$$

where j_k – current in kth bar of inner cage. Thus, the electromagnetic torque of induction motor with a double cage rotor is equal to

$$M_{em} = M_1 + M_2.$$

The dynamics of rotating induction motor is described by the equations of electric circuits (voltage equations) and the equation of moments of forces, acting on a motor rotor (torque equation).

Let us consider an electrical circuit of double cage rotor winding, shown in Fig. 9. Note that this electric circuit is equivalent to that, shown in Fig.(Fig. 4, b). In the included article PI, the currents flowing in the bars of single cage were defined. Since in terms of design the winding of double cage rotor is presented as two independent cages, using the results obtained in PI for each (outer and inner) cage, we get the system of differential equations for currents in bars of double cage rotor

$$\begin{aligned} L_1 \dot{i}_k + R_1\, i_k &= -l_1\, lB \cos(\theta + \frac{2k\pi}{n_1})\dot{\theta}, \quad k = 1, ..., n_1, \\ L_2 \dot{j}_k + R_2\, j_k &= -l_2\, lB \cos(\theta + \frac{2k\pi}{n_2})\dot{\theta}, \quad k = 1, ..., n_2, \end{aligned} \tag{4}$$

where R_1, L_1 – resistance and inductance of bar of outer cage; R_2, L_2 – resistance and inductance of bar of inner cage.

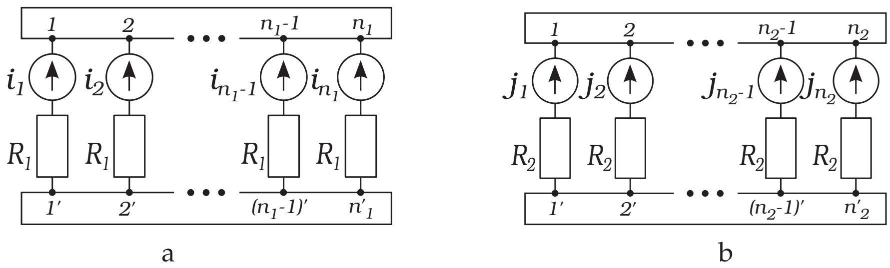


FIGURE 9 Electrical circuit of double cage rotor winding: a – electrical circuit of outer cage; b – electrical circuit of inner cage

The motion of double cage rotor of induction motor about shaft in the chosen coordinate system is described by the torque equation:

$$J\ddot{\theta} = M_{em} - M_l,$$

where θ – mechanical angle of rotor rotation; J – the moment of inertia of the rotor; M_{em} – electromagnetic torque; M_l – load torque.

Thus, the dynamics of induction motor with a double cage rotor is described by the following system of differential equations

$$\begin{aligned} &L_1\dot{i}_k + R_1\, i_k = -l_1\, lB\cos(\theta + \frac{2k\pi}{n_1})\dot{\theta}, \quad k = 1, ..., n_1, \\ &L_2\dot{j}_k + R_2\, j_k = -l_2\, lB\cos(\theta + \frac{2k\pi}{n_2})\dot{\theta}, \quad k = 1, ..., n_2, \\ &J\ddot{\theta} = l_1\, lB\sum_{k=1}^{n_1}\cos(\theta + \frac{2k\pi}{n_1})i_k + l_2\, lB\sum_{k=1}^{n_2}\cos(\theta + \frac{2k\pi}{n_2})j_k - M_l. \end{aligned} \tag{5}$$

Let us transform system (5) to a form more convenient for the further study. For this purpose we introduce the additional assumption that $n_1 = 4m_1$ and $n_2 = 4m_2$. This assumption is justified, since the number of bars used in modern double cage rotors is usually divisible by four. The nonsingular transformation of coordinates (the proof of nonsingularity of this transformation can be found in

Appendix 1)

$$
\begin{aligned}
\vartheta &= -\theta, \\
s &= \dot{\vartheta}, \\
x &= -\frac{2L_1}{n_1 l_1 l B}\sum_{k=1}^{n_1}\sin(\vartheta - \frac{2k\pi}{n_1}) i_k, \\
y &= -\frac{2L_1}{n_1 l_1 l B}\sum_{k=1}^{n_1}\cos(\vartheta - \frac{2k\pi}{n_1}) i_k, \\
z_k &= \sum_{p=-m}^{m} i_{(k+p)\, mod n_1} - \mathrm{ctg}(\frac{\pi}{n_1}) i_k, \qquad k = 2, ..., n_1 - 1, \\
\mu &= -\frac{2L_2}{n_2 l_2 l B}\sum_{k=1}^{n_2}\sin(\vartheta - \frac{2k\pi}{n_2}) j_k, \\
\nu &= -\frac{2L_2}{n_2 l_2 l B}\sum_{k=1}^{n_2}\cos(\vartheta - \frac{2k\pi}{n_2}) j_k, \\
v_k &= \sum_{p=-m}^{m} j_{(k+p)\, mod n_2} - \mathrm{ctg}(\frac{\pi}{n_2}) j_k, \qquad k = 2, ..., n_2 - 1,
\end{aligned}
\tag{6}
$$

reduces system (5) to the form

$$
\begin{aligned}
&\dot{\theta} = s, \\
&\dot{s} = a_1 y + a_2 \nu + \gamma, \\
&\dot{x} = -c_1 x + ys, \\
&\dot{y} = -c_1 y - xs - s, \\
&\dot{z}_k = -c_1 z_k, \qquad k = 2, ..., n_1 - 1, \\
&\dot{\mu} = -c_2 \mu + \nu s, \\
&\dot{\nu} = -c_2 \nu - \mu s - s, \\
&\dot{v}_k = -c_2 v_k, \qquad k = 2, ..., n_2 - 1,
\end{aligned}
\tag{7}
$$

where $a_1 = \frac{n_1 (l l_1 B)^2}{2 J L_1}$, $c_1 = \frac{R_1}{L_1}$, $a_2 = \frac{n_2 (l l_2 B)^2}{2 J L_2}$, $c_2 = \frac{R_2}{L_2}$, $\gamma = \frac{M_l}{J}$. From now on we assume $\frac{R_1}{L_1} = \frac{R_2}{L_2}$, hence, $c_1 = c_2 = c$.

In system (7) variables x, y, μ, ν, z_k, v_k determine electric quantities in the rotor bars, and variable s determines the rotor slip speed. Note that the equations with the variables z_k and v_k can be integrated independently of the rest of the system and have no effect on its stability. The remaining equations, except the first one, do not depend on θ; therefore, it is sufficient to consider the system of fifth order differential equations

$$
\begin{aligned}
&\dot{s} = a_1 y + a_2 \nu + \gamma, \\
&\dot{x} = -cx + ys, \\
&\dot{y} = -cy - xs - s, \\
&\dot{\mu} = -c\mu + \nu s, \\
&\dot{\nu} = -c\nu - \mu s - s.
\end{aligned}
\tag{8}
$$

If we exclude from consideration one of two cages, we obtain the following equations of induction motor with squirrel-cage rotor (see PI, PII, PV)

$$\begin{aligned} &L\dot{i}_k + R\, i_k = -l_0 l B \cos(\theta + \frac{2k\pi}{n})\dot{\theta}, \quad k = 1, ..., n, \\ &J\ddot{\theta} = l_0 l B \sum_{k=1}^{n} \cos(\theta + \frac{2k\pi}{n}) i_k - M_l, \end{aligned} \tag{9}$$

where i_k – the current in the kth bar; R – the bar resistance, L – the bar inductance, l and l_0 –the radius and the length of the squirrel cage rotor, respectively; θ – the angle between the magnetic field vector B and a radius vector directed to the nth bar; J – the rotor moment of inertia; M_l is the load torque. Using nonsingular changes of coordinates, presented in PI, PII, PV, the investigation of system (9) in the context of stability reduces to the investigation of the following system (see PV for more details)

$$\begin{aligned} &\dot{s} = ay + \gamma, \\ &\dot{x} = -cx + ys, \\ &\dot{y} = -cy - xs - s, \end{aligned} \tag{10}$$

where $a = \frac{n(l_0 l B)^2}{2JL}$, $\gamma = \frac{M_l}{J}$, $c = \frac{R}{L}$.

Derivation of equations for induction motors with wound rotor, the rotor winding of which consists of three coils, is given in details in (PV; Leonov et al., 2014). In the case when no external devices are connected to slip rings, the initial equations describing such motor have the following form

$$\begin{aligned} &J\ddot{\theta} = nBS \sum_{k=1}^{n} i_k \cos\left(\frac{\pi}{2} - \theta - \frac{2(k-1)\pi}{3}\right) - M_l, \\ &L\dot{i_1} + Ri_1 = -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta\right), \\ &L\dot{i_2} + Ri_2 = -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta - \frac{2\pi}{3}\right), \\ &L\dot{i_3} + Ri_3 = -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta - \frac{4\pi}{3}\right), \end{aligned} \tag{11}$$

where n – the number of turns in each coil; B – an induction of magnetic field; S – an area of one turn of coil, θ – a mechanical angle of rotation of rotor; i_k – currents in coils; R – resistance of each coil; L – inductance of each coil; J – the moment of inertia of the rotor; M_l – a load torque. Using nonsingular transformation of coordinates (the proof of nonsingularity can be found in Appendix 1)

$$
\begin{array}{c}
\vartheta = \frac{\pi}{2} - \theta, \\
s = -\dot{\theta}, \\
x = -\frac{2}{3}\frac{L}{nSB}\left(i_1 \sin(\frac{\pi}{2} - \theta) + i_2 \sin(\frac{\pi}{2} - \theta - \frac{2\pi}{3}) + i_3 \sin(\frac{\pi}{2} - \theta - \frac{4\pi}{3})\right), \\
y = -\frac{2}{3}\frac{L}{nSB}\left(i_1 \cos(\frac{\pi}{2} - \theta) + i_2 \cos(\frac{\pi}{2} - \theta - \frac{2\pi}{3}) + i_3 \cos(\frac{\pi}{2} - \theta - \frac{4\pi}{3})\right), \\
z = i_1 + i_3 - i_2
\end{array}
\tag{12}
$$

and reasoning similarly to the case of double cage rotor we obtain that the investigation of stability of system (11) can be reduced to stability analysis of system (10), where

$$
a = \frac{3(nSB)^2}{2JL}, \quad \gamma = \frac{M_l}{J}, \quad c = \frac{R}{L}. \tag{13}
$$

In the case when the wound rotor winding is connected to variable external resistances r (or to variable external inductances l), the dynamics of induction motor with wound rotor is described by the same system of equation (11) but with other parameter R (L): replace R by $R + r$ (replace L by $L + l$) (see (PIV;PV; Leonov et al., 2014)). These resistances (rheostat) and inductances (inductor) are used to control the motor speed.

As the result we obtained three different dynamical systems, which describe induction machines with various types of rotors: the system of 6 order for the wound rotor, the system of $n + 2$ order for the squirrel-cage rotor and the system of $n + n_1 + 2$ order for the double-cage rotor. However, due to suggested changes of coordinates the systems for wound and squirrel-cage rotors reduce to third order system. The variables x, y correspond to currents flown in the rotor windings (i_1, i_2, i_3 in the case wound rotor and $i_k, k = 1, ..., n$ in the case squirrel-cage rotor). Using inverse transformations of coordinates, we can estimate the force of currents in each of rotors. Thus, investigation of induction motors with different types of rotor is reduced to study systems (5) and (10).

In contrast to the well-known mathematical models (see, for example, (White and Woodson, 1959; Leonhard, 2001; Khalil, 2002; Marino et al., 2010)), in which on the basis of engineering approaches the geometry of rotors is simplified (the winding of any rotor is reduced to the winding of stator), the mathematical models of induction motors constructed in this thesis completely take into account rotor geometry. This allows one to determine real values of currents in bars or coils of rotor windings. Hence, mathematical models (5), (9) and (11) compare favorably with the well-known models.

3 STABILITY ANALYSIS OF INDUCTION MOTORS

The investigation of stability is one of the major scientific and technological problems in the design of electrical machines. By the stability of a machine we mean the ability of the machine to re-establish a steady-state mode after disturbances of the initial mode, for example changes of the external load, changes of supply voltage, etc. The process of pull into synchronism of the machine after asynchronous start-up is also a property of stability of the machine. Stability is an important qualitative characteristic of an electrical machine, providing the reliability of its work.

In this chapter on the basis of the developed mathematical models two types of stability of an induction motor are studied: steady-state and dynamic stability. Also, the limit load problem with different types of loads is formulated and estimations of the limit permissible load are obtained by the author.

3.1 Steady-state stability analysis of induction motors

The ability of an induction motor to re-establish a steady-state mode after its arbitrarily small disturbances is called steady-state (static, local) stability. The term of steady-state stability in the theory of electrical machines is matched by the term of asymptotic stability (stability "in small") in the theory of differential equations. An asymptotically stable equilibrium state corresponds to a stable steady-state mode of operation of an induction motor. This means that after an arbitrarily small short-term disturbances of a steady-state mode the motor returns to this mode. Such mode is called the operating mode. An unstable equilibrium state corresponds to an unstable mode of motor operation, that is, an arbitrarily small disturbance does not lead to a steady-state mode. Such mode is called a physically unrealizable.

Steady-state stability is a necessary condition for normal operation of the induction motor. In addition, the stability or instability of the mode does not depend on the type of disturbance and is uniquely determined by the parameters of this mode.

The local stability of modes of induction motors is studied by use of the classical theorem on stability in the first approximation (Anderson and Fouad, 1977; Gorev, 1985; Ushakov, 1988; Merkin, 1997).

Let us first study the steady-state stability of modes of induction motors with double cage rotor. The equilibrium states of the system

$$\begin{aligned} \dot{s} &= a_1 y + a_2 \nu + \gamma, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \\ \dot{\mu} &= -c\mu + \nu s, \\ \dot{\nu} &= -c\nu - \mu s - s, \end{aligned} \tag{8}$$

under condition $\gamma < \frac{a_1+a_2}{2}$ are points $(x = \mu = -\frac{s_i^2}{c^2+s_i^2},\ y = \nu = -\frac{cs_i}{c^2+s_i^2},\ s = s_i)$. Here s_i are roots of the equation

$$\frac{(a_1 + a_2)cs}{s^2 + c^2} = \gamma. \tag{14}$$

System (8) does not have the equilibrium states under condition $\gamma > \frac{a_1+a_2}{2}$

Consider the characteristic polynomial of the Jacobian matrix of the right-hand side of system (8) in stationary points:

$$f(\lambda) = \det \begin{pmatrix} \gamma' - \lambda & 0 & a_1 & 0 & a_2 \\ -\frac{cs_i}{c^2+s_i^2} & -c-\lambda & s_i & 0 & 0 \\ -\frac{c^2}{c^2+s_i^2} & -s_i & -c-\lambda & 0 & 0 \\ -\frac{cs_i}{c^2+s_i^2} & 0 & 0 & -c-\lambda & s_i \\ -\frac{c^2}{c^2+s_i^2} & 0 & 0 & -s_i & -c-\lambda \end{pmatrix} =$$

$$\begin{aligned} = -((c+\lambda)^2 + s_i^2)[\lambda^3 + (2c - \gamma')\lambda^2 + (c^2 + s_i^2 + \tfrac{(a_1+a_2)c^2}{c^2+s_i^2} - 2c\gamma')\lambda + \\ +(a_1 + a_2)c - c^2\gamma' - s_i^2\gamma']. \end{aligned} \tag{15}$$

The second-order polynomial situated in round brackets of (15), has two complex roots with negative real parts. Hence, stability of the characteristic polynomial is determined by stability of polynomial of the third order situated in square brackets of (15).

It is known that for the third order polynomial

$$p^3 + b_2 p^2 + b_1 p + b_0$$

the necessary and sufficient conditions of stability are the following

$$b_0 > 0, \quad b_1 > 0 \quad b_2 > 0, \quad b_1 b_2 - b_0 > 0,$$

Sometimes they are called conditions of Vyshnegradskii (Leonov, 2001, 2006b). Check these conditions in stationary points. Taking into account $\gamma'(s) \leq 0$ (since functions of considered below loads is nonincreasing), we obtain

$$b_0 = (a_1 + a_2)c - c^2\gamma'(s) - s^2\gamma'(s) > 0, \tag{16}$$

$$b_1 = c^2 + s^2 + \frac{(a_1+a_2)c^2}{c^2+s^2} - 2c\gamma'(s) > 0, \quad (17)$$

$$b_2 = 2c - \gamma'(s) > 0, \quad (18)$$

$$b_2b_1 - b_0 = (2c-\gamma'(s))(c^2+s^2+\tfrac{(a_1+a_2)c^2}{c^2+s^2} - 2c\gamma'(s)) - (a_1+a_2)c + c^2\gamma'(s) +$$

$$+s^2\gamma'(s) = 2c(c^2+s^2-2c\gamma'(s)) + \tfrac{2c(a_1+a_2)c^2}{c^2+s^2} - ac - \gamma'(s)(\tfrac{(a_1+a_2)c^2}{c^2+s^2} - 2c\gamma'(s)) =$$

$$= 2c(c^2+s^2-2c\gamma'(s)) - \gamma'(s)(\tfrac{(a_1+a_2)c^2}{c^2+s^2} - 2c\gamma'(s)) + \tfrac{(a_1+a_2)c(c^2-s^2)}{c^2+s^2}. \quad (19)$$

It is obvious that conditions (16),(17),(18) are satisfied for any stationary point of the system (8). The condition (19) is fulfilled in the case $s_i < c$ and is not fulfilled in the case $s_i > c$. Hence, the equilibrium states $(x = \mu = -\frac{s_i^2}{c^2+s_i^2},\ y = \nu = -\frac{cs_i}{c^2+s_i^2},\ s = s_i \leqslant c)$ are asymptotically stable and correspond to operating modes. The equilibrium states $(x = \mu = -\frac{s_i^2}{c^2+s_i^2},\ y = \nu = -\frac{cs_i}{c^2+s_i^2},\ s = s_i > c)$ are unstable and correspond to physically unrealizable modes.

Study next the steady-state stability of modes of induction motors with cage and wound rotors. In the case when $\gamma < \frac{a}{2}$ system

$$\begin{aligned} \dot{s} &= ay + \gamma, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s \end{aligned} \quad (10)$$

which describes the dynamics of induction motors with cage and wound rotors, has several equilibrium points $(x = -\frac{s_i^2}{c^2+s_i^2},\ y = -\frac{cs_i}{c^2+s_i^2},\ s = s_i)$, where s_i are roots of the equation $\frac{acs}{s^2+c^2} = \gamma$. The characteristic polynomial of the Jacobian matrix of the right-hand side of system (10) in stationary states is as follows:

$$f(\lambda) = \lambda^3 + (2c-\gamma')\lambda^2 + (c^2 + s_i^2 + \frac{ac^2}{c^2+s_i^2} - 2c\gamma')\lambda + ac - c^2\gamma' - s_i^2\gamma'.$$

Analysis of this polynomial was carried out above ($a = a_1 + a_2$). Hence, operating modes of induction motors with cage and wound rotor are determined by asymptotically stable equilibrium states $(x = -\frac{s_i^2}{c^2+s_i^2},\ y = -\frac{cs_i}{c^2+s_i^2},\ s = s_i \leqslant c)$. The equilibrium states $(x = -\frac{s_i^2}{c^2+s_i^2},\ y = -\frac{cs_i}{c^2+s_i^2},\ s = s_i > c)$ are unstable and correspond to physically unrealizable modes.

For $\gamma_0 = 0$ system (8) has an unique asymptotically stable equilibrium state $(x = \mu = y = \nu = s = 0)$ and system (10) also has an unique asymptotically stable equilibrium state $(x = y = s = 0)$. These states correspond to operating modes of the induction motors under no-load conditions.

The function

$$\varphi(s) = \frac{\alpha cs}{c^2+s^2}$$

is called static characteristic of an induction machine (for double cage rotor $\alpha = a_1 + a_2$, for cage and wound rotor $\alpha = a$). It is presented for a wide range of slip, including all possible modes of an induction machine, in the Fig. 10.

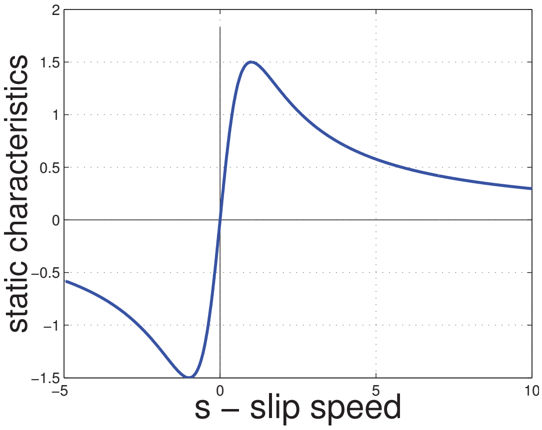


FIGURE 10 The static characteristic of an induction machine

The static characteristic $\varphi(s)$ allows one to describe the behaviour of systems (8) and (10), changing the parameters of load γ or slip speed s, also allows one to find the critical values of these parameters and to define the stability region (Panovko and Gubanova, 1967; Blehman, 1994). Operating modes of an induction motor correspond to the ascending part of static characteristic ($s \in (0, c)$). Physically unrealizable modes correspond to the descending part of static characteristic ($s > c$) (Panovko and Gubanova, 1967; Blehman, 1994).

3.2 Dynamical stability of induction motors under no-load conditions

In practice of induction motor operation the disturbances that occur on the system may be not only small, but also large. The property of an electrical machine to return to an operating mode after large disturbances is called the dynamical stability (Venikov, 1977; Pai, 1989; Xue and Pavella, 1989; Kimbark, 1995; Boldea and Nasar, 2006). The special case of dynamical stability, when machine returns to an operating mode after any disturbances, is called the global stability. The terms of dynamic stability and global stability in the theory of electrical machines correspond to the terms of stability "in large" and stability "in whole" in the theory of differential equations, respectively. Dynamic stability or instability depends not only on the parameters of the initial mode, but also on the value and character of disturbances (Langsdorf, 1955; Chang and Wang, 1992; Ong, 1998; Natarajan, 2002; Toliyat and Kliman, 2004; Leonov, 2006a; Gross, 2007; Begamudre, 2007).

The global stability investigation of induction motors under no-load conditions is carried out by the well-known theorem of Barbashin–Krasovsky (Barbashin and Krasovskiy, 1952) on stability "in whole". In addition, the Lyapunov functions are used.

Following (Leonov and Kondrat'eva, 2009), let us introduce the following definitions.

Definition 1. *The region of attraction of asymptotically stable equilibrium point $x = p$ of the autonomous system*

$$\dot{x} = f(x), \quad x \in R^n \tag{20}$$

is the set of all points in phase space, attracting to point $x = p$ as $t \to +\infty$, that is, the set of points $x_0 \in R^n$ such that

$$\lim_{t \to +\infty} x(t, x_0) = p.$$

Definition 2. *If the attraction region of asymptotically stable equilibrium point of autonomous system (20) is the same as a space of points, then the system is called stable "in whole".*

The following theorem is a modification of the Barbashin–Krasovsky theorem.

Theorem 1. *(Leonov, 2006a; Leonov and Kondrat'eva, 2009) Let $x = 0$ be an unique Lyapunov stable equilibrium point of system (20) and there exists a continuos positive definite function V such that*

1. $\lim |V(x)| = \infty$ *as* $|x| \to \infty$;
2. *for any solution $x(t, x_0)$ of system (20) the function $V(x(t, x_0))$ is nonincreasing;*
3. *if $V(x(t)) \equiv V(x(0))$, then $x(t) \equiv 0$.*

Then system (20) is stable "in whole".

By applying theorem 1, we prove stability "in whole" of the systems, describing the dynamics of induction motors with different types of rotors under no-load conditions.

Theorem 2. *The system of equations of induction motor with double cage rotor under no-load conditions*

$$\begin{aligned} \dot{s} &= a_1 y + a_2 \nu, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \\ \dot{\mu} &= -c\mu + \nu s, \\ \dot{\nu} &= -c\nu - \mu s - s \end{aligned} \tag{21}$$

is stable "in whole".

Proof. It was shown that $s = 0,\ y = 0,\ x = 0,\ \mu = 0,\ \nu = 0$ is a unique asymptotically stable equilibrium point of system (21). Let us consider the continuous positive definite function

$$V(s, x, y, \mu, \nu) = a_1(x^2 + y^2) + a_2(\mu^2 + \nu^2) + \frac{1}{2}s^2.$$

It is obvious that

$$\lim_{|z| \to \infty} |V(z)| = \infty,$$

that is, condition 1 of theorem 1 is fulfilled.

The derivative of V by virtue of system (21) is as follows:

$$\dot{V} = -a_1cx^2 - a_1cy^2 - a_2c\nu^2 - a_2c\mu^2 - a_1ys - a_2\nu s + a_1ys + a_2\nu s =$$

$$= -a_1c(x^2 + y^2) - a_2c(\nu^2 + \mu^2) \leq 0,$$

that is, condition 2 of theorem 1 is satisfied.

It follows that if $z(t) = (s(t), x(t), y(t), \nu(t), \mu(t))$ is a solution of system (21) such that $V(z(t)) \equiv V(z(0))$, then we have $x(t) \equiv 0,\ y(t) \equiv 0,\ \nu(t) \equiv 0,\ \mu(t) \equiv 0$. Hence, from system (21) we get $s(t) \equiv 0$. Thus, condition 3 of theorem 1 is fulfilled.

Finally, the application of theorem 1 yields that system (21) is stable "in whole" □

Theorem 3. *The system of equations of induction motor with cage and wound rotors under no-load conditions*

$$\begin{aligned} \dot{s} &= ay, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s \end{aligned} \tag{22}$$

is stable "in whole".

The proof of theorem 3 is identical to the proof of theorem 2 with the use of the continuous positive definite function

$$V(x, y, s) = x^2 + y^2 + \frac{1}{a}s^2.$$

Thus, these theorems claim that the induction motors under no-load conditions are globally stable.

3.3 Types of loads

Nowadays electrical machines operates under different loading conditions. The most widespread loads (load torques), which occur during operation of induction motors, can be classified depending on speed in the following groups (Pillai, 1989; Sarkar, 2012) :

1. constant loads M_c, which provide active torques. This type of load is typical for hoisting mechanisms (hoists, lifts, elevators, cranes), as well as for all machines operating on gradients (locomotives, trains, escalator).

   Let us consider the constant load through the example of a hoist. When a load is moved upwards, the developed torque opposes the action, i.e. it acts in a direction opposite to speed of rotation. On the other hand, when the load is moved downwards, the developed torque aids the action, i.e. it acts in the same direction as speed of rotation. Thus, it can be seen that load torques continue to act in the same direction even after the direction of rotation speed has been reversed. Also the load torque is independent of magnitude of speed due to the gravitational forces. Hence,

$$M_c = \text{const} > 0.$$

   The speed-torque curve for this type of load is shown in Fig. 11, a.

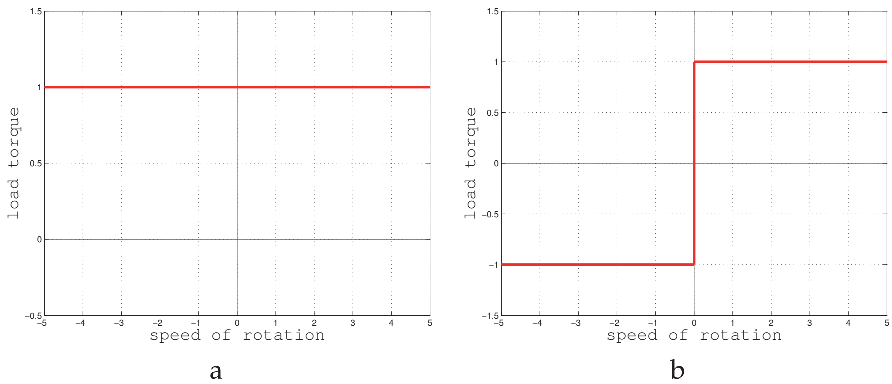


FIGURE 11 Speed torque curves of different types of loads: a – constant load; b – dry friction load

2. dry friction load M_d, which provide passive torques. Such type of load is characteristic for mechanisms with rotational or translational motion in a horizontal plane, for example, for drills, boring machines, conveyors, electric saw.

   The characteristics of dry friction load may be described using the example of a drill. When the drill bit enters the hard material, load torque occurs due to friction and oppose motion, retarding the rotation speed. Moreover, if the direction of speed is changed by the reversing switch, then the direction of torque also changes. However, magnitude of load torque remains constant. During the drilling process there are situations when the motor stops, that correspond to sticking the system. In this case the speed is equal to zero and load torque equals electromagnetic torque. Such effects often happen in practice of drill operation.

   Thus, dry friction torque is constant in magnitude and is directed against motion. Hence, this type of load is given by formula

$$M_d = \begin{cases} M & \text{if } \omega_2 > 0, \\ [-M, M] & \text{if } \omega_2 = 0, \\ -M & \text{if } \omega_2 < 0, \end{cases}$$

where $M = \text{const}$, ω_2 is rotation speed of rotor. The dry friction torque speed curve is illustrated in Fig. 11, b.

3. viscous friction load M_v, which also provide passive torques. It is directly proportional to the speed in magnitude and depends on the direction of motion. Hence,

$$M_v = k\omega_2,$$

where k is a coefficient. The viscous friction torque speed curve is presented in Fig. 12, a.

4. fan type of load M_f, which also provide passive torques. It is a load whose magnitude is proportional to some power of the speed. Hence, this load is defined as

$$M_f = k\omega_2^p,$$

where k is a coefficient. The value of power p depends on type of system: $p = 2$ for fans, blowers, propellers in shop or aeroplanes; $p \cong 2,5$ for pumps; $p \cong 1,25$ for compensators; $p = -1$ for certain type of lathers, boring machines, milling machines, etc. The speed torque curves for the fan type of load in the case a fan are shown in Fig. 12, b ($p = 2$) and Fig. 12, c ($p = -1$).

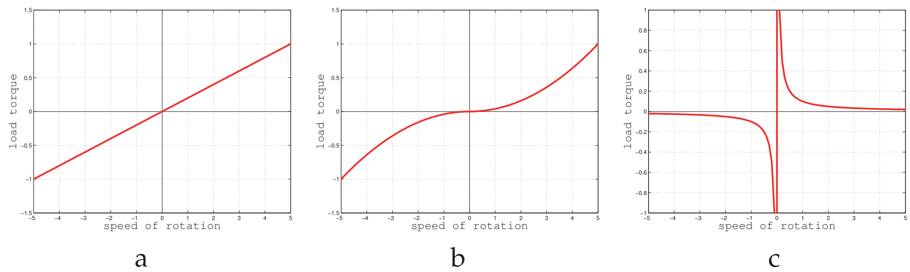


FIGURE 12 Speed torque curves of different types of loads: a – viscous friction load; b – fan type load ($p = 2$); c – fan type load ($p = -1$)

In general, the load torque consists of linear combination of the above mentioned loads.

Further in the work it is considered only the first two loads, because unlike the other ones they are essentially independent of speed. The last condition allows one to obtain analytical results in the case of constant and dry friction loads. Moreover, specifics of both loads is needed to take into account for investigating the stability of induction motors and for their computer modeling.

3.4 Dynamical stability of induction motors under various loads. The limit load problem

The problem of dynamic stability of an induction motor consists of not only of checking whether the motor maintain synchronism after a given dynamic disturbances, but also finding the limit permissible disturbance, corresponding to the boundary of dynamic stability. Therefore, the problem of dynamic stability is closely related to the limit (ultimate) load problem (PI; PII; PIII; PV; Annett, 1950; Haque, 1995; Das, 2002; Bianchi, 2005) and the speed control problem (PIV; PV; Vajnov, 1969; Marino et al., 2010).

In practice of operation of induction motors there are situations, in which a sudden change of external load or change of voltage occurs. A problem of calculation of a limit load, under which the motor does not pull out of synchronism, arises.

A typical situation for induction motor is as follows: the motor is started without load, then in transient process it pulls in synchronism and only after that a load-on occurs (for example, when a motor is used in a driver of metal-cutting machine tool).

Let us describe the limit load problem by the example of an electric circular saw (Fig. 13). The simplest model consist of a blade with cutting teeth, which is driven by an induction motor. In the simplest model the interaction of gears is not considered. Therefore, the electric circular saw in the simplest case can be described by equations of the induction motor.

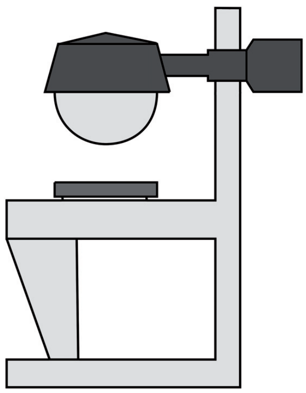


FIGURE 13 Scheme of metal cutting saw under no-load condition

As was shown above the dynamics of induction motors may be described by the autonomous system of the form

$$\dot{z} = f(z), \quad z \in R^n. \tag{23}$$

While the blade does not come in contact with the workpiece, the load is equal to zero (Fig. 13). It can be assumed that the motor operates in a synchronous mode, because its global stability under no-load condition was proved above. This operating mode correspond to the zero solution of the system $z = 0$.

Further at time $t = \tau_0$, when the blade comes into contact with the workpiece, the instantaneous load-on occurs (Fig.14). Thus, for $t > \tau_0$ the load torque is not already zero. Hence, the operating mode of the motor changes. i.e., a new operating mode of the motor under load γ_0 corresponds to an asymptotically stable equilibrium state of the system $z = z_0$. In this case the limit load problem is as follows: to find loads, under which the induction motor pulls in the new operating mode after transient processes. Thus, a mathematical formulation of the limit load problem for induction motors is the following: to find conditions, under which the solution of the system $z = z(t)$ with the initial data $z = 0$ belongs to the attraction domain of the stationary solution $z = z_0$. The latter means that the following relations

$$\lim_{t \to +\infty} z(t) = z_0, \tag{24}$$

must be satisfied. A load is called permissible if after transient processes the motor pulls in a new operating mode, i.e., if relation (24) is satisfied.

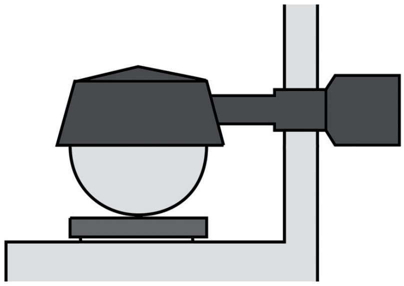


FIGURE 14 Occurrence of the load

Let us consider generalized problem when motor is already operating under load γ_* (the system has asymptotically stable equilibrium state z_*) and at some instant $t = \tau_0$ parameters of motor operation are changed. Then transients processes occur, as the result of which either the motor pulls in synchronism or stops. In the first case it is necessary that the solution $z = z(t)$ of the system with initial data $z = z_*$ tends to asymptotically stable equilibrium state z_0 as $t \to +\infty$:

$$\lim_{t \to +\infty} z(t) = z_0.$$

Thus, the limit load problem is reduced to the problem of defining the attraction region of the stable equilibrium state.

Let us find asymptotically stable equilibrium points of systems (8) and (10) under constant load $\gamma = \text{const} = M$ ($\gamma' = 0$), using the results obtained in section 3.1:

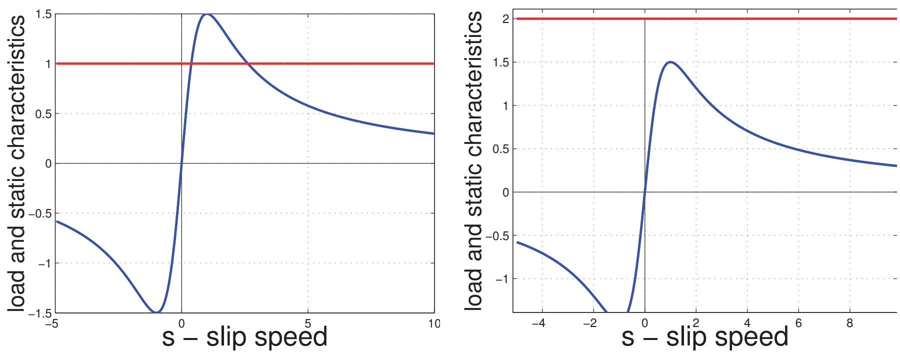
– system (8) under the condition $0 < \gamma_c \leq \frac{a_1 + a_2}{2}$ has one asymptotically stable equilibrium point (Fig. 15, a)

$$(x_0, \mu_0, y_0, \nu_0, s_0) = (x = \mu = -\frac{s_0^2}{c^2 + s_0^2},\ y = \nu = -\frac{cs_0}{c^2 + s_0^2},$$
$$s = s_0 = \frac{c(a_1 + a_2 - \sqrt{(a_1 + a_2)^2 - 4\gamma_c^2})}{2\gamma_c})$$

and one unstable equilibrium point

$$(x_1, \mu_1, y_1, \nu_1, s_1) = (x = \mu = -\frac{s_1^2}{c^2 + s_1^2},\ y = \nu = -\frac{cs_1}{c^2 + s_0^2},$$
$$s = s_1 = \frac{c(a_1 + a_2 + \sqrt{(a_1 + a_2)^2 - 4\gamma_c^2})}{2\gamma_c});$$

under the condition $\gamma_c > \frac{a_1 + a_2}{2}$ system (8) does not have equilibrium points, i.e., its stationary set is empty (Fig. 15, b).



a b

FIGURE 15 Graphical solution of the equation $\frac{\alpha cs}{s^2 + c^2} = \gamma$ in the case $\gamma = \text{const}$ (for double cage rotor $\alpha = a_1 + a_2$, for cage and wound rotor $\alpha = a$)

– system (10) under the condition $0 < \gamma_c \leq \frac{a}{2}$ has one asymptotically stable equilibrium point (Fig. 15, a)

$$(x_0, y_0, s_0) = (x = -\frac{s_0^2}{c^2 + s_0^2},\ y = -\frac{cs_0}{c^2 + s_0^2},\ s = s_0 = \frac{c(a - \sqrt{a^2 - 4\gamma_c^2})}{2\gamma_c})$$

and one stable equilibrium point

$$(x_1, y_1, s_1) = (x = -\frac{s_1^2}{c^2 + s_1^2},\ y = -\frac{cs_1}{c^2 + s_0^2},\ s = s_1 = \frac{c(a + \sqrt{a^2 - 4\gamma_c^2})}{2\gamma_c});$$

under the condition $\gamma_c > \frac{a}{2}$ system (8) does not have equilibrium points (Fig. 15, b).

The limit load problem on induction motors with different types of rotors and the generalized problem in the case of constant load $\gamma = \text{const}$ were formulated and solved with help of modified non-local reduction method in the applied articles (PI, PII, PIII, PV). Analytical estimations of limit permissible load on induction motors, operating under no-load conditions, were found. The main results are given below. In theorem 4 and corollaries 1 and 2 the following notations are used:

$$\begin{aligned} \Gamma &= 2 \max_{\lambda \in (0,c)} \left[\lambda \left(c - \lambda - \frac{\gamma^2}{4c^2(c-\lambda)} \right) \right]^{1/2}, \\ \psi(s) &= -\frac{\gamma}{c} s^2 + \alpha s - c\gamma, \\ s_1 &= \frac{c(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{2\gamma}, \end{aligned} \tag{25}$$

where $\alpha = a$ in the case cage and wound rotors (system (10)) and $\alpha = a_1 + a_2$ in the case of double cage rotor (system (8)).

Theorem 4. *Suppose that $\gamma < 2c^2$, $s_* < s_1$ and the solution of the equation*

$$F(\omega)F'(\omega) = -\Gamma F(\omega) - \psi(\omega)$$

with initial data $F(s_1) = 0$ fulfils the condition

1. *for system (8)*

$$F(s_*) > \sqrt{(a_1 y_* + a_2 \mu_* + \gamma)^2 + (a_1 x_* + a_2 \nu_* + \frac{\gamma}{c} s_*)^2},$$

2. *for system (10)*

$$F(s_*) > \sqrt{(a y_* + \gamma)^2 + (a x_* + \frac{\gamma}{c} s_*)^2}.$$

Then the solution of system (8) with initial data $s = s_*$, $x = x_*$, $\mu = \mu_*$, $y = y_*$, $\nu = \nu_*$ satisfies the relations

$$\lim_{t\to\infty} s(t) = s_0, \lim_{t\to\infty} x(t) = x_0, \lim_{t\to\infty} \mu(t) = \mu_0, \lim_{t\to\infty} y(t) = y_0, \lim_{t\to\infty} \nu(t) = \nu_0 \tag{26}$$

and the solution of system (10) with initial data $s = s_*$, $x = x_*$, $y = y_*$ satisfies the relations

$$\lim_{t\to\infty} s(t) = s_0, \lim_{t\to\infty} x(t) = x_0, \lim_{t\to\infty} y(t) = y_0. \tag{27}$$

The proof of this theorem for three-dimensional system (10) can be found in PV, similarly the proof for five-dimensional system (8) can be carried out.

Corollary 1 (PI, PII, PIII, PV). *Suppose that $\gamma < 2c^2$ and the solution of the equation*

$$F(\omega)F'(\omega) = -\Gamma F(\omega) - \psi(\omega),$$

with initial data $F(s_1) = 0$ *fulfils the condition*

$$F(0) > \gamma. \tag{28}$$

Then the load $\gamma = \text{const}$ *is a permissible load, i.e., the solution of system (8) with* $\gamma = \text{const}$ *and zero initial data* $x = \mu = y = \nu = s = 0$ *satisfies the relations* (26) *and the solution of system (10) with* $\gamma = \text{const}$ *and zero initial data* $x = y = s = 0$ *satisfies the relations* (27)*.*

Corollary 2 (PII, PIII)**.** *If the conditions*

$$5c^2 \geq 2\alpha, \qquad \gamma \leq \frac{\sqrt{3}}{4}\alpha$$

are satisfied, then the load $\gamma = \text{const}$ *is a permissible load.*

From analytical estimates (28) and $\gamma < 2c^2$, we obtain the numerical estimate of the limit permissible constant load on induction motors, presented in the Fig.16.

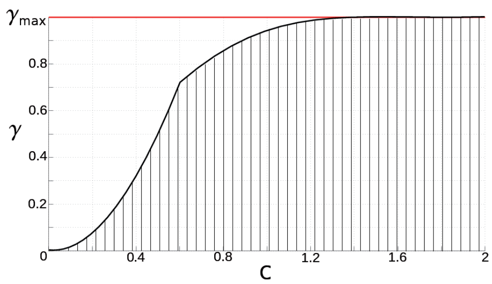


FIGURE 16 Numerical estimate of the limit permissible constant load on induction motors ($\gamma_{max} = \frac{a_1+a_2}{2}$ for system (8), $\gamma_{max} = \frac{a}{2}$ for system (10))

Now consider the case of dry friction load. In this case equations of induction motors with double cage rotor are written as

$$\begin{aligned} \dot{s} &= a_1 y + a_2 \nu + \gamma_l, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \\ \dot{\mu} &= -c\mu + \nu s, \\ \dot{\nu} &= -c\nu - \mu s - s, \end{aligned} \tag{29}$$

where

$$\gamma_l = \begin{cases} \frac{M_l}{J} & \text{if } s < \omega_1, \\ [-\frac{M_l}{J}, \frac{M_l}{J}] & \text{if } s = \omega_1, \\ -\frac{M_l}{J} & \text{if } s > \omega_1. \end{cases}$$

We see that system (29) is a differential inclusion, i.e., this system is of the form

$$\dot{x} \in F(t, x), \tag{30}$$

where $x \in \mathbb{R}^n$, $F(t, x)$ is a multivalued function. Following the works (Filippov, 1960; Gelig et al., 1978; Filippov, 1985, 1988), we define a solution of differential inclusion (30) as absolutely continuous vector function $x(t)$ satisfying $\dot{x} \in F(t, x(t))$ almost everywhere on some interval $[t_1, t_2]$.

Let us show how it is possible to determine system (29) for $s = \omega_1$ such that the solution of the new extended system will be the solution of the initial system. Introduce the notations

$$\eta = a_1 y + a_2 \nu, \qquad M = \frac{M_l}{J}.$$

For $s \neq \omega_1$ the right-hand side of system (29) is continuous and we can use the classical definition of the solution. Namely, for $s < \omega_1$ system (29) have the form

$$\begin{aligned} \dot{s} &= a_1 y + a_2 \nu + M, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \\ \dot{\mu} &= -c\mu + \nu s, \\ \dot{\nu} &= -c\nu - \mu s - s \end{aligned} \tag{31}$$

and for $s > \omega_1$ system (29) have the form

$$\begin{aligned} \dot{s} &= a_1 y + a_2 \nu - M, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \\ \dot{\mu} &= -c\mu + \nu s, \\ \dot{\nu} &= -c\nu - \mu s - s \end{aligned} \tag{32}$$

Assume now that at some instant of time t_0 the trajectory of continuous system (31) (or (32)) falls into the point D belonging to $s = \omega_1$. If $\eta(t_0) < -M$, then clearly the trajectory can be continued due to system (31) in the subspace $s < \omega_1$, since the value $\dot{s}$ for $t = t_0$, found from the first equation of system (31), is positive. Similarly, if $s(t_0) = \omega_1$ and $\eta > M$, then the trajectory be continued due to system (32) in the subspace $s > \omega_1$.

Constructed such ways the trajectories of system (29) pass through a part of the plane $s = \omega_1$, defining by the inequality $\eta > M$, in the side of increase s, and pass through a part of the plane $s = \omega_1$, defining by the inequality $\eta < M$,in the side of decrease s (Fig.17).

However, discussions carried out above is not convenient, if at time $t = t_0$ the following relations are satisfied

$$s = \omega_1, \qquad -M < \eta < M. \tag{33}$$

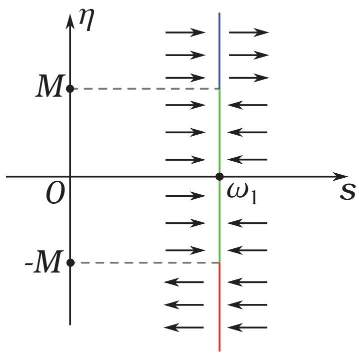


FIGURE 17 Field of directions of trajectory for system (29) in neighborhood of domain $s = \omega_1$

Indeed from the point D, coordinates of which are satisfied relations (33), it is impossible to start the trajectory in subspace $s < \omega_1$ due to system (31), since in the point D the derivation $\dot{s}$, defining from the first equation of system (31), is more zero. Similarly, it is impossible to start the trajectory from point D in subspace $s > \omega_1$ due to system (32), since $\dot{s}$ in point D, defining by the first equation of system (32), is less zero.

The trajectories of system (31) and (32) does not pass trough the strip (33), but slide over it (Fig.17). Hence, the trajectory started from the point D must stay in the strip (33) until $-1 < \eta < 1$. Along arc of this trajectory, belonging the strip (33), $\dot{s} = 0$, i.e., the trajectory slides over the strip (33). Therefore, if we want that the solution is satisfied system (29), then it follows to set equal the right-hand side of the first equation to zero in (29). From this

$$\gamma_l = -\eta.$$

The trajectories of system (29) leave the slip strip (33) either through the space $\eta = M$, $s = \omega_1$, moving away in subspace $s > \omega_1$, or through the space $\eta = -M, s = \omega_1$, moving away in subspace $s < \omega_1$. (For definiteness, we join the boundary $|\eta| = M, s = \omega_1$ to slip strip.)

According to determination γ_l presented above, system (29) takes the following form

$$\begin{aligned}
\dot{s} &= a_1 y + a_2 \nu + \gamma_l, \\
\dot{x} &= -cx + ys, \\
\dot{y} &= -cy - xs - s, \\
\dot{\mu} &= -c\mu + \nu s, \\
\dot{\nu} &= -c\nu - \mu s - s,
\end{aligned} \tag{34}$$

where

$$\gamma_l = \begin{cases} M & \text{if } \ s < \omega_1 \ \text{ or } \ s = \omega_1, a_1 y + a_2 \nu > M, \\ -a_1 y - a_2 \nu & \text{if } \ s = \omega_1, -M \leqslant a_1 y + a_2 \nu \leqslant M, \\ -M & \text{if } \ s > \omega_1 \ \text{ or } \ s = \omega_1, a_1 y + a_2 \nu < -M. \end{cases}$$

Let us find equilibrium points of system (34), using results obtained in section 3.1. Note that $\gamma_l' = 0$. Depending on the values s and γ_l, the number of equilibrium points is equal to either three or two or one (see table 1). By z_0 denote the asymptotically stable equilibrium point which corresponds to an operating mode of motor, by z_1 denote the unstable equilibrium point and by z_2 denote the asymptotically stable equilibrium point which corresponds to rotor stop. The values z_0 and z_1 are calculated as in the case of a constant load: $z_0 = (s_0, x_0, \mu_0, y_0, \nu_0)$, $z_1 = (s_1, x_1, \mu_1, y_1, \nu_1)$, $z_2 = (s_2,\ x_2, \mu_2, y_2, \nu_2) = (\omega_1,\ -\frac{\omega_1^2}{c^2+\omega_1^2},\ -\frac{\omega_1^2}{c^2+\omega_1^2},\ -\frac{c\omega_1}{c^2+\omega_1^2},\ -\frac{c\omega_1}{c^2+\omega_1^2})$.

In the same way we determine system (10), describing an induction motor with cage and wound rotors in the case of dry friction load $\gamma = \gamma_l$:

$$\begin{aligned} \dot{s} &= ay + \gamma_l, \\ \dot{x} &= -cx + ys, \\ \dot{y} &= -cy - xs - s, \end{aligned} \tag{35}$$

where

$$\gamma_l = \begin{cases} M & \text{if} \quad s < \omega_1 \quad \text{or} \quad s = \omega_1, ay > M, \\ -ay & \text{if} \quad s = \omega_1, -M \leqslant ay \leqslant M, \\ -M & \text{if} \quad s > \omega_1 \quad \text{or} \quad s = \omega_1, ay < -M. \end{cases}$$

Equilibrium points of system (35) can be found in table 1 and are calculated from formulas: the asymptotically stable equilibrium point $z_0 = (x_0, y_0, s_0)$, the unstable equilibrium point $z_1 = (s_1, x_1, y_1)$, the asymptotically stable equilibrium point, corresponding the rotor stop $z_2 = (s_2, x_2, y_2) = (\omega_1,\ -\frac{\omega_1^2}{c^2+\omega_1^2},\ -\frac{c\omega_1}{c^2+\omega_1^2})$.

Now when the asymptotically stable equilibrium points of systems (34) and (35) are found, we can solve the generalized problem and the limit load problem for induction motors in the case of dry friction load. We will be used notations (25), where $\gamma = M$.

Theorem 5. *Suppose that $s_0 < \omega_1$, $M < \min\left\{ 2c^2,\ \dfrac{a_1 + a_2}{2} \right\}$ and the solution of the equation*

$$F(s)F'(s) = -\Gamma F(s) - \psi(s)$$

with initial data $F(s_1) = 0$ fulfils the conditions

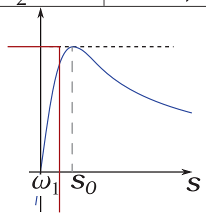
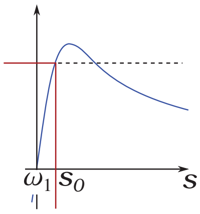
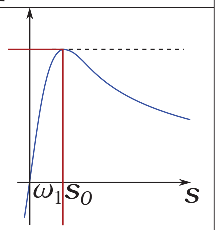
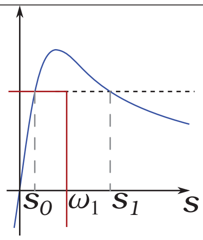
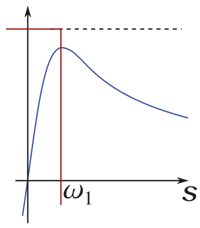
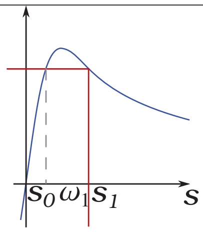
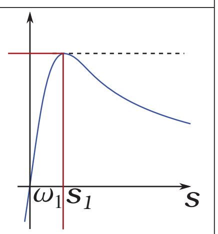
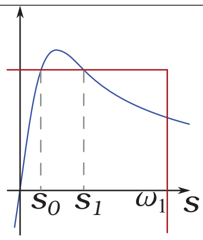
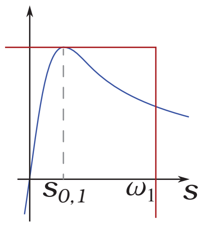
$$F(s_*) > \sqrt{(a_1 y_* + a_2 \mu_* + M)^2 + (a_1 x_* + a_2 \nu_* + \frac{M}{c} s_*)^2}\ , \tag{36}$$

$$F(\omega_1) < M, \qquad \textit{if} \quad s_1 > \omega_1.$$

Then the solution of system (34) with initial data $s = s_*$, $x = x_*$, $\mu = \mu_*$, $y = y_*$, $\nu = \nu_*$ satisfies the relations

$$\lim_{t\to\infty} s(t) = s_0, \lim_{t\to\infty} x(t) = x_0, \lim_{t\to\infty} \mu(t) = \mu_0, \lim_{t\to\infty} y(t) = y_0, \lim_{t\to\infty} \nu(t) = \nu_0. \tag{37}$$

TABLE 1 Equilibrium points of systems (34) and (35), graphical solution of the equation $\frac{\alpha cs}{s^2+c^2} = \gamma$ in the case of dry friction load (for double cage rotor $\alpha = a_1 + a_2$, for cage and wound rotor $\alpha = a$)

	$\gamma < \frac{\alpha}{2}$	$\gamma = \frac{\alpha}{2}$	$\gamma > \frac{\alpha}{2}$
$\omega_1 < s_0$	 z_2		
$\omega_1 = s_0$	 z_2	 z_2	
$s_0 < \omega_1 < s_1$	 z_0		 z_2
$\omega_1 = s_1$	 z_0, z_2	 z_2	
$\omega_1 > s_1$	 z_0, z_1, z_2	 z_0, z_2	

Theorem 6. *Suppose that $s_0 < \omega_1$, $M < \min\{2c^2, a/2\}$ and the solution of the equation*

$$F(s)F'(s) = -\Gamma F(s) - \psi(s)$$

with initial data $F(s_1) = 0$ fulfils the conditions

$$F(s_*) > \sqrt{(ay_* + M)^2 + (ax_* + \frac{M}{c}s_*)^2}\,, \tag{38}$$

$$F(\omega_1) < M, \quad \text{if} \quad s_0 < \omega_1 < s_1.$$

Then the solution of system (35) with initial data $s = s_*$, $x = x_*$, $y = y_*$ satisfies the relations

$$\begin{aligned} \lim_{t\to\infty} s(t) &= s_0, \\ \lim_{t\to\infty} x(t) &= x_0, \\ \lim_{t\to\infty} y(t) &= y_0. \end{aligned} \tag{39}$$

The proof of theorems 5 and 6 is carried out in appendix 2.

Corollary 3. *Suppose that $s_0 < \omega_1$, $M < \min\{2c^2, \alpha/2\}$ and the solution of the equation*

$$F(s)F'(s) = -\Gamma F(s) - \psi(s),$$

with initial data $F(s_1) = 0$ fulfils the conditions

$$F(0) > M, \tag{40}$$

$$F(\omega_1) < M, \quad \text{if} \quad s_0 < \omega_1 < s_1. \tag{41}$$

Then the load

$$\gamma_l = \begin{cases} M & \text{if } s < \omega_1, \\ [-M, M] & \text{if } s = \omega_1, \\ -M & \text{if } s > \omega_1. \end{cases}$$

is a permissible load, i. e., the solution of system (34) with zero initial data $x = \mu = y = \nu = s = 0$ satisfies relations (37) *and the solution of system (35) with zero initial data $x = y = s = 0$ satisfies relations* (2).

Proof. It is not too difficult to see that condition (40) of corollary 3 is equivalent condition (36) of theorem 5 for

$$s_* = x_* = y_* = \mu_* = \nu_* = 0$$

and condition (38) of theorem 6 for

$$s_* = x_* = y_* = 0.$$

Thus, all the conditions of theorems 5 and 6 are satisfied. □

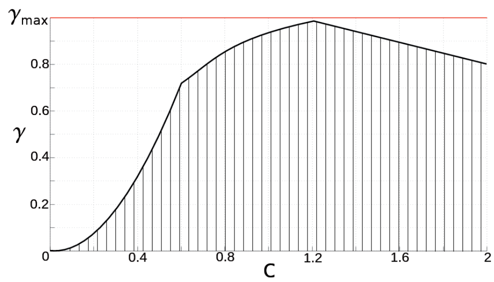


FIGURE 18 Numerical estimate of the limit permissible dry friction load on induction motors ($\gamma_{max} = \frac{a_1+a_2}{2}$ for system (34), $\gamma_{max} = \frac{a}{2}$ for system (35))

From conditions of corollary 3 we obtain the numerical estimate of the limit permissible dry friction load on induction motors, presented in the Fig. 18.

As is seen from Fig. 16 and Fig. 18 the regions of permissible loads are similar, but at certain time the value of permissible dry friction load become less than the value of permissible constant load. It is due to the rotor stop in the case of dry friction load.

In the engineering practice to determine the estimates of limit load it is used so called the the equal-area criterion. The estimates obtained in this section substantially improve estimates obtained by the equal-area criterion.

4 NUMERICAL MODELING OF INDUCTION MOTOR UNDER SUDDEN CHANGES OF LOAD

In this chapter theoretical results are verified by numerical experiments. The systems describing the dynamics of induction motors under different sudden changes of external loads are simulated by the standard computational tools of Matlab and the event-driven method (Piiroinen and Kuznetsov, 2008). The numerical results are presented and analyzed.

Simulation of an induction motor under constant load

In what follows we assume that an induction motor operates in synchronism without load. Our concern is with the behaviour of the motor when a load-on occurs. In the case of constant load the following behaviors of the motor are possible after loading:

1. the motor continues to work in an operating mode, i.e., the rotor continues to rotate in the same direction as that of the rotating magnetic field. It means the solutions of systems (8) and (10), after some oscillations, tend to the stable equilibrium states for $s < \omega_1$ (Figs.20, 21). In this case a constant load is said to be permissible.
2. the motor stops. It means the solutions of systems (8) and (10), after some oscillations, tend to the stable equilibrium states in the special case $s = \omega_1$. In this case a constant load is said to be permissible, but undesirable.
3. the motor continues to operate, but in a braking mode, i.e., the rotor continues to rotate in the opposite direction to that of the rotating magnetic field. It means the solutions of systems (8) and (10), after some oscillations, tend to the stable equilibrium states for $s > \omega_1$ (Figs.22, 23). In this case a constant load is also said to be permissible, but undesirable.
4. the rotor start to rotate in the opposite direction to that of the rotating magnetic field with acceleration. It means the solutions of systems (8) and (10)

tends to infinity (Figs.24, 25). It leads to breakdown of the motor. In this case a constant load is said to be impermissible.

In section 3.4 the numerical estimations of limit permissible constant load on idle induction motors were obtained. In Fig. 19 regions 1 and 2 correspond to permissible load found by corollary 1. Curve 6 corresponds to loads under which the rotor stops. Region 5 correspond to impermissible load according to steady-state stability analysis of induction motors. Thus, regions 3 and 4 remain not investigated analytically. Thereby numerical modeling of systems (8) and (10) is carried out for constant loads, taken from regions 3 and 4.

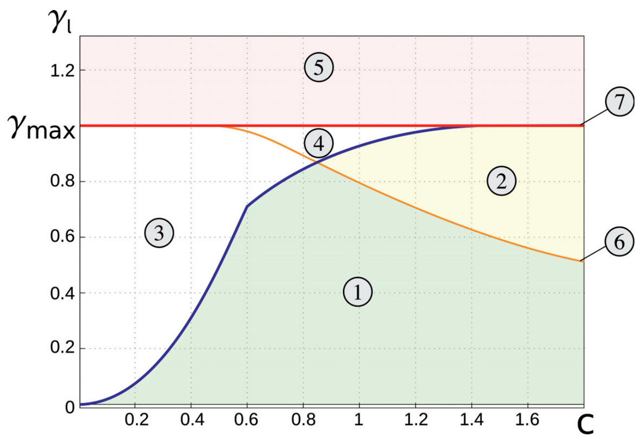


FIGURE 19 Parameter space of systems (8) and (10): regions 1, 2 are permissible loads, obtained by theorems; regions 3, 4 are not investigated analytically; region 5 is impermissible loads; line 6 corresponds to the loads under which the rotor stops; line 7 corresponds to maximum loads under which the operating modes exist

Since idle induction motors are globally stable, then zero initial data are taken. In the case of constant load for computer modeling the standard computational tools of Matlab for systems of ODE's are used. The results of computer modeling shown that for all parameters γ_l, taken from region 3 and 4, the trajectories of systems (8) and (10) tend to asymptotically stable equilibrium states (see, e.g., Figs. 20 – 23). The latter means that considered constant loads are permissible and induction motors pull into synchronism.

Conclusion: the limit permissible constant loads coincide with the values of maximum constant loads under which systems (8) and (10) have stationary solutions, in other words, effects of instability were not found. For loads more than critical values the rotor starts to rotate with acceleration. This can lead to breakdown of the motor.

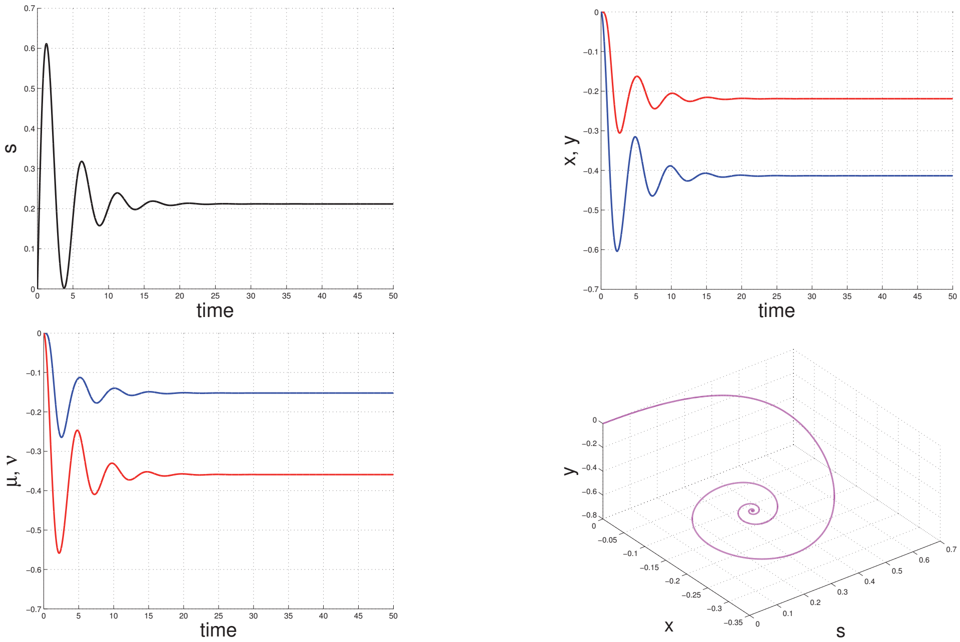


FIGURE 20 Constant load case. Modeling system (8) with parameters from region 3: $a_1 = 1,5$, $a_2 = 0,5$, $c = 0,5$, $\gamma = 0,8$

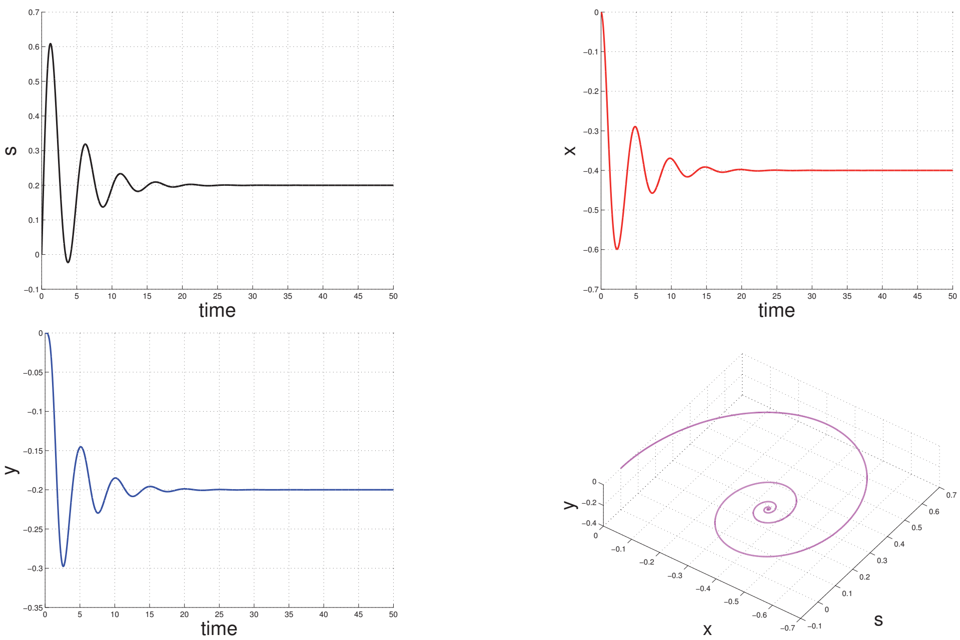


FIGURE 21 Constant load case. Modeling system (10) with parameters from region 3: $a = 2$, $c = 0,4$, $\gamma = 0,8$

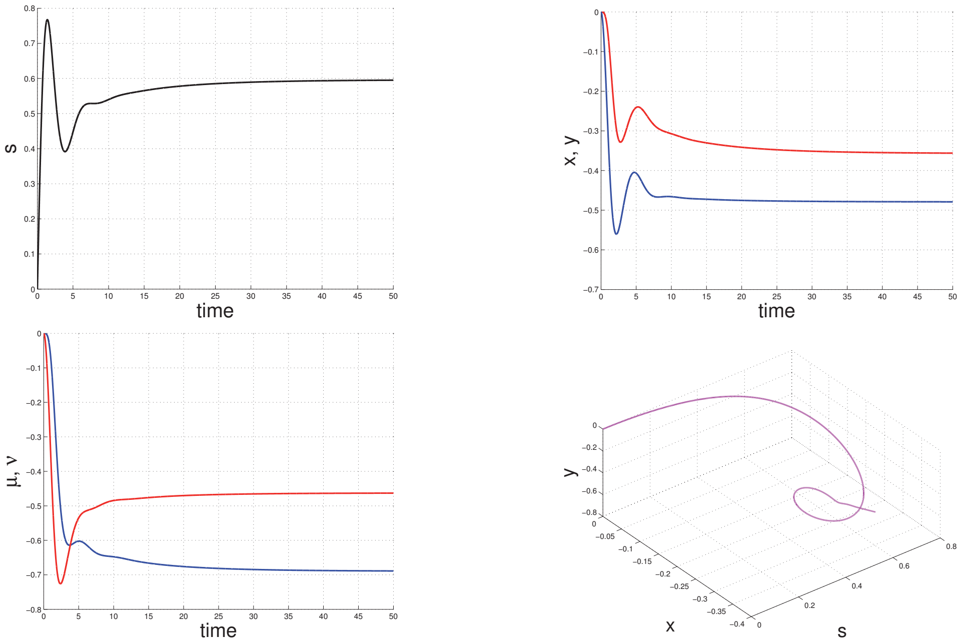


FIGURE 22 Constant load case. Modeling system (8) with parameters from region 4: $a_1 = 1,5$, $a_2 = 0,5$, $c = 0,9$, $\gamma = 0,95$

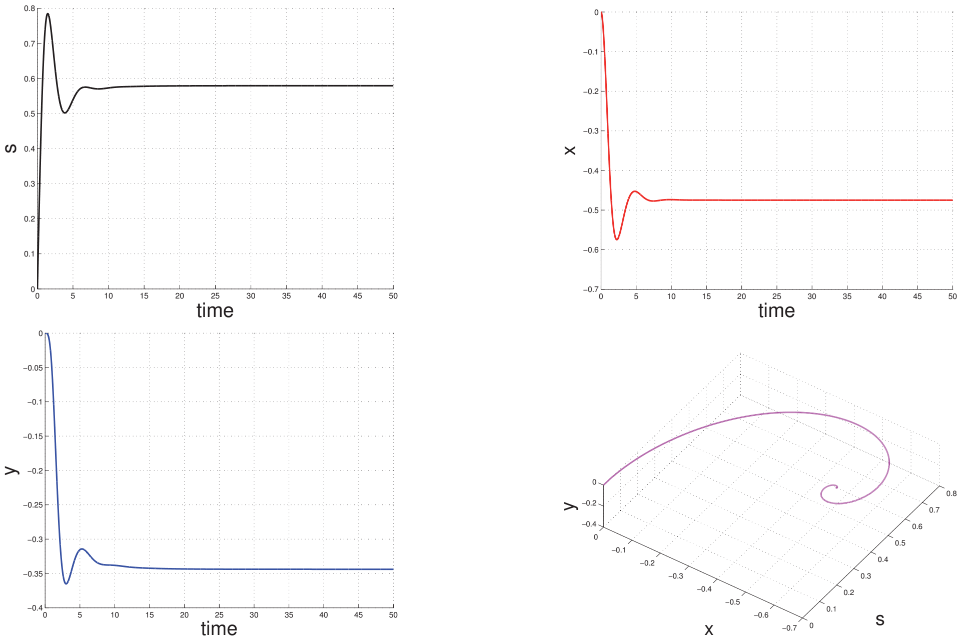


FIGURE 23 Constant load case. Modeling system (10) with parameters from region 5: $a = 2$, $c = 0,8$, $\gamma = 0,95$

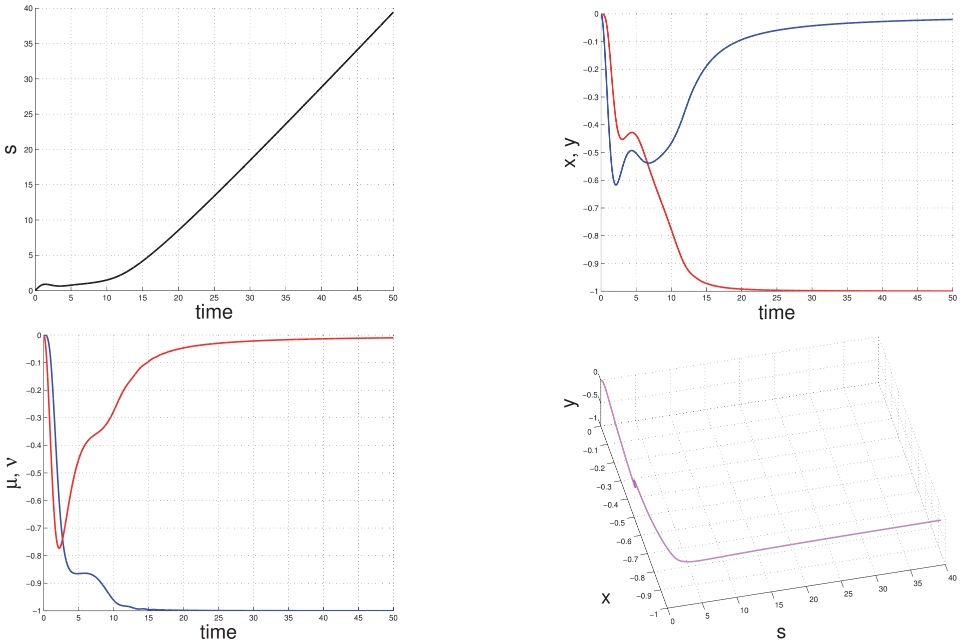


FIGURE 24 Constant load case. Modeling system (8) with parameters from region 5: $a_1 = 1,5$, $a_2 = 0,9$, $c = 0,5$, $\gamma = 1,1$

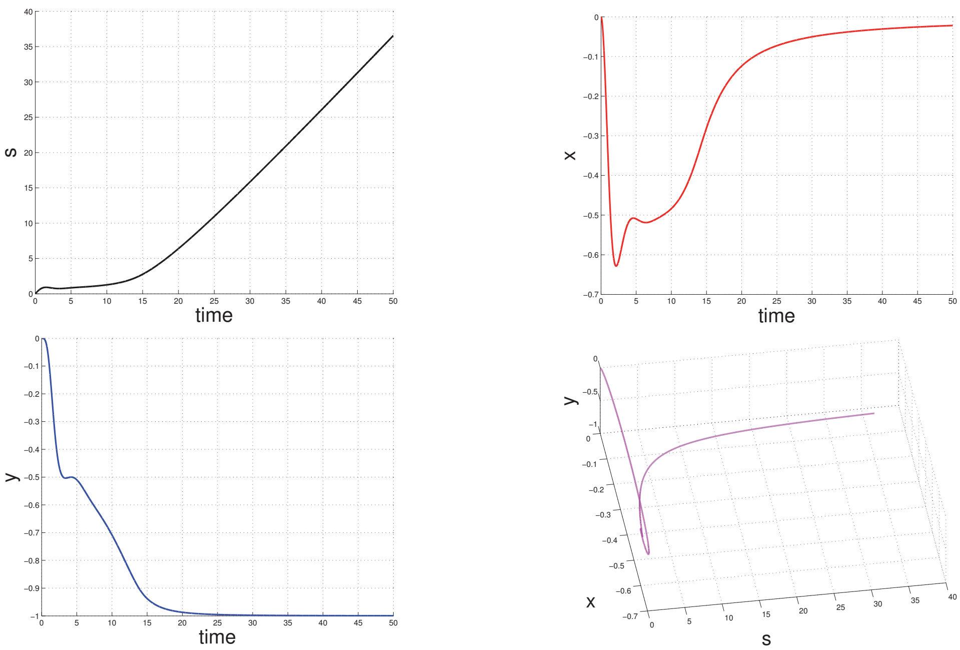


FIGURE 25 Constant load case. Modeling system (10) with parameters from region 5: $a = 2$, $c = 0,8$, $c_2 = 0,5$, $\gamma = 1,1$

Simulation of an induction motor under dry friction load

In the case of dry friction load the following behaviors of the motor are possible after loading:

1. the motor continues to work in an operating mode. It means the solutions of systems (8) and (10), after some oscillations, tend to the stable equilibrium states for $s < \omega_1$. In this case a dry friction load is said to be permissible.
2. the motor stops. It means the solutions of systems (8) and (10), after some oscillations, tend to the stable equilibrium states in the special case $s = \omega_1$. In the first case a dry friction load is said to be impermissible.

In section 3.4 the numerical estimations of limit permissible dry friction load on idle induction motors were found. In Fig. 26 region 1 corresponds to permissible loads obtained by corollary 3. Curve 4 corresponds to loads under which the rotor stops. Thus, regions 2 and 3 remain not investigated analytically. Thereby numerical modeling of systems (34) and (35) is done for dry friction loads, taken from these regions.

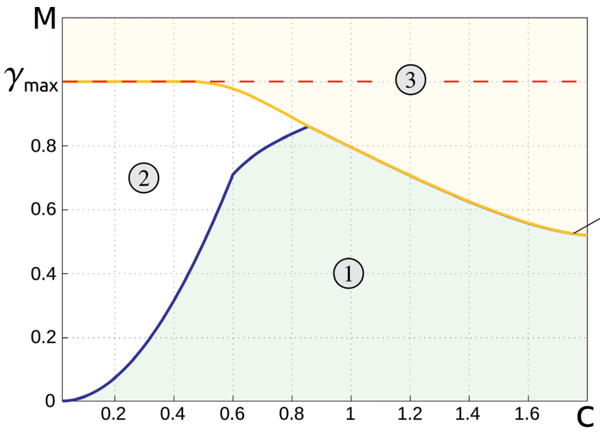


FIGURE 26 Parameter space of systems (34) and (35): region 1 is permissible loads, obtained by theorems; regions 2, 3 are not investigated analytically; line 4 corresponds to the loads under which the rotor stops

The dynamics of induction motors under dry friction load is described by equations (29) and (35), which have discontinuous right hand-sides. Therefore a special method for numerical computation of their solutions is necessary. One of such methods is the event-driven method (Piiroinen and Kuznetsov, 2008). For computer modeling of systems (29) and (35) we use a modified algorithm and Matlab code from (Piiroinen and Kuznetsov, 2008). Introduce the following notations:

$$D_1 = \left\{ (s,\ x,\ y,\ \mu,\ \nu)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle|\ s < \omega_1 \right\},$$

$$D_2 = \left\{ (s,\ x,\ y,\ \mu,\ \nu)^{\mathrm{T}} \in \mathbb{R}^5 \quad \middle| \quad s > \omega_1 \right\},$$

$$H = \left\{ (s,\ x,\ y,\ \mu,\ \nu)^{\mathrm{T}} \in \mathbb{R}^5 \quad \middle| \quad s = \omega_1 \right\},$$

$$\Delta = \left\{ (s,\ x,\ y,\ \mu,\ \nu)^{\mathrm{T}} \in \mathbb{R}^5 \quad \middle| \quad s = \omega_1, \quad -\gamma < a_1 y + a_2 \nu < \gamma \right\}.$$

For three order system (35) similar notations can be introduced.

Algorithm: INIT. Initialize the program with all parameters.

1. Find the initial state (state$_i$ for $i = 1, ..., 5$).
2. Solve the current ODE until an event occurs or if the final simulation time has been reached.
3. Check if the current time is equal to the final time else check in which region the state is in by using the event variables.

4-6. Check which event that occurred.

7. Set state$_1$ $= -$state$_1$ and state$_2$ $= -$state$_2$ since we are moving from region $\{D_1 \cup (H \setminus \Delta)\}$ to $\{D_2 \cup (H \setminus \Delta)\}$ (or $\{D_2 \cup \ (H \setminus \Delta)\}$ to $\{D_1 \cup (H \setminus \Delta)\}$), and continue to step 2.
8. Set state$_4$ $= -$state$_4$ and state$_5$ $= -$state$_5$ since we are moving from region $\{D_1 \ \cup \ (H \ \setminus \ \Delta)\}$ to $\{D_1 \ \cup \ \partial\Delta\}$ (or $\{D_2 \ \cup \ (H \ \setminus \ \Delta)\}$ to $\{D_2 \ \cup \ \partial\Delta\}$), and continue to step 2.
9. Set state$_1$ $= -1$, state$_2$ $= -1$ and state$_3$ $= -$state$_3$ since we are moving from region $\{D_1 \ \cup \ \partial\Delta\}$ to Δ (or $\{D_2 \ \cup \ \partial\Delta\}$ to Δ), and continue to step 2.
10. Set state$_4$ $= -$state$_4$ and state$_5$ $= -$state$_5$ since we are moving from region $\{D_1 \ \cup \ \partial\Delta\}$ to $\{D_1 \ \cup \ (H \ \setminus \ \Delta)\}$ (or $\{D_2 \ \cup \ \partial\Delta\}$ to $\{D_2 \ \cup \ (H \ \setminus \ \Delta)\}$), and continue to step 2.
11. Set state$_1$ $= -$state$_1$, state$_3$ $= -$state$_3$, state$_4$ $= -$state$_4$ and state$_5$ $= -$state$_5$ since we are moving from region Δ to $\{D_1 \ \cup \ (H \ \setminus \ \Delta)\}$ and continue to step 2.
12. Set state$_1$ $= -$state$_1$, state$_3$ $= -$state$_3$, state$_4$ $= -$state$_4$ and state$_5$ $= -$state$_5$ since we are moving from region Δ to $\{D_2 \ \cup \ (H \ \setminus \ \Delta)\}$ and continue to step 2.

The algorithm is presented in Fig. 27.

The results of computer modeling shown that

1. for all parameters γ_l, taken from region 2, the trajectories of systems (8) and (10) tend to asymptotically stable equilibrium states ($s < \omega_1$) (Figs. 28, 29). It means that considered dry friction loads are permissible and induction motors pull into synchronism.
2. for all parameters γ_l, taken from region 3, the trajectories of systems (8) and (10) tend to another asymptotically stable equilibrium states corresponding $s = \omega_1$ (Figs. 30–33). It means that considered dry friction loads are impermissible and induction motors stop.

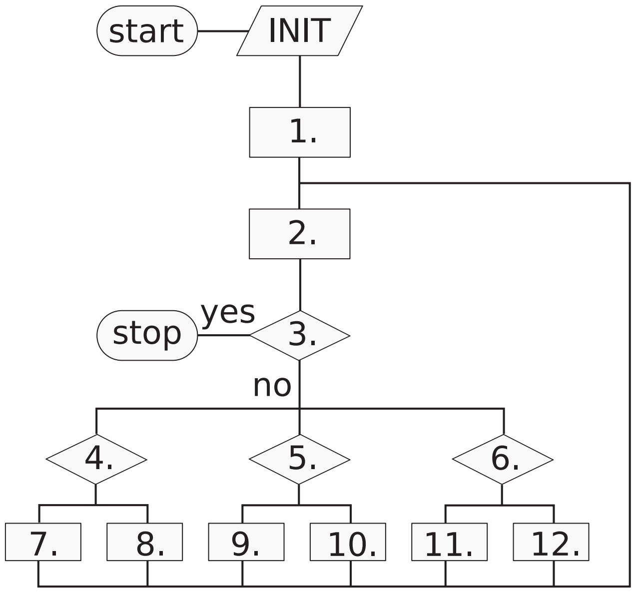


FIGURE 27 A schematic chart of the algorithm for modeling systems (29) and (35)

Conclusion: the limit permissible dry friction loads coincide with the values of maximum dry friction loads under which systems (8) and (10) have equilibrium points, which is in the region D_1. In other words, effects of instability were not found as in the case of a constant load. For loads more than critical values the rotor stops.

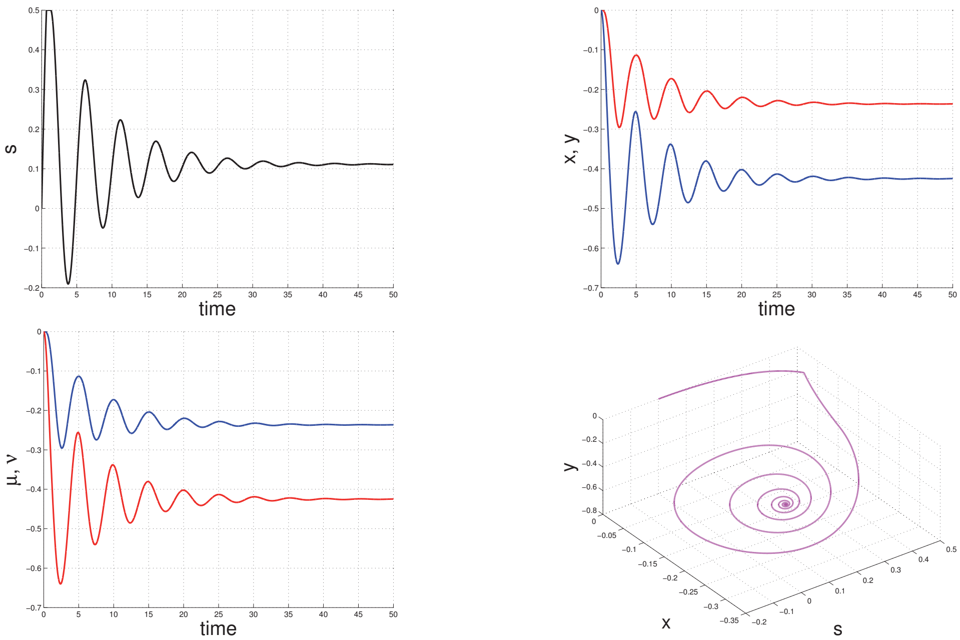


FIGURE 28 Dry friction load case. Modeling system (34) with parameters from region 2: $a_1 = 1,5$, $a_2 = 0,5$, $c = 0,4$, $M = 0,85$, $\omega_1 = 0,5$

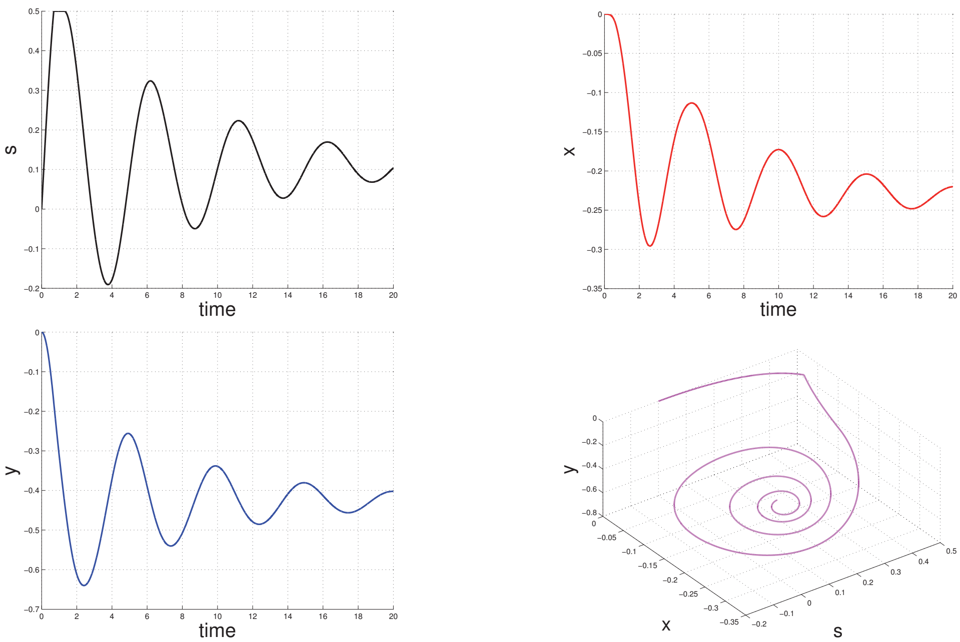


FIGURE 29 Dry friction load case. Modeling system (35) with parameters from region 2: $a = 2$, $c = 0,2$, $M = 0,85$, $\omega_1 = 0,5$

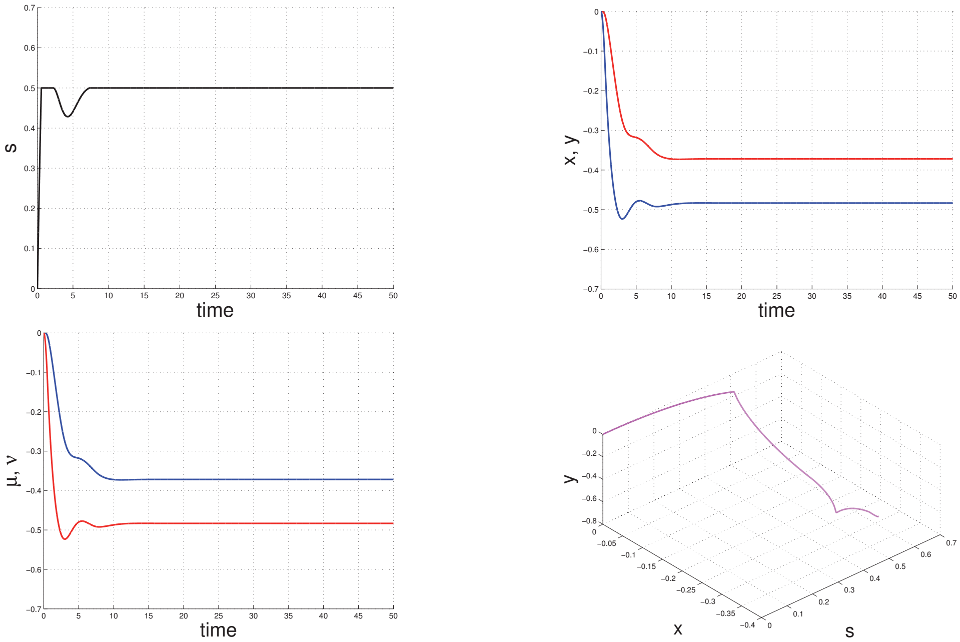


FIGURE 30 Dry friction load case. Modeling system (34) with parameters from region 3: $a_1 = 1,5$, $a_2 = 0,5$, $c = 0,65$, $M = 0,99$, $\omega_1 = 0,5$

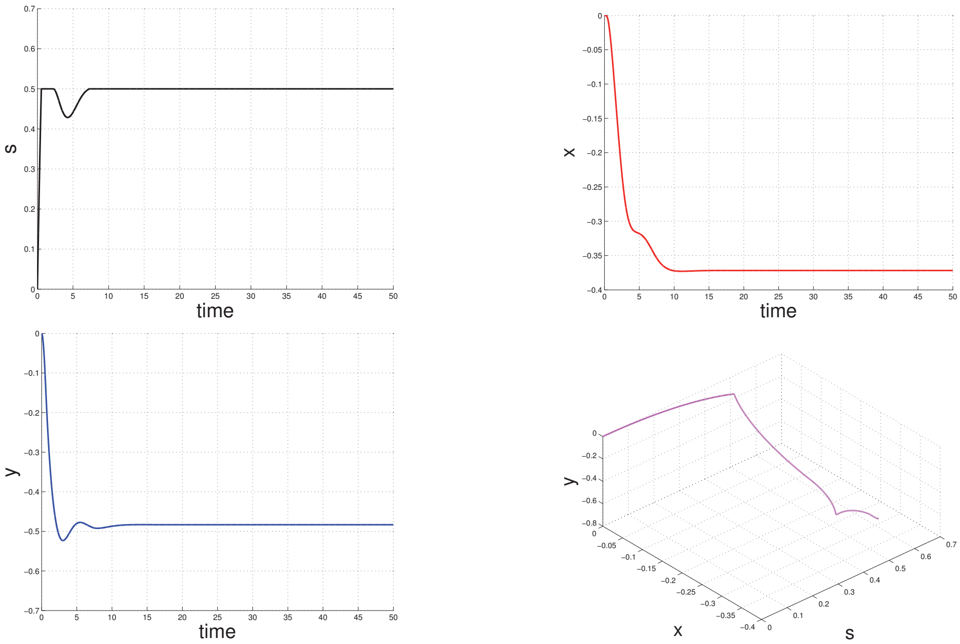


FIGURE 31 Dry friction load case. Modeling system (35) with parameters from region 3: $a = 2$, $c = 0,65$, $M = 0,99$, $\omega_1 = 0,5$

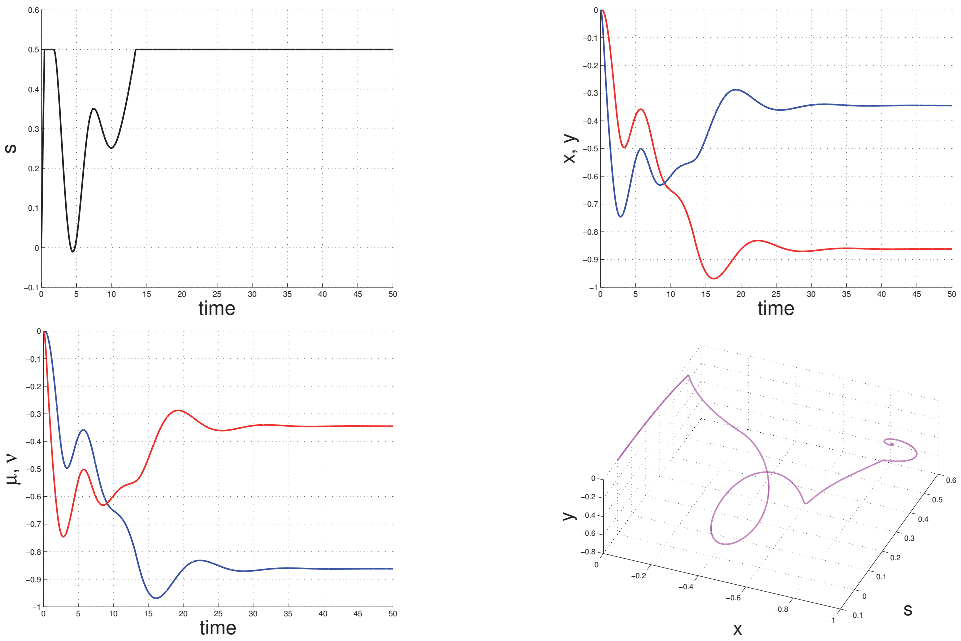


FIGURE 32 Dry friction load case. Modeling system (34) with parameters from region 3: $a_1 = 1,5$, $a_2 = 0,5$, $c = 0,2$, $M = 1,2$, $\omega_1 = 0,5$

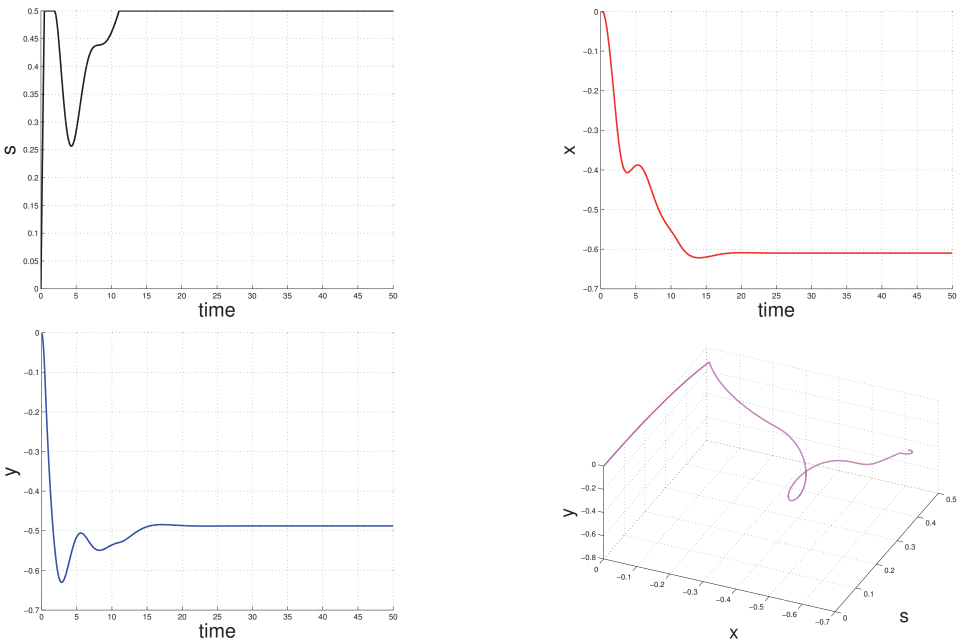


FIGURE 33 Dry friction load case. Modeling system (35) with parameters from region 3: $a = 2$, $c = 0,4$, $M = 1,1$, $\omega_1 = 0,5$

5 CONCLUSIONS

In this chapter the main results of this thesis and further research steps are presented and discussed briefly. Note that this work is a direct continuation and extension of Master's thesis of the author (Solovyeva, 2011).

In this work the induction motors with squirrel-cage, double squirrel-cage and wound rotors have been studied. The original mathematical models of considered motors were developed by the author. These models are explained and described in Chapter 2.2. By introducing a coordinate system, rigidly connected with rotating magnetic field, the dynamics of rotors was described in details. Due to suggested nonsingular changes of coordinates the investigation of stability of these models was reduced to stability analysis of third and fifth order differential equations.

On the basis of the new models developed, steady-state and dynamic stability analysis is performed in Chapter 3. Global stability of induction motors under no-load condition is proved analytically in Chapter 3.2.

Another contribution of the author is the solution of the limit load problem for induction motors, presented in Chapter 3.3. Moreover, two types of loads were considered: constant load and dry friction load. For both cases the estimations of the limit permissible load were obtained by the modified non-local reduction method. These estimations substantially improve engineering estimations, obtained by the equal-area criterion. However, results of the numerical simulations, demonstrated in Chapter 4, shown that found estimations are not optimal. In other words, the estimations obtained by the theorems are smaller than the actual limit loads. Hence, the estimations can be improved and a more careful analysis is required.

Further research work is intended to develop analytical methods of analysis of mathematical models of induction motors and to study the models of induction motors with viscous friction and fan type loads.

Note that the basic assumption behind the modelling that stator windings are fed from a powerful source of sinusoidal voltage was made according to classical theoretical works (Adkins, 1957; White and Woodson, 1959; Skubov and Khodzhaev, 2008). In this case the influence of rotor currents on stator currents can be ignored. Obviously such assumption is allowable for description the dynamics of induction machines working in operating modes and during transient processes, but does not allow one to describe the start-up of these machines if high currents which flow through the rotor windings. Further we need to verify the results obtained in this thesis in the practice and engineering experts are required. That is the next step of our work.

The main results of this thesis could be useful for electrical engineers.

YHTEENVETO (FINNISH SUMMARY)

Tässä työssä mallinnetan matemaattisesti erilaisia roottoreita käyttäviä asynkronimoottoreita ja analysoidaan näiden värähtelyominaisuuksia. Aihe on ajankohtainen, sillä asynkronimoottorien käyttö voimansiirtolaitteina teollisuudessa lisääntyy jatkuvasti. Työssä on kehitetty matemaattisia malleja asynkronimoottoreille, joissa on oravanhäkki- tai kaksoisoravanhäkki-rakenne, kiinteästi magneettikenttään liitetyssä pyörivässä koordinaattijärjestelmässä. Erotuksena tunnetuista asynkronimoottorien matemaattisista malleista kyseessä olevissa malleissa on täysin huomioitu roottorin ulkomuoto. Käyttäen erityisiä koordinaattien ei-singulaarisia muutoksia voidaan näiden järjestelmien stabiilisuutta tutkia analysoimalla kolmannen ja viidennen asteen differentiaaliyhtälöiden stabiilisuutta. Näin ollen kyseiset mallit ovat kuvattavissa melko yksinkertaisten differentiaaliyhtälöiden avulla, mikä mahdollistaa mallien syvällisen laadullisen analyysin. Malleille on luotu ehdot lokaalille ja globaalille stabiilisuudelle. Lisäksi on tarkasteltu asynkronimoottorien dynaamista stabiilisuutta erilaisten kuormitusmomenttityyppien yhteydessä. Vakiokuormitusmomentin tapauksessa asynkronimoottorien käyttäytymistä mallinnetaan tavallisilla differentiaaliyhtälöillä. Kuivana kitkana esiintyvän kuormitusmomentin tapauksessa asynkronimoottorien käyttäytymistä puolestaan mallinnetaan differentiaali-inkluusioilla. Järjestelmien laadullinen analyysi mahdollistaa sen, että asynkronimoottorin parametreille, kuten vastukselle, induktiivisuudelle ja vääntömomentille, voidaan määrittää turvalliset muutosikkunait, joiden sisällä asynkronimoottori säilyttää toimintakykynsä.

Lisäksi tarkastellaan asynkronimoottorien maksimaalista kuormitusta ja johdetaan teoreettiset rajat turvalliselle kuormitukselle. Saadut teoreettiset tulokset on vahvistettu numeerisilla testeillä.

Tietokonesimuloinnin avulla on tutkittu myös erikoistapausta, jossa syntyy äkillinen kuormitus tyhjäkäynnillä olevalle asynkronimoottorille.

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APPENDIX 1 PROOF OF NONSINGULARITY OF COORDINATES TRANSFORMATION

Proposition 1. *A transformation of coordinates*

$$
\begin{aligned}
\vartheta &= -\theta, \\
s &= -\dot{\theta}, \\
x &= -\frac{2L_1}{n_1 l_1 l B} \sum_{k=1}^{n_1} \sin(-\theta - \frac{2k\pi}{n_1}) i_k, \\
y &= -\frac{2L_1}{n_1 l_1 l B} \sum_{k=1}^{n_1} \cos(-\theta - \frac{2k\pi}{n_1}) i_k, \\
z_k &= \sum_{p=-m}^{m} i_{(k+p)\, mod n_1} - \mathrm{ctg}(\frac{\pi}{n_1}) i_k, \qquad k = 2, ..., n_1 - 1, \\
\mu &= -\frac{2L_2}{n_2 l_2 l B} \sum_{k=1}^{n_2} \sin(-\theta - \frac{2k\pi}{n_2}) j_k, \\
\nu &= -\frac{2L_2}{n_2 l_2 l B} \sum_{k=1}^{n_2} \cos(-\theta - \frac{2k\pi}{n_2}) j_k, \\
\upsilon_k &= \sum_{p=-m}^{m} j_{(k+p)\, mod n_2} - \mathrm{ctg}(\frac{\pi}{n_2}) j_k, \qquad k = 2, ..., n_2 - 1
\end{aligned}
\tag{42}
$$

is a non-singular transformation.

Proof. For brevity, we introduce the following notations:

$$C_1 = \frac{2L_1}{n_1 l_1 l B}; \qquad C_2 = \frac{2L_2}{n_2 l_2 l B};$$

$$\vartheta_k^1 = -\theta - \frac{2k\pi}{n_1}, \qquad k = 1, ..., n_1;$$

$$\vartheta_k^2 = -\theta - \frac{2k\pi}{n_2}, \qquad k = 1, ..., n_2;$$

$$
\alpha = \begin{pmatrix} C_1 \sum_{k=1}^{n_1} i_k \cos \vartheta_k^1 \\ -C_1 \sum_{k=1}^{n_1} i_k \sin \vartheta_k^1 \\ 0 \\ \dots \\ 0 \end{pmatrix} \in \mathbb{R}^{n_1}; \qquad
\beta = \begin{pmatrix} C_2 \sum_{k=1}^{n_2} j_k \cos \vartheta_k^2 \\ -C_2 \sum_{k=1}^{n_2} j_k \sin \vartheta_k^2 \\ 0 \\ \dots \\ 0 \end{pmatrix} \in \mathbb{R}^{n_2};
$$

$$
D_1 = \begin{pmatrix}
C_1 \sin \vartheta_1^1 & \dots & C_1 \sin \vartheta_{k-m}^1 & \dots & C_1 \sin \vartheta_k^1 & \dots & C_1 \sin \vartheta_{k+m}^1 & \dots & C_1 \sin \vartheta_{n_1}^1) \\
C_1 \cos \vartheta_1^1 & \dots & C_1 \cos \vartheta_{k-m}^1 & \dots & C_1 \cos \vartheta_k^1 & \dots & C_1 \cos \vartheta_{k+m}^1 & \dots & C_1 \cos \vartheta_{n_1}^1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 1 & \dots & 1 - \mathrm{ctg}(\frac{\pi}{n_1}) & \dots & 1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix};
$$

$$D_2 = \begin{pmatrix} C_2 \sin \vartheta_1^2 & \dots & C_2 \sin \vartheta_{k-m}^2 & \dots & C_2 \sin \vartheta_k^2 & \dots & C_2 \sin \vartheta_{k+m}^2 & \dots & C_2 \sin \vartheta_{n_2}^2) \\ C_2 \cos \vartheta_1^2 & \dots & C_2 \cos \vartheta_{k-m}^2 & \dots & C_2 \cos \vartheta_k^2 & \dots & C_2 \cos \vartheta_{k+m}^2 & \dots & C_2 \cos \vartheta_{n_2}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 1 - \mathrm{ctg}(\frac{\pi}{n_2}) & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The transformation of coordinates (42) is nonsingular one if the Jacobian is nonzero:

$$\det M = \det \begin{pmatrix} -1 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & -1 & \mathbf{0}^T & \mathbf{0}^T \\ \alpha & \mathbf{0} & D_1 & \mathbf{0} \\ \beta & \mathbf{0} & \mathbf{0} & D_2 \end{pmatrix} = \det D_1 \det D_2 \neq 0.$$

The matrices D_1 and D_2 differs from each other only in their degrees. In PV it was shown that the determinants of such matrices are independent of θ and, hence, the determinant of the matrix M is constant, which depends only on the parameters n_1 and n_2. Using results obtained in PV we get that $\det D_{1,2} = 4\beta^2$ for $n_{1,2} = 4$, $\det D_{1,2} = 241.137\beta^2$ for $n_{1,2} = 8$, $\det D_{1,2} = 3.4 \cdot 10^5 \beta^2$ for $n_{1,2} = 12$, $\det D_{1,2} = 2.8 \cdot 10^9 \beta^2$ for $n_{1,2} = 16$, $\det D_{1,2} = 8.24 \cdot 10^{13} \beta^2$ for $n_{1,2} = 20$, $\det D_{1,2} = 6.02 \cdot 10^{18} \beta^2$ for $n_{1,2} = 24$, $\det D_{1,2} = 9.42 \cdot 10^{23} \beta^2$ for $n_{1,2} = 28$, $\det D_{1,2} = 2.8 \cdot 10^{29} \beta^2$ for $n_{1,2} = 32$. The numerical calculation of $\det D_{1,2}$ can be continued further; however, generally the number of bars in outer or inner squirrel-cage is less than 32. Thus, the nonsingularity of coordinates transformation (42) is proved in the case $n_1 \leqslant 32$ and $n_2 \leqslant 32$. □

Proposition 2. *A transformation of coordinates*

$$\begin{aligned} \vartheta &= \tfrac{\pi}{2} - \theta, \\ s &= -\dot{\theta}, \\ x &= -\tfrac{2}{3}\tfrac{L}{nSB}\left(i_1 \sin(\tfrac{\pi}{2} - \theta) + i_2 \sin(\tfrac{\pi}{2} - \theta - \tfrac{2\pi}{3}) + i_3 \sin(\tfrac{\pi}{2} - \theta - \tfrac{4\pi}{3})\right), \\ y &= -\tfrac{2}{3}\tfrac{L}{nSB}\left(i_1 \cos(\tfrac{\pi}{2} - \theta) + i_2 \cos(\tfrac{\pi}{2} - \theta - \tfrac{2\pi}{3}) + i_3 \cos(\tfrac{\pi}{2} - \theta - \tfrac{4\pi}{3})\right), \\ z &= i_1 + i_3 - i_2 \end{aligned}$$

is a non-singular transformation.

Proof. For brevity, we introduce the following notations:

$$C = \frac{2}{3}\frac{L}{nSB}, \qquad \vartheta_k = \frac{\pi}{2} - \theta - \frac{2(k-1)\pi}{3}, \quad k = 1, 2, 3.$$

Then we obtain

$$\det\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & C\sum\limits_{k=1}^{3} i_k \cos\vartheta_k & -C\sin\vartheta_1 & -C\sin\vartheta_2 & -C\sin\vartheta_3 \\ 0 & -C\sum\limits_{k=1}^{3} i_k \sin\vartheta_k & -C\cos\vartheta_1 & -C\cos\vartheta_2 & -C\cos\vartheta_3 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} =$$

$$= \det\begin{pmatrix} -C\sin\vartheta_1 & -C\sin\vartheta_2 & -C\sin\vartheta_3 \\ -C\cos\vartheta_1 & -C\cos\vartheta_2 & -C\cos\vartheta_3 \\ 1 & -1 & 1 \end{pmatrix} = C^2(\sin\vartheta_2\cos\vartheta_3 -$$

$$- \sin\vartheta_3\cos\vartheta_2 + \sin\vartheta_1\cos\vartheta_2 - \sin\vartheta_2\cos\vartheta_1 + \sin\vartheta_1\cos\vartheta_3 - \sin\vartheta_3\cos\vartheta_1) =$$

$$= C^2(\sin\vartheta_2\cos\vartheta_3 - \sin\vartheta_3\cos\vartheta_2 - 2\sin\vartheta_1\cos\vartheta_1\cos\frac{\pi}{3} + 2\cos\vartheta_1\sin\vartheta_1\cos\frac{\pi}{3}) =$$

$$= \frac{C^2}{2}(\sin(2\vartheta_1 - 2\pi) + \sin\frac{2\pi}{3} - \sin(2\vartheta_1 - 2\pi) + \sin\frac{2\pi}{3}) = C^2\sin\frac{2\pi}{3} \neq 0.$$

□

APPENDIX 2 PROOF OF THEOREMS

Recall notations (25) introduced in section 3.4 ($\gamma = M$):

$$\Gamma = 2 \max_{\lambda \in (0,c)} \left[\lambda \left(c - \lambda - \frac{M^2}{4c^2(c-\lambda)} \right) \right]^{1/2},$$
$$\psi(s) = -\frac{M}{c} s^2 + \alpha s - cM,$$
$$s_1 = \frac{c(\alpha + \sqrt{\alpha^2 - 4M^2})}{2M},$$

where $\alpha = a$ in the case cage and wound rotors (system (35)) and $\alpha = a_1 + a_2$ in the case of double cage rotor (system (34)).

Theorem 5. *Suppose that* $s_0 < \omega_1$, $M < \min \left\{ 2c^2, \ \frac{a_1 + a_2}{2} \right\}$ *and the solution of the equation*

$$F(s)F'(s) = -\Gamma F(s) - \psi(s) \tag{43}$$

with initial data $F(s_1) = 0$ *fulfils the conditions*

$$F(s_*) > \sqrt{(a_1 y_* + a_2 \mu_* + M)^2 + (a_1 x_* + a_2 \nu_* + \frac{M}{c} s_*)^2}, \tag{44}$$

$$F(\omega_1) < M, \qquad \text{if} \ \ s_1 > \omega_1. \tag{45}$$

Then the solution of system (34) with initial data $s = s_*$, $x = x_*$, $\mu = \mu_*$, $y = y_*$, $\nu = \nu_*$ *satisfies the relations*

$$\lim_{t\to\infty} s(t) = s_0, \lim_{t\to\infty} x(t) = x_0, \lim_{t\to\infty} \mu(t) = \mu_0, \lim_{t\to\infty} y(t) = y_0, \lim_{t\to\infty} \nu(t) = \nu_0. \tag{46}$$

Proof. In the case $s_1 < \omega_1$ system (34) completely coincides with system (8) and the solution of system (34) is in the region $\left\{ (s,\ x,\ y,\ \mu,\ \nu)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle| \ s < \omega_1 \right\}$. Therefore, the proof in this case is the same as the proof of theorem 4.

Consider the case $\omega_1 < s_1$. The nonsingular transformation of coordinates

$$\eta = a_1 y + a_2 \nu,$$

$$z = -a_1 x - a_2 \mu,$$

$$\zeta = y - \nu,$$

$$\xi = x - \mu,$$

reduces system (34) to the form

$$\begin{aligned}
\dot{s} &= \eta + \gamma_l, \\
\dot{\eta} &= -c\eta - zs - as, \\
\dot{z} &= -cz + \eta s, \\
\dot{\zeta} &= -c\zeta - \xi s, \\
\dot{\xi} &= -c\xi + \zeta s.
\end{aligned} \tag{47}$$

Here

$$\gamma_l = \begin{cases} M & \text{if } \ s < \omega_1 \ \text{ or } \ s = \omega_1, \eta > M, \\ \eta & \text{if } \ s = \omega_1, -M \leq \eta \leq M, \\ -M & \text{if } \ s > \omega_1 \ \text{ or } \ s = \omega_1, \eta < -M. \end{cases}$$

Let us introduce the regions:

$$D_1 = \left\{ (s,\ \eta,\ z,\ \zeta,\ \xi)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle| \ s < \omega_1 \right\},$$

$$D_2 = \left\{ (s,\ \eta,\ z,\ \zeta,\ \xi)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle| \ s > \omega_1 \right\},$$

$$H = \left\{ (s,\ \eta,\ z,\ \zeta,\ \xi)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle| \ s = \omega_1 \right\},$$

$$\Delta = \left\{ (s,\ \eta,\ z,\ \zeta,\ \xi)^{\mathrm{T}} \in \mathbb{R}^5 \ \middle| \ s = \omega_1, \quad -\gamma < \eta < \gamma \right\}.$$

In the slip region Δ system (47) is determined as follows:

$$\begin{aligned}
\dot{s} &= 0, \\
\dot{\eta} &= -c\eta - z\omega_1 - a\omega_1, \\
\dot{z} &= -cz + \eta\omega_1, \\
\dot{\zeta} &= -c\zeta - \xi\omega_1, \\
\dot{\xi} &= -c\xi + \omega_1\zeta,
\end{aligned} \tag{48}$$

This system has one asymptotically stable equilibrium state

$$s = \omega_1,\ \eta = -\frac{ac\omega_1}{\omega^2 + c^2},\ z = -\frac{a\omega_1^2}{\omega^2 + c^2},\ \zeta = 0,\ \xi = 0,$$

which is below the lower bound of the slip region Δ and, hence, is never achieved. Thus, system (47) has one asymptotically stable equilibrium state corresponding to an operating mode

$$s = s_0, \eta = -\frac{cas_0}{s_0^2 + c^2}, z = -\frac{as_0^2}{s_0^2 + c^2}, \zeta = 0, \xi = 0.$$

Note that the initial data $s_*, x_*, y_*, \mu_*, \nu_*$ of system (34) transforms into the initial data

$$s = s_*, \eta = a_1 y_* + a_2 \nu_*, z = -a_1 x_* - a_2 \mu_*, \zeta = y_* - \nu_*, \xi = x_* - \mu_*$$

of new system (47). Show that the solution of system (47) with such initial data is bounded. For this purpose we consider the function

$$W(s,\eta,z,\zeta,\xi) = \frac{1}{2}\left[(\eta+M)^2 + (z+\frac{M}{c}s)^2 + \zeta^2 + \xi^2 - F^2(s)\right]$$

and the region Ω

$$\Omega = \{(s,\eta,z,\zeta,\xi) \mid \quad W(s,\eta,z,\zeta,\xi) \le 0\}\,.$$

Taking into account $c\Gamma > g$ and

$$\psi(\sigma) > 0, \qquad \forall \sigma \in [\omega_1; s_1)\,,$$

the function $W(s,\eta,z,\zeta,\xi)$ on the solutions of system (47) satisfies the relations in the region Ω

$$\dot{W}(s,\eta,z,\zeta,\xi) = (\eta+M)[-c\eta - zs - as] + \left(z+\frac{M}{c}s\right)\left[-cz+\eta s+\frac{M}{c}(\eta+\gamma_l)\right] +$$
$$+\zeta[-c\zeta - \xi s] + \xi[-c\xi + \zeta s] - F(s)F'(s)\,[\eta+\gamma_l] = -c(\eta+M)^2 - c(z+\frac{M}{c}s)^2 -$$
$$-c\zeta^2 - c\xi^2 + \frac{M}{c}\left(z+\frac{M}{c}s\right)(\eta+\gamma_l) - \psi(s)[\eta+\gamma_l] - F(s)F'(s)[\eta+\gamma_l] = -c\zeta^2 -$$
$$-c\xi^2 - c(\eta+M)^2 - c(z+\frac{M}{c}s)^2 + \frac{M}{c}\left(z+\frac{M}{c}s\right)(\eta+M) - \Gamma F(s)[\eta+M] -$$
$$-(M-\gamma_l)\left(\frac{M}{c}\left[z+\frac{M}{c}s\right] + \psi(s) + \Gamma F(s)\right) \le -c(\eta+M)^2 - c(z+\frac{M}{c}s)^2 - c\zeta^2 -$$
$$-c\xi^2 + \frac{M}{c}\left(z+\frac{M}{c}s\right)(\eta+M) - \Gamma F(s)[\eta+M].$$

From here and using Gurvic criterion, we obtain

$$\dot{W}(s,\eta,z,\zeta,\xi) + 2\lambda\, W(s,\eta,z,\zeta,\xi) = -(c-\lambda)(\eta+M)^2 - (c-\lambda)\left(z+\frac{M}{c}s\right)^2 -$$
$$-(c-\lambda)\zeta^2 - (c-\lambda)\xi^2 - \lambda\, F^2(s) + \frac{M}{c}\left(z+\frac{M}{c}s\right)(\eta+M) + \Gamma F(s)(\eta+M) \le 0$$

Thus, the region Ω is the invariant set.

Next, equation (43) has either a solution $F(s)$ (curve 1 in Fig. 34) such that

$$F(s_2) = F(s_1) = 0, \qquad s_2 < \min s_*, s_0$$

$$F(s) > 0, \qquad \forall s \in (s_2, s_1)$$

or a solution $F(s)$ (curve 2 in Fig. 34) such that

$$F(s) > 0, \qquad \forall s \in (-\infty, s_1).$$

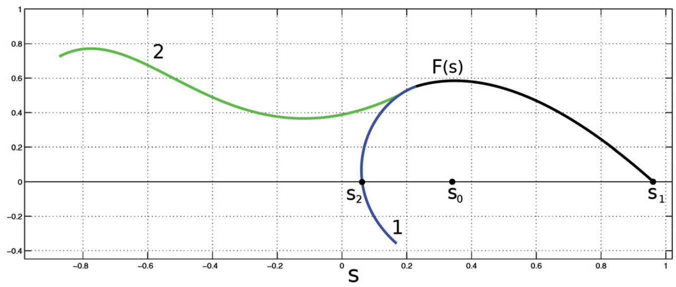

FIGURE 34 Two possible cases of behavior of the solution of equation (43)

In the first case the region Ω is bounded. In the second case we introduce the function

$$V(s,\eta,z,\zeta,\xi) = \frac{1}{2}(\eta+M)^2 + \frac{1}{2}\left(z + \frac{M}{c}s\right)^2 + \frac{1}{2}\zeta^2 + \frac{1}{2}\xi^2 + \int\limits_{s_1}^{s} \psi(\sigma)\mathrm{d}\sigma.$$

In the region D_1 this function on the solutions of system (47) satisfies the relation

$$\dot{V}(s,\eta,z,\zeta,\xi) \le 0.$$

Hence, the region Ω can be bounded by

$$\Omega_2 = \left\{(s,\eta,z,\zeta,\xi) \quad \middle| \quad V(s,\eta,z,\zeta,\xi) < C, \quad s \in [s_3, +\infty)\right\},$$

where s_3 is the solution of the equation

$$\int\limits_{x}^{s_1} \psi_M(\sigma)\mathrm{d}\sigma = C, \quad x < s_1,$$

$$C > \max\left\{\frac{1}{2}\gamma^2 - \int\limits_{s_*}^{s_1} \psi_M(\sigma)\mathrm{d}\sigma, 0\right\}.$$

Consequently, the region Ω is the bounded invariant set.

From condition (44) of the theorem and expressions for C it follows that

$$W(s_*, \eta_*, z_*, \zeta_*, \xi_*) < 0,$$

$$V(s_*, \eta_*, z_*, \zeta_*, \xi_*) < 0.$$

As the result we proved that the trajectory of system (47) with initial data $s_*, \eta_*, z_*, \zeta_*, \xi_*$ is bounded.

Let us study the behaviour of the solution of system (47) for $s = \omega_1$. The region Ω in this case takes the following form (see Fig. 35)

$$\Omega\big|_H = \left\{ (\eta, z, \zeta, \xi) \in H \quad \middle| \quad (\eta + M)^2 + \left(z + \frac{M}{c}s\right)^2 + \zeta^2 + \xi^2 \leq 4M^2 \right\}.$$

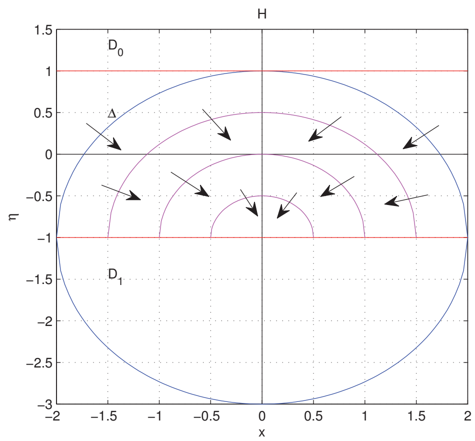

FIGURE 35 Restriction of the region Ω to the region H

Condition (45) guaranties that the region $\Omega\big|_H$ is below the upper bound of the slip region $\Omega\big|_H$ and crosses the lower bound. Consider the function in slip region

$$V_H(\eta, z, \zeta, \xi) = \frac{1}{2}\left[(\eta + M)^2 + \left(z + \frac{M}{c}s\right)^2 + \zeta^2 + \xi^2\right],$$

For the function $V_H(\eta, z, \zeta, \xi)$, the following relation is valid:

$$\dot{V}_H(\eta, z, \zeta, \xi) \leq 0. \tag{49}$$

The surfaces $V_H(\eta, z, \zeta, \xi) = \text{const}$ are non-contact. Moreover, they are the restriction of non-contact surfaces $V(s, \eta, z, \zeta, \xi) = \text{const}$ to the region H. Let be

$$\Omega_1 = \Omega \cap (D_1 \cup \Delta).$$

If the solution is in the region Ω_1, then there exists two cases:

1. either the solution never achieves the region Δ, i.e., it is in the subspace D_1. It follows from the determination γ_l;
2. or the solution falls into the slip region Δ. Then, taking into account (49), the solution falls into the lower bound of the slip region $\eta = -M$ and goes out through it in the region $\Omega \cap D_1$.

Suppose that the solution falls into the region $\Omega \backslash \Omega_1$, then consider the function

$$U(s,\eta,z,\zeta,\xi) = \frac{1}{2}\left[(\eta - M)^2 + \left(z - \frac{M}{c}s\right)^2 + \zeta^2 + \xi^2 + \right.$$

$$\left. + \int_{*}^{s} \left(\frac{M}{c}\sigma^2 + a\sigma + Mc\right) d\sigma \right].$$

For any solution of system (47) from the subspace D_2 the following relation is satisfied:

$$\dot{U}(s,\eta,z,\zeta,\xi) \leq 0.$$

The function $U(s,\eta,z,\zeta,\xi)$ also defines the non-contact surfaces in the region D_2 (see Fig. 36). Since the function $U(s,\eta,z,\zeta,\xi)$ decreases in the region D_2, then the trajectory falls into the region Ω_1.

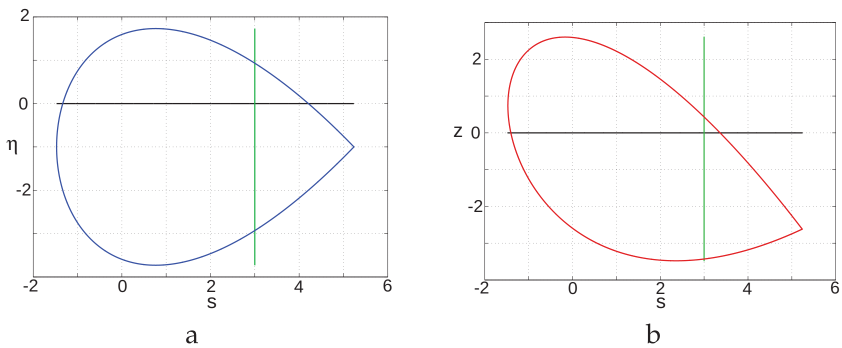

FIGURE 36 Non-contact surfaces, defined by the function $U = \text{const}$

Thus, the solution of system (47) with initial data

$$s_*, \eta_*, z_*, \zeta_*, \xi_*$$

falls into the region Ω_1 and remains in it.

The function $V(s,\eta,z,\zeta,\xi)$ decreases in the region D_1 and in the slip region Δ. Hence, it decreases in the region Ω_1:

$$\dot{V}(s,\eta,z,\zeta,\xi) \leq 0. \tag{50}$$

Let $x(t) = (s(t), \eta(t), z(t), \zeta(t), \xi(t))$ be a solution of system (47) for $t \geq 0$ in the bounded region Ω_1. Then the function $V(x(t))$ is also bounded for $t \geq 0$.

This and the decrease of the function $V(x(t))$ for $t \geq 0$ imply the existence of a finite limit $\lim_{t\to+\infty} V(x(t)) = L$.

Since the trajectory $x(t)$ is bounded, then the set Ω_0 of its ω-limit points is not empty. Let $x_* \in \Omega_0$. The set Ω_0 is invariant and, hence, the trajectory, starting from the point x_* is contained in Ω_0, for all $t \in \mathbb{R}$. Therefore for all $t \in \mathbb{R}$ we get

$$V(x(t, x_*)) \equiv L.$$

Using (50), we obtain $\eta(t, x_*) \equiv 0$, $z(t, x_*) \equiv 0$, $\zeta(t, x_*) \equiv 0$ and $\xi(t, x_*) \equiv 0$. Relations (47) and (50) imply $\dot{s}(t, x^*) \equiv 0$. Consequently, $s(t, x_*) \equiv$ const and the set Ω_0 is a subset of the stationary set of system (47).

Thus, any solution of system (47) in Ω_1 tends to an equilibrium point. This fact, the positive invariance of the set Ω_1 and the inclusions

$$(s_*, a_1 y_* + a_2 \nu_*, -a_1 x_* - a_2 \mu_*, y_* - \nu_*, x_* - \mu_*) \in \Omega_1,$$

$$(s_0, -\frac{cas_0}{s_0^2 + c^2}, -\frac{as_0^2}{s_0^2 + c^2}, 0, 0) \in \Omega_1$$

imply (46). □

Theorem 6. *Suppose that* $s_0 < \omega_1$, $M < \min\left\{ 2c^2, \ \frac{a}{2} \right\}$ *and the solution of the equation*

$$F(s)F'(s) = -\Gamma F(s) - \psi(s)$$

with initial data $F(s_1) = 0$ *fulfils the conditions*

$$F(s_*) > \sqrt{(ay_* + M)^2 + (ax_* + \frac{M}{c} s_*)^2},$$

$$F(\omega_1) < M, \qquad \text{if} \ \ s_1 > \omega_1.$$

Then the solution of system (35) with initial data $s = s_*$, $x = x_*$, $y = y_*$ *satisfies the relations*

$$\lim_{t\to\infty} s(t) = s_0, \lim_{t\to\infty} x(t) = x_0, \lim_{t\to\infty} y(t) = y_0.$$

Proof. The proof of the theorem is similar to the proof of theorem 5 and is based on the functions

$$W(s, x, y) = \frac{1}{2}\left[(ay + M)^2 + a^2(x + \frac{M}{c}s)^2 - F^2(s)\right],$$

$$V(s, x, y) = \frac{1}{2}\left[(ay + M)^2 + a^2(x + \frac{M}{c}s)^2 + \int_{s_1}^{s} \psi(\sigma)\mathrm{d}\sigma\right].$$

□

APPENDIX 3 COMPUTER MODELING OF SYSTEMS DESCRIBING INDUCTION MOTORS UNDER VARIOUS LOADS (MATLAB IMPLEMENTATION)

Program code in Matlab for simulating the behavior of induction motors under constant loads and dry friction loads.

```
function [F1, F2, H, dH] = system3d(z, load)
global params3d;                        % System parameters
  a = params3d(1); c = params3d(2);
  %% Values of the phase vector
  s = z(1); x = z(2); y = z(3);
  %% Vector field without load
  F = [ a*y;                    ... % \dot{s} - \gamma_l
       -c*x + y*s;              ... % \dot{x}
       -c*y - x*s - s;  ]; ... % \dot{y}
  %% Vector fields in regions
  [H, f1, f2, j1, j2] = feval(load, s);
  F1 = F + [f1; 0; 0; ];        % (n1 < s)
  F2 = F + [f2; 0; 0; ];        % (n1 > s)
  %% Vector normal of discontinuity surface
  dH = [-1, 0, 0];              % dH = grad(H)
end
```

```
function [F1, F2, H, dH] = system5d(z, load)
global params5d;                          % System parameters
  a1 = params5d(1); a2 = params5d(2);
  c1 = params5d(3); c2 = params5d(4);
  %% Values of the phase vector
  s = z(1); x = z(2); y = z(3); mu = z(4); nu = z(5);
  %% Vector field without load
  F = [ a1*y + a2*nu;               ... % \dot{s} - \gamma
       -c1*x + y*s;                 ... % \dot{x}
       -c1*y - x*s - s;             ... % \dot{y}
       -c2*mu + nu*s;               ... % \dot{\mu}
       -c2*nu - mu*s - s;     ]; ... % \dot{\nu}
  %% Vector fields in regions
  [H, f1, f2, j1, j2] = feval(load, s);
  F1 = F + [f1; 0; 0; 0; 0; ];      % (n1 < s)
  F2 = F + [f2; 0; 0; 0; 0; ];      % (n1 > s)
  %% Vector normal of discontinuity surface
  dH = [-1, 0, 0, 0, 0];    % dH = grad(H)
end
```

```
function [J1,J2,d2H] = jacobian3d(z, gamma)
global params3d;             % System parameters
  a = params3d(1); c = params3d(2);
  %% Values of the phase vector
  s = z(1); x = z(2); y = z(3);
  %% Jacobian wuthout load
  J = [ 0,   0,  a;      ... % s
        y, -c,  s;       ... % x
     -x-1, -s, -c;  ]; ... % y
  %% Jacobians in regions
  [H, f1, f2, j1, j2] = feval(gamma, s);
  J1 = J; J1(1,1) = J1(1,1) + j1;  % (n1 > s)
  J2 = J; J2(1,1) = J1(1,1) + j2;  % (n1 < s)
  %% (d2/dx2)H(x) - second derivative of H(x)
  d2H = zeros(3, 3);
end
```

```
function [J1,J2,d2H] = jacobian5d(z, gamma)
global params5d;                                % System parameters
  a1 = params5d(1); a2 = params5d(2);
  c1 = params5d(3); c2 = params5d(4);
  %% Values of the phase vector
  s = z(1); x = z(2); y = z(3); mu = z(4); nu = z(5);
  %% Jacobian wuthout load
  J = [    0,    0,   a1,    0,   a2;      ... % s
           y, -c1,    s,    0,    0;      ... % x
        -x-1,   -s, -c1,    0,    0;      ... % y
          nu,    0,    0, -c2,    s;      ... % mu
      -mu-1,    0,    0,   -s, -c2;   ]; ... % nu
  %% Jacobians in regions
  [H, f1, f2, j1, j2] = feval(gamma, s);
  J1 = J; J1(1,1) = J1(1,1) + j1;          % (n1 > s)
  J2 = J; J2(1,1) = J1(1,1) + j2;          % (n1 < s)
  %% (d2/dx2)H(x) - second derivative of H(x)
  d2H = zeros(5, 5);
end
```

```
function [ H, f1, f2, j1, j2 ] = load(s)
global params;
  M = params(1); k = params(2); p = params(3); n1 = params(4);
  if n1 == -1
    H = 1;
  else
    H = n1 - s;
  end
  f1 =  M + k*(n1 - s)^p; f2 = -f1;
  j1 = -k*p*(n1-s)^(p-1); j2 = -j2;
end
```

```
function [tvect, yvect] = filippov(system, jacobian, load, time, z0)
yvect = []; tvect = [];
%% Set ODE options
options = odeset('RelTol', 1e-2, 'AbsTol', 1e-3, ...
                 'MaxStep', 0.01, 'Events', @fevents);
[state, dir] = findstate(system, jacobian, z0, load);
stopit = 0; t0 = 0;
while ~stopit
    [t,z,TE,YE,IE] = feval('ode45', @filippovfunc, [t0 time], z0, ...
                         options, system, jacobian, state, dir, load);
    z0 = z(end,:); t0 = t(end);
    yvect = [yvect;z];
    tvect = [tvect;t];

    if ~isempty(IE) && (t0~=time)
        for k = 1:length(IE)
            if IE(k) ~= 4
                switch 1
                    case state(3)
                        switch IE(k)
                            case {2,3}
                                state(IE(k) - 1) = - state(IE(k) - 1);
                                state([3, 4, 5]) = - state([3, 4, 5]);
                                dir([1, IE(k)])  = - [1, dir(IE(k))];
                            case 5,

                            otherwise
                                disp('ERROR: Wrong event in filippov')
                        end
                    case state(4)
                        switch IE(k)
                            case 1,
```

```
                                state([1, 2]) = - state([1, 2]);
                                dir(IE(k))    = - dir(IE(k));
                            case {2,3}
                                state([4, 5]) = - state([4, 5]);
                                dir(IE(k))    = - dir(IE(k));
                            case 5,

                            otherwise
                                disp('ERROR: Wrong event in filippov')
                        end
                    case state(5)
                        switch IE(k)
                            case 1,
                               state([1, 2, 3]) = - [1, 1, state(3)];
                               dir(IE(k))       = - dir(IE(k));
                            case {2,3}
                                state([4, 5]) = -state([4, 5]);
                                dir(IE(k))    = -dir(IE(k));
                            case 5,

                           otherwise
                                disp('ERROR: Wrong event in filippov')
                        end
                    otherwise
                        disp('ERROR: Wrong state vector in filippov')
                end
            end
        end
    else
       stopit =1;
    end
end

end

%-------------------- findstate ------------------------

function [state,dir] = findstate(vfields, jacobians, z0, load)
    state = -1*ones(1, 5); % [-1, -1, -1, -1, -1]
    [F1, F2, H, dH] = feval(vfields, z0, load);
    dHF1 = dH*F1;  dHF2 = dH*F2;
    dir = [-sign(H), -sign(real(dHF1)), -sign(real(dHF2))];
    %% Find current state
    if      H > 0
        state(1) = -state(1);
    elseif H < 0
        state(2) = -state(2);
    elseif sign(dHF1)*sign(dHF2) < 0
        state(3) = -state(3);
    else
        if sign(dHF1) > 0
            state(1) = -state(1);
        else
            state(2) = -state(2);
        end
    end
    %% Difficalty function
    if sign(dHF1)*sign(dHF2) > 0
        state(4) = -state(4);
    elseif sign(dHF1)*sign(dHF2) < 0
        state(5) = -state(5);
    else
        if isempty(jacobians)
            state(4) = -state(4);
        else
```

```
            [J1,J2,d2H] = feval(jacobians, z0, load);
            if dHF1 == 0
                HxF1x_F1Hxx = dH*J1 + F1'*d2H;
                sig = sign(HxF1x_F1Hxx*F1)*sign(dHF2);
                dir(2) = -sign(HxF1x_F1Hxx*F1);
            elseif dHF2 == 0
                HxF2x_F2Hxx = dH*J2 + F2'*d2H;
                sig = sign(HxF2x_F2Hxx*F2)*sign(dHF1);
                dir(3) = -sign(HxF2x_F2Hxx*F2);
            else
                disp('ERROR: Something is wrong in filippov:findstate')
                sig = 1;
            end

            if sig < 0
                state(5) = -state(5);
            else
                state(4) = -state(4);
            end
        end
    end
end

%-------------------- filippovfunc ------------------------

function [dz] = filippovfunc(t, z, vfields, jacobians, state, dir, load)
    [F1, F2, H, dH] = feval(vfields, z, load);
    switch 1
        case state(1) % s < n1
            dz = F1;
        case state(2) % s > n1
            dz = F2;
        case state(3) % s == n1 (sliding region)
            Fa   = 0.5*F1; Fb   = 0.5*F2;
            dHF1 = dH*F1;  dHF2 = dH*F2;
            Hu   = -((dHF1+dHF2)/(dHF2-dHF1));
            dz    = (Fa + Fb) + Hu*(Fb - Fa) - H*dH';
        otherwise
            disp('ERROR: Wrong state vector in filippov:filippovfunc')
    end
end
%-------------------- filippovevents ------------------------

function [value, isterminal, direction] = fevents(t,z,vfields, ...
                                          jacobians,state,dir,load)
    [F1, F2, H, dH] = feval(vfields, z, load);

    value      = [H, dH*F1, dH*F2];
    direction  = dir;
    isterminal = [1, 1, 1, 0];

    switch 1
        case {state(1),state(2)}     % s != n1
            direction(1) = -state(1);

            value     = [value,     1];
            direction = [direction, 0];
        case state(3)                % s == n1
            value(1) = 1;

            [J1,J2,d2H] = feval(jacobians, z, load);

            F1p2 = F1 + F2;
            F2m1 = F2 - F1;

```

```
            dHF1p2 = value(2) + value(3);
            dHF2m1 = value(2) - value(2);

            Hu   = - dHF1p2/dHF2m1;
            dHu  = - (...
                      ((F1p2')*d2H + dH*(J1 + J2))*(dHF2m1) - ...
                      ((F2m1')*d2H + dH*(J2 - J1))*(dHF1p2)   ...
                    )/(dHF2m1^2);
            dz   = 0.5*(F1p2 + F2m1*Hu) - H*dH';
            value      = [value,     dHu*dz];
            direction  = [direction, 0];
        otherwise
            disp('ERROR: Wrong event in filippov:fevents')
    end
    value = real(value);
end
```

```
global params  ... %% Parameters for load [M, k, p, n1]
    params3d   ... %% Parameters for 3d systems [a1, c1]
    params5d;  ... %% Parameters for 5d systems [a1, a2, c1, c2]
%% System parameters
a1 = 0.1:0.1:2;
c1 = 0.5:0.5:6;
a2 = 0.1:0.1:2;
c2 = 0.1:0.1:2;
%% Load parameters
M  = 0.9:0.05:1;
k  = [0, 1];
p  = [0, 1, 2];
n1 = [1, 3, 6];          %% n1 = -1 for model without discontinuity
%% Modelling parameters
time = 20;               %% Time of modelling
z0 = [0, 0, 0, 0, 0];    %% Initial point
format = 'png';
for i1=1:length(a1)
  for j1=1:length(c1)
    for i2=1:length(a2)
      for j2=1:length(c2)
        params3d = [ a1(i1), c1(j1)];
        params5d = [ a1(i1), a2(i2), c1(j1), c2(j2) ];
        for iM=1:length(M)
          for iK=1:length(k)
            for iP=1:length(p)
              for iN1=1:length(n1)
                params = [M(iM), k(iK), p(iP), n1(iN1)];
                [t, z] = filippov('system5d','jacobian5d','load',...
                                                      time,z0);
                figure(1);
                    plot3(z(:, 1), z(:, 2), z(:, 3), '-b');
                    xlabel('s'); ylabel('x'); zlabel('y');
                    grid on;
                figure(2);
                    plot(t, z(:, 1), '-b');
                    xlabel('time'); ylabel('s');
                    grid on;
                save_and_close(params5d, format);
              end
            end
          end
        end
      end
    end
  end
end
```

```
function [ ] = save_and_close(p, format)
global params;
    mkdir(format);
    name = strcat(mat2str(p), '_', mat2str(params));
    name = strrep(strrep(name, ' ', '|'), '.', ',');
    file1 = strcat(num2str(length(p)+1), 'd_phase_', name, '.', format);
    file2 = strcat(num2str(length(p)+1), 'd_time_', name, '.', format);
    saveas(figure(1), strcat(format,'/',file1), format);
    saveas(figure(2), strcat(format,'/',file2), format);
end
```