Supersymmetric Technicolor

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Abstract

In this thesis, we consider a supersymmetrized version of the Standard Model extended with a technicolor sector. We specialize in the regime of the model in which the technicolor sector is fully responsible of the electroweak symmetry breaking and the natural elementary Higgses of the supersymmetric model only transmit the breaking to the Standard Model fermions, thereby giving them masses.

In particular, we consider the vacuum structure of the effective theory, find it to be different to that of a mere technicolor model and study the contributions of the new sector to the oblique electroweak parameters. Relative to the current experimental limits, we find the model viable.

Preface

I would like to express my gratitude to my supervisor Dr. Kimmo Tuominen for enthusiastic guidance and for introducing me to this interesting subject.

Many thanks belong also to all my friends for a great atmosphere and coffee breaks, without which the writing of this thesis would have been a lot more difficult.

Tiivistelmä

Hiukkasfysiikan standardimalli on tarkin tähän mennessä rakennettu fysikaalinen teoria, eikä suoria ristiriitoja (jos ei huomioida massiivisia neutriinoja) kokeellisten tulosten kanssa ole. Epäsuoria todisteita siitä, että standardimalli ei kuitenkaan ole riittävä, on olemassa. Ensinnäkin standardimallista ei löydy pimeän aineen kandidaattia eikä materian ja antimaterian välistä epäsymmetriaa voida selittää standardimallin avulla. Toisaalta standardimallin Higgs-sektori on teoreettisesti ongelmallinen.

Näistä ongelmista johtuen suurilla energioilla tarvitaan jokin perustavanlaatuisempi teoria selittämään hiukkasten vuorovaikutuksia. Viimeisten 50 vuoden aikana monia ehdotuksia on esitetty, mutta kokeellisesti saavutettavan energiaskaalan vasta lähestyessä standardimallin pätevyysalueen rajaa tietoa oikeanlaisesta laajennuksesta ei ole.

Tämä pro gradu -tutkielma käsittelee kahden tunnetun laajennusehdotuksen, teknivärin ja supersymmetrian, yhdistämistä siten, että standardimalliin ensin lisätään teknivärisektori, minkä jälkeen saatu kokonaisuus supersymmetrisoidaan. Tässä tutkielmassa keskitytään erityisesti sellaiseen malliin, jossa skaala, jolla tekniväri dominoi on supersymmetristä skaalaa matalampi ja tästä syystä sähköheikon symmetriarikon aiheuttaa täysin uusi teknivärisektori. Tämän symmetriarikon välittämiseen standardimallin fermioneille sitä vastoin tarvitaan supersymmetriaa.

Tämän mallin pohjalta muodostetaan sähköheikon skaalan efektiivinen teoria ja tutkitaan saadun teorian tyhjiörakennetta, joka osoittautuu erilaiseksi kuin pelkän teknivärin tapauksessa. Lisäksi tarkastellaan erityisesti saadun mallin pätevyyttä sähköheikon teorian tarkkuusmittausten valossa ja selvitetään, aiheuttavatko kosmologiset havainnot mallille lisärajoituksia.

Näiden tarkastelujen perusteella malli on hyvin sopusoinnussa nykyisten havaintojen kanssa.

Contents

1 Introduction

The Standard Model of particle physics (SM) has proven to be the most accurate physical theory thus far. Leaving non-zero neutrino masses out, no direct experimental discrepancies have come up. Yet, the particle physicists, in more or less unison, are ready to leave the non-sinking ship for something more profound. Why is this?

First of all, there is no dark matter candidate nor can the matter-antimatter asymmetry be explained within the realm of the SM. Moreover, there are some disturbing theoretical aspects in the SM that make it clear that the SM cannot be the final truth, even in the non-gravitational world.

This leads us to strive for a more profound theory, but in which direction to proceed? The first possibility is to take the top-down approach: to shake the whole foundations and build up something brand new. Maybe the most appealing alternative would be the way of supersymmetry (SUSY). The Poincaré symmetry can be extended to include a symmetry between fermions and bosons as well. This would solve some of the faults of the SM description, but would bring some new problems. Moreover, the minimal way to include the spectrum of the SM seems rather narrowed by experiments.

Another possibility would be approaching the problem from bottom up: to accept the SM as part of the solution, the entirety of which is still unclear. We would then hope to extend the SM in a way that would fix the inconsistencies. After realising that the most severe problems are related to the Higgs sector of the SM, the most obvious way would be to replace this sector with something else. Arguably, the most straight-forward way would be considering a composite Higgs field instead of an elementary one. This leads us to technicolor (TC). The appeal of TC is that the similar mechanisms are already present in Nature, superconductivity being the best-known example, and that due to the dynamical breaking of the electroweak symmetry, no significant fine-tuning is needed. Technicolor, however, has its difficulties as well. While perfectly capable of producing the needed electroweak symmetry breaking pattern, the requirement of simultaneously producing masses for SM fermions is more problematic. Indeed, the TC sector must be somehow extended for this purpose.

How about, then, trying to combine these two and hoping to get the best of both of them? That is, first extend the SM with a TC sector and then supersymmetrize the whole thing. We would like the electroweak symmetry breaking to be entirely due to the TC sector, and SUSY would then play the role of the extension responsible of generating the fermion masses.

Assuming we now have the SM with the TC extension, we are left with a question:

In which way to carry out the supersymmetrization? One particularly appealing scenario arises if we include a TC sector with one left-handed doublet of techniquarks and their right-handed partners all transforming under the adjoint representation of SU(N) of TC. If we then identify one of the right-handed techniquarks with the fermionic partner of the technigluon, we obtain (after including the necessary SUSY partners for all the fields) exactly the field content of an $\mathcal{N} = 4$ superYang-Mills theory in TC sector. Studying this scenario is the goal of this thesis.

The thesis is structured as follows. In section 2 we introduce the building blocks of the model, the Standard Model, technicolor and supersymmetry. Section 3 concentrates on combining these into one model and finally in section 4 we study the so-called strong regime of the model in more detail. A summary of the notations and conventions used in this thesis can be found in Appendix A.

2 Preliminaries

2.1 The Standard Model

To recap, the Standard Model is a gauge theory of strong and electroweak interactions described by the gauge group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times U(1)_Y$, with $\mathrm{SU}(3)_\mathrm{C}$ describing the strong and $\mathrm{SU}(2)_\mathrm{L} \times U(1)_Y$ the electroweak interaction. While $\mathrm{SU}(3)_\mathrm{C}$ mediated by massless gluons is confining gauge theory, $\mathrm{SU}(2)_\mathrm{L} \times U(1)_Y$ is in the Higgs phase; $W^\pm, Z^0$ mediating the weak interaction are massive. The spontaneous breaking of this symmetry is driven by introducing an additional complex scalar field, the Higgs field, which acquires a non-zero vacuum expectation value (VEV) [1, 2, 3].

The problems of the SM are mainly related to this Higgs sector. The difficulties arise with the addition of a fundamental scalar field, the mass of which gets quadratically divergent contributions within the renormalization. The success of the SM is related to the fact that, even though it must be regarded as a low-energy theory, it is only logarithmically dependent on the energy scale. Therefore, the SM is a very good approximation at energies within reach. Now, the mass of the Higgs particle is quadratically divergent and, thus, there is no natural way (i.e. without high level of fine-tuning) of explaining why the Higgs particle should be light and not having a mass of the order of the Planck scale (or some other high scale beyond which the SM is not applicable any more, e.g. the scale of Grand Unified Theory, or GUT for short). This is called the hierarchy problem.

2.1.1 Electroweak constraints

While trying to augment the view beyond the SM, it is important to have some, preferably model-independent, guidelines based on experiments to maintain some correspondence with the observed world and the new model. The oblique electroweak constraints are playing this guiding role for models we consider here.

If we consider beyond-SM (BSM) physics and in which way it ought to affect the electroweak (EW) precision data, we are lead to oblique electroweak corrections. That is, when considering processes with light fermions in the final state (processes from which the precision data has been collected), the new physics in EW sector should have the largest contribution in the self-energies of the weak gauge bosons. The oblique corrections usually dominate even though there would be new states coupling directly to SM fermions since usually only few particles couple to a specific fermion flavour whereas all the charged particles couple to the vector bosons [4].

At the lowest order, these corrections depend on three input parameters which then should be eliminated by three observables. The most obvious choice for these observables is the fine structure constant, α, the Fermi coupling constant, G_{F}, and the mass of the Z boson, m_Z, since these are the most accurately measured.

Now, if the mass scale of the new particles is (much) higher than the Z mass, the vacuum polarisation amplitudes can be Taylor expanded and the effect of new physics can be parameterized by the so-called STU parameters. Of these parameters, U is rather unimportant and all the neutral current and low-energy observables depend only on S and T [5]. These are defined as [6]

$$\alpha S := 4 s_{\mathrm{W}}^2 c_{\mathrm{W}}^2 \frac{\Pi_{ZZ}^{\mathrm{new}}(m_Z^2) - \Pi_{ZZ}^{\mathrm{new}}(0)}{m_Z^2}, \tag{1}$$

$$\alpha T := \frac{\Pi_{WW}^{\mathrm{new}}(0)}{m_W^2} - \frac{\Pi_{ZZ}^{\mathrm{new}}(0)}{m_Z^2}, \tag{2}$$

where $s_{\mathrm{W}} = \sin\theta_{\mathrm{W}},\ c_{\mathrm{W}} = \cos\theta_{\mathrm{W}}$ and θ_{W} is the weak mixing angle, Π_{ij}^{new} are the self-energies and the superscript *new* points out that the origin of the (S, T) plane correspond to SM with Higgs mass m_{ref}.

Alternatively, T can be represented by the so-called ρ parameter. Let us denote the low-energy ratio of charged and neutral current interactions by $\rho_*(0)$. At tree level in the SM this ratio is given by [7],

$$\rho = \frac{m_W^2}{m_Z^2 \cos\theta_W}, \tag{3}$$

and equals to one. This relation is satisfied to better than 1% by experiments [5].

The deviation from unity is then parameterized by the T parameter [5]

$$\alpha T = \rho_*(0) - 1. \tag{4}$$

The experimental limit gives strong restrictions for the Higgs sector. If, however, there is an unbroken, global SU(2) symmetry, the so-called *custodial* symmetry in the Higgs sector, this condition is naturally valid up to electroweak radiative corrections [8].

As an example of the oblique electroweak parameters, let us write down the oblique corrections due to the SM Higgs. These read [5]

$$S_H \approx \frac{1}{12\pi} \ln \frac{m_H^2}{m_{\rm ref}^2} \tag{5}$$

$$T_H \approx -\frac{3}{16\pi c_{\rm W^2}} \ln \frac{m_H^2}{m_{\rm ref}^2}, \tag{6}$$

where m_H is the mass of the Higgs boson and $m_{\rm ref}$ the reference mass at which the S and T parameters have been defined, i.e.

$$(S_{\rm SM}(m_{\rm ref}), T_{\rm SM}(m_{\rm ref})) = (0,0). \tag{7}$$

The current experimental limits for reference mass $m_{\rm ref} = 117$ GeV read [6]

$$S = 0.03 \pm 0.09, \quad \text{and} \quad T = 0.07 \pm 0.08. \tag{8}$$

2.2 Technicolor

2.2.1 Prelude

Technicolor is a common name given to theories where the electroweak symmetry breaking (EWSB) is due to the dynamics of a new gauge sector instead of an elementary Higgs field. That is, one introduces a new gauge interaction coupled to new massless fermions some of which, at least, are weakly coupled. Especially in the early models, a new sector similar to Quantum Chromodynamics (QCD) was suggested, and thereby the name technicolor. After that, a variety of models, with more or less QCD-like dynamics, have been proposed, but the common factor is that albeit asymptotically free at high energies, the new gauge interaction must become confining at electroweak scale of around 250 GeV in order to produce the fermion condensate that breaks spontaneously the global chiral symmetry (able to contain the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ symmetry group of the SM) of the massless fermions and, thus, triggers the EWSB.

Whereas a fundamental Higgs is capable of producing masses for SM fermions via Yukawa interactions as well, TC must be extended with new gauge interactions (ETC interactions, for short) to generate fermion masses. As opposed to a fundamental Higgs, the dynamical breaking due TC is natural, and therefore avoids the hierarchy problem.

The idea of TC, a spontaneous breaking of the electroweak symmetry driven by a new gauge symmetry, was first proposed in 1979 by Weinberg [9] and Susskind [10], although the idea of dynamical symmetry breaking itself was already widely known that time.

Whereas much of the appeal of supersymmetry is due to aesthetics, technicolor is highly motivated by the fact that the very mechanism is already present in Nature. The best-known example of a similar phenomenon is superconductivity, where the formation of Cooper pairs is analogous to the formation of the fermion condensate in TC. Albeit maybe the best-known, this is not the only example. Perhaps even more motivating is that dynamical symmetry breaking is already present in the SM. Namely, the quark-antiquark condensate of QCD breaks the electroweak symmetry with pions as Goldstone bosons. The EWSB due to QCD is not, however, enough on its own to account for the whole EWSB in the SM but would give the W bosons a mass of approximately 29 MeV and, Z a mass of 33 MeV only (note, however, that the ratio $M_{\mathrm{W}}/M_{\mathrm{Z}}$ would be correct) [11]. Therefore, when considering the whole EWSB in the SM, the QCD driven part is usually negligible.

The stumbling stone of many TC models is the ETC sector which, if capable of producing large enough masses for SM fermions, tends to bring about too large flavour-changing neutral current (FCNC) effects to be compatible with experimental constraints. For a cure, a *walking* technicolor was proposed in the early 1980's [12, 13]. In walking TC, the coupling constant is QCD-like in both the infrared (IR) and the ultraviolet (UV) but in between from TC to ETC scale it evolves logarithmically, or walks. This makes the theory able to provide a large anomalous dimension for the fermion mass operator (needed for producing large enough SM fermion masses) and at the same time, reduces the FCNC contributions.

2.2.2 Fermions in higher representations

In order an SU(3) TC model, say, with fermions in the fundamental representation of the TC gauge group to exhibit walking behaviour, one needs to include at least around eight technifermions [14].

Adding fermions in higher representations to reduce the number of needed fermion

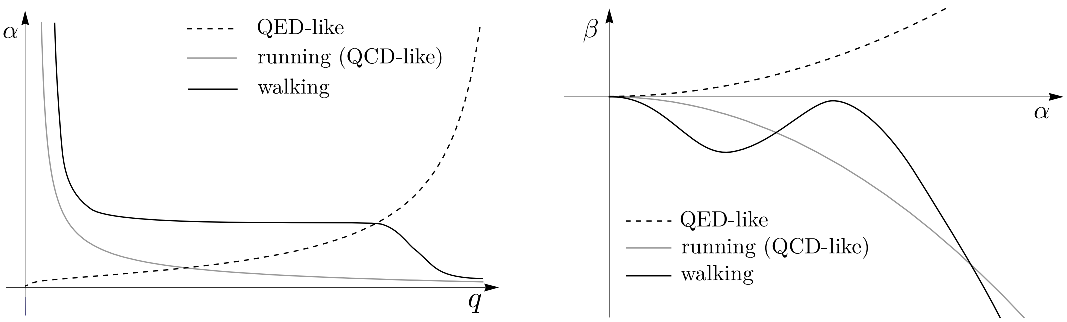


Figure 1. Left panel: Coupling constant evolvement as a function of energy for a QED-like (dashed line), a QCD-like (solid gray line) and a walking type (solid black line) coupling. **Right panel:** The associated β functions as functions of the couplings.

flavours was first proposed by Eichten and Lane [15] in 1989. Sannino and Tuominen [16] reintroduced and systematized the idea in 2005 and formulated the so-called Minimal Walking Technicolor (MWT) model.

Technicolor with fermions in the adjoint representation of SU(N) should be walking (if not already conformal, as some recent lattice simulations imply) with only two flavours of techniquarks. Thus, adding fermions into the higher representation can highly reduce the amount of needed fermion flavours for walking behaviour.

2.2.3 Coupling constant development and phase diagrams

Let us use the β functions of the SM interactions as a launch pad to the subject. For Quantum Electrodynamics (QED) we know that the coupling runs to infinity in the UV, i.e. the β function is positive whereas with QCD we know the development to be exactly the opposite with a negative β function. Since the coupling of QCD tends to zero at high energy, QCD is asymptotically free. Moreover, in the IR, the coupling runs to infinity making QCD confining.

For a walking theory, the issue is a somewhat more complex. By walking we mean a theory in which the coupling, instead of rapidly varying, i.e. *running*, evolves very slowly from one scale to another, i.e. *walks*. Between these scales, the coupling lies near a fixed-point value, α_*, for which $\beta(\alpha_*) = 0$. Examples of QED-, QCD- and a walking-type coupling constant developments with the associated β functions are depicted in fig. 1.

Now, in order to produce a large enough anomalous dimension for the fermion mass operator and simultaneously, to reduce the FCNC contributions, we would like the TC coupling to walk between the scales Λ_{TC} and Λ_{ETC}. This means that the theory should be near conformal, i.e. $\alpha \simeq \alpha_*$, between these scales. This, however, brings about a delicate interplay between the fixed-point value, α_*, and

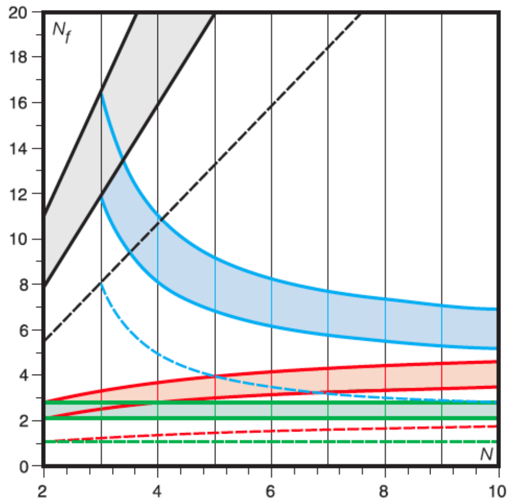


Figure 2. Phase diagram for theories with fermions in the (from top to bottom in the plot): *i*) fundamental representation (grey), *ii*) two-index antisymmetric (blue), *iii*) two-index symmetric (red), *iv*) adjoint representation (green) as a function of the number of flavours and the number of colours. The shaded areas depict the corresponding conformal windows. The upper solid curve represents N_f^{I} (loss of asymptotic freedom), the lower N_f^{II} (loss of chiral symmetry breaking). The dashed curves show N_f^{III} (existence of a Banks-Zaks fixed point). The figure has been taken from [14].

the critical value of the coupling, α_c, at which the chiral symmetry breaking (CSB) occurs. Namely, if $\alpha_* < \alpha_\mathrm{c}$, then, due to the renormalization group flow, the theory reaches the fixed-point α_*, but does not evolve any further (since $\beta(\alpha_*) = 0$) and, thus, the chiral symmetry breaking cannot occur. On the other hand, if $\alpha_* > \alpha_\mathrm{c}$, chiral symmetry breaking is first achieved due to which the fermions condense, their screening effect is lost and fixed-point cannot be attained.

Consider a gauge group $\mathrm{SU}(N)$ for a fixed $N \geq 2$. Depending on the number of fermion flavours, N_f, and, moreover, on the representation in which the fermions lie, the theory exhibits different behaviours, i.e. there are different phases to be identified. These different regions are usually depicted in the form of a phase diagram in (N, N_f) plane. First of all, there is a number of fermion flavours, denoted by N_f^{I} above which the theory loses asymptotic freedom. This corresponds to a QED with non-abelian gauge group and is called the free electric phase. Just below this limit there should be an IR fixed point, the so-called Banks-Zaks fixed point [17]. Thus, another interesting limit is the number of fermion flavours, N_f^{III}, required for a Banks-Zaks fixed point to appear. For $N_\mathrm{f}^{III} < N_\mathrm{f} < N_\mathrm{f}^{I}$ there is, therefore, an IR fixed point, α_*, in the theory. It depends on the critical value of the coupling, α_c, at which the CSB occurs, whether the theory with an IR fixed point is de facto conformal. Let us, thus, denote the number of fermions, above which the theory is conformal, i.e. $\alpha_* < \alpha_c$, by N_f^{II}. For $N_f^{II} < N_f < N_f^I$, the theory is, then, conformal and this region is called the conformal window. The limits, N_f^I, N_f^{II}, and N_f^I depend drastically on the representation in which the fermions lie. A thorough study about the limits for different representations of $\mathrm{SU}(N)$ can be found in [14]. A phase diagram of $\mathrm{SU}(N)$ with fermions in different representations is depicted in figure 2.

2.2.4 Minimal Walking Technicolor

In MWT, one generation of techniquarks is added, a left-handed doublet and their right-handed singlet partners transforming in the adjoint representation of the TC gauge group $\mathrm{SU}(2)_{\mathrm{TC}}$, i.e.

$$Q_{\mathrm{L}}^a = \left(\begin{array}{c} U_{\mathrm{L}}^a \\ D_{\mathrm{L}}^a \end{array} \right), \quad U_{\mathrm{R}}^a, \quad D_{\mathrm{R}}^a, \quad a = 1, 2, 3. \tag{9}$$

Moreover, we must add another generation of leptons to cancel the topological Witten anomaly [18]

$$L_{\mathrm{L}} = \left(\begin{array}{c} N_{\mathrm{L}} \\ E_{\mathrm{L}} \end{array} \right), \quad N_{\mathrm{R}}, \quad E_{\mathrm{R}}. \tag{10}$$

Gauge anomalies, in turn, cancel with the following hypercharge assignments

$$\begin{aligned} Y(Q_{\mathrm{L}}) &= \frac{y}{2}, \quad Y(U_{\mathrm{R}}, D_{\mathrm{R}}) = \left(\frac{y+1}{2}, \frac{y-1}{2} \right), \\ Y(L_{\mathrm{L}}) &= -\frac{3y}{2}, \quad Y(N_{\mathrm{R}}, E_{\mathrm{R}}) = \left(\frac{-3y+1}{2}, \frac{-3y-1}{2} \right), \end{aligned} \tag{11}$$

where y is a real-valued parameter (we will consider gauge anomaly cancellations in more detail for a general $\mathrm{SU}(N)$ TC group in section 3.1, but let us here just take the above hypercharge assignment without further justification).

A thorough construction of a low-energy theory for MWT can be found in [19], and we will follow that here with the exception that we will not fix any specific values for N and N_f but carry on with the general case with adjoint fermions since this does not make things any more difficult.

Before doing that, however, let us first study the chiral symmetry in more detail.

2.2.5 $\mathrm{SU}(N)$ technicolor with N_f adjoint techniquarks

Let us consider next a theory with N_f fermions in the adjoint representation of TC gauge group $\mathrm{SU}(N)$. First thing to note is that the adjoint representation is real implying the relation

$$(T^a)^* = (T^a)^{\mathrm{T}} = -T^a \tag{12}$$

for the generators of the gauge group, $a = 1, \dots, N^2-1$. Moreover, the fundamental representation of SU(2) is pseudoreal implying the condition

$$(\sigma^i)^* = (\sigma^i)^{\mathrm{T}} = -\sigma^2\sigma^i\sigma^2 \tag{13}$$

for the Pauli matrices, or in somewhat more useful form

$$(\sigma^\mu)^* = (\sigma^\mu)^{\mathrm{T}} = \sigma^2\bar{\sigma}^\mu\sigma^2, \tag{14}$$

where $\bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma})$.

Let us start with massless fermions and write the left- and right-handed components of them in the form of N_f component vectors q_L and q_R, respectively. The Lagrangian then reads

$$\begin{aligned}\mathcal{L}_{\text{chiral}} =&(q_\mathrm{R}^\dagger\ q_\mathrm{L}^\dagger) \begin{pmatrix} 0 & \mathrm{i}\sigma^\mu \\ \mathrm{i}\bar{\sigma}^\mu & 0 \end{pmatrix} D_\mu \begin{pmatrix} q_\mathrm{L} \\ q_\mathrm{R} \end{pmatrix} \\ &= q_\mathrm{L}^\dagger\, \mathrm{i}\bar{\sigma}^\mu D_\mu q_\mathrm{L} + q_\mathrm{R}^\dagger\, \mathrm{i}\sigma^\mu D_\mu q_\mathrm{R},\end{aligned} \tag{15}$$

where the sigma matrices should be understood as $\sigma^\mu \otimes \mathbb{1}_{N_\mathrm{f}\times N_\mathrm{f}}$ and the covariant derivative reads $D_\mu = \partial_\mu - \mathrm{i}gT^aA^a_\mu$. The Lagrangian is then invariant under the chiral group $\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R}$. There is, however, even larger global symmetry if the representation of the gauge group in which the fermions lie is either real or pseudoreal. Let us consider below only adjoint fermions, i.e. a real representation. For pseudoreal representation the computation is similar. Thus, to be able to utilise the reality of the adjoint representation (12) and pseudoreality of the fundamental representation of SU(2) (14), let us write $q \coloneqq q_\mathrm{L}$ and $\tilde{q} \coloneqq -\mathrm{i}\sigma^2 q_\mathrm{R}^*$. With these new fields, the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{chiral}} =&q\, \mathrm{i}\bar{\sigma}^\mu D_\mu q + \tilde{q}^{\mathrm{T}}\sigma^2\, \mathrm{i}\sigma^\mu D_\mu \sigma^2 \tilde{q}^* \\ &= q\, \mathrm{i}\bar{\sigma}^\mu D_\mu q - \tilde{q}^\dagger \sigma^2 D_\mu^{\mathrm{T}}\, \mathrm{i}(\sigma^\mu)^{\mathrm{T}}\sigma^2\tilde{q} \\ &= q\, \mathrm{i}\bar{\sigma}^\mu D_\mu q + \tilde{q}^\dagger D_\mu\, \mathrm{i}\bar{\sigma}^\mu \tilde{q},\end{aligned} \tag{16}$$

where the minus sign on the second line is due to transposing the second term and, hence, interchanging two Grassmannian variables. In addition, note that D_μ^{T} is a formal expression and is to be understood in the following manner:

$$q^{\mathrm{T}} D_\mu^{\mathrm{T}} = (D_\mu q)^{\mathrm{T}} = \partial_\mu q^{\mathrm{T}} + q^{\mathrm{T}}(-\mathrm{i}g(T^a)^{\mathrm{T}} A^a_\mu), \tag{17}$$

which after an integration by parts (and dropping of the surface terms, as usual) and the use of the reality of the adjoint representation yields

$$q^{\mathrm{T}} D_\mu = q^{\mathrm{T}}(-\partial_\mu + \mathrm{i} g T^a A_\mu^a) \tag{18}$$

implying a formal relation $D_\mu^{\mathrm{T}} = -D_\mu$. This relation along with the pseudoreality condition for the sigma matrices (14) has brought us to the last line of eq. (16). If we now assign the fields q and $\tilde{q}$ into one $2N_{\mathrm{f}}$ component field

$$Q = \begin{pmatrix} q \\ \tilde{q} \end{pmatrix}, \tag{19}$$

we can write the chiral Lagrangian in the form

$$\mathcal{L}_{\mathrm{chiral}} = Q^\dagger \, \mathrm{i}\bar{\sigma}^\mu D_\mu Q, \tag{20}$$

thereby making the global $\mathrm{SU}(2N_{\mathrm{f}})$ symmetry explicit.

Adding mass terms for the fermions breaks this global symmetry. To find out the invariant subgroup, let us next consider Dirac mass terms of the form $q_{\mathrm{R}}^\dagger q_{\mathrm{L}} + q_{\mathrm{L}}^\dagger q_{\mathrm{R}}$. Writing these in terms of q and $\tilde{q}$ yields

$$\begin{aligned} q_{\mathrm{R}}^\dagger q_{\mathrm{L}} + \text{h.c.} &= -\frac{\mathrm{i}}{2}\left(\tilde{q}^{\mathrm{T}}\sigma^2 q + \tilde{q}^{\mathrm{T}}\sigma^2 q\right) + \text{h.c.} \\ &= -\frac{\mathrm{i}}{2}\left(\tilde{q}^{\mathrm{T}}\sigma^2 q - q^{\mathrm{T}}(\sigma^2)^{\mathrm{T}}\tilde{q}\right) + \text{h.c.} \\ &= -\frac{i}{2}\left(\tilde{q}^{\mathrm{T}}\sigma^2 q + q^{\mathrm{T}}\sigma^2 \tilde{q}\right) + \text{h.c.} \\ &= -\frac{\mathrm{i}}{2} Q^{\mathrm{T}}\sigma^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q + \text{h.c.} = -\frac{\mathrm{i}}{2} Q^{\mathrm{T}}\sigma^2 E Q + \text{h.c.}, \end{aligned} \tag{21}$$

where on the second line we have again transposed the second term and on the third line used the pseudoreality condition for the sigma matrices. On the last line, we have written the terms with the help of the $2N_{\mathrm{f}}$ component vector Q and introduced an $2N_{\mathrm{f}} \times 2N_{\mathrm{f}}$ matrix

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

A similar computation for a pseudoreal representation leads to

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

Now, since Q transforms under $\mathrm{SU}(2N_\mathrm{f})$ as

$$Q \to gQ, \quad g \in \mathrm{SU}(2N_\mathrm{f}), \tag{24}$$

the transformations that leave the mass terms invariant satisfy

$$g^\mathrm{T} E g = E. \tag{25}$$

For a real representation, E is symmetric, and the above relation gives the invariant subgroup $\mathrm{SO}(2N_\mathrm{f})$. For a pseudoreal representation the invariant subgroup is the sympletic group $\mathrm{Sp}(2N_\mathrm{f})$. For a complex representation, the mass terms break the global group to $\mathrm{SU}(N_\mathrm{f})$.

2.2.6 Low-energy theory for the Higgs sector

Recall that the SM fermion masses arise from the Yukawa interaction terms between the fermions and the Higgs field which after the EWSB are (in the unitary gauge) of the form

$$-\frac{y_f}{\sqrt{2}}(v+h)(f_\mathrm{R}^\dagger f_\mathrm{L} + \mathrm{h.c}), \tag{26}$$

where y_f is the Yukawa coupling for fermion f, v is the VEV of the Higgs field and h the physical Higgs field.

Now, instead of the fundamental Higgs field, we would like to break the symmetry by a techniquark condensate. The above breaking of $\mathrm{SU}(2N_\mathrm{f})$ to $\mathrm{SO}(2N_\mathrm{f})$ is driven by the condensate

$$\langle Q_i^\alpha Q_j^\beta \epsilon_{\alpha\beta} E_{ij} \rangle. \tag{27}$$

In order to build a low-energy theory, let us define an effective variable M in which the information of the composite Higgs, its pseudoscalar partner and the Goldstone bosons arising from the spontaneous breaking of the symmetry along with their scalar partners is encoded. This $2N_\mathrm{f} \times 2N_\mathrm{f}$ matrix $M \sim QQ^\mathrm{T}$ transforms as

$$M \to gMg^\mathrm{T}, \quad g \in \mathrm{SU}(2N_\mathrm{f}). \tag{28}$$

and obtains a VEV

$$\langle M \rangle = \frac{v}{2} E. \tag{29}$$

We can now write down the general $\mathrm{SU}(2N_\mathrm{f})$ conserving potential for M (up to dimension four operators) as

$$\begin{aligned}\mathcal{V}_M = &- \frac{m_M^2}{2}\mathrm{Tr}\left[M^\dagger M\right] + \frac{\lambda_M}{4}\mathrm{Tr}\left[M^\dagger M\right]^2 + \lambda'_M\mathrm{Tr}\left[M^\dagger M M^\dagger M\right] \\ &- 2\lambda''_M\left[\mathrm{det}M + \mathrm{det}M^\dagger\right].\end{aligned} \tag{30}$$

For a positive m_M^2, this potential triggers the chiral symmetry breaking.

Let us then consider the electroweak sector in more detail. First of all, as the global symmetry $\mathrm{SU}(2N_\mathrm{f})$ breaks into $\mathrm{SO}(2N_\mathrm{f})$ via the introduction of mass terms for the chiral fermions, the chiral symmetry $\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$ breaks into the vectorial subgroup $\mathrm{SU}(N_\mathrm{f})_\mathrm{V}$. The vectorial subgroup, thus, lies within $\mathrm{SO}(2N_\mathrm{f})$ whereas the generators of the axial group are broken. It is, therefore, beneficial to identify the generators of $\mathrm{SU}(N_\mathrm{f})_\mathrm{V}$, S^a, and those of $\mathrm{SU}(N_\mathrm{f})_\mathrm{A}$, X^i, and, with help of these, to find out the generators of the chiral group $\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$. The choice of the generators in [19] readily generalises to the case of general N_f. The derivation of explicit realisations of the generators can be found in Appendix B.

From this point on, for notational simplicity, let us only consider the case $N_\mathrm{f} = 2$. We can assemble the techniquarks into a four-component vector

$$Q = \begin{pmatrix} U_\mathrm{L} \\ D_\mathrm{L} \\ -\mathrm{i}\,\sigma^2 U_\mathrm{R}^* \\ -\mathrm{i}\,\sigma^2 D_\mathrm{R}^* \end{pmatrix}, \tag{31}$$

allowing us to consider the global symmetry SU(4).

The left- and right-handed generators then read (see Appendix B)

$$L^a = \frac{S^a + X^a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{2}\sigma^a & 0 \\ 0 & 0 \end{pmatrix}, \quad R^a = \frac{X^{a\mathrm{T}} - S^{a\mathrm{T}}}{\sqrt{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma^a \end{pmatrix}, \tag{32}$$

and the generator for $\mathrm{U}(1)_\mathrm{V}$ reads

$$S^4 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33}$$

Thus, to embed the electroweak gauge group into the global chiral symmetry group, let us formally gauge the $\mathrm{SU}(2)_\mathrm{L}$ and the subgroup $\mathrm{U}(1)_Y$ of $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$ generated by

$$Y = -R^{3\mathrm{T}} + \sqrt{2}Y_\mathrm{V} S^4, \tag{34}$$

where Y_{V} is the $U(1)_{\mathrm{V}}$ charge, by introducing the covariant derivative for matrix M

$$D_\mu M = \partial_\mu M - ig\left[G_\mu M + M G_\mu^{\mathrm{T}}\right], \tag{35}$$

where

$$gG_\mu = gW_\mu^a L^a + g' B_\mu Y, \tag{36}$$

and Y is given by eq. (34). Now, as SU(4) spontaneously breaks into SO(4) causing $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ to break into $\mathrm{SU}(2)_{\mathrm{V}}$, the electroweak symmetry breaks, consequently, into $\mathrm{U}(1)_Q$ generated by

$$Q = L^3 + Y = \sqrt{2}S^3 + \sqrt{2}Y_{\mathrm{V}}S^4. \tag{37}$$

The further three unbroken generators of $\mathrm{SU}_{\mathrm{L}} \times \mathrm{SU}(2)\mathrm{R} \times \mathrm{U}(1)_{\mathrm{V}}$, which, together with Q, generate the unbroken $\mathrm{SU}(2)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{V}}$, act as a custodial isospin insuring the ρ parameter to be equal to one at tree level.

The effective Lagrangian for the Higgs sector then, in its full glory, reads

$$\mathcal{L}_{\mathrm{Higgs}} = \frac{1}{2}\mathrm{Tr}\left[D_\mu M D^\mu M^\dagger\right] + \mathcal{V}_M + \mathcal{L}_{\mathrm{ETC}}, \tag{38}$$

where $\mathcal{V}_M$ is given by

$$\begin{aligned}\mathcal{V}_M = & -\frac{m_M^2}{2}\mathrm{Tr}\left[M^\dagger M\right] + \frac{\lambda_M}{4}\mathrm{Tr}\left[M^\dagger M\right]^2 + \lambda'_M \mathrm{Tr}\left[M^\dagger M M^\dagger M\right] \\ & - 2\lambda''_M\left[\mathrm{det}M + \mathrm{det}M^\dagger\right],\end{aligned} \tag{39}$$

where $\mathcal{L}_{ETC}$ contains terms from ETC interactions.

2.3 Supersymmetry

2.3.1 Prelude

The birth of supersymmetry dates back to early 1970's, and the history of the discovery is intriguing already per se; a thorough historical review can be found e.g. in [20]. The development took place on multiple fronts; the very first notion of SUSY arose within the context of string theory in the turn of 60's and 70's whereas the first proposal in the framework of four dimensional quantum field theory emerged in the early 70's. The rise of supersymmetric field theory was interestingly bifurcate as well for SUSY was first proposed as an extension of Poincaré symmetry by Golfand and Likhtman [21] in the Soviet Union in 1971 but roused no extensive interest until 1973 when Wess and Zumino (independent of Golfand and Likhtman) published their famous paper [22] in the 'West'. From that on, SUSY has been subject to considerable study in context of both the string theory and the field theory as well as their interplay via dualities.

The interest in SUSY in the string theoretical environment is above all related to the description of gravitation since SUSY seems to at least tame the divergences related to quantum gravity and, moreover, makes the calculations more manageable. [23]

Here, we are not particularly interested in string theory (except for AdS/CFT correspondence, on which we will shortly comment later) but in supersymmetric field theories, so let us concentrate on those from now on.

The extensive interest in SUSY during the past decades and still at present explains itself with a few extremely appealing features that SUSY brings forth. Much of the appeal is aesthetics. Supersymmetry gives a fundamental symmetry between fermions and bosons, matter and forces, and this very symmetry provides a cure for many of the disturbing theoretical problems in the SM. First of all, it provides natural fundamental scalars and removes the hierarchy problem. While in the SM the masses of fundamental scalars are quadratically divergent implying them to be heavy (mass of the order of the Planck or the GUT scale), there is no natural explanation for a light Higgs needed to complete the SM. The problem that the vast hierarchy between the electroweak and the Planck/GUT scale brings about in the SM is absent in a supersymmetric theory due to the equal amount of bosonic and fermionic degrees of freedom, resulting in cancellation of the quadratic divergences.

These remarkable cancellations of the ultraviolet divergences make SUSY extremely attractive also outside the SM environment. Indeed, some supersymmetric theories ($\mathcal{N} = 4$ superYang-Mills as the best example) are the only known theories that are finite to all orders in perturbation theory. Moreover, SUSY provides possible candidates for Grand Unified Theories.

After this list of virtues, we must, however, confront the one major draw back of SUSY: To date it has evaded all experimental verification leaving the followers to only hope for the best and fear for the worst. Moreover, aesthetics on one front must be paid back on another. The (minimal) supersymmetrization of the Standard Model comes with a huge amount of extra parameters, most of them associated with the SUSY breaking mechanism. Indeed, if present in Nature, SUSY must anyway be broken, since we do not see the equally massive SUSY partners of the SM fermions. Whether there really is supersymmetry remains still a very much open question. Hopefully, LHC will change this status quo in the near future.

The following presentation of supersymmetry is not in any way even trying to be an all-covering introduction to the subject; on the contrary. Some preliminary exposure to supersymmetry is expected of the reader, and in the following only the main aspects of the theory are presented without dwelling too much on technical details, yet only listing of formulas is tried to be highly avoided.

The material presented here is adapted mainly from three sources: the classic by Wess and Bagger [24], which we mainly follow with our notations and conventions (note, however, that we have dropped the extra overall minus sign in the definition of σ^0), the nice summary of the basics in Srednicki's quantum field theory book [25] and fine pedagogical lecture notes on the subject by Christian Sämann [26]. Another pedagogical introductions, which have been very much helpful in the writing process, can be found in [27] and [28], but with the latter the reader should be careful with the slightly unorthodox conventions. For a more hands-on introduction, especially to the Minimal Supersymmetric Standard Model, the reader is referred to [29]. Those who prefer 4-component spinors probably find the book by Weinberg [30] extremely useful.

The discussion is structured as follows. After presenting the SUSY transformations and the algebra, the theory is formulated in the so-called superspace formalism. After the technical preliminaries, two examples of supersymmetric theories, namely the Wess-Zumino model and the superYang-Mills theory (SYM), are considered. We close the generic discussion of SUSY by presenting the Minimal Supersymmetric Standard Model (MSSM). An excessive list of spinor identities extremely useful with SUSY manipulations have been collected in Appendix C and the Grassmannian derivatives shortly discussed in Appendix D.

2.3.2 Supersymmetry algebra and its representations

Coleman and Mandula proved in 1967 [31] that under some rather general assumptions the most general symmetry group of the S-matrix is (locally isomorphic to) a direct product of the Poincaré group and an internal symmetry group (symmetries related to conserved quantum numbers like electric charge, for example). There was no going around this restriction without loosening some of the assumptions of this theorem, and indeed there was a loophole to be found: the theorem assumes that the symmetry algebra involves only commutators of the symmetry generators. Allowing also anticommutators opened the door to SUSY.

An important further no-go theorem followed in 1975 when Haag, Łopuszański and Sohnius proved that supersymmetry was the only possible additional symmetry with this loosened set of assumptions [32]. Supersymmetry is, then, (to date, at the very least) the only possible extension of the Poincaré symmetry.

Let us start with the classical Poincaré algebra. The algebra reads

$$[P_\mu, P_\nu] = 0, \tag{40}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \tag{41}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}, \tag{42}$$

where P_μ is the momentum operator generating the spacetime translations and $M_{\mu\nu}$ generate the Lorentz boosts and the spatial rotations.

By introducing anticommuting symmetry generators (supercharges) Q_α and $\bar{Q}_{\dot{\alpha}}$, we can then extend the above Poincaré algebra to the Poincaré superalgebra. See e.g. the first chapter of [24] for a nice derivation of the superalgebra using Coleman-Mandula theorem. The result (in addition to the classical Poincaré algebra) is

$$[Q_{\alpha A}, P^\mu] = 0, \tag{43}$$
$$[\bar{Q}_{\dot{\alpha} A}, P^\mu] = 0, \tag{44}$$
$$[Q_{\alpha A}, M^{\mu\nu}] = \frac{1}{2} (\sigma^{\mu\nu})_\alpha{}^\beta Q_{\beta A}, \tag{45}$$
$$[\bar{Q}^{\dot{\alpha} A}, M^{\mu\nu}] = \frac{1}{2} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}_A, \tag{46}$$
$$\{Q_{\alpha A}, Q_{\beta B}\} = Z_{AB} \epsilon_{\alpha\beta}, \tag{47}$$
$$\{\bar{Q}_{\dot{\alpha} A}, \bar{Q}_{\dot{\beta} B}\} = Z_{AB} \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}, \tag{48}$$
$$\{Q_{\alpha A}, \bar{Q}_{\dot{\alpha} B}\} = 2\delta_{AB} \sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{49}$$

where the capital indices run from 1 to $\mathcal{N}$, $\mathcal{N} = 1, 2$ or 4, and the central charges $Z_{AB} = -Z_{BA}$ commute with $Q_{\alpha A}, \bar{Q}_{\dot{\alpha} A}, P^\mu$ and $M^{\mu\nu}$. Notice, that the antisymmetry of the central charges imply that $Z_{AB} = 0$ for $\mathcal{N} = 1$.

Before entering the realm of supersymmetric model building further, let us first consider the particle content of supersymmetric theories. Now, P^2 is a Casimir operator of a SUSY representation yielding equal mass to all particles belonging to the same irreducible representation. We call these irreducible representations supermultiplets. We should then distinguish two cases: the massless and massive supermultiplets. Let us consider here only the massless case, see e.g. [24, pp. 12–16] for the massive case. Moreover, consider here only theory without central charges, i.e. $Z_{AB} = 0$, yielding anticommutation relations

$$\{Q_{\alpha A}, Q_{\beta B}\} = \{\bar{Q}_{\dot{\alpha} A}, \bar{Q}_{\dot{\beta}, B}\} = 0. \tag{50}$$

Notice first that in the massless case we can always boost into a reference frame, where $P_\mu = (E, 0, 0, E)$, so that

$$\sigma^\mu P_\mu = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}, \tag{51}$$

and

$$\{Q_{\alpha A}, \bar{Q}_{\dot{\alpha} B}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta_{AB}. \tag{52}$$

In particular, $\{Q_{2A}, \bar{Q}_{\dot{2}B}\} = 0$ for all A, B. Now,

$$0 = \langle \Phi | \{Q_{2A}, \bar{Q}_{\dot{2}A}\} | \Phi \rangle = ||Q_{2A}\Phi\rangle||^2 + ||\bar{Q}_{\dot{2}A}\Phi\rangle||^2, \tag{53}$$

for all states $|\Phi\rangle$ yielding $Q_{2A} = 0$ and $\bar{Q}_{\dot{2}A} = 0$. Rescaling

$$a_A = \frac{1}{\sqrt{4E}} Q_{1A} \quad \text{and} \quad a_A^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{1}A}, \tag{54}$$

we obtain N creation and annihilation operators obeying indeed the algebra

$$\{a_A, a_B^\dagger\} = \delta_{AB}, \; \{a_A, a_B\} = \{a_A^\dagger, a_B^\dagger\} = 0. \tag{55}$$

These $a_A^\dagger$ and a_A raise and lower the helicity of the state by $\frac{1}{2}$, respectively, see e.g. [27]. Thus, a_A annihilates the lowest helicity state, the so-called Clifford vacuum, denoted by $|\Omega_{\lambda_0}\rangle$.

The supermultiplet is, then, of the form

$$\begin{aligned} &|\Omega_{\lambda_0}\rangle \\ a_A^\dagger |\Omega_{\lambda_0}\rangle =& |\Omega_{\lambda_0 + \frac{1}{2}, A}\rangle \\ \frac{1}{\sqrt{2}} a_B^\dagger a_A^\dagger |\Omega_{\lambda_0}\rangle =& |\Omega_{\lambda_0 + 1, AB}\rangle \\ &\vdots \\ \frac{1}{\sqrt{N!}} a_{A_N}^\dagger \cdots a_{A_1}^\dagger |\Omega_{\lambda_0}\rangle =& |\Omega_{\lambda_0 + \frac{1}{2}N, A_1 \cdots A_N}\rangle. \end{aligned} \tag{56}$$

The states with helicity $\lambda = \lambda_0 + \frac{1}{2}n$ must be antisymmetric in $A_1, \cdots, A_n$ and are, thus, $\begin{pmatrix} N \\ n \end{pmatrix}$ times degenerate [24]. Consistent theories with particles of spin larger than two are to date unknown and, moreover, since we are not interested in gravity here (graviton being a spin-2 particle and its SUSY partner gravitino a spin-$\frac{3}{2}$ particle), we restrict ourselves to particles of spin less than or equal to 1. Hence, for $\mathcal{N} = 1$ there are two possible supermultiplets: the chiral supermultiplet with one helicity-0 and one helicity-$\frac{1}{2}$ field, and the vector supermultiplet with one field of helicity $\frac{1}{2}$ and one of helicity 1. Note, however, that to preserve $\mathcal{CPT}$ invariance, we must add a $\mathcal{CPT}$-conjugate multiplet if the multiplet itself is not a $\mathcal{CPT}$-selfconjugate. For chiral supermultiplet the $\mathcal{CPT}$-conjugate consists of fields

Table 1. Number of fields of given helicity in the possible supermultiplets for $\mathcal{N} = 1, 2, 4$.

	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$\mathcal{N} = 1$					
chiral multiplet			1	1	
$\mathcal{CPT}$-conjugate		1	1		
vector multiplet				1	1
$\mathcal{CPT}$-conjugate	1	1			
$\mathcal{N} = 2$					
vector multiplet			1	2	1
$\mathcal{CPT}$-conjugate	1	2	1		
hypermultiplet		1	2	1	
$\mathcal{N} = 4$	1	4	6	4	1

of helicities 0 and $-\frac{1}{2}$, one of each, and for vector supermultiplet $-\frac{1}{2}$ and -1 helicity fields, again one of each, of course.

The possible supermultiplets for cases $\mathcal{N} = 1, 2, 4$ are gathered in table 1. The case $\mathcal{N} = 3$ is usually not considered separately since the $\mathcal{N} = 3$ supermultiplet together with its $\mathcal{CPT}$-conjugate forms an $\mathcal{N} = 4$ supermultiplet.

Let us concentrate on $\mathcal{N} = 1$ SUSY for now. We will return to the $\mathcal{N} = 4$ case later.

2.3.3 R-Symmetry

Invariance of the superalgebra under a global transformation of the supercharges is called an R-symmetry. The $\mathcal{N} = 1$ superalgebra is invariant under global U(1) transformations of the supercharges whereas the extended supersymmetries are invariant under global SU($\mathcal{N}$) transformations mixing the supercharges. In particular, the $\mathcal{N} = 4$ SUSY has a global SU(4) R-symmetry.

2.3.4 Superspace and supertranslations

Supersymmetry is most elegantly formulated with superfields in the superspace. The great advantage of this approach is that the superfield formalism treats the related bosonic and fermionic degrees of freedom at once and is, thus, manifestly supersymmetric. This is a welcome feature since checking whether a given theory is supersymmetric, or not, tends to be rather tedious.

The idea, in order to accommodate both bosonic and fermionic degrees of freedom into one and the same superfield, is to first upgrade the space with anticommuting coordinates. The superfield is then not only a function of the spacetime coordinates, x^μ, but also of an anticommuting left-handed spinor coordinate, θ, and its right-handed complex conjugate, $\bar{\theta}$. This is, however, rather loosely put, but we will not go here deeper into the mathematical rigour of the very problem of introducing anticommuting coordinates. An interested reader is encouraged to acquaint himself with non-commutative geometry for deeper understanding. However, to see that this is well motivated, notice the relativistic mechanics analogue: the superspace is to Poincaré superalgebra what Minkowski space is to Poincaré algebra. Since Lorentz transformations leave the origin of the Minkowski space invariant, the Minkowski space (with origin fixed) can be identified with the space of right cosets of the Poincaré group modulo the Lorentz group. Thus, there is a one-to-one correspondence between the right cosets and spacetime translations, i.e. $x^\mu \leftrightarrow \mathrm{e}^{-\mathrm{i}x^\mu P_\mu}$, which allows us to treat the action of the Poincaré group on Minkowski space as a left multiplication in the group (the minus sign in the exponential does have a purpose that will become clear in a while).

Analogously, we would like to identify the superspace as the space of right cosets of the Poincaré supergroup modulo Lorentz group. To avoid the (possible) problem of getting from the superalgebra to the supergroup elements, we would like to make some modifications so that we would be able to work with a traditional Lie algebra, i.e. with only commutators. To this end, we introduce anticommuting spinorial parameters θ^α and $\bar{\theta}_{\dot{\alpha}}$. With these parameters, the supersymmetry algebra can be written in terms of commutators only, namely

$$[\theta Q, \bar{\theta}\bar{Q}] = 2\,\theta\sigma^\mu\bar{\theta}\,P_\mu, \tag{57}$$

$$[\theta Q, \theta Q] = [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}] = 0, \tag{58}$$

$$[P^\mu, \theta Q] = [P^\mu, \bar{\theta}\bar{Q}] = 0. \tag{59}$$

Now, a general group element can be written as an exponential

$$G(x, \theta, \bar{\theta}, \omega) = \mathrm{e}^{\mathrm{i}(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})}\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\mu\nu}M_{\mu\nu}}. \tag{60}$$

Thus, there is a correspondence $(x^\mu, \theta, \bar{\theta}) \leftrightarrow \mathrm{e}^{\mathrm{i}(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})} =: G(x^\mu, \theta, \bar{\theta})$, and we can treat the action of the Poincaré supergroup on the superspace as a left multiplication in the group.

We can use Baker-Campbell-Hausdorff formula

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{61}$$

to multiply the group elements, and the first three terms above are enough because all the higher order commutators vanish. Thus,

$$G(\epsilon^\mu, 0, 0)G(x^\mu, \theta, \bar{\theta}) = G(x^\mu + \epsilon^\mu, \theta, \bar{\theta}), \tag{62}$$

$$G(0, \xi, \bar{\xi})G(x^\mu, \theta, \bar{\theta}) = G(x^\mu - \mathrm{i}\xi\sigma^\mu\bar{\theta} + \mathrm{i}\theta\sigma^\mu\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}). \tag{63}$$

Linearizing the group elements (assume ϵ^μ and $\xi, \bar{\xi}$ infinitesimal) we obtain

$$g(\epsilon^\mu, 0, 0) = 1 - \mathrm{i}\epsilon^\mu P_\mu, \tag{64}$$

$$g(0, \xi, \bar{\xi}) = 1 + \mathrm{i}(\xi Q + \bar{\xi}\bar{Q}) = 1 + (\xi\mathcal{Q} + \bar{\xi}\bar{\mathcal{Q}}), \tag{65}$$

where we have defined differential operators $Q_\alpha = -\mathrm{i}\mathcal{Q}_\alpha$ and $\bar{Q}_{\dot{\alpha}} = -\mathrm{i}\bar{\mathcal{Q}}_{\dot{\alpha}}$ (cf. momentum operator and spacetime derivative). These linearized group elements thus induce the following translations in the coordinate space (see eqs. (62) and (63)):

$$g(\epsilon^\mu, 0, 0) : (x^\mu, \theta, \bar{\theta}) \mapsto (x^\mu + \epsilon^\mu, \theta, \bar{\theta}), \tag{66}$$

$$g(0, \xi, \bar{\xi}) : (x^\mu, \theta, \bar{\theta}) \mapsto (x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\xi} - \mathrm{i}\xi\sigma^\mu\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}). \tag{67}$$

By comparing eqs. (65) and (67), we obtain the differential operators

$$\mathcal{Q}_\alpha = \partial_\alpha - \mathrm{i}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{68}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + \mathrm{i}\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{69}$$

Note, however, that eqs. (64) and (66) imply an unconventional sign for the momentum operator, i.e. $P_\mu = +\mathrm{i}\partial_\mu$. This is a consequence of choosing the sign of the $x^\mu P_\mu$ term to be minus in the group element (eq. (60)). This choice is important, though, since with this choice the differential operators $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ obey the SUSY algebra

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0, \tag{70}$$

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 2\mathrm{i}(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu = 2(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu. \tag{71}$$

Another possibility would have been starting with the opposite algebra, more on this can be found in [28].

Had we, however, considered the right actions, we would have disentangled an another set of differential operators, namely

$$\mathcal{D}_\alpha = \partial_\alpha + \mathrm{i}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{72}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{73}$$

These obey the following anticommutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0, \tag{74}$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}\dot\alpha\} = -2\mathrm{i}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = -2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu, \tag{75}$$

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot\beta}\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{Q}}_{\dot\beta}\} = 0. \tag{76}$$

We, thus, notice that if we would have begun with the right action, we could have kept the normal sign convention in the momentum operator. The history, however, chose another path, and we will cling on that. This does not mean that we would not make any use of these differential operators. On the contrary, they turn out to be very important and, therefore, are worthy of a special appellation. From now on, we will call these the supercovariant derivatives for they anticommute with the SUSY transformations.

Having defined the needed operators, we are ready to move on to superfields.

2.3.5 Chiral and vector superfields

Let us start with the component field expansion of a general superfield. The spinor θ is a two-component anticommuting object and, therefore, the highest possible power of θ's is two. In particular, a series expansion in θ always terminates after the second order in θ. Hence, by expanding in powers of θ, we obtain the component form of a general superfield (note that since θ is complex, θ and $\bar\theta$ are treated as independent degrees of freedom)

$$\begin{aligned}\Phi(x,\theta,\bar\theta) =& \phi(x) + \theta\psi + \bar\theta\bar\psi'(x) + \theta\sigma^\mu\bar\theta\, v_\mu(x) \\ &+ \theta^2 F(x) + \bar\theta^2 F'(x) + \bar\theta^2\theta\xi(x) + \theta^2\bar\theta\bar\xi'(x) + \theta^2\bar\theta^2 D(x).\end{aligned} \tag{77}$$

In the end of the previous section, we found out that the supercovariant derivatives anticommute with the generators of SUSY transformations. Thus, the special class of superfields whose supercovariant derivatives vanish is preserved by SUSY transformations. This leads to the definition of a chiral superfield.

A chiral superfield is defined to satisfy

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0, \tag{78}$$

and similarly one defines an antichiral superfield to satisfy

$$\mathcal{D}_\alpha\Phi = 0. \tag{79}$$

In order to find out the most general chiral superfield in terms of the component fields, let us introduce another set of even (or bosonic) coordinates

$$y^\mu := x^\mu + \mathrm{i}\theta\sigma^\mu\bar\theta. \tag{80}$$

The advantage of this change of coordinates is that

$$\begin{aligned}\bar{\mathcal{D}}_{\dot{\alpha}} y^\mu &= (-\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^\alpha (\sigma^\nu)_{\alpha\dot{\alpha}} \partial_\nu)(x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\theta}) \\ &= \mathrm{i}\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} - \mathrm{i}\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} = 0,\end{aligned} \tag{81}$$

where the plus sign of the first term on the last line is due to an extra minus sign from the anticommutativity of the Grassmannian derivative. Moreover, after noting that in addition

$$\bar{\mathcal{D}}_{\dot{\alpha}} \theta^\alpha = 0, \tag{82}$$

we can deduce that a superfield Φ which is a function only of y and θ (and not of $\bar{\theta}$) is chiral. Luckily, this turns out to be a necessary condition of a chiral superfield as well. To see this, change from $(x^\mu, \theta, \bar{\theta})$ to $(y^\mu, \theta, \bar{\theta})$ coordinates; we will refer to these in the following by subscripts x and y, respectively. Chain rule gives us

$$(\partial_x)_\mu = (\partial_y)_\mu, \tag{83}$$

$$(\partial_x)_\alpha = (\partial_y)_\alpha + \frac{\partial y^\mu}{\partial \theta^\alpha} \frac{\partial}{\partial y^\mu} = (\partial_y)_\alpha + \mathrm{i}(\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} (\partial_y)_\mu, \tag{84}$$

$$(\bar{\partial}_x)_{\dot{\alpha}} = (\partial_y)_{\dot{\alpha}} + \frac{\partial y^\mu}{\partial \bar{\theta}^{\dot{\alpha}}} \frac{\partial}{\partial y^\mu} = (\bar{\partial}_y)_{\dot{\alpha}} - \mathrm{i}\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} (\partial_y)_\mu. \tag{85}$$

Hence, the differential operators in the y coordinates read (since everything is in the y system, we drop the subscripts for simplicity)

$$\mathcal{Q}_\alpha = \partial_\alpha, \tag{86}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2\mathrm{i}\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu, \tag{87}$$

$$\mathcal{D}_\alpha = \partial_\alpha + 2\mathrm{i}(\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu, \tag{88}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}. \tag{89}$$

The chirality condition of Φ, then, becomes

$$\bar{\partial}_{\dot{\alpha}} \Phi(y, \theta, \bar{\theta}) = 0, \tag{90}$$

and further

$$\Phi(y, \theta, \bar{\theta}) = \Phi(y, \theta). \tag{91}$$

Hence, by expanding in powers of theta, we obtain the most general chiral superfield

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \tag{92}$$

whereas all the higher powers of θ vanish due to the anticommutativity. In the above, the factor $\sqrt{2}$ is purely conventional. Returning to $(x, \theta, \bar{\theta})$ coordinates leads to a component expansion (after some use of Fierz and spinor identities)

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) =&\phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) - \mathrm{i}\,\theta\sigma^\mu\bar{\theta}\,\partial_\mu\phi(x)\\ &- \frac{\mathrm{i}}{\sqrt{2}}\theta^2\,\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x).\end{aligned} \tag{93}$$

The next question arising is the following: If we have a set of chiral superfields, can we somehow combine them and produce a new chiral superfield? The answer is, as anticipated, yes, and in fact any holomorphic function of chiral superfields is a chiral superfield [27]. This function is called the superpotential and is denoted by $W(\Phi)$ (note that, for simplicity, we denote the argument of the superpotential by a generic superfield Φ regardless of the actual number of the superfields in the argument). However, if we want to build a renormalizable theory, there is a further very restrictive constraint: on dimensional grounds, it turns out that the superpotential should be, at most, a cubic polynomial of chiral superfields (see e.g. [28]).

Another kind of superfield we will still be needing is the so-called vector superfield which will, not very surprisingly, give us the vector degrees of freedom. The defining condition of a vector superfield is the hermiticity of the field, i.e. a superfield $V(x, \theta, \bar{\theta})$ is a vector superfield if it is hermitian,

$$[V(x,\theta,\bar{\theta})]^\dagger = V(x,\theta,\bar{\theta}). \tag{94}$$

The hermiticity, then, implies the following component form of a vector superfield

$$\begin{aligned}V(x,\theta,\bar{\theta}) =&C(x) + \theta\chi + \bar{\theta}\bar{\chi} + \theta\sigma^\mu\bar{\theta}\,v_\mu(x) + \theta^2 G(x) + \bar{\theta}^2 G^\dagger(x)\\ &+ \bar{\theta}^2\theta\eta(x) + \theta^2\bar{\theta}\bar{\eta}(x) + \frac{1}{2}\theta^2\bar{\theta}^2 E(x),\end{aligned} \tag{95}$$

where C, v_μ and E are real fields and the factor $\frac{1}{2}$ in front of the last term is conventional.

We now have all the ingredients ready for building proper supersymmetric field theories. Let us illustrate further details of constructing supersymmetric Lagrangians with two main examples.

2.3.6 Wess-Zumino model

Let us start by noting how the different components of a chiral superfield transform under an infinitesimal SUSY transformation δ_ϵ,

$$\delta_\epsilon\Phi = (\epsilon Q + \bar{\epsilon}\bar{Q})\cdot\Phi = (\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\Phi, \tag{96}$$

where ϵ is an infinitesimal anticommuting parameter. For component fields, we obtain

$$\delta_\epsilon\phi = \sqrt{2}\epsilon\psi, \tag{97}$$

$$\delta_\epsilon\psi_\alpha = \mathrm{i}\sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F, \tag{98}$$

$$\delta_\epsilon F = \mathrm{i}\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi. \tag{99}$$

The last of these leads to an important observation: the $\theta\theta$ component of a chiral superfield transforms into a total derivative, and, thus, the spatial integral over the term is invariant under a SUSY transformation. This makes it a possible term in the supersymmetric action. More generally, since the superpotential is a chiral superfield, the $\theta\theta$ component of the superpotential, denote this by

$$W(\Phi)|_{\theta\theta}\,, \tag{100}$$

serves as a building block of a supersymmetric theory. One can get this component by differentiating $W(\phi)$, a function one gets by replacing the superfields in the superpotential by their scalar components (note that also here the argument ϕ is generic and may contain multiple fields), with respect to the scalar fields (see e.g. [29] for more detailed derivation),

$$W(\Phi)|_{\theta\theta} = \frac{\partial W(\phi)}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2 W(\phi)}{\partial\phi_i\partial\phi_j}\psi_i\psi_j. \tag{101}$$

Summation over repeated indices is assumed in the above. The superpotential is not hermitian, though, and the hermitian conjugate of the above should be, thereby, added in order the action to be hermitian.

Another combination of superfields one might be tempted to include in the action is $\Phi^\dagger\Phi$. This, however, is not chiral but since it is manifestly hermitian, it is a vector superfield. Applying the SUSY transformation to a vector superfield, we find that the $\theta^2\bar{\theta}^2$ component of the vector superfield transforms as

$$\delta_\epsilon E = \frac{\mathrm{i}}{2}\epsilon\sigma^\mu\partial_\mu\bar{\eta} + \frac{\mathrm{i}}{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\eta. \tag{102}$$

That is, the $\theta^2\bar{\theta}^2$ component transforms into a total derivative and is, thus, a possible term in the supersymmetric Lagrangian.

The simplest supersymmetric model is the so-called Wess-Zumino model, which is a theory of a single chiral superfield Φ. In order to start building a supersymmetric Lagrangian consisting of terms involving only the field Φ, let us first look at the mass dimensions of the component fields. The lowest component of the chiral superfield (a commonly used nomenclature to refer to the component lowest order in θ; similarly the highest component) being a scalar field has mass dimension 1. Now, the spinorial coordinates θ must have mass dimension $-\frac{1}{2}$ and, thus, the highest component of a chiral superfield must have mass dimension two higher than the lowest component, i.e. 3. Hence, while the Lagrangian must have mass dimension 4, we are to look at terms at least quadratic in Φ.

Moreover, since renormalizability prevents terms higher-order than cubic, we are lead to consider superpotential with quadratic and cubic terms.

Using the great virtue of foresight, let us fix the coefficients for the Wess-Zumino superpotential as follows:

$$W_{\mathrm{WZ}}(\Phi) = \frac{1}{2} m\Phi^2 + \frac{1}{6} g\Phi^3, \tag{103}$$

with the parameters m and g real. Now, the full Lagrangian for the Wess-Zumino model reads

$$\mathcal{L}_{\mathrm{WZ}} = \Phi^\dagger\Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(W_{\mathrm{WZ}}(\Phi)|_{\theta\theta} + \mathrm{h.c.}\right). \tag{104}$$

In terms of component fields, the first term becomes

$$\begin{aligned}\Phi^\dagger\Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} =& \frac{1}{4}\phi^\dagger\Box\phi + \frac{1}{4}\phi\Box\phi^\dagger - \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi \\ &+ \frac{\mathrm{i}}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \frac{\mathrm{i}}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + F^\dagger F,\end{aligned} \tag{105}$$

which after some integrations by parts (and dropping of total derivatives) simplifies to

$$\Phi^\dagger\Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} = -\partial_\mu\phi^\dagger\partial^\mu\phi + \mathrm{i}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F. \tag{106}$$

We, thus, identify the first two terms as the kinetic terms of a complex scalar field ϕ and a left-handed Weyl fermion ψ. Moreover, we have a term for a complex scalar field F without derivatives. This field is called an auxiliary field, and it ensures the off-shell closure of the SUSY algebra.

Moving on to the component field expansion of the superpotential, we find using eq. (101) that

$$\begin{aligned}W_{\mathrm{WZ}}(\Phi)|_{\theta\theta} &= \frac{\partial W(\phi)}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W(\phi)}{\partial\phi\partial\phi}\psi\psi \\ &= m\phi F + \frac{1}{2}g\phi^2 F - \frac{1}{2}m\psi\psi - \frac{1}{2}g\phi\psi\psi.\end{aligned} \tag{107}$$

Now, since the auxiliary field F appears quadratically and without derivatives, we can perform the path integral over it. This amounts to solving the classical equation of motion for F,

$$0 = \frac{\partial \mathcal{L}}{\partial F} - \partial \mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu F)} = F^\dagger + \frac{\partial W(\phi)}{\partial \phi}. \tag{108}$$

Substituting this to eq. (107), we obtain

$$W_{\mathrm{WZ}}(\Phi)|_{\theta\theta} + \mathrm{h.c.} = -2\left|\frac{\partial W(\phi)}{\partial \phi}\right|^2 - \frac{1}{2}\left[\frac{\partial^2 W(\phi)}{\partial \phi^2}\psi\psi + \mathrm{h.c}\right]. \tag{109}$$

Hence, the full component expansion of the Wess-Zumino Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\mathrm{WZ}} =& -\partial_\mu \phi^\dagger \partial^\mu \phi + \mathrm{i}\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \left|\frac{\partial W(\phi)}{\partial \phi}\right|^2 - \frac{1}{2}\left[\frac{\partial^2 W(\phi)}{\partial \phi^2}\psi\psi + \mathrm{h.c}\right] \\ =& -\partial_\mu \phi^\dagger \partial^\mu \phi + \mathrm{i}\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - m^2 \phi^\dagger \phi + \frac{1}{2} g m \phi(\phi^\dagger \phi^2 + \phi(\phi^\dagger)^2) \\ & + \frac{1}{4} g^2 \phi^2 (\phi^\dagger)^2 - \frac{1}{2}(m\psi\psi + g\phi\psi\psi + \mathrm{h.c}).\end{aligned} \tag{110}$$

Writing $\phi = \frac{1}{2}(S + \mathrm{i}P)$, where S and P are real fields, and $\xi_a = (\psi_\alpha, \bar{\psi}^{\dot{\alpha}})$, we conclude that the WZ model describes three fields of equal mass m: a real scalar, S, a real pseudoscalar, P and a Majorana fermion, ξ.

2.3.7 $\mathcal{N} = 1$ SuperYang-Mills theory

We begin by noting that the real part of a chiral superfield Λ is a special kind of a vector superfield

$$\begin{aligned}\Lambda + \Lambda^\dagger =& (\phi + \phi^\dagger) + \theta\psi + \bar{\theta}\bar{\psi} + \theta^2 F + \bar{\theta}^2 F^\dagger - \mathrm{i}\,\theta\sigma^\mu\bar{\theta}\,\partial_\mu(\phi - \phi^\dagger) \\ & - \frac{\mathrm{i}}{2}\theta^2\,\bar{\theta}\bar{\sigma}^\mu \partial_\mu \chi - \frac{\mathrm{i}}{2}\bar{\theta}^2\,\theta\sigma^\mu \partial_\mu \bar{\chi} + \frac{1}{4}\theta^2\bar{\theta}^2 \Box(\phi + \phi^\dagger),\end{aligned} \tag{111}$$

where the vector component is a total derivative of a real scalar field. This implies that the transformation

$$V \mapsto V + (\Lambda + \Lambda^\dagger) \tag{112}$$

would generalize a $U(1)$ gauge transformation for superfields. The above transformation has the following effect on the component fields:

$$C \mapsto C + (\phi + \phi^\dagger), \tag{113}$$
$$\chi \mapsto \chi + \psi, \tag{114}$$
$$G \mapsto G + F, \tag{115}$$
$$v_\mu \mapsto v_\mu - \mathrm{i}\partial_\mu(\phi - \phi^\dagger), \tag{116}$$
$$\eta_\alpha \mapsto \eta_\alpha - \frac{\mathrm{i}}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}}, \tag{117}$$
$$E \mapsto E + \frac{1}{4}\Box(\phi + \phi^\dagger). \tag{118}$$

From this result we conclude that the combinations

$$\lambda_\alpha := \eta_\alpha + \frac{\mathrm{i}}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}}, \tag{119}$$
$$D := E + \frac{1}{4}\Box C \tag{120}$$

are gauge invariant. Moreover, since C, χ and G transform only by shifts, we can choose a gauge in which they vanish. This gauge is called the Wess-Zumino gauge (or WZ gauge for short) and in this gauge the vector superfield reads

$$V = \theta\sigma^\mu\bar{\theta}\, v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{121}$$

The great advantage of the WZ gauge is that the powers of V are now extremely simple, namely

$$V^2 = -\frac{1}{2}\theta^2\bar{\theta}^2 v^\mu v_\mu, \tag{122}$$

and all higher powers vanish. This will become handy when calculating the exponentials of vector superfields, which we will be having to do soon. In WZ gauge the exponential e^V reads simply

$$\mathrm{e}^V = 1 + \theta\sigma^\mu\bar{\theta}\, v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \theta^2\bar{\theta}^2(D - \frac{1}{4}v^\mu v_\mu). \tag{123}$$

Next, define a spinorial superfield W_α,

$$W_\alpha := -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V. \tag{124}$$

This superfield is both chiral and gauge invariant (see e.g. [24, p. 38]), so we can work in the WZ gauge. Due to the chirality of W_α, we know that the $\theta\theta$ component of $W^\alpha W_\alpha$ transforms into a total derivative under a SUSY transformation. It, therefore, serves as a term in the supersymmetric Lagrangian. Writing in terms of the component fields, we arrive (after some lines of manipulation) at

$$W^\alpha W_\alpha|_{\theta\theta} = 2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}f^{\mu\nu}f_{\mu\nu} - \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma} + D^2, \tag{125}$$

where $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. We, thus, obtain the kinetic term for a vector superfield,

$$\begin{aligned}\mathcal{L}_{V,\mathrm{kin}} &= \frac{1}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \mathrm{h.c.}\\ &= \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial\lambda + \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{1}{2}D^2.\end{aligned} \tag{126}$$

We are still short of gauge invariant interaction terms, so let us concentrate next on those. Consider still a U(1) gauge symmetry with one chiral superfield Φ having charge e under U(1). The gauge transformation, then, takes the form

$$\Phi \mapsto \mathrm{e}^{-e\Lambda}\Phi, \text{ and } \Phi^\dagger \mapsto \Phi^\dagger\mathrm{e}^{-e\Lambda^\dagger}.[1] \tag{127}$$

It turns out that it is not the transformation law of the vector superfield itself that one wants to generalize to non-abelian case but transformation law of the exponential of it. Having this premonition in mind, let us write down the transformation law of exponential of the vector superfield as

$$\mathrm{e}^{eV} \mapsto \mathrm{e}^{e\Lambda^\dagger}\mathrm{e}^{eV}\mathrm{e}^{e\Lambda}, \text{ and } \mathrm{e}^{-eV} \mapsto \mathrm{e}^{-e\Lambda}\mathrm{e}^{-eV}\mathrm{e}^{-e\Lambda^\dagger}. \tag{128}$$

If we now consider the kinetic terms for chiral fields that we had before, i.e. terms of the form $\Phi^\dagger\Phi$, we find that these are not gauge invariant, but transform as

$$\Phi^\dagger\Phi \mapsto \Phi^\dagger\mathrm{e}^{-e(\Lambda+\Lambda^\dagger)}\Phi, \tag{129}$$

so we conclude that terms of the form

$$\Phi^\dagger\mathrm{e}^{eV}\Phi \tag{130}$$

are gauge invariant. We can, therefore, write down the gauge invariant gauge-matter coupled Lagrangian:

$$\mathcal{L}_{\text{matter-coupled}} = \Phi^\dagger\mathrm{e}^{eV}\Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left[\frac{1}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \mathrm{h.c.}\right]. \tag{131}$$

[1]Note that even though Wess and Bagger change the form of the gauge transformation at this point to $V \mapsto V + \mathrm{i}(\Lambda - \Lambda^\dagger)$, we will stick to our previous convention $V \mapsto V + \Lambda + \Lambda^\dagger$ following [26] here.

This model does not, however, allow massive charged matter since the mass term in the superpotential is not gauge invariant. In order to allow massive charged matter, one needs to include two oppositely charged chiral superfields $\Phi_\pm$ transforming under the U(1) gauge group as

$$\Phi_\pm \mapsto \mathrm{e}^{\pm e\Lambda}\Phi_\pm. \tag{132}$$

The supersymmetric Lagrangian becomes then (see e.g. [28])

$$\mathcal{L}_{\text{massive}} = \left(\Phi_+^\dagger \mathrm{e}^{eV}\Phi_+ + \Phi_-^\dagger \mathrm{e}^{eV}\Phi_-\right)\Big|_{\theta\theta\bar\theta\bar\theta} + \left[\left(\frac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-\right)\Big|_{\theta\theta} + \text{h.c.}\right]. \tag{133}$$

The coupling to matter generalizes straight-forwardly to a non-abelian gauge theory by writing

$$V \coloneqq V^a T^a, \tag{134}$$

where T^a's are the generators of the gauge group and $a = 1, \ldots, d_{\mathrm{G}}$, where d_{G} is the dimension of the gauge group, is the adjoint index. To obtain the kinetic terms of the previous form, we upgrade the definition of W_α a bit. First of all, notice that in the abelian case we can write the definition of W_α (eq. (124)) in the form

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2 \mathrm{e}^{-gV}\mathcal{D}_\alpha \mathrm{e}^{gV}. \tag{135}$$

Now, in the non-abelian case, in order to maintain the conventional couplings of the component fields, we have to rescale $V \to 2gV$ and, in addition, to rescale the spinorial field strength W_α by factor $\frac{1}{2g}$, see e.g. [28]. Thus, the upgraded definition reads

$$W_\alpha = -\frac{1}{8g}\bar{\mathcal{D}}^2 \mathrm{e}^{-2gV}\mathcal{D}_\alpha \mathrm{e}^{2gV}. \tag{136}$$

It then follows that W_α transforms under the gauge group as

$$W_\alpha \mapsto \mathrm{e}^{-2g\Lambda} W_\alpha \mathrm{e}^{2g\Lambda} \tag{137}$$

Thus,

$$\mathrm{Tr}[W^\alpha W_\alpha]|_{\theta\theta} \tag{138}$$

is gauge invariant and we obtain the kinetic terms

$$\mathcal{L}_{V,\text{kin}} = \frac{1}{4}\mathrm{Tr}[W^\alpha W_\alpha]\Big|_{\theta\theta} + \text{h.c.} \tag{139}$$

Analogously, we obtain gauge-matter coupling terms

$$\mathcal{L}_{\text{gauge}-\text{matter}} = \Phi^\dagger \mathrm{e}^{2gV} \Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}}, \tag{140}$$

where matter superfield Φ is now on some representation of the gauge group.

Having then the kinetic and matter-coupling terms, the final issue is to find out the gauge invariant superpotential. The result is that the most general renormalisable superpotential is a cubic polynomial

$$W(\mathbf{\Phi}) = a^i \Phi^i + \frac{1}{2} m^{ij} \Phi^i \Phi^j + \frac{1}{3} g^{ijk} \Phi^i \Phi^j \Phi^k, \tag{141}$$

where the chiral superfields Φ^i are the components of $\mathbf{\Phi}$ in some representation, if and only if a^i, m^{ij}, and g^{ijk} are symmetric, invariant tensors in the representation of $\mathbf{\Phi}$ [28].

As a final curiosity, we mention that the so-called Fayet-Iliopoulos term, which is important when considering spontaneous supersymmetry breaking, and is of the form

$$\mathrm{Tr}[\kappa V]|_{\theta\theta\bar{\theta}\bar{\theta}} = \mathrm{Tr}[\kappa D], \tag{142}$$

where κ is a constant element in the centre of the Lie algebra of the gauge group, is also both supersymmetric and gauge-invariant.

2.3.8 Some remarks on $\mathcal{N} = 4$ SYM

The $\mathcal{N} = 4$ supermultiplet in 4 spacetime dimensions consists of a gauge field, 4 Weyl fermions and 6 scalars. These degrees of freedom can be implanted in one vector and three chiral $\mathcal{N} = 1$ superfields transforming in the adjoint representation of the gauge group. Then, the Lagrangian written in terms of these $\mathcal{N} = 1$ superfields takes the form [33]

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4\,\mathrm{SYM}} =& \mathrm{Tr}\left[\Phi^\dagger \mathrm{e}^{2gV} \Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \frac{1}{4}\left(W^\alpha W_\alpha|_{\theta\theta} + \text{h.c.}\right)\right] \\ &- \frac{\mathrm{g}}{3\sqrt{2}}\left(\epsilon_{ijk} f^{abc} \Phi_i^a \Phi_j^b \Phi_k^c\Big|_{\theta\theta} + \text{h.c.}\right), \end{aligned} \tag{143}$$

where $i, j, k = 1, 2, 3$ are the flavour indices, $a, b, c = 1, \ldots, N^2 - 1$ are the adjoint gauge indices and f^{abc} are the $\mathrm{SU}(N)$ structure constants.

The $\mathcal{N} = 4$ SYM has a global $\mathrm{SU}(4)_\mathrm{R}$ symmetry although it is not explicitly apparent when written in terms of $\mathcal{N} = 1$ superfields. The gaugino and the three adjoint fermios transform as **4** of $\mathrm{SU}(4)_{\mathrm{R}}$ whereas the real adjoint scalars transform as **6** of $\mathrm{SU}(4)_{\mathrm{R}}$ [33].

The N = 4 SYM is special in many ways. First of all, it is known to be perturbatively finite, i.e. the β function vanishes, to all orders, and this is believed to hold also non-perturbatively. Secondly, it provides a link between field theory and string theory via the so-called AdS/CFT correspondence [34, 35, 36]. Via this correspondence, a supergravity theory in a $d+1$ dimensional anti-de-Sitter space is dual to d dimensional (super)conformal field theory. In this case, a type IIB string theory in $\mathrm{AdS}_5 \times S^5$ background (five dimensions compactified on a sphere) is dual $\mathcal{N} = 4$ SYM in four-dimensional Minkowski space.

2.3.9 Supersymmetry breaking

Before entering the finale of this section, i.e. introducing the MSSM, we must confront yet another obstacle. Namely, if there is supersymmetry in the Nature, it must, at best, be somehow broken since we have not yet observed the supersymmetric partners of the SM particles. Here, we will briefly address ourselves to the main aspects of the problem.

Let us start with an observation that the Hamiltonian of a supersymmetric theory can be written in the form (see e.g. [28])

$$H = \frac{1}{4}(\bar{Q}_{\dot{1}} Q_1 + Q_1 \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{2}} Q_2 + Q_2 \bar{Q}_{\dot{2}}), \tag{144}$$

which implies that

$$\langle\psi|H|\psi\rangle = \frac{1}{4}(\|Q_1|\psi\rangle\|^2 + \|\bar{Q}_{\dot{1}}|\psi\rangle\|^2 + \|Q_2|\psi\rangle\|^2 + \|\bar{Q}_{\dot{2}}|\psi\rangle\|^2). \tag{145}$$

This relatively simple looking expression is crucial: The energy in a supersymmetric theory is positive-semidefinite. This plays an essential role in the spontaneous breaking of SUSY.

Choosing $|\psi\rangle$ to be the vacuum, we notice that the vacuum energy is zero if and only if the vacuum of the theory is supersymmetric, i.e. it is annihilated by the supercharges. This further implies that the SUSY is spontaneously broken if and only if the vacuum energy is strictly positive.

With chiral superfields, the only source of spontaneous SUSY breaking comes with non-zero vacuum expectation values (VEVs) of auxiliary fields and is known as F-term (or O'Raifeartaigh) SUSY breaking. Similarly with vector superfields, only the non-zero VEV of the auxiliary D field can act as a source of spontaneous SUSY breaking. This is known as the D-term SUSY breaking. However, due to gauge invariance, the D-term SUSY breaking is a bit more complicated since a non-zero VEV of D field breaks gauge invariance unless the (Lie algebra valued) D field belongs to the centre of the gauge group. Now, since the centre of a semisimple Lie group is discrete, the gauge invariance requires the gauge group to have abelian factors. With a suitable gauge group, the D-term SUSY breaking can be brought about by adding Fayet-Iliopoulos terms; see eq. (142). Further details of these both types of SUSY breaking can be found e.g. in [28] and more thoroughly in [29].

With the MSSM, the required breaking pattern can be achieved by a Planck-scale-mediated SUSY breaking, which at low energies manifests itself by so-called soft terms.

The soft terms (couplings are of positive mass dimension) consist of gaugino mass terms, scalar mass terms, three- and two-scalar couplings and scalar tadpole couplings (see e.g. [29]). These can be explicitly written (with generic gauginos λ^a and scalars ϕ_i) as

$$\mathcal{L}_{\text{soft}} = -\left(\frac{1}{2}M^a\lambda^a\lambda^a + \frac{1}{6}a_{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b_{ij}\phi_i\phi_j + t_i\phi_i + \text{h.c.}\right) - m_{ij}^2\phi_i^*\phi_j. \tag{146}$$

2.3.10 Minimal Supersymmetric Standard Model (MSSM)

We close our generic discussion of SUSY by introducing the MSSM, the minimal extension of the SM in the realm of supersymmetry. In other words, we want to build a supersymmetric model with gauge group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ that contains the particle content of the SM. To this end, we need to introduce the vector superfields $V_\mathrm{C}^a, V_\mathrm{L}^b$ and V_Y, $a = 1, \ldots, 8$ and $b = 1, 2, 3$, to include the gauge bosons (along with their fermionic partners, the *gauginos*). Moreover, to include SM fermions and their bosonic partners, the *sleptons* and the *squarks*, we need the five left-handed chiral superfields for each generation of leptons and quarks, namely two $\mathrm{SU}(2)_\mathrm{L}$ doublets Φ_{ℓ^I} and $\Phi_{q^I}^c$ and three $\mathrm{SU}(2)_\mathrm{L}$ singlets $\Phi_{\bar{e}^I}$, $\Phi_{\bar{u}^I}^c$ and $\Phi_{\bar{d}^I}^c$, where $c = 1, 2, 3$ is the colour index and $I = 1, 2, 3$ denotes the matter generations. In addition, to obtain the proper electroweak symmetry breaking pattern and to generate fermion masses, we must introduce two Higgs doublet superfields, Φ_{H_u} and Φ_{H_d}. The exact particle content and transformation properties under the gauge group of all of these superfields are presented in table (2). Now, following the treatment of the SYM in section 2.3.7 we immediately obtain the kinetic terms for the gauge superfields from eq. (139),

$$\mathcal{L}_{\text{MSSM, kin}} = \frac{1}{4}\sum_X \operatorname{Tr}\left[W_X W_X|_{\theta\theta} + \text{h.c.}\right], \tag{147}$$

Table 2. Quantum numbers and particle content of the MSSM superfields. The representation indices run as follows: $a = 1,\ldots,8$, $b = 1,2,3$ and $c = 1,2,3$. Moreover, the matter generation is denoted by the capital index $I = 1,2,3$. To avoid excess notational clutter, the index referring $\mathrm{SU}(2)_\mathrm{L}$ doublets has been omitted. Note that tilde *always* refers to the spin-0 component and *not* on the SUSY partner of a SM field

Superfield	$\mathrm{SU}(3)_\mathrm{C}$	$\mathrm{SU}(2)_\mathrm{L}$	$\mathrm{U}(1)_Y$	spin 0	spin 1/2	spin 1	auxiliary
V_C^a	$\mathbf{8}$	$\mathbf{1}$	0		λ_C^a	G_μ^a	D_C^a
V_L^b	$\mathbf{1}$	$\mathbf{3}$	0		λ_L^b	W_μ^b	D_L^b
V_Y	$\mathbf{1}$	$\mathbf{1}$	0		λ_Y	B_μ	D_Y
Φ_{ℓ^I}	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$\tilde{\ell}^I$	ℓ^I		F_{ℓ^I}
$\Phi_{\bar{e}^I}$	$\mathbf{1}$	$\mathbf{1}$	1	$\tilde{\bar{e}}^I$	$\bar{e}^I := (e_\mathrm{R}^I)^\dagger$		$F_{\bar{e}^I}$
$\Phi_{q^I}^a$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$\tilde{q}^{Ia}$	q^{Ia}		$F_{q^I}^a$
$\Phi_{\bar{u}^I}^a$	$\mathbf{3}$	$\mathbf{1}$	$-\frac{2}{3}$	$\tilde{\bar{u}}^{Ia}$	$\bar{u}^{Ia} := (u_\mathrm{R}^{Ia})^\dagger$		$F_{\bar{u}^I}^a$
$\Phi_{\bar{d}^I}^a$	$\mathbf{3}$	$\mathbf{1}$	$\frac{1}{3}$	$\tilde{\bar{d}}^{Ia}$	$\bar{d}^{Ia} := (d_\mathrm{R}^{Ia})^\dagger$		$F_{\bar{d}^I}$
Φ_{H_u}	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	$\tilde{H}_\mathrm{u}$	H_u		F_{H_u}
Φ_{H_d}	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$\tilde{H}_\mathrm{d}$	H_d		F_{H_d}

where $(W_X)_\alpha = -\frac{1}{8g_X}\bar{\mathcal{D}}^2\mathrm{e}^{-2g_XV_X}\mathcal{D}_\alpha\mathrm{e}^{2g_XV_X}$, and $X = \mathrm{C},\ \mathrm{L},\ Y$.

Similarly, we obtain the gauge-matter coupling terms from eq. (140),

$$\begin{aligned}\mathcal{L}_{\mathrm{MSSM,\,gauge-matter}} = \Big[&\Phi_{q^I i}^{a\,\dagger}\exp\Big[2g_\mathrm{C}V_\mathrm{C} + 2g_\mathrm{L}V_\mathrm{L} + 2g_YY_{q^I}V_Y\Big]\,\Phi_{q^I i}^a\\ &+ \Phi_{\bar{u}^I}^{a\,\dagger}\exp\left[2g_\mathrm{C}V_\mathrm{C} + 2g_YY_{\bar{u}^I}V_Y\right]\Phi_{\bar{u}^I}^a\\ &+ \Phi_{\bar{d}^I}^{a\,\dagger}\exp\left[2g_\mathrm{C}V_\mathrm{C} + 2g_YY_{\bar{d}^I}V_Y\right]\Phi_{\bar{d}^I}^a\\ &+ \Phi_{\ell^I i}^\dagger\exp\left[2g_\mathrm{L}V_\mathrm{L} + 2g_YY_{\ell^I}V_Y\right]\Phi_{\ell^I i}\\ &+ \Phi_{\bar{e}^I}^\dagger\exp\left[2g_YY_{\bar{e}^I}V_Y\right]\Phi_{\bar{e}^I}\\ &+ \Phi_{H_\mathrm{u}i}^\dagger\exp\left[2g_\mathrm{L}V_\mathrm{L} + 2g_YY_{H_\mathrm{u}}V_Y\right]\Phi_{H_\mathrm{u}i}\\ &+\Phi_{H_\mathrm{d}i}^\dagger\exp\left[2g_\mathrm{L}V_\mathrm{L} + 2g_YY_{H_\mathrm{d}}V_Y\right]\Phi_{H_\mathrm{d}i}\Big]\Big|_{\theta\theta\bar{\theta}\bar{\theta}}\,,\end{aligned} \tag{148}$$

where, for completeness, we have also included the $\mathrm{SU}(2)_\mathrm{L}$ doublet index $i = 1,2$. The MSSM superpotential reads

$$\begin{aligned}W_\mathrm{MSSM} = &- y_{IJ}^\mathrm{e}\,\Phi_{\ell^I}\cdot\Phi_{H_\mathrm{d}}\,\Phi_{\bar{e}^J} - y_{IJ}^\mathrm{d}\,\Phi_{q^I}\cdot\Phi_{H_\mathrm{d}}\,\Phi_{\bar{d}^J}\\ &+ y_{IJ}^\mathrm{u}\,\Phi_{q^I}\cdot\Phi_{H_\mathrm{u}}\,\Phi_{\bar{u}^J} + \mu\,\Phi_{H_\mathrm{u}}\cdot\Phi_{H_\mathrm{d}},\end{aligned} \tag{149}$$

where $y^\mathrm{e},\ y^\mathrm{d}$ and y^u are 3×3 matrices containing the Yukawa couplings and μ the coupling between the Higgses and is called μ parameter. Note that the products, denoted by a dot, between two $\mathrm{SU}(2)_\mathrm{L}$ doublets stand for contractions with the antisymmetric epsilon symbol, i.e. for arbitrary $\mathrm{SU}(2)_\mathrm{L}$ doublets Φ_1 and Φ_2 we have $\Phi_1 \cdot \Phi_2 \coloneqq \epsilon_{ij}\Phi_{1i}\Phi_{2j}$. The signs have been chosen such that they produce positive mass terms for the SM fermions, see e.g. [29]. From the superpotential, we see that we indeed need two Higgs doublet superfields to generate the masses for all of the SM fermions: Since the superpotential must be holomorphic function of the superfields, a term of the form $\Phi_{\bar{u}}\, \Phi^*_{H_\mathrm{d}} \cdot \Phi_q$, for example, is not allowed. In addition, we still need to break the supersymmetry explicitly with the so-called soft SUSY breaking terms. From eq. (146) we then obtain

$$
\begin{aligned}
\mathcal{L}_{\mathrm{MSSM,\,soft}} = &-\frac{1}{2}\Big(M_\mathrm{C}\lambda^a_\mathrm{C}\lambda^a_\mathrm{C} + M_\mathrm{L}\lambda^b_\mathrm{L}\lambda^b_\mathrm{L} + M_Y\lambda_Y\lambda_Y + \text{h.c.}\Big) \\
&- \Big(-(a_u)^{IJ}\tilde{\bar{u}}^{Ia}\tilde{H}_\mathrm{u}\cdot\tilde{q}^{Ja} + (a_d)^{IJ}\tilde{\bar{d}}^{Ia}\tilde{H}_\mathrm{d}\cdot\tilde{q}^{Ja} \\
&\qquad + (a_\ell)^{IJ}\tilde{\bar{e}}^{I}\tilde{H}_\mathrm{d}\cdot\tilde{\ell}^{J} - b\tilde{H}_\mathrm{u}\cdot\tilde{H}_\mathrm{d} + \text{h.c.}\Big) \\
&- (m_q^2)^{IJ}\tilde{q}^{Ia\dagger}\tilde{q}^{Ja} - (m_\ell^2)^{IJ}\tilde{\ell}^{I\dagger}\tilde{\ell}^{J} - (m_{\bar{u}}^2)^{IJ}\tilde{\bar{u}}^{Ia\dagger}\tilde{\bar{u}}^{Ja} \\
&- (m_{\bar{d}}^2)^{IJ}\tilde{\bar{d}}^{Ia\dagger}\tilde{\bar{d}}^{Ja} - m_\mathrm{u}^2\tilde{H}_\mathrm{u}^\dagger\tilde{H}_\mathrm{u} - m_\mathrm{d}^2\tilde{H}_\mathrm{d}^\dagger\tilde{H}_\mathrm{d},
\end{aligned}
\tag{150}
$$

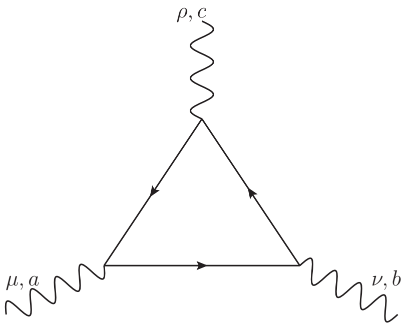
where $M_\mathrm{C}, M_\mathrm{L}$ and M_Y are the gluino, wino and bino masses, respectively. Moreover, a_u, a_d and a_ℓ are complex 3×3 matrices in generation space containing couplings of dimensions of mass, $m_q, m_\ell, m_{\bar{u}}$ and $m_{\bar{d}}$ are 3×3 matrices (with possibly complex entries) in generation space with restriction they be hermitian in order the Lagrangian to be real, and b, m_u^2 and m_d^2 are complex of dimension mass-squared. Again, the dot products denote contractions with the two-index epsilon symbol.

3 Supersymmetric Technicolor

3.1 Anomaly cancellation for general $\mathrm{SU}(N)$

Before entering the actual topic of the this section, i.e. building a supersymmetric technicolor model, we first need to look into a phenomenon that already the chiral fermions per se bring about, namely the possible chiral gauge anomalies.

When considering loop corrections of a chiral theory with massless fermions, one encounters a subtlety with severe consequences. Although the Lagrangian be chirally invariant and the axial current at tree level conserved, loop corrections can produce anomalous non-conservation of the axial current. These possibly anomalous contributions can be evaluated by computing the so-called triangle diagrams with three external gauge bosons attached to a fermion loop. A detailed discussion of chiral anomalies can be found in [37]. For these triangle diagrams, we obtain



$$\propto \mathcal{A}^{abc} = \mathrm{Tr}[t^a\{t^b, t^c\}], \tag{151}$$

where t^a, t^b and t^c are the group theoretical factors related to the gauge bosons A_μ^a, A_ν^b and A_ρ^c, respectively.

Theories for which $\mathcal{A}^{abc} = 0$ are said to be *anomaly-free*. Possible anomalous non-zero contributions may arise if different chiral components couple differently in the given theory as is the case with weak interaction, for example. Therefore, QCD (as well as TC coupling the chiral partners democratically) on its own is anomaly free, so triangle diagrams with three (techni)gluons give a zero contribution. Likewise, $\mathrm{SU}(2)_\mathrm{L}$ itself is anomaly free, although it couples left- and right-handed fermions differently. This is due to the fact that the fundamental representation of SU(2) is pseudoreal implying an extra minus sign if one changes all the fermions in the loop of a triangle diagram to their antiparticles (see eq. (13)). Thus, diagrams with three $\mathrm{SU}(2)_\mathrm{L}$ bosons can be omitted as well. What remains, are the diagrams with at least one $U(1)_Y$ boson. The hypercharge assignment of the SM along with equal number of quark and lepton generations makes the SM as a whole anomaly-free.

Let us now figure out the anomaly free hypercharge assignment of an $\mathrm{SU}(N)$ TC section with N_f techniquarks in the adjoint of the TC group. Consider only the case where the techniquarks are assembled in left-handed $\mathrm{SU}(2)_\mathrm{L}$ doublets, $N_\mathrm{f} = 2N_\mathrm{Q}$, and right-handed singlets. To avoid the topological Witten anomaly, we need to supplement the new sector with N_L new lepton doublets, $L_{\mathrm{L}j} = \begin{pmatrix} N_{\mathrm{L}j} \\ E_{\mathrm{L}j} \end{pmatrix}$, and their right-handed partners, $N_{\mathrm{R}j}, E_{\mathrm{R}j}$, such that $N_\mathrm{Q} + N_\mathrm{L}$ is even. For simplicity, let us denote $Y(Q_{\mathrm{L}i}^a) =: y_i,\ Y(L_{\mathrm{L}j}) =: x_j$, where $i = 1, \dots, N_\mathrm{Q},\ j = 1, \dots, N_\mathrm{L}$ and

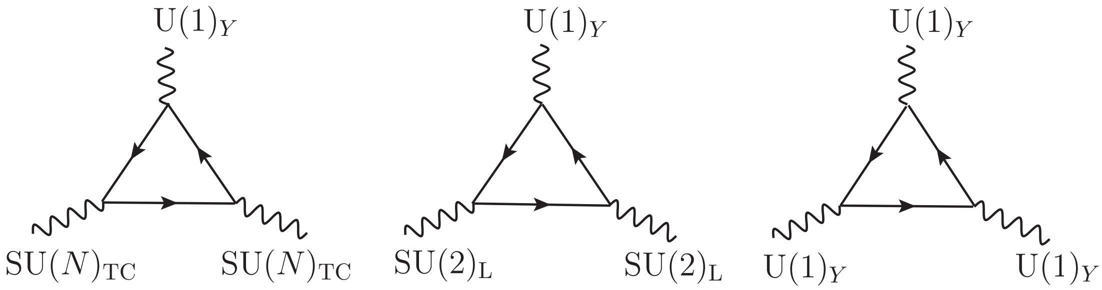


Figure 3. The possible anomalous triangle diagrams for TC sector.

$a = 1, \ldots, N^2 - 1 =: d_{\rm A}$. Then, we have the following hypercharge assignments:

$$Y(U^a_{{\rm L}i}) = Y(D^a_{{\rm L}i}) = y_i, \quad Y(U^a_{{\rm R}i}) = y_i + \frac{1}{2}, \quad Y(D^a_{{\rm R}i}) = y_i - \frac{1}{2}, \tag{152}$$

$$Y(N_{{\rm L}j}) = Y(N_{\rm L}j) = x_j, \quad Y(N_{{\rm R}j}) = x_j + \frac{1}{2}, \quad Y(E_{{\rm R}j}) = x_j - \frac{1}{2}. \tag{153}$$

Next, consider all the possible triangle diagrams and demand $\mathcal{A}^{abc} = 0$ for all the diagrams. Note first that since the generators of SU(N), $N \geq 2$, are traceless, diagrams with only one SU(N) boson give zero. Thus, it suffices to consider only diagrams with one or three U(1)$_Y$ bosons; the three remaining cases are depicted in fig. 3.

The first case with two SU(N)$_{\rm TC}$ bosons gives

$$\begin{aligned}\mathcal{A}^{abc} &\propto \sum_{\text{techniquarks}} \mathrm{Tr}\left[Y_f\{T^b, T^c\}\right] \propto \sum_{\text{techniquarks}} Y_f \delta^{bc} \\ &= \sum_i \left[-2d_{\rm A}y_i + d_{\rm A}\left(y_i + \frac{1}{2}\right)\right] + d_{\rm A}\left(y_i - \frac{1}{2}\right)\delta^{bc} = 0.\end{aligned} \tag{154}$$

The first diagram, thereby, does not bring any potential anomalous contributions regardless of the hypercharge assignment.

The second case with two SU(2)$_{\rm L}$, in turn, gives

$$\begin{aligned}\mathcal{A}^{abc} &\propto \sum_{\text{left}-\text{handed}} \mathrm{Tr}\left[Y_f\frac{1}{2}\{\sigma^b, \frac{1}{2}\sigma^c\}\right] \propto \sum_{\text{left-handed}} Y_f \delta^{bc} \\ &= \sum_i -2d_{\rm A}y_i + \sum_j -2x_j.\end{aligned} \tag{155}$$

Setting this equal to zero gives

$$\sum_j x_j = -\sum_i y_i d_{\rm A}. \tag{156}$$

The final case with three $\mathrm{U}(1)_Y$ bosons results in

$$\begin{aligned}\sum_{\text{all new fermions}} Y_f^3 &= \sum_i \left[-2d_\mathrm{A}y_i^3 + d_\mathrm{A}\left(y_i+\frac{1}{2}\right)^3 + d_\mathrm{A}\left(y_i-\frac{1}{2}\right)^3\right] \\ &\quad + \sum_j \left[-2x_j^3 + \left(x_j+\frac{1}{2}\right)^3 + \left(x_j-\frac{1}{2}\right)^3\right] \\ &= \frac{3}{2}d_\mathrm{A}\sum_i y_i + \frac{3}{2}\sum_j x_j,\end{aligned} \tag{157}$$

and when set to zero, this gives the previous condition. Thus, to build an anomaly-free TC sector, the hypercharges of the new fermions must be chosen to fulfil eq. (156).

Having done this, let us redo the calculations in a slightly more specific setting. First, supplement SM with another generation of QCD quarks and then add an $\mathrm{SU}(N)$ TC sector. To keep things simple, let us only consider the MWT sector with one left-handed doublet of techniquarks and their right-handed partners, since then the Witten anomaly is absent even without any extra leptons.

We have now one more triangle diagram to consider, namely the one with two gluons and one $U(1)_Y$ boson. Since SM is anomaly free, we only need to consider the fourth generation of QCD quarks. Let us denote the hypercharges by

$$Y(q_{\mathrm{L4}}^a) = w, \quad Y(u_{\mathrm{R4}}^a) = w + \frac{1}{2}, \quad Y(d_{\mathrm{R4}}^a) = w - \frac{1}{2}, \tag{158}$$

where $q_{\mathrm{L4}}^a = (u_{\mathrm{L4}}^a \; d_{\mathrm{L4}}^a)^\mathrm{T}$ is the weak doublet. Moreover, denote the hypercharges of the techniquarks by

$$Y(Q_\mathrm{L}^a)) = y, \; Y(U_\mathrm{R}^a) = y + \frac{1}{2}, \; Y(D_\mathrm{R}^a) = y - \frac{1}{2}. \tag{159}$$

Then, the diagram with two technigluons is a special case of eq. (154) and is, thus, anomaly-free.

Similarly, the case with two gluons (remembering that SM with three generations of quarks is anomaly free) is similar to eq. (154) (with d_A replaced by 3, the dimension of the fundamental representation **3**) and is anomaly free.

The second case with two $\mathrm{SU}(2)_\mathrm{L}$ bosons gives

$$\mathcal{A}^{abc} \propto \sum_{\text{left-handed}} Y_f = -2 \cdot 3y - 2 \cdot 3w. \tag{160}$$

Setting this to zero, we have

$$y = -w. \tag{161}$$

The third case with three $U(1)_Y$ bosons gives

$$\begin{aligned}
\sum_{\text{all new fermions}} Y_f^3 &= \left[-2 \cdot 3y^3 + 3\left(y + \frac{1}{2}\right)^3 + 3\left(y - \frac{1}{2}\right)^3\right] \\
&+ \left[-2 \cdot 3w^3 + 3\left(w + \frac{1}{2}\right)^3 + 3\left(w - \frac{1}{2}\right)^3\right] \\
&= \frac{9}{2}y + \frac{9}{2}w,
\end{aligned} \tag{162}$$

resulting in the condition of eq. (161).

3.2 Minimal SuperConformal Technicolor (MSCT)

Let us start by extending the Standard Model with Minimal Walking Technicolor. The fermion content of the new sector, then, consists of one weak doublet of techniquarks with their right-handed partners transforming in the adjoint of the TC gauge group $\mathrm{SU}(2)_{\mathrm{TC}}$ along with a new weak doublet of leptons with their right-handed partners to avoid the Witten anomaly (see section 2.2.4). From eq. (156) we deduce that the anomaly-free hypercharge assignment reads

$$\begin{aligned}
&Y(Q_{\mathrm{L}}) = \frac{y}{2}, \quad Y(U_{\mathrm{R}}, D_{\mathrm{R}}) = \left(\frac{y+1}{2}, \frac{y-1}{2}\right), \\
&Y(L_{\mathrm{L}}) = -\frac{3y}{2}, \quad Y(N_{\mathrm{R}}, E_{\mathrm{R}}) = \left(\frac{-3y+1}{2}, \frac{-3y-1}{2}\right),
\end{aligned} \tag{163}$$

with a real-valued parameter y. Up to this point, nothing is new, but now we are ready to make the key observation. If we identify $\bar{D}_{\mathrm{R}}$ techniquark as the supersymmetric fermionic partner of the tehnigluon, the fermionic and gluonic content (without the new leptons) of this new sector is exactly that of $\mathcal{N} = 4$ supermultiplet! Moreover, the chiral SU(4) symmetry can be identified with the SU(4) R-symmetry of $\mathcal{N} = 4$ SUSY.

Therefore, by adding the needed scalar fields (always distinguished by an extra tilde from the fermionic partner), we obtain the following $\mathcal{N} = 1$ superfields forming the $\mathcal{N} = 4$ superYang-Mills theory:

$$\left(\tilde{U}_{\mathrm{L}}, U_{\mathrm{L}}\right) \in \Phi_1, \ \left(\tilde{D}_{\mathrm{L}}, D_{\mathrm{L}}\right) \in \Phi_2, \ \left(\tilde{\bar{U}}_{\mathrm{R}}, \bar{U}_{\mathrm{R}}\right) \in \Phi_3, \ \left(G, \bar{D}_{\mathrm{R}}\right) \in V, \tag{164}$$

Table 3. Superfields of the supersummetrised MWT sector and their quantum numbers. The technicolor multiplets are denoted by superscripts $a = 1, 2, 3$ and the weak doublets by subscripts $i = 1, 2$.

Superfield	$\mathrm{SU}(2)_{\mathrm{TC}}$	$\mathrm{SU}(3)_{\mathrm{C}}$	$\mathrm{SU}(2)_{\mathrm{L}}$	$\mathrm{U}(1)_Y$
Φ_i^a	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$
Φ_3^a	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	-1
V^a	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	0
Λ_i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{3}{2}$
N	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	1
E	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	2

where Φ_i, $i = 1, 2, 3$ are chiral superfields, and V is a vector superfield. The adjoint indices of the TC gauge group have been omitted.

To fully supersymmetrize the entire MWT sector, we need four additional chiral superfields

$$\left(\tilde{N}_{\mathrm{L}}, N_{\mathrm{L}}\right) \in \Lambda_1, \; \left(\tilde{E}_{\mathrm{L}}, E_{\mathrm{L}}\right) \in \Lambda_2, \; \left(\tilde{\bar{N}}_{\mathrm{R}}, \bar{N}_{\mathrm{R}}\right) \in N, \; \left(\tilde{\bar{E}}_{\mathrm{R}}, \bar{E}_{\mathrm{R}}\right) \in E. \tag{165}$$

The identification of $\bar{D}_{\mathrm{R}}$ as the technigluino fixes the hypercharge assignment to $y = 1$ since the technigluinos must be singlets under the SM gauge group and, thus, have hypercharge 0. The superfields of the supersymmetrized MWT sector with the associated quantum numbers have been collected in table 3.

We next couple this MWT sector to the MSSM (taken invariant under the TC gauge group) including also the both Higgses of the MSSM. The superpotential of this model consists then of two parts,

$$W = W_{\mathrm{MSSM}} + W_{\mathrm{TC}}, \tag{166}$$

where W_{MSSM} is the MSSM superpotential of eq. (149) and the gauge invariant, renormalizable superpotential of the TC sector, W_{TC}, which additionally is $\mathcal{N} = 4$ invariant in the regime where TC coupling, g_{TC}, is much larger than the other couplings, reads

$$\begin{aligned} W_{\mathrm{TC}} = &- \frac{g_{\mathrm{TC}}}{3\sqrt{2}} \epsilon_{ijk} \epsilon^{abc} \Phi_i^a \Phi_j^b \Phi_k^c + y_U \epsilon_{ij3} \Phi_i^a H_{\mathrm{u}j} \Phi_3^a + y_N \epsilon_{ij3} \Lambda_i H_{\mathrm{u}j} N \\ &- y_E \epsilon_{ij3} \Lambda_i H_{\mathrm{d}j} E + y_R \Phi_3^a \Phi_3^a E. \end{aligned} \tag{167}$$

If $\mathcal{N} = 4$ invariance is not required, the coupling in front of the first term would be a general Yukawa coupling y_{TC}, which, however, has been found to tend towards g_{TC} at low energies in [38], thereby justifying the above choice.

Now, depending on the strength of the TC coupling, two basic regimes are to be identified. First, if the coupling is sufficiently small, the new sector can be treated within the perturbation theory. This model is denoted pMSCT and is further investigated in [38]. The second regime is the one in which the supersymmetric technicolor is strongly coupled and is denoted by sMSCT. In this thesis, we will concentrate on the latter.

For completeness, yet another regime can be distinguished. This is the one in which SUSY is not broken and the TC sector is strongly coupled at EW scale. This regime must be analysed with so-called unparticle methods introduced by Georgi in [39] and is beyond our scope here.

Before taking the step to the strong regime, let us first briefly discuss the generalization of the TC section to gauge group $\mathrm{SU}(N)_{\mathrm{TC}}$ with one left-handed techniquark doublet and their right-handed partners in the adjoint of the TC gauge group.

3.3 General N

First note that the low-energy theory for the Higgs sector in the non-supersymmetric case is independent of the number of technicolors, N, with one weak doublet of techniquarks in the adjoint of the TC gauge group since the adjoint representation is real; see section 2.2.6. Thus, all the above (within TC sector) generalises for $N = 3, 4, \ldots$ since the theory is still walking with only one techniquark doublet. The interesting feature is that it is the 'lepton sector'[2] that at electroweak scale sees the actual TC gauge group. The only difference to the setting comes with the new TC singlets. To avoid Witten anomaly, the number of weak doublets must be even. Now, whereas for odd N the dimension of the adjoint representation, $N^2 - 1$, is even, for even N the dimension is odd. Thus, for odd N we must add an even number (or zero, for that matter) of weak doublets and for even N an odd number of weak doublets.

If we add N_{L} lepton doublets, then after fixing the hypercharges of the techniquarks, i.e. identifying $\bar{D}_{\mathrm{R}}$ as the technigaugino, the hypercharges of the leptons must fulfil (see eq. (156))

$$\sum_j x_j = -\frac{1}{2} d_{\mathrm{A}}. \tag{168}$$

[2]Note that 'lepton sector' should not be taken literally here; here with 'lepton sector' we actually mean those beyond-Standard-Model (BSM) fields that are singlets under the technicolor group but not under $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$, i.e. those that play a role in assigning the techniquarks anomaly-free hypercharges. The lepton sector in this wider meaning here includes a fourth generation of QCD quarks, for example.

where $x_j = Y(L_{\mathrm{L}j})$ is the hypercharge of the j^{th} lepton doublet and d_{A} the dimension of the adjoint representation of the TC group.

As an interesting special case, if we consider MWT and add, not a lepton doublet, but the fourth generation of QCD quarks, then fixing the hypercharges of the techniquarks as above sets $Y(q_{\mathrm{L}4}) = -1$ (see eq. (161)).

4 Strong regime

4.1 General overview of the model

The general setting is that of sec. 3.2 with strongly coupled supersymmetric technicolor sector at the electroweak scale but with general N. We, therefore, assume the electroweak symmetry breaking to be entirely due to the TC condensate and the MSSM Higgses only transmit the EWSB to the fermion sector, thereby obviating an additional ETC sector. We assume the Higgses, as well as the squarks and the technisquarks, to be heavy and, thus, decoupled from the low-energy theory.

4.2 The effective Lagrangian

Let us now construct the effective Lagrangian of the model at the electroweak scale. We follow here the treatment discussed in [40].

The effective theory is described in terms of the composite field

$$M \sim \eta^a \eta^{a\mathrm{T}}, \tag{169}$$

where $\eta^a = (U_{\mathrm{L}}^a \; D_{\mathrm{L}}^a \; \bar{U}_{\mathrm{R}}^a \; \bar{D}_{\mathrm{R}}^a)^{\mathrm{T}}$ and a is the technicolor index, which from now on will be omitted to avoid notational clutter. This multiplet transforms under $\mathrm{SU}(4)_R$ as

$$\eta \to g\eta, \tag{170}$$

where $g \in \mathrm{SU}(4)_R$. This composite field is the only spin-zero TC singlet field made out of two techniquarks. It transforms as

$$M \to gMg^{\mathrm{T}} \tag{171}$$

under the global symmetry group $\mathrm{SU}(4)_{\mathrm{R}}$. The effective Lagrangian can then be divided into two parts: the one conserving the $\mathrm{SU}(4)_{\mathrm{R}}$ symmetry and the symmetry breaking part. The symmetry conserving part is the effective Lagrangian of pure MWT sector and can be read off from sec. 2.2.6. To recap,

$$\mathcal{L}_{\rm SC} = \frac{1}{2}{\rm Tr}\left[DM^\dagger DM\right] - \mathcal{V}_M, \tag{172}$$

where $D_\mu M = \partial_\mu M - ig\left[G_\mu M + MG_\mu^{\rm T}\right]$, $gG_\mu = gW_\mu^a L^a + g' B_\mu Y$, and

$$\begin{aligned}\mathcal{V}_M =& -\frac{m_M^2}{2}{\rm Tr}\left[M^\dagger M\right] + \frac{\lambda_M}{4}{\rm Tr}\left[M^\dagger M\right]^2 + \lambda'_M {\rm Tr}\left[M^\dagger M M^\dagger M\right] \\ & - 2\lambda''_M\left[{\rm det}M + {\rm det}M^\dagger\right].\end{aligned} \tag{173}$$

Let us then include supersymmetry. Now, the Yukawa couplings in the superpotential induce some symmetry breaking terms in the effective Lagrangian, and the full Lagrangian can be written as

$$\mathcal{L}_{\rm eff} = \frac{1}{2}{\rm Tr}\left[DM^\dagger DM\right] - \mathcal{V}_M - \mathcal{V}_{\rm SB}. \tag{174}$$

To tackle the symmetry breaking part, $\mathcal{V}_{\rm SB}$, let us consider the Yukawa coupling terms of the superpotential in more detail. They can be conveniently written in the form

$$W_{\rm Yukawa} = \tilde{H}_{\rm u}\cdot F_{\rm u} + \tilde{H}_{\rm d}\cdot F_{\rm d} + {\rm h.c.}, \tag{175}$$

where

$$F_{\rm u} = - q_{{\rm L}u}^I y_{\rm u}^I \bar{u}^I - y_U Q_{\rm L}\bar{U}_{\rm R} - y_N^J L_{\rm L}^J \bar{N}_{\rm R}^J, \tag{176}$$

$$F_{\rm d} = q_{{\rm L}d}^I y_d^I \bar{d}^I + \ell^I y_e^I \bar{e}^I + y_E^J L_{\rm L}^J \bar{E}_{\rm R}^J, \tag{177}$$

and the flavour indices $I = 1, 2, 3$ and $J = 1, \ldots, N_{\rm L}$ are to be summed over. The Yukawa coupling matrices y_u, y_d, y_e, y_N and y_E are diagonal and the Cabibbo-Kobayashi-Maskawa matrix V has been built in to the definition of the vectors $q_{{\rm L}u}^I = (u_{\rm L},\ V^{IJ} d_{\rm L}^J)$ and $q_{{\rm L}d}^I = (V^{\dagger IJ} u_{\rm L}^J,\ d_{\rm L}^I)$. Moreover, it should be noted, that the contraction between the ${\rm SU}(2)_{\rm L}$ vectors by the antisymmetric ϵ tensor is again denoted by the dot product.

Due to the soft SUSY breaking b term (see eq. (150)) the MSSM Higgses $\tilde{H}_{\rm u}$ and $\tilde{H}_{\rm d}$ mix. They can be diagonalized by writing

$$\begin{pmatrix} \tilde{H}_{\rm u} \\ \tilde{H}_{\rm d}^c \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{H}_1 \\ \tilde{H}_2 \end{pmatrix}, \tag{178}$$

where $\tilde{H}_{\rm d}^c = \epsilon \tilde{H}_{\rm d}^*$. It turns out to be convenient to write the masses of the Higgses $\tilde{H}_1$ and $\tilde{H}_2$, denoted by m_1 and m_2, respectively, in terms of two new parameters m_s and θ as

$$\frac{1}{2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)=\frac{c_\theta^2}{m_s^2}, \quad \frac{1}{2}\left(\frac{1}{m_1^2}-\frac{1}{m_2^2}\right)=\frac{c_\theta s_\theta}{m_s^2}, \tag{179}$$

where $c_\theta \coloneqq \cos\theta$ and $s_\theta \coloneqq \sin\theta$. Now, taking $\tilde{H}_1$ and $\tilde{H}_2$ heavy, we can integrate them out from the effective Lagrangian resulting in effective four-fermion interactions. There are three types of four-fermion interaction terms arising: First, the interactions of two SM fermions with two techniquarks. From these we then obtain the mass terms for SM fermions as the techniquark condensate acquires VEV. Second, there are four-technifermion interaction terms, which give masses to pseudo-Goldstone bosons arising from the chiral symmetry breaking. Last, the third category consists of the interaction terms of four SM fermions, four new leptons or two SM fermions and two new leptons.

In terms of the new parameters m_s and θ and these simply read

$$\mathcal{L}_{4-\text{fermion}} = \frac{c_\theta^2}{m_s^2}(F_{\rm u}^\dagger F_{\rm u} + F_{\rm d}^\dagger F_{\rm d}) + \frac{c_\theta s_\theta}{m_s^2}(F_{\rm u}\cdot F_{\rm d} + \text{h.c.}). \tag{180}$$

To obtain the effect of these $\mathrm{SU}(4)_R$ breaking terms on the effective Lagrangian at EW scale, we use the so-called spurion technique. That is, we come up with spurious fields, or spurions, and assign them fictional transformation properties that make the symmetry breaking terms invariant under the largest possible global symmetry. If we then include terms lowest order in these spurions to the effective Lagrangian, we obtain the correct effect of the explicit symmetry breaking assuming the explicit breaking to be small.

Let us start with mass terms of SM fermions and the new leptons. After integrating out the Higgses, the four-fermion interaction terms between two techniquarks and two non-techniquarks can be written as

$$\begin{aligned}\frac{y_U c_\theta \omega}{m_s^2}\left(Q_{{\rm L}i}\bar{U}_{\rm R}\right)\Big[&c_\theta q_{{\rm Lu}i}^{*I} y_u^{*I}\bar{u}_{\rm R}^{*I} + c_\theta y_N^{J*} L_{{\rm L}i}^{J*}\bar{N}_{\rm R}^{J*} + s_\theta\epsilon_{ij} q_{{\rm Ld}j}^I y_d^I \bar{d}_{\rm R}^I \\ &+ s_\theta\epsilon_{ij}\ell_{{\rm Ld}j}^I y_e^I \bar{e}_{\rm R}^I + s_\theta y_E^J \epsilon_{ij} L_{{\rm L}j}^J \bar{E}_{\rm R}^J\Big]\end{aligned} \tag{181}$$

where ω is a dimensionless techniquark renormalization constant. Now, by introducing a spurion Z,

$$\begin{aligned}Z_{ij} = \frac{y_U c_\theta \omega}{m_s^2}\Big[&\delta_{ik} c_\theta\left(q_{{\rm Lu}k}^* y_u^* \bar{u}_{\rm R}^* + y_N^* L_{{\rm L}k}^* \bar{N}_{\rm R}^*\right) \\ &+\epsilon_{ik} s_\theta\left(q_{{\rm Ld}k} y_d \bar{d}_{\rm R} + \ell_{{\rm Ld}k} y_e \bar{e}_{\rm R} + y_E L_{{\rm L}k}\bar{E}_{\rm R}\right)\Big]\delta_{3j},\end{aligned} \tag{182}$$

where we have omitted the flavour indices to simplify the notation, we can write the mass terms in the form

$$\eta^{\mathrm{T}} Z \eta. \tag{183}$$

Note that only Z_{13}, Z_{23} are non-zero. In order these terms to obey the full global symmetry, the spurion Z ought to transform under $\mathrm{SU}(4)_R$ as

$$Z \to g^* Z g^\dagger. \tag{184}$$

This implies that we should include term of the form $\mathrm{Tr}\,[MZ]$ into the effective Lagrangian, since this is invariant under $\mathrm{SU}(4)_R$:

$$\mathrm{Tr}\,[MZ] \to \mathrm{Tr}\left[gMg^{\mathrm{T}} g^* Z g^\dagger\right] = \mathrm{Tr}\,[MZ]\,. \tag{185}$$

Another spurion, W, is related to the four-technifermion terms. These can be written as

$$-\frac{|y_U|^2 c_\theta^2 \omega^2}{m_s^2} \left(Q_{\mathrm{L}i} \bar{U}_{\mathrm{R}}\right) \left(Q_{\mathrm{L}i}^* \bar{U}_{\mathrm{R}}^*\right) = -W_{ijkl} \eta_i^\alpha \eta_j^\alpha \eta_k^{*\beta} \eta_l^{*\beta}, \tag{186}$$

with

$$W_{ijkl} = \frac{|y_U|^2 c_\theta^2 \omega^2}{m_s^2} (\delta_{ik1} + \delta_{ik2}) \delta_{jl3}, \tag{187}$$

and α, β are spin indices. In order the above terms to be invariant under $\mathrm{SU}(4)_R$, W must transform as

$$W_{ijkl} \to g_{im}^* g_{jn}^* g_{ko} g_{lp} W_{mnop}. \tag{188}$$

This, in turn, implies that we should include a term of the form $W_{ijkl} M_{ij} M_{kl}^*$ into the effective Lagrangian, since

$$\begin{aligned} W_{ijkl} M_{ij} M_{kl}^* &\to (g_{im}^* g_{jn}^* g_{ko} g_{lp} W_{mnop})(g_{iq} M_{qr} g_{jr})(g_{ks}^* M_{st}^* g_{lt}^*) \\ &= W_{mnop} M_{qr} M_{st}^* (g^\dagger g)_{mq} (g^\dagger g)_{nr} (g^{\mathrm{T}} g^*)_{os} (g^{\mathrm{T}} g^*)_{pt} \\ &= W_{ijkl} M_{ij} M_{kl}^*. \end{aligned} \tag{189}$$

In addition to the superpotential Yukawa couplings, there is still another source of $\mathrm{SU}(4)_R$ symmetry breaking, namely the soft SUSY breaking mass term of the gaugino $\bar{D}_{\mathrm{R}}$. Introducing a spurion X, this term can be written as

$$\frac{1}{2} M_D \bar{D}_{\mathrm{R}} \bar{D}_{\mathrm{R}} = \eta^{\mathrm{T}} X \eta, \tag{190}$$

where $X = \mathrm{diag}(0, 0, 0, \frac{1}{2} M_D)$. This spurion, then, transforms as

$$X \to g^* X g^\dagger, \tag{191}$$

and induces a term similar to the one related to the spurion Z (see eq. (185)) into the effective Lagrangian.

Finally, note that in the $\mathcal{N} = 4$ limit all the couplings of the TC sector can be unified into one operator. Decoupling this term induces a four-techniquark operator, which, however, is invariant under the full global group. Therefore, it can be embedded into the symmetry conserving potential (eq. (173)) by redefinition of the coefficients and, for simplicity, we exclude it from the analysis here. The full analysis has been carried out in [40].

The full symmetry breaking potential then reads

$$\mathcal{V}_{\mathrm{SB}} = c_1 \Lambda_{\mathrm{TC}}^2 \mathrm{Tr}\left[M X \right] + c_2 \Lambda_{\mathrm{TC}}^2 \mathrm{Tr}\left[M Z \right] + c_3 \Lambda_{\mathrm{TC}}^4 W_{ijkl} M_{ij} M_{kl}^* + \mathrm{c.c.} \tag{192}$$

In the above, we have inserted powers of Λ_{TC}, the TC scale, to make the constants c_1, c_2 and c_3 dimensionless.

Now, to get some estimates for the coefficients in the effective Lagrangian, we can use naive dimensional analysis (NDA) [41, 42]. In terms of constant $g \coloneqq \Lambda_{\mathrm{TC}}/v_{\mathrm{w}}$, where $v_{\mathrm{w}} = 246$ GeV is the weak scale, we obtain the following estimates:

$$m_M^2 \sim \Lambda_{\mathrm{TC}}^2,\ \lambda_M \sim \lambda'_M \sim \lambda''_M \sim g^2,\ c_1 \sim c_2 \sim \frac{1}{g},\ c_3 \sim \frac{1}{g^2}. \tag{193}$$

We assume this NDA parameter g to be of order 4π from which we expect $\Lambda_{\mathrm{TC}} \sim 3$ TeV.

4.3 Vacuum

The vacuum structure is ignorant of the new lepton sector. Hence, we can find out the vacuum for the model with general N and without fixing any particular lepton sector.

Let us start by getting more insight into the effective variable, M. Using the result of [19], we can write M in terms of mass eigenstates as

$$M = \begin{pmatrix} \mathrm{i}\Pi_{UU} + \tilde{\Pi}_{UU} & \frac{\mathrm{i}\Pi_{UD} + \tilde{\Pi}_{UD}}{\sqrt{2}} & \frac{\sigma + \mathrm{i}\Theta + \mathrm{i}\Pi^0 + A^0}{2} & \frac{\mathrm{i}\Pi^+ + A^+}{\sqrt{2}} \\ \frac{\mathrm{i}\Pi_{UD} + \tilde{\Pi}_{UD}}{\sqrt{2}} & \mathrm{i}\Pi_{DD} + \tilde{\Pi}_{DD} & \frac{\mathrm{i}\Pi^- + A^-}{\sqrt{2}} & \frac{\sigma + \mathrm{i}\Theta - \mathrm{i}\Pi^0 - A^0}{2} \\ \frac{\sigma + \mathrm{i}\Theta + \mathrm{i}\Pi^0 + A^0}{2} & \frac{\mathrm{i}\Pi^- + A^-}{\sqrt{2}} & \mathrm{i}\Pi_{\bar{U}\bar{U}} + \tilde{\Pi}_{\bar{U}\bar{U}} & \frac{\mathrm{i}\Pi_{\bar{U}\bar{D}} + \tilde{\Pi}_{\bar{U}\bar{D}}}{\sqrt{2}} \\ \frac{\mathrm{i}\Pi^+ + A^+}{\sqrt{2}} & \frac{\sigma + \mathrm{i}\Theta - \mathrm{i}\Pi^0 - A^0}{2} & \frac{\mathrm{i}\Pi_{\bar{U}\bar{D}} + \tilde{\Pi}_{\bar{U}\bar{D}}}{\sqrt{2}} & \mathrm{i}\Pi_{\bar{D}\bar{D}} + \tilde{\Pi}_{\bar{D}\bar{D}} \end{pmatrix}, \tag{194}$$

where σ is the 'composite Higgs' responsible for the EWSB and Θ its pseudoscalar partner, Π_I, $I = UU, DD, \bar{U}\bar{U}, \bar{D}\bar{D}$, the six pseudoscalar Goldstone bosons related to technibaryons and $\tilde{\Pi}_I$ their scalar partners, and Π^J, $J = 0, +, -$, the pseudoscalar Goldstone bosons related to the technimesons and A^J their scalar partners. More explicitly (see [19] for further details),

$$\begin{array}{ll} \sigma \sim \bar{U}U + \bar{D}D, & \theta \sim \mathrm{i}(\bar{U}\gamma^5 U + \bar{D}\gamma^5 D), \\ A^0 \sim \bar{U}U - \bar{D}D, & \Pi^0 \sim \mathrm{i}(\bar{U}\gamma^5 U - \bar{D}\gamma^5 D), \\ A^+ \sim \bar{D}U, & \Pi^+ \sim \mathrm{i}\bar{D}\gamma^5 U, \\ A^- \sim \bar{U}D, & \Pi^- \sim \mathrm{i}\bar{U}\gamma^5 D, \\ \Pi_{UU} \sim U^{\mathrm{T}}CU, & \tilde{\Pi}_{UU} \sim \mathrm{i}U^{\mathrm{T}}C\gamma^5 U, \\ \Pi_{DD} \sim D^{\mathrm{T}}CD, & \tilde{\Pi}_{DD} \sim \mathrm{i}D^{\mathrm{T}}C\gamma^5 D, \\ \Pi_{UD} \sim U^{\mathrm{T}}CD, & \tilde{\Pi}_{UD} \sim \mathrm{i}U^{\mathrm{T}}C\gamma^5 D, \end{array} \tag{195}$$

where $U = (U_{\mathrm{L}},\, U_{\mathrm{R}})$ and $D = (D_{\mathrm{L}},\, D_{\mathrm{R}})$ are the Dirac technifermions and C the charge conjugation matrix, $C^{\mathrm{T}} = C^\dagger = C^{-1} = -C$.

Using these, we can now write down the most general candidate for CP-invariant, electrically neutral vacuum by giving VEVs to electrically neutral real fields. Remember here that according to the hypercharge assignment of table 3, $Q(U) = 1$ and $Q(D) = 0$ and, therefore, the only electrically neutral combinations of these techniquarks are $\sigma, A^0, \Pi_{DD}, \Pi_{\bar{D}\bar{D}}$ and their pseudoscalar partners. Moreover, to preserve the CP invariance, we only assign VEVs for real fields. With technimeson-related scalars this is simple; by looking at the forms of eq. (195) we see that σ and A^0 are real whereas their pseudoscalar partners are imaginary. With technibaryon-related Goldstone bosons we cannot say that e.g. Π_{DD} is real and its pseudoscalar partner imaginary, or vice versa, and hence, we assign a VEV for the real parts of the sums $\mathrm{i}\Pi_{DD} + \tilde{\Pi}_{DD}$ and $\mathrm{i}\Pi_{\bar{D}\bar{D}} + \tilde{\Pi}_{\bar{D}\bar{D}}$.

Thus, denoting

$$\begin{array}{ll} \langle\sigma\rangle =: v_1, & \langle \mathrm{Re}\left(\mathrm{i}\Pi_{DD} + \tilde{\Pi}_{DD}\right)\rangle =: v_2 \\ \langle A^0\rangle =: v_3, & \langle \mathrm{Re}\left(\mathrm{i}\Pi_{\bar{D}\bar{D}} + \tilde{\Pi}_{\bar{D}\bar{D}}\right)\rangle =: v_4, \end{array} \tag{196}$$

we can search for the most general CP-invariant, electrically neutral vacuum in the form of

$$\langle M\rangle = \frac{1}{2}\begin{pmatrix} 0 & 0 & v_1 + v_3 & 0 \\ 0 & \sqrt{2}v_2 & 0 & v_1 - v_3 \\ v_1 + v_3 & 0 & 0 & 0 \\ 0 & v_1 - v_3 & 0 & \sqrt{2}v_4 \end{pmatrix}. \tag{197}$$

We now obtain the actual vacuum structure by minimizing the potential with respect to the scalar fields, i.e. we are to solve the minimum equations

$$\left.\frac{\partial \mathcal{V}}{\partial \phi_i}\right|_{\phi_i=\langle\phi_i\rangle} = 0. \tag{198}$$

The explicit minimum equations can be found in the Appendix E. Let us first make a couple of important observations from the minimum equations:

(i) The mass terms for SM fermions (spurion Z) results a term constant in the VEVs in the minimum equations for σ and A^0. As a consequence, $v_3 = 0$ (or $v_1 = 0$) gives no longer an extremum of the potential which in turn indicates that the vacuum is substantially different from that of a pure TC model (cf. eq. (29)).

(ii) Non-zero gaugino mass M_D tilts the potential in the v_4 direction implying non-zero value for v_4 at the minimum. However, the minimum equation for v_2 requires that for $v_2 = 0$ also $v_4 = 0$. Thus, a non-zero M_D implies not only non-zero v_4 but also non-zero v_2.

(iii) The minimum equations for A^0 is obtained from the equation for σ with replacement $v_1 \leftrightarrow v_3$. This implies that $v_1 = v_3$ gives a minimum.

We, thus, find a vacuum of the form

$$\langle M\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 0 & \sqrt{2}v_2 & 0 & 0 \\ v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}v_4 \end{pmatrix}, \tag{199}$$

where the square roots of two are conventional. There are then three Higgs-like composite scalars acquiring VEVs v_1, v_2 and v_4. From now on, we denote these by h_1, h_2 and h_3, respectively.

From the vacuum structure, we deduce the squared masses of the $W^{\pm}$ and Z bosons:

$$m_Z^2 = \frac{1}{4}(g_{\mathrm{L}}^2 + g_Y^2)(v_1^2 + 4v_2^2), \quad m_W^2 = \frac{1}{4}g_{\mathrm{L}}^2(v_1^2 + 2v_2^2). \tag{200}$$

We can then identify

$$(246\ \mathrm{GeV})^2 = v_{\mathrm{w}} = \frac{1}{\sqrt{2}G_{\mathrm{F}}} = v_1^2 + 2v_2^2. \tag{201}$$

We chose to identify the VEV contribution from the W mass with the SM VEV, v_{w} since the Fermi coupling constant, G_{F}, is experimentally determined by measuring the muon decay, which is a W-exchange process.

4.4 EW precision tests

While the vacuum structure is restricted by the ability of generating correct masses for the SM fermions and weak gauge bosons, it is independent of the lepton sector. The lepton sector, in turn, is mainly constrained by the S and T parameters (see sec. 2.1.1). The full S parameter can be written as

$$S = S_{\rm SM}(m_{\rm ref}) - S_{\rm H}(m_{\rm ref}) + S_{\rm new} = S_{\rm new} - S_{\rm H}(m_{\rm ref}), \tag{202}$$

where $S_{\rm SM}(m_{\rm ref})$ is the SM contribution at SM Higgs mass $m_{\rm ref}$ and is zero by definition and $S_{\rm new}$ contains the contribution from the full BSM sector. However, since we consider a TC model, we have replaced the SM Higgs sector by a TC condensate and, to avoid double counting, we must also subtract the contribution from the Higgs boson, $S_{\rm H}(m_{\rm ref})$. A similar expression holds for T.

The one-loop contributions from a Higgs-like scalar read [5]

$$S_H(m_{\rm ref}) \approx \frac{1}{12\pi}\ln\frac{m_H^2}{m_{\rm ref}^2}, \quad T_H(m_{\rm ref}) \approx -\frac{3}{16\pi c_{\rm W}^2}\ln\frac{m_H^2}{m_{\rm ref}^2}, \tag{203}$$

where m_H is the mass of the of the Higgs boson and $m_{\rm ref}$ the reference mass at which the S and T parameters have been defined, i.e.

$$(S_{\rm SM}(m_{\rm ref}), T_{\rm SM}(m_{\rm ref})) = (0,0). \tag{204}$$

The problem now is the following: If there is not one fundamental SM Higgs boson, but three composite ones, h_1, h_2 and h_3, what do we really mean by the Higgs reference mass, $m_{\rm ref}$? First of all, the S parameter comes, by definition, from the ZZ self-energies (see eq. (1)) and, therefore, at one loop the Higgs contribution is proportional to its linear coupling to ZZ, denoted by g_{ZZH}. We should somehow take this into account with the fact that the couplings related to these new composite Higgses, g_{ZZh_i}, $i = 1, 2, 3$, are different. Moreover, from the estimate (203) we see that the Higgs contribution to S is, in addition, proportional to the logarithm of the Higgs mass. Having these two points in mind, we define the geometrical mean of the masses m_{h_i}, $i = 1, 2, 3$, weighted by the ratios $g^2_{ZZh_i}/g^2_{ZZH}$:

$$m_{H_g}^2 := \prod_{i=1}^{3} m_{h_i}^{2w_i}, \quad w_i = \frac{g^2_{ZZh_i}}{g_{ZZH}}. \tag{205}$$

We then take this m_{H_g} to be the reference mass at the limit $v_2/v_1 \to 0$, so that the sum of the weights equals to one.

The contribution from the TC sector can be approximated by the so-called naive estimate, which takes into account the one-loop contribution from heavy techniquarks, which are mass-degenerate. These naive estimates read

$$S_{\text{naive}} = \frac{d(R)}{6\pi}, \quad T_{\text{naive}} = 0, \tag{206}$$

where $d(R)$ is the dimension of the representation of the techniquarks.

However, with the explicit soft SUSY breaking mass for the technigluino $\bar{D}_{\text{R}}$, the assumption of mass-degeneracy is not particularly well justified. Moreover, since the soft SUSY breaking mass term only involves the right-handed component of the techniquark D, the D quark has, in addition to the Dirac type dynamical mass, a Majorana-type mass. Note that this is allowed since D is neutral. This also affects the S and T contributions.

One way of improving this estimate would be taking account the contributions from all the composite states of the effective theory. This, however, is not very well controlled and in addition we should also take into account the vector resonances, which we have not included (see [19]). Another possibility would be improving the naive estimate, i.e. estimating the contributions to the oblique parameters from the underlying gauge theory.

Let us, therefore, take a closer look at the mass terms of the techniquarks. Taking into account the soft SUSY breaking term $-\frac{1}{2}M_D\bar{D}_{\text{R}}\bar{D}_{\text{R}}$ the mass terms can be written as (see e.g. [43])

$$\mathcal{L}_{Q,\,\text{mass}} = -m_U\bar{U}U - \frac{1}{2}\left[\begin{pmatrix}\overline{D_{\text{L}}} & \overline{(D_{\text{R}})^c}\end{pmatrix}\begin{pmatrix} 0 & m_D \\ m_D & M_D \end{pmatrix}\begin{pmatrix} (D_{\text{L}})^c \\ D_R \end{pmatrix} + \text{h.c.}\right], \tag{207}$$

where $U = (U_{\text{L}}\ U_{\text{R}})^{\text{T}}$, m_U and m_D the Dirac masses of U and D, respectively, and the superscript c denotes the charge conjugate of the field.

Diagonalization of the D mass matrix results in two Majorana eigenstates, N_1 and N_2, with real and positive masses, M_1 and M_2. In terms of m_D and M_D these read

$$M_1 = \frac{M_D}{2}\left(\sqrt{1+4\frac{m_D^2}{M_D^2}} - 1\right), \quad M_2 = \frac{M_D}{2}\left(\sqrt{1+4\frac{m_D^2}{M_D^2}} + 1\right), \tag{208}$$

or the other way around,

$$m_D^2 = M_1M_2, \quad M_D = M_2 - M_1. \tag{209}$$

The chiral states, D_L and D_R can be written in terms of the Majorana eigenstates as

$$\begin{pmatrix} D_\mathrm{L} \\ (D_\mathrm{R})^c \end{pmatrix} = \begin{pmatrix} \mathrm{i}\cos\varphi & \sin\varphi \\ -\mathrm{i}\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} P_\mathrm{L}N_1 \\ P_\mathrm{L}N_2 \end{pmatrix}, \quad \tan 2\varphi = \frac{2m_D}{M_D}. \tag{210}$$

The contributions of the U techniquark with only Dirac mass and D techniquark with both Dirac and Majorana masses, as discussed above, to the S and T parameters are given in [43] and read

$$\begin{aligned} S_{U,D} =& \frac{d(R)}{6\pi}\left[-Y\left(c_\varphi^2 \ln\frac{M_1^2}{m_U^2} + s_\varphi^2 \ln\frac{M_2^2}{m_U^2}\right) + \frac{3}{2}\right. \\ & \left. -s_\varphi^2 c_\varphi^2\left(\frac{8}{3} + f_1(M_1,M_2) - f_2(M_1,M_2)\ln\frac{M_1^2}{M_2^2}\right)\right], \end{aligned} \tag{211}$$

$$\begin{aligned} T_{U,D} =& \frac{d(R)}{16\pi s_\mathrm{W}^2 c_\mathrm{W}^2 m_Z^2}\left[c_\varphi^2\left(M_1^2 + m_U^2 - \frac{2M_1^2 m_U^2}{M_1^2 - m_U^2}\ln\frac{M_1^2}{m_U^2}\right)\right. \\ & +s_\varphi^2\left(M_2^2 + m_U^2 - \frac{2M_2^2 m_U^2}{M_2^2 - m_U^2}\ln\frac{M_2^2}{m_U^2}\right) \\ & -s_\varphi^2 c_\varphi^2\Big(M_1^2 + M_2^2 - 4M_1M_2 \\ & \left.\left. +2\frac{M_1^3M_2 - M_1^2M_2^2 + M_1M_2^3}{M_1^2 - M_2^2}\ln\frac{M_1^2}{M_2^2}\right)\right], \end{aligned} \tag{212}$$

where $d(R)$ is the dimension of the techniquark representation, $Y(=\frac{1}{2})$ is the hypercharge of the techniquark doublet, $s_\varphi = \sin\varphi$, $c_\varphi = \cos\varphi$ and the functions f_1 and f_2 are given by

$$f_1(M_1,M_2) = \frac{3M_1M_2^3 + 3M_1^3M_2 - 4M_1^2M_2^2}{(M_1^2 - M_2^2)^2}, \tag{213}$$

$$f_2(M_1,M_2) = \frac{M_1^6 - 3M_1^4M_2^2 + 6M_1^3M_2^3 - 3M_1^2M_2^4 + M_2^6}{(M_1^2 - M_2^2)^3}. \tag{214}$$

Let us then compare this improved analysis to the naive estimate. To this end, we make the following estimates: $v_2 \sim 10$ GeV and $c_\theta^2/m_s^2 \sim 1/m_\mathrm{SUSY}^2$. We plot the values of $S_{U,D}$ and $T_{U,D}$ as functions of M_D within range $M_D \leq 5$ TeV and vary m_SUSY from 5 to 15 TeV. The plot is depicted in fig. 4. We have scaled $S_{U,D}$ with factor 100 to fit the both $S_{U,D}$ and $T_{U,D}$ in the same plot. The curves from darker to lighter correspond to m_SUSY values from 5 TeV to 15 TeV, respectively. We have chosen $N = 2$, since the number of technicolors only affects the overall factor and does not change the form of the curves. We notice that the most significant change to the naive estimate happens with the contribution to T. The requirement of a small T parameter constrains the mass M_D to be small. This, however, is a desired result, since the minimum equations for the scalar potential of the effective theory imply a rough estimate of $M_D \sim v_{\mathrm{w}}$ [40]. We, thus, focus only on the reasonably small values of $|T_{U,D}|$. The values of $S_{U,D}$ and $T_{U,D}$ as a function of M_D, $M_D \leq 400$ GeV, are depicted in upper panel of fig. 5 for $N = 2$ and, for a comparison, for $N = 3$ in the lower panel of fig. 5. We have included explicitly the naive estimate S_{naive} and the curves from darker to lighter again represent the values $5 \ldots 15$ TeV of m_{SUSY}, respectively.

Moreover, the special vacuum alignment gives a tree level contribution to the T parameter [40]

$$\alpha T_{\mathrm{tree}} = -\frac{2v_2^2}{v_{\mathrm{w}}^2}, \tag{215}$$

where $\alpha \approx 1/137$ is the fine structure constant and $v_{\mathrm{w}} = 246$ GeV the VEV of the SM Higgs. The tree level contribution to S is zero.

On top of these contributions arising from the new strongly interacting sector, there are contributions to S and T parameters from the new leptons. The one-loop contributions from weak doublet of fermions, $f = (f_1, f_2)$, with Dirac masses M_1 and M_2, respectively, to the oblique parameters read [44]

$$\begin{aligned} S_f =& \frac{N_{\mathrm{c}}}{6\pi} \left\{ 2(4Y+3)x_1 + 2(-4Y+3)x_2 - 2Y \ln \frac{x_1}{x_2} \right. \\ & \left. + \left[\left(\frac{3}{2} + 2Y \right) x_1 + Y \right] G(x_1) + \left[\left(\frac{3}{2} - 2Y \right) x_1 - Y \right] G(x_2) \right\}, \end{aligned} \tag{216}$$

$$T_f = \frac{N_{\mathrm{c}}}{8\pi} s_{\mathrm{W}}^2 c_{\mathrm{W}}^2 F(x_1, x_2), \tag{217}$$

where $x_i = (M_i/m_Z)^2$, $i = 1, 2$, Y is the hypercharge of f and N_{c} is the colour factor, i.e. the dimension of the colour representation of the fermion f; for leptons $N_{\mathrm{c}} = 1$. The functions F and G are defined as

$$F(x_1, x_2) = \frac{x_1 + x_2}{2} - \frac{x_1 x_2}{x_1 - x_2} \ln \frac{x_1}{x_2}, \tag{218}$$

$$G(x) = -4\sqrt{4x-1} \arctan \frac{1}{\sqrt{4x-1}}. \tag{219}$$

At the limit $M_{1,2}^2 \gg m_Z^2$ we obtain

$$\begin{aligned} S_f = \frac{N_{\mathrm{c}}}{6\pi} \Bigg[& 1 - 2Y \ln \left(\frac{M_1}{M_2} \right)^2 + \frac{1+8Y}{20} \left(\frac{m_Z}{M_1} \right)^2 \\ & + \frac{1-8Y}{20} \left(\frac{m_Z}{M_2} \right)^2 + \mathcal{O}\left(\left(\frac{m_Z}{M_i} \right)^4 \right) \Bigg]. \end{aligned} \tag{220}$$

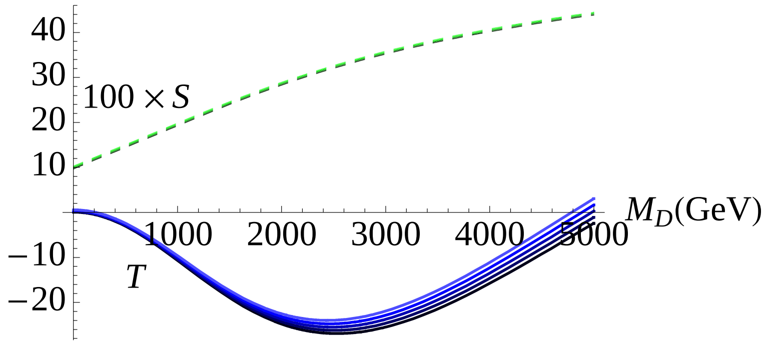


Figure 4. The values of $T_{U,D}$ (solid curves) and $100 \times S_{U,D}$ (dashed curves) as functions of M_D. The curves from darker to lighter represent the different values of $m_{\rm SUSY}$ from 5 TeV to 15 TeV, respectively.

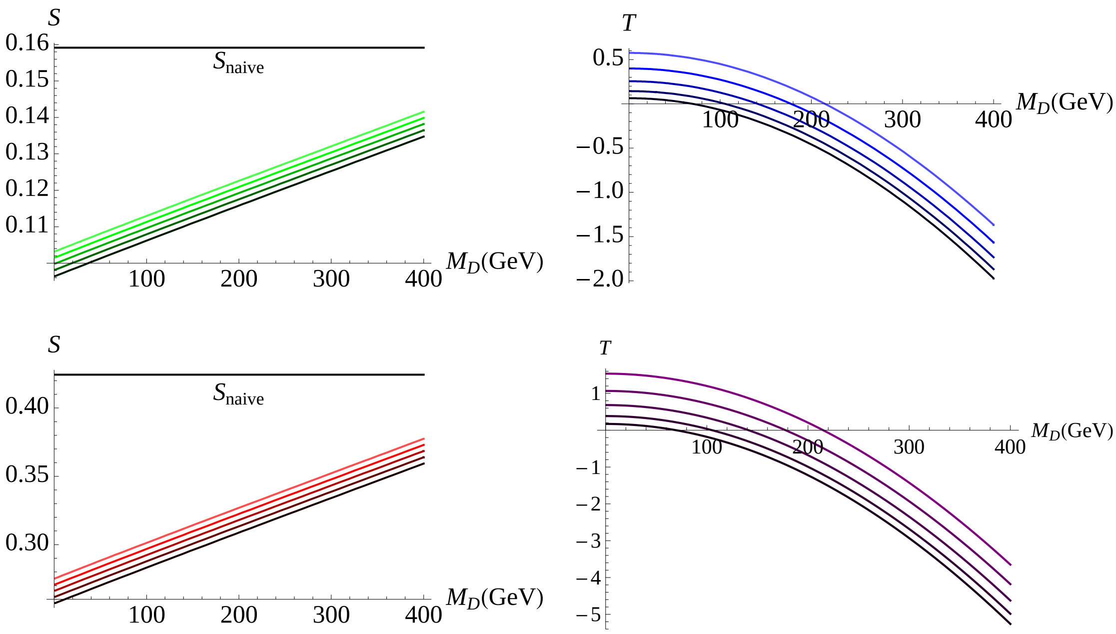


Figure 5. Upper left panel: The values of $S_{U,D}$ for $N = 2$ as a function of M_D. The curves from darker to lighter represent the different values of $m_{\rm SUSY}$ from 5 TeV to 15 TeV, respectively, and the constant line shows the naive estimate. **Upper right panel:** The values of $T_{U,D}$ for $N = 2$ as a function of M_D. The curves from darker to lighter represent the different values of $m_{\rm SUSY}$ from 5 TeV to 15 TeV, respectively. **Lower panels:** Corresponding plots for $N = 3$.

If the mass splitting within the fermion doublet is small, we obtain the naive estimate $N_c/6\pi$.

We are now ready to add the pieces together to obtain the full S and T parameters for the model. To this end, we choose $m_{\rm ref} = 117$ GeV. The current experimental values for S and T (with $U = 0$) assuming a 117 GeV Higgs boson reference mass are [6]

$$S = 0.03 \pm 0.09, \quad \text{and} \quad T = 0.07 \pm 0.08. \tag{221}$$

Hence, we obtain estimates

$$\begin{aligned} S &= S_{\rm U,D} + S_{\rm leptons} \\ T &= T_{U,D} + T_{\rm tree} + T_{\rm leptons}. \end{aligned} \tag{222}$$

Having these general expressions, let us then focus on particular choices of the lepton sector.

4.4.1 Technicolor group $\mathrm{SU}(N)_{\rm TC}$ with $N_{\rm L}$ lepton doublets

Depending on whether N is even or odd, we must add an odd or even number of lepton doublets, respectively, in order to avoid the Witten anomaly. After choosing the number of the doublets, we must choose the hypercharges of the leptons such that they fulfil eq. (168). Note here, however, that in the simplest case where the lepton doublets are mass degenerate, i.e. $m_{N_i} = m_N$ and $m_{E_i} = m_E$ for all $i = 1, \ldots, N_{\rm L}$, we see from eqs. (216) and (217) that the particular hypercharge assignment (i.e. whether we take three doublets of hypercharges -3/2, -1/2 and 1/2 or three doublets each of hypercharge -1/2, for example) is irrelevant when considering the contributions to the S and T parameters; only the number of lepton doublets matters.

The full S and T (of eq. (222)) for a model with $N = 2, M_D = 100$ GeV and $m_{\rm SUSY} = 5$ TeV and either one or three lepton doublets are depicted in the upper left panel of figure 6 as a function of m_N and m_E varying in the range $m_Z \leq m_N \leq m_E \leq 5m_Z$. We denote $x_1 = (m_N/m_Z)^2$ and $x_2 = (m_E/m_Z)^2$, x_1 thus varying from 1 to 25 and x_2 from x_1 to 25. We have explicitly marked the $x_1 = 1$ and $x_2 = 25$ contours for $N_{\rm L} = 1$ case; for $N_{\rm L} = 3$ the behaviour is similar with x_1 increasing to the right and x_2 decreasing downward. We have chosen $m_E \leq m_N$ since with that choice the contribution to S is negative and can, therefore, be used to cancel the positive $S_{U,D}$. The ellipse in the figures shows the current experimental limit for S and T at 95% C.L. For one lepton doublet we have also included the behaviour when increasing M_D to 200 GeV (red curve) or $m_{\rm SUSY}$ to 10 TeV (purple curve).

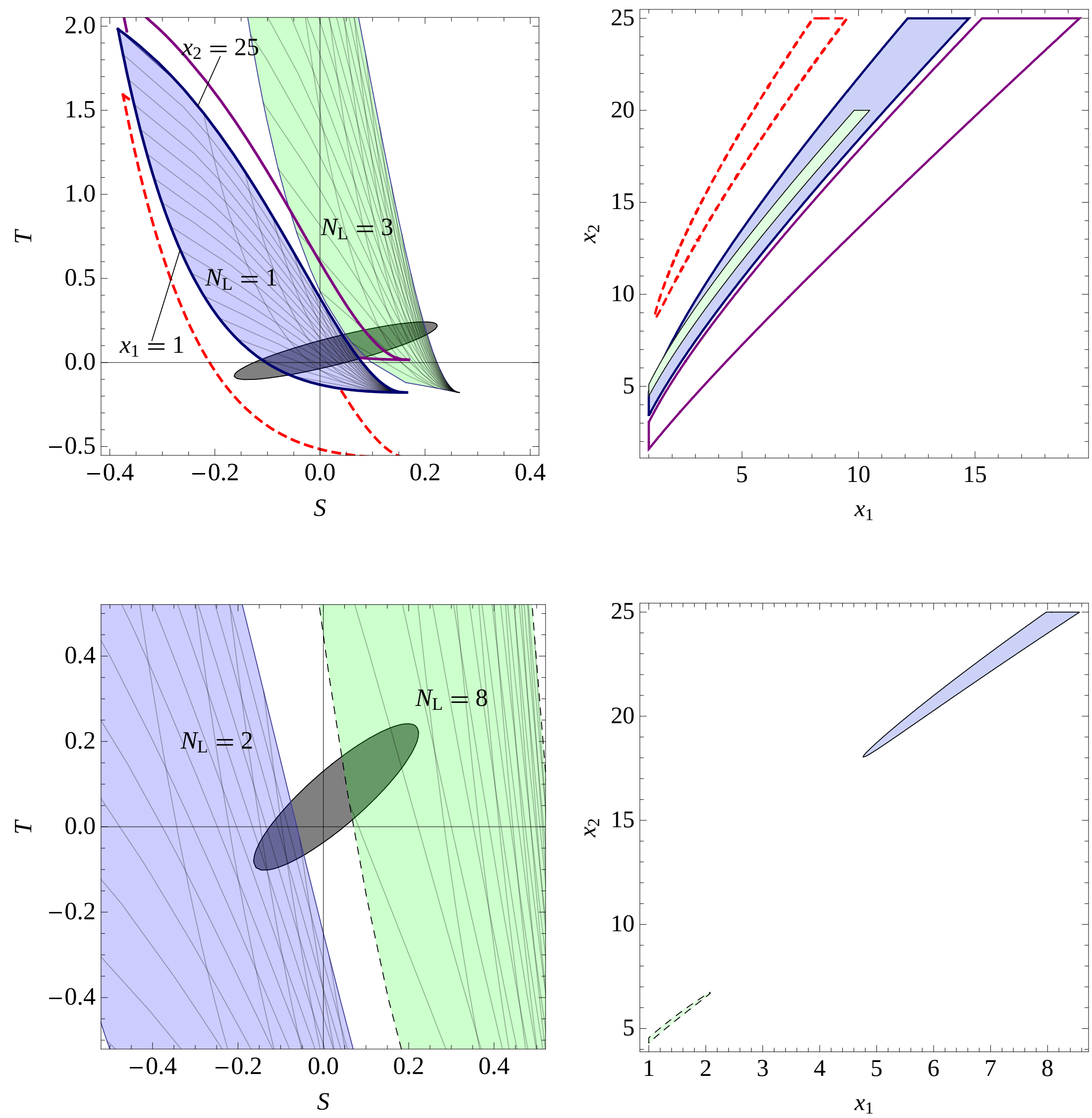


Figure 6. Upper left panel: The blue and green areas show the full S and T parameters for $N = 2, M_D = 100$ GeV, $m_{\text{SUSY}} = 5$ TeV and $N_{\text{L}} = 1, 3$ as a function of the lepton masses, $m_N^2 = x_1 m_Z^2$ and $m_E^2 = x_2 m_Z^2$, $1 \leq x_1 \leq x_2 \leq 25$. In $N_{\text{L}} = 3$ case we assume $m_{N_i} = m_N$ and $m_{E_i} = m_E$, $i = 1, 2, 3$. The ellipse corresponds to the 95% C.L. region from experimental data. The dashed red and solid purple curves represents the shifts of the $N_{\text{L}} = 1$ region when, respectively, M_D is increased to 200 GeV or m_{SUSY} is increased to 10 TeV. **Lower left panel:** Similar plot for $N = 3, M_D = 200$ GeV, $m_{\text{SUSY}} = 5$ Tev and $N_{\text{L}} = 2, 8$. Again, $m_{N_i} = m_N$ and $m_{E_i} = m_E$, $i = 1, \ldots, N_{\text{L}}$. **Right panels:** The allowed parametric regions (points mapping inside the ellipse) corresponding to the left panels.

The (x_1, x_2) regions allowed by experiments (i.e. mapped inside the ellipse) are shown in the upper right panel of fig. 6.

Similar plots for $N = 3$ with $M_D = 200$ GeV and $m_{\text{SUSY}} = 5$ TeV, are depicted in the lower panel of fig. 6. We conclude that even for higher N there exists a region of the parameter space consistent with current limits. Moreover, due to the rapid increment of $T_{U,D}$ with increasing M_D, the total oblique parameters, even for large number of leptons, remain reasonably small. Actually, for M_D of about 200 GeV or larger, the preferred lepton sector is that of a larger number of doublets with $(N^2 - 1)$ SM like doublets being a good candidate. Note that the results would be somewhat opposite (the small number of lepton doublets preferred) if we would have estimated the TC sector contributions only with the naive estimates due to $T_{\text{naive}} = 0$.

4.4.2 $\mathrm{SU}(2)_{\mathrm{TC}}$ with fourth generation of QCD quarks

According to eq. (161) the anomaly-free hypercharge assignment for the new doublet of QCD quarks is $Y(q_{\mathrm{L}4}) = -1$. We get the contribution from this doublet similarly from eqs. (216) and (217) now remembering to include the colour factor $N_{\mathrm{c}} = 3$. The current lower limit for the masses of these fourth generation quarks lies around 300 GeV [45] and accordingly, we set $10 \leq x_1 \leq x_2 \leq 50$, where again $x_i = (m_i/m_Z)^2$. The full S and T parameters of this model are shown in fig. 7 as functions of the quark masses with $M_D = 200$ GeV and $m_{\text{SUSY}} = 5$ TeV. Again, we conclude that from EW precision test point of view a fourth generation of QCD quarks would be a viable option as well.

4.5 Problems with a charged lepton N?

As we have seen above, even for larger N there is a parameter region to be found that is in agreement with the current experimental constraints. However, there is a subtlety we have not yet discussed. Namely, some of the models above, e.g. the sMCST, suggest a doubly charged lepton and its singly charged 'neutrino' partner. The problem is that within the model, there is no channel for N decay (because its doubly charged partner E is expected to be heavier) leading to a stable singly charged lepton. This, in turn, is not preferred by results from cosmology.

However, the situation is not exactly that grim. If we include a Yukawa coupling between, say, $N, \tilde{H}_1$ and τ (which is perfectly acceptable if we allow the lepton number violation), we obtain a decay channel for N.

This coupling, however, gives a possibly disfavoured contribution to the anomalous

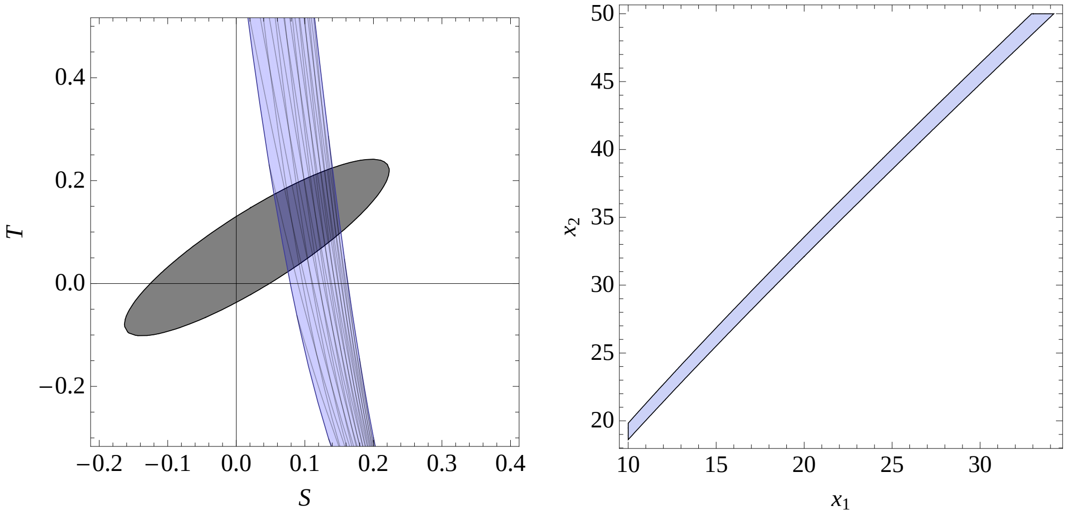


Figure 7. Left panel: The full S and T parameters for $N = 2, M_D = 200$ GeV and $m_{\text{SUSY}} = 5$ TeV with fourth generation of QCD quarks as functions of quark masses, $m_1^2 = x_1 m_Z^2$ and $m_2^2 = x_2 m_Z^2$, $10 \leq x_1 \leq x_2 \leq 50$. The ellipse corresponds to the 95% C.L. region from experimental data. **Right panel:** The allowed parametric region.

magnetic moment of τ. After a lengthy calculation, we obtain

$$a_\tau^{H_1} = F_2^{H_1}(0) = -\frac{y^2}{16\pi^2} \int_0^1 \mathrm{d}z \, \frac{m_\tau(m_N - z m_\tau)(1-z)^2}{z(z-1)m_\tau^2 + (1-z)m_N^2 + z m_1^2}, \tag{223}$$

where y is the Yukawa coupling, α the fine structure constant and m_τ, m_N, m_1 the masses of τ, N and $\tilde{H}_1$, respectively. The detailed calculation of the anomalous magnetic moment can be found in in Appendix F.

The Yukawa coupling ought to be of order one so let us approximate $y \approx 1$ and $m_1 \approx m_{\text{SUSY}}$ to get a numerical estimate of the anomalous magnetic moment. A plot of the obtained values as a function of m_N with $m_{\text{SUSY}} = 5 \ldots 15$ TeV is given in fig. 8.

The current experimental limits for the anomalous magnetic moment of τ are [46]

$$-0.052 \leq a_\tau \leq 0.013, \tag{224}$$

at 95% C.L. so within the current accuracy, no further constraints arise from here.

Having seen that the Yukawa coupling term at issue is not excluded due to the contribution to a_τ, we can give an estimate for the resulting lifetime as well. Assuming that the Higgs decays dominantly to $b\bar{b}$ pair (now that the MSSM Higgses

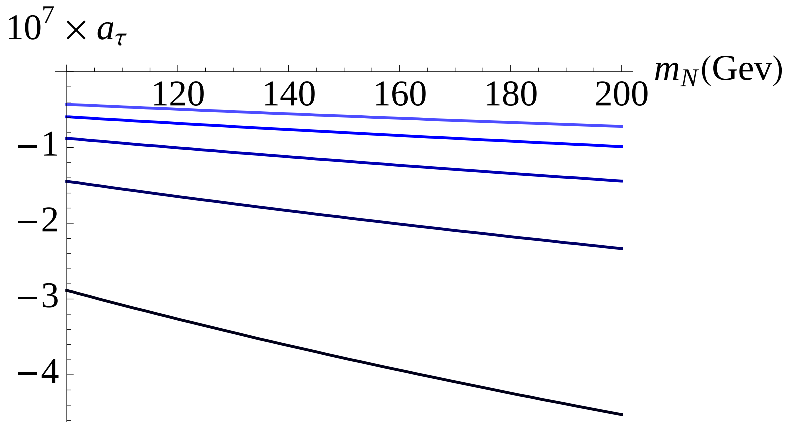


Figure 8. The contribution to the anomalous magnetic moment of τ as function of N mass, m_N. The different curves (from darker to lighter) represent the different values of m_{SUSY} in the range $5 \ldots 15$ TeV.

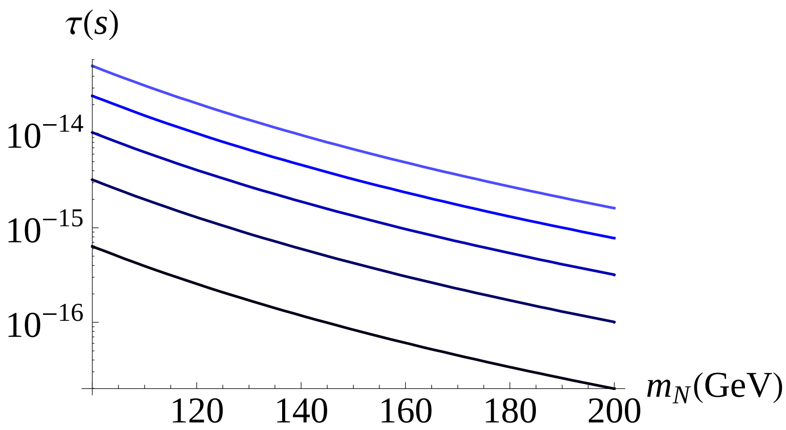


Figure 9. The lifetime of N as a function of its mass, m_N. The different curves (from darker to lighter) represent the different values of m_{SUSY} in the range $5 \ldots 15$ TeV.

do not acquire a VEV there are no tree level vertices with one Higgs and two weak bosons), we obtain for the decay width

$$\Gamma(N \to b\bar{b}\tau) = \int\limits_{(m_{12}^2)_{\min}}^{(m_{12}^2)_{\max}} \mathrm{d}m_{12}^2 \int\limits_{(m_{23}^2)_{\min}}^{(m_{23}^2)_{\max}} \mathrm{d}m_{23}^2 \, \frac{1}{(2\pi)^3} \frac{3y^2 y_b^2}{16 m_N^3 (-m_{12}^2 + m_H^2)^2} \times \tag{225}$$

$$\left[\left((m_N + m_\tau)^2 - m_{12}^2 \right) m_{12}^2 - 4 m_b^2 \left(m_N^2 + m_\tau^2 - m_{12}^2 \right) \right],$$

where y, y_b are the $\tilde{H}_1 N \tau$ and $\tilde{H}_1 \bar{b} b$ Yukawa couplings, respectively, m_b the mass of the bottom quark and

$$(m_{23}^2)_{\min}^{\max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_b^2} \mp \sqrt{E_3^{*2} - m_\tau^2} \right)^2, \tag{226}$$

$$(m_{12}^2)_{\min} = 4 m_b^2, \quad (m_{12}^2)_{\max} = (m_N - m_\tau)^2, \tag{227}$$

with

$$E_2^* = \frac{1}{2} m_{12}, \quad E_3^* = \frac{1}{2 m_{12}} (m_N^2 - m_{12}^2 - m_\tau^2). \tag{228}$$

A more detailed calculation of the lifetime can be found in Appendix G. To get some numerical results, we approximate $y = y_b = 1$ and $m_H = m_{\text{SUSY}}$. The lifetime of N as a function of m_N is depicted in fig. 9 with $m_{\text{SUSY}} = 5 \ldots 15$ TeV. We conclude that the lifetime obtained for N is of the same order than that of the neutral pion, $\tau_{\pi^0} = (8.4 \pm 0.4) \times 10^{-17}$ s [6]. The lifetime is well short enough not to interfere the Big Bang nucleosynthesis and is also from this point of view in accordance with the cosmological observations.

5 Conclusions

In this thesis, we have discussed the union of supersymmetry and technicolor as a candidate for the path beyond the Standard Model of particle physics with emphasis in the strong regime of the model in which the energy scale of SUSY is higher than that of TC. This way TC is solely responsible of the EWSB and the Higgs fields of MSSM only transmit it to the fermion sector giving the fermions their masses. Hence, there is no need for a separate ETC sector while SUSY is playing this role.

We considered specific supersymmetrization scheme, in which the right-handed partner of the D techniquark is identified with the fermionic partner of the technigluon. The resulting model is particularly interesting: the bosonic and fermionic content of the TC sector fits exactly in a $\mathcal{N} = 4$ supermultiplet leading to a possible connection with string theory via the AdS/CFT duality.

More precisely, we have considered the effective theory at the EW scale, studied the vacuum structure and in particular the contributions of this BSM sector to the oblique EW parameters.

Including SUSY introduces new symmetry breaking terms into the effective Lagrangian when compared to a pure TC leading to a novel vacuum structure. Moreover, the minimum equations imply a small soft SUSY-breaking mass, M_D, for the technigluino.

Moreover, we have used an improved estimate for the contributions from the TC sector to the oblique parameters (in comparison to the famous naive estimate) which takes into account the mass splitting between the techniquarks. Using this estimate we find that even for higher number of technicolors, the model is not excluded by current experimental limits if the soft SUSY-breaking mass of the technigluino is small. Hence, the constraint from the oblique parameters for M_D matches the constraint already obtained from the vacuum considerations.

Finally, since a stable charged lepton appearing in some variants of the model is disfavoured by cosmological observations, we have considered adding a lepton number violating Yukawa coupling term between the charged stable lepton, SM τ lepton and one of the MSSM Higgses to give a decay channel for the lepton. This, a priori, could result in a disfavoured contribution to the anomalous magnetic moment of τ but we have found this contribution negligible within the current experimental accuracy.

All in all, this novel marriage of SUSY and TC gives viable and interesting models for BSM physics.

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Appendix A Notations and conventions

Let us begin with fixing some notations and conventions, in which we will mainly follow those of Wess and Bagger [24]. First of all, we will be working in a Minkowski space with the mostly-plus metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{229}$$

implying $p^2 = -m^2$ for an on-shell particle. For concreteness, we will also be working in the chiral representation of the gamma matrices, i.e.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{230}$$

where

$$\sigma^0 = \bar{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{231}$$

and

$$\begin{aligned} &\sigma^1 = -\bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = -\bar{\sigma}^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\ &\sigma^3 = -\bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{232}$$

are the Pauli matrices.

In this basis, one can write the four-component Dirac spinor in terms of two-component Weyl spinors, and these turn out to be very handy both in the context of SUSY and TC. Let us, therefore, concentrate on these objects a bit more carefully.

In order to lay a solid foundation, and not to sink in the endless swamp of different conventions, let us take a short bypath on the representations of the Lorentz group.

The double cover of the proper orthochronous Lorentz group $\mathrm{SO}^+(3,1)$, i.e. the spin group $\mathrm{Spin}(3,1)$, is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ and, thus, has a natural two-dimensional complex representation, denoted by $\mathbb{W}$, in which $\mathrm{SL}(2,\mathbb{C})$ acts by matrix multiplication when the elements of $\mathrm{SL}(2,\mathbb{C})$ are identified with complex 2×2 matrices. That is, if $M \in \mathrm{SL}(2,\mathbb{C})$ and $w \in \mathbb{W}$, then

$$(M \cdot w)_\alpha = M_\alpha{}^\beta w_\beta. \tag{233}$$

The placement of indices is not arbitrary and will be explained shortly. A spinor w_α transforming this way is a left-handed, or a $(\frac{1}{2}, 0)$ spinor. This implies that the hermitian conjugate spinor $(w_\alpha)^\dagger =: \bar{w}_{\dot\alpha}$ transforms as

$$(M \cdot \bar{w})_{\dot\alpha} = \bar{w}_{\dot\beta} (M^\dagger)^{\dot\beta}{}_{\dot\alpha} = (\bar{M})_{\dot\alpha}{}^{\dot\beta} \bar{w}_{\dot\beta}. \tag{234}$$

The spinor $\bar{w}_{\dot{\alpha}}$ that transforms with the complex conjugate matrix, $\bar{M}$, is a right-handed, or a $(0, \frac{1}{2})$ spinor. This corresponds to the conjugate representation, $\overline{\mathbb{W}}$. The same group multiplication rule can, however, be satisfied by other choices of action as well. For example, the multiplication by the inverse transpose of M satisfies the same group multiplication since $((MN)^T)^{-1} = (M^T)^{-1}(N^T)^{-1}$. Correspondingly, multiplication by the inverse hermitian conjugate, $(M^\dagger)^{-1}$, is a possible action. We call these the dual ($\mathbb{W}^*$) and the conjugate dual ($\overline{\mathbb{W}}^*$) representations, respectively. These are not, however, inequivalent to the previous two but are in fact isomorphic to $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively.

Following the usual convention, we will distinguish the conjugate representations by using dotted indices and will adopt the following placement of indices:

$$w_\alpha \in \mathbb{W},\ \bar{w}_{\dot{\alpha}} \in \overline{\mathbb{W}},\ w^\alpha \in \mathbb{W}^* \text{ and } \bar{w}^{\dot{\alpha}} \in \overline{\mathbb{W}}^*.$$

In building Lorentz tensors out of these spinors, it is useful to regard w^α as a row-vector and $\bar{w}^{\dot{\alpha}}$ as a column vector.

In terms of Weyl spinors, a Dirac spinor then becomes

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{235}$$

We use two-index antisymmetric epsilon symbols to raise and lower spinor indices in the following manner:

$$w^\alpha = \epsilon^{\alpha\beta} w_\beta,\ w_\alpha = \epsilon_{\alpha\beta} w^\beta,\ \bar{w}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{w}_{\dot{\beta}},\ \bar{w}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{w}^{\dot{\beta}}. \tag{236}$$

We fix the components of the epsilon symbols as

$$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1. \tag{237}$$

Moreover, we suppress descending contracted indices and ascending contracted dotted indices, i.e. contractions

$${}^{\alpha}{}_{\alpha} \quad \text{and} \quad {}_{\dot{\alpha}}{}^{\dot{\alpha}}$$

can be suppressed. In practice, this implies

$$\psi\chi \coloneqq \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi, \tag{238}$$
$$\bar{\psi}\bar{\chi} \coloneqq \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \tag{239}$$

Further conventions and an excessive list of spinor identities have been collected in Appendix C. Grassmannian derivatives are, in addition, shortly discussed in Appendix D.

Finally, a couple of words on indices. We denote the spacetime indices running from 0 to 3 with Greek letters, μ, ν, ρ, etc. and the spatial indices running from 1 to 3 with Roman letters, i, j, k, etc. The two-component spinor indices are denoted by Greek letters from the beginning of the alphabet, α, β, γ, etc. Summation over repeated indices of any kind is always assumed.

Appendix B The generators of $\mathrm{SU}(2N_\mathrm{f})$

According to eq. (25), the generators of $\mathrm{SO}(2N_\mathrm{f})$ satisfy

$$S^{a\mathrm{T}}E + ES^a = 0. \tag{240}$$

After writing the generators in a block form

$$S^a = \begin{pmatrix} A^a & B^a \\ C^a & D^a \end{pmatrix}, \tag{241}$$

the condition (240) reads

$$\begin{pmatrix} C^{a\mathrm{T}} & A^{a\mathrm{T}} \\ D^{a\mathrm{T}} & B^{a\mathrm{T}} \end{pmatrix} + \begin{pmatrix} C^a & D^a \\ A^a & B^a \end{pmatrix} = 0, \tag{242}$$

thereby implying

$$C^{a\mathrm{T}} = -C^a, \quad B^{a\mathrm{T}} = -B^a \quad \text{and} \quad D^{a\mathrm{T}} = -A^a. \tag{243}$$

The hermiticity of the generators further implies

$$A^a = A^{a\dagger} \quad \text{and} \quad C^a = B^{a\dagger}. \tag{244}$$

The generators S^a take then the form

$$S^a = \begin{pmatrix} A^a & B^a \\ B^{a\dagger} & -A^{a\mathrm{T}} \end{pmatrix}, \tag{245}$$

where $A^{a\dagger} = A$ and $B^{a\mathrm{T}} = -B$.

Denote the broken generators by X^i, which, after imposing the hermiticity, can be written as

$$X^i = \begin{pmatrix} A^i & C^i \\ C^{i\dagger} & D^i \end{pmatrix}. \tag{246}$$

Now, in order them to reside in the complement of $\mathrm{SO}(2N_\mathrm{f})$, set $A^{i\mathrm{T}} = D^i$ and $C^{i\mathrm{T}} = C^i$. We have to, in addition, demand D^i to be traceless to obtain traceless generators. The broken generators then read

$$X^i = \begin{pmatrix} D^i & C^i \\ C^{i\dagger} & D^{i\mathrm{T}} \end{pmatrix}, \tag{247}$$

where D^i is hermitian and traceless and $C^{i\mathrm{T}} = C^i$. To check that this gives a complete set of broken generators, let us count the free parameters of X^i. A complex, hermitian, traceless $N_\mathrm{f} \times N_\mathrm{f}$ matrix D^i can be parameterized by $N_\mathrm{f} - 1$ real parameters whereas a complex symmetric matrix C^i has $N_\mathrm{f}(N_\mathrm{f}+1)$ free real parameters, therefore summing up to

$$2N_\mathrm{f}^2 + N_\mathrm{f} - 1 = (4N_\mathrm{f}^2 - 1) - \frac{2N_\mathrm{f}(2N_\mathrm{f}-1)}{2} = \dim \mathrm{SU}(2N_\mathrm{f}) - \dim \mathrm{SO}(2N_\mathrm{f}). \tag{248}$$

Moreover, it is useful to identify S^a, and X^a, $a = 1, \ldots, N_{\mathrm{f}^2} - 1$, as the generators of $\mathrm{SU}(N_\mathrm{f})_\mathrm{V}$ and $\mathrm{SU}(N_\mathrm{f})_\mathrm{A}$, respectively, and $S^{N_\mathrm{f}^2}$ as the generator of $U(1)_\mathrm{V}$. These then read

$$\begin{aligned} S^a &= \frac{1}{\sqrt{2}} \begin{pmatrix} \tau^a & 0 \\ 0 & -\tau^{a\mathrm{T}} \end{pmatrix}, \quad S^{N_\mathrm{f}^2} = \frac{1}{2\sqrt{N_\mathrm{f}}} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \\ X^a &= \frac{1}{\sqrt{2}} \begin{pmatrix} \tau^a & 0 \\ 0 & \tau^{a\mathrm{T}} \end{pmatrix}, \end{aligned} \tag{249}$$

where τ^a, $a = 1, \ldots, N_\mathrm{f}^2 - 1$, are the generators of $\mathrm{SU}(N_\mathrm{f})$. The generators are normalised such that

$$\mathrm{Tr}[S^a S^b] = \frac{1}{2}\delta^{ab}, \ \mathrm{Tr}[X^i X^j] = \frac{1}{2}\delta^{ij}. \tag{250}$$

In term of these, we can then write the generators of $\mathrm{SU}(N_\mathrm{f})_\mathrm{L}$ and $\mathrm{SU}(N_\mathrm{f})_\mathrm{R}$ as

$$L^a = \frac{S^a + X^a}{\sqrt{2}}, \quad R^a = \frac{X^{a\mathrm{T}} - S^{a\mathrm{T}}}{\sqrt{2}}, \tag{251}$$

respectively.

Appendix C Two-component spinor identities

Let us collect the useful identities for manipulating spinors here. A good collection of needed identities and two-component spinor techniques can be found in e.g. [47]

(note especially the reference therein to an otherwise identical paper but with different metric convention).

First of all, recall the basic relations and conventions

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{252}$$
$$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1, \tag{253}$$
$$\epsilon^{0123} = 1, \tag{254}$$
$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta,\ \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta,\ \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}},\ \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}, \tag{255}$$
$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \tag{256}$$
$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}, \tag{257}$$
$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha) = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}, \tag{258}$$
$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha{}_\gamma,\ \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma,\ \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}{}_{\dot{\gamma}},\ \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}{}^{\dot{\gamma}}, \tag{259}$$
$$\sigma^\mu = (\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \vec{\sigma}),\ \bar{\sigma}^\mu = (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (1, -\vec{\sigma}), \tag{260}$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta\ ^3 \tag{261}$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{262}$$

Start with some σ matrix identities.

$$\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] = \mathrm{Tr}[\bar{\sigma}^\mu\sigma^\nu] = -2\eta^{\mu\nu}, \tag{263}$$
$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}, \tag{264}$$
$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)_{\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}, \tag{265}$$
$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = -2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{266}$$
$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = -2\eta^{\mu\nu}\delta_\alpha{}^\beta, \tag{267}$$
$$(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{268}$$

Alternatively, we can combine eqs. (261) and (267), and eqs. (262) and (268) to obtain

$$(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta = -\eta^{\mu\nu}\delta_\alpha{}^\beta + (\sigma^{\mu\nu})_\alpha{}^\beta, \tag{269}$$
$$(\bar{\sigma}^\mu\sigma^\nu)^{\dot{\alpha}}{}_{\dot{\beta}} = -\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} + (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{270}$$

[3] The reader is encouraged to pay extra attention in the definitions of $\sigma^{\mu\nu}$ in the literature since a variety of overall factors is to be encountered, including at least $\frac{1}{2}$ ([28]), $\frac{i}{2}$ ([48]), $\frac{1}{4}$ ([24]), and $\frac{i}{4}$ ([27, 25, 47]). We chose the factor $\frac{1}{2}$ since for our purposes the motivation of defining $\sigma^{\mu\nu}$ is merely to cut down unnecessary notational burden and not a group theoretical as such. Thus, a mere antisymmetrized product of the matrices serves as a natural choice.

The $\sigma^{\mu\nu}$'s obey the following useful trace identities:

$$\mathrm{Tr}[\sigma^{\mu\nu}] = \mathrm{Tr}[\bar{\sigma}^{\mu\nu}] = 0, \tag{271}$$
$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = 2(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}), \tag{272}$$
$$\mathrm{Tr}[\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}] = 2(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma} - \mathrm{i}\epsilon^{\mu\nu\rho\sigma}). \tag{273}$$

Let us then list some useful identities for manipulating spinors.

Start with the (anti)symmetry relations with respect to changing the order of spinors

$$\psi\chi = \chi\psi,\ \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}, \tag{274}$$
$$\psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi, \tag{275}$$
$$\psi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\psi, \tag{276}$$
$$\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi} = \bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\psi}. \tag{277}$$

Consider then identities involving product of two spinors. These are proportional to the epsilon symbols in the following manner

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \tag{278}$$
$$\theta_\alpha\theta_\beta = +\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \tag{279}$$
$$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = +\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \tag{280}$$
$$\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}. \tag{281}$$

The validity of these can be easily checked by exhausting all the possible index combinations.

The following Fierz rearrangement identities are also excessively useful:

$$\phi\chi\,\psi\xi = -\phi\psi\,\chi\xi - \phi\xi\,\chi\psi, \tag{282}$$
$$\bar{\phi}\bar{\chi}\,\bar{\psi}\bar{\xi} = -\bar{\phi}\bar{\psi}\,\bar{\chi}\bar{\xi} - \bar{\phi}\bar{\xi}\,\bar{\chi}\bar{\psi}, \tag{283}$$
$$\phi\sigma^\mu\bar{\chi}\,\bar{\psi}\bar{\sigma}_\mu\xi = 2\phi\xi\,\bar{\chi}\bar{\psi}, \tag{284}$$
$$\bar{\phi}\bar{\sigma}^\mu\chi\,\bar{\psi}\bar{\sigma}_\mu\xi = -2\bar{\phi}\bar{\psi}\,\chi\xi, \tag{285}$$
$$\phi\sigma^\mu\bar{\chi}\,\psi\sigma_\mu\bar{\xi} = -2\phi\psi\,\bar{\chi}\bar{\xi}, \tag{286}$$
$$\phi\sigma^{\mu\nu}\chi\,\psi\sigma_{\mu\nu}\xi = 8\phi\xi\,\chi\psi + 4\phi\chi\,\psi\xi, \tag{287}$$
$$\bar{\phi}\bar{\sigma}^{\mu\nu}\bar{\chi}\,\bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\xi} = 8\bar{\phi}\bar{\xi}\,\bar{\chi}\bar{\psi} + 4\bar{\phi}\bar{\chi}\,\bar{\psi}\bar{\xi}, \tag{288}$$
$$\phi\sigma^{\mu\nu}\chi\,\bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\xi} = 0. \tag{289}$$

The Fierz identities can also be used to compute more general relations for products of spinors,

$$\epsilon_\alpha \theta_\beta = \frac{1}{2}\epsilon\theta\epsilon_{\alpha\beta} + \frac{1}{8}\epsilon\sigma^{\mu\nu}\theta(\sigma_{\mu\nu})_{\alpha\beta} \tag{290}$$

$$\bar{\epsilon}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \frac{1}{2}\bar{\epsilon}\bar{\theta}\epsilon_{\dot{\alpha}\dot{\beta}} + \frac{1}{8}\bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\theta}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}, \tag{291}$$

and further, to figure out the following identities:

$$\theta\phi\,\theta\psi = -\frac{1}{2}\phi\psi\,\theta\theta, \tag{292}$$

$$\bar{\theta}\bar{\phi}\,\bar{\theta}\bar{\psi} = -\frac{1}{2}\bar{\phi}\bar{\psi}\,\bar{\theta}\bar{\theta} \tag{293}$$

$$\theta\sigma^\mu\bar{\theta}\,\theta\sigma^\nu\bar{\psi} = -\frac{1}{2}\theta^2\,\bar{\theta}\bar{\psi}\eta^{\mu\nu} + \frac{1}{2}\theta^2\,\bar{\theta}\bar{\sigma}^{\mu\nu}\bar{\psi}, \tag{294}$$

$$\bar{\theta}\bar{\sigma}^\mu\theta\,\bar{\theta}\bar{\sigma}^\nu\psi = -\frac{1}{2}\bar{\theta}^2\,\theta\psi\eta^{\mu\nu} + \frac{1}{2}\bar{\theta}^2\,\theta\sigma^{\mu\nu}\psi. \tag{295}$$

Finally, let us consider hermitian conjugation. First of all, note that complex conjugation should always reverse the order of Grassmannian variables to be consistent with hermitian conjugation. Using the hermiticity of Pauli matrices we obtain relations for hermitian conjugation (for quantum operators) or complex conjugation (for classical fields)

$$(\psi\sigma^\mu\bar{\chi})^\dagger = \chi\sigma^\mu\bar{\psi}, \tag{296}$$

$$(\bar{\psi}\bar{\sigma}^\mu\chi)^\dagger = \bar{\chi}\bar{\sigma}^\mu\psi, \tag{297}$$

$$(\psi\sigma^\mu\bar{\sigma}^\nu\chi)^\dagger = \bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\psi}, \tag{298}$$

$$(\psi\sigma^{\mu\nu}\chi)^\dagger = -\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\psi}. \tag{299}$$

Notice that we have an extra minus sign in the last equation in comparison with [47] due to the different convention of $\sigma^{\mu\nu}$.

Appendix D Some remarks about Grassmannian derivatives

Let us write down the basic definitions and identities of Grassmannian derivatives. The defining relations are

$$\partial_\alpha \theta^\beta := \frac{\partial \theta^\beta}{\partial \theta^\alpha} = \delta_\alpha{}^\beta, \tag{300}$$

$$\partial^\alpha \theta_\beta := \frac{\partial \theta_\beta}{\partial \theta_\alpha} = \delta^\alpha{}_\beta, \tag{301}$$

$$\bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} := \frac{\partial \bar{\theta}^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}, \tag{302}$$

$$\bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} := \frac{\partial \bar{\theta}_{\dot{\beta}}}{\partial \bar{\theta}_{\dot{\alpha}}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{303}$$

From these it follows that

$$2 = \partial_\alpha \theta^\alpha = \partial^\alpha \theta_\alpha \text{ and} \tag{304}$$

$$2 = \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}, \tag{305}$$

thus implying

$$\partial^\alpha = -\epsilon^{\alpha\beta} \partial_\beta \text{ and } \bar{\partial}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\beta}}. \tag{306}$$

As an immediate consequence, one obtains

$$\partial^\alpha \theta^\beta = -\epsilon^{\alpha\beta}, \tag{307}$$

$$\partial_\alpha \theta_\beta = -\epsilon_{\alpha\beta}, \tag{308}$$

$$\bar{\partial}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{309}$$

$$\bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{310}$$

The Grassmannian derivatives anticommute with the Grassmannian variables, leading to a modified version of the Leibniz rule

$$\partial_\alpha \theta^\beta \theta^\gamma = \delta_\alpha{}^\beta \theta^\gamma - \theta^\beta \delta_\alpha{}^\gamma. \tag{311}$$

Utilizing that we obtain

$$\partial_\alpha \theta^2 = \partial_\alpha \theta^\beta \theta_\beta = \delta_\alpha{}^\beta \theta_\beta - \theta^\beta(-\epsilon_{\alpha\beta}) = 2\theta_\alpha \text{ and} \tag{312}$$

$$\bar{\partial}_{\dot{\alpha}} \bar{\theta}^2 = \bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\beta}} - \bar{\theta}_{\dot{\beta}} \delta_{\dot{\alpha}}{}^{\dot{\beta}} = -2\bar{\theta}_{\dot{\alpha}}, \tag{313}$$

and further

$$\partial^2 \theta^2 = \partial^\alpha \partial_\alpha \theta^\beta \theta_\beta = 2\partial^\alpha \theta_\alpha = 4 \text{ and} \tag{314}$$

$$\bar{\partial}^2 \bar{\theta}^2 = \bar{\partial}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} = \bar{\partial}_{\dot{\alpha}} \left(\bar{\theta}^{\dot{\alpha}} - \bar{\theta}_{\dot{\beta}} \left(-\epsilon^{\dot{\alpha}\dot{\beta}} \right) \right) = 4. \tag{315}$$

Appendix E The vacuum structure

The full potential of the effective Lagrangian reads

$$\mathcal{V} = \mathcal{V}_M + \mathcal{V}_{\mathrm{SB}}, \tag{316}$$

where

$$\begin{aligned}\mathcal{V}_M = &- \frac{m_M^2}{2}\mathrm{Tr}\left[M^\dagger M\right] + \frac{\lambda_M}{4}\mathrm{Tr}\left[M^\dagger M\right]^2 + \lambda'_M \mathrm{Tr}\left[M^\dagger M M^\dagger M\right] \\ &- 2\lambda''_M\left[\mathrm{det}M + \mathrm{det}M^\dagger\right]\end{aligned} \tag{317}$$

and

$$\mathcal{V}_{\mathrm{SB}} = c_1\Lambda^2\mathrm{Tr}\left[MX\right] + c_2\Lambda^2\mathrm{Tr}\left[MZ\right] + c_3\Lambda^4 W_{ijkl}M_{ij}M^*_{kl} + \mathrm{c.c.} \tag{318}$$

Using

$$\langle M\rangle = \frac{1}{2}\begin{pmatrix} 0 & 0 & v_1+v_3 & 0 \\ 0 & \sqrt{2}v_2 & 0 & v_1-v_3 \\ v_1+v_3 & 0 & 0 & 0 \\ 0 & v_1-v_3 & 0 & \sqrt{2}v_4 \end{pmatrix}, \tag{319}$$

we obtain

$$\begin{aligned}\langle\mathcal{V}_M\rangle = \frac{1}{16}\Big[&-4m_M^2(2v_1^2+v_2^2+2v_3^2+v_4^2) + \lambda_M(2v_1^2+v_2^2+2v_3^2+v_4^2)^2 \\ &+ \lambda'_M\Big(2v_2^4 + 4v_2^2(v_1-v_3)^2 + (v_1-v_3)^4 + (v_1+v_3)^4 \\ &\qquad +4v_2(v_1-v_3)^2v_4 + 4(v_1-v_3)^2v_4^2 + 2v_4^4\Big) \\ &-4\lambda''_M(v_1+v_3)^2\Big((v_1-v_3)^2 - 2v_2v_4\Big)\Big],\end{aligned} \tag{320}$$

and

$$\langle\mathcal{V}_{SB}\rangle = \sqrt{2}c_1\Lambda^2 M_D v_4 + c_2\Lambda^2 Z_{13}(v_1+v_3) + \frac{1}{2}c_3\Lambda^4 W(v_1+v_3)^2, \tag{321}$$

with $W_{ijkl} =: W(\delta_{ik1}+\delta_{ik2})\delta_{jl3}$.

The minimum equations read

$$
\begin{aligned}
0 =&\frac{\partial\langle\mathcal{V}\rangle}{\partial v_1}\\
=&(\lambda_M+\lambda'_M-\lambda''_M)v_1^3\\
&+\frac{1}{2}\Big[-2m_M^2+2c_3\Lambda^4W+(\lambda_M+2\lambda'_M)(v_2^2+v_4^2)\\
&\qquad+2(\lambda'_M+\lambda''_M)v_2v_4+2(\lambda_M+3\lambda'_M+\lambda''_M)v_3^2\Big]\,v_1\\
&+\Big[\lambda''_Mv_2v_4-\lambda'_M(v_2^2+v_2v_4+v_4^2)+c_3\Lambda^4W\Big]\,v_3+c_2\Lambda^2Z_{13},
\end{aligned}
\tag{322}
$$

$$
\begin{aligned}
0 =&\frac{\partial\langle\mathcal{V}\rangle}{\partial v_2}\\
=&\frac{1}{4}(\lambda_M+4\lambda'_M)v_2^3\\
&+\frac{1}{4}\Big[-2m_M^2+\lambda_M\Big(2(v_1^2+v_3^2)+v_4^2\Big)+4\lambda'_M(v_1-v_3)^2\Big]\,v_2\\
&+\frac{1}{2}\Big[\lambda'_M(v_1-v_3)^2v_4+\lambda''_M(v_1+v_3)^2v_4\Big],
\end{aligned}
\tag{323}
$$

$$
\begin{aligned}
0 =&\frac{\partial\langle\mathcal{V}\rangle}{\partial v_3}\\
=&(\lambda_M+\lambda'_M-\lambda''_M)v_3^3\\
&+\frac{1}{2}\Big[-2m_M^2+2c_3\Lambda^4W+(\lambda_M+2\lambda'_M)(v_2^2+v_4^2)\\
&\qquad+2(\lambda'_M+\lambda''_M)v_2v_4+2(\lambda_M+3\lambda'_M+\lambda''_M)v_1^2\Big]\,v_3\\
&+\Big[\lambda''_Mv_2v_4-\lambda'_M(v_2^2+v_2v_4+v_4^2)+c_3\Lambda^4W\Big]\,v_1+c_2\Lambda^2Z_{13},
\end{aligned}
\tag{324}
$$

$$
\begin{aligned}
0 =&\frac{\partial\langle\mathcal{V}\rangle}{\partial v_4}\\
=&(\lambda_M+4\lambda'_M)v_4^3\\
&+\Big[-2m_M^2+\lambda_M(2v_1^2+v_2^2+2v_3^2)+4\lambda'_M(v_1-v_3)^2\Big]\,v_4\\
&+\frac{1}{2}\Big[2\sqrt{2}c_1\Lambda^2M_D+\lambda'_Mv_2(v_1-v_3)^2+\lambda''_Mv_2(v_1+v_3)^2\Big].
\end{aligned}
\tag{325}
$$

Appendix F Contribution to anomalous magnetic moment of τ lepton

The invariant amplitude corresponding to the Feynman digram in fig. 10 reads

$$
\mathrm{i}\mathcal{M}=\bar{u}_{s'}(p')\,\mathrm{i}V^{\mu}_{h,\,\text{1-loop}}(p',p)\,u_s(p)\epsilon^*_{\lambda\,\mu}(q),
\tag{326}
$$

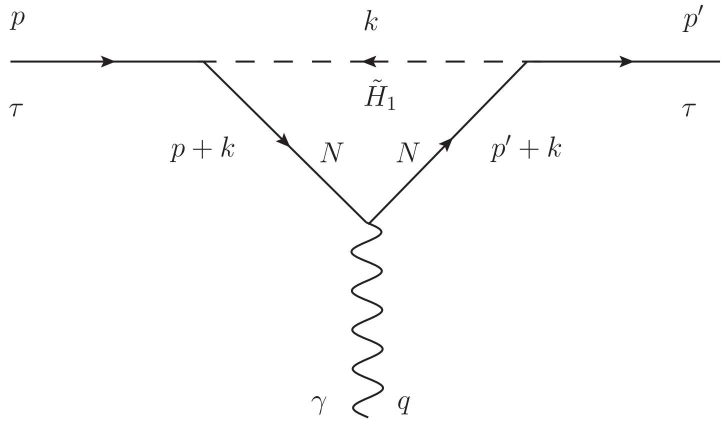

Figure 10. The Feynman diagram giving the one-loop contribution to the anomalous magnetic moment of τ lepton from the $\tilde{H}_1 N \tau$ Yukawa coupling term.

where

$$
\begin{aligned}
&\mathrm{i}V^{\mu}_{\tilde{H}_1,\,\text{1-loop}}(p', p) \\
&= (-\mathrm{i}y)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \frac{-\mathrm{i}(\not{p}' + \not{k} + m_N)}{(p'+k)^2 + m_N^2 - \mathrm{i}\epsilon} (\mathrm{i}e\gamma^{\mu}) \frac{-\mathrm{i}(\not{p} + \not{k} + m_N)}{(p+k)^2 + m_N^2 - \mathrm{i}\epsilon} \, \frac{-\mathrm{i}}{k^2 + m_H^2 - \mathrm{i}\epsilon} \\
&= y^2 e \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \frac{\overbrace{(\not{p}' + \not{k} + m_N)\gamma^{\mu}(\not{p} + \not{k} + m_N)}^{=:N^{\mu}}}{\underbrace{\left((p'+k)^2 + m_N^2 - \mathrm{i}\epsilon\right)}_{=:A}\underbrace{\left((p+k)^2 + m_N^2 - \mathrm{i}\epsilon\right)}_{=:B}\underbrace{\left(k^2 + m_H^2 - \mathrm{i}\epsilon\right)}_{=:C}},
\end{aligned}
\tag{327}
$$

$\bar{u}_{s'}(p')$ and $u_s(p)$ the spinors of the outgoing τ with momentum p' and spin s' and the incoming τ with momentum p and spin s, respectively, $\epsilon^{*}_{\lambda\,\mu}(q)$ is the polarisation vector of the photon with momentum q and polarisation λ, e is the electron charge, y the Yukawa coupling constant between $\tilde{H}_1$, N and τ, m_N and m_H the masses of N and $\tilde{H}_1$, respectively and k the momentum of $\tilde{H}_1$.

Using the Feynman parameterization, we obtain

$$
\frac{1}{ABC} = \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(x+y+z-1) \frac{2!}{(\underbrace{xA + yB + zC}_{=:D})^3} =: \int \mathrm{d}F_3 \, \frac{1}{D^3}, \tag{328}
$$

where A, B and C are as is eq. (327) and

$$\begin{aligned} D =& x(p'^2 + k^2 + 2p' \cdot k + m_N^2) + y(p^2 + k^2 + 2p \cdot k + m_N^2) \\ &+ 2(k^2 + m_H^2) + (x + y + z)(-\mathrm{i}\epsilon), \end{aligned} \tag{329}$$

which after the change of variables $l = k + xp' + yp$ and the use of $x + y + z \to 1$ can be written as

$$\begin{aligned} D \to & l^2 + x(1-x)p'^2 + y(1-y)p^2 - 2xy\, p' \cdot p + (1-z)m_N^2 + zm_H^2 - \mathrm{i}\epsilon \\ =:& \, l^2 + \Delta - \mathrm{i}\epsilon. \end{aligned} \tag{330}$$

After the change of variables, the numerator of eq. (327) takes the form

$$\begin{aligned} N^\mu :=& \not{p}'\gamma^\mu\not{p} + \not{p}'\gamma^\mu\not{k} + \not{k}\gamma^\mu\not{p} + \not{k}\gamma^\mu\not{k} + m_N(\not{p}' + \not{k})\gamma^\mu + m_N\gamma^\mu(\not{p} + \not{k}) + m_N^2\gamma^\mu \\ =& \not{l}\gamma^\mu\not{l} + x(x-1)\not{p}'\gamma^\mu\not{p}' + y(y-1)\not{p}\gamma^\mu\not{p} + (1 + xy - x - y)\not{p}'\gamma^\mu\not{p} \\ &+ xy\,\not{p}\gamma^\mu\not{p}' + m_N(\not{p}'\gamma^\mu + 2xp'^\mu + \gamma^\mu\not{p} + 2yp^\mu + m_N\gamma^\mu) + \text{ terms linear in } l \\ =:& \not{l}\gamma^\mu\not{l} + \tilde{N}^\mu + \text{ terms linear in } l \\ & \xrightarrow{4\to d \text{ dim}} \left(1 - \frac{2}{d}\right) l^2\gamma^\mu + \tilde{N}^\mu, \end{aligned} \tag{331}$$

where on the last line we have changed from 4 to d dimensional case and used first $\not{l}\gamma^\mu\not{l} = l^2\gamma^\mu - 2\not{l}l^\mu$ and then used the identity (see e.g. [25, p. 379])

$$\int \mathrm{d}^d l \, l^\mu l^\nu f(l^2) = \frac{1}{d}\eta^{\mu\nu} \int \mathrm{d}^d l \, l^2 f(l^2). \tag{332}$$

Moreover, we have dropped the terms linear in l since they integrate to zero. Thus, by denoting $\epsilon = 4 - d$ we obtain

$$\begin{aligned} \mathrm{i}V^\mu_{h,\,\text{1-loop}}(p', p) &\xrightarrow{4\to d} ey^2\tilde{\mu}^\epsilon \int \mathrm{d}F_3 \int \frac{\mathrm{i}\,\mathrm{d}^d l_\mathrm{E}}{(2\pi)^d} \frac{\left(1 - \frac{2}{d}\right) l_\mathrm{E}^2\gamma^\mu + \tilde{N}^\mu}{\left(l_\mathrm{E}^2 + \Delta\right)^3} \\ &\overset{i)}{=} \mathrm{i}ey^2 \int \mathrm{d}F_3 \left[(2-\epsilon)\frac{\Gamma(\frac{\epsilon}{2})}{4(4\pi)^{\frac{d}{2}}}\Delta^{-\frac{\epsilon}{2}}\gamma^\mu + \frac{\epsilon\Gamma(\frac{\epsilon}{2})}{4(4\pi)^{\frac{d}{2}}}\Delta^{-\frac{\epsilon}{2}}\frac{\tilde{N}^\mu}{\Delta}\right] \\ &\overset{ii)}{=} \frac{\mathrm{i}ey^2}{32\pi^2} \int \mathrm{d}F_3 \left[1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{\Delta}\right)\right] \\ &\qquad \cdot \left[\left(1 - \frac{\epsilon}{2}\right)\left(\frac{2}{\epsilon} - \gamma_\mathrm{E}\right)\gamma^\mu + \frac{\epsilon}{2}\left(\frac{2}{\epsilon} - \gamma_\mathrm{E}\right)\frac{\tilde{N}^\mu}{\Delta}\right] + \mathcal{O}(\epsilon) \\ &\overset{iii)}{=} \frac{\mathrm{i}ey^2}{16\pi^2}\left[\left(\frac{1}{\epsilon} - \frac{1}{2} - \frac{1}{2}\int \mathrm{d}F_3 \ln\frac{\Delta}{\mu^2}\right)\gamma^\mu + \frac{1}{2}\int \mathrm{d}F_3 \frac{\tilde{N}^\mu}{\Delta}\right] + \mathcal{O}(\epsilon), \end{aligned} \tag{333}$$

where, after the transition to d dimensions, an extra i emerges from the Wick rotation to Euclidean space, i.e. $\mathrm{d}^d l = \mathrm{i}\,\mathrm{d}^d l_{\mathrm{E}}$, where the subscript E denotes the corresponding Wick-rotated Euclidean vector. Similarly, the subscript E in l_{E}^2 denotes the Euclidean inner product but with our metric convention $l^2 = l_{\mathrm{E}}^2$, so this does not result in any extra minus signs. In *i)* we have performed the d-dimensional integral over l and in *ii)* we have expanded in powers of ϵ; γ_{E} is the Euler-Mascheroni constant. Finally, in *iii)* we have dropped all terms at least linear in ϵ.

Now, the total vertex function up to one-loop order can be written as

$$\mathrm{i}V^{\mu}_{\text{1-loop}}p', p = \mathrm{i}Z_1 e\gamma^{\mu} + \mathrm{i}V^{\mu}_{\text{SM, 1-loop}}(p', p) + \mathrm{i}V^{\mu}_{h,\text{1-loop}}(p', p), \tag{334}$$

where $\mathrm{i}Z_1 e\gamma^{\mu}$ is the original tree-level vertex and $\mathrm{i}V^{\mu}_{\text{SM, 1-loop}}(p', p)$ denotes the SM 1-loop corrections.

Now, demanding $V^{\mu}(p', p)$ to be finite, we obtain the following expression for Z_1:

$$Z_1 = 1 - K_{\mathrm{SM}} - \frac{ey^2}{16\pi^2}\left(\frac{1}{\epsilon} + \text{ finite }\right) + \text{ higher order terms}, \tag{335}$$

where K_{SM} contains the contribution cancelling the divergent terms in $V^{\mu}_{\text{SM, 1-loop}}$.

Next, impose on-shell renormalization conditions by demanding (see e.g. [25, p. 385])

$$\begin{aligned} \bar{u}_{s'}(p')\, V^{\mu}_{h,\text{1-loop}}(p', p)\, u_s(p)\Big|_{\substack{p^2=p'^2=-m_\tau \\ q^2=0}} &= e\bar{u}_{s'}(p')\,\gamma^{\mu}\, u_s(p)\big|_{\substack{p^2=p'^2=-m_\tau \\ q^2=0}} \\ &= 2ep^{\mu}\delta_{ss'}, \end{aligned} \tag{336}$$

where the last equation follows from the fact that the on-shell conditions actually imply not only that $p^2 = p'^2$ but also that $p^{\mu} = p'^{\mu}$. Due to the delta function on the spins, we, thus, set $s = s'$ and drop the subscript s in the u and $\bar{u}$ spinors in what follows.

Using the freedom to choose the finite part of Z_1 at will, we can write the total vertex function as

$$V^{\mu}(p', p) = e\gamma^{\mu} + \tilde{V}^{\mu}_{\mathrm{SM}}(p', p) - \frac{ey^2}{32\pi^2}\int \mathrm{d}F_3\, \left[\left(\ln\frac{\Delta}{\Delta_0} + \kappa\right)\gamma^{\mu} - \frac{\tilde{N}^{\mu}}{2\Delta}\right], \tag{337}$$

where

$$\begin{aligned} \Delta_0 &= \Delta|_{\substack{p^2=p'^2=-m_\tau \\ q^2=0}} \\ &= -x(1-x)m_\tau^2 - y(1-y)m_\tau^2 + 2xym_\tau^2 + (1-z)m_N^2 + zm_H^2 \\ &\to (z-1)(m_N^2 - zm_\tau^2) + zm_H^2. \end{aligned} \tag{338}$$

Now, fix the constant κ in $V^\mu(p',p)$ by imposing equation (336). Assuming the SM part to be already renormalised, we obtain

$$0 = -\frac{ey^2}{32\pi^2}\bar{u}(p)\int \mathrm{d}F_3 \left[\kappa\gamma^\mu - \frac{\tilde{N}_0^\mu}{2\Delta_0}\right] u(p), \tag{339}$$

with

$$\tilde{N}_0^\mu = \tilde{N}^\mu\Big|_{\substack{p^2=p'^2=-m_\tau \\ q^2=0}} . \tag{340}$$

Using Dirac equation, i.e. $\not{p}u(p) = -m_\tau u(p)$ and $\bar{u}(p)\not{p} = -m_\tau\bar{u}(p)$ and relations $\bar{u}(p)\gamma^\mu u(p) = 2p^\mu$ and $\bar{u}(p)u(p) = 2m_\tau$, we can simplify

$$\bar{u}(p)\tilde{N}_0^\mu u(p) \to 2p^\mu(m_N - zm_\tau)^2. \tag{341}$$

Equation (339), accordingly, simplifies to

$$2\kappa p^\mu = \int \mathrm{d}F_3\, \frac{p^\mu(m_N - zm_\tau)^2}{(1-z)(m_N^2 - zm_\tau^2) + zm_H^2}, \tag{342}$$

giving

$$\kappa = \int \mathrm{d}F_3\, \frac{(m_N - zm_\tau)^2}{2(1-z)(m_N^2 - zm_\tau^2) + zm_H^2}. \tag{343}$$

We have now fixed the vertex function. The next step towards the contribution to the anomalous magnetic moment of τ lepton is to to achieve the decomposition

$$\bar{u}_{s'}(p')V^\mu_{h,\,\text{1-loop}}(p',p)u_s(p) = e\bar{u}_{s'}(p')\left[F_1^h(q^2)\gamma^\mu - \frac{\mathrm{i}}{m_\tau}F_2^h(q^2)S^{\mu\nu}q_\nu\right]u_s(p), \tag{344}$$

where $F_1^h(q^2)$ and $F_2^2(q^2)$ are the contributios to the so-called *form factors* from the $hN\tau$ Yukawa coupling to one-loop order and

$$S^{\mu\nu} = \frac{\mathrm{i}}{4}\left[\gamma^\mu,\gamma^\nu\right].$$

Therefore, let now $p^2 = p'^2 = -m_\tau^2$ but $q^2 = (p'-p)^2$ arbitrary and denote $\bar{u}' \coloneqq \bar{u}_{s'}(p')$ and $u \coloneqq u_s(p)$ in the following.

Now, using $\not{p}u = -m_\tau u$ and $\bar{u}'\not{p}' = -m_\tau\bar{u}'$ (and $x+y+z-1 \to 0$) we can simplify

$$\begin{aligned}\bar{u}'N^\mu u \to& \bar{u}'\left[(z(2-z)m_\tau^2 - 2m_N m_\tau + m_N^2 + xyq^2)\gamma^\mu\right.\\ &\left.+(m_N - zm_\tau)\left[(x-y)(p'^\mu - p^\mu) + (x+y)(p'^\mu + p^\mu)\right]\right]u\end{aligned} \tag{345}$$

and

$$\Delta \to z(z-1)m_\tau^2 + (1-z)m_N^2 + zm_H^2 + xyq^2. \tag{346}$$

Now, since Δ is symmetric under the exchange $x \leftrightarrow y$, the $(x-y)$ term in eq. (345) integrates to zero and we have

$$\begin{aligned} \bar{u}' N^\mu u \to & \bar{u}' \Big[(z(2-z)m_\tau^2 - 2m_N m_\tau + m_N^2 + xyq^2)\gamma^\mu \\ & + (m_N - zm_\tau)(x+y)(p'^\mu + p^\mu) \Big] u. \end{aligned} \tag{347}$$

Then, using Gordon identity, $\bar{u}'(p'^\mu + p^\mu)u = \bar{u}'(2m_\tau\gamma^\mu + 2\mathrm{i}\, S^{\mu\nu}q_\nu)u$, we get

$$\begin{aligned} \tilde{N}^\mu \to & \Big[z(2-z)m_\tau^2 - 2m_N m_\tau + m_N^2 + xyq^2 + 2m_\tau(m_N - zm_\tau)(1-z) \Big] \gamma^\mu \\ & + 2\mathrm{i}\,(m_N - zm_\tau)(1-z)S^{\mu\nu}q_\nu. \end{aligned} \tag{348}$$

Thus, we obtain

$$F_2^h(q^2) = -\frac{y^2}{32\pi^2} \int \mathrm{d}F_3 \, \frac{m_\tau(m_N - zm_\tau)(1-z)}{z(z-1)m_\tau^2 + (1-z)m_N^2 + zm_H^2 + xyq^2} \tag{349}$$

and further

$$\begin{aligned} F_2^h(0) = & -\frac{y^2}{32\pi^2} \int \mathrm{d}F_3 \, \frac{m_\tau(m_N - zm_\tau)(1-z)}{z(z-1)m_\tau^2 + (1-z)m_N^2 + zm_H^2} \\ = & -\frac{y^2}{16\pi^2} \int_0^1 \mathrm{d}z \, \frac{m_\tau(m_N - zm_\tau)(1-z)^2}{z(z-1)m_\tau^2 + (1-z)m_N^2 + zm_H^2}. \end{aligned} \tag{350}$$

Appendix G Lifetime of the N lepton

After the introduction of the $\tilde{H}_1 N\tau$ Yukawa coupling, the N lepton is no longer stable but can decay via the Higgs boson, $\tilde{H}_1$. Taking the mass of N, m_N, between 100 and 200 GeV, the dominant decay channels for the Higgs are the decay into bottom–anti-bottom pair. The corresponding Feynman diagram is depicted in fig. 11

Let us first compute the invariant amplitude for $b\bar{b}$ channel. The invariant amplitude of the decay reads

$$\begin{aligned} \mathrm{i}\mathcal{M}(N \to b\bar{b}\tau) = & \bar{u}_{s_1}(p_3)(-\mathrm{i}y)u_s(P) \frac{-\mathrm{i}}{k^2 + m_H^2 - \mathrm{i}\epsilon} \bar{u}_{s_3}(p_1)\,(-\mathrm{i}y_b)\,v_{s_2}(p_2) \\ = & \mathrm{i}yy_b \frac{1}{k^2 + m_H^2 - \mathrm{i}\epsilon} \bar{u}_{s_1}(p_3)u_s(P)\,\bar{u}_{s_3}(p_1)v_{s_2}(p_2), \end{aligned} \tag{351}$$

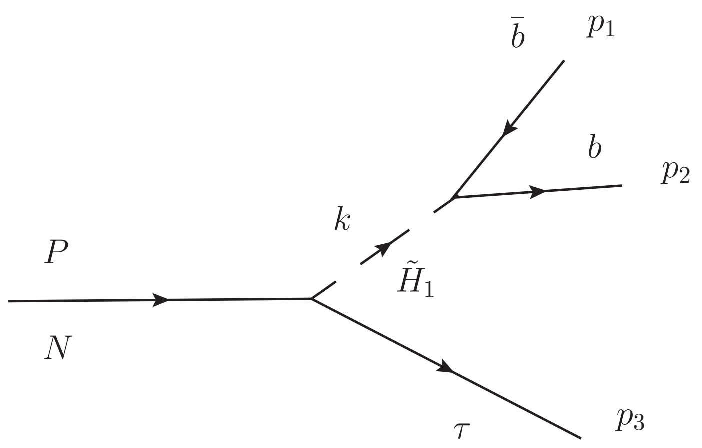

Figure 11. The Feynman diagram of N decay into $b\bar{b}$ and τ

where $u_s(p)$, $\bar{u}_{s_1}(p_3)$, $v_{s_2}(p_2)$ and $\bar{u}_{s_3}(p_1)$ are the spinors of an incoming N with momentum P and spin s, an outgoing τ with momentum p_3 and spin s_1, an incoming $\bar{b}$ with momentum p_2 and spin s_2 and an outgoing b with momentum p_1 and spin s_3, respectively, y the Yukawa coupling constant between h, N and τ, y_b the Yukawa coupling between the $b, \bar{b}$ and h, m_H the mass of h and $k = P - p_3$ the momentum of h.

Now, squaring, averaging over the incoming spin and summing over the outgoing spins and colour yields

$$\begin{aligned}\overline{|\mathcal{M}(N \to b\bar{b}\tau)|^2} &= \frac{1}{2}\sum_{\text{spins}}\sum_{\text{colors}} |\mathcal{M}(N \to b\bar{b}\tau)|^2 \\ &= \frac{3y^2y_b^2}{2}\frac{16}{(k^2+m_H^2)^2}\left[(P\cdot p_3)(p_1\cdot p_2) + m_b^2\, P\cdot p_3 - m_\tau m_N(p_1\cdot p_2 - m_b^2)\right],\end{aligned} \tag{352}$$

where m_τ and m_b are the masses of τ and b, respectively.

Consider then a general three-body decay, see fig.and denote the mass and the momentum of the decaying particle by M and P, respectively, and the masses and momenta of the decay products by m_i and p_i, $i = 1, 2, 3$, respectively. Moreover, define m_{ij}^2, $i, j = 1, 2, 3$ by setting

$$m_{ij}^2 = -(p_i + p_j)^2. \tag{353}$$

Then, the differential decay width of a three-body decay (if the decaying particle is scalar or if its spin is averaged over) can be written as (see e.g. [6])

$$\mathrm{d}\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}\overline{|\mathcal{M}|^2}\mathrm{d}m_{12}^2\mathrm{d}m_{23}^2. \tag{354}$$

To obtain the total decay width, we need the limits of integration. If the m_{23}^2 integral is first performed, the limits read [6]

$$(m_{23}^2)_{\min}^{\max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} \mp \sqrt{E_3^{*2} - m_3^2}\right)^2, \quad \text{and} \tag{355}$$

$$(m_{12}^2)_{\min} = (m_1 + m_2)^2, \quad (m_{12}^2)_{\max} = (M - m_3)^2, \tag{356}$$

where

$$E_2^* = \frac{1}{2m_{12}}(m_{12}^2 - m_1^2 + m_2^2), \quad E_3^* = \frac{1}{2m_{12}}(M^2 - m_{12}^2 - m_3^2). \tag{357}$$

Let us then return to the N decay. After the change of variables to m_{12}^2 and m_{23}^2 the invariant amplitude reads

$$\begin{aligned}
&\overline{|\mathcal{M}(N \to b\bar{b}\tau)|^2} \\
&= \frac{3y^2y_b^2}{2}\frac{16}{(-m_{12}^2 + m_H^2)^2}\left[\frac{1}{4}(m_{12}^2 - m_N^2 - m_\tau^2)(2m_b^2 - m_{12}^2)\right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{1}{2}m_b^2(m_{12}^2 - m_N^2 - m_\tau^2) + \frac{1}{2}m_\tau m_N m_{12}^2\right] \\
&= \frac{6y^2y_b^2}{(-m_{12}^2 + m_H^2)^2}\left[\left((m_N + m_\tau)^2 - m_{12}^2\right)m_{12}^2 - 4m_b^2\left(m_N^2 + m_\tau^2 - m_{12}^2\right)\right].
\end{aligned} \tag{358}$$

Further, the differential decay width takes the form

$$\begin{aligned}
\mathrm{d}\Gamma =& \mathrm{d}m_{12}^2\, \mathrm{d}m_{23}^2\, \frac{1}{(2\pi)^3}\frac{3y^2y_b^2}{16m_N^3(-m_{12}^2 + m_H^2)^2}\times \\
&\left[\left((m_N + m_\tau)^2 - m_{12}^2\right)m_{12}^2 - 4m_b^2\left(m_N^2 + m_\tau^2 - m_{12}^2\right)\right],
\end{aligned} \tag{359}$$

and finally after carrying out the integrations, we obtain the lifetime

$$\tau = \frac{1}{\Gamma}. \tag{360}$$