Supersymmetry, Supergravity and the AdS/CFT Correspondence

Ilkka Mäkinen

Master's Thesis
University of Jyväskylä
Supervisor: Kimmo Tuominen

2011

Abstract

This thesis is an overview of the AdS/CFT correspondence. My main objective has been to understand the chain of ideas that leads from ordinary quantum field theory and general relativity to the formulation of the conjecture that certain string theories are equivalent to lower-dimensional gauge theories. The applications of the correspondence to the building of models are of secondary importance in this work.

We will first give an overview of supersymmetry in Chapter 1. After a brief historical motivation, we introduce the algebra of supersymmetry transformations. We study the representations of the algebra, both from a more abstract point of view, and through an example of a supersymmetric field theory. We then introduce the superspace formalism, and show how supersymmetry transformations are interpreted as coordinate transformations in superspace, and that integration of functions defined on superspace produces Lagrangians which define supersymmetric field theories. We discuss the construction of supersymmetric gauge theories, including a supersymmetrized version of QCD, using the superspace formalism. Finally, we look briefly into the spontaneous breaking of supersymmetry, and into the quantization of supersymmetric field theories.

In Chapter 2, we promote the global supersymmetry transformations into local ones, which leads us to supergravity theories. After an overview of the vielbein formulation of general relativity, the use of which is necessary for introducing spinors into a theory of gravity, we argue by use of an example that making the supersymmetry transformations local automatically introduces gravity into the theory. We move on to give a rather detailed discussion of the so-called simple supergravity in four-dimensional spacetime. We then look into how the algebra of supersymmetry transformations is properly generalized into higher-dimensional spacetimes. We conclude the chapter with a description of the ten- and eleven-dimensional supergravity theories and their symmetries.

With the ideas introduced in Chapters 1 and 2, we are ready to describe the ingredients of the AdS/CFT duality in Chapter 3. We give a brief overview of string theory, including a description of D-branes. We note that string theories reduce to supergravity theories in the low-energy limit. We discuss the so-called large N limit of gauge theories, and argue that the properties of the large N expansion suggest a connection between gauge theories and string theories. We construct the p-brane solutions of ten-dimensional supergravity, and explain how they are related to the D-branes of string theory. After an overview of conformally invariant field theories, we are ready to introduce Maldacena's conjecture on the AdS/CFT duality. We examine a system of open strings and D-branes from two different points of view, leading to the conclusion that a four-dimensional supersymmetric gauge theory likely is equivalent to a string theory defined on a five-dimensional spacetime. We briefly discuss various aspects of the duality, after which we conclude the thesis by a brief introduction into how the duality could be used to construct models for QCD.

Tiivistelmä

Tämä tutkielma on johdatus AdS/CFT-dualiteettiin. Pääasiallinen tavoitteeni on ollut ymmärtää se ajatusten ketju, joka alkaa tavallisesta kvanttikenttäteoriasta ja yleisestä suhteellisuusteoriasta ja päätyy konjektuuriin, jonka mukaan tietyt säieteoriat ovat ekvivalentteja alempiulotteisten mittateorioiden kanssa. Dualiteetin sovellukset mallien rakentamisessa jäävät tässä työssä vähälle huomiolle.

Kappale 1 sisältää katsauksen supersymmetriaan. Lyhyen historiallisen johdatuksen jälkeen esittelemme supersymmetriamuunnosten algebran. Tämän algebran esityksiä tutkimme sekä abstraktein menetelmin että tarkastelemalla esimerkkiä supersymmetrisestä kenttäteoriasta. Tämän jälkeen esittelemme superavaruusformalismin, ja näytämme miten supersymmetriamuunnokset tulkitaan superavaruuden koordinaattimuunnoksina, ja että superavaruudessa määriteltyjen funktioiden integroiminen tuottaa supersymmetrisiä teorioita määritteleviä Lagrangen funktioita. Näytämme miten superavaruusformalismia käytetään supersymmetristen mittateorioiden, mm. supersymmetrisoidun QCD:n, rakentamiseen. Lopuksi tarkastelemme lyhyesti supersymmetrian spontaania rikkoutumista ja supersymmetristen kenttäteorioiden kvantisointia.

Kappaleessa 2 korotamme globaalit supersymmetriamuunnokset lokaaleiksi, päätyen tätä kautta supergravitaatioteorioihin. Esittelemme aluksi yleisen suhteellisuusteorian monijalkamuotoilun (vielbein formulation), jonka käyttäminen on välttämätöntä spinorien tuomiseksi gravitaatioteoriaan. Sen jälkeen perustelemme esimerkin avulla miksi supersymmetrian korottaminen lokaaliksi tuo automaattisesti gravitaation osaksi teoriaa. Käsittelemme melko perusteellisesti niin sanotun yksinkertaisen neliulotteisen supergravitaation. Tutkimme myös, miten supersymmetria-algebra kuuluu yleistää korkeampiulotteisiin avaruusaikoihin. Kappaleen lopuksi kuvailemme 10- ja 11- ulotteisia supergravitaatioteorioita ja niiden symmetrioita.

Kappaleiden 1 ja 2 sisältämien tietojen avulla olemme valmiit esittelemään loput AdS/ CFT-dualiteetin rakenneosat kappaleessa 3. Annamme aluksi lyhyen johdatuksen säieteoriaan ja D-braaneihin. Toteamme, että matalan energian rajalla säieteoriat palautuvat supergravitaatioteorioiksi. Tutkimme niin sanottua suuren N:n rajaa mittateorioissa, ja perustelemme miksi suuren N:n kehitelmän muoto viittaa mittateorioiden ja säieteorioiden välillä olevaan yhteyteen. Tarkastelemme kymmenulotteisen supergravitaation p-braaniratkaisuja ja selitämme, miten ne liittyvät säieteorian D-braaneihin. Annamme katsauksen konformi-invariantteihin kenttäteorioihin, jonka jälkeen olemme valmiit esittelemään Maldacenan konjektuurin AdS/CFT-dualiteetista. Tarkastelemme avoimien säikeiden ja D-braanien muodostamaa systeemiä kahdella eri tavalla, päätellen tätä kautta että eräs neliulotteinen supersymmetrinen mittateoria on todennäköisesti ekvivalentti tietyn viisiulotteisessa avaruudessa määritellyn säieteorian kanssa. Esittelemme lyhyesti dualiteetin eri puolia, jonka jälkeen lopetamme tutkielman tarkastelemalla lyhyesti, miten dualiteettia voidaan käyttää QCD:ta kuvaavien mallien rakentamiseen.

Contents

Introduction **1**

1 Supersymmetry **5**
1.1 The supersymmetry algebra 5
1.2 Representations of the supersymmetry algebra 7
1.3 An example: the Wess-Zumino model 13
1.4 Superspace and superfields 15
1.5 Lagrangians from the superspace formalism 19
1.6 Supersymmetric gauge theories 21
1.7 Spontaneous supersymmetry breaking 27
1.8 Quantization of supersymmetric field theories 29

2 Supergravity **33**
2.1 The vielbein formulation of general relativity 33
2.2 Local supersymmetry 38
2.3 Simple supergravity 40
2.4 Tensor calculus for supergravity 45
2.5 Off-shell formulation of simple supergravity 47
2.6 Supersymmetry in higher dimensions 52
2.7 Eleven-dimensional supergravity 54
2.8 Ten-dimensional supergravities 57

3 Duality **63**
3.1 Elements of superstring theory 63
3.2 Large N field theory 69
3.3 Geometry of anti de Sitter spaces 71
3.4 Branes in supergravity 75
3.5 Conformal field theories 80
3.6 Maldacena's conjecture 84
3.7 Holographic models for QCD 90

Conclusions **97**

Appendices **101**
A.1 Conventions 101
A.2 The Lorentz group. Spinors 103
A.3 The Poincaré group and the conformal group 107
A.4 Differential geometry in superspace 109

References **113**

Introduction

In 1998, Maldacena put forward a conjecture [59] that a string theory defined on a particular kind of anti de Sitter space is equivalent to a supersymmetric gauge theory which, in some sense, is defined on the boundary of the anti de Sitter space. This conjectured equivalence, or duality, goes by the name of AdS/CFT correspondence. Since then, other examples have been developed of presumed dualities between string theories and gauge theories, and the term AdS/CFT correspondence may refer more generally to any of these dualities. The AdS/CFT correspondence has been called one of the most significant results that string theory has produced [60]. It has even inspired a comparison with the work of the philosophers of ancient Greece [61].

The correspondence intriguingly brings together many ideas developed in theoretical physics during the last forty years. The equivalence between string theories and gauge theories is motivated by considering a system of branes and open strings, from the point of view of D-branes in superstring theory, and from the point of view of p-branes in classical supergravity. The string theory–gauge theory duality is a materialization of 't Hooft's old idea that gauge theories are related to string theories in the so-called large N limit. The correspondence also seems closely related to the holographic principle, as it provides a concrete realization of the claim that in a gravity theory, the degrees of freedom inside a given region should be associated with the boundary of the region.

Besides being theoretically attractive, the AdS/CFT correspondence also has potential to be practically very useful. As soon as a precise correspondence is established between the relevant quantities, such as the fields and the operators, of the two theories, one could hope that calculations which have previously been intractable in one theory could be carried out by going to the dual theory, where the calculation might be considerably simpler. This seems particularly relevant for QCD, where calculations are extremely difficult to perform in the limit of strong coupling. As the strong coupling limit in the gauge theory side of the duality generally corresponds to the weak coupling limit in the string theory side, there is a hope that progress could be made in QCD by making use of the duality. However, as of now the string theory dual to QCD is not known, which severely limits the efficiency of the duality approach to QCD.

It has been my goal in this work to thoroughly explore the ideas which form the basis of the AdS/CFT correspondence. The applications of the correspondence are mostly outside of the scope of this work. We begin with an outline of supersymmetry in Chapter 1. Subsequently, in Chapter 2 we describe the main features of supergravity theories. I chose to give a rather thorough treatment of four-dimensional supergravity, which is arguably the most tractable of the supergravity theories, and many features of the higher-dimensional theories can be understood by analogy with the four-dimensional theory.

Chapters 1 and 2 contain a lot of material which will be only minimally relevant to the rest of the work. The justification for including this material comes in part from my desire to write as self-contained a work as possible, and to understand why supersymmetry and supergravity have been so widely judged to be worthy of extremely serious research, even in the face of a complete absence of support from experiments during the nearly forty years that supersymmetry has been in existence.

While Chapters 1 and 2 form a more or less coherent whole, where things logically follow one another in a reasonable sequence, the same is perhaps not true of Chapter 3, in which many ideas will be introduced which may seem unrelated to each other and to the ideas of the preceding chapters. These ideas will, however, find their purpose towards the end of the chapter as the building blocks of the AdS/CFT correspondence. Furthermore, Chapter 3 will contain many statements which are justified only incompletely, or not at all, in contrast to Chapters 1 and 2, where the majority of the statements are given a reasonably detailed justification. In part, this is a reflection of my extremely incomplete understanding of these matters, though it is certainly necessary to disregard many details in order to reduce the vast subject of AdS/CFT duality to a single chapter.

A complete description of our conventions and notations will be given in Appendix 1. We use the metric $\eta_{\mu\nu} = (-1, +1, \ldots, +1)$. In four spacetime dimensions – that is, from the beginning of the work up to section 2.5 – we treat spinors using the two-component Weyl notation. This greatly simplfies things, compared to the formalism of four-component Dirac spinors, when explicit calculations need to be performed. Perhaps an even more convincing argument for using the two-component notation is that it is the formalism which is used by Nature, as indicated by the completely different weak interactions of left-handed and right-handed particles. Appendices 2 and 3 describe certain group-theoretical ideas relevant to the main part of the work.

A few words about our references: I learned supersymmetry from the books [1] and [2], and the reviews [3, 4, 5, 6, 7]. All of what appears in Chapter 1 originates in some way from one or more of these references. Only whenever a particular passage is clearly influenced by certain references, has it been indicated by an explicit reference in the text. The references which I used in writing Chapters 2 and 3 are adequately described within the text. The discussion of simple supergravity given in sections 2.1 through 2.5 closely follows the references [20] and [21], which use the two-component notation for spinors. The references [22, 23, 24, 25] have also contributed to my understanding of supergravity. For the AdS/CFT duality, good general references are given by [56, 69, 72, 75, 77].

I have generally tried to make references to relevant original papers whenever I have been aware of their existence. I did this for two reasons: On one hand, to acknowledge the work of the people who have discovered the ideas which form the contents of this work; and on the other hand, because I think one should read the original papers in one's field as much as possible, since what they sometimes lack in pedagogical clarity is more than sufficiently compensated by their distinctive point of view and the unconventional insights that they often contain.

While I have made a reasonable effort to find errors and remove them, it is very likely that some sign errors, wrong numerical factors, inconsistencies, and misleading or incorrect statements still remain. I will not present any scientifically significant original findings in this thesis, and so maybe it is best regarded as a documentation of my attempts to make sense out of certain exciting areas of theoretical physics. The value of this work perhaps lies e.g. in its potential to act as a guide for students who may wish to embark on a journey towards understanding these things in the future.

Chapter 1

Supersymmetry

1.1 The supersymmetry algebra

In the 1960's, physicists were becoming increasingly aware of the importance of internal symmetries in quantum field theories. There were many attempts to find a symmetry which would combine an internal symmetry group with the Poincaré group of spacetime symmetries in some nontrivial way. This led to a series of theorems on what kind of symmetries are possible in a quantum field theory, the most powerful of which was the theorem of Coleman and Mandula in 1967 [8]. On very general assumptions, among which were that the S-matrix is based on a relativistic four-dimensional quantum field theory [9], the scattering amplitudes $\langle\psi_{\text{out}}|S|\psi_{\text{in}}\rangle$ are analytic and nontrivial functions of the Mandelstam variables s and t, and that all particles have a positive energy, with only a finite number of particles having a mass less than m_0 for any m_0, they proved that the only possible Lie algebra of the symmetries of the S-matrix is a direct product of the Poincaré algebra and the algebra of an internal symmetry group. That is, the algebra contains the momentum operators P_μ, the Lorentz rotation generators $M_{\mu\nu}$, and a finite number of internal symmetry generators T_r, which commute with P_μ and $M_{\mu\nu}$. In the case where all the particles are massless, the symmetry group is improved to the direct product of the conformal group with an internal symmetry group, but the conclusion still stands that it is not possible to nontrivially intertwine a spacetime symmetry with an internal symmetry.

An important point is that the theorem is concerned with the symmetries of the S-matrix, and it is possible for the theory itself to have more (or less) symmetry than the S-matrix. A notable class of such symmetries is given by spontaneously broken symmetries, which are not symmetries of the S-matrix despite being symmetries of the action. Therefore, the Coleman–Mandula theorem does not apply to them.

Sohnius [10] points out that Witten has given an elegant explanation of the meaning of the Coleman-Mandula theorem: invariance under spacetime translations and rotations already places the maximum number of constraints on the scattering amplitudes, in the sense that an additional spacetime symmetry would further constrain the amplitudes so that scattering could occur only at discrete scattering angles. Then the assumption that the amplitudes are analytic functions of the scattering angle allows no scattering at all [11].

In 1971, Golfand and Likhtman showed that it is possible to find a symmetry group which nontrivially mixes internal symmetries with Poincaré symmetries, provided that one of the assumptions in the Coleman–Mandula theorem is relaxed [12]. The essential idea is to consider, instead of a Lie algebra, a more general algebraic structure – usually called by mathematicians a graded algebra, and by physicists a superalgebra – which contains both commuting and anticommuting generators. The commuting generators C and the anticommuting generators A are required to satisfy certain rules, which are expressed schematically as

$$[C, C] = C, \qquad [C, A] = A, \qquad \{A, A\} = C. \tag{1.1}$$

The most general set of symmetries, under the new assumption of anticommuting generators, was found by Haag, Lopuszanski and Sohnius in 1975 [13]. It consists of the operators P_μ, $M_{\mu\nu}$, T_r, ${Q_\alpha}^i$, where T_r are the generators of internal symmetries, and ${Q_\alpha}^i$ are anticommuting. The index i enumerates the different anticommuting operators and goes from 1 to some number N. The index α enumerates the components of ${Q_\alpha}^i$, which is a Weyl spinor. (Naturally the algebra contains also the conjugate spinors $\bar{Q}_{\dot{\alpha}i}$.) The commuting operators P_μ, $M_{\mu\nu}$, T_r must still obey the Coleman–Mandula theorem.

We will only briefly outline the Haag–Lopuszanski–Sohnius calculation here. Its details are discussed for example in [1] and [2]. The calculation is based on the observation that the identities

$$\begin{aligned}
[[C_1, C_2], C_3] + [[C_3, C_1], C_2] + [[C_2, C_3], C_1] = 0,\\
[[C_1, C_2], A_3] + [[A_3, C_1], C_2] + [[C_2, A_3], C_1] = 0,\\
\{[C_1, A_2], A_3\} + \{[A_3, C_1], A_2\} + [\{A_2, A_3\}, C_1] = 0,\\
[\{A_1, A_2\}, A_3] + [\{A_3, A_1\}, A_2] + [\{A_2, A_3\}, A_1] = 0,
\end{aligned} \tag{1.2}$$

where C_i are commuting and A_i are anticommuting, place very strong constraints on the possible form of the algebra satisfied by the generators.

In particular, it follows from these identities that ${Q_\alpha}^i$ is a Weyl spinor. Using the second identity with $(C_1, C_2, A_3) = (M_{\mu\nu}, M_{\rho\sigma}, {Q_\alpha}^i)$, together with $[{Q_\alpha}^i, M_{\mu\nu}] = i{(b_{\mu\nu})_\alpha}^\beta {Q_\beta}^i$, which follows from Eq. (1.1) because Q is the only anticommuting generator, one can show that ${Q_\alpha}^i$ carries a representation of the Lorentz group [2]. If this is the (m, n) representation, then the anticommutator $\{Q, \bar{Q}\}$ will contain the $(m+n, m+n)$ representation. The only available operator that belongs to such a representation is P_μ, which is in the $(\frac{1}{2}, \frac{1}{2})$ representation. Therefore the most general possibility is that all the Q-operators belongs to the $(\frac{1}{2}, 0)$ or the $(0, \frac{1}{2})$ representation [10].

The form of the supersymmetry algebra can be found by repeatedly applying the identities (1.2) to derive conditions on the commutators and anticommutators between the operators P_μ, $M_{\mu\nu}$, T_r, ${Q_\alpha}^i$. The algebra of the Poincaré group and the internal symmetry group remains naturally unchanged. The algebra satisfied by the new generators ${Q_\alpha}^i$, which are called the supercharges, turns out to be

$$\begin{aligned}
&\{{Q_\alpha}^i, \bar{Q}_{\dot\beta j}\} = 2\delta^i_j(\sigma_\mu)_{\alpha\dot\beta}P^\mu, \\
\{{Q_\alpha}^i, {Q_\beta}^j\} &= \epsilon_{\alpha\beta}Z^{ij}, \qquad && \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} = \bar\epsilon_{\dot\alpha\dot\beta}\bar{Z}_{ij}, \\
[P_\mu, {Q_\alpha}^i] &= 0, && [P_\mu, \bar{Q}_{\dot\alpha j}] = 0, \\
[M_{\mu\nu}, {Q_\alpha}^i] &= i(\sigma_{\mu\nu})_\alpha{}^\beta {Q_\beta}^i, && [M_{\mu\nu}, \bar{Q}_{\dot\alpha j}] = i(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar{Q}_{\dot\beta i} \\
[{Q_\alpha}^i, T_r] &= (S_r)^i{}_j {Q_\alpha}^j, && [\bar{Q}_{\dot\alpha i}, T^r] = -(S^{*r})_i{}^j \bar{Q}_{\dot\alpha j}.
\end{aligned} \tag{1.3}$$

The objects Z_{ij} are called central charges. They are antisymmetric in the indices i, j, and they commute with every element of the algebra, including themselves. The central charges can be nonzero only if there is more than one supercharge. If there are central charges, they must be of the form

$$Z^{ij} = (a^r)^{ij}T_r. \tag{1.4}$$

The matrices $(S_r)^i{}_j$ form a representation of the internal symmetry group. The matrix $(a^r)^{ij}$ is an intertwiner of the representation S_r with the complex conjugate representation $-S_r^*$. Central charges can only exist if the algebra of the internal symmetry group allows such an intertwiner [1].

The form of the supersymmetry algebra remains unchanged under unitary transformations of the supercharges among themselves – that is,

$${Q_\alpha}^i \to U^i{}_j {Q_\alpha}^j, \qquad \bar{Q}_{\dot\alpha i} \to \bar{Q}_{\dot\alpha j}(U^{-1})^j{}_i, \qquad Z_{ij} \to U^i{}_k U^j{}_l Z_{kl}.$$

These transformations go by the name of R-symmetry. In the case of a single supercharge, the R-symmetry group is $U(1)$. Through a suitable rescaling, the commutators of the supercharges with the generator R of this group can be taken as

$$[R, Q_\alpha] = -Q_\alpha, \qquad [R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}. \tag{1.5}$$

1.2 Representations of the supersymmetry algebra

We saw in the previous section that the supercharges ${Q_\alpha}^i$ belong to the spin-$\frac{1}{2}$ representation of the Lorentz group. Acting on a state of spin j, a supercharge therefore gives a linear combination of states of $j \pm \frac{1}{2}$. Thus, supersymmetry is a symmetry which relates particles of integer spin and particles of half-integer spin – in other words, it relates bosons and fermions.

We will first prove two general properties of the representations of the supersymmetry algebra. In any finite-dimensional linear representation of the algebra, there are equally many bosonic and fermionic degrees of freedom. To prove this, consider the operator $(-1)^{N_F}$, which gives $+1$ on bosonic states and -1 on fermionic states. Therefore

$$(-1)^{N_F} Q_\alpha = -Q_\alpha (-1)^{N_F}. \tag{1.6}$$

Now, consider $\mathrm{Tr}\big[(-1)^{N_F}\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\}\big]$, where the trace is taken over the states belonging to the representation. We have

$$\begin{aligned}\mathrm{Tr}\left[(-1)^{N_F}\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\}\right] &= \mathrm{Tr}\left[(-1)^{N_F}(Q_\alpha{}^i \bar{Q}_{\dot\beta j} + \bar{Q}_{\dot\beta j} Q_\alpha{}^i)\right] \\ &= \mathrm{Tr}\left[-Q_\alpha{}^i (-1)^{N_F} \bar{Q}_{\dot\beta j} + Q_\alpha{}^i (-1)^{N_F} \bar{Q}_{\dot\beta j}\right] = 0.\end{aligned}$$

On the other hand, we know from the supersymmetry algebra what $\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\}$ is, and so we find

$$0 = \mathrm{Tr}\left[(-1)^{N_F}\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\}\right] = 2\delta^i_j (\sigma_\mu)_{\alpha\dot\beta} \mathrm{Tr}\big[(-1)^{N_F} P_\mu\big].$$

On taking $i = j$ and choosing any nonzero momentum P_μ, this implies

$$\mathrm{Tr}\,(-1)^{N_F} = 0, \tag{1.7}$$

showing that there are equally many bosonic and fermionic degrees of freedom in the representation. An exception to this rule is given by the so-called non-linear realizations (see e.g. [1]). In this case, the supercharges do not act as linear operators, and so the above argument does not apply, since a trace operation cannot be defined.

Another general feature of the representations of the supersymmetry algebra is that all the particles in an irreducible representation have the same mass. This follows immediately because P_μ commutes with $Q_\alpha{}^i$, and so P^2 is a Casimir operator of the algebra. Obviously not all the particles in Nature have the same mass. Therefore one needs to find a way to break supersymmetry while not losing all the attractive features of the theory, before one can attempt to use supersymmetry to give a reasonable description of Nature.

As an aside, an interesting alternative way to discover the above features of supersymmetry is to consider a general field theory, and require that the theory is free of ultraviolet divergences, which is one of the nice properties of supersymmetric field theories. The vacuum energy of a particle of spin j and mass m_j is given by

$$\begin{aligned}&\frac{1}{2}(-1)^{2j}(2j+1)\int d^3k\,\sqrt{k^2+m_j^2} \\ &\qquad = \frac{1}{2}(-1)^{2j}(2j+1)\int d^3k\,\sqrt{k^2}\left[1 + \frac{1}{2}\frac{m_j^2}{k^2} - \frac{1}{8}\left(\frac{m_j^2}{k^2}\right)^2 + \dots\right].\end{aligned}$$

The theory does not have the quartic, quadratic and logarithmic divergences provided that

$$\sum_j (-1)^{2j}(2j+1) = 0$$

and

$$\sum_j (-1)^{2j}(2j+1)m_j^2 = 0, \qquad \sum_j (-1)^{2j}(2j+1)m_j^4 = 0.$$

The first condition requires that there are equally many bosonic and fermionic degrees of freedom in the theory. If the first condition is satisfied, then the other conditions are also satisfied, provided that all the particles have the same mass. According to West [2], this was observed a long time ago by Pauli.

We will now construct the representations of the supersymmetry algebra, and so we will find the particle content of supersymmetric field theories. For the moment, we assume that the central charges vanish. The case where they do not vanish will be discussed briefly in the end. We first consider representations corresponding to massless particles. We go to the frame where $P_\mu = (-E, 0, 0, E)$. Then the supersymmetry algebra is

$$\begin{aligned} \{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\} &= \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^i_j, \\ \{Q_\alpha{}^i, Q_\beta{}^j\} = 0, &\qquad \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} = 0. \end{aligned} \tag{1.8}$$

We see that $Q_1{}^i$ and $\bar{Q}_{\dot{1}i}$ anticommute with all the Q's and $\bar{Q}$'s, so they must be equal to zero. We are left with N supercharges, and we rescale them according to

$$a^i = \frac{1}{2\sqrt{E}} Q_2{}^i, \qquad (a^i)^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{2}}{}^i, \tag{1.9}$$

and use Eq. (1.8) to show that they obey the algebra of N fermionic creation and annihilation operators,

$$\begin{aligned} \{a^i, (a^j)^\dagger\} &= \delta^{ij}, \\ \{a^i, a^j\} = 0, &\qquad \{(a^i)^\dagger, (a^j)^\dagger\} = 0. \end{aligned} \tag{1.10}$$

Now we would like to show that $(a^i)^\dagger$ raises the helicity of a state by $\frac{1}{2}$, while a^i lowers it by $\frac{1}{2}$. With our choice of coordinate system, the helicity operator is just $J_3 = M_{12}$. Using the relevant commutators from Eq. (1.3) we indeed find

$$[J_3, a^i] = -\tfrac{1}{2} a^i, \qquad [J_3, (a^i)^\dagger] = +\tfrac{1}{2}(a_i)^\dagger. \tag{1.11}$$

The representations of this algebra are constructed as follows. We choose a state of lowest helicity $|\Omega_{\lambda_-}\rangle$, having helicity λ_-, which is annihilated by all the a^i. Then we construct all the states by applying the creation operators on the state of lowest helicity:

$$|\lambda_- + \tfrac{1}{2}n;\, i_1 \dots i_n\rangle = \frac{1}{\sqrt{n!}} (a^{i_1})^\dagger \dots (a^{i_n})^\dagger |\Omega_{\lambda_-}\rangle. \tag{1.12}$$

The states which are constructed using n creation operators have helicity $\lambda_- + \frac{1}{2}n$, and they are antisymmetric in the indices $i_1 \dots i_n$. There are $\binom{N}{n}$ such states. Because there are N different creation operators, the state of highest helicity has helicity $\lambda_+ = \lambda_- + \frac{1}{2}N$, and so the total number of states is

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N.$$

Moreover, there are equally many bosonic and fermionic degrees of freedom, since the binomial coefficients satisfy the identity

$$\sum_{n=0}^{N/2} \binom{N}{2n} = \sum_{n=0}^{N/2} \binom{N}{2n+1}.$$

If we require a CPT-invariant theory, we will usually have twice as many states, because CPT reverses the helicity, and so we need to also take the multiplet with opposite helicities. The multiplets with $(N, \lambda_-) = (2, -\frac{1}{2}), (4, -1)$ and $(8, -2)$ are CPT-complete by themselves. The theory with $N = 8$ has particles of all spins between 0 and 2. It is called the maximally extended theory of supergravity. It is, in some sense, the largest possible supersymmetric field theory, because in a theory with $N > 8$, there would be particles having spin greater than 2, and it is believed to be impossible to construct a consistent quantum field theory involving interacting particles of spin 5/2 or greater.

For $N = 1$, each multiplet consists of only the two states $|\Omega_{\lambda_-}\rangle$ and $a^\dagger|\Omega_{\lambda_-}\rangle$. We will denote this multiplet by $(\lambda_-, \lambda_- + \frac{1}{2})$. Then we can have the following multiplets:

- The chiral multiplet, which consists of $(0, \frac{1}{2})$ and its CPT conjugate $(-\frac{1}{2}, 0)$. This corresponds to a complex scalar field and a Weyl fermion.
- The vector multiplet, consisting of $(\frac{1}{2}, 1)$ and $(-1, -\frac{1}{2})$, which corresponds to a Weyl fermion and a massless vector boson.
- The gravitino multiplet consists of $(1, \frac{3}{2})$ and $(-\frac{3}{2}, -1)$, corresponding to a massless vector boson and a spin-3/2 gravitino.
- The supergravity multiplet, which contains $(\frac{3}{2}, 2)$ and $(-2, -\frac{3}{2})$, that is, the graviton and the gravitino.

Consider then representations corresponding to massive particles. Now $P^2 = -M^2$, and we may go into the rest frame where $P_\mu = (-M, 0, 0, 0)$. The supersymmetry algebra becomes

$$\begin{aligned} \{Q_\alpha{}^i, \bar{Q}_{\dot{\beta}j}\} &= 2M\delta_{\alpha\dot{\beta}}\delta^i_j, \\ \{Q_\alpha{}^i, Q_\beta{}^j\} = 0, &\qquad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0. \end{aligned} \tag{1.13}$$

We again define the scaled supercharges

$$a_\alpha{}^i = \frac{1}{\sqrt{2M}} Q_\alpha{}^i, \qquad (a_\alpha{}^i)^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha}i}, \tag{1.14}$$

and find

$$\{a_\alpha{}^i, (a_\beta{}^j)^\dagger\} = \delta_{\alpha\beta}\delta^{ij},$$
$$\{a_\alpha{}^i, a_\beta{}^j\} = 0, \qquad \{(a_\alpha{}^i)^\dagger, (a_\beta{}^j)^\dagger\} = 0. \tag{1.15}$$

To construct the representations of this algebra, we start from a vacuum state $|\Omega\rangle$, which is annihilated by all of the $a_\alpha{}^i$, and which satisfies $P^2|\Omega\rangle = -M^2|\Omega\rangle$. We then apply the creation operators $(a_\alpha{}^i)^\dagger$ to the vacuum, so finding the states

$$|\alpha_1\, i_1\, \ldots\, \alpha_n i_n\rangle = \frac{1}{\sqrt{n!}}(a_{\alpha_1}{}^{i_1})^\dagger \ldots (a_{\alpha_n}{}^{i_n})^\dagger|\Omega\rangle. \tag{1.16}$$

We have $2N$ different creation operators, so n cannot be larger than $2N$. For a given n, we can construct $\binom{2N}{n}$ different states. Therefore the total number of states in the representation is 2^{2N}.

In general, we may take the vacuum to be a spin-s state $|\Omega_s\rangle$, so it is a $2s+1$-dimensional $SU(2)$ multiplet. We can again use the commutators from Eq. (1.3) to show that

$$[J_3, (a_1{}^i)^\dagger] = -\tfrac{1}{2}(a_1{}^i)^\dagger, \qquad [J_3, (a_2{}^i)^\dagger] = +\tfrac{1}{2}(a_2{}^i)^\dagger. \tag{1.17}$$

If the vacuum has spin s, the multiplet will then consist of states with spins $j - \frac{1}{2}, j, j + \frac{1}{2}$. For $N = 1$, these are given by

$$|\Omega_s\rangle, \qquad (a_\alpha)^\dagger|\Omega_s\rangle, \qquad \frac{1}{\sqrt{2}}(a_1)^\dagger(a_2)^\dagger|\Omega_s\rangle. \tag{1.18}$$

The vacuum state and the state with two creation operators have the same Bose–Fermi parity, which is different from that of the states with one creation operator. Since the vacuum has $2s+1$ components, there are $2(2s+1)$ bosonic states as well as $2(2s+1)$ fermionic states, so the balance between bosonic and fermionic states is satisfied.

For example, if $N = 1$ and $s = 0$, there will be two states with spin 0 and two states with spin 1/2, one of them having $J_3 = \frac{1}{2}$ and the other having $J_3 = -\frac{1}{2}$. The spin-0 state $\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle$ is a pseudoscalar, because parity interchanges $a_1^\dagger$ and $a_2^\dagger$. Therefore this representation contains a real scalar field, a real pseudoscalar field (or, equivalently, a complex scalar field), and a Weyl fermion.

As a final example, consider $N = 2$ and $s = 0$. We have four different creation operators, and there will be a total of 16 states. They are

- One spin-0 state $|\Omega\rangle$,
- Four spin-1/2 states $(a_\alpha{}^i)^\dagger|\Omega\rangle$,
- Six states $(a_{\alpha_1}{}^{i_1})^\dagger(a_{\alpha_2}{}^{i_2})^\dagger|\Omega\rangle$, which come out to be three spin-0 states and three spin-1 states [5],

- Four spin-1/2 states $(a_{\alpha_1}{}^{i_1})^\dagger (a_{\alpha_2}{}^{i_2})^\dagger (a_{\alpha_3}{}^{i_3})^\dagger |\Omega\rangle$,
- One spin-0 state $(a_1{}^1)^\dagger (a_1{}^2)^\dagger (a_2{}^1)^\dagger (a_2{}^2)^\dagger |\Omega\rangle$.

So, in total there are five spin-0 states, eight spin-1/2 states and three spin-1 states. These are the degrees of freedom of five real scalar fields, four Weyl spinors and one massive vector boson.

We will then consider the case with nonvanishing central charges. We will shortly see that in the case of massless representations the central charges must vanish, so we take $P_\mu = (-M, 0, 0, 0)$. The supersymmetry algebra then reads

$$\begin{gathered} \{Q_\alpha{}^i, \bar{Q}_{\dot{\beta}j}\} = 2M\delta_{\alpha\dot{\beta}}\delta^i_j, \\ \{Q_\alpha{}^i, Q_\beta{}^j\} = \epsilon_{\alpha\beta} Z^{ij}, \qquad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij}. \end{gathered} \tag{1.19}$$

Through a suitable unitary transformation on the supercharges, the central charges can be put in the form

$$Z^{ij} = \begin{pmatrix} 0 & z_1 & & & \\ -z_1 & 0 & & & \\ & & 0 & z_2 & \\ & & -z_2 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{1.20}$$

where the elements outside of the diagonal are all zero, and all the z_i are positive. Without any real loss of generality, we may assume that N is even. Then i takes values from 1 to $N/2$. The case of N odd can be treated analogously, since there will merely be an extra zero on the diagonal of the matrix Z^{ij}.

We now define the operators

$$\begin{aligned} a^i_\alpha &= \frac{1}{\sqrt{2}} \left(Q^{2i-1}_\alpha + \epsilon_{\alpha\beta} \left(Q^{2i}_\beta \right)^\dagger \right), \\ b^i_\alpha &= \frac{1}{\sqrt{2}} \left(Q^{2i-1}_\alpha - \epsilon_{\alpha\beta} \left(Q^{2i}_\beta \right)^\dagger \right), \end{aligned} \tag{1.21}$$

which obey the algebra

$$\begin{aligned} \{a^i_\alpha, (a^j_\beta)^\dagger\} &= (2M - z_i)\delta^{ij}\delta_{\alpha\beta}, \\ \{b^i_\alpha, (b^j_\beta)^\dagger\} &= (2M + z_i)\delta^{ij}\delta_{\alpha\beta}, \end{aligned} \tag{1.22}$$

where there is no sum over i, and all the anticommutators between an a operator and a b operator vanish. The operator $\{a^i_\alpha, (a^j_\beta)^\dagger\}$ is positive definite, and so the condition

$$z_i \leq 2M \tag{1.23}$$

must be satisfied for all i. In particular, in the massless case all the central charges vanish. If there are k of the z_i's which obey the equality in (1.23), then the corresponding operators must be equal to zero. There will then remain $2(N-k)$ creation and annihilation operators, giving rise to a so-called short multiplet, which contains $2^{2(N-k)}(2s+1)$ states, assuming the vacuum has spin s.

For the rest of this chapter, we will only consider theories with $N=1$, postponing the discussion of the case $N>1$ to later chapters.

1.3 An example: the Wess-Zumino model

In this section, we will illustrate a different, perhaps less abstract method for constructing a representation of the supersymmetry algebra. We will consrtuct a supersymmetric field theory which contains at least a scalar field ϕ. In order to ensure that the fields in the theory transform correctly under supersymmetry transformations, we will introduce more fields into the theory, until we have enough fields so that they form a complete multiplet, whose fields transform among themselves under supersymmetry [1].

We define the supersymmetry transformation of any field φ as

$$\delta_\epsilon \varphi = -\big(\epsilon Q + \bar{\epsilon}\bar{Q}\big)\varphi.$$

The parameter ϵ of supersymmetry transformations is a Majorana spinor, which appears in the above equation in the equivalent form of two Weyl spinors. We now consider supersymmetry transformations on the scalar field ϕ. The supersymmetry algebra requires that the commutator of two such supersymmetry transformations is given by

$$\begin{aligned}\big(\delta_\epsilon\delta_\xi - \delta_\xi\delta_\epsilon\big)\phi &= \big[\epsilon Q, \bar{\xi}\bar{Q}\big]\phi - \big[\xi Q, \bar{\epsilon}\bar{Q}\big]\phi \\ &= -2i\big(\epsilon\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\epsilon}\big)\partial_\mu\phi. \end{aligned} \tag{1.24}$$

From the algebra $\{Q,\bar{Q}\} \sim P$, the supercharge Q has mass dimension $\frac{1}{2}$. Therefore, Q may transform a field of dimension d into a field of dimension $d+\frac{1}{2}$ and into derivatives of fields of lower dimension. We define a fermion field ψ to be the field into which the scalar field gets transformed,

$$\delta_\epsilon\phi = \epsilon\psi, \qquad \delta_\epsilon\bar{\phi} = \bar{\epsilon}\bar{\psi}. \tag{1.25}$$

To satisfy Eq. (1.24), we must take

$$\begin{aligned}\delta_\epsilon\psi_\alpha &= 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu\phi + 2\epsilon_\alpha F, \\ \delta_\epsilon\bar{\psi}_{\dot{\alpha}} &= -2i\epsilon^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\phi} + 2\bar{\epsilon}_{\dot{\alpha}}\bar{F}.\end{aligned} \tag{1.26}$$

Here we introduced a third field F. We may do so because the terms involving F cancel in the commutator (1.24). Now we must choose the transformation of F so that the commutator $(\delta_\epsilon\delta_\xi - \delta_\xi\delta_\epsilon)\psi$ obeys the supersymmetry algebra. We have

$$\begin{aligned}\big(\delta_\epsilon\delta_\xi - \delta_\xi\delta_\epsilon\big)\psi_\alpha =& -2i\big(\epsilon\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\epsilon}\big)\partial_\mu\psi_\alpha \\ & - i(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\partial_\nu\psi_\beta\big(\epsilon\sigma_\mu\bar{\xi} - \xi\sigma_\mu\bar{\epsilon}\big) + 2\big(\xi_\alpha\delta_\epsilon F - \epsilon_\alpha\delta_\xi F\big).\end{aligned}$$

The terms in the second line cancel if we take

$$\delta_\epsilon F = i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \psi, \qquad \delta_\epsilon \bar{F} = i\epsilon\sigma^\mu \partial_\mu \bar{\psi}. \tag{1.27}$$

Using Eqs. (1.26) and (1.27) we can now show that

$$\big(\delta_\epsilon \delta_\xi - \delta_\xi \delta_\epsilon\big) F = -2i\big(\epsilon\sigma^\mu \bar{\xi} - \xi\sigma^\mu \bar{\epsilon}\big)\partial_\mu F. \tag{1.28}$$

So we have found a multiplet (ϕ, ψ, F) on which the supersymmetry algebra closes. At this point there is no need to introduce any more fields. Note that in a supersymmetry transformation F transforms by a total derivative. This will always be the case for the field with highest mass dimension in any multiplet.

The field F has mass dimension 2. The Lagrangian for this theory cannot have a kinetic term for F, because F^2 already has mass dimension 4. Therefore F will be an auxiliary field – it has an algebraic equation of motion, which can be used to eliminate F from the Lagrangian. Using the transformation laws for the different fields, we can show that the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m \tag{1.29}$$

with

$$\begin{aligned} \mathcal{L}_0 &= -\partial_\mu \bar{\phi}\partial^\mu \phi + \frac{i}{2}\partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi + \bar{F}F, \\ \mathcal{L}_m &= m\big(\phi F + \bar{\phi}\bar{F}\big) - \frac{m}{4}\big(\psi\psi + \bar{\psi}\bar{\psi}\big), \end{aligned} \tag{1.30}$$

transforms into a total derivative under a supersymmetry transformation. Note the unconventional normalization of the fermion terms. One can also take the standard normalization, provided that Eq. (1.26) is replaced by $\delta_\epsilon \phi = \sqrt{2}\epsilon\psi$, and the transformation laws of ψ and F are adjusted accordingly.

The Lagrangian gives rise to the following equations of motion:

$$\begin{aligned} \Box\phi + m\bar{F} &= 0, \\ i\bar{\sigma}^\mu \partial_\mu \psi + m\bar{\psi} &= 0, \\ F + m\bar{\phi} &= 0. \end{aligned} \tag{1.31}$$

Using the last equation, we can eliminate F from the Lagrangian, which then becomes

$$\mathcal{L} = -\partial_\mu \bar{\phi}\partial^\mu \phi - m^2 \bar{\phi}\phi + \frac{i}{2}\partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi - \frac{m}{4}\left(\psi\psi + \bar{\psi}\bar{\psi}\right). \tag{1.32}$$

The theory defined by (1.29) or (1.32) is known as the Wess–Zumino model. It was constructed by Wess and Zumino in 1974 [14]. It is possible to include interaction terms in the Lagrangian, but we will not do so yet.

Under a supersymmetry transformation, the Lagrangian (1.32) transforms into a total derivative only if the equation of motion of the fermion is satisfied. We say that supersymmetry is only realized on-shell, and the fields (ϕ, ψ) give an on-shell representation of supersymmetry. To obtain (1.32), we set $F = -m\bar{\phi}$, and for this to be consistent under a supersymmetry transformation, we must require $\delta_\epsilon F = -m\delta_\epsilon\bar{\phi}$. From Eqs. (1.26) and (1.27) we see that this is equivalent to requiring that $i\bar{\sigma}^\mu\partial_\mu\psi = -m\bar{\psi}$, which is exactly the fermion equation of motion.

In the on-shell formulation of the Wess–Zumino model, the supersymmetry transformation of the fermion is given by

$$\delta_\epsilon\psi_\alpha = 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu\phi - 2m\bar{\phi}\epsilon_\alpha \tag{1.33}$$

The commutator of two supersymmetry transformations on the fermion will then read

$$\begin{aligned}\big(\delta_\epsilon\delta_\xi - \delta_\xi\delta_\epsilon\big)\psi_\alpha = & -2i\big(\epsilon\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\epsilon}\big)\partial_\mu\psi_\alpha \\ & - \big(\epsilon\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\epsilon}\big)(\sigma_\mu)_{\alpha\dot{\alpha}}\left(i\bar{\sigma}^\nu\partial_\nu\psi + m\bar{\psi}\right)^{\dot{\alpha}}.\end{aligned}$$

The supersymmetry transformations therefore close on the fields (ϕ, ψ) only if the fermion equation of motion is satisfied. In many supersymmetric field theories, in particular when $N > 1$, one has to be satisfied with having to use the equations of motion to close the supersymmetry transformations, because the appropriate set of auxiliary fields for the theory may not be known.

1.4 Superspace and superfields

The Wess–Zumino model is arguably the simplest example of a supersymmetric field theory. Even so, supersymmetry is not at all manifest in the Lagrangian (1.29), and it is not a trivial calculation to show that the Lagrangian does transform into a total derivative under a supersymmetry transformation. We would therefore like to have a notation where invariance under supersymmetry becomes manifest. Such a notation is provided by the superspace formalism, which was introduced by Salam and Strathdee in 1974 [15, 16]. The superspace notation is in many ways superior to the notation used in the previous section, much like the manifestly Lorentz invariant notation $\partial_\mu\bar{\phi}\partial^\mu\phi$ is superior to something like $|\dot{\phi}|^2 - |\nabla\phi|^2$. In particular, it is easy to consruct Lagrangians for supersymmetric field theories using the superspace formalism.

A point $(x, \theta, \bar{\theta})$ in superspace is represented by the element of the supersymmetry group

$$G(x, \theta, \bar{\theta}) = \exp\big(-ix^\mu P_\mu - i\theta Q - i\bar{\theta}\bar{Q}\big). \tag{1.34}$$

(This is in complete analogy with the correspondence between the point x^μ in Minkowski space and the group element $\exp(-ix^\mu P_\mu)$ of the Poincaré group, as expressed by Eq. (A.29) in Appendix 3.) Using the identity $e^A e^B = e^{(A+B+[A,B]/2+\dots)}$, we can show that left multiplication in the group has the effect

$$G(0, \xi, \bar{\xi})G(x, \theta, \bar{\theta}) = G\big(x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}\big). \tag{1.35}$$

The operators Q and $\bar{Q}$ generate translations in superspace. From Eq. (1.35) we read off that they are represented by the differential operators

$$\begin{aligned} Q_\alpha &= \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \\ \bar{Q}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \end{aligned} \tag{1.36}$$

They obey the algebra

$$\begin{gathered} \{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \end{gathered} \tag{1.37}$$

So we see that left multiplication in the group leads to P_μ being represented by $+i\partial_\mu$. The reason for this is that group multiplication corresponds to a superspace translation with a reversed order of multiplication. For example, $G(0,\xi,\bar{\xi})G(0,\eta,\bar{\eta})$ corresponds to the translation $(x,\theta,\bar{\theta}) \to (x+\ldots,\theta+\xi,\bar{\theta}+\bar{\xi}) \to (x+\ldots,\theta+\xi+\eta,\bar{\theta}+\bar{\xi}+\bar{\eta})$. This is more or less a historical accident. Had superspace translations been chosen to be represented by right multiplication, they would have been generated by the operators

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \\ \bar{D}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu, \end{aligned} \tag{1.38}$$

which satisfy the algebra

$$\begin{gathered} \{D_\alpha, D_\beta\} = 0, \qquad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \\ \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \end{gathered} \tag{1.39}$$

Furthermore, all of the Q operators anticommute with all of the D operators. We will soon find that the operators D and $\bar{D}$ have a task to fulfill as the so-called supercovariant derivatives.

Superfield are functions defined on superspace. A superfield $\Phi(x,\theta,\bar{\theta})$ can be expanded as a series in θ and $\bar{\theta}$, and because θ and $\bar{\theta}$ both have only two independent components, the expansion has only a finite number of terms:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x) + \theta\chi(x) + \bar{\theta}\bar{\xi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu(x) \\ &\quad + \theta^2 F(x) + \bar{\theta}^2 G(x) + \bar{\theta}^2\theta\eta(x) + \theta^2\bar{\theta}\bar{\zeta}(x) + \theta^2\bar{\theta}^2 D(x). \end{aligned} \tag{1.40}$$

The action of a supersymmetry transformation on superfields is given by

$$\delta_\epsilon\Phi = (\epsilon Q + \bar{\epsilon}\bar{Q})\Phi = (\epsilon^\alpha Q_\alpha - \bar{\epsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}})\Phi. \tag{1.41}$$

The transformation laws for the various fields appearing in Eq. (1.40) can be derived by carrying out the calculation on the right using Eq. (1.36), and comparing powers of θ and $\bar{\theta}$ on each side. In particular, we have

$$\delta_\epsilon D(x) = \frac{i}{2}\partial_\mu\big(\epsilon\sigma^\mu\bar{\zeta}(x) - \bar{\epsilon}\bar{\sigma}^\mu\eta(x)\big)\,. \tag{1.42}$$

We see that $D(x)$, the component field of highest mass dimension in Eq. (1.40), transforms by a total derivative, as already advertized in section 1.3. This enables one to use superfields to construct Lagrangians that are invariant under supersymmetry transformations with relatively little effort. We will return to this point in the next section.

Linear combinations of superfields, as well as their products, are also superfields. However, the derivative of a superfield with respect to θ or $\bar{\theta}$ is not a superfield: it does not transform according to Eq. (1.41) in supersymmetry transformations, because the derivatives ∂_α and $\bar{\partial}_{\dot{\alpha}}$ do not anticommute with the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$. However, the operators D_α and $\bar{D}_{\dot{\alpha}}$ anticommute with Q_α and $\bar{Q}_{\dot{\alpha}}$, and so if Φ is a superfield, then $D_\alpha \Phi$ and $\bar{D}_{\dot{\alpha}}\Phi$ are also superfields. For this reason, D_α and $\bar{D}_{\dot{\alpha}}$ are called supercovariant derivatives.

Because linear combinations and products of superfields are also superfields, superfields form representations of the supersymmetry algebra. Usually these representations are highly reducible. We may reduce the size of the representation by introducing various constraints. These constraints need to be invariant under supersymmetry transformations, in order for the reduced set of fields to still form a representation of the supersymmetry algebra. Such constraints are, for example, $\bar{D}_{\dot{\alpha}}\Phi = 0$, and $\bar{\Phi} = \Phi$.

Superfields satisfying the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{1.43}$$

are called chiral superfields, or scalar superfields. Any function of the variables $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and θ satisfies Eq. (1.43), because $\bar{D}_{\dot{\alpha}} y^\mu = 0$ and $\bar{D}_{\dot{\alpha}}\theta = 0$. Therefore

$$\begin{aligned}\Phi &= \phi(y) + \theta\psi(y) + \theta^2 F(y) \\ &= \phi(x) + \theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) \\ &\qquad + \frac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box F(x).\end{aligned} \tag{1.44}$$

This is in fact the most general chiral superfield. It contains precisely the fields which appear in the Wess–Zumino model. Using Eqs. (1.36) and (1.41), one can reproduce the transformation laws (1.24), (1.26) and (1.27). It simplifies the calculation somewhat if we express the supercharges in terms of the y coordinate as

$$Q_\alpha = \partial_\alpha, \qquad \bar{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu,$$

where ∂_μ now denotes the derivative with respect to y^μ.

The conjugate superfield $\bar{\Phi}$ satisfies $D_\alpha\bar{\Phi} = 0$. Superfields satisfying this condition are called antichiral superfields. Any superfield which is a function of

$\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$ and $\bar{\theta}$ is an antichiral superfield. The most general antichiral superfield is given by the complex conjugate of Eq. (1.44).

Superfields which satisfy the constraint

$$\bar{V} = V \tag{1.45}$$

are called vector superfields, or real superfields. The most general vector superfield is given by

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu(x) \\ &+ \theta^2 G(x) + \bar{\theta}^2\bar{G}(x) + \bar{\theta}^2\theta\eta(x) + \theta^2\bar{\theta}\bar{\eta}(x) + \theta^2\bar{\theta}^2 E(x), \end{aligned} \tag{1.46}$$

where the fields C, v_μ and E are real. Among its components there is the vector field v_μ, after which the whole superfield is named. The real part of a chiral superfield Λ,

$$\begin{aligned} \Lambda + \bar{\Lambda} = \phi + \bar{\phi} + \theta\psi + \bar{\theta}\bar{\psi} + \theta^2 F + \bar{\theta}^2\bar{F} + i\theta\sigma^\mu\bar{\theta}\partial_\mu(\phi - \bar{\phi}) \\ + \frac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi + \frac{i}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{4}\theta^2\bar{\theta}^2\Box(\phi + \bar{\phi}), \end{aligned}$$

is a special kind of a vector superfield, whose vector component is a spacetime derivative. This suggests that we define the superspace generalization of the usual $U(1)$ gauge transformation as

$$V \to V + \Lambda + \bar{\Lambda}, \tag{1.47}$$

where Λ is a chiral superfield. The effect of this transformation on the various fields is

$$\begin{aligned} C &\to C + \phi + \bar{\phi}, \\ \chi &\to \chi + \psi, \\ G &\to G + F, \\ v_\mu &\to v_\mu - i\partial_\mu(\phi - \bar{\phi}), \\ \eta &\to \eta + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\psi}, \\ E &\to E + \frac{1}{4}\Box(\phi + \bar{\phi}). \end{aligned} \tag{1.48}$$

First of all, we see that the fields

$$\lambda = \eta - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}, \qquad D = E - \frac{1}{4}\Box C \tag{1.49}$$

are gauge invariant. Secondly, we see that it is possible to choose a gauge where the fields C, χ and G all vanish. In this gauge, known as the Wess–Zumino gauge, the vector superfield takes the form

$$V = \bar{\theta}\bar{\sigma}^\mu\theta v_\mu(x) + \bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \theta^2\bar{\theta}^2 D(x). \tag{1.50}$$

Powers of V are now easy to calculate:

$$V^2 = -\frac{1}{2}\theta^2\bar{\theta}^2 v^\mu v_\mu, \qquad V^3 = 0. \tag{1.51}$$

The Wess–Zumino gauge still allows us to make the normal gauge transformations in the vector field, since a gauge transformation (1.47) with $\phi = -\bar{\phi}$, and $\psi = 0$, $F = 0$, preserves the Wess–Zumino gauge, while inducing on the vector field the transformation

$$v_\mu \to v_\mu + \partial_\mu \alpha \tag{1.52}$$

with $\alpha = -i(\phi - \bar{\phi})$ being a real field.

1.5 Lagrangians from the superspace formalism

In this section, we will see how superfields can be used to write down Lagrangians which are invariant under supersymmetry transformations. The important fact which enables us to do so is that the highest component of any superfield transforms by a total derivative in a supersymmetry transformation. In fact, all possible renormalizable supersymmetric Lagrangians can be constructed using only chiral superfields and vector superfields.

To begin with, we will define integration in superspace. Integration involving a single anticommuting variable θ is conventionally defined as

$$\int d\theta = 0, \qquad \int d\theta\, \theta = 1. \tag{1.53}$$

This can be generalized to the superspace variables θ^α and $\bar{\theta}_{\dot{\alpha}}$ without too much trouble. If we define

$$d^2\theta = -\frac{1}{4}\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta, \quad d^2\bar{\theta} = -\frac{1}{4}\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}, \quad d^4\theta = d^2\theta d^2\bar{\theta}, \tag{1.54}$$

we then find that we can integrate in superspace according to the rules

$$\int d^2\theta\, \theta^2 = 1, \qquad \int d^2\bar{\theta}\, \bar{\theta}^2 = 1, \qquad \int d^4\theta\, \theta^2\bar{\theta}^2 = 1. \tag{1.55}$$

In what follows, the use of superspace integration will be to pick out components of superfields that are invariant under supersymmetry, which can then be used to build invariant Lagrangians, remembering that products of multiple superfields are still superfields. For example, for any superfield Ψ, the Lagrangian $\mathcal{L} = \int d^4\theta\, \Psi$ is invariant under supersymmetry. We may then write the action in the aesthetically pleasing form as an integral over all of superspace,

$$S = \int d^4x\, \mathcal{L} = \int d^4x\, d^4\theta\, \Psi.$$

Let us now recover the Lagrangian for the Wess–Zumino model using superfields. We will need a chiral superfield Φ and its conjugate $\bar{\Phi}$. The kinetic term is given by

$$\mathcal{L}_0 = \int d^4\theta\, \bar{\Phi}\Phi = \bar{\phi}\Box\phi + \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi + \bar{F}F, \tag{1.56}$$

in agreement with Eq. (1.30) up to a partial integration. More generally, we can take a set of chiral superfields Φ^i, and consider the Lagrangian

$$\mathcal{L} = \int d^4\theta\, \bar{\Phi}_i \Phi^i + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W(\Phi)}, \tag{1.57}$$

where the so-called superpotential is

$$W(\Phi) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k, \tag{1.58}$$

and the coefficients m_{ij} and g_{ijk} are symmetric. Renormalizability requires that the superpotential cannot contain terms with more than three superfields, because the θ^2 component of W has mass dimension of one more than W itself, while the mass dimension of Φ is 1. The Lagrangian (1.57) will contain mass terms, as well as interaction terms, which we did not consider in section 1.3. Using

$$\begin{aligned}
\Phi^i \Phi^j &= \phi^i \phi^j + \theta(\phi^i \psi^j + \phi^j \psi^i) + \theta^2(\phi^i F^j + \phi^j F^i - \psi^i \psi^j), \\
\Phi^i \Phi^j \Phi^k &= \phi^i \phi^j \phi^k + \theta(\phi^i \phi^j \psi^k + \phi^i \phi^k \psi^j + \phi^j \phi^k \psi^i) \\
&\quad + \theta^2(\phi^i \phi^j F^k + \phi^i \phi^k F^j + \phi^j \phi^k F^i - \psi^i \psi^j \phi^k - \psi^i \psi^k \phi^j - \psi^j \psi^k \phi^i)
\end{aligned}$$

where all the component fields are functions of $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, we find that (1.57) becomes, up to total derivatives,

$$\begin{aligned}
\mathcal{L} &= \bar{\phi}_i \Box \phi^i + \frac{i}{2} \partial_\mu \bar{\psi}_i \sigma^\mu \psi^i + \bar{F}_i F^i \\
&\quad + \left[m_{ij} \left(\phi^i F^j - \frac{1}{4} \psi^i \psi^j \right) + g_{ijk} \left(\phi^i \phi^j F^k - \frac{1}{2} \psi^i \psi^j \phi^k \right) + a_i F^i + \text{h.c.} \right].
\end{aligned} \tag{1.59}$$

The equations of motion for the auxiliary fields F_i are

$$F^i + a^{*i} + m^{*ij} \bar{\phi}_j + g^{*ijk} \bar{\phi}_j \bar{\phi}_k = 0,$$

$$\bar{F}_i + a_i + m_{ij} \phi^j + g_{ijk} \phi^j \phi^k = 0.$$

With the help of these, we can put the Lagrangian into the form

$$\begin{aligned}
\mathcal{L} = \bar{\phi}_i \Box \phi^i &+ \frac{i}{2} \partial_\mu \bar{\psi}_i \sigma^\mu \psi^i - \frac{1}{4} m_{ij} \psi^i \psi^j - \frac{1}{4} m^{*ij} \bar{\psi}_i \bar{\psi}_j \\
&- \frac{1}{2} g_{ijk} \psi^i \psi^j \phi^k - \frac{1}{2} g^{*ijk} \bar{\psi}_i \bar{\psi}_j \bar{\phi}_k - V(\phi, \bar{\phi})
\end{aligned} \tag{1.60}$$

where

$$V(\phi, \bar{\phi}) = \bar{F}_i F^i = \sum_i |a_i + m_{ij} \phi^j + g_{ijk} \phi^j \phi^k|^2. \tag{1.61}$$

A still more general theory can be obtained through dropping the requirement of renormalizability, as well as allowing a more general form of the kinetic term

[1, 3]. We therefore take a set of chiral superfields Φ^i, and take the Lagrangian to be

$$\mathcal{L} = \int d^4\theta\, K(\Phi, \bar{\Phi}) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W(\Phi)}, \tag{1.62}$$

where $K(\Phi, \bar{\Phi})$ is a real function of its arguments. To express this Lagrangian in terms of the component fields, the functions K and W are expanded in power series as

$$K(\Phi, \bar{\Phi}) = \sum a_{i_1 \ldots i_m, j_1 \ldots j_n} \Phi^{i_1} \cdots \Phi^{i_m} \bar{\Phi}^{j_1} \cdots \bar{\Phi}^{j_n},$$

$$W(\Phi) = \sum b_{i_1 \ldots i_m} \Phi^{i_1} \cdots \Phi^{i_m}.$$

We introduce the geometrical notation

$$\begin{gathered}
g_{i\bar{j}} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^i \partial \bar{\phi}^j} \equiv \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}), \\
\Gamma^i_{jk} = g^{i\bar{l}} \partial_k g_{\bar{l}j}, \qquad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{i}l} \partial_{\bar{k}} g_{l\bar{j}}, \\
\nabla_\mu \bar{\psi}^{\bar{i}} = \partial_\mu \bar{\psi}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \partial_\mu \bar{\phi}^{\bar{j}} \bar{\psi}^{\bar{k}}, \\
\nabla_i W = \partial_i W, \qquad \nabla_i \nabla_j W = \partial_i \partial_j W - \Gamma^k_{ij} \partial_k W, \\
R_{ij\bar{k}\bar{l}} = \partial_j \partial_{\bar{l}} g_{i\bar{k}} - g^{m\bar{n}} \partial_i g_{j\bar{n}} \partial_{\bar{k}} g_{m\bar{l}}.
\end{gathered} \tag{1.63}$$

This allows us to express the result of a lengthy calculation, some of whose details are given in [1] and [3], as

$$\begin{aligned}
\mathcal{L} = &- g_{i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^j + \frac{i}{2} g_{i\bar{j}} \psi^i \sigma^\mu \nabla_\mu \bar{\psi}^j + \frac{1}{16} R_{ij\bar{k}\bar{l}} \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l \\
&- g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W} - \frac{1}{4} \psi^i \psi^j \nabla_i \nabla_j W - \frac{1}{4} \bar{\psi}^i \bar{\psi}^j \nabla_{\bar{i}} \nabla_{\bar{j}} \overline{W}.
\end{aligned} \tag{1.64}$$

This Lagrangian has the following geometrical interpretation. The scalar fields ϕ^i are interpreted as the coordinates of the so-called target space, which is a Riemannian manifold with the metric $ds^2 = g_{i\bar{j}} d\phi^i d\bar{\phi}^{\bar{j}}$. The spinor fields ψ^i play the role of vectors in the tangent space. A manifold whose metric is derived from a function K – called in this context a Kähler potential – according to Eq. (1.63), is called a Kähler manifold. The Lagrangian (1.64) therefore gives a supersymmetric generalization of a non-linear sigma model whose target space is a Kähler manifold.

1.6 Supersymmetric gauge theories

The vector superfield V is, in some sense, the supersymmetric generalization of the gauge field A_μ which appears in the usual gauge theories of particle physics. In this section, we will see how supersymmetric gauge theories are constructed with the help of vector superfields. We will first take up the case where the gauge group is $U(1)$, leading to a supersymmetric version of QED.

Then we will consider the generalization to non-abelian gauge groups. This will give us a supersymmetric generalization of Yang–Mills theory.

In a gauge theory, we have a gauge group G, and a gauge field A_μ, which takes values in the Lie algebra of G:

$$A_\mu = A_\mu^a T^a, \tag{1.65}$$

where T^a are the generators of G. The covariant derivative is defined as

$$\mathcal{D}_\mu = \partial_\mu - igA_\mu. \tag{1.66}$$

The field strength tensor then is

$$F_{\mu\nu} = -\frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{1.67}$$

Under a gauge transformation $U(x)$ in G, the covariant derivative should transform covariantly. This requirement fixes the transformation of the gauge field as

$$A_\mu \to \frac{i}{g}U^{-1}\mathcal{D}_\mu U = U^{-1}A_\mu U + \frac{i}{g}U^{-1}(\partial_\mu U).$$

The field strength then transforms covariantly: $F_{\mu\nu} \to U^{-1}F_{\mu\nu}U$.

Using the vector superfield V, we may define a spinor-valued superfield, which can be thought of as a supersymmetric field strength tensor, as

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V. \tag{1.68}$$

The superfield W_α is chiral – this follows immediately from its definition because $\bar{D}^3 = 0$. Moreover it is gauge invariant under the supersymmetric gauge transformation (1.52). Using the chirality of Λ, we have

$$W_\alpha \to -\frac{1}{4}\bar{D}^2 D_\alpha\big(V + \Lambda + \bar{\Lambda}\big) = W_\alpha - \frac{1}{4}\bar{D}^{\dot{\alpha}}\big(\bar{D}_{\dot{\alpha}}D_\alpha + D_\alpha\bar{D}_{\dot{\alpha}}\big)\Lambda = W_\alpha,$$

since $\bar{D}^{\dot{\alpha}}$ commutes with $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu$.

Since W_α is gauge invariant, we may calculate its component expansion in any gauge, in particular, in the Wess–Zumino gauge. We use Eq. (1.50), and write $V(x,\theta,\bar{\theta}) = V(y - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})$, where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ is the natural variable in which the chiral superfield W_α should be expanded. We find

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\theta_\beta F_{\mu\nu}(y) + \theta^2(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}}(y), \tag{1.69}$$

where $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$.

The Wess–Zumino gauge does not respect supersymmetry, in the sense that a vector superfield originally in the Wess–Zumino gauge does not remain in that gauge under a supersymmetry transformation. A supersymmetry transformation should therefore be accompanied with a compensating gauge transformation of the form (1.47), with Λ chosen so that the vector superfield gets transformed back into the Wess–Zumino gauge. So, the complete supersymmetry transformation of a vector superfield V now is

$$\delta_\epsilon V = -\big(\epsilon Q + \bar{\epsilon}\bar{Q}\big)V + \Lambda_\epsilon + \bar{\Lambda}_\epsilon. \tag{1.70}$$

To fix the parameters of Λ_ϵ, we note that

$$-\big(\epsilon Q + \bar{\epsilon}\bar{Q}\big)V = \theta\sigma^\mu\bar{\epsilon}v_\mu - \bar{\theta}\bar{\sigma}^\mu\epsilon v_\mu - \theta^2\bar{\epsilon}\bar{\lambda} - \bar{\theta}^2\epsilon\lambda + \dots,$$

where the dots represent additional terms which are in the Wess–Zumino gauge. If we therefore take

$$\Lambda_\epsilon = \phi + \theta\psi + \theta^2 F \tag{1.71}$$

with

$$\phi = 0, \qquad \psi^\alpha = (\sigma^\mu)^{\alpha\dot{\alpha}}\bar{\epsilon}_{\dot{\alpha}}v_\mu, \qquad F = \bar{\epsilon}\bar{\lambda}, \tag{1.72}$$

the superfield V remains in the Wess–Zumino gauge under the transformation (1.70).

Let us now turn to the construction of the Lagrangian for supersymmetric QED. Using W_α and $\overline{W}_{\dot{\alpha}}$, we may write down the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar{\theta}\,\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}\right)\\ &= -i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}D^2.\end{aligned} \tag{1.73}$$

This will be the kinetic part of our supersymmetric QED Lagrangian. To add mass terms and interaction terms, we introduce two chiral superfields Φ_+ and Φ_-, which transform under $U(1)$ gauge transformations as

$$\Phi_+ \to e^{ie\Omega}\Phi_+, \qquad \Phi_- \to e^{-ie\Omega}\Phi_-,$$

where Ω is a chiral superfield. The reason why Ω must appear here is that the superfield $e^{ie\omega(x)}\Phi$, with $\omega(x)$ a real function on Minkowski space, would no longer be a chiral superfield. However, now a kinetic term like $\bar{\Phi}_+\Phi_+$ is no longer gauge invariant – instead, it transforms into $\bar{\Phi}\Phi e^{i(\Omega-\bar{\Omega})}$. To remedy this, we use the vector superfield V, and require that it transforms under gauge transformations as

$$V \to V - i(\Omega - \bar{\Omega}), \tag{1.74}$$

in order to make the terms $\bar{\Phi}_\pm e^{\pm eV}\Phi_\pm$ gauge invariant. The superfield Ω is therefore related to Λ of Eq. (1.47) by $\Omega = i\Lambda$. With this, we write the full supersymmetric QED Lagrangian as

$$\begin{aligned}\mathcal{L}_{\text{QED}} = \frac{1}{4}&\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar{\theta}\,\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}\right)\\ &+ \int d^4\theta\,\left(\bar{\Phi}_+ e^{eV}\Phi_+ + \bar{\Phi}_- e^{-eV}\Phi_-\right)\\ &+ m\left(\int d^2\theta\,\Phi_+\Phi_- + \int d^2\bar{\theta}\,\bar{\Phi}_+\bar{\Phi}_-\right).\end{aligned} \tag{1.75}$$

Carrying out the integrals in $\mathcal{L}_{\text{QED}}$, and eliminating the auxiliary fields D and $F_{\pm}$, one would find that the Lagrangian describes a charged fermion of mass m and a neutral gauge boson – the electron the photon – as well as their supersymmetric partners, a charged scalar particle of mass m and a neutral massless fermion – the selectron and the photino [1].

We now go on to consider the case of a non-abelian gauge group G [3]. Take a chiral superfield Φ belonging to a unitary representation of G, which we take to be the adjoint representation. Under a gauge transformation, Φ transforms as

$$\Phi \to e^{i\Omega}\Phi, \qquad \bar{\Phi} \to \bar{\Phi}e^{-i\Omega}. \tag{1.76}$$

The proper generalization of the transformation law (1.74) reads

$$e^V \to e^{i\bar{\Omega}}e^V e^{-i\Omega}. \tag{1.77}$$

In these equations, Ω and V now take values in the Lie algebra of G – that is, they are the matrices

$$(\Omega)_{ij} = (T^a)_{ij}\Omega^a, \qquad (V)_{ij} = (T^a)_{ij}V^a,$$

with the generators T^a belonging to the adjoint representation of G:

$$[T^a, T^b] = if^{abc}T^c, \qquad \text{Tr}\, T^aT^b = \delta^{ab}.$$

To first order in Ω, the transformation (1.77) reduces to (1.74). As long as (1.77) is a symmetry of our Lagrangian, we may therefore work in the Wess–Zumino gauge, using (1.77) to put a vector superfield V in the form (1.50).

The definition of the field strength W_α is generalized in the non-abelian case to

$$W_\alpha = -\frac{1}{4}\bar{D}^2\big(e^{-V}D_\alpha e^V\big). \tag{1.78}$$

Under the gauge transformation (1.74), W_α transforms as

$$W_\alpha \to e^{i\Omega}W_\alpha e^{-i\Omega}. \tag{1.79}$$

The combination $\text{Tr}\, W^\alpha W_\alpha$ is therefore gauge invariant, and we may use it to construct the Lagrangian for supersymmetric Yang–Mills theory.

In the Wess–Zumino gauge, we have $V^3 = 0$, and consequently

$$e^V = 1 + V + \tfrac{1}{2}V^2.$$

Using this in Eq. (1.78), we find

$$W_\alpha = -\frac{1}{4}\bar{D}^2D_\alpha V + \frac{1}{8}\bar{D}^2[V, D_\alpha V],$$

which results in the component expansion

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu})_\alpha{}^\beta\theta_\beta F_{\mu\nu}(y) + \theta^2(\sigma^\mu)_{\alpha\dot{\alpha}}\mathcal{D}_\mu\bar{\lambda}^{\dot{\alpha}}(y), \tag{1.80}$$

where we now have

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu], \tag{1.81}$$

$$\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}} = \partial_\mu \bar{\lambda}^{\dot{\alpha}} - \frac{i}{2}[v_\mu, \bar{\lambda}^{\dot{\alpha}}]. \tag{1.82}$$

As of now, the Yang–Mills coupling constant is absent from the theory. To introduce the coupling constant g, we scale the vector superfield V by $2g$, so that the field strength W_α becomes

$$W_\alpha = -\frac{1}{4}\bar{D}^2\big(e^{-2gV} D_\alpha e^{2gV}\big). \tag{1.83}$$

Then the gauge transformation of Eq. (1.77) needs to be changed to

$$e^{2gV} \to e^{2ig\bar{\Omega}} e^{2gV} e^{-2ig\Omega}. \tag{1.84}$$

Furthermore, Eqs. (1.81) and (1.82) are replaced by

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu], \tag{1.85}$$

$$\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}} = \partial_\mu \bar{\lambda}^{\dot{\alpha}} - ig[v_\mu, \bar{\lambda}^{\dot{\alpha}}]. \tag{1.86}$$

We also define the quantity

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}, \tag{1.87}$$

where Θ is the theta parameter of Yang–Mills theory. The supersymmetric Yang–Mills Lagrangian is then given by

$$\begin{aligned} \mathcal{L}_{\rm SYM} &= \frac{1}{32\pi}\mathrm{Im}\left[\tau \int d^2\theta \,\mathrm{Tr}\, W^\alpha W_\alpha\right] \\ &= \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu \mathcal{D}_\mu\bar{\lambda} + \frac{1}{2}D^2\right) + \frac{\Theta g^2}{32\pi^2}\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}, \end{aligned} \tag{1.88}$$

where the dual field strength tensor is

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{1.89}$$

The last term in the Lagrangian can be written as $\partial_\mu K^\mu$, with

$$K^\mu = \frac{\Theta g^2}{16\pi}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma\right).$$

As a total derivative, this term can be ignored when doing perturbation theory. However, it does give rise to non-perturbative effects through so-called instanton configurations, for which the boundary term $\oint d^3\Sigma_\mu \, K^\mu$ has a nonzero value.

We may add matter fields to the theory through a multiplet of chiral superfields Φ^i, which transform under G as

$$\Phi^i \to \left(e^{2ig\Omega}\right)^i{}_j \Phi^j, \qquad \bar{\Phi}_i \to \bar{\Phi}_i \left(e^{-2ig\bar{\Omega}}\right)^i{}_j, \tag{1.90}$$

where $\Omega = \Omega^a T^a$. A kinetic term of the form

$$\bar{\Phi}_i \left(e^{2gV^aT^a}\right)^i{}_j \Phi^j \equiv \bar{\Phi} e^{2gV} \Phi$$

is then gauge invariant, and so the Lagrangian for the matter fields is

$$\mathcal{L}_{\mathrm{m}} = \int d^4\theta\, \bar{\Phi} e^{2gV} \Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W(\Phi)}. \tag{1.91}$$

The form of the superpotential $W(\Phi)$ is constrained by the requirement of its gauge invariance. Let R be the representation to which the Φ^i belong to. Then a term of the form $\gamma_{i_1 \ldots i_N} \Phi^{i_1} \ldots \Phi^{i_N}$ may appear in the expansion of $W(\Phi)$ only if the N-fold product $R \times \cdots \times R$ contains the trivial representation, and if $\gamma_{m_1 \ldots m_N}$ is an invariant tensor of the group G. Renormalizability further restricts $W(\Phi)$ to be at most a cubic polynomial of Φ [5].

There is one more type of term which can appear in the Lagrangian, if the group G contains $U(1)$ factors. Let $\kappa = \kappa^a T^a$ be a central element of the Lie algebra of G – that is, the index a takes nonzero values only when T^a is a commuting element of the algebra. The so-called Fayet–Iliopoulos term,

$$\mathcal{L}_{\mathrm{FI}} = g \int d^4\theta\, \mathrm{Tr}\, \kappa V = g\, \mathrm{Tr}\, \kappa D, \tag{1.92}$$

is then supersymmetric and gauge invariant, and so it may be included in the Lagrangian.

The component expansion of the complete Lagrangian, after one has eliminated the auxiliary fields, comes out to be

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_{\mathrm{FI}} \\
&= \mathrm{Tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda \sigma^\mu \mathcal{D}_\mu \bar{\lambda} + \frac{\Theta g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \\
&\quad - \mathcal{D}_\mu \bar{\phi}_i \mathcal{D}^\mu \phi^i - \frac{i}{2} \psi^i \sigma^\mu \mathcal{D}_\mu \bar{\psi}_i + \mathrm{Tr}\left(ig\bar{\phi}_i \lambda \psi^i - ig\bar{\lambda} \bar{\psi}_i \phi^i \right) \\
&\quad - \frac{1}{4} \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i \psi^j - \frac{1}{4} \frac{\partial^2 \overline{W}}{\partial \bar{\phi}^i \partial \bar{\phi}^j} \bar{\psi}^i \bar{\psi}^j - V(\phi, \bar{\phi}),
\end{aligned} \tag{1.93}$$

where

$$V(\phi, \bar{\phi}) = \bar{F}_i F^i + \frac{1}{2} \mathrm{Tr}\, D^2 = \sum_i \left| \frac{\partial W}{\partial \phi^i} \right|^2 + \frac{g^2}{2} \sum_a \left| \bar{\phi}_i T^a \phi^i + \kappa^a \right|, \tag{1.94}$$

and

$$D_\mu \phi = \partial_\mu \phi - \frac{i}{2} v_\mu^a T^a \phi, \qquad D_\mu \psi = \partial_\mu \psi - \frac{i}{2} v_\mu^a T^a \psi. \tag{1.95}$$

To obtain the supersymmetric version of QCD, the gauge group G is taken to be $SU(3)$. The chiral superfields belong to the $\mathbf{3} \oplus \bar{\mathbf{3}}$ representation of $SU(3)$. That is, we have chiral superfields

$$\begin{aligned} \Phi^i &= \phi^i + \theta q^i + \theta^2 F^i, \\ \tilde{\Phi}_i &= \tilde{\phi}_i + \theta \tilde{q}_i + \theta^2 \tilde{F}^2, \end{aligned} \tag{1.96}$$

where $i = 1, 2, 3$, and Φ^i transforms in the $\mathbf{3}$ while $\tilde{\Phi}_i$ transforms in the $\bar{\mathbf{3}}$. The q^i and $\tilde{q}_i$ are the quarks and antiquarks, while the ϕ^i and $\tilde{\phi}_i$ are their spin-0 supersymmetric partners, the squarks and antisquarks. (Note that here $\tilde{q}$ denotes the antiparticle of q, contrary to the standard notation used in particle physics, where the quarks are denoted by u, d, etc., and the squarks are denoted by $\tilde{u}, \tilde{d}$, etc.) There are also the gluons v^a_μ, with $a = 1, \ldots 8$, as well as the spin-1/2 gluinos λ^a. The Lagrangian cannot have a Fayet–Iliopoulos term, because G does not contain any $U(1)$ factors.

1.7 Spontaneous supersymmetry breaking

Before we conclude our chapter on supersymmetry, we will consider two topics, which will not be relevant to the rest of this work, but nevertheless are important and interesting enough to be worthy of at least a brief mention. In this section, we will look into how supersymmetry may be broken. To have mechanisms for breaking supersymmetry is important for the business of applying supersymmetry to Nature, because if supersymmetry is a symmetry of Nature at all, it needs to be broken, as the degeneracy in mass between bosons and fermions, which is a consequence of unbroken supersymmetry, is not observed in Nature even approximately. In the next section, we will discuss some of the features of supersymmetric quantum field theories, thereby gaining a little more insight into the reasons for why supersymmetry is so widely considered to be such an attractive idea.

Even though there are other ways to break supersymmetry, we will only discuss spontaneous supersymmetry breaking, that is, the breaking of supersymmetry through choosing a vacuum which is not invariant under supersymmetry. The action of the supersymmetry anticommutator $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}$ on an eigenstate of momentum is

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}|p\rangle = 2(\sigma^\mu)_{\alpha\dot{\beta}}\, p_\mu |p\rangle = 2 \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix}_{\alpha\dot{\beta}} |p\rangle.$$

In particular, the energy is

$$p^0|p\rangle = \tfrac{1}{4}\left(\{Q_1, \bar{Q}_1\} + \{Q_2, \bar{Q}_2\}\right)|p\rangle,$$

so the Hamiltonian is the positive semidefinite operator

$$H = \frac{1}{4}\left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right). \tag{1.97}$$

Therefore every state, including the vacuum $|\Omega\rangle$, has an energy of at least zero. Moreover, the vacuum has zero energy if and only if it is annihilated by all the supercharges. Therefore a vacuum with zero energy is supersymmetric, whereas a vacuum with a positive energy spontaneously breaks supersymmetry. Spontaneous breaking of supersymmetry thus depends only on the minimum value of the potential, in contrast to the spontaneous breaking of a gauge symmetry, which depends only on the shape of the potential.

To spontaneously break supersymmetry, a theory must therefore have a field φ whose supersymmetry variation has a nonzero vacuum expectation value:

$$\langle\Omega|\delta_\epsilon\varphi|\Omega\rangle > 0.$$

In order to not break Lorentz invariance, the variation $\delta_\epsilon\varphi$ must be a Lorentz scalar. The field φ must thus be fermionic, because the supersymmetry variation of a bosonic field would be proportional to a fermionic field, which would transform nontrivially under a Lorentz transformation. The fermion fields ψ and λ contained in chiral and vector superfields, respectively, transform under supersymmetry as

$$\delta_\epsilon\psi_\alpha = -2\epsilon_\alpha F + \ldots, \qquad \delta_\epsilon\lambda_\alpha = -2\epsilon_\alpha D + \ldots \tag{1.98}$$

We see that to spontaneously break supersymmetry, it is enough if one of the auxiliary fields F and D has a positive vacuum expectation value. The corresponding fermion ψ or λ is then called a Goldstone fermion, in analogy with the Goldstone boson which appears when a continuous gauge symmetry is broken.

An example [17] of theory which breaks supersymmetry through a positive vacuum expectation value of a F field was given by O'Raifeartaigh [18]. It has the Lagrangian (1.57), with three chiral superfields Φ_0, Φ_1 and Φ_2. The superpotential is given by

$$W(\Phi) = \Phi_1 g_1(\Phi_0) + \Phi_2 g_2(\Phi_0), \tag{1.99}$$

where g_1 and g_2 are polynomials which are chosen so that the equations

$$g_1(\phi_0) = 0, \qquad g_2(\phi_0) = 0$$

do not have a simultaneous solution for ϕ_0. The auxiliary fields are eliminated from the Lagrangian by setting $F_i = -\partial\overline{W}/\partial\bar{\phi}_i$. This introduces into the Lagrangian the potential term

$$V = \sum_i |F_i|^2 = \left|\phi_1 g_1'(\phi_0) + \phi_2 g_2'(\phi_0)\right|^2 + |g_1(\phi_0)|^2 + |g_2(\phi_0)|^2. \tag{1.100}$$

By assumption, the last two terms cannot simultaneously vanish. Therefore this theory has a positive vacuum energy, and so it spontaneously breaks supersymmetry. Eq. (1.100) explicitly shows that the supersymmetry breaking is due to one of the fields F_1 and F_2 having a positive vacuum expectation value.

A further example of spontaneous supersymmetry breaking, which involves an interesting intertwining of supersymmetry and gauge symmetry, is given by a supersymmetric gauge theory whose gauge group is $U(1)$, and which contains a single chiral superfield Φ having a nonzero charge under the gauge group [19]. A gauge invariant superpotential is therefore not possible. However, there can be a Fayet–Iliopoulos term, whose contribution to the Lagrangian is

$$V = \frac{g^2}{2} \left| \bar{\phi}\phi + \kappa \right| . \tag{1.101}$$

Two different outcomes are now possible. If $\kappa > 0$, the vacuum has $\phi = 0$. The vacuum energy is then positive, and so supersymmetry is spontaneously broken. However, if $\kappa < 0$, the vacuum satisfies $\bar{\phi}\phi = |\kappa|$. Then the vacuum energy is zero, so supersymmetry remains unbroken. On the other hand, the gauge symmetry is now spontaneously broken, as indicated by the nonzero vacuum expectation value of ϕ.

1.8 Quantization of supersymmetric field theories

We briefly outline the quantization of a theory with a single chiral superfield Φ [1, 6]. Just as in usual field theory, we are supposed to consider the object

$$Z[J, \bar{J}] = \int \mathcal{D}\Phi \, \mathcal{D}\bar{\Phi} \, \exp \Big[i(S_0 + S_J) \Big], \tag{1.102}$$

where

$$S_0 = \int d^4x \, d^4\theta \, \bar{\Phi}\Phi + \int d^4x \left(\int d^2\theta \, \frac{1}{2} m\Phi^2 + \int d^2\bar{\theta} \, \frac{1}{2} m\bar{\Phi}^2 \right), \tag{1.103}$$

$$S_J = \int d^4x \left(- \int d^2\theta \, J\Phi - \int d^2\bar{\theta} \, \bar{J}\bar{\Phi} \right). \tag{1.104}$$

Before we can make further progress, we should express S_0 and S_J entirely as integrals over the whole superspace. Using the identities

$$\bar{D}^2 D^2 \Phi = -16 \Box \Phi,$$

which is valid for a chiral superfield Φ, and

$$\int d^4x \, d^2\theta \, \Psi = -\frac{1}{4} \int d^4x \, \bar{D}^2 \Psi,$$

which holds for any superfield Ψ, we can rewrite S_0 and S_J as

$$S_0 = \int d^4x \, d^4\theta \left[\bar{\Phi}\Phi + \frac{m}{2} \Phi \left(\frac{D^2}{4\Box} \right) \Phi + \frac{m}{2} \bar{\Phi} \left(\frac{\bar{D}^2}{4\Box} \right) \bar{\Phi} \right], \tag{1.105}$$

$$S_J = \int d^4x \, d^4\theta \left[J \left(-\frac{D^2}{4\Box} \right) \Phi + \bar{J} \left(-\frac{\bar{D}^2}{4\Box} \right) \bar{\Phi} \right]. \tag{1.106}$$

We may then express $Z[J,\bar{J}]$, interpreting the integral inside the exponential as a sum, in the form

$$Z[J,\bar{J}] = \int \mathcal{D}\Phi\, \mathcal{D}\bar{\Phi}\, \exp\big(- X_i^* M_{ij} X_j + X_i^* K_i + X_i K_i^* \big),$$

where $X = (\Phi, \bar{\Phi})$, and M is an appropriate 2-by-2 matrix and K an appropriate two-component vector. By analogy with the integral

$$\int \prod_{k=1}^{N} dz_k\, dz_k^* \exp\left(-z_i^* A_{ij} z_j + u_i^* z_i + u_i z_i^*\right) = \frac{(2\pi i)^N}{\det A} \exp(u_i^* A_{ij}^{-1} u_j),$$

we find

$$Z[J,\bar{J}] = \exp\left[i \int d^4x\, d^4\theta\, \left(\bar{J} \frac{1}{\Box - m^2} J + \frac{1}{2} J \frac{m}{\Box - m^2} \frac{D^2}{4\Box} J + \frac{1}{2} \bar{J} \frac{m}{\Box - m^2} \frac{\bar{D}^2}{4\Box} \bar{J} \right) \right].$$

Introducing the notation $z = (x, \theta, \bar{\theta})$, we find that the propagators are given in momentum space by

$$\begin{aligned}
\big\langle \Phi(z_1) \bar{\Phi}(z_2) \big\rangle &= \frac{1}{p^2 + m^2} \delta^{(4)}(\theta_1 - \theta_2), \\
\big\langle \Phi(z_1) \Phi(z_2) \big\rangle &= \frac{m/4p^2}{p^2 + m^2} D^2(p,1) \delta^{(4)}(\theta_1 - \theta_2), \\
\big\langle \bar{\Phi}(z_1) \bar{\Phi}(z_2) \big\rangle &= \frac{m/4p^2}{p^2 + m^2} \bar{D}^2(p,1) \delta^{(4)}(\theta_1 - \theta_2)
\end{aligned} \tag{1.107}$$

where

$$D(p,i) = \frac{\partial}{\partial \theta^i} + \sigma^\mu \bar{\theta}_i\, p_\mu. \tag{1.108}$$

The vertex rules are derived by taking functional derivatives of $Z[J,\bar{J}]$ with respect to J, according to the rule

$$\frac{\delta J(z_2)}{\delta J(z_1)} = -\frac{1}{4} \bar{D}^2 \delta^{(4)}(x_1 - x_2) \delta^{(4)}(\theta_1 - \theta_2).$$

The Feynman rules for the theory then are the following:

- For each external line, write a $\Phi(p,\theta)$ or a $\bar{\Phi}(p,\theta)$.
- For internal lines, use the propagators given by (1.107).
- For a Φ^n vertex with k internal lines, insert $k-1$ operators $-\frac{1}{4}\bar{D}^2$ acting on the propagators. For a $\bar{\Phi}^n$ vertex, use the operators $-\frac{1}{4}D^2$.
- Include integrals over all the external momenta and the loop momenta. For each vertex, include an $\int d^4\theta$.
- Multiply by the symmetry factor, according to the rules of the usual ϕ^n theory.

The absence of ultraviolet divergences in supersymmetric field theories was already pointed out in section 1.2. We are now in a position to understand the improved quantum behaviour of supersymmetric theories in a little more detail.

One of the consequences of the above Feynman rules is that the contribution from any graph can be expressed in terms of a single superspace integral $\int d^4\theta$. Consider a graph containing some number of loops, and focus on one of them. This part of the diagram gives a multitude of delta functions with D^2's and $\bar{D}^2$'s acting on them, and an integral over $d^4x_1\, d^4\theta_1 \cdots d^4x_n\, d^4\theta_n$. Using partial integrations, the delta functions can be made free of the D^2's and $\bar{D}^2$'s one by one. Having done so, we may use the delta functions carry out all of the $d^4\theta_i$ integrations except one. This leaves us with a term like

$$\int d^4\theta_n \; (D^2)^k (\bar{D}^2)^l \delta^{(4)}(\theta_n - \theta_1)\Big|_{\theta_1=\theta_n},$$

or a similar term with D^2 and $\bar{D}^2$ reversed. The numbers k and l are either zero or one. If either of them is zero, the term vanishes; otherwise it equals $16 \int d^4\theta_n$. We may continue this process one loop at a time, until we have reduced the whole graph to an expression of the form

$$\int d^4\theta \int d^4x_1 \ldots d^4x_n \, F_1(x_1, \theta, \bar{\theta}) \ldots F_n(x_n, \theta, \bar{\theta}) G(x_1, \ldots, x_n), \qquad (1.109)$$

where the F's are products of superfields and their derivatives.

A remarkable consequence of the above observation is the so-called nonrenormalization theorem. The theorem states that in a renormalizable theory, the superpotential is not renormalized at any order in perturbation theory. This is so because the superpotential is an integral over only half of superspace, while all the counterterms are necessarily integrals over all of superspace. The superpotential contains both the mass and the coupling constant, and so the tree-level masses and couplings will not receive loop corrections from higher orders of perturbation theory. Supersymmetry therefore provides a solution to the hierarchy problem of particle physics, since a set of tree-level masses is protected from ever receiving any radiative corrections. This is one of the most attractive of the phenomenological features of supersymmetry.

Chapter 2

Supergravity

2.1 The vielbein formulation of general relativity

As supersymmetry is a symmetry between bosons and fermions, it is clear that spinor fields are an essential component of supersymmetric theories. However, it is difficult to generalize the transformation properties of spinors to curved spacetimes, since (in D spacetime dimensions) spinors form a representation of the Lorentz group $SO(1, D-1)$, rather than the group $GL(D)$ of general coordinate transformations. The usual formulation of general relativity is therefore not ideal for dealing with spinors. The purpose of this section is to introduce an alternative formalism, the so-called the vielbein formalism, which is better suited for introducing spinors into a theory of gravity [21, 26, 27].

Let us begin with a brief review of the standard formulation of general relativity [21, 26]. It consists of the metric $g_{\mu\nu}$, and the connection $\Gamma^{\lambda}_{\mu\nu}$. The connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ are often called the Christoffel symbols in the literature. The connection is used to construct the covariant derivative

$$\nabla_\mu = \partial_\mu - \Gamma^{\lambda}_{\mu\nu}\Delta_\lambda{}^\nu. \tag{2.1}$$

The operator $\Delta_\lambda{}^\nu$ acts on covariant and contravariant indices as

$$\Delta_\lambda{}^\nu V_\mu = \delta^\nu_\mu V_\lambda, \qquad \Delta_\lambda{}^\nu V^\mu = \delta^\mu_\lambda V^\nu. \tag{2.2}$$

On a tensor with multiple indices, $\Delta_\lambda{}^\nu$ acts additively on each index.

The commutator of two covariant derivatives measures, roughly speaking, the extent to which two parallel transports in different directions fail to commute. The commutator is given by

$$[\nabla_\mu, \nabla_\nu] = -T_{\mu\nu}{}^\rho \nabla_\rho - R_{\mu\nu\rho}{}^\sigma \Delta_\sigma{}^\rho.$$

It defines two tensors – the torsion tensor $T_{\mu\nu}{}^\rho$ and the curvature tensor $R_{\mu\nu\rho}{}^\sigma$ – as

$$T_{\mu\nu}{}^\rho = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}, \tag{2.3}$$

and

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}. \tag{2.4}$$

The Jacobi identity $[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] +$ cyclic permutations $= 0$ gives rise to identities involving the torsion and the curvature. These are called Bianchi identities.

Let us now consider the requirement that the covariant derivative of a tensor should transform as a tensor under general coordinate transformations. Under the coordinate transformation $x \to x'(x)$, a tensor $T(x)$ transforms into $T'(x')$. Let $\delta T(x)$ denote the variation of a tensor at a fixed spacetime point x: $\delta T(x) \equiv T'(x) - T(x)$. Infinitesimal coordinate transformations, $x \to x - \xi(x)$, are generated on tensors by the Lie derivative. That is, $\delta_\xi T = \mathcal{L}_\xi T$, where the Lie derivative is

$$\mathcal{L}_\xi = \xi^\mu \partial_\mu + \partial_\mu \xi^\nu \, \Delta_\nu{}^\mu. \tag{2.5}$$

The requirement that the covariant derivative of a tensor is a tensor is now expressed as $\delta_\xi \nabla_\mu T = \mathcal{L}_\xi \nabla_\mu T$, and it leads to the transformation law

$$\delta_\xi \Gamma^\lambda_{\mu\nu} = \mathcal{L}_\xi \Gamma^\lambda_{\mu\nu} + \partial_\mu \partial_\nu \xi^\lambda$$

for the connection under an infinitesimal coordinate transformation.

In general relativity, one usually requires the covariant derivative to satisfy $\nabla_\lambda g_{\mu\nu} = 0$. Geometrically, this requirement guarantees that the length of a vector remains unchanged under parallel transport. This condition determines the symmetric part of $\Gamma^\lambda_{\mu\nu}$ in terms of the metric and the torsion tensor. In the usual formulation of general relativity the torsion tensor is taken to vanish. The connection is then symmetric in μ and ν and is given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \big(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \big). \tag{2.6}$$

From the curvature tensor, we can construct the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$, and the curvature scalar $R = R_\mu{}^\mu$. The field equations for general relativity follow from the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g}\, R + S_{\rm m} \tag{2.7}$$

where $S_{\rm m}$ denotes the action for the matter fields. (For now we will set $8\pi G = 1$. Had we not done so, the numerical factor in the first term would have been $1/16\pi G$.) This action gives rise to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}, \tag{2.8}$$

where the energy-momentum tensor is constructed from the matter field action as $T_{\mu\nu} = -(2/\sqrt{-g})(\delta S_{\rm m}/\delta g_{\mu\nu})$.

In the vielbein formalism, we introduce a locally inertial – that is, flat – coordinate system at each spacetime point. Such a coordinate system is spanned by the orthnormal set of basis vectors ${e_m}^\mu(x)$, where the index m takes the values $0, \ldots, D-1$ in a D-dimensional spacetime. The orthonormality of the basis vectors is expressed as

$$g_{\mu\nu}(x){e_m}^\mu(x){e_n}^\nu(x) = \eta_{mn}. \tag{2.9}$$

We also have the "inverse" vectors ${e_\mu}^m(x)$, which satisfy

$${e_\mu}^m(x){e_m}^\nu(x) = \delta_\mu^\nu, \qquad {e_m}^\mu(x){e_\mu}^n(x) = \delta_m^n. \tag{2.10}$$

The set of vectors ${e_\mu}^m(x)$ is called the vielbein. The metric can be expressed in terms of the vielbein as

$$g_{\mu\nu}(x) = \eta_{mn}{e_\mu}^m(x){e_\nu}^n(x). \tag{2.11}$$

Using the vielbein, we may transform local Lorentz indices $m, n, \ldots$ into general coordinate indices $\mu, \nu, \ldots$ and vice versa:

$$V^m = {e_\mu}^m V^\mu, \qquad V^\mu = {e_m}^\mu V^m. \tag{2.12}$$

The vielbein is clearly not unique: it is determined only up to local Lorentz transformations ${e_\mu}^m(x) \to {e_\mu}^n(x){\Lambda_n}^m(x)$, with ${\Lambda_n}^l(x){\Lambda_l}^m(x) = \delta_n^m$.

In the vielbein formalism, assuming one wishes to make a distinction between fundamental and non-fundamental objects, the vielbein is considered to be the fundamental object representing the gravitational field, while the metric is constructed from the vielbein and is therefore not fundamental.

We can now introduce spinors. Under general coordinate transformations they transform as scalars, while under local Lorentz transformations they transform in the spinor representation of the Lorentz group. On a Dirac spinor ψ, the action of the generators M_{mn} of local Lorentz transformations is given by

$$M_{mn}\psi = \frac{1}{2}\gamma_{mn}\psi,$$

while on vectors the generators act as

$$M_{mn}V^l = \delta_m^l V_n - \delta_n^l V_m.$$

The generators M_{mn} satisfy the Lorentz algebra

$$[M_{mn}, M_{pr}] = \eta_{mp}M_{nr} - \eta_{np}M_{mr} + \eta_{nr}M_{mp} - \eta_{mr}M_{np}. \tag{2.13}$$

To define covariant derivatives of Lorentz tensors, we introduce the so-called spin connection ${\omega_\mu}^{mn}$, in terms of which the Lorentz covariant derivative is given by

$$D_\mu = \partial_\mu - \frac{1}{2}{\omega_\mu}^{mn}M_{mn}, \qquad \gamma_{mn} = \frac{1}{2}[\gamma_m, \gamma_n]. \tag{2.14}$$

We now have two covariant derivatives, D_μ for Lorentz tensors and ∇_μ for $GL(D)$ tensors. We would like the two covariant derivatives to be equivalent, in the sense that $\nabla_\mu V_\nu = e_\nu{}^m D_\mu V_m$. This requirement implies that the full covariant derivative of the vielbein vanishes,

$$D_\mu e_\nu{}^m - \Gamma^\lambda_{\mu\nu} e_\lambda{}^m = \partial_\mu e_\nu{}^m - \omega_\mu{}^n{}_m e_\nu{}^n - \Gamma^\lambda_{\mu\nu} e_\lambda{}^m = 0. \tag{2.15}$$

(Note that this is consistent with the condition $\nabla_\lambda g_{\mu\nu} = 0$.) From this equation, an explicit expression for the spin connection in terms of the vielbein and its derivatives can be found. In the general case, with nonvanishing torsion, we have

$$\omega_\mu{}^{mn} = \frac{1}{2} e_{l\mu} \left(\Omega^{mnl} - \Omega^{nlm} - \Omega^{lmn} \right) + K_\mu{}^{mn}. \tag{2.16}$$

where

$$\Omega_{mnl} = e_m{}^\mu e_n{}^\nu \big(\partial_\mu e_{\nu l} - \partial_\nu e_{\mu l} \big) \tag{2.17}$$

and

$$K_\mu{}^{mn} = \frac{1}{2} \left(T^{mn}{}_\mu + T_\mu{}^{mn} + T^m{}_\mu{}^n \right). \tag{2.18}$$

The commutator of two Lorentz covariant derivatives defines a curvature tensor through

$$[D_\mu, D_\nu] = -\frac{1}{2} R_{\mu\nu}{}^{mn} M_{mn}$$

as

$$R_{\mu\nu}{}^{mn} = \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} - \omega_\mu{}^{ml} \omega_{\nu l}{}^n + \omega_\nu{}^{ml} \omega_{\mu l}{}^n. \tag{2.19}$$

This is related to the usual spacetime curvature tensor through $R_{\mu\nu}{}^{mn} = R_{\mu\nu\rho}{}^\sigma e^{\rho m} e_\sigma{}^n$.

Using the vielbein, we can define gamma matrices in curved spacetime. Let

$$\gamma_\mu(x) = e_\mu{}^m(x) \gamma_m, \tag{2.20}$$

where γ_m are the standard gamma matrices which obey $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. The matrices $\gamma_\mu(x)$ then satisfy

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2 g_{\mu\nu}(x).$$

This enables us to write down Lagrangians for spinor fields in curved spacetime. We replace partial derivatives with the Lorentz covariant derivatives, and the usual gamma matrices with the matrices (2.20), and insert a factor of $\sqrt{-g}$ to obtain a proper integration measure. For example, for a Dirac spinor we would have

$$S = -\int d^4x\, |e| \bar{\psi} \big(i\gamma^\mu(x) D_\mu + m \big) \psi,$$

where $|e| = \det(e_\mu{}^a) = \sqrt{-g}$. In four dimensions, we can also define a curved space version of the Pauli matrices, $\sigma^\mu(x) = e_m{}^\mu(x) \sigma^m$, and similarly for $\bar{\sigma}^\mu(x)$, To conclude this section, we give the formulation of general relativity in the vielbein formalism. The Einstein–Hilbert Lagrangian is expressed in terms of the vielbein as

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{2}\sqrt{-g}\,R = -\frac{1}{2}|e|e_m{}^\mu e_n{}^\nu R_{\mu\nu}{}^{mn}. \tag{2.21}$$

We should now choose whether to not impose the condition (2.15) and regard the vielbein and the spin connection as independent, or to impose (2.15) and treat the spin connection as a function of the vielbein. The first choice, where the Lagrangian is $\mathcal{L}[e,\omega]$, is known as the first-order formulation, or the Palatini formulation. The latter choice, with the Lagrangian $\mathcal{L}[e(\omega)]$ is called the second-order formulation.

In the Palatini formulation, we have the Lagrangian

$$\mathcal{L}_{\mathrm{P}}[e,\omega] = -\frac{1}{2}|e|e_m{}^\mu e_n{}^\nu R_{\mu\nu}{}^{mn}[\omega]. \tag{2.22}$$

Its variation is given by

$$\delta\mathcal{L}_{\mathrm{P}} = -|e|\big(R_\mu{}^m - \tfrac{1}{2}e_\mu{}^m R\big)\delta e_m{}^\mu - 3|e|\big(D_\mu e_\nu{}^m\big)e_{[m}{}^\mu e_n{}^\nu e_{l]}{}^\lambda \delta\omega_\lambda{}^{nl}. \tag{2.23}$$

(Some details of the calculation of the variation are given in [23].) This gives two equations, the latter of which is equivalent to Eq. (2.15) in the case of zero torsion, and the first gives Einstein's equation when the solution for ω in terms of e is inserted into it.

The Palatini formulation gives rise to a trick, sometimes called the 1.5 order formulation, which significantly simplifies the variation of the action, and which is often used in constructing supergravity theories. We will illustrate the trick with the Einstein–Hilbert Lagrangian, but it can be used with any Lagrangian which contains two or more different fields, of which one has an algebraic equation of motion, and so can be expressed in terms of the other fields. The Einstein–Hilbert Lagrangian and the Palatini Lagrangian are connected by the relation $\mathcal{L}_{\mathrm{EH}}[e] = \mathcal{L}_{\mathrm{P}}[e,\omega]|_{\omega=\omega[e]}$. We can then calculate the variation of $\mathcal{L}_{\mathrm{EH}}$ as

$$\delta\mathcal{L}_{\mathrm{EH}}[e] = \left(\left.\frac{\delta\mathcal{L}_{\mathrm{P}}}{\delta e_\mu{}^m}\right|_{\omega=\omega[e]} + \left.\frac{\delta\mathcal{L}_{\mathrm{P}}}{\delta\omega_\nu{}^{nl}}\right|_{\omega=\omega[e]}\frac{\delta\omega_\nu{}^{nl}}{\delta e_\mu{}^m}\right)\delta e_\mu{}^m.$$

However, the second term in the above expression vanishes, because $\omega[e]$ is a solution to its equation of motion. The variation of the Einstein–Hilbert Lagrangian can therefore be calculated by varying the Palatini Lagrangian with respect to the vielbein only, using for the spin connection the expression found by solving its equation of motion. The variation is explicitly given by the first term of Eq. (2.23).

2.2 Local supersymmetry

Before we go on to discuss simple supergravity, we would like to give an example [21] of the general proposition that if we promote supersymmetry into a local symmetry, then gravity automatically becomes introduced into the theory. We start from the free, massless Wess–Zumino model, make the parameter ϵ of supersymmetry transformations spacetime dependent, and find what adjustments we have to make to preserve supersymmetry.

Our starting point is the Lagrangian

$$\mathcal{L}^{(0)} = -\partial^\mu \bar{\phi} \partial_\mu \phi - \frac{i}{2} \partial_\mu \bar{\chi} \bar{\sigma}^\mu \chi, \tag{2.24}$$

where the fields ϕ and χ now transform under supersymmetry as

$$\delta_\epsilon \phi = \epsilon(x) \chi, \qquad \delta_\epsilon \bar{\phi} = \bar{\epsilon}(x) \bar{\chi}, \tag{2.25}$$

and

$$\delta_\epsilon \chi = 2i \sigma^\mu \bar{\epsilon}(x) \partial_\mu \phi, \qquad \delta_\epsilon \bar{\chi} = -2i \epsilon \bar{\sigma}^\mu \partial_\mu \bar{\phi}. \tag{2.26}$$

We now find that the Lagrangian (2.24) is no longer invariant under supersymmetry, its supersymmetry variation being given (up to a total derivative) by

$$\delta_\epsilon \mathcal{L}^{(0)} = \frac{1}{2} \partial_\nu \bar{\phi} \, \chi \sigma^\mu \bar{\sigma}^\nu \partial_\mu \epsilon + \text{c.c.}$$

In order to cancel this term, and so to preserve supersymmetry, we introduce the gauge field ψ_μ, which is a Weyl spinor, and which transforms as

$$\delta_\epsilon \psi_\mu = \frac{1}{\kappa} \partial_\mu \epsilon + \ldots, \qquad \delta_\epsilon \bar{\psi}_\mu = \frac{1}{\kappa} \partial_\mu \bar{\epsilon} + \ldots \tag{2.27}$$

On dimensional grounds, we introduced the constant κ, which has mass dimension -1, and which will eventually be identified with $\sqrt{8\pi G}$. Using the field ψ_μ, we construct the term

$$\mathcal{L}^{(1)}_\psi = -\frac{\kappa}{2} \partial_\nu \bar{\phi} \, \chi \sigma^\mu \bar{\sigma}^\nu \psi_\mu + \text{c.c.}, \tag{2.28}$$

and add it to the Lagrangian, so that now the combination $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ is invariant to order κ^0. To order κ, we have

$$\begin{aligned} \delta_\epsilon \big(\mathcal{L}^{(0)} + \mathcal{L}^{(1)} \big) &= i \partial_\nu \bar{\phi} \partial_\lambda \phi \, \bar{\epsilon} \bar{\sigma}^\lambda \sigma^\mu \bar{\sigma}^\nu \psi_\mu + \ldots \\ &= -i\kappa \big(\epsilon \sigma_\nu \bar{\psi}_\mu - \psi_\mu \sigma_\nu \bar{\epsilon} \big) \big(\partial^\mu \bar{\phi} \partial^\nu \phi + \partial^\nu \bar{\phi} \partial^\mu \phi - \eta^{\mu\nu} \partial^\rho \bar{\phi} \partial_\rho \phi \big) \\ &\quad - \kappa \, \epsilon^{\mu\nu\rho\sigma} \partial_\mu \bar{\phi} \partial_\nu \phi \big(\epsilon \bar{\sigma}_\rho \bar{\psi}_\sigma - \bar{\psi}_\sigma \sigma_\rho \bar{\epsilon} \big), \end{aligned}$$

where the dots indicate a term quartic in the fermion fields, which we will ignore. For the moment, we will also ignore the second term of the final expression. In the term on the middle line, there appears the energy-momentum tensor of the scalar field ϕ. To cancel this term, we have to introduce a symmetric tensor $h_{\mu\nu}$, which appears in the Lagrangian through the term

$$\mathcal{L}_h^{(1)} = \frac{1}{2}\kappa h_{\mu\nu} T^{\mu\nu}, \tag{2.29}$$

and whose variation under supersymmetry is given by

$$\delta_\epsilon h_{\mu\nu} = 2i\big(\epsilon\sigma_{(\mu}\bar{\psi}_{\nu)} - \psi_{(\mu}\sigma_{\nu)}\bar{\epsilon}\big). \tag{2.30}$$

Our total Lagrangian, to order κ, now reads

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}_\psi^{(1)} + \mathcal{L}_h^{(1)} = -\big(1 + \tfrac{1}{2}\kappa h_\rho{}^\rho\big)\big(\eta^{\mu\nu} - \kappa h^{\mu\nu}\big)\partial_\mu\bar{\phi}\partial_\nu\phi. \tag{2.31}$$

We do not continue our calculation to higher orders in κ, since it should now seem plausible that if we were to complete the calculation to all orders, we would be led to introduce a metric $g_{\mu\nu}$, and the kinetic term of the scalar field in Eq. (2.24) would get replaced by $\sqrt{-g}\, g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi$. Explicitly, to order κ, we have

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad \text{so} \qquad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu},$$

and

$$\sqrt{-g} = 1 + \frac{1}{2}\kappa h_\rho{}^\rho.$$

This process of iteratively modifying the Lagrangian and the transformation laws of the fields in order to promote a global symmetry into a local one is known as the Noether method. The parameter, in whose powers the calculation is organized, will take the role of a coupling constant in the end. During the calculation, one has to introduce additional fields, which couple in the Lagrangian to the conserved currents corresponding to global symmetries. In our calculation, ψ_μ couples to the current of global supersymmetry, while $h_{\mu\nu}$ couples to the energy-momentum tensor, which is the current of spacetime translations. The Noether procedure was the method first used to construct both the on-shell and off-shell formulations of four-dimensional supergravity [29, 31, 32, 33], see also [2].

We are still to deal with the second term in the variation $\delta_\epsilon\big(\mathcal{L}^{(0)} + \mathcal{L}^{(1)}\big)$. It can be expressed in the form

$$\frac{\kappa}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\big(\phi\partial_\nu\bar{\phi} - \bar{\phi}\partial_\mu\phi\big)\big(\epsilon\bar{\sigma}_\rho\bar{\psi}_\sigma - \bar{\psi}_\sigma\sigma_\rho\bar{\epsilon}\big).$$

Supersymmetry is now completely restored to order κ if we introduce for the field ψ_μ the kinetic term

$$\mathcal{L}_\psi^{(0)} = \epsilon^{\mu\nu\rho\sigma}\big(\partial_\mu\psi_\nu\sigma_\rho\bar{\psi}_\sigma + \psi_\sigma\sigma_\rho\partial_\mu\bar{\psi}_\nu\big), \tag{2.32}$$

and modify the supersymmetry variation of ψ_μ to read

$$\delta_\epsilon\psi_\mu = \frac{1}{\kappa}\partial_\mu\epsilon - \frac{\kappa}{4}\epsilon\big(\phi\partial_\mu\bar{\phi} - \bar{\phi}\partial_\mu\phi\big). \tag{2.33}$$

Eq. (2.32) is the Lagrangian of a free, massless spin-3/2 particle. It was first constructed by Rarita and Schwinger in 1941 [28].

The field ψ_μ is the gravitino, the supersymmetric partner of the graviton. It has spin 3/2, as we can see by considering the object $\psi_{\beta\dot{\beta}\alpha} = (\sigma^\mu)_{\beta\dot{\beta}}\psi_{\mu\alpha}$ and decomposing it as

$$\psi_{\beta\dot{\beta}\alpha} = \frac{1}{2}\left(\psi_{\beta\dot{\beta}\alpha} + \psi_{\alpha\dot{\beta}\beta}\right) + \frac{1}{2}\left(\psi_{\beta\dot{\beta}\alpha} - \psi_{\alpha\dot{\beta}\beta}\right),$$

that is,

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},0\right) = \left(1,\tfrac{1}{2}\right) \oplus \left(0,\tfrac{1}{2}\right).$$

In a theory with N supersymmetry generators, there is a conseved current for each generator. The result of a Noether method calculation would be to couple each of the currents to a gravitino in the Lagrangian, and so in a theory with N supersymmetries there will be N gravitinos.

2.3 Simple supergravity

We will now move on to discuss $N = 1$ supergravity in four dimensions [20, 21, 23], which commonly goes by the name of simple supergravity. This theory contains the graviton and the gravitino, and no additional fields. In this section, we will give the on-shell formulation of the theory and show its invariance under supersymmetry, deferring the construction of the off-shell formulation, including the question of finding a proper set of auxiliary fields, to the following sections.

For the graviton, a kinetic term is readily given by the Einstein–Hilbert Lagrangian. For the gravitino, we use the Rarita–Schwinger Lagrangian (2.32), and replace the spacetime derivatives by Lorentz covariant derivatives. We do not introduce a connection $\Gamma^\lambda_{\mu\nu}$ for the spacetime index in ψ_μ, since it can be shown that an antisymmetric combination of spacetime derivatives transforms as a tensor under general coordinate transformations by itself, even without a connection. We therefore take the Lagrangian to be

$$\mathcal{L} = -\frac{1}{2}|e|{e_m}^\mu {e_n}^\nu {R_{\mu\nu}}^{mn} + 2\epsilon^{\mu\nu\rho\sigma}\Big(D_\mu\psi_\nu\sigma_\rho\bar{\psi}_\sigma + \psi_\sigma\sigma_\rho D_\mu\bar{\psi}_\nu\Big). \qquad (2.34)$$

The form of the spin connection ${\omega_\mu}^{mn}$ is found by solving its equation of motion, which is obtained by varying the Lagrangian with respect to the spin connection taken as an independent field. We again have an unconventional normalization, but one could equally well use the standard normalization, provided that compensating adjustments are made in the supersymmetry variations of the fields.

Before we determine the spin connection, we put the Lagrangian into a form from which its variation is easier to calculate. We use the identity

$$|e|{e_m}^\mu {e_n}^n = -\tfrac{1}{4}\epsilon_{mnrs}\epsilon^{\mu\nu\rho\sigma}{e_\rho}^r {e_\sigma}^s$$

to write

$$\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{8}\epsilon_{mnrs}e_\rho{}^r e_\sigma{}^s R_{\mu\nu}{}^{mn} + 2D_\mu\psi_\nu\sigma_\rho\bar{\psi}_\sigma + 2\psi_\sigma\sigma_\rho D_\mu\bar{\psi}_\nu\right). \qquad (2.35)$$

Under the variation $\omega \to \omega + \delta\omega$, the curvature tensor $R_{\mu\nu}{}^{mn}$ transforms by $D_\mu\delta\omega_\nu{}^{mn} - D_\nu\delta\omega_\mu{}^{mn}$, and so we have

$$\delta\mathcal{L} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Big(\epsilon_{mnrs}e_\rho{}^r e_\sigma{}^s D_\nu\delta\omega_\mu{}^{mn} + 4\psi_\nu(\sigma_{mn}\sigma_r + \sigma_r\bar{\sigma}_{mn})\bar{\psi}_\sigma e_\rho{}^r\,\delta\omega_\mu{}^{mn}\Big).$$

A partial integration, together with the identity $\sigma_{mn}\sigma_r + \sigma_r\bar{\sigma}_{mn} = i\epsilon_{mnrs}\sigma^s$, now leads to

$$\delta\mathcal{L} = \frac{1}{2}\epsilon_{mnrs}\epsilon^{\mu\nu\rho\sigma}e_\rho{}^r\left(D_\nu e_\sigma{}^s - 2i\psi_\nu\sigma^s\bar{\psi}_\sigma\right)\delta\omega_\mu{}^{mn}.$$

Therefore

$$D_\mu e_\nu{}^m - D_\nu e_\mu{}^m = 2i\left(\psi_\mu\sigma^m\bar{\psi}_\nu - \psi_\nu\sigma^m\bar{\psi}_\mu\right). \qquad (2.36)$$

It is possible to solve this equation explicitly for the spin connection. The result is

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e) + 2i\Big(\psi_\mu\sigma^{[m}\bar{\psi}^{n]} + \psi^{[m}\sigma^{n]}\bar{\psi}_\mu + \psi^{[m}\sigma_\mu\bar{\psi}^{n]}\Big), \qquad (2.37)$$

where $\omega_\mu{}^{mn}(e)$ is the spin connection from Eq. (2.16), with $K^m{}_\mu{}^n$ set to zero.

Equations of motion for the vielbein and the gravitino are derived by varying (2.34) or (2.35) with respect to the fields. Variation with respect to the vielbein gives the equation $R_\mu{}^m - \frac{1}{2}e_\mu{}^m R = T_\mu{}^m$, where the energy-momentum tensor is given by an expression involving the gravitino and its derivatives. The curvature tensor contains terms involving the gravitino, because the spin connection now depends on ψ. Therefore, to put the equation of motion into the standard form of the Einstein equation, one would have to split the curvature tensor into terms containing the gravitino, and terms depending only on the vielbein, the latter of which would give the usual curvature tensor of general relativity.

For the gravitino we find, varying (2.35) with respect to ψ_μ,

$$\epsilon^{\mu\nu\rho\sigma}\sigma_\nu D_\rho\bar{\psi}_\sigma = 0. \qquad (2.38)$$

If we define $\bar{\psi}_{\mu\nu} = D_\mu\bar{\psi}_\nu - D_\nu\bar{\psi}_\mu$ and $G^\mu = |e|^{-1}\epsilon^{\mu\nu\rho\sigma}\sigma_\nu D_\rho\bar{\psi}_\sigma$, we can prove several identities which we will make use of later:

$$\begin{gathered}\varepsilon_{\mu\nu\rho\sigma}G^\sigma = 3\sigma_{[\mu}\bar{\psi}_{\nu\rho]},\\ \bar{\psi}_\mu G^\mu = 2i\bar{\sigma}^{\mu\nu}\bar{\psi}_{\mu\nu}, \qquad \sigma_\mu\bar{\sigma}_\nu G^\mu = 2i\sigma^\mu\bar{\psi}_{\nu\mu},\\ \bar{\sigma}_\rho\sigma_{\mu\nu}G^\rho = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar{\psi}^{\rho\sigma} - i\bar{\psi}_{\mu\nu},\end{gathered} \qquad (2.39)$$

where the tensor $\varepsilon_{\mu\nu\rho\sigma}$ is related to the numerically invariant $\epsilon_{\mu\nu\rho\sigma}$-symbol by $\varepsilon_{\mu\nu\rho\sigma} = |e|\epsilon_{\mu\nu\rho\sigma}$.

We are now going to show that the Lagrangian (2.34), or equivalently (2.35), is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon e_\mu{}^m &= 2i\big(\epsilon\sigma^m\bar\psi_\mu - \psi_\mu\sigma^m\bar\epsilon\big), \\ \delta_\epsilon\psi_\mu &= D_\mu\epsilon, \qquad \delta_\epsilon\bar\psi_\mu = D_\mu\bar\epsilon. \end{aligned} \tag{2.40}$$

Note that the transformation law of the vielbein implies Eq. (2.30). We will calculate the supersymmetry variation of the Lagrangian in the form (2.35), using the 1.5 order formalism. That is, we vary only the vielbein and the gravitino, while for the spin connection we use Eq. (2.37). We further divide the variation into two parts, the one proportional to ϵ and the one proportional to $\bar\epsilon$. Then it is enough to consider one of the parts, say the one proportional to ϵ, because the Lagrangian is real, and each of the parts is the complex conjugate of the other.

In (2.35), we therefore have to vary only e and ψ, while $\bar\psi$ and ω are not varied. The vielbein enters also through the σ_ρ–matrices, which are obtained by converting the index m of the constant σ_m–matrices into the index ρ using $e_\rho{}^m$. The variation of σ_ρ is given by

$$\delta_\epsilon(\sigma_\rho)_{\alpha\dot\alpha} = \big(\delta_\epsilon e_\rho{}^m\big)\,(\sigma_m)_{\alpha\dot\alpha} = \big(2i\epsilon\sigma^m\bar\psi_\rho\big)\,(\sigma_m)_{\alpha\dot\alpha} = -4i\epsilon_\alpha\bar\psi_{\rho\dot\alpha},$$

where we neglected the part proportional to $\bar\epsilon$, and the last form is obtained from the one preceding it through a Fierz rearrangement. We then have for the ϵ-part of the variation of (2.35)

$$\begin{aligned} \delta_\epsilon\mathcal{L} = 2\epsilon^{\mu\nu\rho\sigma}\bigg(&\frac{i}{4}\epsilon_{mnrs}\epsilon\sigma^s\bar\psi_\sigma e_\rho{}^r R_{\mu\nu}{}^{mn} + D_\mu D_\nu\epsilon\sigma_\rho\bar\psi_\sigma \\ &- 4iD_\mu\psi_\nu\epsilon\bar\psi_\rho\bar\psi_\sigma + D_\sigma\epsilon\sigma_\rho D_\mu\bar\psi_\nu - 4i\psi_\sigma\epsilon\bar\psi_\rho D_\mu\bar\psi_\nu\bigg). \end{aligned}$$

Due to the overall $\epsilon^{\mu\nu\rho\sigma}$-symbol, we are free to antisymmetrize in suitable indices inside the brackets. First of all, this makes the first term in the second line vanish, because it contains the factor $\bar\psi_\rho\bar\psi_\sigma$, which is symmetric in ρ and σ. In the fourth term we integrate by parts to bring both of the covariant derivatives to act on $\bar\psi_\nu$. In the terms with two derivatives, we then antisymmetrize in their indices, which allows us to write

$$D_{[\mu}D_{\nu]}\epsilon = -\frac{1}{4}R_{\mu\nu}{}^{mn}\epsilon\sigma_{mn}, \qquad D_{[\sigma}D_{\mu]}\bar\psi_\nu = -\frac{1}{4}R_{\mu\nu}{}^{mn}\bar\sigma_{mn}\bar\psi_\nu.$$

We then find

$$\begin{aligned} \delta_\epsilon\mathcal{L} = 2\epsilon^{\mu\nu\rho\sigma}\bigg(&\frac{1}{4}\epsilon\big(i\epsilon_{mnrs}\sigma^s - \sigma_{mn}\sigma_r - \sigma_r\bar\sigma_{mn}\big)\bar\psi_\sigma\epsilon_\rho{}^r R_{\mu\nu}{}^{mn} \\ &- 4i\psi_\sigma\epsilon\bar\psi_\rho D_\mu\bar\psi_\nu - D_\sigma e_\rho{}^m\epsilon\sigma_m D_\mu\bar\psi_\nu\bigg), \end{aligned}$$

where the last term was generated by the partial integration. The first term vanishes because of the identity $\sigma_{mn}\sigma_r + \sigma_r\bar{\sigma}_{mn} = i\epsilon_{mnrs}\sigma^s$. Using Eq. (2.36) in the last term, we find

$$\delta_\epsilon \mathcal{L} = -4i\epsilon^{\mu\nu\rho\sigma}\psi_\sigma \left(2\epsilon\bar{\psi}_\rho + \sigma^m\bar{\psi}_\rho\epsilon\sigma_m\right) D_\mu\bar{\psi}_\nu.$$

Using a Fierz rearrangement, we can now show that the two terms inside the brackets cancel each other.

To complete the demonstration of the supersymmetry of the Lagrangian, we still need to show that the algebra of the symmetry transformations closes on-shell. We do not expect it to close off-shell, since as of now we have not introduced any auxiliary fields into the theory. We have to show that the commutator of any two symmetry transformations of the Lagrangian, which include general coordinate transformations, local Lorentz transformations, and local supersymmetry transformations, can be expressed as a combination of symmetry transformations.

We denote the variations under the various transformations by δ_G, δ_L and δ_S. For the vielbein, the commutator of two successive supersymmetry transformations is given by

$$\left[\delta_S(\epsilon_1), \delta_S(\epsilon_2)\right] e_\mu{}^m = 2iD_\mu\left(\epsilon_2\sigma^m\bar{\epsilon}_1 - \epsilon_1\sigma^m\bar{\epsilon}_2\right) \equiv D_\mu\xi^m. \tag{2.41}$$

We now define $\xi^m = \xi^\nu e_\nu{}^m$, so that

$$D_\mu\xi^m = \xi^\nu\partial_\nu e_\mu{}^m + \partial_\mu\xi^\nu e_\nu{}^m + \xi^\nu\left(\partial_\mu e_\nu{}^m - \partial_\nu e_\mu{}^m\right) + \xi^\nu\omega_{\mu\nu}{}^m,$$

where we have added and subtracted $\xi^\nu\partial_\nu e_\mu{}^m$. In the third term, we use Eq. (2.36), leading to

$$D_\mu\xi^m = \xi^\nu\partial_\nu e_\mu{}^m + \partial_\mu\xi^\nu e_\nu{}^m + \xi^\nu\omega_{\nu\mu}{}^m - i\xi^\nu\left(\psi_\nu\sigma^m\bar{\psi}_\mu - \psi_\mu\sigma^m\bar{\psi}_\nu\right).$$

From Eq. (2.5) we see that the first two terms represent a general coordinate transformation with the parameter ξ^ν. The third term gives a local Lorentz transformation with the parameter $\lambda^{nm} = -\xi^\nu\omega_\nu{}^{nm}$, and the last term is a local supersymmetry transformation with the parameter $\epsilon_{12} = -\xi^\nu\psi_\nu$. We therefore have

$$\left[\delta_S(\epsilon_1), \delta_S(\epsilon_2)\right] e_\mu{}^m = \left(\delta_G(\xi) + \delta_L(\lambda) + \delta_S(\epsilon_{12})\right) e_\mu{}^m. \tag{2.42}$$

We see that this commutator closes off-shell.

On the gravitino, the same commutator is

$$\left[\delta_S(\epsilon_1), \delta_S(\epsilon_2)\right]\psi_\mu = \frac{1}{2}\Big(\delta_S(\epsilon_1)\omega_\mu{}^{mn}\sigma_{mn}\epsilon_2 - \delta_S(\epsilon_2)\omega_\mu{}^{mn}\sigma_{mn}\epsilon_1\Big). \tag{2.43}$$

To determine the supersymmetry variation of the spin connection, we apply to Eq. (2.36) the operator δ_+, which gives the part of the supersymmetry variation which is proportional to ϵ. On the left-hand side, we find

$$\delta_+\left(D_\mu e_\nu{}^m - D_\nu e_\mu{}^m\right) = iD_{[\mu}\left(\epsilon\sigma^m\bar{\psi}_{\nu]}\right) - e_{[\mu}{}^n\delta_+\omega_{\nu]n}{}^m,$$

while on the right-hand side we get $iD_{[\mu}\epsilon\sigma^m\bar{\psi}_{\nu]}$. Comparing the two sides, we see that

$$e_{[\mu}{}^n\delta_+\omega_{\nu]n}{}^m = \frac{i}{2}\epsilon\sigma^m\bar{\psi}_{\mu\nu}.$$

We then use the identity

$$\omega_{\nu mn} = e_m{}^\rho e_n{}^\sigma\big(\omega_{[\nu\rho]\sigma} - \omega_{[\rho\sigma]\nu} + \omega_{[\sigma\nu]\rho}\big),$$

to find

$$\delta_+\omega_{\mu mn} = \frac{i}{2}e_m{}^\nu e_n{}^\rho\epsilon\big(\sigma_\nu\bar{\psi}_{\mu\rho} - \sigma_\rho\bar{\psi}_{\mu\nu} - \sigma_\nu\bar{\psi}_{\rho\nu}\big). \tag{2.44}$$

Using Eq. (2.39), we can write the final result as

$$\delta_S(\epsilon)\omega_{\mu mn} = i\big(\epsilon\sigma_\mu\bar{\psi}_{mn} - \psi_{mn}\sigma_\mu\bar{\epsilon}\big) - \frac{i}{2}\epsilon_{mnrs}e_\mu{}^r\big(\epsilon G^s - \bar{\epsilon}\bar{G}^s\big). \tag{2.45}$$

We will now eliminate all of the terms that vanish on-shell. According to the gravitino equation of motion, we can immediately drop the terms involving G and $\bar{G}$. Furthermore. Eq. (2.39) shows that on-shell $\bar{\psi}_{mn}$ is anti-self-dual, and consequently ψ_{mn} self-dual. In Eq. (2.43), $\delta_S(\epsilon)\omega_\mu{}^{mn}$ gets multiplied with σ_{mn}, which is self-dual, and so of the first two terms we need to keep only one which involves the self-dual object ψ_{mn}. On-shell, we therefore have

$$\big[\delta_S(\epsilon_1),\delta_S(\epsilon_2)\big]\psi_\mu = \frac{i}{2}\Big((\psi_{mn}\sigma_\mu\bar{\epsilon}_2)\sigma^{mn}\epsilon_1 - (\psi_{mn}\sigma_\mu\bar{\epsilon}_1)\sigma^{mn}\epsilon_2\Big).$$

Using a Fierz rearrangement and Eq. (2.39), we can put this in the form

$$\begin{aligned}\big[\delta_S(\epsilon_1),\delta_S(\epsilon_2)\big]\psi_\mu &= \xi^\nu D_\nu\psi_\mu - \xi^\nu D_\mu\psi_\nu\\ &= \xi^\nu\partial_\nu\psi_\mu + \partial_\mu\xi^\nu\psi_\nu + \frac{1}{2}\big(\xi^\nu\omega_\nu{}^{mn}\big)\sigma_{mn}\psi_\mu - D_\mu\big(\xi^\nu\psi_\nu\big),\end{aligned} \tag{2.46}$$

where we have the same vector ξ^ν as we had previously. Thus

$$\big[\delta_S(\epsilon_1),\delta_S(\epsilon_2)\big]\psi_\mu = \big(\delta_G(\xi) + \delta_L(\lambda) + \delta_Q(\epsilon_{12})\big)\psi_\mu, \tag{2.47}$$

with the same parameters λ and ϵ_{12} as for the vielbein, showing that the commutator closes on the gravitino, even though only on-shell.

All of the remaining commutators are straightforward to calculate. They all close off-shell, and are given by

$$\begin{aligned}\big[\delta_G(\xi),\delta_G(\eta)\big] &= \delta_G(\zeta) & \qquad & \zeta^\mu = \eta^\nu\partial_\nu\xi^\mu - \xi^\nu\partial_\nu\eta^\mu\\ \big[\delta_L(\lambda),\delta_L(\kappa)\big] &= \delta_L(\omega) & & \omega = [\kappa,\lambda]\\ \big[\delta_G(\xi),\delta_L(\lambda)\big] &= \delta_L(\omega) & \text{where} \qquad & \omega = -\xi^\mu\partial_\mu\lambda,\\ \big[\delta_G(\xi),\delta_S(\epsilon)\big] &= \delta_S(\chi) & & \chi = -\xi^\mu\partial_\mu\epsilon,\\ \big[\delta_L(\lambda),\delta_S(\epsilon)\big] &= \delta_S(\chi) & & \chi = -\tfrac{1}{2}\lambda^{mn}\epsilon\sigma_{mn}.\end{aligned} \tag{2.48}$$

Four-dimensional supergravity was first constructed by Freedman, van Nieuwenhuizen, and Ferrara [29, 30]. They took the spin connection to be a function of the vielbein, and performed a tedious calculation, which involved introducing additional terms in the Lagrangian and adjusting the transformation laws of the fields until the invariance of the action had been established. In hindsight, the effect of all the adjustments was just to replace the $\omega(e)$ of Eq. (2.16) by the $\omega(e, \psi)$ of Eq. (2.37). Not long after, Deser and Zumino demonstrated the invariance of the action using an independent spin connection [31]. Both of these analyses are rather complicated compared to the one provided by the 1.5 order formalism, which combines the advantages of both the first-order and the second-order formalism.

2.4 Tensor calculus for supergravity

We now proceed to find a suitable set of auxiliary fields needed to construct an off-shell formulation of simple supergravity. It is possible, but rather complicated, to do this directly in the component field formulation [32, 33]. We prefer to construct the off-shell formulation using a tensor calculus, outlined in [20] and [21], which applies not only to supergravity, but to a more general class of gauge theories. When applied to supergravity, this formalism is equivalent to the superspace tensor calculus [34], which is briefly described in Appendix 4, and which was originally developed for supergravity by Wess and Zumino [1, 35, 36, 37], and by others, e.g. [38].

We begin by introducing gauge transformations and exterior derivatives for tensor fields. Under a gauge transformation, a tensor field transforms by a term which involves only the gauge parameters ξ^M and not their derivatives. We can therefore write

$$\delta_\xi T = \xi^M \Delta_M T; \tag{2.49}$$

this essentially defines an operator Δ_M on tensor fields. For the exterior derivative $d = dx^\mu \partial_\mu$, we write

$$dT = A^M \Delta_M T. \tag{2.50}$$

This introduces the gauge fields A_μ^M through $A^M = dx^\mu A_\mu^M$. We now assume that the gauge fields A_μ^M include the components of an invertible matrix $e_\mu{}^m$, and we denote the rest of the gauge fields by $A_\mu^{\hat{M}}$. Eq. (2.50) now allows us to define gauge covariant derivatives as

$$\Delta_m = e_m{}^\mu \big(\partial_\mu - A_\mu^{\hat{M}} \Delta_{\hat{M}}\big) T. \tag{2.51}$$

In order for this to be gauge covariant, we must have that $\Delta_M T$ is a tensor field, if T is a tensor field. This requires that the operators Δ_M obey

$$\Delta_M(T_1 T_2) = (\Delta_M T_1) T_2 + (-1)^{|M||T_1|} T_1 (\Delta_M T_2), \tag{2.52}$$

where $|O|$ is the Grassmann parity, or the grading, of the object O, and $|M| = |\Delta_M| = |\xi^M|$. For the gauge fields we have $|A_\mu^M| = |M|$.

In order to ensure that our formalism really provides an off-shell formulation of a gauge theory, we must introduce certain consistency requirements. The first consistency condition is that the algebra of gauge transformations must close:

$$[\delta_{\xi_1}, \delta_{\xi_2}]T = \delta_{\xi_{12}}T, \tag{2.53}$$

with possibly field-dependent parameters ξ_{12}. From Eq. (2.49), we find that this implies

$$\xi_2^N \xi_1^M \big|\Delta_M, \Delta_N\big| T = \xi_{12}^P \Delta_P T, \tag{2.54}$$

where the bars denote the graded commutator,

$$\big|A, B\big| = AB - (-1)^{|A||B|} BA. \tag{2.55}$$

Therefore, $|A, B| = [A, B]$, unless both of A and B are odd, in which case $|A, B| = \{A, B\}$.

Since the right-hand side of Eq. (2.54) contains only the parameters ξ_{12} and not their derivatives, we may introduce the tensor fields $\mathcal{F}_{NM}{}^P$ by writing $\xi_{12}^P = \xi_1^M \xi_2^N \mathcal{F}_{NM}{}^P$. We then have

$$\big|\Delta_M, \Delta_N\big| = -\mathcal{F}_{NM}{}^P \Delta_P. \tag{2.56}$$

The $\mathcal{F}$'s have the same symmetry properties as the graded commutator, namely

$$\mathcal{F}_{MN}{}^P = -(-1)^{|M||N|} \mathcal{F}_{NM}{}^P. \tag{2.57}$$

The graded Jacobi identity

$$\sum_{(MNP)} \big|\Delta_M, |\Delta_N, \Delta_P|\big| = 0, \tag{2.58}$$

where the cyclic sum is defined as

$$\sum_{(MNP)} O_{MNP} = (-1)^{|M||P|} O_{MNP} + (-1)^{|N||M|} O_{NPM} + (-1)^{|P||N|} O_{PMN},$$

now gives constraints on the structure functions $\mathcal{F}$. Assuming that the Δ_M are linearly independent, we find that Eqs. (2.56) and (2.58) imply the Bianchi identities

$$\sum_{(MNP)} \Delta_M \mathcal{F}_{NP}{}^Q + \mathcal{F}_{MN}{}^R \mathcal{F}_{RP}{}^Q = 0. \tag{2.59}$$

These identities give rise to nontrivial restrictions when this formalism is applied to a gauge theory.

We will introduce two further consistency requirements. We require that exterior differentiation commutes with gauge transformations, and that the exterior derivative satisfies $d^2 = 0$. A detailed analysis of these conditions is given in [20]. The result is that the first requirement gives the transformation law of the gauge fields as

$$\delta_\xi A_\mu^M = \partial_\mu \xi^M + A_\mu^P \xi^N \mathcal{F}_{NP}{}^M, \tag{2.60}$$

while from the second requirement there results the equation

$$\partial_\mu A_\nu^M - \partial_\nu A_\mu^M + A_\mu^P A_\nu^N \mathcal{F}_{NP}{}^M = 0. \tag{2.61}$$

This equation can be solved for the structure functions $\mathcal{F}_{mn}{}^M$, which can be interpreted as curvatures corresponding to the derivatives Δ_m, since $[\Delta_m, \Delta_n]T = -\mathcal{F}_{mn}{}^M \Delta_M T$. We have

$$\begin{aligned}\mathcal{F}_{mn}{}^M = e_m{}^\mu e_n{}^\nu \Big(\partial_\mu A_\nu^M - \partial_\nu A_\mu^M + A_\mu^{\hat{P}} A_\nu^{\hat{N}} \mathcal{F}_{\hat{N}\hat{P}}{}^M\Big) \\ + A_\mu^{\hat{N}} \Big(e_m{}^\mu \mathcal{F}_{n\hat{N}}{}^M - e_n{}^\mu \mathcal{F}_{m\hat{N}}{}^M\Big).\end{aligned} \tag{2.62}$$

For the gauge fields A_μ^M, one can use Eqs. (2.59) and (2.60) to show that all of the consistency requirements are satisfied on them.

We note that we can put the transformation laws of the various fields into a more standard form by defining the new transformation parameters

$$\xi^\mu = e_m{}^\mu \xi^m, \qquad \epsilon^{\hat{M}} = \xi^{\hat{M}} - \xi^\mu A_\mu^{\hat{M}}.$$

The transformation laws of the fields are then given by

$$\begin{aligned}\delta_\xi T &= \xi^\nu \partial_\nu T + \epsilon^{\hat{M}} \Delta_{\hat{M}} T, \\ \delta_\xi e_\mu{}^m &= \xi^\nu \partial_\nu e_\mu{}^m + \partial_\mu \xi^\nu e_\nu{}^m + A_\mu^P \epsilon^{\hat{N}} \mathcal{F}_{\hat{N}P}{}^m, \\ \delta_\xi A_\mu^{\hat{M}} &= \xi^\nu \partial_\nu A_\mu^{\hat{M}} + \partial_\mu \xi^\nu A_\nu^{\hat{M}} + \partial_\mu \epsilon^{\hat{M}} + A_\mu^P \epsilon^{\hat{N}} \mathcal{F}_{\hat{N}P}{}^{\hat{M}}.\end{aligned} \tag{2.63}$$

Now the transformations proportional to ξ^μ are generated by the Lie derivative, and so they correspond to general coordinate transformations.

Finally, we would like to remark that even though Eq. (2.60) looks like the transformation law of the gauge field in a Yang–Mills theory, the algebra (2.56) is not a graded Lie algebra, but instead a more general algebraic structure, because in the place of the structure constants there are the objects $\mathcal{F}_{MN}{}^P$, which need not be constant.

2.5 Off-shell formulation of simple supergravity

We now apply the above formalism to construct an off-shell formulation of simple supergravity [20, 21]. This amounts to choosing a suitable set of structure functions $\mathcal{F}_{MN}{}^P$, which have to satisfy the Bianchi identity (2.59). The invertible matrix $e_\mu{}^m$, which is contained among the gauge fields A_μ^M, is now identified with the vielbein. The gravitino and the spin connection constitute the rest of the gauge fields $A_\mu^{\hat{M}}$. The gauge transformations corresponding to these fields are given by general coordinate transformations, local supersymmetry transformations, and local Lorentz transformations.

We denote the generators of general coordinate transformations and local supersymmetry transformations with $\mathcal{D}_A$. Thus,

$$\Delta_M = \big(\mathcal{D}_A, M_{mn}\big), \qquad \mathcal{D}_A = \big(\mathcal{D}_m, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}\big).$$

We adopt the summation conventions

$$\begin{aligned} X^M Y_M &= X^A Y_A + \frac{1}{2} X^{mn} Y_{mn}, \\ X^A Y_A &= X^m Y_m + X^\alpha Y_\alpha + X_{\dot\alpha} Y^{\dot\alpha} = X^m Y_m + X^{\underline{\alpha}} Y_{\underline{\alpha}}. \end{aligned} \tag{2.64}$$

From Eq. (2.50) we then have

$$\begin{aligned} \mathcal{D}_m &= e_m{}^\mu \Big(\partial_\mu - \psi_\mu^\alpha \mathcal{D}_\alpha - \psi_{\mu\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} - \tfrac{1}{2}\omega_\mu{}^{mn} M_{mn}\Big) \\ &= e_m{}^\mu \big(D_\mu - \psi_\mu^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}}\big). \end{aligned} \tag{2.65}$$

The derivative $\mathcal{D}_m$ is called a supercovariant derivative, since in addition to the spin connection it involves the gravitino, and so it is covariant also with respect to local supersymmetry transformations. It does not contain the Christoffel connection $\Gamma^\lambda_{\mu\nu}$, because in the present formalism, tensor fields must be scalars with respect to general coordinate transformations, since we have required that the gauge transformation of a tensor field must not contain derivatives of the transformation parameters. Supercovariant tensor fields therefore cannot carry curved spacetime indices, and so $\mathcal{D}_m$ does not need a term with $\Gamma^\lambda_{\mu\nu}$.

We now turn to study the restrictions that we should put on the structure functions $\mathcal{F}$. The commutators of the generators Δ_M with M_{mn} involve the functions $\mathcal{F}_{[mn]P}{}^R$, and so these functions should form representations of the Lorentz group. We therefore take

$$\begin{aligned} \mathcal{F}_{[mn]r}{}^s &= -\eta_{mr}\delta_n^s + \eta_{nr}\delta_m^s, \\ \mathcal{F}_{[mn]\alpha}{}^\beta &= (\sigma_{mn})_\alpha{}^\beta, \qquad \mathcal{F}_{[mn]\dot\alpha}{}^{\dot\beta} = (\bar\sigma_{mn})_{\dot\alpha}{}^{\dot\beta}, \\ \mathcal{F}_{[mn][pr]}{}^{[st]} &= -2\Big(\eta_{mp}\delta_n^{[s}\delta_r^{t]} - \eta_{np}\delta_m^{[s}\delta_r^{t]} + \eta_{nr}\delta_m^{[s}\delta_p^{t]} - \eta_{mr}\delta_n^{[s}\delta_p^{t]}\Big), \end{aligned} \tag{2.66}$$

corresponding to the vector, spinor and adjoint representations, respectively. All the functions $\mathcal{F}_{[mn]P}{}^R$ where P and R are not the same type of index are taken to vanish. The remaining functions $\mathcal{F}_{AB}{}^M$ can be interpreted as the torsion and curvature of superspace:

$$\mathcal{F}_{AB}{}^C = T_{AB}{}^C, \qquad \mathcal{F}_{AB}{}^{[mn]} = R_{AB}{}^{mn}. \tag{2.67}$$

The algebra of the operators $\Delta_M = (\mathcal{D}_A, M_{mn})$ now reads

$$\begin{aligned} \big|\mathcal{D}_A, \mathcal{D}_B\big| &= -T_{AB}{}^C \mathcal{D}_C - \frac{1}{2} R_{AB}{}^{mn} M_{mn}, \\ [M_{mn}, \mathcal{D}_r] &= \eta_{mr}\mathcal{D}_n - \eta_{nr}\mathcal{D}_m, \\ [M_{mn}, \mathcal{D}_\alpha] = -(\sigma_{mn})_\alpha{}^\beta \mathcal{D}_\beta, &\qquad [M_{mn}, \bar{\mathcal{D}}^{\dot\alpha}] = -(\bar\sigma_{mn})^{\dot\alpha}{}_{\dot\beta} \bar{\mathcal{D}}^{\dot\beta}. \end{aligned} \tag{2.68}$$

In contrast to Minkowski space, there is nonvanishing torsion even in flat superspace, where ${e_m}^{\mu} = \delta_m^{\mu}$. From the anticommutator $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2i(\sigma^m)_{\alpha\dot{\beta}} P_m$ we can read off that ${T_{\alpha\dot{\beta}}}^m = 2i(\sigma^m)_{\alpha\dot{\beta}}$.

We still need to place further constraints on the above introduced torsion and curvature tensors, since they contain a great number of superfluous components, and we would like to eliminate as many of them as possible. The constraints need to respect the Bianchi identities (2.59). It turns out that the only nontrivial identities are

$$\sum_{(ABC)} \mathcal{D}_A {T_{BC}}^D + {T_{AB}}^E {T_{EC}}^D - {R_{ABC}}^D = 0, \tag{2.69}$$

$$\sum_{(ABC)} \mathcal{D}_A {R_{BC}}^{mn} + {T_{AB}}^D {T_{DC}}^{mn} = 0, \tag{2.70}$$

where

$${R_{ABC}}^D = -\frac{1}{2} {R_{AB}}^{mn} {\mathcal{F}_{[mn]C}}^D. \tag{2.71}$$

The remaining identities are automatically satisfied due to the functions ${\mathcal{F}_{[mn]P}}^R$ giving representations of the Lorentz group and the torsion and curvature being Lorentz tensors. We would now have to solve the above identities for the components of the torsion and curvature tensors in terms of as few independent fields as possible. Some of the components of ${T_{AB}}^C$ and ${R_{ABC}}^D$ can be expressed in terms of their remaining components and the gauge fields using (2.62) Furthermore, it can be shown that (2.70) is a consequence of (2.69), and so it suffices to solve only the identities (2.69) [40].

We will only give a brief outline of the long analysis required to solve the Bianchi identities. To begin with, a field redefinition of the form $\Delta_M \to {O_M}^N \Delta_N$, with the corresponding adjustments on the gauge transformation parameters, can be used to make as many as possible of the structure functions vanish. A detailed analysis [39] shows that we may choose

$$\begin{gathered} {T_{\alpha\dot{\beta}}}^m = {T_{\dot{\beta}\alpha}}^m = 2i(\sigma^m)_{\alpha\dot{\beta}}, \\ {T_{mn}}^r = 0, \quad {T_{\alpha\beta}}^\gamma = {T_{\dot{\alpha}\dot{\beta}}}^{\dot{\gamma}} = 0, \quad {T_{\alpha\dot{\beta}}}^{\underline{\gamma}} = {T_{\alpha\dot{\beta}}}^{\underline{\gamma}} = 0, \end{gathered} \tag{2.72}$$

where $\underline{\gamma}$ denotes either γ or $\dot{\gamma}$. Different further restrictions now lead to different formulations of simple supergravity. The so-called minimal formulation is obtained by imposing the conditions

$${T_{\alpha\beta}}^m = {T_{\dot{\alpha}\dot{\beta}}}^m = 0, \qquad {T_{\underline{\alpha}m}}^n = -{T_{m\underline{\alpha}}}^n = 0. \tag{2.73}$$

With these constraints, it follows from the Bianchi identity with $ABCD = \alpha\beta\gamma\delta$ that

$${T_{\alpha\beta}}^{\dot{\gamma}} = 0, \quad \text{and} \quad {T_{\dot{\alpha}\dot{\beta}}}^\gamma = 0. \tag{2.74}$$

With these constraints, there remain thirteen independent Bianchi identities which have to be solved. The result of a complicated analysis, whose details are given e.g. in [1] and [41], shows that all of the components of the torsion and the curvature can be expressed in terms of a complex scalar field C and a real vector field b_m, or can be written down using Eq. (2.62). The Bianchi identities give

$$\begin{aligned} T_{\dot{\alpha}m}{}^{\beta} &= \frac{i}{8} C (\sigma_m)^{\beta}{}_{\dot{\alpha}}, \qquad T_{\dot{\alpha}m}{}^{\dot{\beta}} = -i\Big(\delta^{\dot{\beta}}_{\dot{\alpha}} b_m + b^n (\bar{\sigma}_{nm})^{\dot{\beta}}{}_{\dot{\alpha}}\Big), \\ R_{\dot{\alpha}\dot{\beta}}{}^{mn} &= -C(\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}, \qquad R_{\alpha\dot{\beta}}{}^{mn} = 2i\epsilon^{mnrs}(\sigma_r)_{\alpha\dot{\beta}} b^s, \\ R_{\dot{\alpha}m}{}^{rs} &= i\Big((\sigma_m)_{\alpha\dot{\alpha}} T^{rs\alpha} - (\sigma^r)_{\alpha\dot{\alpha}} T^s{}_m{}^{\alpha} + (\sigma^s)_{\alpha\dot{\alpha}} T^r{}_m{}^{\alpha}\Big), \end{aligned} \tag{2.75}$$

while from Eq. (2.62) we find

$$\begin{aligned} T_{mn}{}^{r} &= e_m{}^{\mu} e_n{}^{\nu} \Big(D_\mu e_\nu{}^r - D_\nu e_\mu{}^m - 2i\psi_\mu \sigma^r \bar{\psi}_\nu + 2i\psi_\nu \sigma^r \bar{\psi}_\mu\Big), \\ T_{mn}{}^{\alpha} &= e_m{}^{\mu} e_n{}^{n} D_\mu \psi_\nu^\alpha + \psi_m^\alpha T_{n\beta}{}^{\alpha} + T_{m\dot{\beta}}{}^{\alpha} \bar{\psi}_n^{\dot{\beta}} - (m \leftrightarrow n), \\ R_{mn}{}^{rs} &= e_m{}^{\mu} e_n{}^{\nu} R_{\mu\nu}{}^{rs}(\omega) + \psi_m^{\underline{\alpha}} \psi_n^{\underline{\beta}} R_{\underline{\alpha\beta}}{}^{rs} - \psi_m^{\underline{\alpha}} R_{\underline{\alpha}n}{}^{rs} + \psi_n^{\underline{\alpha}} R_{\underline{\alpha}m}{}^{rs} \end{aligned} \tag{2.76}$$

where $R_{\mu\nu}{}^{rs}(\omega)$ is the curvature tensor given in Eq. (2.19) with $K = 0$. The first equation shows that from the constraint $T_{mn}{}^{r} = 0$ we recover the condition (2.36).

All of the remaining torsions and curvatures either vanish or are related by complex conjugation to those that are already known. The fields C and b_μ will fulfill the task of auxiliary fields in the off-shell formulation of four-dimensional supergravity.

The supersymmetry transformations of the fields are found either from Eq. (2.63), or from the Bianchi identities, which give the action of $\mathcal{D}_\alpha$ on the auxiliary fields as

$$\begin{aligned} \mathcal{D}_\alpha C &= \frac{16}{3} (\sigma^{mn})_{\alpha\beta} T_{mn}{}^{\beta}, \qquad \mathcal{D}_\alpha \bar{C} = 0, \\ \mathcal{D}_\alpha b_{\beta\dot{\beta}} &= \frac{1}{3} \epsilon_{\beta\alpha} (\bar{\sigma}^{mn})_{\dot{\beta}\dot{\alpha}} T_{mn}{}^{\dot{\alpha}} - (\sigma^{mn})_{\alpha\beta} T_{mn\dot{\beta}}. \end{aligned} \tag{2.77}$$

The supersymmetry generators now satisfy the algebra

$$\begin{aligned} \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} &= -2i(\sigma^m)_{\alpha\dot{\alpha}} + 2b_{\beta\dot{\alpha}} M_\alpha{}^\beta - 2B_{\alpha\dot{\beta}} \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} \\ \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= \bar{C} M_{\alpha\beta}, \qquad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = -C \bar{M}_{\dot{\alpha}\dot{\beta}}, \end{aligned} \tag{2.78}$$

where

$$M_{\alpha\beta} = \frac{1}{2} (\sigma^{mn})_{\alpha\beta} M_{mn}, \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}} M_{mn}. \tag{2.79}$$

These operators give the self-dual and anti-self-dual parts of M_{mn}, as explicitly shown by the decomposition

$$M_{mn} = (\sigma_{mn})^{\alpha\beta} M_{\alpha\beta} - (\bar{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}.$$

It is clear that the algebra of supersymmetry transformations now closes by construction. This is guaranteed by the closure of the algebra of the Δ_M operators. In the commutator

$$\big[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\big] = \delta_G(\xi_{12}) + \delta_L(\lambda_{12}) + \delta_Q(\epsilon_{12})$$

the auxiliary fields C and b_μ appear only in the parameter λ_{12} of local Lorentz transformations.

A way to construct an invariant Lagrangian for supergravity can now be found by investigating how the chiral multiplet of section 1.3 should be generalized to the case of local supersymmetry. It turns out that a proper generalization of the chiral multiplet is given by the fields (ϕ, χ, F), where the chiral scalar field ϕ satisfies the constraints

$$\bar{\mathcal{D}}_{\dot\alpha}\phi = 0, \qquad \bar{M}_{\dot\alpha\dot\beta}\phi = 0, \tag{2.80}$$

and the remaining fields in the multiplet are given in terms of ϕ by

$$\chi_\alpha = \mathcal{D}_\alpha\phi, \qquad F = -\frac{1}{2}\mathcal{D}^2\phi. \tag{2.81}$$

The requirement that the algebra (2.78) is satisfied on the fields determines the action of the supercovariant derivatives on the fields as

$$\begin{aligned}
\mathcal{D}_\alpha\phi &= \chi_\alpha, & \bar{\mathcal{D}}_{\dot\alpha}\phi &= 0,\\
\mathcal{D}_\alpha\chi_\beta &= -\epsilon_{\alpha\beta}F, & \bar{\mathcal{D}}_{\dot\alpha}\chi_\beta &= -2i(\sigma^m)_{\beta\dot\alpha}\mathcal{D}_m\phi,\\
\mathcal{D}_\alpha F &= -\frac{1}{2}\bar{C}\chi_\alpha, & \bar{\mathcal{D}}_{\dot\alpha}F &= -2i(\sigma^m)_{\alpha\dot\alpha}\mathcal{D}_m\chi^\alpha + (\sigma^m)_{\alpha\dot\alpha}b_m\chi^\alpha,
\end{aligned} \tag{2.82}$$

where

$$\begin{aligned}
\mathcal{D}_m\phi &= e_m{}^\mu\big(\partial_\mu\phi - \psi_\mu\chi\big),\\
\mathcal{D}_m\chi &= e_m{}^\mu\big(D_\mu\chi - \psi_\mu F - 2i\sigma^\nu\bar\psi_\mu\mathcal{D}_\nu\phi\big).
\end{aligned} \tag{2.83}$$

Using the above transformation laws, one can calculate the supersymmetry variation of the field F and find that it is not a total derivative. However, it can be shown [20, 21] that the combination

$$|e|\Big(2F + 4i\psi_\mu\sigma^\mu\chi - 3\bar{C}\phi - 16\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu\phi\Big)$$

transforms by a total derivative under supersymmetry. We therefore have that if we take any chiral scalar field ϕ, and apply to it the operator

$$\Delta = -\mathcal{D}^2 + 4i\bar\psi_\mu\bar\sigma^\mu\mathcal{D} - 3\bar{C} - 16\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu + \text{c.c.}, \tag{2.84}$$

we obtain an expression which is real, and invariant under supersymmetry when multiplied by $|e|$ and integrated over the whole spacetime.

The Lagrangian for simple supergravity is obtained by taking $\phi = \frac{3}{32}C$. Using the Bianchi identities, one can show that

$$\mathcal{D}^2 C = -\frac{8}{3} R_{mn}{}^{mn} + 2\bar{C}C - 16 b_m b^m + 16i\mathcal{D}_m b^m,$$

where $R_{mn}{}^{mn}$ can be calculated using Eq. (2.76). The Lagrangian is then given by

$$\mathcal{L}_0 = |e|\Big(-\frac{1}{2} e_m{}^\mu e_n{}^\nu R_{\mu\nu}{}^{mn} - 3b_\mu b^\mu - \frac{3}{16}\bar{C}C\Big) + \epsilon^{\mu\nu\rho\sigma}\Big(D_\mu \psi_\nu \sigma_\rho \bar{\psi}_\sigma + \psi_\sigma \psi_\rho D_\mu \bar{\psi}_\nu\Big). \tag{2.85}$$

On-shell, the auxiliary fields C and b_μ vanish, provided that we do not have couplings to matter or other generalizations, so that (2.85) is the whole Lagrangian. After the elimination of the auxiliary fields, the Lagrangian becomes equal to that of Eq. (2.34). In the present formalism, the action is supersymmetric by construction, and so there is no need to show its supersymmetry by calculation.

To conclude this section, we point out that any complex number λ is trivially a chiral field. Even so, we can construct from λ the non-trivial Lagragian

$$\mathcal{L}_\lambda = -|e|\lambda\Big(3\bar{C} + 16\bar{\psi}_\mu \bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\Big) + \text{c.c.} \tag{2.86}$$

The theory defined by the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\lambda$ has a massive gravitino, as well as a negative cosmological constant. The vacuum of this theory is therefore anti de-Sitter space, instead of a Minkowski space.

2.6 Supersymmetry in higher dimensions

We would now like to move on to discuss supergravity theories in spacetimes of higher dimension. We begin by generalizing the analysis of the representations of the supersymmetry algebra to higher-dimensional spacetimes, leading to a knowledge of all possible supergravity theories in spacetimes of various dimensions [22, 23]. Because the generators of supersymmetry transformations are spinors, we will have to include a discussion on which kind of spinors can exist in different spacetime dimensions [42, 43]. At this point, we will abandon the two-component notation for spinors, as it does not generalize to arbitrary dimensions.

In a D-dimensional spacetime, a representation of the Clifford algebra

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn} \tag{2.87}$$

is given by a set of $2^{\lfloor D/2 \rfloor}$ gamma matrices, which can be taken to be

$$\begin{aligned}\Gamma_0 &= i\sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \cdots ,\\ \Gamma_1 &= \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \cdots ,\\ \Gamma_2 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots ,\\ \Gamma_3 &= \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \cdots ,\\ \Gamma_4 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \cdots ,\\ &\text{etc.}\end{aligned} \tag{2.88}$$

Under hermitian conjugation, they behave as

$$\Gamma_0^\dagger = -\Gamma_0, \qquad \Gamma_i^\dagger = \Gamma_i. \tag{2.89}$$

In even dimensions, the generalization of γ_5 is given by the object

$$\bar{\Gamma} = -(-i)^{\lfloor D/2 \rfloor}\Gamma_0\Gamma_1\ldots\Gamma_{D-1}, \tag{2.90}$$

which satisfies $\bar{\Gamma}^2 = 1$, and anticommutes with all of the other gamma matrices. In the representation (2.88) we have $\bar{\Gamma} = \sigma_3 \otimes \sigma_3 \cdots$.

In both even and odd dimensions, there exists a charge conjugation matrix C with the properties

$$C^T = -\epsilon C, \qquad \Gamma_m^T = -\eta C\Gamma_m C^T, \tag{2.91}$$

where $\epsilon = \pm 1$ and $\eta = \pm 1$. The values of ϵ and η depend on D, and are given in [42].

In D dimensions, an unconstrained spinor has $2^{\lfloor D/2 \rfloor}$ components. There are two ways to reduce the number of components and so obtain irreducible representations of the Clifford algebra. In even dimensions, we can define the projection operators

$$P_L = \frac{1}{2}\big(1 + \bar{\Gamma}\big), \qquad P_R = \frac{1}{2}\big(1 - \bar{\Gamma}\big), \tag{2.92}$$

and use them to cut the number of spinor components in half by defining Weyl spinors through the conditions $P_L\psi = \psi$ and $P_R\psi = \psi$.

In some dimensions, it is possible to put on a complex spinor a reality condition of the form

$$\psi^* = B\psi, \tag{2.93}$$

where

$$B = e^{i\alpha}\big(CA^{-1}\big)^T, \qquad A = \Gamma_0 \ldots \Gamma_{D-1}.$$

A consistency analysis, based on examining the Lorentz transformations of the both sides of (2.93), then shows that it is possible to impose a reality condition when $D = 0, 1, \ldots, 4 \bmod 8$. Spinors satisfying such a condition are called

Majorana spinors. (We will not make a distinction between Majorana spinors and pseudo-Majorana spinors, contrary to e.g. the references [23] and [24].)

Even if a reality condition of the form (2.93) is not possible, a so-called symplectic condition can still be introduced, provided that we are dealing with a multiplet of spinors. The condition reads

$$\psi_i^* = B\Omega_{ij}\psi_j, \tag{2.94}$$

where Ω is an antisymmetric matrix satisfying $\Omega^*\Omega = 1$.

If the reality condition (2.93) can be consistently enforced on Weyl spinors, then both the Weyl and Majorana conditions can be imposed on spinors simultaneously. This is possible when $D = 2 \bmod 8$. A simultaneous imposition of the Weyl condition and the symplectic condition is possible when $D = 6 \bmod 8$.

The particle content of supergravity theories is determined by the representations of the supersymmetry algebra. A detailed analysis is given in [44], and its results are summarized in [24].

If we consider massless representations, we have as in section 1.2 that half of the supercharges vanish (see e.g. [22]), and of the remaining supercharges half act as raising operators and half as lowering operators with respect to helicity. The states in the representations are constructed by choosing a state of lowest helicity and acting on it with the raising operators. We must not have too many raising operators in order to not have states of spin higher than 2. In eleven spacetime dimensions, a single unconstrained spinor supercharge has 32 components, while in twelve dimensions, the number of components is already 64. Thus, we find that eleven is the highest number of dimensions in which a supergravity theory can be constructed, as in twelve dimensions a single Majorana spinor would give rise to 16 raising operators. Furthermore, the eleven-dimensional theory is unique, since there is only one $N = 1$ multiplet which contains a spin-2 particle but is free of a spin-5/2 particle. A table of all possible theories of pure supergravity – that is, theories which are free of matter fields – is given in [24]; there are 26 of them in total.

2.7 Eleven-dimensional supergravity

In eleven dimensions, the most general form of the supersymmetry algebra is given by

$$\{Q_a, \bar{Q}_b\} = 2(\Gamma^\mu)_{ab}P_\mu + \left(\Gamma^{\mu\nu}\right)_{ab}Z_{\mu\nu} + \left(\Gamma^{\mu_1\cdots\mu_5}\right)_{ab}Z_{\mu_1\cdots\mu_5}. \tag{2.95}$$

In contrast to four dimensions, it is possible to have nonvanishing central charges even though there is only one supercharge. For the moment, we will set the central charges to zero; we will return to them at the end of this section.

The fields which appear in the eleven-dimensional theory of supergravity are determined by a suitable massless representation of the supersymmetry algebra. As we now have $P^2 = 0$, we can take $P_0 = P_{10}$ and set the rest of the components of P_μ to zero. The supersymmetry algebra can then be put into the form

$$\{Q_a, Q_b^\dagger\} = 2\big(\Gamma^\mu\Gamma^0\big)_{ab}P_\mu = 2\big(1+\Gamma^{10}\Gamma^0\big)_{ab}P_0, \tag{2.96}$$

in which the object on the right-hand side is a projection operator. This shows explicitly that half of the supercharges act as zero on states in the representation.

The field content of eleven-dimensional supergravity could now be constructed by group-theoretical arguments. From Eq. (2.96) one can show that the sixteen non-vanishing supercharges obey a Clifford algebra of the form $\{Q_a, Q_b\} \sim \delta_{ab}$, and so form a representation of the group $SO(16)$. On the other hand, the physical degrees of freedom of a massless particle in eleven dimensions give a representation of $SO(9)$. The fields are then determined by how the various representations decompose under the embedding $SO(16) \supset SO(9)$ [22, 23].

We choose to not go into any further into the group-theoretical analysis here, because the proper set of fields can also be deduced by counting the transverse degrees of freedom of the fields. The vielbein $e_\mu{}^m$ has $\frac{1}{2}(D-1)(D-2)-1 = 44$ components, while the Majorana spinor $(\psi_\mu)_a$ has $\frac{1}{2}2^{\lfloor D/2\rfloor}(D-2-1) = 128$ components. The -1 in these expressions is due to a tracelessness requirement on $e_\mu{}^m$ and ψ_μ, i.e. $\gamma^\mu\psi_\mu = 0$, and similarly for the vielbein. The remaining bosonic degrees of freedom can be assigned to an antisymmetric scalar field $A_{\mu\nu\rho}$, which has $\binom{9}{3} = 84$ components, as required.

The Lagrangian of eleven-dimensional supergravity was found by Cremmer, Julia and Scherk in 1978 [45]. Not long after, a superspace formulation of the theory was constructed [46, 47], but an off-shell formulation of the theory is, to the best of my knowledge, not known.

The Lagrangian has the form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}|e|R(\omega) - |e|\bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\Big(\tfrac{1}{2}(\omega+\hat\omega)\Big)\psi_\rho - \frac{1}{24}|e|F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\\ &-\frac{1}{96}|e|\Big(\bar\psi_{\mu_1}\Gamma^{\mu_1\mu_2\nu_1\cdots\nu_4}\psi_{\mu_2} + 12\bar\psi^{\nu_1}\Gamma^{\nu_2\nu_3}\psi^{\nu_4}\Big)\big(F_{\nu_1\cdots\nu_4} + \hat F_{\nu_1\cdots\nu_4}\big)\\ &+\frac{4}{144^2}\epsilon^{\mu_1\cdots\mu_{11}}F_{\mu_1\cdots\mu_4}F_{\mu_5\cdots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}.\end{aligned} \tag{2.97}$$

Here $F_{\mu\nu\rho\sigma}$ is the field strength of the gauge field $A_{\mu\nu\rho}$,

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]}, \tag{2.98}$$

and the covariant derivative of the gravitino is given by

$$\begin{aligned} D_\mu(\omega)\psi_\nu &= \partial_\mu\psi_\nu - \frac{1}{4}\omega_\mu{}^{mn}\Gamma_{mn}\psi_\nu,\\ \omega_\mu{}^{mn} &= \omega_\mu{}^{mn}(e) + K_\mu{}^{mn}, \end{aligned} \tag{2.99}$$

where $K_\mu{}^{mn}$ is a sum of terms involving the ψ_μ and the gamma matrices. The Lagrangian contains terms up to fourth order in the fermions. These are absorbed in the supercovariant objects

$$\begin{aligned}\hat{\omega}_\mu{}^{mn} &= \omega_\mu{}^{mn} + \frac{1}{4}\bar{\psi}_\nu \Gamma_\mu{}^{mn\nu\rho}\psi_\rho, \\ \hat{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma} + 3\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]}.\end{aligned} \tag{2.100}$$

That they are supercovariant means that their variations under supersymmetry do not contain derivatives of the transformation parameter ϵ.

The transformation laws of the fields under supersymmetry are given by

$$\begin{aligned}\delta e_\mu{}^m = \bar{\epsilon}\Gamma^m \psi_\mu, \qquad \delta A_{\mu\nu\rho} = -\frac{3}{2}\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \\ \delta\psi_\mu = D_\mu(\hat{\omega})\epsilon + \frac{1}{144}\Big(\Gamma_\mu{}^{\nu_1\cdots\nu_4} + 8\Gamma^{\nu_1\nu_2\nu_3}\delta_\mu^{\nu_4}\Big)\epsilon\hat{F}_{\nu_1\cdots\nu_4} \equiv \hat{D}_\mu\epsilon.\end{aligned} \tag{2.101}$$

In [45], a description is given of how the Lagrangian (2.97) is obtained by a brute force calculation. One begins with a Lagrangian which contains a kinetic term for each of the fields e, ψ and A. The supersymmetry variation of the Lagrangian can be put into a form where similar types of terms are grouped together. Some types of terms cancel by themselves. To make possible the cancellation of the remaining terms, additional terms are introduced into the Lagrangian. These eventually take the form of the interaction term and the topological term in Eq. (2.97). In the course of the calculation, the covariant objects $\hat{\omega}$ and $\hat{F}$ are introduced. Once the Lagrangian has been constructed, its invariance under supersymmetry, and the on-shell closure of the supersymmetry algebra, are demonstrated by an explicit calculation.

In addition to its invariance under supersymmetry transformations, as well as general coordinate transformations and local Lorentz transformations, the Lagrangian (2.97) is invariant under the gauge transformations $\delta A_{\mu\nu\rho} = \partial_{[\mu}\Lambda_{\nu\rho]}$. Furthermore, the effect of the so-called Weyl transformations,

$$e_\mu{}^m \to e^\alpha e_\mu{}^m, \qquad \psi_\mu \to e^{\alpha/2}\psi_\mu, \qquad A_{\mu\nu\rho} \to e^{3\alpha}A_{\mu\nu\rho}, \tag{2.102}$$

is to multiply the Lagrangian by the constant factor $e^{9\alpha}$. We will restore the gravitational constant κ for a moment; the Lagrangian is then given by the expression in Eq. (2.97) multiplied with κ^{-2}. While this is not a symmetry of the Lagrangian, it is a symmetry of the equations of motion, because two Lagrangians which differ by a constant factor give rise to the same equations of motion. Furthermore, the effect of a Weyl transformation can then be cancelled through a scaling of κ by $e^{9\alpha/2}$.

To conclude this section, we will give an interpretation of the central charges appearing in the algebra (2.95) [22]. By analogy with electrodynamics, where the magnetic and the electric fluxes are given by the surface integrals $\oint d\Sigma \cdot F$

and $\oint d\Sigma \cdot \star F$, respectively, we could think of calculating fluxes corresponding to the field strength $F_{\mu_1 \cdots \mu_4}$, and its dual

$$\star F_{\mu_1 \cdots \mu_7} = \frac{1}{7!} \epsilon_{\mu_1 \cdots \mu_7 \nu_1 \cdots \nu_4} F^{\nu_1 \cdots \nu_4} - F_{[\mu_1 \cdots \mu_4} A_{\mu_5 \mu_6 \mu_7]},$$

The fluxes would be given by surface integrals of F over the boundary of a 5-dimensional region, and of $\star F$ over the boundary of an 8-dimensional region. These regions are orthogonal to a 5-brane and a 2-brane (branes will be explained in Chapter 3), and so the charges corresponding to the fluxes are Lorentz tensors of rank 5 and rank 2. These can be identified with the central charges $Z_{\mu_1 \cdots \mu_5}$ and $Z_{\mu\nu}$ of Eq. (2.95).

2.8 Ten-dimensional supergravities

In ten dimensions, it is possible to simultaneously impose the Majorana and the Weyl condition on spinors, and so the supercharges are Majorana–Weyl spinors. As such, each supercharge has eight components, which constitute an eight-dimensional Clifford algebra. We therefore have three different representations of the group $SO(8)$, one of which contains the supercharges, and the other two contain the bosonic and fermionic fields. For more details, including how the particle content of ten-dimensional supergravities is deduced from these representations, see [22].

There are three different theories of supergravity in ten dimensions. The $N = 1$ theory, which is based on a single Majorana–Weyl supercharge, has the name of type I supergravity [48, 49]. There are two different $N = 2$ theories. The so-called type IIA supergravity [50, 51, 52] has two Weyl supercharges of opposite chirality, which arise from a single Majorana supercharge. We therefore say that it is a $N = (1, 1)$ theory. The third ten-dimensional theory is known as type IIB supergravity [53, 54, 55]. It has two Majorana–Weyl supercharges of the same chirality, and so it is a $N = (2, 0)$ theory.

Each of the ten-dimensional supergravities is the low energy limit of a corresponding string theory: type IIA supergravity of the type IIA superstring theory, and likewise for the type IIB theories, while for type I supergravity there correspond the type I superstring theory and the heterotic string theory.

In the rest of this section, we will give an outline of the IIA and IIB supergravity theories. We will not discuss the type I theory in any more detail, except to note that it was the first ten-dimensional theory of supergravity to be constructed.

Type IIA supergravity can be derived from eleven-dimensional supergravity by compactification [25]. We take the coordinate x^{10} to describe a circle of radius R, and so the points x^{10} and $\tilde{x}^{10}$ are equivalent if $\tilde{x}^{10} = x^{10} + 2\pi n R$ for some integer n. The fields can then be expanded in Fourier series in the coordinate

$x^{10} \equiv R\theta$. The expansion of a generic field φ has the form

$$\varphi(x^\mu, \theta) = \varphi(x^\mu) + \sum_{n \neq 0} \varphi_n(x^\mu) e^{in\theta}. \tag{2.103}$$

(From now on, the indices $\mu, \nu, \ldots$ take only the values $0, \ldots, 9$; indices which take values from 0 to 10 are denoted by $\hat{\mu}, \hat{\nu}, \ldots$) The fields φ_n with $n \neq 0$ are massive, their masses being inversely proportional to the radius R. To give an example [56], consider a scalar field ϕ. The kinetic part of the action is given by

$$S_\phi = \int d^D x\, \phi(-\Box_D + m^2)\phi.$$

The derivative operator decomposes as

$$\Box_D = \Box_{D-1} + \frac{\partial^2}{\partial x_{10}^2}.$$

Using the expansion (2.103), the action then becomes

$$S_\phi = 2\pi R \sum_n \int d^{D-1}x\, \phi_n \left(-\Box_{D-1} + m^2 + \frac{n^2}{R^2} \right) \phi_n.$$

As $R \to 0$, the fields with $n \neq 0$ become infinitely massive. Therefore we keep only the massless fields and discard the massive ones. The discarding of the massive fields amounts to taking all of the fields in eleven dimensions to be independent of x^{10}.

Under the reduction from eleven dimensions to ten, the eleven-dimensional fields decompose schematically as

$$e_{\hat{\mu}}{}^{\hat{m}} \to e_\mu{}^m,\ B_\mu,\ \phi; \qquad \psi_{\hat{\mu}a} \to \psi_{\mu a},\ \lambda_a; \qquad A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3} \to A_{\mu_1\mu_2\mu_3},\ A_{\mu_1\mu_2}.$$

The specific decomposition [25], from which type IIA supergravity is obtained, is given by

$$\begin{aligned} A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3} &= \big(A_{\mu_1\mu_2\mu_3},\ A_{\mu_1\mu_2}\big), \\ e_{\hat{\mu}}{}^{\hat{m}} &= \begin{pmatrix} e^{-\sigma/12} e_\mu{}^m & 2e^{2\sigma/3} B_\mu \\ 0 & e^{2\sigma/3} \end{pmatrix}, \\ \psi_{\hat{m}} &= \left(e^{-\sigma/24} e_m{}^\mu \psi'_\mu,\ \frac{2\sqrt{2}}{3} e^{17\sigma/24} \lambda \right), \end{aligned} \tag{2.104}$$

where

$$\psi'_\mu = e^{-\sigma/24} \left(\psi_\mu - \frac{1}{6\sqrt{2}} \Gamma_\mu \bar{\Gamma} \lambda \right) - \frac{4\sqrt{2}}{3} e^{3\sigma/4} B_\mu \lambda. \tag{2.105}$$

This decomposition gives rise to the Lagrangian

$$\begin{aligned} \mathcal{L} = &- \frac{1}{2}|e| R(\omega) - \frac{1}{24}|e| e^{\sigma/2} F'_{\mu_1\cdots\mu_4} F'^{\mu_1\cdots\mu_4} + \frac{1}{6}|e| e^{-\sigma} F_{\mu\nu\rho} F^{\mu\nu\rho} \\ &- \frac{1}{2} e^{3\sigma/2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{144^2} \epsilon^{\mu_1\cdots\mu_{10}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} A_{\mu_9\mu_{10}} \\ &- \frac{1}{4} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{4}|e| \bar{\lambda} \Gamma^\mu D_\mu \lambda + \ldots \end{aligned} \tag{2.106}$$

where the dots denote numerous further terms involving the fermion fields, and

$$
\begin{aligned}
F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad F_{\mu\nu\rho} = 3\partial_{[\mu} A_{\nu\rho]},\\
F'_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A'_{\nu\rho\sigma]} + 12 A_{[\mu_1\mu_2} G_{\mu_3\mu_4]},\\
A'_{\mu\nu\rho} = A_{\mu\nu\rho} - 6 B_{[\mu} A_{\nu\rho]}.
\end{aligned}
\tag{2.107}
$$

The general coordinate transformations in the coordinate x^{10} of the eleven-dimensional theory become gauge transformations, whose gauge field the field B_μ is, in the ten-dimensional theory. To see how this comes about, we note that the decomposition (2.104) corresponds to rearranging the eleven-dimensional metric into the form

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{4\sigma/3}\big(dx^{10} + B_\mu\, dx^\mu\big)\big(dx^{10} + B_\nu\, dx^\nu\big).
$$

From this expression, we see explicitly that the coordinate transformation $\big(x^\mu, x^{10}\big) \to \big(x^\mu, x^{10} - \xi(x)\big)$ induces the transformation $B_\mu \to B_\mu + \partial_\mu \xi$ in the field B_μ [22].

Besides its invariance under supersymmetry, the Lagrangian (2.106) is invariant under the following transformations:

$$
\sigma \to \sigma + \alpha, \quad B_\mu \to e^{-3\alpha/4} B_\mu, \quad A_{\mu\nu} \to e^{\alpha/2} A_{\mu\nu}, \quad A_{\mu\nu\rho} \to e^{-\alpha/4} A_{\mu\nu\rho}.
$$

This symmetry arises from the Weyl transformation of Eq. (2.102), which can be made into a symmetry of the ten-dimensional theory, provided that it is accompanied with a transformation $x^{10} \to e^{-9\alpha} x^{10}$ in the eleventh coordinate. The effect of the latter is to replace $\int dx^{10}$ with $e^{-9\alpha} \int dx^{10}$, and so the ten-dimensional action, which is obtained by carrying out the trivial integral $\int dx^{10}$ in the eleven-dimensional action, is invariant under the combined transformation. As the gravitational constant κ occurs in the Lagrangian only as an overall multiplicative factor κ^{-2}, we see that the ten-dimensional and the eleven-dimensional gravitational constants are connected by the relation

$$
\frac{1}{\kappa_{10}^2} = \frac{2\pi R}{\kappa_{11}^2}. \tag{2.108}
$$

It is clear that κ_{10} is invariant under the combined transformation, since the Weyl transformation (2.102) effectively replaces κ_{11} by $e^{-9\alpha/2}\kappa_{11}$.

Type IIB supergravity is based on two Majorana–Weyl spinors of the same chirality. It contains the bosonic fields ${e_\mu}^m$, $A_{\mu\nu}$, a and $B_{\mu\nu\rho\sigma}$, and the fermionic fields $(\psi_\mu)_a^I$ and λ_a^I. The fermionic fields ψ_μ and λ are complex. Both of the gravitinos have the same chirality, which is opposite to the chirality of the λ^I fields. Instead of the complex fields $A_{\mu\nu}$ and a, we can equivalently use the real fields C, σ, $B_{\mu\nu}$ and $C_{\mu\nu}$. The four-form gauge field $B_{\mu\nu\rho\sigma}$ is real.

The field strength $G_{\mu_1\cdots\mu_5}$ of the four-form field satisfies a self-duality condition of the form

$$
(\star G)_{\mu_1\cdots\mu_5} \equiv \frac{1}{5!}\varepsilon^{\mu_1\cdots\mu_{10}} G_{\mu_6\cdots\mu_{10}} = G_{\mu_1\cdots\mu_5}. \tag{2.109}
$$

This complicates the construction of a Lagrangian, as a standard kinetic term

$$G_{\mu_1\cdots\mu_5}G^{\mu_1\cdots\mu_5}$$

for the four-form field now vanishes on the account of the self-duality condition. Nevertheless, one can give a formulation of the theory in terms of an action principle, using an action in which $G_{\mu_1\cdots\mu_5}$ is not self-dual and imposing the self-duality condition as an additional field equation [56]. This action is given by

$$\begin{aligned} S = \int d^{10}x \Bigg[&\frac{1}{2}|e|e^{2\sigma}\Big(R + 4(\partial\sigma)^2 - \frac{1}{2}|H^{(3)}|^2\Big) \\ &- \frac{1}{4}|e|\Big(|F^{(1)}|^2 + |G^{(3)}|^2 + \frac{1}{2}|G^{(5)}|^2\Big)\Bigg] - \frac{1}{4}\int B^{(4)} \wedge H^{(3)} \wedge F^{(3)}, \end{aligned} \tag{2.110}$$

where all the terms involving the fermions have been omitted. The various field strengths are defined as

$$\begin{aligned} G^{(3)} &= F^{(3)} - CH^{(3)}, \\ G^{(5)} &= F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)}, \end{aligned} \tag{2.111}$$

where

$$\begin{aligned} F^{(1)} &= dC, \\ H^{(3)} &= dB^{(2)}, \\ F^{(3)} &= dC^{(2)}, \\ F^{(5)} &= dB^{(4)}, \end{aligned} \tag{2.112}$$

and for a p-form, we have defined

$$|F^{(p)}|^2 = \frac{1}{p!}\bar{F}_{\mu_1\cdots\mu_p}F^{\mu_1\cdots\mu_p}. \tag{2.113}$$

The self-duality condition $\star G^{(5)} = G^{(5)}$ does not follow from the action, and so it has to be enforced by hand. There is also an alternative formulation of the theory, in which an auxiliary fields is introduced, and the self-duality condition appears not as an equation of motion, but through a gauge fixing requirement on the auxiliary field [57].

The scalar fields C and σ can be interpreted as giving a parametrization of a $SU(1,1)/U(1)$ coset space. In the $SU(1,1)/U(1)$ formulation of the theory [58], we have the scalar fields V_+^α and V_-^α, where α is a $SU(1,1)$ index, and the $+$ and $-$ denote the charge of the fields under $U(1)$. The coset space $SU(1,1)/U(1)$ is described by the matrix

$$U = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix}, \tag{2.114}$$

with the constraints

$$(V_-^1)^* = V_+^2, \qquad V_-^1 V_+^2 - V_+^1 V_-^2 = 1.$$

The $U(1)$ invariant complex variable

$$z = \frac{V_-^2}{V_-^1} \tag{2.115}$$

can be used as a coordinate of the coset space, and as such it represents the physical degrees of freedom of the scalar fields. The $SU(1,1)$ transformation

$$\begin{pmatrix} V_-^1 \\ V_-^2 \end{pmatrix} \to \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix} \begin{pmatrix} V_-^1 \\ V_-^2 \end{pmatrix} \tag{2.116}$$

corresponds to the transformation

$$z \to \frac{\bar{\mu} z + \bar{\nu}}{\mu z + \nu}$$

in the variable z. The $SU(1,1)$ transformations can be expressed as $SL(2,\mathbb{R})$ transformations through the change of variables

$$z = \frac{1 + i\tau}{1 - i\tau}.$$

Under the transformation (2.116), the variable τ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{2.117}$$

where the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the transformation parameters is an element of $SL(2,\mathbb{R})$. The scalar fields C and σ are related to τ by

$$\tau = C + ie^{-\sigma}. \tag{2.118}$$

The action in the form (2.110) arises naturally from the low energy limit of type IIB string theory. However, the $SL(2,\mathbb{R})$ symmetry of the theory is not manifest in Eq. (2.110). The symmetry can be made manifest by performing a transformation from the so-called string frame, in which the action has the form (2.110), to the so-called Einstein frame [56]. The transformation is achieved by setting

$$(e_\mu{}^m)_E = e^{\sigma/4} e_\mu{}^m, \qquad \tau = C + ie^{\sigma}, \qquad D^{(3)} = \frac{F^{(3)} - \tau H^{(3)}}{\sqrt{\operatorname{Im}\tau}}. \tag{2.119}$$

In the Einstein frame, the action then has the form

$$\begin{aligned} S = {} & \frac{1}{2} \int d^{10}x \, |e_E| \left(R_E - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{2(\operatorname{Im}\tau)^2} \right) \\ & - \frac{1}{4} \int d^{10}x \, |e_E| \left(\frac{1}{2} |F^{(1)}|^2 - |D^{(3)}|^2 - \frac{1}{2} |G^{(5)}|^2 \right) + \frac{i}{4} \int B^{(4)} \wedge \bar{D}^{(3)} \wedge D^{(3)}. \end{aligned} \tag{2.120}$$

The $SL(2,\mathbb{R})$ transformations leave the vielbein and the field $B^{(4)}$ invariant. The field τ transforms according to Eq. (2.117). The fields $B^{(2)}$ and $C^{(2)}$ transform among each other so that the field $D^{(3)}$ transforms as

$$D^{(3)} \to \frac{c\bar{\tau}+d}{|c\tau+d|} D^{(3)}. \tag{2.121}$$

The action in the form (2.120) is manifestly invariant under these transformations.

In addition to the gravitational constant κ, both of the type II supergravities contain another coupling constant, which is the expectation value $\langle e^{\sigma} \rangle$. These are related to the string length l_s and the string coupling constant g_s of the respective string theories. In particular, we have $g_s = \langle e^{\sigma} \rangle$. The $SL(2,\mathbb{R})$ transformations can now be seen as duality transformations which take one from a strong string coupling into a weak string coupling, or vice versa. In particular, the transformation which takes $\tau = C + ie^{-\sigma}$ into $\tau' = -1/\tau$ transforms the coupling constant $g_s = \langle e^{\sigma} \rangle$ into $g_s' = 1/\langle e^{\sigma} \rangle$ [25].

Chapter 3

Duality

3.1 Elements of superstring theory

In the 1960's, the understanding of the strong interaction was very incomplete, and physicists were charged with the task of organizing and interpreting a large amount of experimental data. An ad hoc scattering amplitude, which reproduced several experimental facts, was proposed by Veneziano in 1968 [62]. String theory [63, 64, 65] got started soon after, when Nambu and Goto showed that such an amplitude is predicted by a theory in which elementary particles are realized as the vibrational states of relativistic strings.

However, such theories were initially found to be not satisfactory. The primary difficulties were the following: All such theories contained a tachyon, and several contained a massless spin-2 particle, which were not possible to get rid of. Furthermore, it was not possible to introduce fermions into the theory, and the quantum theory was Lorentz invariant only when the number of spacetime dimensions is 26. All of these difficulties were resolved during the 1970's. A way to include fermions was found by Neveu and Schwarz [66], and by Ramond [67]. Gliozzi, Scherk and Olive were able to eliminate the tachyons and in the process introduced supersymmetry into string theory [68]. Moreover, a way to reduce the theory into four dimensions by compactifying the extra dimensions was found, and it was understood that the spin-2 particle should be viewed as a manifestation of the fact that gravity is naturally included in string theory.

String theory is formulated in terms of an action principle, where the action is given by the area of the worldsheet, i.e. the two-dimensional surface traced into spacetime by the string, in analogy with the action principle of a relativistic point particle, where the action is given by $S = -m \int dt \sqrt{1 - v^2} = -m \int dt \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = -m \int ds$. For bosonic string theory, the so-called

Nambu–Goto action is given by

$$S = -\frac{1}{2\pi\alpha'}\int d^2\xi\,\sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)}$$
$$= -\frac{1}{2\pi\alpha'}\int d^2\xi\,\sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2}. \tag{3.1}$$

The worldsheet is parametrized by the coordinates (ξ^0, ξ^1). The functions $X^\mu(\xi)$ give the location of the worldsheet in spacetime; they can be viewed as the spacetime coordinates of the worldsheet. We have denoted $\dot{X} = \partial X^\mu/\partial\xi^0$ and $X' = \partial X^\mu/\partial\xi^1$. The indices μ, ν are spacetime indices, while the indices a, b refer to the worldsheet coordinates. The parameter α' is related to the string length scale l_s by $\alpha' = l_s^2$. The string tension is given by $T = 1/2\pi\alpha'$.

An alternative action for bosonic string theory is given by the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\,\sqrt{-h}\,h_{ab}\partial^a X^\mu \partial^b X_\mu. \tag{3.2}$$

The advantage of the Polyakov action is that the square root in the Nambu–Goto action has been gotten rid of. This comes at a cost of having to introduce the worldsheet metric h_{ab}. The metric h_{ab} can be eliminated using its equation of motion. The action then reduces to the form (3.1).

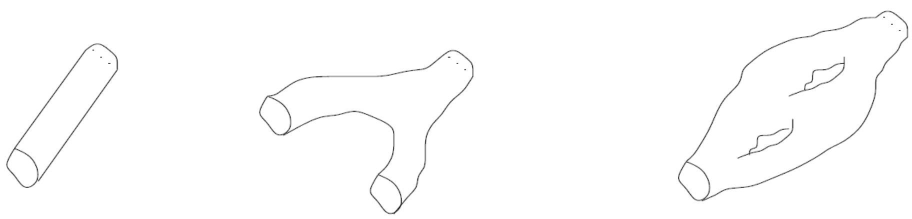


Figure 1. Closed strings propagating in spacetime. The string on the left is free, while the other two drawings represent interactions between strings [56].

The Neveu–Schwarz–Ramond formulation of superstring theory is based on a generalization of the Polyakov action. In addition to the bosonic worldsheet coordinates X^μ and the metric h_{ab}, the action involves the fermionic worldsheet coordinates ψ^μ and $\bar{\psi}^\mu$, and the gravitino χ_a. The fields (X^μ, ψ^μ) form D sets of chiral supermultiplets, and the metric h and the gravitino χ also combine into a supermultiplet. The action has the form

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-h}\Big[h_{ab}\partial^a X^\mu \partial^b X_\mu + \frac{i}{2}\bar{\psi}^\mu \partial\!\!\!/ \psi_\mu$$
$$+ \frac{i}{2}\left(\bar{\chi}_a\gamma^b\gamma^a\psi^\mu\right)\left(\partial_b X^\mu - \frac{i}{4}\bar{\chi}\psi^\mu\right)\Big]. \tag{3.3}$$

The spacetime index μ takes values from 0 to $D-1$; the quantum theory is consistent only if $D = 10$. The indices $a, b = 1, 2$ refer to the worldsheet coordinates ξ^a.

The fields on the worldsheet are separated into so-called left-moving and right-moving fields. The fields ψ^μ are right-moving, while $\bar{\psi}^\mu$ are left-moving. The action is invariant under both left-moving and right-moving supersymmetry transformations on the worldsheet. Boundary conditions have to be specified which relate the left-moving and right-moving fields into each other. There are two different choices for the boundary conditions. These split the fields into two different sectors, the NS sector and the R sector. The quantum ground state of the NS sector is a boson while the ground state of the R sector is a fermion.

In the case of closed strings, the massless particle states are obtained as tensor products of left-moving and right-moving fields. The massless states are thus divided into four sectors. The bosons belong to the NS-NS and the R-R sectors while the fermions are in the NS-R and R-NS sectors. The particle content of the various sectors is as follows:

- The NS-NS sector contains the graviton, an antisymmetric rank-2 tensor field and a scalar field.
- The R-R sector contains antisymmetric tensor fields of various ranks, and possibly a scalar field.
- The NS-R and R-NS sectors each contain a Majorana–Weyl gravitino and a spin-1/2 particle.

The two gravitinos may either have opposite chiralities, or they may both have the same chirality. The theories where these possibilities are realized are known as type IIA and IIB superstring theories, respectively. In the IIA theory, the R-R sector contains a vector field and a rank-3 antisymmetric tensor field, while in the IIB theory, there are a scalar field, a rank-2 antisymmetric tensor field and a self-dual rank-4 antisymmetric tensor field. Both of the type II theories contain two gravitinos, and so they have local $N = 2$ supersymmetry, with 32 supercharges.

In addition to the type IIA and IIB superstring theories, which are theories of oriented closed strings, there is the type I superstring theory, which involves unoriented closed strings and unoriented open strings, and which only has $N = 1$ supersymmetry. In addition, there is the heterotic string theory, which is somehow a mixture of superstring theory and bosonic string theory, as it has ten left-moving fields but 26 right-moving fields on the worldsheet. The type I and heterotic theories have a gauge symmetry; the gauge group is dictated by the requirement that the theory must be free of quantum anomalies. The gauge group of the type I theory is uniquely determined as $SO(32)$, while for the heterotic theory one may take either $SO(32)$ or $E_8 \times E_8$.

The low-energy limits of superstring theories are supergravity theories. The low-energy action of a given superstring theory can be found by the following counting: A scalar field has weight 0, a fermion field has weight 1/2, and a derivative has weight 1. The low-energy action is then obtained by discarding terms which have a weight greater than 2. In particular, the type IIA and IIB superstring theories reduce to type IIA and IIB supergravity in the low-energy limit.

There are numerous connections, or dualities, which relate the five ten-dimensional superstring theories to each other. The dualities give one a reason to suspect that all the superstring theories are just different limits of a single underlying theory. One can view them as perturbative expansions of the underlying theory about different points (in the space of physically acceptable vacua). The underlying theory is named M-theory. A proper definition or a formulation for M-theory is not known. The theory is presumably eleven-dimensional, and eleven-dimensional supergravity is believed to be its low-energy limit. A point in favour of this interpretation is that there is a unique eleven-dimensional supergravity theory, so it is plausible that it could be the low-energy limit of a well-defined fundamental theory. However, it should be borne in mind that the uniqueness of eleven-dimensional supergravity is due to the requirement that spin-5/2 particles be absent from the theory, which in turn is dictated by the difficulties in constructing a quantum field theory involving interacting spin-5/2 particles. However, it is not clear whether one can deduce from these difficulties that a fundamental theory of Nature should be free of spin-5/2 particles. It is conceivable that the difficulties could be a reflection of some limitation in the standard methods of quantum field theory, and that a theory of spin-5/2 particles could be constructed by other means.

One of the string dualities is T-duality [65], which relates to each other two string theories having a dimension which is compactified on a circle. If the radius of the compact dimension is R, then the momentum of the string in the compact direction is quantized, and has one of the values $p = n/R$, with n an integer. With closed strings, an additional excitation results from the possibility that the string may be wound around the compact dimension a certain number of times. If m is the winding number, then the energy related to this excitation is given by $E_m = m(2\pi RT) = mR/\alpha'$, where $T = 1/2\pi\alpha'$ is the string tension. In the lower-dimensional theory, the two kinds of excitations appear as corrections to mass, as the squared energy is now given by

$$E^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \ldots \tag{3.4}$$

The important point is that this remains unchanged when R is replaced by α'/R, and m and n are interchanged. T-duality is the statement of equivalence between two theories, whose compact dimensions have radii R_1 and R_2 which are related by $R_1R_2 = \alpha'$. Excitations along the compact dimension in one theory correspond to winding excitations in the dual theory. The type IIA and IIB theories are related to each other by T-duality, as are the two heterotic theories. Note that the limit $R \to 0$ in one theory corresponds to the decompactification of a dimension in the dual theory.

Another string duality is the so-called S-duality, which relates the strong coupling limit of a theory into the weak coupling limit of another theory. The type I superstring theory and the heterotic $SO(32)$ theory are related to each other by S-duality. Furthermore, type IIB superstring theory is self-dual under S-duality. Under quantization, the $SL(2,\mathbb{R})$ invariance of the classical theory is reduced to a $SL(2,\mathbb{Z})$ invariance. The $SL(2,\mathbb{Z})$ transformations are the S-duality transformations. A particular S-duality transformation replaces the coupling constant g_s with $1/g_s$, and the invariance of the theory under $SL(2,\mathbb{Z})$ translates to the prediction that the strong coupling and weak coupling limits of the theory are equivalent. This is significant as it allows one to extract information about the strong coupling limit from perturbative calculations.

The different string dualities combine in various ways into an intricate web of dualities which indirectly relates all of the superstring theories to each other and to M-theory. The dualities give rise to a number of highly nontrivial predictions, and the topic is certainly not adequately understood as of now. We will not explore this topic any further, but turn instead to other matters.

The antisymmetric tensor fields which appear in superstring theories can be identified with differential forms, the rank-$p+1$ tensor field $A_{\mu_0\cdots\mu_p}$ being identified with the $p+1$-form

$$A^{(p+1)} = \frac{1}{(p+1)!} A_{\mu_0\cdots\mu_p}\, dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_p}. \tag{3.5}$$

A $p+1$-form naturally couples to geometrical objects of dimension $p+1$. The $p+1$-dimensional objects which are charged under $A^{(p+1)}$ are called p-branes [56, 63, 65]. A 0-brane is a point particle, and a 1-brane is a string, but p may have any value up to 9. The coupling is given by

$$iQ_p \int_{\Sigma_{p+1}} A^{(p+1)} = iQ_p \int_{\Sigma_{p+1}} A_{\mu_0\cdots\mu_p}\, dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_p}.$$

This can be thought of as the generalization of the coupling $A_\mu \dot{x}^\mu$ in electrodynamics.

The field $A^{(p+1)}$ has the so-called magnetic dual $A_{\mathrm{m}}^{(D-3-p)}$, which is related to $A^{(p+1)}$ by

$$dA_{\mathrm{m}}^{(D-3-p)} = \star dA^{(p+1)}. \tag{3.6}$$

Consequently, the magnetic dual of a p-brane is a $D-4-p$ brane to which the field $A_{\mathrm{m}}^{(D-3-p)}$ couples.

In type II string theories, there are p-branes which couple to the R-R fields $A^{(p+1)}$, and their magnetic duals. In the IIA theory p has even values, while in the IIB theory p has odd values. The NS-NS field $B_{\mu\nu}$ couples to a 1-brane, which is the fundamental string. The magnetic dual of the string is a 5-brane. In M-theory, the only antisymmetric tensor field is the three-form field $A_{\mu\nu\rho}$, and consequently the theory only allows a 2-brane, and its magnetic dual, which is a 5-brane.

On the other hand, in string theories there are so-called Dp-branes [63, 70, 71], which arise when Dirichlet boundary conditions are specified on open strings. If Dirichlet boundary conditions are imposed on p spatial coordinates, this constrains the p coordinates to have certain values, which makes the most sense physically if there are $p+1$-dimensional objects in spacetime on which the endpoints of open strings may attach themselves. These objects are the Dp-branes.

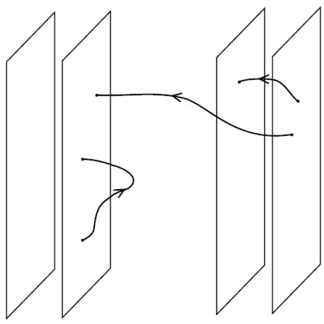


Figure 2. Open strings ending on Dp-branes [71].

Dp-branes are lower-dimensional defects in the ten-dimensional spacetime. Their fluctuations are described by the excitations of the open strings which are attached to them. The open string boundary conditions identify the left-moving and right-moving supercharges with each other. Open strings therefore have only half as many supersymmetries as closed strings, and so in type II superstring theories, Dp-branes preserve only a half of the 32 supersymmetries of the theory. Dp-branes may interact with closed strings which propagate in the whole spacetime. On hitting the brane, the closed string splits into two open strings, whose endpoints become attached to the brane. Conversely, a pair of open strings may become a closed string and leave the brane. Dp-branes therefore act as sources of closed strings (see Fig. 3).

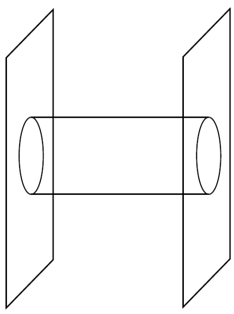


Figure 3. A D-brane emits a closed string, which is then absorbed by another D-brane [71].

It can be shown that a Dp-brane carries one unit of charge under the R-R field $A^{(p+1)}$. This suggests that Dp-branes are related to the p-branes described earlier. All Dp-branes are in fact p-branes, but there are p-branes which are not Dp-branes. In particular, the fundamental string is not a D1-brane. The charge of a Dp-brane can be related to the tension of the brane, which is given by

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{3.7}$$

This is inversely proportional to the coupling constant, which shows that the brane is a non-perturbative effect. In string perturbation theory, when $g_s \ll 1$, the tension, or the energy density, of the brane becomes very large, and so the brane can be thought of as some kind of a heavy, semiclassical object.

3.2 Large N field theory

String theory was motivated by the desire to find a theory for strong interactions. A satisfactory theory for strong interactions was later provided by QCD, which is a non-abelian gauge theory with the gauge group $SU(3)$. However, while QCD is extremely suitable for studying strong interactions at high energies, it is a less ideal tool for low-energy phenomena. The so-called large N limit of QCD [69, 72] strongly suggests that QCD might be related to a string theory, as the perturbative expansion of QCD in the large N limit is identical in structure to the perturbative expansion which appears in string perturbation theory. It would be extremely desirable to uncover the duality connection between QCD and a string theory, if there indeed is one, as the dual theory might provide a more suitable description of the physics at low energies. An additional motivation for considering the large N limit of QCD, is the hope that an exact solution could be found for the $N = \infty$ theory. If such a solution were found, the $N = 3$ theory could then be treated by an expansion in powers of $1/N$.

The large N limit of gauge theories was first considered by 't Hooft [73]. The idea is very general, and applies to any non-abelian gauge theory with the gauge group $U(N)$ or $SU(N)$. The order N of the gauge group is considered as a parameter, and physical quantities are expanded in powers of $1/N$. We consider a generic $U(N)$ or $SU(N)$ gauge theory, which has the fields φ_i^a in the adjoint representation of the gauge group. The index a is the gauge group index, and i is some additional index (say, a Lorentz index) which labels the fields. We assume that the Lagrangian has the form

$$\mathcal{L} = \mathrm{Tr}\Big[(d\varphi_i)(d\varphi^i) + g_{\mathrm{YM}} a_{ijk}\big(\varphi^i\varphi^j\varphi^k\big) + g_{\mathrm{YM}}^2 b_{ijkl}\big(\varphi^i\varphi^j\varphi^k\varphi^l\big)\Big]. \tag{3.8}$$

The prescription for how the $N \to \infty$ limit should be taken is dictated by the equation

$$\frac{dg_{\mathrm{YM}}}{d\ln E} = -\frac{11}{48\pi^2} N g_{\mathrm{YM}}^3 + \mathcal{O}\big(g_{\mathrm{YM}}^5\big), \tag{3.9}$$

which describes the running of the coupling constant in an $SU(N)$ Yang–Mills theory. The leading terms are of the same order if we let $N \to \infty$ and at the same time $g_{\rm YM} \to 0$ in such a way that $\lambda \equiv g_{\rm YM}^2 N$ remains fixed. The quantity λ is often called the 't Hooft coupling.

If we scale all the fields φ_i in the Lagrangian (3.8) by $1/g_{\rm YM}$, it becomes

$$\mathcal{L} = \frac{N}{\lambda}\mathrm{Tr}\Big[\big(d\varphi_i\big)\big(d\varphi^i\big) + a_{ijk}\big(\varphi^i\varphi^j\varphi^k\big) + b_{ijkl}\big(\varphi^i\varphi^j\varphi^k\varphi^l\big)\Big]. \tag{3.10}$$

The adjoint field φ^a can be written as a direct product φ_i^j of a fundamental and an anti-fundamental field. The Feynman diagrams of the theory may then be written in a double line notation, where the lines correspond to the indices i and j. This notation was also introduced by 't Hooft.

From the Lagrangian (3.10), we may read off that in a Feynman diagram each vertex gives a factor of N/λ, and each propagator gives a factor of λ/N. (Strictly speaking, the propagator depends on whether the gauge group is $U(N)$ or $SU(N)$. In the $SU(N)$ case, we have $\langle\varphi_i^j\varphi_k^l\rangle \sim \delta_i^l\delta_k^j - (1/N)\delta_i^j\delta_k^l$, while for $U(N)$ only the first term is present. In the $SU(N)$ case, the propagator therefore has an additional term, but it can be neglected in the $N \to \infty$ limit.) Furthermore, each loop gives an additional factor of N, which arises from $\sum \delta_a^a = N$. These rules allow us to deduce the N-dependence of a general Feynman diagram, and find which diagrams dominate in the large N limit.

Consider first vacuum diagrams. In the double line representation, a general vacuum diagram can be viewed as a surface, which may in general be multiply connected. In this picture, the propagators are the edges of the surface and the loops are its faces. A diagram with V vertices, E propagators and F loops will be proportional to

$$N^{V-E+F}.$$

There is a theorem due to Euler, which relates the combination $V - E + F$ to the number of holes H in the surface by

$$V - E + F = 2 - 2H. \tag{3.11}$$

We therefore have that the N-dependence of a general Feynman diagram is given by

$$N^{2-2H}. \tag{3.12}$$

The perturbative expansion for a particular quantity then has the form

$$\sum_{H=0}^{\infty} N^{2-2H} \sum_{k=0}^{\infty} c_{H,k}\lambda^k \equiv \sum_{H=0}^{\infty} N^{2-2H} f_H(\lambda). \tag{3.13}$$

In the large N limit, the expansion is dominated by the diagrams for which $H = 0$. These are the so-called planar diagrams. They are proportional to N^2, while all the other diagrams will be proportional to lower powers of N. In the double line representation, the planar diagrams can be drawn onto a plane so that none of the lines lie on the top of each other.

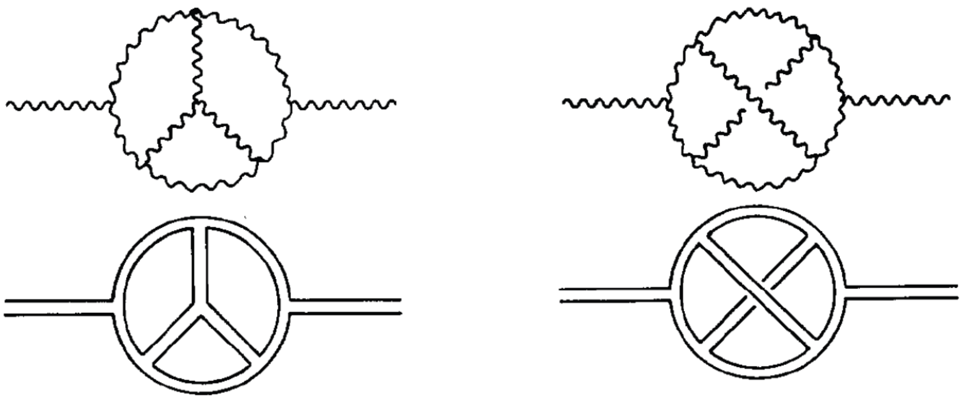


Figure 4. The diagram on the left is planar; the one on the right is not [74].

The topological nature of the expansion (3.13) is a strong indication that large N field theories may be related to string theories. In a theory of closed, oriented strings, the perturbative expansion has the form of Eq. (3.13), but with the inverse string coupling constant $1/g_s$ in the place of N; thus g_s should be identified with $1/N$. Apparently the double-line diagrams of the field theory are some kind of deformations of diagrams which represent the interactions of strings.

The form of the expansion (3.13) actually is the same also for diagrams with external lines, even though we derived it only for vacuum diagrams [69]. In the double line notation, each external line becomes a vertex of the surface, and so the analysis leading to Eq. (3.12) is valid for diagrams with external lines, too. It can be shown that each external line reduces the power of N by one, and so in the large N limit, the dominant contribution to an n-point function will be proportional to N^{2-n}. The two-point functions is therefore independent of N, the three-point function is proportional to $1/N$, and so on. This shows that the coupling constant is indeed $1/N$.

3.3 Geometry of anti de Sitter spaces

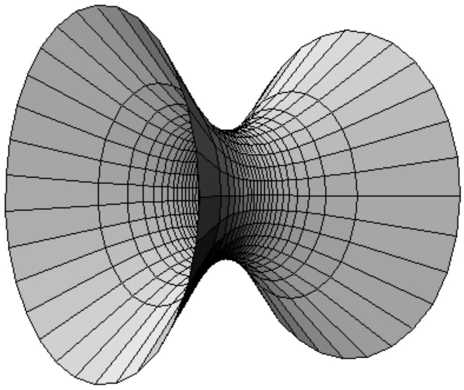
The $n+1$-dimensional anti de Sitter space AdS_{n+1} [63, 69] may be defined as the hyperboloid

$$x_0^2 + x_{n+1}^2 - \sum_{i=1}^{n} x_i^2 = R^2 \tag{3.14}$$

in the flat $n+2$ dimensional space whose metric is

$$ds^2 = -dx_0^2 - dx_{n+1}^2 + \sum_{i=1}^{n} dx_i^2. \tag{3.15}$$

From this definition, it is clear that the isometry group of AdS_{n+1} is $SO(2,n)$.



(Picture from Wikipedia.)

Figure 5. The space AdS_2 can be represented as a hyperboloid in a three-dimensional embedding space.

A possible parametrization of AdS_{n+1} is given by

$$\begin{aligned} x_0 &= R\cosh\rho\cos\tau, \\ x_{n+1} &= R\cosh\rho\sin\tau, \\ x_i &= R\sinh\rho\,\Omega_i, \end{aligned} \tag{3.16}$$

where the coordinates Ω_i parametrize the sphere S^{n-1}, and satisfy

$$\sum_{i=1}^{n}\Omega_i = 1.$$

In these coordinates, the metric of AdS_{n+1} reads

$$ds^2 = R^2\Big(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega^2\Big). \tag{3.17}$$

The whole hyperboloid is covered as the parameters ρ and τ take values from 0 to ∞, and from 0 to 2π, respectively. Near $\rho = 0$, the metric has the form

$$ds^2 = R^2\Big(-d\tau^2 + d\rho^2 + \rho^2\, d\Omega^2\Big).$$

This shows that AdS_{n+1} has the topology $S^1 \times \mathbb{R}^n$. The sphere S^1 represents closed timelike curves. These are eliminated by taking the universal cover of τ, which amounts to allowing τ to have values from $-\infty$ to ∞, and points with different values of τ are not identified with each other. From now on, we take AdS_{n+1} to mean this universal covering space.

Physically, anti de Sitter space is the maximally symmetric solution of the empty space Einstein equation with a constant negative curvature [75]. That is, it is a solution of the equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}\Lambda g_{\mu\nu} \tag{3.18}$$

with $\Lambda > 0$. That the solution is maximally symmetric means that it has the largest possible number of Killing vectors. For such a solution, the curvature tensor is given by

$$R_{\mu\nu\rho\sigma} = \text{const.} \times \big(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\big).$$

Furthermore, the near-horizon geometry of the so-called extremal charged black hole is described by an anti de Sitter space [63, 72]. The metric describing a black hole of mass M and charge Q is given by

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) dt^2 + \left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)\right]^{-1} dr^2 + r^2\, d\Omega^2, \tag{3.19}$$

where the horizons are located at $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The extremal black hole has $Q = M$, and so $r_+ = r_-$. Defining the coordinate $\rho = M^2/(r - M)$, the metric can be written near $r = r_+$ as

$$ds^2 = \frac{M^2}{\rho^2}\big(-dt^2 + d\rho^2\big) + M^2\, d\Omega^2. \tag{3.20}$$

The first term is the metric of AdS_2 in the so-called Poincaré coordinates. The near-horizon geometry is thus described by the direct product $AdS_2 \times S^2$.

The space AdS_{n+1} has a certain kind of a boundary [75], which can be described using the coordinates

$$\xi = x_0 + ix_{n+1}, \qquad \eta = x_0 - ix_{n+1}, \tag{3.21}$$

so that the condition (3.14) now reads

$$\xi\eta - \vec{x}^2 = R^2. \tag{3.22}$$

Roughly speaking, the boundary arises when the length of the vector $\vec{x} \in AdS_{n+1}$ becomes very large. If we define $(\tilde{\xi}, \tilde{\eta}, \tilde{x}_i)$ by $(\xi, \eta, x_i) = (a\tilde{\xi}, a\tilde{\eta}, a\tilde{x}_i)$, then Eq. (3.22) shows that in the limit $R \to \infty$ we have $\tilde{\xi}\tilde{\eta} - \tilde{\vec{x}}^2 = 0$. But instead of a we could have used ta with any real number t, and so the boundary should more properly be considered as the set of points which satisfy

$$\xi\eta - \vec{x}^2 = 0, \tag{3.23}$$

with the prescription that the points $(\xi, \eta, \vec{x})$ and $t(\xi, \eta, \vec{x})$ for any $t \in \mathbb{R}$ are considered identical.

We would now like to show explicitly that the isometry group $SO(2, n)$ acts on the boundary of AdS_{n+1} as the conformal group on Minkowski space [75]. An infinitesimal $SO(2, n)$ transformation acts on the boundary point $(\xi, \eta, \vec{x})$ as

$$(1 + \omega)\begin{pmatrix} \xi \\ \eta \\ \vec{x} \end{pmatrix}.$$

The requirement that the transformation must preserve the metric of the embedding space dictates that the matrix ω has the form

$$\omega = \begin{pmatrix} a & 0 & \vec{\alpha}^T \\ 0 & -a & \vec{\beta}^T \\ \frac{1}{2}\vec{\beta} & \frac{1}{2}\vec{\alpha} & \omega_n \end{pmatrix}, \tag{3.24}$$

where ω_n is an antisymmetric $n \times n$ matrix. We then have

$$(1+\omega)\begin{pmatrix} \xi \\ \eta \\ \vec{x} \end{pmatrix} = \begin{pmatrix} \xi' \\ \eta' \\ \vec{x}' \end{pmatrix} = \begin{pmatrix} (1+a)\xi + \vec{\alpha}\cdot\vec{x} \\ (1-a)\eta + \vec{\beta}\cdot\vec{x} \\ \vec{x} + (\xi/2)\vec{\beta} + (\eta/2)\vec{\alpha} + \omega_n\vec{x} \end{pmatrix}.$$

We can use the scaling $(\xi, \eta, \vec{x}) \sim t(\xi, \eta, \vec{x})$ to put any point on the boundary (except the point $\eta = 0$) into the form $(\vec{x}^2, 1, \vec{x})$, so that the boundary point is being represented by the vector $\vec{x}$. The image point $(\xi', \eta', \vec{x}')$ is transformed to this form by dividing each component with $(1-a)\eta + \vec{\beta}\cdot\vec{x}$. After these adjustments, we have that the infinitesimal $SO(2,n)$ transformation acts on the boundary points as

$$\vec{x} \to \big(1 + a - \vec{\beta}\cdot\vec{x}\big)\vec{x} + \frac{\vec{x}^2}{2}\vec{\beta} + \frac{1}{2}\vec{\alpha} + \omega_n\vec{x}. \tag{3.25}$$

This is indeed a combination of (i) a translation through $\vec{\alpha}/2$, (ii) a Lorentz rotation with the parameters given by ω_n, (iii) a dilation, i.e. a scaling of $\vec{x}$ by $1+a$, and (iv) a special conformal transformation with the parameter $\vec{b} = \vec{\beta}/2$. The only nontrivial point is (iv); taking the b^μ in Eq. (A.32) to be infinitesimal, we have

$$x^\mu \to x^\mu + x^2 b^\mu - 2b^\nu x_\nu x^\mu.$$

On the other hand, if only $\vec{\beta} \neq 0$, then Eq. (3.25) becomes

$$\vec{x} \to \vec{x} - (\vec{\beta}\cdot\vec{x})\vec{x} + \frac{\vec{x}^2}{2}\vec{\beta};$$

a special conformal transformation with the parameter $b^\mu = \vec{\beta}/2$.

An alternative parametrization for AdS_{n+1} is given by the coordinates u, t and $\vec{y} \in \mathbb{R}^{n-1}$, which are defined by

$$\begin{aligned} x_0 &= \frac{1}{2u}\Big(1 + u^2\big(R^2 + \vec{y}^2 - t^2\big)\Big), \qquad & x^i &= Ruy^i, \\ x^{n+1} &= \frac{1}{2u}\Big(1 - u^2\big(R^2 - \vec{y}^2 + t^2\big)\Big), \qquad & x^{n+2} &= Rut. \end{aligned} \tag{3.26}$$

The coordinate u takes values from 0 to ∞. The coordinates (3.26) cover a half of the hyperboloid (3.14). In these coordinates, the metric is

$$ds^2 = R^2\bigg(\frac{du^2}{u^2} + u^2\big(-dt^2 + d\vec{y}^2\big)\bigg). \tag{3.27}$$

The boundary of AdS_{n+1} is located at $u = \infty$, and this boundary is the four-dimensional Minkowski space.

Yet a different version of the metric is obtained by defining $r = R^2/u$. We then have

$$ds^2 = \frac{R^2}{r^2}\Big(dr^2 - dt^2 + d\vec{y}^2\Big). \tag{3.28}$$

The boundary is now at $r = 0$. This form of the metric makes it clear that AdS_{n+1} is conformally equivalent to flat space. Both of the metrics (3.27) and (3.28) are sometimes called the Poincaré metric in the literature.

There is also a version of AdS_{n+1} with a Euclidean time coordinate. This is obtained by replacing x_{n+1} with $x_E = -ix_{n+1}$. AdS_{n+1} then is defined by

$$x_0^2 - x_E^2 - \vec{x}^2 = R^2 \tag{3.29}$$

in an embedding space whose metric is

$$ds_E^2 = -dx_0^2 + dx_E^2 + d\vec{x}^2. \tag{3.30}$$

Equivalently, one can replace the Poincaré t coordinate by $-it$. The various metrics of AdS_{n+1} given above are written in Euclidean form as

$$\begin{aligned} ds_E^2 &= R^2\Big(\cosh^2\!\rho\, d\tau_E^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega^2\Big) \\ &= R^2\left(\frac{du^2}{u^2} + u^2\big(dt_E^2 + d\vec{y}^2\big)\right) \\ &= \frac{R^2}{r^2}\Big(dr^2 + dt_E^2 + d\vec{y}^2\Big). \end{aligned} \tag{3.31}$$

The Euclidean AdS_{n+1} can be mapped into a $(n+1)$-dimensional disk. The boundary – an n-dimensional disk – corresponds to $u = \infty$, except for one point. The full boundary includes the point $u = 0$, corresponding to the point $\vec{y} = \infty$.

3.4 Branes in supergravity

We will now look in some detail into the p-brane solutions in type II supergravities [63, 69, 76]. In the string frame, the relevant part of the low-energy action reads

$$S = \frac{1}{16\pi G}\int d^{10}x\,\sqrt{-g}\,\left(e^{-2\phi}\Big[R + 4(\partial\phi)^2\Big] - \frac{1}{2(p+2)!}F_{(p+2)}^2\right) + \ldots \tag{3.32}$$

where $F^{(p+2)}$ is the field strength of the field $A^{(p+1)}$. The gravitational constant G is related to the string length l_s by

$$16\pi G = (2\pi)^7 l_s^8. \tag{3.33}$$

In the type IIA theory, p is odd, and in the type IIB theory, p is even. Alternatively, one can derive the brane solution by starting from the action written in the Einstein frame, where it reads

$$S = \frac{1}{16\pi G}\int d^{10}x\,\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(p+2)!}e^{-(p-3)\phi/2}F^2_{(p+2)}\right),$$

and which is obtained from the string frame metric by the Weyl rescaling $g_{\mu\nu} \to e^{-\phi/2}g_{\mu\nu}$. This approach is spelled out in great detail in [75].

The equations of motion which arise from the action (3.32) are

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi &= \frac{e^{2\phi}}{2(p+1)!}\left(F^2_{\mu\nu} - \frac{g_{\mu\nu}}{2(p+2)}F^2\right), \\ d\star F^{(p+2)} = 0, \qquad & R = 4(\partial\phi)^2 - 4\Box\phi. \end{aligned} \tag{3.34}$$

In the first equation, $F^2_{\mu\nu}$ is a shorthand notation for $F_{\mu\mu_1\cdots\mu_{p+1}}F_\nu{}^{\mu_1\cdots\mu_{p+1}}$. It turns out that the solution which describes a p-brane is given by

$$ds^2 = H^{-1/2}(r)\Big(-f(r)\,dt^2 + \sum_{i=1}^{p}\big(dx^i\big)^2\Big) + H^{1/2}(r)\left(\frac{dr^2}{f(r)} + r^2\,d\Omega^2_{8-p}\right). \tag{3.35}$$

The coordinates t and x^i are the coordinates on the volume of the brane, while the coordinate r and the angular coordinates are perpendicular to the brane. The equations of motion imply that

$$e^{2\phi} = g_s^2\,H^{(3-p)/2}(r), \tag{3.36}$$

and that the functions H and f are harmonic in the transverse coordinates. A p-brane is pointlike in the transverse subspace, and so it is reasonable to expect that its fields are spherically symmetric in the transverse coordinates. The requirement of spherical symmetry constrains the functions H and r to have the form

$$H(r) = 1 + \left(\frac{L}{r}\right)^{7-p}, \qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^{7-p}. \tag{3.37}$$

One can also have more general solutions. For example, the so-called multicenter solution is given by

$$H(\vec{r}) = 1 + \sum_{i=1}^{M}\frac{N_i\xi_i}{|\vec{r} - \vec{r}_i|^{7-p}},$$

where

$$\xi_i = 2^{5-p}\pi^{(5-p)/2}\Gamma\big(\tfrac{1}{2}(7-p)\big)g_s l_s^{7-p}.$$

This solution corresponds to M parallel p-branes located at the positions $\vec{r}_i$, the R-R charge of the i-th brane being N_i.

Some features of the geometry of the p-brane solution become perhaps more transparent when a new coordinate ρ is defined by

$$\rho^{7-p} = L^{7-p} + r^{7-p}.$$

The metric then takes the form

$$ds^2 = -\frac{f_+(\rho)}{f_-^{1/2}(\rho)}\, dt^2 + f_-^{1/2}(\rho)\sum_{i=1}^{p}(dx^i)^2 + f_-(\rho)^{-1/2-\frac{5-p}{7-p}}\left(\frac{d\rho^2}{f_+(\rho)} + f_-(\rho)\,\rho^2\, d\Omega_{8-p}^2\right). \tag{3.38}$$

where

$$f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p}, \tag{3.39}$$

and

$$r_- = L, \qquad r_+^{7-p} = r_0^{7-p} + L^{7-p}. \tag{3.40}$$

The metric (3.38) has singularities at $\rho = r_\pm$. One can show that at $\rho = r_-$, there is a curvature singularity except when $p = 3$. Moreover, for $p \leq 6$ there is a horizon at $\rho = r_+$. If $r_+ > r_-$, the singularity is hidden inside the horizon, and the solution represents a black hole.

The p-brane solution has mass (per unit volume) M and R-R charge N, which are given by

$$M = \frac{(8-p)f_+^{7-p} - r_-^{7-p}}{(7-p)(2\pi)^7 d_p l_{\mathrm{P}}^8}, \qquad N = \frac{\left(r_+ r_-\right)^{(7-p)/2}}{d_p g_s l_s^{7-p}}. \tag{3.41}$$

Here $l_{\mathrm{P}} = g_s^{1/4} l_s$ is the Planck length in ten dimensions, and the numerical factor d_p is

$$d_p = 2^{5-p}\pi^{(5-p)/2}\Gamma\big(\tfrac{1}{2}(7-p)\big). \tag{3.42}$$

The condition $r_+ > r_-$, which guarantees the absence of a naked singularity, leads to the following inequality between the mass and the R-R charge:

$$M \geq \frac{N}{(2\pi)^p g_s l_s^{p+1}}. \tag{3.43}$$

The solution whose mass satisfies the lower bound is called an extremal p-brane. The extremal brane presumably corresponds to the brane being in the ground state in the quantum description. The solutions whose masses are greater than the lower bound are non-extremal p-branes. The non-extremal branes are sometimes called black branes, because of the event horizon at r_+.

The case $p = 3$ is special, as it is completely free of singularities. The $p = 3$ brane therefore represents a smooth, localized defect in spacetime. At the classical level, the singular $p \neq 3$ branes are maybe best regarded as unphysical objects analogous to the Dirac monopole in electrodynamics. However, when the relevant string theory is quantized, one is forced by symmetry considerations to include these in the theory; however, at least in the limit of weak string coupling, the singularities will be removed [63].

In the extremal limit $r_0 \to 0$, the 3-brane solution (3.35) becomes

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-dt^2 + d\vec{x}^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2\, d\Omega_5^2\right). \quad (3.44)$$

As $r \to \infty$, the metric reduces to that of flat Minkowski space. Furthermore, as $r \to 0$, the metric takes the form

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) + L^2\, d\Omega_5^2 \quad (3.45)$$

in terms of the variable $z = L^2/r$. This is a direct product of the space AdS_5, whose radius of curvature is L, and the sphere S^5 of radius L. The geometry can be visualized as an infinitely deep "throat", which opens up into flat ten-dimensional Minkowksi space when $r \gg L$ (see Fig. 6).

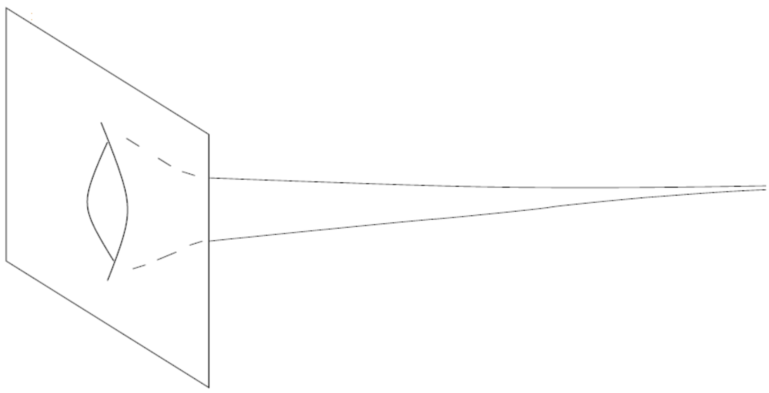


Figure 6. The geometry of the extremal 3-brane solution [56].

The maximal symmetry of the spaces AdS_5 and S^5 implies that the curvature tensor in both spaces has the form

$$R_{ijkl} = \pm\frac{1}{L^2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \quad (3.46)$$

where the $-$ sign belongs to AdS_5 and the $+$ sign to the sphere. This shows that the geometry is non-singular even as $r \to 0$, and that all the curvatures are inversely proportional to L. The classical supergravity description of the p-branes is appropriate as long as the curvature of the p-brane geometry is small in comparison with the string length l_s. When the curvature becomes comparable to l_s, the full string theory description should be used. In the case of extremal 3-branes, the requirement that the curvature is small enough translates to $L \gg l_s$. As we have

$$L^{7-p} = d_p g_s N l_s^{7-p}, \tag{3.47}$$

we see that the supergravity description is valid when

$$g_s N \gg 1. \tag{3.48}$$

The p-branes of type II supergravity are charged under the R-R gauge field $A^{(p+1)}$, and it can be shown that the extremal p-branes preserve half of the 32 supersymmetries of the supergravity theory. This suggests that the extremal p-branes should be identified with the Dp-branes of superstring theory – apparently they are just two different descriptions of the same physical objects. It was shown by Polchinski [70] that that a stack of N Dp-branes on top of each other has N units of charge under the field $A^{(p+1)}$. In string perturbation theory, in the case of N coincident Dp-branes, the expansion parameter is given by $g_s N$, and so the Dp-brane description can be used when

$$g_s N \ll 1. \tag{3.49}$$

This is the opposite of where the supergravity description is appropriate.

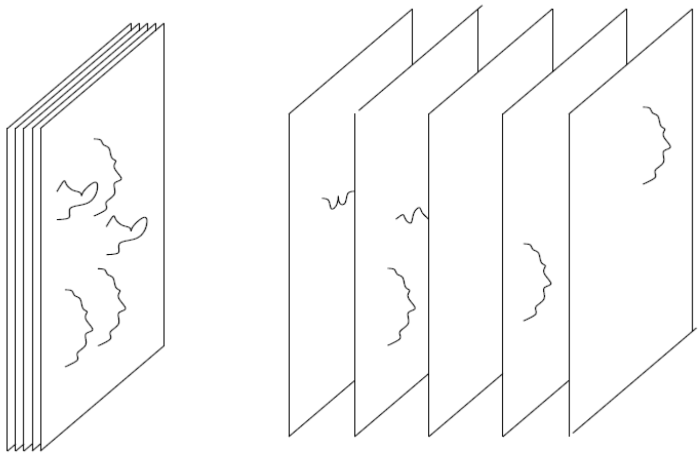


Figure 7. Stacks of D-branes with open strings ending on them. On the left, the branes are on the top of each other, while on the right they are separated by finite distances [56].

An interesting, and extremely relevant, feature of D-branes is that the effective low-energy theory of open strings and D-branes is a supersymmetric gauge theory on the world volume of the branes [69, 77]. This was first worked out by Witten [78]. The massless excitations of open strings on a single Dp-brane reproduce the spectrum of a $U(1)$ gauge theory with maximal supersymmetry in $p+1$ dimensions. In the case of N parallel Dp-branes located on top of each other, the gauge group is improved to $U(N)$, as suggested by the fact that the endpoints of a string can be located on the branes in N^2 different ways, and N^2 is the dimension of the adjoint representation of $U(N)$. There are $9-p$ massless scalar fields, which are interpreted as the Goldstone bosons related to the transverse excitations of the branes. When all the branes are on top of each other, all the scalar fields have vanishing vacuum expectation values. The case where the vacuum expectation values do not vanish corresponds to the branes being at different locations, and their relative distances are proportional to the expectation values.

3.5 Conformal field theories

In this section, we will describe some properties of conformally invariant field theories [69]. The conformal group, which is briefly outlined in Appendix 3, is a generalization of the Poincaré group. In addition to the spacetime transformations of the Poincaré group, it includes scale transformations. Theories invariant under the conformal group are therefore scale invariant. Many field theories, for example the four-dimensional Yang–Mills theory, are classically scale invariant. The scale invariance generally does not carry over to the quantum theory, though sometimes, such as in the $D = 4$, $N = 4$ supersymmetric Yang–Mills theory, it does. Although it has not been proven, there are no known counterexamples to the proposition that unitary, interacting scale invariant field theories are always invariant under the full conformal group.

The physically most relevant representations of the conformal group consist of fields, or operators, which are eigenfunctions of the dilation operator D with eigenvalues $-i\Delta$. The number Δ is the conformal dimension, or the scaling dimension, of the field. Under the transformation $x^\mu \to \lambda x^\mu$, a field of conformal dimension Δ transforms as $\phi(x) \to \lambda^\Delta \phi(x)$.

The commutation relations

$$[D, P_\mu] = -iP_\mu, \qquad [D, K_\mu] = iK_\mu, \tag{3.50}$$

show that P_μ raises the conformal dimension of a field by one unit, while K_μ lowers it by one unit. The requirement of unitarity places a lower bound on the conformal dimensions of fields; for example, a scalar field ϕ must have $\Delta_\phi \geq (D-2)/2$, where the equality can only be achieved by a free scalar field. Each representation of the conformal group therefore has to have a certain operator of lowest dimension, called a primary operator, which is annihilated by K_μ.

A representation of the conformal group can be constructed by starting from a primary operator, and including all the operators which can be constructed by acting on the primary operator with the generators of the conformal group. Such a representation is labeled by the conformal dimension of the primary field, and by the representation of the Lorentz group to which it belongs. As the operators in the representation are eigenfunctions of D, they generally are not eigenfunctions of $H = P_0$ or $M^2 = -P^\mu P_\mu$. A representation corresponding to massless fields has $M^2 = 0$, but otherwise the representation has a continuous mass spectrum, with M^2 taking values from 0 to ∞.

The properties of a conformal field theory are strongly constrained by the requirement of conformal invariance, as the conformal group is much larger than the Poincaré group. For example, the form of correlation functions between primary operators is very restricted, as the operators must be invariant under conformal transformations. It can be shown (see e.g. [79]) that the two-point function vanishes unless both of the fields have the same conformal dimension, and for two scalar fields of conformal dimension Δ, the two-point function is restricted to have the form

$$\big\langle \phi_i(x_1)\phi_j(x_2)\big\rangle = \frac{c_{ij}}{(r_{12})^{2\Delta}}, \tag{3.51}$$

where $r_{12} = |x_1 - x_2|$. Similarly, for the three-point function we have

$$\big\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\big\rangle = \frac{c_{ijk}}{(r_{12})^{\Delta_1+\Delta_2-\Delta_3}(r_{13})^{\Delta_1+\Delta_3-\Delta_2}(r_{23})^{\Delta_2+\Delta_3-\Delta_1}}. \tag{3.52}$$

Even though the two-point function and the three-point function are determined up to a constant by conformal invariance, the same is not true of the general n-point function, which may depend on an arbitrary function of the so-called cross ratios $r_{ij}r_{kl}/r_{ik}r_{jk}$, which are invariant under conformal transformations.

In certain spacetime dimensions, and with a suitable number of supercharges, it is possible to include the conformal generators to the supersymmetric Poincaré algebra to form a so-called superconformal algebra. All possible such algebras were classified by Nahm [80]. In particular, a superconformal algebra is only possible if $D \leq 6$. In addition to the Poincaré and conformal generators, the superconformal algebra includes further fermionic generators S, one for each supercharge Q, and possibly the generators of an R-symmetry. The generators S arise from the anticommutator $[K, Q] \sim S$, and their appearance doubles the number of fermionic generators compared to the supersymmetric Poicaré algebra.

A particular example of a conformal field theory is given by the $N = 4$ supersymmetric Yang–Mills theory in four dimensions [56, 81]. The theory can be constructed by dimensional reduction from the ten-dimensional $N = 1$ supersymmetric Yang–Mills theory [82, 83], which is the unique supersymmetric theory in ten dimensions which does not include gravity.
The ten-dimensional theory is desrcibed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}\,\big(F_{\mu\nu}F^{\mu\nu} - 2i\bar{\lambda}\Gamma^\mu D_\mu\lambda\big), \tag{3.53}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \qquad D_\mu = \partial_\mu + igA_\mu. \tag{3.54}$$

The field λ is a Majorana–Weyl spinor. The Lagrangian is invariant under the supersymmetry transformations

$$\delta_\epsilon A_\mu = -i\bar{\epsilon}\Gamma_\mu\lambda, \qquad \delta_\epsilon\lambda = \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon. \tag{3.55}$$

Under the reduction from ten dimensions to four, the ten-dimensional gauge field A_μ gives rise to the four-dimensional gauge field, and to six scalar fields X^i. The ten-dimensional spinor λ splits into four Majorana spinors λ_a, which are subject to an $SU(4)$ symmetry. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\bigg(&- \frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - D_\mu X_i D^\mu X^i \\ &+ gC_i^{ab}\lambda_a[X^i,\lambda_b] + g\bar{C}_{iab}\bar{\lambda}^a[X^i,\bar{\lambda}^b] + \frac{g^2}{2}\sum_{i,j}[X^i,X^j]^2\bigg). \end{aligned}\tag{3.56}$$

where the matrices C_i^{ab} are related to the gamma matrices belonging to the six reduced dimensions.

It is possible to formulate the four-dimensional theory in terms of superfields. In the superfield description, one of the λ_a spinors combines with the gauge field A_μ into a vector superfield, while the remaining three spinors each combine with two of the scalar fields into three chiral superfields.

The single ten-dimensional Majorana–Weyl supercharge splits into four Majorana supercharges in four dimensions, and the Lagrangian (3.56) is invariant under supersymmetry by construction. The supersymmetry transformation laws of the fields could be deduced from Eq. (3.55). The Lagrangian is also invariant under the conformal group, and under conformal supersymmetries, which are generated by the conformal supercharges $S_{a\alpha}$ and $\bar{S}_{\dot{\alpha}}{}^a$ arising from the commutator of K_μ with the Q supercharges.

The $SU(4)$ symmetry of the λ_a fermions is a remnant of the Lorentz invariance of the ten-dimensional theory, as the ten-dimensional Lorentz group decomposes as $SO(1,9) \to SO(1,3) \times SO(6) \sim SO(1,3) \times SU(4)$. The fermions λ_a transform in the fundamental representation $\mathbf{4}$ of $SU(4)$, while the scalars X^i transform in the rank 2 antisymmetric tensor representation $\mathbf{6}$. The gauge field A_μ is an $SU(4)$ singlet. The algebra of the $SU(4)$ transformations commutes with the conformal algebra. The supercharges $Q_\alpha{}^a$ and $\bar{S}_{\dot{\alpha}}{}^a$ transform in the $\mathbf{4}$ of the $SU(4)$, whereas $\bar{Q}_{a\dot{\alpha}}$ and $S_{a\dot{\alpha}}$ transform in the $\bar{\mathbf{4}}$. All of the global continuous symmetries of the theory combine together to form the so-called supergroup $SU(2,2|4)$.

The Lagrangian (3.56) is subject to a further discrete symmetry, which is best expressed by making use of the complex coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{3.57}$$

The classical theory is invariant under $\theta \to \theta + 2\pi$, which corresponds to $\tau \to \tau + 1$. According to the Montonen–Olive conjecture [84], the transformation $\tau \to -1/\tau$ is also an invariance of the quantum theory. When $\theta = 0$, this transformation replaces g with $1/g$. Together, these transformations form the S-duality group $SL(2, \mathbb{Z})$, which consists of the transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \text{with} \qquad ad - bc = 1, \tag{3.58}$$

and where the numbers a, b, c, d are integers.

The quantized $D = 4$, $N = 4$ SYM theory is free of ultraviolet divergences in perturbation theory. The theory is presumably ultraviolet finite, as instanton configurations also give only finite contributions. The exact conformal invariance of the theory carries over to the quantum theory, since no dependence on a scale defined by a cutoff Λ is ever introduced; consequently, the beta function of the theory is identically zero.

We will briefly consider the construction and classification of operators in the super-Yang–Mills theory [56]. We restrict ourselves to operators that are local, gauge invariant, and polynomial in the gauge invariant fields X^i, λ_a and $F_\pm^{\mu\nu}$. All such operators can be constructed from the so-called superconformal primary operators, which are defined by the condition

$$[S, \mathcal{O}] = 0, \tag{3.59}$$

where the brackets denote either the commutator or the anticommutator, depending on whether $\mathcal{O}$ is a bosonic or a fermionic operator. The operator $\mathcal{O}_0$ of lowest dimension in a given superconformal multiplet is a superconformal primary operator. The remaining operators in the multiplet are obtained by repeatedly applying one of the Q supercharges to the primary operator $\mathcal{O}_0$. This implies that the commutator of any operator with Q cannot be a primary operator. The relations

$$\begin{aligned} &[Q, X] \sim \lambda, \qquad &&\{Q, \lambda\} \sim F^+ + [X, X], \\ &[Q, F] \sim D\lambda, \qquad &&\{Q, \bar{\lambda}\} \sim DX, \end{aligned}$$

therefore show that superconformal primary operators may not involve the fields λ or F, nor the derivatives or commutators of the fields X. The primary operators will be gauge invariant objects involving only the fields X^i. The simplest primary operators are the so-called single trace operators, which have the form

$$\mathrm{str}\big(X^{i_1} \cdots X^{i_n}\big), \tag{3.60}$$

where str denotes a symmetrized trace over the $SO(6)$ algebra. There are also multiple trace operators, which are constructed as products of the single trace operators.

3.6 Maldacena's conjecture

The first concrete example of the idea that string theories and gauge theories could be related to each other was given by Maldacena [59], who argued that a particular compactification of type IIB superstring theory is equivalent to the $N = 4$ supersymmetric Yang–Mills theory in four dimensions. More details on how the theories are related to each other were soon supplied by Gubser, Klebanov and Polyakov [85], and by Witten [86].

The conjectured equivalence between the two theories by the name of AdS/CFT correspondence, or duality, since the non-compact dimensions of the IIB theory belong to an anti de Sitter space. Since Maldacena's discovery of the duality between the two theories specified above, several other examples have been found of dualities between various superstring or supergravity theories and gauge theories. The term AdS/CFT duality is often used to refer to any or all of these dualities. The term gauge–gravity duality, which also is often used, is perhaps more appropriate, since it allows for the possibility that the string theory is defined on some other kind of a geometry instead of an anti de Sitter space.

We will now desrcibe the argument on which the Maldacena conjecture is based [69]. We consider type IIB superstring theory in flat spacetime, and we take a stack of N parallel D3-branes, which are located very close to each other. In the string theory, there are closed strings, and open strings which end on the D-branes. The open strings describe the excitations of the branes, while the closed strings describe the excitations of empty space. If we consider the system in the low-energy limit, where the relevant energies are small compared to $1/l_s$, then only massless excitations can appear. The massless excitations of the closed strings are described by the Lagrangian of type IIB supergravity, while the Lagrangian describing the massless excitations of the open strings is that of the $N = 4$ $U(N)$ supersymmetric Yang–Mills theory in four dimensions, as we explained in section 3.4. (We hope that no confusion arises from the fact that the number of supercharges and the order of the gauge group are both denoted by N.)

We may now consider the above construction from two different points of view. On the one hand, we can write down the effective action describing the massless excitations. It has the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}, \tag{3.61}$$

where S_{bulk} contains the type IIB supergravity action, S_{brane} contains the $N = 4$ super-Yang–Mills action, and S_{int} describes the interactions between the brane excitations and the empty space excitations. The action S_{bulk} can be split into the part which describes freely propagating massless particles, and the remaining terms, which are proportional to (positive) powers of $\kappa = \sqrt{8\pi G}$. The interaction term S_{int} is also proportional to positive powers of κ.

It is more convenient to take the low-energy limit by letting $l_s \to 0$, while keeping the energies and all other parameters fixed. We then have, from Eq. (3.33), $\kappa \sim l_s^4 \to 0$. This means in particular that the interactions between the brane excitations and the other excitations vanish in the low-energy limit. The bulk action reduces to one that describes free IIB supergravity, while the brane action reduces to the action of the four-dimensional $N = 4$ $U(N)$ super-Yang–Mills theory. The system of strings and D-branes has been reduced to a free supergravity theory in the full spacetime, and a gauge theory in four dimensions. Furthermore, these theories are decoupled – that is, they do not interact with each other.

Now, on the other hand, we may use the extremal p-brane solution of supergravity to give an alternative description of the above system. For extremal 3-branes, we have

$$ds^2 = H^{-1/2}(r)\Big(-dt^2 + dx^2 + dy^2 + dz^2\Big) + H^{1/2}(r)\Big(dr^2 + r^2\, d\Omega_5^2\Big), \quad (3.62)$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s N l_s^4. \quad (3.63)$$

From the metric, we read off that the energy of an object, measured by a stationary observer at distance r, is related to the energy measured at infinity by

$$E(r) = H^{1/4}(r)E_\infty. \quad (3.64)$$

For an observer at infinity, there are therefore two different kinds of low-energy excitations: massless excitations of long wavelength, which may propagate anywhere at all in spacetime, and excitations of any mass and any wavelength, located very close to $r = 0$. In the low-energy limit, the excitations near $r = 0$ do not have enough energy to escape to large distances, and it can be shown that the cross-section for absorption of particles by the branes becomes negligible in the low-energy limit [77, 87, 88]. The two kinds of excitaions therefore become decoupled in the low-energy limit. The low-energy theory is again reduced to two decoupled components: a free supergravity theory in the whole spacetime, and the whole IIB superstring theory in the so-called near-horizon region of the metric (3.62). The geometry of the near-horizon region is given by the $r \to 0$ limit of the metric as

$$ds^2 = \frac{r^2}{L^2}\Big(-dt^2 + dx^2 + dy^2 + dz^2\Big) + L^2\left(\frac{dr^2}{r^2} + d\Omega_5^2\right). \quad (3.65)$$

We recognize this as the metric of $AdS_5 \times S^5$.

We have now shown that two different descriptions of the system of strings and D3-branes both give rise to two decoupled theories in the low-energy limit. One of the theories is type IIB supergravity in flat space in either case, but the other theory is different in the two cases. These considerations led Maldacena to put forward the conjecture that the other theories, despite appearing to be completely different, should actually be identified with each other. The conjecture thus amounts to the seemingly surprising statement that type IIB superstring theory on $AdS_5 \times S^5$ is equivalent to the four-dimensional $N = 4$ $U(N)$ supersymmetric Yang–Mills theory.

The relation between the parameters of the two theories is given by

$$g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}, \qquad L^4 = 4\pi g_s N l_s^4, \tag{3.66}$$

where L is the radius of curvature of the AdS_5, and the radius of the S^5. The relation between the string coupling constant and the Yang–Mills coupling constant can be justified by considering the so-called Dirac–Born–Infeld action [63], which describes the dynamics of Dp-branes. The relevant part of the action is given by

$$S_p = -T_p \int d^{p+1}\xi \, e^{-\phi} \sqrt{-\det\bigl(G_{ab} + 2\pi l_s^2 F_{ab}\bigr)}, \tag{3.67}$$

where the indices a, b refer to the coordinates along the brane, and G_{ab} is the metric on the world volume of the brane. One can show that in the low-energy limit, the action takes the form

$$S_p = \int d^{p+1}\xi \, \frac{1}{4}(2\pi l_s^2)^2 \, T_p \, \mathrm{Tr}\bigl(F_{\mu\nu}F^{\mu\nu}\bigr) + \ldots \tag{3.68}$$

In particular, for D3-branes we find, using Eq. (3.7),

$$S_3 = \int d^4\xi \, \frac{1}{2(4\pi g_s)} \, \mathrm{Tr}\bigl(F_{\mu\nu}F^{\mu\nu}\bigr) + \ldots \tag{3.69}$$

This shows that g_{YM}^2 should be identified with $4\pi g_s$, which reproduces the relation in Eq. (3.66).

The conjecture, as it was originally formulated by Maldacena, states that the equivalence between the IIB string theory and the $D = 4$, $N = 4$ super-Yang–Mills theory holds for all values of the parameters N and g_s or g_{YM}. This statement is now known as the strong form of the conjecture. It is possible to formulate weaker forms of the conjecture, which sacrifice generality in order to make the implications of the conjecture more tractable [56]. One of the weaker forms is obtained by taking the large N limit, keeping $\lambda \equiv g_s N \sim g_{\mathrm{YM}}^2 N$ fixed, as specified in section 3.2. The interpretation of this limit in the gauge theory side was elaborated on in section 3.2. In the string theory side, the limit implies that $g_s \to 0$, so it corresponds to doing string perturbation theory in the limit of weak string coupling. Thus, the correspondence now is between the classical limit of a string theory and the large N limit of a gauge theory. The other weak form of the conjecture is obtained in the limit $\lambda \to \infty$, and it corresponds to the strong coupling limit on the gauge theory side, and to the classical supergravity limit on the string theory side.

A natural question to ask at this point is that how can the type IIB theory and the supersymmetric Yang–Mills theory be equivalent to each other, despite appearing completely different. An answer to this question is given by the following considerations [69]. In the super-Yang–Mills theory, it is valid to do perturbation theory as long as $g_{\rm YM}^2 N \ll 1$. On the other hand, the classical supergravity description can be trusted when the radius of curvature of the AdS_5 and S^5 is very large compared to the string length l_s. We have already shown in Eq. (3.48) that this corresponds to $g_s N \gg 1$, and thus to $g_{\rm YM}^2 N \gg 1$. This shows why there is no obvious contradiction in the conjecture even though the two theories do not look the same at all – it is because the weak coupling limit of each theory corresponds to the strong coupling limit of the other.

A nontrivial test of the Maldacena conjecture is that the global symmetries of the IIB theory and the SYM theory match exactly [56, 75]. This is an important test, because it suggests that the conjecture could be valid for all values of N, and not just in the large N limit. The isometry group of $AdS_5 \times S^5$ is $SO(2,4) \times SO(6)$, where the factors are the isometry groups of AdS_5 and S^5, respectively. The SYM theory is conformally invariant, and the $SO(2,4)$ factor is to be identified with the group of conformal transformations in four dimensions, as we have argued in section 3.3 and in Appendix 3. The $SO(6)$ factor is identified with the $SU(4)$ symmetry of the supercharges and the scalar fields of the SYM theory. The number of fermionic symmetry generators is also the same in both theories. Even though the IIB theory has 32 supercharges and the SYM theory only 16, the SYM theory has in addition the 16 conformal supercharges, and so both of the theories have 32 fermionic generators in total.

In addition, both the IIB theory and the SYM theory have a discrete $SL(2,\mathbb{Z})$ symmetry, which acts on the complex coupling constant τ as

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{3.70}$$

In the IIB theory and in the SYM theory we have, respectively

$$\tau_{\rm IIB} = C + ie^{-\phi}, \qquad \text{and} \qquad \tau_{\rm SYM} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2}. \tag{3.71}$$

As the closed string coupling constant is given by $g_s = e^{\langle\phi\rangle}$, this gives us an alternative reason to identify g_s with $g_{\rm YM}^2/4\pi$.

Naturally, in addition to identifying the symmetries of the two theories, it should be possible to also identify the actual representations of the symmetry group on both sides of the corrrespondence with each other. That is, operators in the SYM theory should be related to the fields of the IIB theory on $AdS_5 \times S^5$. The general prescription is given by [69, 85, 86]

$$\left\langle e^{\int d^4x\, \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \right\rangle_{\rm SYM} = Z_s\Big[\phi(\vec{x}, z)\big|_{z=0}\Big] = Z_s\big[\phi_0(\vec{x})\big]. \tag{3.72}$$

Here the object on the left is the generating function of correlation functions in the gauge theory. Correlation functions are calculated by taking functional derivatives of the generating function with respect to ϕ_0 and setting $\phi_0 = 0$ in the end. On the right, we have the partition function of the full string theory. The field $\phi(\vec{x}, z)$ is defined on AdS_5, and satisfies the boundary condition $\phi(\vec{x}, 0) = \phi_0(\vec{x})$. Eq. (3.72) therefore puts fields on the AdS side into a one-to-one correspondence with operators on the SYM side. More details on this correspondence can be found e.g. in [56, 69, 75].

We will now give a brief outline on how the correlation functions on the two sides of the correspondence are related to each other [56, 69, 85, 86]. On the SYM side, we have a conformal field theory, and so we can immediately write down that the two-point and three-point functions have the form

$$\begin{aligned}\big\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\mathcal{O}_{\Delta_2}(\vec{x}_2)\big\rangle &= \frac{\delta_{\Delta_1\Delta_2}}{(x_1 - x_2)^{2\Delta_1}},\\ \big\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\mathcal{O}_{\Delta_2}(\vec{x}_2)\mathcal{O}_{\Delta_3}(\vec{x}_3)\big\rangle &= \frac{c_{\Delta_1\Delta_2\Delta_3}(g_{\rm YM}, N)}{(r_{12})^{\Delta_1+\Delta_2-\Delta_3}(r_{13})^{\Delta_1+\Delta_3-\Delta_2}(r_{23})^{\Delta_2+\Delta_3-\Delta_1}}.\end{aligned} \tag{3.73}$$

On the AdS side, we may perform an expansion of the supergravity action in powers of the five-dimensional gravitational constant κ_5. The limit of small κ_5 corresponds to the limit of large N, because κ_5 and N are related by $\kappa_5^2 = 4\pi^2/N^2$. The expansion can be represented graphically in terms of so-called Witten diagrams. A Witten diagram is a disk, whose interior represents the bulk of AdS_5, and whose boundary represents the boundary of AdS_5. On the boundary, there are source fields $\varphi_\Delta(\vec{x}_i)$. The source fields are connected to each other, or to interaction points within the bulk, by a boundary-to-bulk propagator. Interaction points in the bulk are connected to each other by a bulk-to-bulk propagator. The interactions in the bulk obey the Feynman rules derived from the supergravity action.

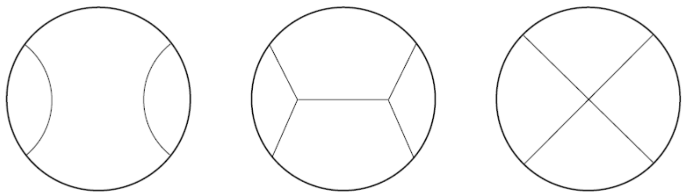


Figure 8. Witten diagrams [69].

If we use for AdS_5 the Euclidean coordinates, where the metric is $ds^2 = (dz_0^2 + d\vec{z}^2)/z_0^2$, then the boundary-to-bulk propagator is given by

$$K_\Delta(z_0, \vec{z}, \vec{x}) = C_\Delta \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta, \tag{3.74}$$

where C_Δ is a numerical factor and $\vec{x}$ is a point on the boundary. The two-point function on the AdS side thus agrees with the two-point function on the SYM side, as we have

$$\lim_{z_0 \to 0} z_0^{-\Delta} K_\Delta(z_0, \vec{z}, \vec{x}) \sim \frac{1}{(\vec{z} - \vec{x})^{2\Delta}}.$$

The evaluation of the three-point function is more complicated, as it involves an integration over an interaction point in the bulk. A calculation, which has been done in [89, 90], nevertheless shows that the three-point function on the AdS side agrees with the three-point function of Eq. (3.73).

Finite temperature field theory on the gauge theory side of the correspondence can be studied by replacing the extremal D-branes with non-extremal ones, thus introducing into the AdS_5 a black hole, whose Hawking radiation can be interpreted as providing the finite temperature [63, 69, 77]. At finite temperature, neither supersymmetry nor conformal invariance is preserved, which complicates the comparison between the two sides of the correspondence. The near-horizon limit of the non-extremal 3-brane geometry is given by

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)\,dt^2 + dx^2 + dy^2 + dz^2\right) + L^2\left(\frac{dr^2}{f(r)r^2} + d\Omega_5^2\right), \qquad (3.75)$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^4. \qquad (3.76)$$

The Euclidean version of the metric is obtained by setting $t = -it_E$, and t_E is related to the temperature by $t_E = 1/T$. There is now a Schwarzschild black hole in the AdS_5; the horizon is at r_0. The metric enables one to calculate the area of the horizon and thus find the entropy $S_{\rm BH}$ of the non-extremal 3-brane at temperature T. This entropy is related to the strong coupling limit of the super-Yang–Mills theory. On the other hand, the entropy S_0 of a free $N = 4$ $U(N)$ supersymmetry multiplet can be calculated from statistical mechanics. These calculations show that $S_{\rm BH} = \frac{3}{4} S_0$. This provides a starting point for an analysis on how the entropy and other thermodynamical quantities depend on the coupling constant in the $D = 4$, $N = 4$ super-Yang–Mills theory; see e.g. [91, 92, 93, 94].

The AdS/CFT correspondence seems to give a concrete implementation of the so-called holographic principle [69, 95, 96]. The holographic principle roughly states that in a quantum theory of gravity, it should be possible to associate the degrees of freedom inside a given region with the boundary of the region. Furthermore, there should be no more than one degree of freedom assigned to each Planck area of the boundary. The holographic principle originates from the study of the thermodynamics of black holes. More specifically, it was shown by Bekenstein and Hawking that a black hole can be viewed as a thermodynamical object, whose entropy is proportional to the area of the event horizon, and is given by $S_{\rm BH} = A/4G$ [97, 98, 99]. This can be taken to suggest that the degrees of freedom of the black hole somehow belong to the event horizon. One can also show that the entropy inside any region can never exceed the value of $S_{\rm BH}$ [69]. As the entropy somehow is a measure of the number of degrees of freedom within the system, it thus seems to be generally true that the degrees of freedom of a physical configuration should be assigned to the boundary of the region in which the configuration is contained.

The holographic principle seems to imply that one should be able to describe the physics within some region by a theory defined on the boundary of the region. This is exactly what happens in the AdS/CFT correspondence, since the conjecture states that the theory which describes the physics in the AdS_5 (we are presumably free to not pay any attention to the compact dimensions) is equivalent to a field theory defined on a space whose dimension is lower by one. A detailed demonstration that the AdS/CFT correspondence is compatible with the holographic principle, including a proof that there is exactly one degree of freedom per each Planck area, was given by Susskind and Witten [100].

3.7 Holographic models for QCD

In its original form, the AdS/CFT duality has only a very limited applicability to questions related to experimental observations. The gauge theory side of the duality has conformal invariance and maximal supersymmetry, neither of which are realized in Nature. In order to make the duality into a tool for doing calculations related to QCD, one has to find a way to reduce the symmetry of the gauge theory by breaking the conformal invariance and possibly bringing the $N = 4$ supersymmetry down to $N = 1$ supersymmetry or to no supersymmetry at all. There are two main approaches to this. One may start from a full string theory and make various adjustments and deformations in order to make the dual gauge theory less symmetrical, with the hope of ending up with a theory resembling QCD. One may also begin with a phenomenologically motivated gauge theory and apply the duality to it, hoping to build up the theory and to eventually be able make a connection with a string theory.

It is not our intention to go into the applications of the duality in any depth; however, before we conclude this work, we will give a brief discussion of a particular class of models, developed by Kiritsis and collaborators [101, 102, 103, 104]. The models are based on a five-dimensional gravity theory, which contains the metric $g_{\mu\nu}$ and a scalar field Φ. The metric is dual to the energy-momentum tensor of the gauge theory, and the scalar field is dual to $\mathrm{Tr}\,F^2$. The effective action for the gravity theory is given by

$$S = -N^2 M_{\rm P}^3 \int d^5x\,\sqrt{-g}\biggl(R - \frac{4}{3}(\partial\Phi)^2 + V(\Phi)\biggr) + 2N^2 M_{\rm P}^3 \int d^4x\sqrt{-h}\,K, \tag{3.77}$$

where K is a curvature scalar associated with the four-dimensional boundary, and the five-dimensional gravitational constant is given by $N^2 M_{\rm P}^3 = 1/16\pi G_5$.

The second term is the so-called Gibbons–Hawking term. It has a role in ensuring consistency of the quantum theory, but it will not be relevant to the calculations which we are going to perform.

By varying the action, we find Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{4}{3}\biggl(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}(\partial\Phi)^2 g_{\mu\nu}\biggr), \tag{3.78}$$

and the equation of motion for the scalar field,

$$\Box\Phi + \frac{3}{8}\frac{\partial V}{\partial\Phi} = 0. \tag{3.79}$$

(The Laplacian is defined as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, where ∇_μ is the covariant derivative.) These equations have vacuum solutions, given by

$$ds^2 = b(r)^2\bigl(-dt^2 + d\vec{x}^2 + dr^2\bigr), \tag{3.80}$$

as well as black hole solutions, which have the form

$$ds^2 = b(r)^2\biggl(-f(r)\,dt^2 + d\vec{x}^2 + \frac{dr^2}{f(r)}\biggr). \tag{3.81}$$

In either case Φ is a function of r only. The boundary is at $r = 0$; furthermore, the black hole solution has a horizon at $r = r_h$, so that $f(r_h) = 0$. The coordinate r is related to an energy scale in the four-dimensional gauge theory, according to $E = E_0 b(r)$. Arguments on why this is a reasonable identification are given in [101]. Moreover, e^Φ is identified with the (running) 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$. These identifications allow us to define the beta function of the gauge theory in terms of quantities belonging to the gravity theory as

$$\beta(\lambda) \equiv \frac{d\lambda}{d\ln E} = \lambda\frac{\dot{\Phi}}{\dot{b}/b}. \tag{3.82}$$

Here, and for the rest of this section, the dot denotes a derivative with respect to r.

Depending on one's point of view, a particular model for QCD can be specified by giving either the potential $V(\Phi)$, or the function $b(r)$. In either case, Eq. (3.79) gives a relation between $V(\Phi)$ and $b(r)$. In principle, the form of $V(\Phi)$ or $b(r)$ can be chosen freely. Of course, one should try to make the choice so that the resulting theory would have as many features as possible resembling those of QCD. This is a distinctly phenomenological approach, because it is not attempted to derive, say, the form of $V(\Phi)$ from first principles. In some sense, $V(\Phi)$ or $b(r)$ is a fitting parameter of the theory. From now on, we will focus on the black hole solutions, which correspond to a finite temperature on the gauge theory side. Even so, the function $f(r)$ does not provide another fitting parameter, because it is determined by the equations of motion once $V(\Phi)$ or $b(r)$ is specified.

An important feature that we require the model to have is that it produces confinement in the low energy limit. This requirement translates into a condition on the asymptotic behaviour of the beta function [102], in a way which we will now briefly describe. The idea is to calculate the energy of a configuration where two quarks are bound together by a string in such a way that the quarks are constrained to remain on the four-dimensional boundary, but the string extends also along the fifth dimension [105, 106]. Confinement then arises if the energy increases linearly with the separation of the quarks, in the limit where the separation is large. Assuming the metric in the string frame has the general form

$$(g_S)_{\mu\nu}dx^\mu dx^\nu = -g_{tt}(r)\,dt^2 + g_{xx}(r)\,d\vec{x}^2 + g_{rr}(r)\,dr^2,$$

the result of the analysis is that the energy is given as a function of the separation L by

$$E(L) = T_s g(r_*)L - 2T_s \int_0^{r_*} dr\,\frac{h(r)}{g(r)}\sqrt{g(r)^2 - g(r_*)^2}, \tag{3.83}$$

where r_* is the turning point of the string in the fifth direction, and the functions $g(r)$ and $h(r)$ are defined as

$$g(r) = \sqrt{g_{tt}(r)g_{xx}(r)}, \qquad h(r) = \sqrt{g_{tt}(r)g_{rr}(r)}. \tag{3.84}$$

In the limit of large L, provided that $g(r_*) \neq 0$, the first term in Eq. (3.83) is dominant over the second term. In this limit, the energy therefore is

$$E(L) = T_s g(r_*)L. \tag{3.85}$$

This shows that a condition which leads to an energy which grows linearly with L when L is large, and thus guarantees confinement, is that $g(r)$ does not vanish at $r = r_*$. Further analysis is required to make a connection between this condition and the properties of the QCD model. The analysis can be found in [102], and we will not go into it here. The result is that if the metric has the form (3.81), and if the function $b(r)$ obeys

$$\ln b(r) \sim -Cr^\alpha \qquad \text{when} \qquad r \to \infty,$$

then the model exhibits confinement if and only if the potential goes as

$$V(\lambda) \sim \lambda^{4/3}\big(\ln\lambda\big)^{1-1/\alpha} \tag{3.86}$$

when λ is large. An equivalent condition for confinement, given in terms of the beta function, is that

$$\lim_{\lambda\to\infty}\left(\frac{\beta(\lambda)}{\lambda} + \frac{3}{2}\right)\ln\lambda \leq 0; \tag{3.87}$$

this includes the case where the limit is equal to $-\infty$.

To conclude this section, we will look more closely into a particular holographic QCD model. The model is defined by setting [102, 107]

$$b(r) = \frac{L}{r} e^{-\Lambda^2 r^2/3} = \frac{L\Lambda}{\sqrt{y}} e^{-y/3}, \tag{3.88}$$

where L is the radius of the anti de Sitter space, Λ is an (in principle arbitrary) energy scale, and we have defined $y = \Lambda^2 r^2$. We then take the black hole metric (3.81), and calculate its Einstein tensor. Doing so, we find that the Einstein equations (3.78) become

$$6\frac{\dot{b}^2}{b^2} - 3\frac{\ddot{b}}{b} = \frac{4}{3}\dot{\Phi}^2, \tag{3.89a}$$

$$\frac{\ddot{f}}{\dot{f}} + 3\frac{\dot{b}}{b} = 0, \tag{3.89b}$$

$$6\frac{\dot{b}^2}{b^2} + 3\frac{\ddot{b}}{b} + 3\frac{\dot{b}}{b}\frac{\dot{f}}{f} = \frac{b^2}{f}V. \tag{3.89c}$$

We also have the equation for the scalar field,

$$\ddot{\Phi} + \left(\frac{\dot{f}}{f} + 3\frac{\dot{b}}{b}\right)\dot{\Phi} + \frac{3b^2}{8f}\frac{dV}{d\Phi} = 0;$$

however, one can show that this equation follows from the three Einstein equations, and so we may disregard it. It proves advantageous to eliminate the second derivatives of b from the above system of equations, at the cost of introducing an additional equation [107]. Defining $W = -\dot{b}/b^2$, we may show that the system of equations (3.89) is equivalent to

$$\dot{W} = 4bW^2 - \frac{1}{f}\left(W\dot{f} + \frac{1}{3}bV\right), \tag{3.90a}$$

$$W = -\frac{\dot{b}}{b^2}, \tag{3.90b}$$

$$\dot{\lambda} = \frac{3}{2}\lambda\sqrt{b\dot{W}}, \tag{3.90c}$$

$$\ddot{f} = 3bW\dot{f}. \tag{3.90d}$$

We are going to solve this system in terms of the variable y. It is straightforward to show that Eq. (3.90c) implies

$$\lambda(y) = \lambda_0 \exp\left(-\frac{1}{2}\sqrt{y(y+\tfrac{9}{2})}\right)\left(\sqrt{y} + \sqrt{y+\tfrac{9}{2}}\right)^{9/4}. \tag{3.91}$$

To solve for $f(r)$, we note that Eq. (3.90d) is equivalent to

$$\dot{f}(r) = \frac{C}{b(r)^3}.$$

With the function $b(r)$ specified in Eq. (3.88), this equation has an analytical solution. It is given by

$$f(y) = \frac{y_h - 1 - (y-1)e^{y-y_h}}{y_h - 1 + e^{-y_h}}. \tag{3.92}$$

where the constants of integration have been adjusted so that $f(0) = 1$ and $f(y_h) = 0$. Finally, from Eq. (3.90a) we find

$$V\big(y(\lambda)\big) = \frac{12}{L^2} e^{2y/3} \left[\left(\frac{1}{3} y^2 + \frac{5}{6} y + 1 \right) f(y) - \left(\frac{1}{3} y^2 + \frac{1}{2} y \right) f'(y) \right], \tag{3.93}$$

where the prime denotes differentiation with respect to y.

Let us now examine the properties of this solution. In the high energy limit, $r \to 0$, and

$$b(r) \sim \frac{L}{r}.$$

For the beta function, we then have

$$\beta(\lambda) = \frac{\dot{\lambda}}{\dot{b}/b} = -r \frac{d\lambda}{dr}.$$

Matching this with the one-loop beta function of QCD,

$$\beta(\lambda) = -b_0 \lambda^2,$$

we find

$$b_0 \lambda(r) = -\frac{1}{\ln(\Lambda r)}$$

with $\Lambda =$ constant. This shows that as $r \to 0$, $\lambda(r) \to 0$, so the model reproduces asymptotic freedom in the high energy limit. To determine whether there is confinement in the low energy limit, we use Eq. (3.91) to calculate the beta function predicted by the model as

$$\beta(\lambda) = -\frac{3}{2} \lambda \frac{\sqrt{1 + 9/2y}}{1 + 3/2y}. \tag{3.94}$$

In the limit $y \to \infty$, we have

$$\beta(\lambda) = -\frac{3}{2} \lambda \left(1 + \frac{3}{4y} + \dots \right) = -\frac{3}{2} \lambda \left(1 + \frac{3}{8 \ln \lambda} + \dots \right). \tag{3.95}$$

This implies that

$$\frac{\beta(\lambda)}{\lambda} + \frac{3}{2} = -\frac{3}{8 \ln \lambda} + \mathcal{O}\left(\frac{1}{(\ln \lambda)^2} \right),$$

from which we see that this model indeed satisfies the criterion (3.87) for confinement. Furthermore, if we set $f = 1$ in Eq. (3.93) and use Eq. (3.91), we find that in the limit of large y the potential has the asymptotic form

$$V(\lambda) \sim \lambda^{4/3} \sqrt{\ln \lambda}, \tag{3.96}$$

again in agreement with Eq. (3.86), because we now have $\alpha = 2$.

The present model can be used to make calculations about the thermodynamics of QCD. Other quantities, such as the masses of the so-called glueballs (bound states of gluons), can also be calculated. Such calculations have been performed e.g. in [103, 104, 107]. The results are in good agreement with data obtained from lattice simulations.

Conclusions

This thesis can be regarded as a documentary about my journey from having a reasonable knowledge of quantum field theory and general relativity to achieving some kind of an understanding of the duality between gauge theories and string theories. I will hopefully be able to shed some light on my reasons for wanting to take on such a journey a little later. A look at the table of contents shows that many things need to be learned in order to be able to trace a path from quantum field theory and general relativity to the AdS/CFT duality. Because of this learning process, I feel that my time working on this thesis was well spent, even though I did not produce an answer to a research question, and my results cannot be quantified by conventional measures, such as the number of papers published.

The contents of this thesis form a fairly accurate representation of what I have studied along my way towards understanding the gauge theory–string theory duality. In retrospect, I now look at some of the topics which I studied as detours, which did not take me any closer to the ultimate goal of my journey, and whose purpose was only to look more closely into a particularly interesting scenery, which was unfolding somewhere along the way. In particular, I feel that the construction of an off-shell formulation for simple supergravity is not a topic important enough to deserve such a detailed treatment. If I were to write this thesis again, I would be inclined leave out most of the material in sections 2.4 and 2.5, replacing it with a more detailed treatment of some aspects of the AdS/CFT duality.

It seems appropriate to give a brief summary of what I feel I have learned during the course of this work. I naturally achieved a basic knowledge feel ideas related to the AdS/CFT duality, including supersymmetry and supergravity, to the point that I feel I am able to read basic research papers on these subjects. Also, my ability to perform technical calculations probably became better; however, this is not really specific to this work, as one can improve one's calculational skill by performing pencil-and-paper calculations related to any field of physics. An incomplete list of my less obvious learning achievements includes the following: Learning to appreciate the merits of supersymmetry and string theory, and so becoming able to understand why they have been in the focus of theoretical research during the last couple of decades despite their seemingly obvious difficulties; becoming aware of the underlying analogy between how general relativity is constructed and how gauge theories are constructed in particle physics; going from knowing practically no group theory at all to knowing a little about group theory; and improving my understanding of some parts of physics which I already knew, in particular general relativity. Instead of trying to give a more detailed account of my learning process, I would now like to bring this work to a conclusion with a few closing thoughts.

The AdS/CFT duality has shown itself to be a useful computational tool. Models such as the one we described in section 3.7 provide one with a method for calculating quantities which would be extremely difficult to calculate directly from QCD. The models have reasonable predictive power, in that a greater number of physical quantities can be calculated from the model than there are adjustable parameters in it. Moreover, the results mostly agree with those derived from lattice calculations, which give one confidence that QCD can be reasonably modeled by making use of the AdS/CFT duality, even though a string theory is not known which could be put into direct correspondence with QCD.

One should keep in mind that the duality offers computational simplicity only as long as one can use the classical supergravity description on the string theory side. This corresponds to going to the large N limit on the gauge theory side. There could very well be situations on the gauge theory side in which the large N approximation is not satisfactory. In such cases, using classical supergravity on the string theory side might yield incorrect results, and one would have to calculate corrections from the full string theory, provided one still wants to make use of the duality. In this case, much of the computational simplicity would be lost, as the string corrections could turn out to be very difficult to calculate.

I am not convinced whether the AdS/CFT duality should be regarded as anything more than a useful computational tool, and how seriously one should take the picture that the observable world is the boundary of a five-dimensional space, and elementary particles are the endpoints of strings living on this five-dimensional space. The validity of this picture naturally depends, among other things, on whether or not string theory has any relevance to Nature. While string theory certainly is an extremely impressive theoretical construction, which has a multitude of remarkable features, I find it quite worrying that practically no connection has been managed to make between string theory and Nature, even though superstring theory has been in existence for almost thirty years. This certainly cannot be blamed on not enough research having been done on string theory.

A perhaps more conventional topic for a master's thesis would have been to take a particular model for QCD motivated by the duality, and examine its consequences in great detail. I would not have felt happy with such a topic, partly because I, like a lot of theoreticians, find pleasure in working with things which I find aesthetically pleasing. Another part of the reason is that I feel that the studying of purely phenomenologically motivated models has only very limited potential to advance our understanding of Nature, due to their quite incomplete theoretical foundations. To gain a more satisfying understanding from such models, one should try to put the model on a more solid theoretical basis; however, it seems extremely difficult to improve the foundations of the model without at the same time making it completely intractable.

In some sense, the choice between theory and phenomenology ultimately is not much more than a matter of taste. Any succesful theory of physics should be able to connect itself with Nature in the end, and one could be able to construct a good theory regardless of whether one takes theoretical considerations, or what is observed in Nature, as one's starting point. However, the way I look at physics is that phenomenology should be viewed not as an end, but as a means to an end, the end being to accomplish a satisfactory fundamental theory of physics. In more concrete terms, I feel that defining a model in an ad hoc manner, by postulating the metric given by Eqs. (3.81) and (3.88), is not really satisfying at all, and in an ideal world one would be able to show that the model follows as a consequence of a more fundamental theory.

To make a connection between a QCD model and a fundamental theory is of course more easily said than done. The task of deriving such a model from string theory, and the task of working one's way up from such a model until one reaches a string theory, both seem close to impossble at the moment. It also seems very difficult to judge which of them might be less impossible, and so it is perfectly justifiable that both approaches are being studied side by side. One should also remain open to the possibility that the fundamental theory might not be a string theory at all.

Appendices

A.1 Conventions

Our conventions do not entirely follow those of any single author. They are a compromise between personal preference and the need to conform to the references which were being used.

We naturally use units in which $c = \hbar = 1$. For Newton's constant, we sometimes set $8\pi G = 1$, but sometimes we keep it explicit, often in the form $\kappa \equiv \sqrt{8\pi G}$. We also set Boltzmann's constant $k = 1$, though this will be relevant only in section 3.6.

For flat Minkowski spacetime, we use the mostly plus metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}. \tag{A.1}$$

We define the ϵ-symbol so that in D-dimensional spacetime

$$\epsilon^{01\cdots D-1} = 1, \qquad \text{and consequently} \qquad \epsilon_{01\cdots D-1} = -1. \tag{A.2}$$

There are not enough letters in the Latin and Greek alphabet to enable one to write down various indices so that there would be absolutely no potential for confusion. Our choices for different indices are summarized below:

- Spacetime indices are denoted by $\mu, \nu, \ldots$
- Indices referring to the locally flat coordinate systems in the vielbein formalism are denoted by $m, n, \ldots$
- Four-component spinor indices are denoted by $a, b, \ldots$
- Two-component spinor indices are denoted by $\alpha, \beta, \ldots$
- Various internal indices are denoted by $a, b, \ldots$, and by $i, j, \ldots$, and sometimes by $r, s, \ldots$

In four spacetime dimensions, we treat spinors using two-component Weyl spinors and the notation of dotted and undotted indices. A four-component spinor contains two two-component Weyl spinors according to

$$\Psi_a = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

In this formalism, a Dirac spinor contains two different Weyl spinors, while a Majorana spinor contains only one. The above spinor therefore is a Dirac spinor; a Majorana spinor has the form

$$\Psi_a = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}.$$

The two-component indices are raised and lowered with the ϵ-matrices

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.3}$$

Undotted indices are contracted from upper left to lower right, and dotted indices are contracted from lower left to upper right:

$$\begin{aligned} \psi\chi &= \psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \chi_\alpha, \\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} \bar{\chi}^{\dot{\alpha}}. \end{aligned}$$

The index structure of the Pauli matrices is

$$(\sigma_\mu)_{\alpha\dot{\alpha}}, \qquad (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma_\mu)_{\beta\dot{\beta}}. \tag{A.4}$$

We also define

$$\begin{aligned} (\sigma_{\mu\nu})_\alpha{}^\beta &= \frac{1}{4}\Big((\sigma_\mu)_{\alpha\dot{\alpha}} (\bar{\sigma}_\nu)^{\dot{\alpha}\beta} - (\sigma_\nu)_{\alpha\dot{\alpha}} (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} \Big), \\ (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}\Big((\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} (\sigma_\nu)_{\alpha\dot{\beta}} - (\bar{\sigma}_\nu)^{\dot{\alpha}\alpha} (\sigma_\mu)_{\alpha\dot{\beta}} \Big). \end{aligned} \tag{A.5}$$

For anticommuting spinors χ and ψ, we have the Fierz identities

$$\begin{aligned} \chi_\alpha \psi_\beta &= -\frac{1}{2} \chi\psi \epsilon_{\alpha\beta} - \frac{1}{4} \chi \sigma^{\mu\nu} \psi (\sigma_{\mu\nu})_{\alpha\beta}, \\ \bar{\chi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} &= -\frac{1}{2} \bar{\chi}\bar{\psi} \epsilon_{\dot{\alpha}\dot{\beta}} - \frac{1}{4} \bar{\chi} \bar{\sigma}^{\mu\nu} \bar{\psi} (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}, \\ \chi_\alpha \bar{\psi}_{\dot{\beta}} &= +\frac{1}{2} \chi \sigma^\mu \bar{\psi} (\bar{\sigma}_\mu)_{\alpha\dot{\beta}}. \end{aligned} \tag{A.6}$$

Using the Fierz identities and identities satisfied by the Pauli matrices, which can be found e.g. in [1], one can derive a number of further identities, such as

$$\bar{\chi}\sigma_\mu\chi \, \bar{\chi}\bar{\sigma}_\nu\chi = -\frac{1}{2}\chi^2\bar{\chi}^2\eta_{\mu\nu},$$

which are often needed in supersymmetry calculations.

A.2 The Lorentz group. Spinors

The Lorentz group [6, 108, 109] is the group $O(1,3)$ of coordinate transformations $x^\mu \to x^{\mu\prime}$, under which the length squared of x^μ,

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2,$$

is invariant. The group is generated by the six operators $M_{\mu\nu}$, which are antisymmetric in μ, ν, and which obey the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\big(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}\big). \tag{A.7}$$

The Lorentz group splits into four components, $L_+^\uparrow$, $L_-^\uparrow$, $L_+^\downarrow$ and $L_-^\downarrow$, which are disconnected from each other, and which contain, respectively, the identity element, the parity operator, the time reversal operator, and the combination of parity and time reversal. The transformations belonging $L_+^\uparrow$ or $L_-^\downarrow$ have determinant $+1$, and so they constitute the group $SO(1,3)$. The set $L_+^\uparrow$ is a group by itself. It is denoted by $SO^+(1,3)$, and is sometimes called the proper orthochronous Lorentz group [6]. Many authors, when they speak of the Lorentz group, actually mean the group $SO(1,3)$ or $SO^+(1,3)$.

The representations of the group $SO(1,3)$ are found by separating the generators $M_{\mu\nu}$ into the generators of rotations J_i and the generators of Lorentz boosts K_i, where i goes from 1 to 3. These generators obey the algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{A.8}$$

We then define the linear combinations

$$J_\pm = \frac{1}{2}\big(J \pm iK\big). \tag{A.9}$$

This splits the algebra into two commuting $SU(2)$ algebras, as the generators $J_\pm$ obey

$$[J_{+i}, J_{+j}] = i\epsilon_{ijk}J_{+k}, \qquad [J_{-i}, J_{-j}] = i\epsilon_{ijk}J_{-k}, \tag{A.10}$$

and

$$[J_{+i}, J_{-j}] = 0. \tag{A.11}$$

As the representations of $SU(2)$ are labeled by a number j, which takes the values $0, \frac{1}{2}, 1, \ldots,$ we have that representations of $SO(1,3)$ are labeled by a pair of numbers (j_+, j_-), each of which is an integer or a half-integer. The representation (j_+, j_-) consists of $(2j_+ + 1)(2j_- + 1)$ objects φ_{mn}, where the index m takes values from $-j_+$ to j_+, while n goes from $-j_-$ to j_-.

The $(0,0)$ representation is the scalar representation. The representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ correspond to Weyl spinors of opposite chiralities. These can be combined into the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation, which is a Dirac spinor. By counting the number of components, one would expect that the representation $(\frac{1}{2}, \frac{1}{2})$ is the vector representation, and this in fact is indeed so.

We would like to discuss the spinor representations of the Lorentz group in some more detail. Our discussion closely parallels that of [6]. It has been shown [110, 111] the unitary irreducible representations of the Lorentz group are given by the irreducible representations of the so-called universal covering group. The universal cover of a group G is a simply connected group which is locally isomorphic to G, and contains all of the elements of G.

We will focus for the moment on the proper orthochronous group $SO^+(1,3)$. Its universal cover is given by the group $SL(2,\mathbb{C})$, as can be shown by defining

$$X(x) = \sigma^\mu x_\mu = \begin{pmatrix} x^0 - x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x_0 - x^3 \end{pmatrix}.$$

The objects $X(x)$ define the so-called carrier space, which contains the vectors x^μ, and on which the representation of Lorentz transformations will act. As we now have $x^\mu x_\mu = -\det X$, we see that Lorentz transformations will be represented by elements of $SL(2,\mathbb{C})$. If we denote the representation matrices by $\rho(L)$, then Lorentz transformations act on X as

$$LX(x) = \rho(L)X(x)\rho^\dagger(L). \tag{A.12}$$

The transformation matrix is explicitly given by

$$\rho(L) = \exp\left[\frac{1}{2}\left(-i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma}\right)\right]. \tag{A.13}$$

If the direction of the boost is the same as the axis of the rotation, then the matrix factorizes as $\rho(L) = R_L B_L$, where R_L represents a rotation and B_L represents a boost. This gives us a two-to-one mapping from $SL(2,\mathbb{C})$ to $SO^+(1,3)$, as we have $R_{\theta+2\pi} = -R_\theta$, but the minus sign cancels in Eq. (A.12), because it has two ρ matrices.

In the fundamental representation, the carrier space W is the space of complex two-component Weyl spinors,

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

on which Lorentz transformations act as $L\chi = \rho(L)\chi$.

In order to define inner products between spinors, we introduce the dual space $\widetilde{W}$, whose elements have the form

$$\tilde{\psi} = \left(\psi^1 \; \psi^2\right).$$

The Lorentz group acts in the dual space as $L\tilde{\psi} = \tilde{\psi}\rho(L^{-1})$.

An inner product is now given by $(\chi, \psi) = \chi^T M \psi$, where M is a mapping which takes a $\psi \in W$ to $\psi^T M \in \widetilde{W}$. The requirement of Lorentz invariance essentially restricts M to have the form

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \epsilon^{\alpha\beta}. \tag{A.14}$$

For each $\chi_\alpha \in W$, we therefore define $\chi^\alpha \equiv \epsilon^{\alpha\beta}\chi_\beta \in \widetilde{W}$. The inner product, which we will denote by $\chi\psi$ from now on, is then given by

$$\chi\psi = \epsilon^{\alpha\beta}\chi_\beta\psi_\alpha. \tag{A.15}$$

The requirement that the inner product is symmetric implies that spinors are anticommuting: $\chi_\beta\psi_\alpha = -\psi_\alpha\chi_\beta$.

We will further define the complex conjugate representation, whose carrier space $\overline{W}$ consists of the elements $\bar{\psi}^{\dot{\alpha}}$, whose Lorentz transformations are given by $L\bar{\psi} = \rho^\dagger(L)\bar{\psi}$. The dual space is now defined by

$$\bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}, \tag{A.16}$$

where

$$\epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.17}$$

We have $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$, and a similar identity for $\epsilon_{\alpha\beta}$.

The representations corresponding to W and $\overline{W}$ are the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, respectively. That the representation $(\frac{1}{2}, \frac{1}{2})$ is the vector representation is explicitly shown by the definition $x_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}x_\mu$. A general representation $(m/2, n/2)$ can be constructed as a tensor product of $m + n$ Weyl spinors. These give objects of the form $\varphi_{\alpha_1 \cdots \alpha_m}{}^{\dot{\alpha}_1 \cdots \dot{\alpha}_n}$.

To conclude this section, we will discuss the universal cover of the full Lorentz group $O(1, 3)$. We are to consider a collection of D objects γ^μ, with $\mu = 0 \ldots D - 1$. The anticommutator of two such objects is

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}. \tag{A.18}$$

We denote by V the vector space spanned by the γ^μ. The Clifford algebra $C(V)$ is now defined as the vector space spanned by all different antisymmetrized combinations of the γ^μ. That is,

$$C(V) = \mathbb{C} + V + V \wedge V + V \wedge V \wedge V \ldots$$

where the n-fold wedge product $V \wedge \cdots \wedge V$ is the vector space spanned by all the antisymmetrized combinations of n γ's. On the account of Eq. (A.18), the sum which defines $C(V)$ terminates after the D-fold wedge product.

We denote by $C^+(V)$ and $C^-(V)$ the parts of the Clifford algebra which contain, respectively, even and odd numbers of the γ^μ. We also define the operation τ as

$$\big(\gamma_{\mu_1}\cdots\gamma_{\mu_n}\big)^\tau = (-1)^n \gamma_{\mu_n}\cdots\gamma_{\mu_1}. \tag{A.19}$$

The pin group and the spin group are now defined as

$$\begin{aligned} \mathrm{Pin}(V) &\equiv \big\{\Lambda \in C(V) \mid \Lambda\Lambda^\tau = 1, \text{ and } \Lambda V \Lambda^\tau \supset V\big\}, \\ \mathrm{Spin}(V) &\equiv \mathrm{Pin}(V) \cap C^+(V). \end{aligned} \tag{A.20}$$

The pin group contains a representation of the group $O(V)$, the action of an $L \in O(V)$ being given by

$$\Lambda_L \gamma_\mu \Lambda_L^\tau = \Lambda_L \gamma_\mu \Lambda_L^{-1} = \gamma_\nu \Lambda^\nu{}_\mu.$$

The group $\mathrm{Pin}(1,3)$ is the so-called double cover of the full Lorentz group $O(1,3)$, while the respective spin group is a double cover of $SO(1,3)$. For example, to the operators of parity and time reversal there correspond the elements of the pin group $\Lambda_{\mathcal{P}} = \gamma_0$ and $\Lambda_{\mathcal{T}} = i\gamma_1\gamma_2\gamma_3$. The operator $\Lambda_{\mathcal{P}}$ reverses the sign of γ_μ when $\mu = 1,2,3$, while leaving γ_0 untouched, while $\Lambda_{\mathcal{T}}$ reverses the sign of γ_0 only. The operator of finite Lorentz transformations is

$$\Lambda_L = \exp\left(\frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right), \tag{A.21}$$

where L is an element of $L_+^\uparrow$, and $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$.

The field content of Lorentz invariant theories is given by the representations of the Clifford algebra. The smallest representation is a four-dimensional one. It is given by

$$\gamma^\mu = \begin{pmatrix} 0 & i\sigma^\mu \\ -i\bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.22}$$

where we have defined

$$(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}}. \tag{A.23}$$

The matrices γ^μ act on a pair of Weyl spinors, which combine into the Dirac spinor

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.24}$$

If we define $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, then the operators

$$P_\pm = \frac{1}{2}\big(1 \pm \gamma_5\big) \tag{A.25}$$

project the different Weyl spinors out of the Dirac spinor.

A.3 The Poincaré group and the conformal group

The Poincaré group is the group of transformations which leave invariant the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

of Minkowski space. The Poincaré group clearly contains the Lorentz group as a subgroup.

The generators of this group are the four generators of translations P_μ, and the six Lorentz generators $M_{\mu\nu}$.The algebra of the generators is

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [P_\mu, M_{\nu\rho}] &= i\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\big(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}\big). \end{aligned} \tag{A.26}$$

A particular representation of the generators is given by

$$P_\mu = -i\partial_\mu, \qquad M_{\mu\nu} = i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big). \tag{A.27}$$

Elementary particles form representations of the Poincaré group. Its irreducible representations are labeled by the eigenvalues of the Casimir operators. These are given by P^2 and W^2, where

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{A.28}$$

is the so-called Pauli–Lubanski vector.

For massive representations, with $P^2 = -m^2$, it can be shown that $W^2 = -m^2 s(s+1)$, where s is an integer of a half-integer; it is the spin of the particle. Massive representations are therefore labeled by their mass and their spin. For massless representations, we have $P^2 = 0$ and $W^2 = 0$. However, one can show that $W_\mu = \lambda P_\mu$. The integer or half-integer λ labels the massless representations. It is the helicity of the particle [6].

To each point in Minkowski space, there corresponds an element of the Poincaré group, according to

$$x^\mu \longleftrightarrow \exp\big(-ix^\mu P_\mu\big). \tag{A.29}$$

This correspondence enables us to express the action of the Poincaré group in Minkowski space as multiplication of the group elements [4]. The multiplication

$$\exp\big(-ia^\mu P_\mu\big)\exp\big(-ix^\mu P_\mu\big) = \exp\big(-i(x^\mu + a^\mu)P_\mu\big)$$

represents a translation through a^μ. Similarly, a Lorentz transformation is represented by

$$\exp\big(-\tfrac{i}{2}\lambda^{\rho\sigma}M_{\rho\sigma}\big)\exp\big(-ix^\mu P_\mu\big) = \exp\big(x^\mu \Lambda_\mu{}^\nu P_\nu\big)\exp\big(-\tfrac{i}{2}\lambda^{\rho\sigma}M_{\rho\sigma}\big),$$

where the matrix $\Lambda_\mu{}^\nu$ is defined through

$$\Lambda_\mu{}^\nu P_\nu = \exp\big(-\tfrac{i}{2}\lambda^{\rho\sigma}M_{\rho\sigma}\big)P_\mu \exp\big(\tfrac{i}{2}\lambda^{\rho\sigma}M_{\rho\sigma}\big).$$

A Poincaré transformation $\exp(-ia^\mu P_\mu)\exp(-\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu})$ therefore has the effect $x^\mu \to x^\nu\Lambda_\nu{}^\mu + a^\mu$.

The conformal group is the group of coordinate transformations which leave the metric $g_{\mu\nu}(x)$ invariant up to a scale factor:

$$g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x). \tag{A.30}$$

Such transformations preserve the angle between any two vectors. In Minkowski space, the conformal group contains the Poincaré group as a subgroup.

The generators of the conformal group can be found [79] by considering an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, and the Minkowski metric, for which we have

$$ds^2 \to ds^2 + \big(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\big)dx^\mu dx^\nu.$$

We must therefore have

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \omega(x)\eta_{\mu\nu}, \tag{A.31}$$

and by taking the trace we find $\omega(x) = (2/D)\partial^\lambda\epsilon_\lambda$. From Eq. (A.31) we can further show that

$$\big(\eta_{\mu\nu}\Box + (D-2)\partial_\mu\partial_\nu\big)\partial^\lambda\epsilon_\lambda = 0.$$

Therefore, in dimensions higher than two, ϵ can be at most quadratic in x. (In two dimensions, one can use Eq. (A.31) to show that conformal transformations amount to analytic coordinate transformations of the form $z \to f(z)$ in the complex coordinate z.) We thus find that the conformal group consists of the following transformations:

- Translations, with $\epsilon^\mu = a^\mu$,
- Lorentz rotations, with $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$,
- Dilations, with $\epsilon^\mu = \lambda x^\mu$,
- Special conformal transformations, with $\epsilon^\mu = x^2b^\mu - 2x^\mu x^\nu b_\nu$.

A finite special conformal transformation has the form

$$x^\mu \to \frac{x^\mu + x^2b^\mu}{1 + 2b^\mu x_\mu + b^2x^2}. \tag{A.32}$$

Under such a transformation, we have

$$\frac{x^{\mu\prime}}{x'^2} = \frac{x^\mu}{x^2} + b^\mu, \tag{A.33}$$

from which we see that the special conformal transformation is a combination of an inversion and a translation, followed by another inversion.

The algebra of the conformal group is given by Eq. (A.26), together with the following additional nonvanishing commutators:

$$\begin{aligned} [P_\mu, K_\nu] &= 2i\big(M_{\mu\nu} - \eta_{\mu\nu} D\big), \\ [M_{\mu\nu}, K_\rho] &= -i\big(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\big), \\ [D, P_\mu] &= iP_\mu, \qquad [D, K_\mu] = -iK_\mu. \end{aligned} \tag{A.34}$$

A representation of the generators of dilations and conformal transformations is given by

$$D = -ix^\mu \partial_\mu, \qquad K_\mu = -i\big(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu\big). \tag{A.35}$$

The conformal group in D spacetime dimensions is isomorphic to the group $SO(2, D)$, as can be shown [63] by considering the generators $M_{\mu\nu}$ together with

$$M_{\mu D} = \frac{1}{2}\big(K_\mu - P_\mu\big), \quad M_{\mu(D+1)} = \frac{1}{2}\big(K_\mu + P_\mu\big), \quad M_{D(D+1)} = D.$$

A.4 Differential geometry in superspace

The purpose of this section is to illustrate the connection between the formalism of section 2.4 and the theory of differential forms in superspace [1, 34]. We will do this by introducing enough of the theory so that we will be able to see how the Bianchi identity (2.69) arises from this formalism. We denote a point in superspace by $z^M = (x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha})$. We also introduce the basis one-forms dz^M, whose multiplication obeys the rules

$$dz^M dz^N = -(-1)^{|M||N|} dz^N dz^M, \qquad dz^M z^N = (-1)^{|M||N|} z^N dz^M. \tag{A.36}$$

We can then define differential forms,

$$\Omega = dz^{M_1} \cdots dz^{M_p} \omega_{M_p \ldots M_1}(z), \tag{A.37}$$

as well as an exterior derivative, which acts on differential forms as

$$d\Omega = dz^{M_1} \cdots dz^{M_p} dz^N \partial_N \omega_{M_p \ldots M_1}(z). \tag{A.38}$$

For the product of a p-form and a q-form, we have

$$d\big(\Omega^{(p)} \Lambda^{(q)}\big) = \Omega^{(p)}\, d\Lambda^{(q)} + (-1)^q d\Omega^{(p)}\, \Lambda^{(q)}. \tag{A.39}$$

The coefficient functions $\omega_{M_p \ldots M_1}$ are odd if they have an odd number of spinor indices; otherwise they are even.

Equations that are written in terms of differential forms and exterior derivatives are covariant under general coordinate transformations

$$z^M \to y^M(z).$$

In order to demonstrate this, we define the mapping ϕ^*, which maps functions of z to functions of y according to

$$\phi^* F(z) = F\big(y(z)\big) = F(y). \tag{A.40}$$

It can then be shown that ϕ^* has the properties

$$\begin{aligned}\phi^*\big(\Omega + \Lambda\big) &= \phi^*\Omega + \phi^*\Lambda,\\ \phi^*\big(\Omega\Lambda\big) &= \big(\phi^*\Omega\big)\big(\phi^*\Lambda\big),\\ d\big(\phi^*\Omega\big) &= \phi^*\big(d\Omega\big).\end{aligned}$$

In analogy with general relativity, we now introduce local coordinate frames, whose basis one-forms

$$E^A = dz^M E_M{}^A(z) \tag{A.41}$$

are called the vielbein. We can then consider both general coordinate transformations and local Lorentz transformations. Under the latter, the vielbein transforms as

$$\delta E_M{}^A = E_M{}^B L_B{}^A(z),$$

while under an infinitesimal general coordinate transformation $z \to z - \xi(z)$ we have

$$\delta E_M{}^A(z) \equiv E'_M{}^A(z) - E_M{}^A(z) = -\xi^L \partial_L E_M{}^A - \big(\partial_M \xi^L\big) E_L{}^A.$$

To define covariant derivatives, we introduce the connection

$$\phi = dz^M \phi_M, \qquad \phi_M = \phi_M{}^{AB}, \tag{A.42}$$

whose transformation law is

$$\delta\phi = L\phi - \phi L - dL. \tag{A.43}$$

The covariant derivative then is given by

$$\mathcal{D}\Omega = d\Omega + \Omega\phi. \tag{A.44}$$

Using the covariant derivative, we construct the torsion tensor

$$\begin{aligned}T^A &= \mathcal{D}E^A = dE^A + E^B \phi_B{}^A\\ &= \frac{1}{2} dz^M dz^N T_{NM}{}^A = \frac{1}{2} E^B E^C T_{CB}{}^A,\end{aligned} \tag{A.45}$$

and the curvature tensor

$$\begin{aligned}R_A{}^B &= \mathcal{D}\phi_A{}^B = d\phi_A{}^B + \phi_A{}^C \phi_C{}^B\\ &= \frac{1}{2} dz^M dz^N R_{NMA}{}^B = \frac{1}{2} E^C E^D R_{DCA}{}^B.\end{aligned} \tag{A.46}$$

The torsion and the curvature are subject to the Bianchi identities

$$\mathcal{D}^2 E^A = E^B R_B{}^A, \qquad \mathcal{D}T^A = E^B R_B{}^A. \tag{A.47}$$

The second identity follows directly from Eq. (A.45), while the first can be derived by showing with a short calculation that $d\mathcal{D}E^A = E^B R_B{}^A - \mathcal{D}E_A{}^B \phi_B$.

In the basis determined by the E^A, we have

$$\begin{aligned}\mathcal{D}T^A &= \frac{1}{2}\mathcal{D}\big(E^B E^C T_{CB}{}^A\big)\\ &= \frac{1}{2}\Big(E^B E^C \mathcal{D}T_{CB}{}^A + E^B T^C T_{CB}{}^A - T^B E^C T_{CB}{}^A\Big).\end{aligned}$$

Consequently, the second Bianchi identity takes the form

$$E^B E^C E^D \Big(\mathcal{D}_D T_{CB}{}^A + T_{DC}{}^E T_{EB}{}^A - R_{DCB}{}^A\Big) = 0. \tag{A.48}$$

This is equivalent to Eq. (2.69).

References

[1] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd edition, Princeton University Press, 1992.

[2] P. C. West, *Introduction to Supersymmetry and Supergravity*, 2nd edition, World Scientific Publishing, 1990.

[3] A. Bilal, *Introduction to Supersymmetry*, hep-th/0101055.

[4] J. M. Figueroa-O'Farrill, *BUSSTEPP Lectures on Supersymmetry*, hep-th/0109172.

[5] J. Lykken, *Introduction to Supersymmetry*, hep-th/9612114.

[6] C. Sämann, *Introduction to Supersymmetry*, http://www.christian-saemann.de/files/LecturesOnSUSY.pdf.

[7] U. Theis, *An Introduction to Supersymmetry*, www.itp.uni-hannover.de/~lechtenf/Events/Lectures/theis.pdf.

[8] S. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. 159, 1251 (1967).

[9] The assumption of four spacetime dimensions is not critical. See e.g. O. Pelc and R. W. Horwitz, *Generalization of the Coleman-Mandula Theorem to Higher Dimension*, J. Math. Phys. 38, 139 (1997), hep-th/9605147.

[10] M. Sohnius, *Introducing Supersymmetry*, Phys. Rep. 128, 39 (1985).

[11] E. Witten, *Introduction to Supersymmetry*, in Proc. Intern. School of Subnuclear Physics, Erice 1981, ed. A. Zichichi, Plenum Press, 1983.

[12] Y. A. Golfand and E. P. Likhtman, *Extension of the algebra of Poincaré group generators and violation of P invariance*, JETP Lett. 13, 323 (1971).

[13] R. Haag, J. T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, Nucl. Phys. B 88, 257 (1975).

[14] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nucl. Phys. B 70, 39 (1974).

[15] A. Salam and J. Strathdee, *Super-gauge transformations*, Nucl. Phys. B 76, 477 (1974).

[16] A. Salam and J. Strathdee, *Superfields and Fermi-Bose symmetry*, Phys. Rev. D 11, 1521 (1975).

[17] Both of our examples of spontaneous supersymmetry breaking are taken from the lecture notes by S. Mukhi, http://theory.tifr.res.in/stringslhc/talks/intro-susy.pdf.

[18] L. O'Raifeartaigh, *Spontaneous symmetry breaking for chiral scalar superfields*, Nucl. Phys. B96, 331 (1975)

[19] P. Fayet and J. Iliopoulos, *Spontaneously broken supergauge symmetries snd Goldstone spinors*, Phys. Lett. B 51, 461 (1974).

[20] F. Brandt, *Lectures on Supergravity*, Fortsch. Phys. 50, 1126 (2002), hep-th/0204035.

[21] U. Theis, *A Beginner's Guide to Supergravity*, http://www.personal.uni-jena.de/~p5thul2/pub/sugra.pdf.

[22] B. de Wit, *Supergravity*, in *Unity from Duality: Gravity, Gauge Theory and Strings*, eds. C. Bachas, A. Bilal, F. David, M. Douglas, and N. Nekrasov, Springer, 2003, hep-th/0212245.

[23] H. Samtleben, *Introduction to Supergravity*, http://www.itp.uni-hannover.de/~lechtenf/Events/Lectures/samtleben.pdf.

[24] Y. Tanii, *Supergravities in Diverse Dimensions*, hep-th/9802138.

[25] P. C. West, *Supergravity, Brane Dynamics and String Duality*, hep-th/9811101.

[26] S. M. Carroll, *Lecture Notes on General Relativity*, gr-qc/9712019.

[27] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.

[28] W. Rarita and J. Schwinger, *On a Theory of Particles with Half-Integral Spin*, Phys. Rev. 60, 61 (1941).

[29] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress Toward a Theory of Supergravity*, Phys. Rev. D 13 3214 (1976).

[30] Some details on the history of supergravity are given by P. van Nieuwenhuizen in *Supergravity as a Yang–Mills Theory*, in *50 Years of Yang–Mills theory*, ed. G. 't Hooft, World Scientific Publishing, Singapore, 2005, hep-th/0408137.

[31] S. Deser and B. Zumino, *Consistent supergravity*, Phys. Lett. B 62, 335 (1976).

[32] K. S. Stelle and P. C. West, *Minimal auxiliary fields for supergravity*, Phys. Lett. B 76, 330 (1978).

[33] S. Ferrara and P. van Nieuwenhuizen, *The auxiliary fields of supergravity*, Phys. Lett. B 76, 333 (1978).

[34] P. Binétruy, G. Girardi and R. Grimm, *Supergravity couplings: a geometric formulation*, Phys. Rept. 343, 255 (2001), hep-th/0005225.

[35] J. Wess and B. Zumino, *Superspace formulation of supergravity*, Phys. Lett. B 66, 361 (1977).

[36] J. Wess and B. Zumino, *Superfield Lagrangian for supergravity*, Phys. Lett. B 74, 51 (1978).

[37] J. Wess and B. Zumino, *The component formalism follows from the superspace formulation of supergravity*, Phys. Lett. B 79, 395 (1978).

[38] S. Ferrara and P. van Nieuwenhuizen, *Tensor calculus for supergravity*, Phys. Lett. B 76, 379 (1978).

[39] F. Brandt, *Lagrangian Densities and Anomalies in Four-dimensional Supersymmetric Theories*, Ph.D. thesis, 1991.

[40] N. Dragon, *Torsion and curvature in extended supergravity*, Z. Phys. C 2, 29 (1979).

[41] R. Grimm, J. Wess and B. Zumino, *A complete solution of the Bianchi identities in superspace with supergravity constraints*, Nucl. Phys. B 152, 255 (1979).

[42] A. Van Proeyen, *Tools for supersymmetry*, hep-th/9910030.

[43] A. Van Proeyen, *Structure of supergravity theories*, hep-th/0301005.

[44] J. Strathdee, *Extended Poincaré supersymmetry*, Int. J. Mod. Phys. A2, 273 (1987).

[45] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. B 76, 409 (1978).

[46] E. Cremmer and S. Ferrara, *Formulation of 11-dimensional supergravity in superspace*, Phys. Lett. B 91, 61 (1980).

[47] L. Brink and B. Howe, *Eleven-dimensional supergravity on the mass shell in superspace*, Phys. Lett. B 91, 384 (1980).

[48] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields*, Nucl. Phys. B 195, 97 (1982).

[49] E. Bergshoeff, M. de Roo and B. de Wit, *Conformal supergravity in ten dimensions*, Nucl. Phys. B 217, 143 (1983).

[50] C. Campbell and P. West, *N = 2 D = 10 nonchiral supergravity and its spontaneous compactification*, Nucl. Phys. B 243, 112 (1984).

[51] F. Giani and M. Pernici, *N = 2 supergravity in ten dimensions*, Phys. Rev. D 30, 325 (1984).

[52] M. Huq and M. Namazie, *Kaluza-Klein supergravity in ten dimensions*, Class. Q. Grav. 2 (1985).

[53] J. Schwarz and P. West, *Symmetries and transformation of chiral N = 2 D = 10 supergravity*, Phys. Lett. B 126, 301 (1983).

[54] J. Schwarz, *Covariant field equations of chiral N = 2 D = 10 supergravity*, Nucl. Phys. B 226, 269 (1983).

[55] P. Howe and P. West, *The complete N = 2 D = 10 supergravity*, Nucl. Phys. B 238, 181 (1984).

[56] E. D'Hoker and D. Z. Freedman, *Supersymmetric Gauge Theories and the AdS/CFT Correspondence*, hep-th/0201253.

[57] G. Dall'Agata, K. Lechner and M. Tonin, *D = 10, N = IIB Supergravity: Lorentz-invariant actions and duality*, hep-th/9806140.

[58] E. A. Bergshoeff, M. de Roo, S. F. Kerstan and F. Riccioni, *IIB Supergravity Revisited*, hep-th/0506013.

[59] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200.

[60] J. de Boer, *Introduction to the AdS/CFT Correspondence*, http://www-library.desy.de/preparch/desy/proc/proc02-02/Proceedings/pl.6/deboer_pr.pdf.

[61] A. M. Polyakov, *The wall of the cave*, Int. J. Mod. Phys. A 14, 645 (1999), hep-th/9809057.

[62] G. Veneziano, *Construction of a crossing-symmetric, Regge-behaved amplitude for linearly rising trajectories*, Nuovo Cim. A 57, 190 (1968).

[63] E. Kiritsis, *String Theory in a Nutshell*, Princeton University Press, 2007.

[64] J. Polchinski, *String Theory*, Cambridge University Press, 1998.

[65] J. H. Schwarz, *Introduction to Superstring Theory*, hep-ex/0008017.

[66] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions*, Nucl. Phys. B 31, 86 (1971).

[67] P. Ramond, *Dual Theory for Free Fermions*, Phys. Rev. D 3, 2415 (1971).

[68] F. Gliozzi, J. Scherk and D. I. Olive, *Supergravity and the spinor dual model*, Nucl. Phys. B 122, 253 (1977).

[69] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys. Rept. 323, 183 (2000), hep-th/9905111.

[70] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, Phys. Rev. Lett. 75, 4724 (1995), hep-th/9510017.

[71] J. Polchinski, *TASI Lectures on D-Branes*, hep-th/9611050.

[72] J. G. Russo, *Large N field theories from Superstrings*, hep-th/0008212.

[73] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. B 72, 461 (1974).

[74] R. Pisarski, http://quark.phy.bnl.gov/~pisarski/talks/PhaseDiagram.pdf.

[75] J. L. Petersen, *Introduction to the Maldacena Conjecture on AdS/CFT*, Int. J. Mod. Phys. A14, 3597 (1999), hep-th/9902131.

[76] G. T. Horowitz and A. Strominger, *Black strings and p-branes*, Nucl. Phys. B 360, 197 (1991).

[77] I. R. Klebanov, *TASI Lectures: Introduction to the AdS/CFT Correspondence*, hep-th/0009139.

[78] E. Witten, *Bound States of Strings and p-Branes*, Nucl. Phys. B 460, (335) 1996, hep-th/9510135.

[79] P. Ginsparg, *Applied Conformal Field Theory*, hep-th/9108028.

[80] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. B 135, 149 (1978).

[81] R. Grimm, M. Sohnius and J. Wess, *Extended supersymmetry and gauge theories*, Nucl. Phys. B 133, 275 (1978).

[82] J. M. Figueroa-O'Farrill, *Electromagnetic Duality for Children*, http://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/EDC.pdf

[83] S. Kovacs, *N=4 supersymmetric Yang-Mills theory and the AdS/SCFT correspondence*, PhD thesis, hep-th/9908171.

[84] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. 72 B, 117 (1977).

[85] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys. Lett. B 428, 105 (1998), hep-th/9802109.

[86] E. Witten, *Anti de Sitter Space and Holography*, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150.

[87] I. R. Klebanov, *World volume approach to absorption by nondilatonic branes*, Nucl. Phys. B 496, 231 (1997), hep-th/9702076.

[88] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, *String theory and classical absorption by three-branes*, Nucl. Phys. B 499, 217 (1997), hep-th/9703040.

[89] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d + 1) correspondence*, Nucl. Phys. B 546, 96 (1999), hep-th/9804058.

[90] H. Liu and A. A. Tseytlin, *D = 4 super Yang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity*, Nucl. Phys. B 533, 88 (1998), hep-th/9804083.

[91] E. Witten, *Anti-de Sitter Space, Thermal Phase Transition, and Confinement in Gauge Theories*, Adv. Theor. Math. Phys. 2, 505 (1998), hep-th/9803131.

[92] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, *Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory*, Nucl. Phys. B 534, 202 (1998), hep-th/9805156.

[93] M. Vázquez-Mozo, *A Note on Supersymmetric Yang-Mills Thermodynamics*, Phys. Rev. D 60, 106010 (1999), hep-th/9905030.

[94] C. Kim and S.-J. Rey, *Thermodynamics of Large-N Super Yang-Mills Theory and AdS/CFT Correspondence*, Nucl. Phys. B 564, 430 (2000), hep-th/9905205.

[95] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, gr-qc/9310026.

[96] L. Susskind, *The World as a Hologram*, J. Math. Phys. 36, 6377 (1995) hep-th/9409089.

[97] J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. D 7, 2333 (1973).

[98] S. Hawking, *Particle Creation by Black Holes*, Comm. Math. Phys. 43, 190 (1975).

[99] S. Hawking, *Black holes and thermodynamics*, Phys. Rev. D 13, 191 (1976).

[100] L. Susskind and E. Witten, *The Holographic Bound in Anti-de Sitter Space*, hep-th/9805114.

[101] U. Gürsoy and E. Kiritsis, *Exploring improved holographic theories for QCD: Part I*, JHEP 0802, 032 (2008), arXiv: 0707.1324 (hep-th).

[102] U. Gürsoy, E. Kiritsis and F. Nitti, *Exploring improved holographic theories for QCD: Part II*, JHEP 0802, 019 (2008), arXiv: 0707.1349 (hep-th).

[103] U. Gürsoy, E. Kiritsis, L. Mazzanti and F. Nitti, *Holography and Thermodynamics of 5D Dilaton-gravity*, JHEP 0905, 033 (2009), arXiv: 0812.0792 (hep-th).

[104] U. Gürsoy, E. Kiritsis, L. Mazzanti and F. Nitti, *Improved Holographic Yang–Mills theories at Finite Temperature: Comparison with Data*, Nucl. Phys. B 820, 148 (2009), arXiv: 0903.2859 (hep-th).

[105] Y. Kinar, E. Schreiber and J. Sonnenschein, *Q anti-Q potential from strings in curved spacetime: Classical results*, Nucl. Phys. B 566, 103 (2000), hep-th/9811192.

[106] J. Maldacena, *Wilson loops in large N field theories*, Phys. Rev. Lett. 80, 4859 (1998), hep-th/9803002.

[107] K. Kajantie, M. Krssak, M. Vepsäläinen and A. Vuorinen, *Frequency and wave number dependence of the shear correlator in strongly coupled hot Yang-Mills theory*, arXiv: 1104.5352 (hep-ph).

[108] S. Weinberg, *The Quantum Theory of Fields, Vol. 1*, Cambridge University Press, 1995.

[109] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd edition, Princeton University Press, 2010.

[110] E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. 40, 149 (1939).

[111] V. Bargmann, *On unitary ray representations of continuous groups*, Ann. of Math. 59, 1 (1954).