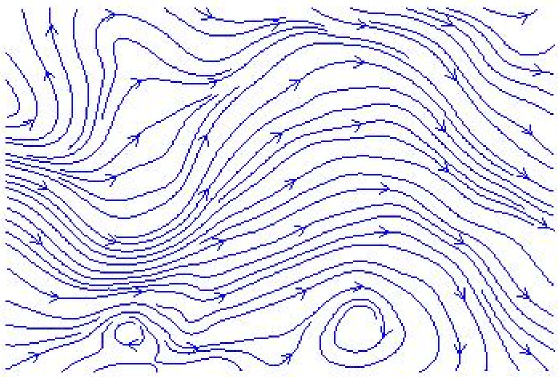
Pro gradu-tutkielma

Navier-Stokes Equations



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Tämän Pro gradu -tutkielman aiheena ovat Navierin ja Stokesin yhtälöt, jotka ovat nesteiden ja kaasujen liikettä kuvaavia osittaisdifferentiaaliyhtälöitä. Fysikaalisesti ne ovat liikkeessä olevan nesteen tai kaasun nopeuskentän ja paineen väliset tasapainoyhtälöt. Mielenkiintoista näissä yhtälöissä on se, että vaikka ne voidaan muotoilla varsin yksinkertaisesti, niihin liittyy useita ratkaisemattomia ongelmia. Navierin ja Stokesin yhtälöt on nimetty ranskalaisen insinöörin Claude-Louis Navierin ja irlantilaisen matemaatikon ja fyysikon Sir George Gabriel Stokesin mukaan. Yhtälöillä on lukuisia sovelluksia. Niitä käytetään mm. ilmatieteessä, merentutkimuksessa ja lääketieteessä. Nesteiden ja kaasujen liikkeiden lisäksi yhtälöillä voidaan mallintaa esimerkiksi lämmön johtumista.

Tässä tutkielmassa keskitytään puristumattomiin, viskooseihin nesteisiin ja kaasuihin. Puristumattomuudella tarkoitetaan sitä, että nesteen tai kaasun tiheys ei muutu ajan tai paikan suhteen. Viskoosilla nesteellä tai kaasulla on sisäistä kitkaa, joka syntyy siitä, että aineen osaset liikkuvat toistensa suhteen. Myös viskoosittomia nesteitä ja kaasuja tarkastellaan tässä tutkielmassa. Viskoosittomassa tapauksessa Navierin ja Stokesin yhtälöt saavat yksinkertaisemman muodon, jolloin yhtälöitä nimitetään usein Eulerin yhtälöiksi.

Tutkielman ensimmäisessä osassa johdetaan Eulerin liikeyhtälö, Navierin ja Stokesin liikeyhtälö sekä näihin liikeyhtälöihin liitettävä massan säilymisen yhtälö. Mainitut liikeyhtälöt seuraavat Newtonin toisesta laista. Myös muutamia esimerkkejä virtauksista, jotka toteuttavat Eulerin tai Navierin ja Stokesin yhtälöt, annetaan. Eulerin yhtälöstä johdetaan Bernoullin yhtälö, joka on nopeuden ja paineen yhtälö tietyllä virtaviivalla.

Tutkielman toisessa osassa tarkastellaan Stokesin yhtälöitä, Navierin ja Stokesin yhtälöiden staattista muotoa sekä täydellisiä Navierin ja Stokesin yhtälöitä. Kutakin mainittua osittaisdifferentiaaliyhtälöryhmää vastaavalle reunaehto-ongelmalle määritellään ns. heikko ratkaisu. Ajatuksena on, että kyseessä oleva yhtälö kerrotaan puolittain sopivalla testifunktiolla, sen jälkeen integroidaan yhtälön kummatkin puolet määrittelyalueen yli ja suoritetaan osittaisintegrointi termeille, jotka sisältävät tuntemattomien derivaattoja. Näin saadaan yhtälö, jossa derivaatat ovat niin ikään siirtyneet tuntemattomilta funktioilta testifunktiolle. Näin voidaan lisäksi lieventää tuntemattomien funktioiden sileysvaatimuksia. Ratkaisua, joka toteuttaa edellä kaavaillun yhtälön, sanotaan alkuperäisen reunaehto-ongelman heikoksi ratkaisuksi.

Tutkielmassa esitetään vaihtoehtoisia ns. heikkoja muotoiluja kullekin reunaehto-ongelmalle. Lisäksi näytetään, että nämä muotoilut todella ovat vaihtoehtoisia, eli jos jokin funktio ratkaisee näistä yhden, se ratkaisee myös toisen, ja päinvastoin. Stokesin yhtälöiden sekä Navierin ja Stokesin yhtälöiden staattisen tapauksen reunaehto-ongelmille osoitetaan heikkojen ratkaisujen olemassaolo, kun reuna-arvona on nolla ja avaruuden dimensio on korkeintaan neljä. Vähintään yhden ratkaisun olemassaolo staattiselle ongelmalle todistetaan Galerkinin metodia käyttäen. Myös joitakin yksikäsitteisyystuloksia osoitetaan mainituille olemassaoleville ratkaisuille. Täydellisten Navierin ja Stokesin yhtälöiden tapauksessa heikoille ratkaisuille esitetään eräs olemassaolotulos. Tämän olemassaolevan heikon ratkaisun yksikäsitteisyys on avoin ongelma. Yksikäsitteisyys tunnetaan sen sijaan sellaisessa funktioiden joukossa, jossa ratkaisun olemassaolosta ei tiedetä.

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1 Introduction

The Navier-Stokes equations, that are a system of partial differential equations describing the movement of liquids and gases, play a great part in mathematical research of today. The interesting point is that despite their simple formulation and the rich variety of their applications many problems related to their solutions still remain open.

The Navier-Stokes equations have their history in fluid mechanics which is a branch of physics exploring liquids and gases. Liquids and gases together are called fluids. The history of fluid mechanics goes back to the ancient time when Archimedes formulated the laws for floatage around 200 B.C. and the Romans built long aquaeducti around 300 B.C. Not much research on the topic were carried out until the Renaissance.

Leonardo da Vinci (1452-1519) derived the equation of conservation of mass in the case of one-dimensional steady flow. Isaac Newton (1642-1727) came up with the laws of motion and the law of viscosity of linear fluids. Then problems with forces affecting on fluid or with velocity could be considered. Newton's work launched the research on ideal fluids without inner friction. Many mathematicians of the 18th century derived beautiful results for this ideal flow. Swiss mathematician and physicist Leonard Euler (1707-1783) developed the differential equations of motion of fluid, known as the Euler equations, in 1755. He also developed the integrated form of them that is now called Bernoulli's equation after Daniel Bernoulli (1700-1782). Meanwhile, engineers developed their own, purely experimental field of hydraulics.

In the end of the 19th century, the theoretical and experimental researches on the motion of fluids converged. Osborne Reynolds published an important report on flow in a pipe in 1883. French engineer and physicist Claude-Louis Navier (1785-1836) and Irish mathematician and physicist Sir George Gabriel Stokes (1819-1903) successfully combined the Newtonian viscosity terms with the equations of motion. During that time, these equations were not highly valued because of their complicity for an arbitrary flow.

German engineer Ludwig Prandtl (1875-1953) observed that in flows with small viscosity the fluid can be divided into two different parts which can be examined separately. Near the boundary, there is a layer where the viscosity is notable. In the other part of fluid, the effect of viscosity is negligible such that the fluid can be considered as an ideal fluid. This so-called boundary layer theory is one of the most important tools in the modern fluid mechanics. It might also give some hints for mathematical approach to the unsolved problems. We refer to [7] for the historical notes of fluid mechanics.

The Navier-Stokes equations have many applications related to fluid mechanics. They are applied for example in meteorology, hydrology, oceanography, and medical research on breathing and blood circulation. They are applied not only in the movement of fluids but also in other phenomena. Heat conduction is often modelled by an incompressible flow. They are also applied in magneto-hydrodynamics combined with the Maxwell equations. The research of today is also focused on phenomena where domains have free boundaries governed by surface tension, for example surface waves and the shape of rising bubbles [4].

In this thesis, we are interested in the incompressible, viscous flow. The Navier-Stokes equations for this flow are represented in the following form

$$\begin{cases} D_t u + u \cdot Du - \eta \Delta u = -Dp + f \\ \operatorname{div} u = 0 \end{cases} \tag{1.1}$$

where u is the velocity vector field, p the pressure, f the external force and η the viscosity constant. The first equation in (1.1) comes from Newton's second law and the second from the conservation law of mass. In the case of inviscid flow, we have $\eta = 0$. In this case, equation (1.1) becomes the Euler equations.

We call equation (1.1) the evolution Navier-Stokes equations, provided that $D_t u \neq 0$. From the computational and analytical point of view, a simpler system of equations, known as the Stokes equations, is important. It is obtained when we set $D_t u = 0$ and $u \cdot Du = 0$ in equation (1.1). The steady-state Navier-Stokes equations are obtained when we set $D_t u = 0$ in equation (1.1).

The preliminaries are given in the second section of this thesis. The preliminaries include the notations, the definitions and the preliminary results. In the third section, we represent the derivation for the law of conservation of mass, the Euler equations and the Navier-Stokes equations. We also study the physical significance of the equations and give some examples of the solutions.

The fourth section is divided into four subsections. In the first subsection, we give the preliminaries. We define the auxiliary spaces for the discussion. In the remaining three subsections, we study the Stokes equations, the steady-state Navier-Stokes equations and the evolution Navier-Stokes equations respectively. We only consider the case where the domain is bounded.

We define the weak solutions to the Stokes equations, the steady-state Navier-Stokes equations and the evolution Navier-Stokes equations. We represent alternative variational problems for these equations. We show the existense of a weak solution to the Stokes equations and to the steady-state Navier-Stokes equations in the case where we have zero boundary value and the dimension of the space is less than or equal to four. The existence of at least one weak solution to the steady-state equations is proved by Galerkin method. For these existing weak solutions, we prove some uniqueness results. For the evolution Navier-Stokes equations, we represent some existence results and discuss the uniqueness.

2 Preliminaries

2.1 Notation

First we introduce some notations that will be used.

In the thesis, we denote by Ω an open, bounded set in $\mathbb{R}^n$. The boundary of Ω is denoted by $\partial\Omega$.

Let $u : \Omega \to \mathbb{R}$, $x \in \Omega$. The partial derivative of u is

$$\frac{\partial}{\partial x_i}u(x) = \lim_{h\to 0}\frac{u(x_1,\ldots,x_i+h,\ldots,x_n) - u(x_1,\ldots,x_i,\ldots,x_n)}{h}.$$

We write $D_i u$ for $\frac{\partial}{\partial x_i}u$. For higher derivatives, we use notation

$$D^\alpha u(x) = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. For the gradient of u, we write

$$Du = (D_1 u, \ldots, D_n u).$$

Throughout the thesis, $Du = D_x u$ denotes the gradient of u with respect to the spatial variable $x \in \mathbb{R}^n$, and $D_t u$ denotes the partial derivative with respect to the time variable t, $t \geq 0$. We denote

$$\Delta u = \sum_{i=1}^n D_i D_i u.$$

Let then $u : \Omega \to \mathbb{R}^3$, $u(x) = (u_1(x), u_2(x), u_3(x))$. The rotor (or curl) of u is

$$\operatorname{rot} u = (D_2u_3 - D_3u_2, D_3u_1 - D_1u_3, D_1u_2 - D_2u_1).$$

The divergence of u is

$$\operatorname{div} u = D_1u_1 + D_2u_2 + D_3u_3.$$

The vector product of the two vectors $\xi = (\xi_1, \ldots, \xi_1)$ and $\eta = (\eta_1, \ldots, \eta_n)$ is

$$\xi \times \eta = (\xi_2\eta_3 - \xi_3\eta_2, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1).$$

We say that $u : \Omega \to \mathbb{R}$ is of class

$$\mathcal{C}^k(\Omega)$$

if all the derivatives $D^\alpha u$, $|\alpha| \leq k$, exist and are continuous. The space of functions that are infinite times differentiable is denoted by $\mathcal{C}^\infty(\Omega)$. We denote

$$\mathcal{C}^k(\overline{\Omega}) = \{u \in \mathcal{C}^k(\Omega) : D^\alpha u \text{ is uniformly continuous for all } |\alpha| \leq k\}.$$

Thus if $u \in \mathcal{C}^k(\overline{\Omega})$, then $D^\alpha u$ continuously extends to $\overline{\Omega}$ for each multi index α, $|\alpha| \leq k$. We denote

$$\mathcal{C}^\infty(\overline{\Omega}) = \bigcap_{k=0}^{\infty} \mathcal{C}^k(\overline{\Omega}).$$

The boundary $\partial\Omega$ is $\mathcal{C}^k$ if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ in $\mathcal{C}^k$ such that we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

A domain Ω is of class $\mathcal{C}^k$ if its boundary $\partial\Omega$ is $\mathcal{C}^k$.

The support of a function ϕ is the set $\operatorname{spt}(\phi) = \overline{\{x \in \Omega : \phi(x) \neq 0\}}$. We denote

$$\mathcal{D}(\Omega) = \{\phi \in \mathcal{C}^\infty(\Omega) : \operatorname{spt}(\phi) \text{ is compact and in } \Omega\},$$

and

$$\mathcal{D}(\overline{\Omega}) = \{\phi \in \mathcal{C}^\infty(\overline{\Omega}) : \operatorname{spt}(\phi) \text{ is compact and in } \Omega\}.$$

We denote the space of linear continuous functions from X to Y by

$$\mathcal{L}(X, Y).$$

The dual space of vector space X is denoted by X' unless we define a special notation. We will write $\langle \cdot, \cdot \rangle$ to denote the duality pairing between X' and X.

2.2 Equations

A partial differential equation is an equation of a function and some of its partial derivatives where the function is unknown. Let us give a precise definition.

Definition 2.1. *The presentation of the form*

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0, \quad x \in \Omega \tag{2.1}$$

is called a partial differential equation *(PDE) of the order k where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is unknown and

$$D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\}.$$

Definition 2.2. *The presentation of the form*

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0, \quad x \in \Omega \tag{2.2}$$

is called a system of partial differential equations *of the order k where*

$$F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n$$

is given and

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, \quad u = (u^1, \dots, u^m)$$

is unknown.

Definition 2.3. *We say that $u \in \mathcal{C}^k(\Omega)$ is a* classical solution *of equation* (2.1) *if the equation holds for all $x \in \Omega$.*

Partial differential equations can be classified by their linearity as follows.

Definition 2.4. *Given functions ϕ_α and f.*

1. *Partial differential equation* (2.1) *is called* linear *if it has the form*

$$\sum_{|\alpha| \le k} \phi_\alpha(x) D^\alpha u = f(x).$$

If $f \equiv 0$ the PDE is said to be homogeneous.

2. *PDE* (2.1) *is called* semilinear *if it has the form*

$$\sum_{|\alpha|=k} \phi_\alpha D^\alpha u + \phi_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

3. *PDE* (2.1) *is* quasilinear *if it has the form*

$$\sum_{|\alpha|=k} \phi_\alpha(D^{k-1}u, \ldots, Du, u, x) D^\alpha u + \phi_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

4. *PDE* (2.1) *is called* fully nonlinear *if it depends nonlinearly upon the highest order derivatives.*

Example 2.1. *Let us give some examples of systems of partial differential equations.*

1. *Maxwell equations*

$$\begin{cases} D_t E = \operatorname{rot} B \\ D_t B = -\operatorname{rot} E \\ \operatorname{div} B = \operatorname{div} E = 0. \end{cases}$$

2. *Euler equations*

$$\begin{cases} D_t u + u \cdot Du = -Dp \\ \operatorname{div} u = 0. \end{cases}$$

3. *Navier-Stokes equations*

$$\begin{cases} D_t u + u \cdot Du - \Delta u = -Dp \\ \operatorname{div} u = 0. \end{cases}$$

The Maxwell equations are linear and the others nonlinear. The Euler equations are quasilinear and the Navier-Stokes equations semilinear systems of partial differential equations.

2.3 Preliminary results

We recall some preliminary results. Let us begin with the Gauss-Green theorem.

Theorem 2.1 (Gauss-Green)**.** *Let $\Omega \subset \mathbb{R}^3$ be bounded. Suppose $\partial\Omega$ is $\mathcal{C}^1$ and $u \in \mathcal{C}^1(\overline{\Omega})$. Then*

$$\int_{\partial\Omega} u\,\nu_i\,dS = \int_{\Omega} D_i u\,dx$$

where $\nu : \partial\Omega \to \mathbb{R}^3$ is the unique outward unit.

Theorem 2.2 (Gauss)**.** *Let $\Omega \subset \mathbb{R}^3$ be bounded. Suppose $\partial\Omega$ is $\mathcal{C}^1$ and $u : \Omega \to \mathbb{R}^3$ with $u_i \in \mathcal{C}^1(\overline{\Omega})$. Then*

$$\int_{\partial\Omega} u \cdot \nu\,dS = \int_{\Omega} \operatorname{div} u\,dx.$$

Theorem 2.3. *Suppose $\partial\Omega$ is $\mathcal{C}^1$. Let $u, v \in \mathcal{C}^1(\overline{\Omega})$. Then*

$$\int_{\Omega} v D_i u\,dx = \int_{\partial\Omega} uv\,\nu_i\,dS - \int_{\Omega} u D_i v\,dx.$$

Theorem 2.4 (Stokes)**.** *Let $S \subset \mathbb{R}^3$ be a bounded $\mathcal{C}^2$ smooth surface with the boundary ∂S that is $\mathcal{C}^2$. Let $u : S \to \mathbb{R}^3$ with $u_i \in \mathcal{C}^1(\overline{S})$. Then*

$$\int_{\partial S} u \cdot d\bar{s} = \int_{S} \operatorname{rot} u \cdot \nu\,dS$$

where

$$\int_{\partial S} u \cdot d\bar{s} = \int_0^1 u(\gamma(t)) \cdot \gamma'(t)\,dt$$

and $\gamma : [0,1] \to \partial S$ is a closed contour.

2.4 Sobolev spaces

Let us start with the Lebesgue space L^p.

Definition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Let $1 \le p < \infty$. We denote*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is Lebesgue measurable, } \|u\|_{L^p(\Omega)} < \infty\}$$

and

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} : u \textit{ is Lebesgue measurable, } \|u\|_{L^\infty(\Omega)} < \infty\}.$$

The norms above are defined as

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} \tag{2.3}$$

and

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega}|u(x)|. \tag{2.4}$$

Theorem 2.5. *The spaces $L^p(\Omega)\,(1 \le p < \infty)$ equipped with norm (2.3) and $L^\infty(\Omega)$ equipped with (2.4) are complete normed spaces (Banach spaces).*

Theorem 2.6. *The space $L^2(\Omega)$ with the scalar product*

$$(u, v) = \int_\Omega u(x)v(x)\, dx$$

is a Hilbert space.

Theorem 2.7 (Minkowski)**.** *Let $1 \le p \le \infty$. If $u \in L^p(\Omega)$ and $v \in L^p(\Omega)$, then $u + v \in L^p(\Omega)$ and*

$$\|u + v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Theorem 2.8 (Hölder)**.** *Let $1 \le p \le \infty$ and $1 \le q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (or $p = 1$ and $q = \infty$ or vice versa). If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)}\, \|v\|_{L^q(\Omega)}.$$

Definition 2.6. *A function u, locally integrable on Ω, is said to have a* weak derivative *of order α if there exists a locally integrable function v such that*

$$(u, D^\alpha\phi) = (-1)^{|\alpha|}(v, \phi) \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

Definition 2.7. *Let m be an integer, and $1 \le p \le \infty$. The* Sobolev space, *denoted by $W^{m,p}(\Omega)$, consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi index α with $|\alpha| \le m$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.*

Theorem 2.9. *$W^{m,p}(\Omega)$ with the norm*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|k|\le m} \|D^k u\|^p_{L^p(\Omega)} \right)^{1/p}$$

is a Banach space.

Theorem 2.10. *$W^{m,2}(\Omega) = H^m(\Omega)$ with the scalar product*

$$((u,v))_{H^m(\Omega)} = \sum_{|k|\le m} (D^k u, D^k v)$$

is a Hilbert space.

Definition 2.8. *The closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ is*

$$\begin{aligned} W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) :\ &\exists\ \{u_i\}_{i=1}^{\infty} \subset \mathcal{D}(\Omega) \\ &\text{such that } \|u - u_i\|_{W^{m,p}(\Omega)} \to 0 \text{ as } i \to \infty\}. \end{aligned}$$

When $p = 2$ we denote $W_0^{m,2}(\Omega) = H_0^m(\Omega)$.

Definition 2.9. *We denote the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$. The norm in $H^{-1}(\Omega)$ is defined by*

$$\|f\|_{H^{-1}(\Omega)} = \sup\{\langle f, u\rangle : u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \le 1\}.$$

We need also the product spaces of the spaces introduced above. We denote them in the usual way $\{L^p(\Omega)\}^n$, $\{W^{m,p}(\Omega)\}^n$, $\{H^m(\Omega)\}^n$ and $\{\mathcal{D}(\Omega)\}^n$. The first three of them are equipped with product norm

$$\sup_i \|u_i\|_X$$

where $u = (u_1, \ldots, u_n) \in X^n$. This norm is equivalent to norm

$$\Big(\sum_{i=1}^{n} \|u_i\|_X^2\Big)^{1/2}. \tag{2.5}$$

We will use norm (2.5) in our discussion.

If X is a scalar product space with scalar product $(\cdot,\cdot)_X$ then

$$\sum_{i=1}^{n}(u_i, v_i)_X, \tag{2.6}$$

is a scalar product in X^n. Then

$$\Big(\sum_{i=1}^{n}(u_i, u_i)_X\Big)^{1/2},$$

gives norm (2.5).

$\mathcal{D}(\Omega)$ is not a normed space, and thus neither is $\{\mathcal{D}(\Omega)\}^n$.

We now give a type of Poincaré inequality.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In the i^{th} direction we have*

$$\|u\|_{L^2(\Omega)} \leq c(\Omega)\,\|D_i u\|_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega). \tag{2.7}$$

For the proof of Theorem 2.11 we refer to the book of Foiaş [3].

Remark 2.1. *When $\Omega \subset \mathbb{R}^n$ is a bounded domain, Poincaré inequality* (2.7) *implies that*

$$\|u\|_{L^2(\Omega)} \leq c(\Omega)\,\|Du\|_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega) \tag{2.8}$$

because $\|D_i u\|_{L^2(\Omega)} \leq \|Du\|_{L^2(\Omega)}$ always. Usually we use (2.8)*.*

From inequality (2.8), it follows that we can define another norm equivalent to $\|\cdot\|_{H^1(\Omega)}$ in $H_0^1(\Omega)$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In $H_0^1(\Omega)$ the norm defined by*

$$\|u\| = \left(\sum_{i=1}^{n}\|D_i u\|^2_{L^2(\Omega)}\right)^{1/2} \tag{2.9}$$

and norm $\|\cdot\|_{H^1(\Omega)}$ are equivalent.

As a consequence of Proposition 2.1, $H_0^1(\Omega)$ is a Hilbert space also with scalar product

$$((u,v)) = \sum_{i=1}^{n}(D_i u, D_i v). \tag{2.10}$$

In the thesis we denote the norm of $H_0^1(\Omega)$ by $\|\cdot\|$ and the scalar product by $((\cdot,\cdot))$.

We have the following inequality.

Lemma 2.1. *Let $u, v \in H_0^1(\Omega)$. Then*

$$|((u,v))| \leq c(n)\|u\|\|v\|.$$

Proof. Let $u, v \in H_0^1(\Omega)$. Then

$$\begin{aligned}
|((u,v))| = |\sum_{i=1}^{n}(D_i u, D_i v)| &\leq \sum_{i=1}^{n}|(D_i u, D_i v)| \\
&\leq \sum_{i=1}^{n}\|D_i u\|_{L^2(\Omega)}\|D_i v\|_{L^2(\Omega)} \\
&\leq \left(\sum_{i=1}^{n}\|D_i u\|_{L^2(\Omega)}\right)\left(\sum_{i=1}^{n}\|D_i v\|_{L^2(\Omega)}\right) \\
&\leq n\left(\sum_{i=1}^{n}\|D_i u\|_{L^2(\Omega)}^2\right)^{1/2} n\left(\sum_{i=1}^{n}\|D_i v\|_{L^2(\Omega)}^2\right)^{1/2} \\
&= n^2\|u\|\|v\|.
\end{aligned}$$

□

We now give the trace theorem.

Theorem 2.12 (Trace Theorem)**.** *Let $1 \leq p < \infty$. Assume that Ω is bounded with $\mathcal{C}^1$ boundary $\partial\Omega$. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

1. *$Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$,*
2. *$\|Tu\|_{L^p(\partial\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)}$, for each $u \in W^{1,p}(\Omega)$, with the constant c depending only on p and Ω.*

For the proof of Theorem 2.12, we refer to [2, p. 258]. The operator T is often called the trace operator and Tu the trace of u on $\partial\Omega$.

Theorem 2.13. *Let $1 \leq p < \infty$. Assume that Ω is bounded and $\partial\Omega$ is $\mathcal{C}^1$ smooth. Let $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \text{ if and only if } Tu = 0 \text{ on } \partial\Omega.$$

For the proof of Theorem 2.13, we refer to [2, p. 259]. Notice that Theorem 2.13 implies that the kernel of T equals to $W_0^{1,p}(\Omega)$:

$$\ker T = W_0^{1,p}(\Omega).$$

We use these theorems in the case when $p = 2$.

Theorem 2.14 (Riesz Representation Theorem)**.** *Let H be a real Hilbert space with inner product $(\cdot,\cdot)$, and let H' denote its dual space. Then for each $u' \in H'$ there exists a unique element $u \in H$ such that*

$$\langle u', v\rangle = (u, v) \quad \forall\, v \in H.$$

The mapping $u' \mapsto u$ is a linear isomorphism of H' onto H.

Definition 2.10. *We say that a sequence $\{u_m\}_{m=1}^{\infty} \subset X$* converges weakly *to $u \in X$, written*

$$u_m \rightharpoonup u,$$

if

$$\langle u', u_m\rangle \to \langle u', u\rangle$$

for each bounded linear functional $u' \in X'$.

Theorem 2.15 (Weak Compactness)**.** *Let X be a reflexive Banach space and suppose that the sequence $\{u_m\}_{m=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ and $u \in X$ such that*

$$u_{m_j} \rightharpoonup u.$$

3 Derivation of Equations

In this section, we derive the Euler equations, the Navier-Stokes equations, and the equation of conservation of mass. We also derive the Bernoulli's equation and give some examples of solutions to the equations, and some physical illustrations. We refer to [5] for the details of the derivation.

3.1 Euler equations

We are studying the motion of fluids. Fluids are considered to be continuous. We are not interested in single molecules but in bigger elements with a great amount of molecules. Then we may perceive the properties of fluid, such as density and pressure, in a realistic way. We are not interested in the density of one molecule and one point has no density because its volume is zero. Considering the fluid as a continuum, we mean that a certain property is defined at every point of the space considered. Then, for example, density in a point x can be illustrated as follows. Consider small x-centered balls with radius ε, $B(x,\varepsilon)$, and the average density of these balls, $\rho_{B(x,\varepsilon)}$, and define the value of density of fluid in x as a limit of values $\rho_{B(x,\varepsilon)}$ as ε tends to zero.

3.1.1 Conservation of mass

We derive the equation of conservation of mass in the case of an arbitrary flow in three dimensions. In the end of the derivation we consider the case of incompressible fluid which means that the density of the fluid does not vary in time or place.

We consider the space $\Omega \subset \mathbb{R}^3$ where the whole fluid is. We now fix a bounded domain $V \subset \Omega$ into which and out of which the fluid may flow freely. We assume that the boundary ∂V of V is smooth.

We will give two separate representations for the loss of mass through the boundary ∂V.

Suppose that the density of the fluid in every point is given by the function $\rho : \Omega \times [0,\infty) \to (0,\infty)$, $\rho(x,t) = \rho(x_1,x_2,x_3,t)$, where $\overline{V} \subset \Omega \subset \mathbb{R}^3$ and $\rho \in \mathcal{C}^1(\Omega \times [0,\infty))$. We know that the mass of substance fragmented by a density function ρ in V is

$$m_V(t) := \int_V \rho(x,t)\,dx$$

where $m_V : [0,\infty) \to \mathbb{R}$ is the mass of fluid inside V at the moment t.

The change of this mass is the time derivative $\frac{d}{dt}m_V$. If the mass increases in time, this change is positive and more substance flows into V. Likewise, if the mass decreases in time, the change is negative, and some substance flows out of V. Thus, the mass flowing out of V in a unit time is $-\frac{d}{dt}m_V$.

Because ρ is supposed to be $\mathcal{C}^1$ and $\overline{V}$ is compact, we obtain by the following lemma that

$$-\tfrac{d}{dt}m_V(t) = -\tfrac{d}{dt}\int_V \rho(x,t)\,dx = -\int_V D_t\rho(x,t)\,dx. \tag{3.1}$$

Lemma 3.1. *Let $V \subset \mathbb{R}^3$ be a bounded domain and let $f \in \mathcal{C}^1(\overline{V} \times [0,\infty))$. Then for all $t \geq 0$,*

$$\tfrac{d}{dt}\int_V f(x,t)\,dx = \int_V D_t f(x,t)\,dx.$$

Proof. The proof is based on the definition of derivative. Let V and f be as in the lemma. We obtain

$$\tfrac{1}{h}\left(\int_V f(x,t+h)\,dx - \int_V f(x,t)\,dx\right) = \int_V \underbrace{\tfrac{1}{h}\left(f(x,t+h) - f(x,t)\right)}_{=:g_h(x,t)}\,dx$$

where $g_h(x,t) \longrightarrow D_t f(x,t)$ uniformly as $h \to 0$. Thus $D_t f(x,t)$ is integrable and

$$\tfrac{d}{dt}\int_V f(x,t)\,dx = \lim_{h\to 0}\int_V g_h(x,t)\,dx = \int_V \lim_{h\to 0} g_h(x,t)\,dx = \int_V D_t f(x,t)\,dx.$$

□

Consider now the mass flowing out of V in another way. Suppose that we know the velocity field at a point x at the moment t as a function $u : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$,

$$u(x,t) = u(x_1,x_2,x_3,t) = (u_1(x_1,x_2,x_3,t), u_2(x_1,x_2,x_3,t), u_3(x_1,x_2,x_3,t))$$

where u is supposed to belong to $\mathcal{C}^1(\mathbb{R}^3 \times [0,\infty))$. Now the product function ρu gives the flow field of the mass of the fluid. The overall mass flowing through ∂V in time t is the flux of vector field ρu through ∂V,

$$\int_{\partial V} \rho u \cdot d\bar{S} := \int_{\partial V} (\rho u)\cdot \nu\,dS.$$

Using Gauss' divergence theorem, Theorem 2.2, we obtain

$$\int_{\partial V} \rho u \cdot d\bar{S} = \int_V \operatorname{div}(\rho u)\,dx. \tag{3.2}$$

Thus, it follows from (3.1) and (3.2) that

$$-\int_V D_t\rho\,dx = \int_V \operatorname{div}(\rho u)\,dx,$$

that is

$$\int_V (D_t\rho + \operatorname{div}(\rho u))\,dx = 0. \tag{3.3}$$

We obtain

$$D_t\rho + \operatorname{div}(\rho u) = 0, \tag{3.4}$$

by the following lemma. We call equation (3.4) the equation of conservation of mass.

Lemma 3.2. *Let $f : \Omega \to \mathbb{R}$ be continuous. If $\int_A f\,dx = 0$ for all measurable subsets $A \subset \Omega$, then*

$$f \equiv 0.$$

Proof. Assume that there exists a point $x_0 \in \Omega$ such that

$$f(x_0) \neq 0.$$

We may assume that $f(x_0) > 0$. Then, by continuity of f, there exists $\varepsilon > 0$ such that $f(x) > \frac{1}{2}f(x_0)$ for all $x \in B(x_0, \varepsilon)$. We obtain that

$$\int_{B(x_0,\varepsilon)} f\,dx > \int_{B(x_0,\varepsilon)} \tfrac{1}{2}f(x_0)\,dx > 0.$$

This contradicts with the assumption. Therefore, $f \equiv 0$. $\square$

In the case of incompressible fluid, the density function ρ is identically a constant and thus its derivatives are zero. For the incompressible fluid, equation (3.4) becomes

$$\operatorname{div} u = 0. \tag{3.5}$$

This is the equation of conservation of mass for the incompressible flow.

3.1.2 Euler's equation

Euler's equation follows from Newton's second law which says that the force affecting on a body equals to its mass times its acceleration.

Suppose that pressure at a point x is given by the function $p : \Omega\times[0,\infty) \to \mathbb{R}$, $p(x,t) = p(x_1, x_2, x_3, t)$, where $p \in \mathcal{C}^1(\Omega \times [0,\infty))$. The force affecting on the volume V by the pressure is then given by

$$-\int_{\partial V} p\nu \, dS.$$

Considering the i^{th} direction, we obtain by integration by parts

$$-\int_{\partial V} p\nu_i \, dS = -\int_V D_i p \, dx = \int_V -D_i p \, dx, \quad i = 1, 2, 3.$$

Thus, in the i^{th} direction, the force affecting on one unit volume is amount of $-D_i p$.

We consider now one unit volume and the i^{th} direction. According to Newton's second law ($F = ma$), we have

$$\rho \tfrac{d}{dt} u_i = -D_i p, \; i = 1, 2, 3 \tag{3.6}$$

where $\frac{d}{dt} u_i$ is the acceleration of one fluid particle moving about space. We are now considering the incompressible fluid, thus we may assume the density to be a constant. Let $\rho = 1$. Then the equation of motion is of the form

$$\tfrac{d}{dt} u_i = -D_i p, \; i = 1, 2, 3. \tag{3.7}$$

The velocity field u is now depending not only on time but also on how the certain particle (point) moves in the space. Assume that the position of the point in the set Ω is given by the function $\phi : \Omega \times [0,\infty) \to \Omega$, $\phi(x,t) = (\phi_1(x,t), \phi_2(x,t), \phi_3(x,t))$ in $\mathcal{C}^1(\Omega \times [0,\infty))$ with $\phi(x,0) = x$. Then the velocity in the i^{th} direction is the same as the rate of change of the position in the same direction

$$u_i(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t) = \tfrac{d}{dt} \phi_i(x,t).$$

The acceleration of a particle is the total derivative of u:

$$
\begin{aligned}
&\tfrac{d}{dt}u_i(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t) \\
&\quad = \sum_{j=1}^{3} D_j u_i(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t)\,\tfrac{d}{dt}\phi_j(x,t) \\
&\qquad + D_t u_i(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t) \\
&\quad = \sum_{j=1}^{3} D_j u_i(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t)\,u_j(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t) \\
&\qquad + D_t u_i(\phi_1(x,t),\phi_2(x,t),\phi_3(x,t),t).
\end{aligned}
$$

Thus, we obtain

$$\tfrac{d}{dt}u_i = D_t u_i + \sum_{j=1}^{3} u_j D_j u_i. \tag{3.8}$$

Substituting this derivative to equation of motion (3.7), we obtain

$$D_t u_i + \sum_{j=1}^{3} u_j D_j u_i = -D_i p,\ i = 1,2,3. \tag{3.9}$$

If there are external forces acting on fluid, they are added to the right hand side. Euler's equation for the incompressible flow is then expressed in the form

$$D_t u_i + \sum_{j=1}^{3} u_j D_j u_i = -D_i p + f_i,\ i = 1,2,3 \tag{3.10}$$

where $f = (f_1, f_2, f_3) : \Omega \times [0,\infty) \to \mathbb{R}^3$ denotes the overall external force field acting on the fluid.

Consider the flow in the gravitational field. In the case of incompressible fluid the density is a positive constant and we can simply divide equation (3.6) by ρ to obtain

$$\tfrac{d}{dt}u_i = -D_i(\frac{p}{\rho}). \tag{3.11}$$

If the fluid is in the gravitational field, it is also affected by force ρg where $g \in \mathbb{R}^3$ is the gravitational constant. The force ρg must be added to the right hand side of the equation of motion. And since equation (3.11) is already divided by ρ, we add only the constant $g = (g_1, g_2, g_3)$ componentwise. Thus, in the gravitational field, Euler's equation for the incompressible fluid is

$$D_t u_i + \sum_{j=1}^{3} u_j D_j u_i = -D_i(\frac{p}{\rho}) + g_i,\ i = 1, 2, 3.$$

We have derived above two types of equalities for the incompressible inviscid flow in three dimensions: the equation of the conservation of mass and Euler's equation. We obtain the following equations, known as the Euler equations:

$$D_t u_1 + \sum_{j=1}^{3} u_j D_j u_1 = -D_1 p + f_1, \tag{3.12a}$$

$$D_t u_2 + \sum_{j=1}^{3} u_j D_j u_2 = -D_2 p + f_2, \tag{3.12b}$$

$$D_t u_3 + \sum_{j=1}^{3} u_j D_j u_3 = -D_3 p + f_3, \tag{3.12c}$$

$$\operatorname{div} u = 0, \tag{3.12d}$$

which in the form of vector is

$$D_t u + u \cdot Du = -Dp + f, \tag{3.13a}$$

$$\operatorname{div} u = 0. \tag{3.13b}$$

In the following, we give some examples of solutions to boundary value problems of the Euler equations. Of course, $u =$ constant and $p =$ constant are solutions to the equations (when boundary values are not set and $f \equiv 0$). Keeping in mind that in the derivation of the Euler equations we did not take into account the inner friction in fluid, we may expect the easy examples not to include such flows where the points (particles) move to each other. For example, if the fluid rotates uniformly around a fixed axis the points do not move to each other.

Example 3.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $u(x_1, x_2) = (-\omega x_2, \omega x_1)$ where ω is a positive constant, and let $p : \mathbb{R}^2 \to \mathbb{R}$, $p = \frac{1}{2}\omega^2|x|^2$ and $f \equiv 0$. Then $\{u, p\}$ is a solution to the Euler equations* (3.13)*.*

Let us show this. Denote $u_1(x_1, x_2) = -\omega x_2$ and $u_2(x_1, x_2) = \omega x_1$. All the functions here are infinite times differentiable. We have

$$\begin{aligned}
&D_t u_i = 0,\ i = 1, 2,\\
&Du_1 = (0, -\omega),\ Du_2 = (\omega, 0),\\
&Dp = (\omega^2 x_1, \omega^2 x_2).
\end{aligned}$$

And thus,

$$u \cdot Du = (-\omega^2 x_1, -\omega^2 x_2) = -Dp(x_1, x_2),$$

and

$$\operatorname{div} u = D_1 u_1 + D_2 u_2 = 0 + 0 = 0.$$

Thus, $\{u, p\}$ is a (classical) solution to equation (3.13).

We could also restrict the problem to a ball with radius R, if we demand that the velocity field is also defined on the boundary ($u(x_1, x_2) = (-\omega x_2, \omega x_1)$ on $\partial B(0, R)$).

Example 3.1 may be extended to a three-dimensional flow where the third component is a constant or linearly increasing.

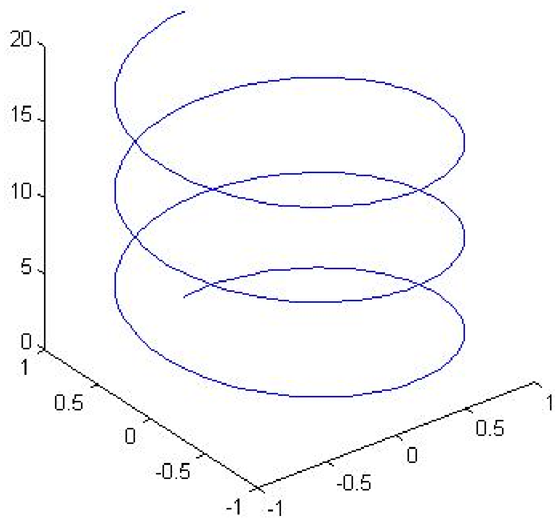
Example 3.2. *Let $u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ be defined as $u(x_1, x_2, x_3) = (-\omega x_2, \omega x_1, u_0 + at)$ where ω and a are positive constants and u_0 is a constant velocity at the moment $t = 0$. Let $p : \mathbb{R}^3 \to \mathbb{R}$, $p = \frac{1}{2}\omega^2(x_1^2 + x_2^2) - ax_3$ and $f(x_1, x_2, x_3) \equiv 0$. Then, $\{u, p\}$ is a solution to the Euler equations (3.13).*

In this case,

$$\begin{aligned}
&D_t u_i = 0,\ i = 1, 2,\ D_t u_3 = a,\\
&Du_1 = (0, -\omega, 0),\ Du_2 = (\omega, 0, 0),\ Du_3 = (0, 0, 0),\\
&Dp = (\omega^2 x_1, \omega^2 x_2, -a).
\end{aligned}$$

And because $D_t u_3 = a = -D_3 p$ and $u \cdot Du_3 = 0$, we see that $\{u, p\}$ is a (classical) solution.

To image the situation where the flow would happen as above, we may think a vertically situated cylinder which is rotating uniformly and falling down at the same time, the fluid being inside of it. The following picture shows the path of a point at distance 1 from the rotating axis when the acceleration a is zero.



We now consider the adiabatic flow. Adiabatic flow is a process where heat energy conserves. In practice there is no heat change between the different parts of the fluid nor between the fluid and the surroundings. Considering the rate of change in entropy in adiabatic flow, we have

$$\tfrac{d}{dt}s = 0$$

where $s : \Omega \times [0, \infty) \to \mathbb{R}$ is the entropy and depends on how the particle moves in the space. We have

$$\begin{aligned}
&\tfrac{d}{dt}s(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t) \\
&\quad = \sum_{j=1}^{3} D_j s(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t)\, \tfrac{d}{dt}\phi_j(x,t) \\
&\qquad + D_t s(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t) \\
&\quad = \sum_{j=1}^{3} D_j s(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t)\, u_j(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t) \\
&\qquad + D_t s(\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), t).
\end{aligned}$$

Thus

$$\tfrac{d}{dt}s = D_t s + \sum_{j=1}^{3} u_j\, D_j\, s.$$

We calculate the equation of continuation for the entropy.

$$\begin{aligned}
D_t(\rho s) + \operatorname{div}(\rho s u) &= D_t s\rho + sD_t + \rho s \operatorname{div} u + v \cdot D(\rho s)\\
&= D_t s\rho + sD_t + \rho s \operatorname{div} u + \rho u \cdot Ds + su \cdot D\rho\\
&= s(\underbrace{D_t\rho + \rho \operatorname{div} u + u \cdot D\rho}_{=0}) + \rho(\underbrace{D_t s + u \cdot Ds}_{=0})\\
&= 0.
\end{aligned}$$

In a usual case, the entropy is constant throughout the fluid and we have

$$s = \text{constant}.$$

3.1.3 Bernoulli's equation

Let us change a little bit the expression of Euler's equation. We start with the following lemma.

Lemma 3.3. *Let $u \in \mathcal{C}^1$. Then*

$$\tfrac{1}{2} D|u|^2 = u \times \operatorname{rot} u + u \cdot Du.$$

Proof. Let $u \in \mathcal{C}^1(\Omega)$ and let $x \in \Omega$. Since

$$\operatorname{rot} u(x) = (D_2u_3(x) - D_3u_2(x), D_3u_1(x) - D_1u_3(x), D_1u_2(x) - D_2u_1(x)).$$

we have

$$\begin{aligned}
u \times \operatorname{rot} u = (&u_2(\operatorname{rot} u)_3 - u_3(\operatorname{rot} u)_2, u_3(\operatorname{rot} u)_1 - u_1(\operatorname{rot} u)_3,\\
&u_1(\operatorname{rot} u)_2 - u_2(\operatorname{rot} u)_1)\\
= (&u_2(D_1u_2 - D_2u_1) - u_3(D_3u_1 - D_1u_3),\\
&u_3(D_2u_3 - D_3u_2) - u_1(D_1u_2 - D_2u_1),\\
&u_1(D_3u_1 - D_1u_3) - u_2(D_2u_3 - D_3u_2)).
\end{aligned}$$

Because $u \cdot Du_i = u_1D_1u_i + u_2D_2u_i + u_3D_3u_i$, we obtain for the i^{th} component of the right hand side

$$\begin{aligned}
(u \times \operatorname{rot} u)_i + u \cdot Du_i &= u_1(x)D_iu_1(x) + u_2(x)D_iu_2(x) + u_3(x)D_iu_3(x)\\
&= \tfrac{1}{2} D_i u^2(x),
\end{aligned}$$

because

$$\begin{aligned}\tfrac{1}{2}D_i u^2(x) &= \tfrac{1}{2}D_i(u_1^2(x)+u_2^2(x)+u_3^2(x)) \\ &= \tfrac{1}{2}\cdot 2(u_1(x)D_iu_1(x)+u_2(x)D_iu_2(x)+u_3(x)D_iu_3(x)) \\ &= u_1(x)D_iu_1(x)+u_2(x)D_iu_2(x)+u_3(x)D_iu_3(x).\end{aligned}$$

□

Proposition 3.1. *If $u \in \mathcal{C}^2$, $p \in \mathcal{C}^2$ and $f \equiv 0$, we have*

$$D_t \operatorname{rot} u - \operatorname{rot}(u \times \operatorname{rot} u) = 0. \tag{3.14}$$

If $D_t u = 0$, we have moreover

$$D_{e_u}(\tfrac{1}{2}|u|^2 + p) = 0,$$

where the partial derivative is taken in the direction of vector u.

Proof. By Lemma 3.3 and from the vector form of Euler's equation (3.9)

$$D_t u + u \cdot Du = -Dp, \tag{3.15}$$

we obtain

$$D_t u + \tfrac{1}{2}D|u|^2 - u \times \operatorname{rot} u = -Dp. \tag{3.16}$$

For a $\mathcal{C}^2$ smooth function f, $\operatorname{rot}(Df) = 0$. Because $u \in \mathcal{C}^2$ and $p \in \mathcal{C}^2$, we may take rot for the both sides of the equation to obtain

$$\operatorname{rot}(D_t u + \tfrac{1}{2}D|u|^2 - u \times \operatorname{rot} u) = \operatorname{rot}(-Dp),$$

that is

$$\operatorname{rot}(D_t u) + \tfrac{1}{2}\underbrace{\operatorname{rot}(D|u|^2)}_{=0} - \operatorname{rot}(u \times \operatorname{rot} u) = -\underbrace{\operatorname{rot}(Dp).}_{=0}$$

Thus, we have

$$D_t \operatorname{rot} u - \operatorname{rot}(u \times \operatorname{rot} u) = 0.$$

Next we consider a steady flow which means, that the velocity does not change in time, that is, $D_t u = 0$. Equation (3.16) takes the form

$$\tfrac{1}{2}D|u|^2 - u \times \operatorname{rot} u = -Dp.$$

Denote a unit vector giving the direction of the velocity in a certain point in the space by e_u. Taking a scalar product the both sides of equation above with this vector e_u, we obtain

$$(\tfrac{1}{2}D|u|^2 - u \times \operatorname{rot} u) \cdot e_u = (-Dp) \cdot e_u$$

that is

$$\tfrac{1}{2}(D|u|^2) \cdot e_u - (u \times \operatorname{rot} u) \cdot e_u = -Dp \cdot e_u$$

Because the vectors $u \times \operatorname{rot} u$ and e_u are perpendicular to each other, their scalar product is zero. By the definition of partial derivative with respect to e_u, we have

$$\tfrac{1}{2}\underbrace{(D|u|^2) \cdot e_u}_{=D_{e_u}|u|^2} - \underbrace{(u \times \operatorname{rot} u) \cdot e_u}_{=0} = -\underbrace{Dp \cdot e_u}_{=D_{e_u}p},$$

and we finally obtain

$$\tfrac{1}{2}D_{e_u}|u|^2 + D_{e_u}p = 0,$$

that is

$$D_{e_u}(\tfrac{1}{2}|u|^2 + p) = 0.$$

□

Thus by Proposition 3.1, we have

$$\tfrac{1}{2}|u|^2 + p = \text{constant} \tag{3.17}$$

along a line determined by the direction of the velocity. This line is called the stream line. Equation (3.17) is called Bernoulli's equation.

Remark 3.1. *In Example 3.1, we introduced the function pair $\{u, p\}$ modelling an uniformly rotating flow where the stream lines are origin-centered circles. We see that Bernoulli's equation* (3.17) *holds in the case of uniform rotation.*

3.1.4 Momentum flux

We now consider the momentum of fluid. Consider again one unit volume. The momentum is given by

$$\rho u. \tag{3.18}$$

The change in the momentum in time t is

$$D_t(\rho u_i) = \rho D_t u_i + D_t \rho\, u_i, \ i = 1, 2, 3. \tag{3.19}$$

Substituting to (3.19) equation of conservation of mass (3.4) and Euler's equation (3.9), we obtain

$$\begin{aligned} D_t(\rho u_i) &= -D_i p - \rho \sum_{j=1}^{3} u_j D_j u_i - \sum_{j=1}^{3} D_j(\rho u_j) u_i \\ &= -D_i p - \sum_{j=1}^{3} (\rho u_j D_j u_i + u_i D_j(\rho u_j)) \\ &= -D_i p - \sum_{j=1}^{3} D_j(\rho u_i u_j). \end{aligned}$$

We now have the following system of equations

$$\begin{aligned} D_t(\rho u_1) &= -D_1 p - D_1(\rho u_1 u_1) && -D_2(\rho u_1 u_2) - D_3(\rho u_1 u_3) \\ D_t(\rho u_2) &= -D_1(\rho u_2 u_1) && -D_2 p - D_2(\rho u_2 u_2) - D_3(\rho u_2 u_3) \\ D_t(\rho u_3) &= -D_1(\rho u_3 u_1) && -D_2(\rho u_3 u_2) - D_3 p - D_3(\rho u_3 u_3) \end{aligned}$$

To use a more simplified notation, we observe that these equations form a matrix-alike system with term $-D_i p$ only on the diagonal. Denote

$$T_{ij} = p\,\delta_{ij} + \rho u_i u_j,$$

Then

$$\sum_{j=1}^{3} D_j T_{ij} = \sum_{j=1}^{3} D_j(p\,\delta_{ij} + \rho u_i u_j) = D_i p + \sum_{j=1}^{3} D_j(\rho u_i u_j),$$

and we may write

$$D_t(\rho u_i) = -\sum_{j=1}^{3} D_j T_{ij}.$$

By integrating over some volume V, we obtain

$$D_t \int_V (\rho u_i)\, dx = \int_V D_t(\rho u_i)\, dx = \int_V -\sum_{j=1}^{3} D_j T_{ij}\, dx = -\int_{\partial V} \sum_{j=1}^{3} T_{ij} \nu_j\, dS,$$

using integration by parts.

From this, we may see that the left hand side of the equation gives the amount of momentum flowing into the fixed volume V in unit time, and the integrand on the right hand side is the flux of the i^{th} component of momentum through unit surface area.

3.1.5 Conservation of circulation

Consider the integral

$$I = \int_\gamma u \cdot d\bar{s} = \int_0^1 u(\gamma(s,t),t) \cdot \tfrac{d}{ds}\gamma(s,t)\, ds \tag{3.20}$$

where $\gamma : [0,1] \times [0,\infty) \to \mathbb{R}^3$ is a closed contour in $\mathcal{C}^2$. We assert that the value of this integral does not depend on time.

Proposition 3.2. *If $p \in \mathcal{C}^2$ and Euler's equation* (3.15) *holds, then we have*

$$\tfrac{d}{dt} \int_\gamma u \cdot d\bar{s} = 0.$$

Proof. By Lemma 3.1, we obtain

$$\begin{aligned}
\tfrac{d}{dt} \int_\gamma u \cdot d\bar{s} &= \int_0^1 D_t(u(\gamma(s,t),t) \cdot \tfrac{d}{ds}\gamma(s,t))\, ds \\
&= \int_0^1 \tfrac{d}{dt}u(\gamma(s,t),t) \cdot \tfrac{d}{ds}\gamma(s,t)\, ds + \int_0^1 u(\gamma(s,t),t) \cdot \tfrac{d}{dt}\tfrac{d}{ds}\gamma(s,t)\, ds \\
&= \int_\gamma \tfrac{d}{dt}u \cdot d\bar{s} + \int_0^1 u(\gamma(s,t),t) \cdot \tfrac{d}{dt}\tfrac{d}{ds}\gamma(s,t)\, ds
\end{aligned}$$

By Euler's equation (3.15) and by Stokes's theorem, Theorem 2.4, we obtain from the first integral on the right hand side that

$$\begin{aligned}
\int_\gamma \tfrac{d}{dt}u \cdot d\bar{s} &= \int_\gamma (D_t u + u \cdot Du) \cdot d\bar{s} \\
&= \int_\gamma (-Dp) \cdot d\bar{s} = \int_{S_\gamma} \underbrace{\mathrm{rot}(-Dp)}_{=0} \cdot d\bar{S}_\gamma = 0.
\end{aligned}$$

Let us calculate the second integral in the equation above.

$$
\begin{aligned}
\int_0^1 u(\gamma(s,t),t)\cdot \tfrac{d}{dt}\tfrac{d}{ds}\gamma(s,t)\,ds &= \sum_{i=0}^{3}\int_0^1 u_i(\gamma(s,t),t)\,\tfrac{d}{ds}\,\underbrace{\tfrac{d}{dt}\gamma(s,t)}_{=u_i(\gamma(s,t),t)}\,ds \\
&= \sum_{i=0}^{3}\int_0^1 u_i(\gamma(s,t),t)\,\tfrac{d}{ds}u_i(\gamma(s,t),t)\,ds \\
&= \sum_{i=0}^{3}\int_0^1 \tfrac{d}{ds}\left(\tfrac{1}{2}u_i^2(\gamma(s,t),t)\right)\,ds \\
&= \sum_{i=0}^{3}\left(\tfrac{1}{2}u_i^2(\gamma(s,t),t)\right)\Big|_0^1 = 0.
\end{aligned}
$$

Thus, we conclude that

$$\tfrac{d}{dt}\int_\gamma u\cdot d\bar{s} = 0.$$

□

Thus the integral (3.20) does not depend on time, and it is constant:

$$\int_\gamma u\cdot d\bar{s} = \text{constant.} \tag{3.21}$$

Equation (3.21) is the equation of conservation of circulation.

3.2 Navier-Stokes equations

In this subsection, we derive the Navier-Stokes equations for the incompressible fluid.

In this case, we need to assume that $u\in\mathcal{C}^2$. Using the same notation as in previous subsection in the case of momentum flux, we express Euler equations in the form

$$D_t(\rho u_i) = -\sum_{j=1}^{3} D_j T_{ij} \tag{3.22}$$

where

$$T_{ij} = p\,\delta_{ij} + \rho u_i u_j$$

is the momentum flux density tensor. We did not take into account the effect of viscosity in the case of the Euler equations. Now we need another term describing the viscosity. This term is added to the momentum flux density tensor. Denote this term first by $-\sigma_{ij}$ and the new momentum flux density tensor by

$$T^*_{ij} = T_{ij} - \sigma_{ij}.$$

The tensor σ_{ij} is to be found out by seeking a suitable tensor. Because of the fact that different particles move with different velocities, there occurs friction between particles which is called viscosity. Viscous forces in a fluid depend on the rate at which the fluid velocity is changing over small distance. Consider the velocity at the moment t at the distance e away from the viewing point x. We may approximate it with a Taylor series

$$u_i(x+e,t) = u_i(x,t) + Du_i(x,t) \cdot e + \dots.$$

Usually the approximation by the first derivatives is enough, and thus

$$u_i(x+e,t) \approx u_i(x,t) + Du_i(x,t) \cdot e.$$

The affecting term is $Du_i(x,t) \cdot e$ because there is no friction when u is a constant. Assuming that the properties are same in every direction we may choose e_1, e_2, e_3. We have indeed

$$\begin{aligned} Du_i(x,t) \cdot e_j &= D_j u_i(x,t) \\ &= \tfrac{1}{2}(D_j u_i(x,t) - D_i u_j(x,t)) + \tfrac{1}{2}(D_j u_i(x,t) + D_i u_j(x,t)). \end{aligned}$$

In the uniform rotation (see Example 3.1) around the x_3-axis, the velocity field is of the form

$$u(x,t) = (-\omega x_2, \omega x_1, 0).$$

We have

$$\begin{aligned} \operatorname{rot} u(x) &= (D_2 u_3(x) - D_3 u_2(x), D_3 u_1(x) - D_1 u_3(x), D_1 u_2(x) - D_2 u_1(x)) \\ &= (0, 0, 2\,\omega). \end{aligned}$$

We obtain the same result in the rotation around any axis and thus the second term is a constant for the uniform rotation. Since the particles do not move to each other in the case of uniform rotation, there is no viscosity. Thus, the second term does not affect. We have left the symmetrical term

$$\tfrac{1}{2}(D_j u_i(x,t) + D_i u_j(x,t)).$$

Taking into account the possibility of compressibility, the most usual tensor to satisfy our demand is of the form

$$\sigma_{ij} = a \operatorname{div} u \, \delta_{ij} + b(D_j u_i + D_i u_j)$$

where a and b are real numbers. (The term $\operatorname{div} u \, \delta_{ij}$ leads in a term Δu which is zero in the case of uniform rotation.)

Generally, the constants a and b are replaced by η and ζ such that

$$\sigma_{ij} = \eta(D_j u_i + D_i u_j - \tfrac{2}{3} \operatorname{div} u \, \delta_{ij}) + \zeta \operatorname{div} u \, \delta_{ij}.$$

The constants η and ζ are called constants of viscosity, and they are both positive (not be shown here). We refer to Landau [5, p. 48] for the details of the constants of viscosity.

Replacing in Euler's equation (3.22) tensor T_{ij} with the tensor T^*_{ij}, we obtain

$$\begin{aligned}
D_t(\rho u_i) &= -\sum_{j=1}^{3} D_j T^*_{ij} = -\sum_{j=1}^{3} D_j(T_{ij} - \sigma_{ij}) \\
&= -\sum_{j=1}^{3} D_j T_{ij} + \sum_{j=1}^{3} D_j \sigma_{ij} \\
&= -D_i p - \sum_{j=1}^{3} D_j(\rho u_i u_j) \\
&\quad + \sum_{j=1}^{3} D_j(\eta(D_j u_i + D_i u_j - \tfrac{2}{3} \operatorname{div} u \, \delta_{ij}) + \zeta \operatorname{div} u \, \delta_{ij}) \\
&= -D_i p - \sum_{j=1}^{3} D_j(\rho u_i u_j) + \eta \underbrace{\sum_{j=1}^{3} D_j D_j u_i}_{=\Delta u_i} \\
&\quad + \eta \underbrace{\sum_{j=1}^{3} D_j D_i u_j}_{=D_i \sum_{j=1}^{3} D_j u_j = D_i \operatorname{div} u} - \tfrac{2}{3}\eta D_i \operatorname{div} u + \zeta D_i \operatorname{div} u \\
&= -D_i p - \sum_{j=1}^{3} D_j(\rho u_i u_j) + \eta \Delta u_i + (\zeta + \tfrac{1}{3}\eta) D_i \operatorname{div} u.
\end{aligned}$$

In the case of incompressible flow, we have $\operatorname{div} u = 0$. Thus

$$\begin{aligned}\sum_{j=1}^{3} D_j(\rho u_i u_j) &= \rho(\sum_{j=1}^{3} u_i D_j u_j + \sum_{j=1}^{3} u_j D_j u_i) \\ &= \rho(u_i \underbrace{\sum_{j=1}^{3} D_j u_j}_{=\operatorname{div} u=0} + \sum_{j=1}^{3} u_j D_j u_i) \\ &= \rho u \cdot D u_i.\end{aligned}$$

We also have $D_t(\rho u_i) = \rho D_t u_i$. Combining these to the equation above we finally obtain

$$\rho D_t u_i = -D_i p - \rho u \cdot D u_i + \eta \Delta u_i.$$

Choosing ρ to be one and rearranging the terms, we obtain

$$D_t u_i + u \cdot D u_i - \eta \Delta u_i = -D_i p, \quad i = 1, 2, 3. \tag{3.23}$$

Equation (3.23) is the Navier-Stokes equation for the viscous incompressible flow. The external forces affecting on the fluid can be added on the right hand side as in the case of Euler's equation.

The pressure can be eliminated from the Navier-Stokes equation in the same way as from Euler's equation, assuming now that $u \in \mathcal{C}^3$. We first write the equation in the form of vector

$$D_t u + u \cdot D u - \eta \Delta u = -Dp. \tag{3.24}$$

By Lemma 3.3, we obtain

$$D_t u + \tfrac{1}{2} D|u|^2 - u \times \operatorname{rot} u - \eta \Delta u = -Dp. \tag{3.25}$$

Taking rot on both sides of equation (3.25), we obtain

$$D_t \operatorname{rot} u - \operatorname{rot}(u \times \operatorname{rot} u) - \eta \Delta \operatorname{rot} u = 0.$$

Combining the Navier-Stokes equation with the equation of conservation of mass, we obtain the following system of partial differential equations, known as the Navier-Stokes equations:

$$
\begin{aligned}
&D_t u_1 + u \cdot Du_1 - \eta\Delta u_1 = -D_1 p + f_1 && \text{(3.26a)}\\
&D_t u_2 + u \cdot Du_2 - \eta\Delta u_2 = -D_2 p + f_2 && \text{(3.26b)}\\
&D_t u_3 + u \cdot Du_3 - \eta\Delta u_3 = -D_3 p + f_3 && \text{(3.26c)}\\
&\operatorname{div} u = 0. && \text{(3.26d)}
\end{aligned}
$$

that is

$$
\begin{aligned}
&D_t u + u \cdot Du - \eta\Delta u = -Dp + f && \text{(3.27a)}\\
&\operatorname{div} u = 0. && \text{(3.27b)}
\end{aligned}
$$

Remark 3.2. *We recall Example 3.1 with* $u(x_1, x_2) = (-\omega x_2, \omega x_1)$ *(*$\omega > 0$ *constant),* $f \equiv 0$ *and* $p = \frac{1}{2}\omega^2(x_1^2 + x_2^2)$*. We took this case of uniform rotation into account while deriving these equations. Observing that* $\Delta u = (0, 0)$*, we see that there is no affection of friction. In this case, the Navier-Stokes equations* (3.27) *are reduced to Euler equations* (3.13)*.*

Similarly, Example 3.2 is an example of a solution to the Navier-Stokes equations (3.27) *in the three-dimensional Euclidean space.*

Example 3.3. *Let* $\Omega = \mathbb{R} \times (0, a) \subset \mathbb{R}^2$*. Let the velocity fields on the boundaries be* $u(x_1, a) = (u_0, 0)$ *and* $u(x_1, 0) = (0, 0)$*, and let* $f \equiv 0$*. If* $p \equiv$ *constant and* $u(x_1, x_2) = (\frac{u_0}{a}x_2, 0)$ *in* $\mathbb{R} \times (0, a)$*, then* $\{u, p\}$ *is a classical solution to the boundary value problem of Navier-Stokes equations* (3.27)*.*

Clearly, $u, p \in C^\infty(\Omega)$*. We have*

$$
\begin{aligned}
&Du_1(0, \frac{u_0}{a}),\ Du_2 = (0, 0),\\
&\Delta u = (0, 0),\\
&Dp = (0, 0).
\end{aligned}
$$

Clearly $u \cdot Du = 0$*. Thus* (3.27a) *is satisfied,* $\operatorname{div} u = 0$ *and on the boundaries*

$$
\begin{aligned}
&u(x_1, 0) = (0, 0),\\
&u(x_1, a) = (u_0, 0).
\end{aligned}
$$

Example 3.3 above illustrates a flow between two parallel planes (or lines) from which the first one is stationary and the second one is moving at the constant velocity u_0 to the direction of x_1-axis.

Example 3.4. *Let $\Omega = \mathbb{R} \times (-a, a) \subset \mathbb{R}^2$, the velocity at the boundary $u(x_1, \pm a) = (0, 0)$ and $f \equiv 0$. Then the function pair $\{u, p\}$, where $u(x_1, x_2) = (\frac{1}{2}(a^2 - x_2^2), 0)$ and $p(x_1, x_2) = (p_1 - \eta x_1, p_2)$ with constants p_1 and p_2, is a classical solution to the the boundary value problem of Navier-Stokes equations* (3.27)*.*

Clearly, $u, p \in C^\infty(\Omega)$. We have

$$\begin{aligned}
&D_t u = (0, 0), \\
&Du_1 = (0, -x_2),\ Du_2 = (0, 0), \\
&\Delta u = (-1, 0), \\
&Dp = (-\eta, 0).
\end{aligned}$$

Clearly, $u \cdot Du = 0$. Thus, (3.27a) *is satisfied,* $\operatorname{div} u = 0$ *and on the boundary*

$$u(x_1, \pm a) = (0, 0).$$

Example 3.4 above models a flow between two parallel planes. The same kind of solutions is obtained in three dimensions illustrating a flow in a pipe.

Example 3.5. *Let $\Omega = \mathbb{R} \times B(0, h) \subset \mathbb{R}^3$, the velocity at the boundary $u(x_1, x_2, x_3) = (0, 0, 0)$ for all $x_2^2 + x_3^2 = a^2$, and $f \equiv 0$. Then the function pair $\{u, p\}$ where $u(x_1, x_2, x_3) = (\frac{1}{4}(a^2 - x_2^2 - x_3^2), 0, 0)$ and $p(x_1, x_2, x_3) = (p_1 - \eta x_1, p_2, p_3)$ (p_1, p_2, p_3 constants) is a classical solution to the Navier-Stokes equations* (3.27)*.*

$u, p \in C^\infty(\Omega)$. We have

$$\begin{aligned}
&D_t u = (0, 0, 0), \\
&Du_1 = (0, -\tfrac{1}{2}x_2, -\tfrac{1}{2}x_3),\ Du_2 = (0, 0, 0),\ Du_3 = (0, 0, 0) \\
&\Delta u = (-1, 0, 0), \\
&Dp = (-\eta, 0, 0).
\end{aligned}$$

Clearly, $u \cdot Du = 0$. Thus (3.27a) *is satisfied,* $\operatorname{div} u = 0$ *and on the boundary, where $x_2^2 + x_3^2 = a^2$, we have*

$$u(x_1, x_2, x_3) = (0, 0, 0).$$

4 Existence of Weak Solutions

In this section, we discuss the solvability of the Navier-Stokes equations in general. We define the weak solutions of the Stokes equations, the steady-state Navier-Stokes equations and the full Navier-Stokes equations. We introduce some results for the existence and smoothness of the solutions. For the details, we refer to [6].

4.1 Preliminaries

First we build up the basic theory on the spaces that we work with.

Consider the following space

$$E(\Omega) = \{u \in \{L^2(\Omega)\}^n \,:\, \operatorname{div} u \in L^2(\Omega)\}. \tag{4.1}$$

We set

$$(u, v)_{E(\Omega)} = (u, v) + (\operatorname{div} u, \operatorname{div} v) \quad \forall\, u, v \in E(\Omega) \tag{4.2}$$

where $(u, v) = \sum_{i=1}^{n}(u_i, v_i)$. Here the derivatives of u are supposed to exist in a weak sense. Our goal is to prove a theorem according to which, if $u \in E(\Omega)$, then the value of the normal component $u \cdot \nu$ can be defined on the boundary $\partial\Omega$.

The space $E(\Omega)$ is an auxiliary space needed in the proofs in this introduction part. We give some elementary results for it. Some results are not proved, but we refer to some references for the proofs.

We denote by $E_0(\Omega)$ the closure of $\{\mathcal{D}(\Omega)\}^n$ in $E(\Omega)$.

Lemma 4.1. *$E(\Omega)$ equipped with $(\cdot, \cdot)_{E(\Omega)}$ is a Hilbert space.*

Proof. First, we show that $(\cdot, \cdot)_{E(\Omega)} : E(\Omega) \times E(\Omega) \to \mathbb{R}$ is a scalar product on $E(\Omega)$. It is clearly linear and symmetric. We have also defined $E(\Omega)$ such that $(u, v)_{E(\Omega)} < \infty$ for all $u, v \in E(\Omega)$ and thus $(\cdot, \cdot)_{E(\Omega)}$ is well-defined.

We need to show that $E(\Omega)$ is a complete normed space with the associated norm

$$\|u\|_{E(\Omega)} = \big((u, u)_{E(\Omega)}\big)^{1/2}. \tag{4.3}$$

Let $(u_m)_m$ be a Cauchy sequence in $E(\Omega)$. Then

$$\begin{aligned}\|u_m - u_n\|^2_{E(\Omega)} &= \sum_{i=1}^{n}(u^i_m - u^i_n, u^i_m - u^i_n) + (\operatorname{div}(u_m - u_n), \operatorname{div}(u_m - u_n)) \\ &= \sum_{i=1}^{n}\|u^i_m - u^i_n\|^2_{L^2(\Omega)} + \|\operatorname{div} u_m - \operatorname{div} u_n\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, $\|u^i_m - u^i_n\|_{L^2(\Omega)} \to 0$ and $\|\operatorname{div} u_m - \operatorname{div} u_n\|_{L^2(\Omega)} \to 0$ as $n, m \to \infty$. Therefore, $\{u_m\}_m$ is a Cauchy sequence in $\{L^2(\Omega)\}^n$and $\{\operatorname{div} u_m\}_m$ is a Cauchy sequence in $L^2(\Omega)$. Because $L^2(\Omega)$ is a Banach space, there exists functions $u = (u^1, \ldots, u^n) \in \{L^2(\Omega)\}^n$ and $g \in L^2(\Omega)$ such that

$$\|u^i_m - u^i\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \|\operatorname{div} u_m - g\|_{L^2(\Omega)} \to 0,$$

as $m \to \infty$.

We want to show that $\operatorname{div} u = g$, that is

$$(u, D\phi) = -(g, \phi) \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

By Hölder's inequality in Theorem 2.8, we have

$$|(u, D\phi) - (u_m, D\phi)| = |(u - u_m, D\phi)| \le \|u - u_m\|_{L^2(\Omega)}\|D\phi\|_{L^2(\Omega)} \to 0,$$

as $m \to \infty$. We obtain

$$(u, D\phi) = \lim_{m\to\infty}(u_m, D\phi) = \lim_{m\to\infty}(-(\operatorname{div} u_m, \phi)) = -(g, \phi).$$

Thus, we have $\operatorname{div} u = g \in L^2(\Omega)$. Thus $u \in E(\Omega)$, and $\|\, u_m - u\|_{E(\Omega)} \to 0$ as $m \to \infty$. □

We state Theorem 4.1 below. We observe that the functions with compact support in $E(\Omega)$ are dense in $E(\Omega)$ and thus it may be assumed that $u \in E(\Omega)$ has a compact support. The rest of proof of Theorem 4.1 is by regularization for $\Omega = \mathbb{R}^n$, and for general set an additional result is needed, too. We refer to [6, p. 6] for the complete proof.

Theorem 4.1. *Let Ω be a $\mathcal{C}^1$ smooth open set in $\mathbb{R}^n$. Then the set of vector functions belonging to $\{\mathcal{D}(\overline{\Omega})\}^n$ is dense in $E(\Omega)$.*

Now we state the trace theorem.

Theorem 4.2. *Let Ω be an open bounded set that is $\mathcal{C}^2$ smooth. Then there exists a linear continuous operator $\gamma_\nu \in \mathcal{L}(E(\Omega), H^{-1/2}(\partial\Omega))$ such that*

$$\gamma_\nu u = \text{ the restriction of } u \cdot \nu \text{ to } \partial\Omega, \ \forall\, u \in \{\mathcal{D}(\overline{\Omega})\}^n. \tag{4.4}$$

In addition, the following formula is true for all $u \in E(\Omega)$ and for all $w \in H^1(\Omega)$:

$$(u, Dw) + (\operatorname{div} u, w) = \langle \gamma_\nu u, \gamma_0 w \rangle \tag{4.5}$$

where $\gamma_0 \in \mathcal{L}(H^1(\Omega), L^2(\partial\Omega))$ such that

$$\gamma_0 w = \text{ the restriction of } w \text{ to } \partial\Omega, \ \forall\, w \in H^1(\Omega) \cap \mathcal{C}^2(\overline{\Omega}),$$

and $H^{1/2}(\partial\Omega) := \gamma_0(H^1(\Omega))$ and $H^{-1/2}(\partial\Omega) = \mathcal{L}(H^{1/2}(\partial\Omega), \mathbb{R})$ is the dual space of $H^{1/2}(\partial\Omega)$.

Remark 4.1. *The space $\gamma_0(H^1(\Omega)) = H^{1/2}(\partial\Omega)$ is a subspace of $L^2(\partial\Omega)$ (actually known to be a dense subspace [6, p. 9]), with the scalar product $(u, v) = \int_{\partial\Omega} u(x)v(x)\, dS$. Theorem 4.2 gives us a functional $\gamma_\nu u$ in the dual space $H^{-1/2}(\partial\Omega)$ for each suitable u such that*

$$\langle \gamma_\nu u, \phi \rangle = \int_{\partial\Omega} u \cdot \nu\, \phi\, dS, \quad \forall\, \phi \in H^{1/2}(\partial\Omega).$$

Proof of Theorem 4.2. We follow the proof of Theorem 1.2 in [6]. Let $\phi \in H^{1/2}(\partial\Omega)$ and let then $w \in H^1(\Omega)$ such that $\gamma_0 w = \phi$ where the function $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is given by Theorem 2.12. We first show that there exists a linear continuous functional of the dual $H^{-1/2}(\partial\Omega)$ that is defined by formula (4.5). Then we check the properties of this functional.

Let $u \in E(\Omega)$, and define $L_u : H^{1/2}(\partial\Omega) \to \mathbb{R}$,

$$L_u(\phi) := (u, Dw) + (\operatorname{div} u, w).$$

The linearity of L_u is clear. We will prove that this functional is independent of the choice of w:

Let w_1 and w_2 be two functions in $H^1(\Omega)$ such that $\gamma_0 w_1 = \gamma_0 w_2 = \phi$. Define $w = w_1 - w_2$. Then $\gamma_0 w = 0$ and thus $w \in \ker \gamma_0$ that is the space $H_0^1(\Omega)$ by Theorem 2.13. Thus there is a sequence of functions $\{w_m\}_{m=1}^\infty \subset$

$\mathcal{D}(\Omega)$ converging to w in $H^1(\Omega)$. We have

$$\begin{aligned}
(\operatorname{div} u, w_m) + (u, Dw_m) &= \int_\Omega \operatorname{div} u(x) w_m(x)\, dx + \sum_{i=1}^n \int_\Omega u_i(x) D_i w_m(x)\, dx \\
&= \int_\Omega \operatorname{div} u(x) w_m(x)\, dx \\
&+ \sum_{i=1}^n \Big[\underbrace{\int_{\partial\Omega} u_i(x) w_m(x) \nu_i \, dS}_{=0,\, w_m \in \mathcal{D}(\Omega)} - \int_\Omega D_i u_i(x) w_m(x)\, dx \Big] \\
&= \int_\Omega \operatorname{div} u(x) w_m(x)\, dx - \int_\Omega \underbrace{\sum_{i=1}^n D_i u_i(x)}_{=\operatorname{div} u(x)} w_m(x)\, dx \\
&= 0.
\end{aligned}$$

Taking the limit, we obtain

$$\begin{aligned}
|(\operatorname{div} u, w_m - w)| &\le \|\operatorname{div} u\|_{L^2(\Omega)} \|w_m - w\|_{L^2(\Omega)} \\
&\le \|\operatorname{div} u\|_{L^2(\Omega)} \|w_m - w\|_{H^1(\Omega)} \to 0,
\end{aligned}$$

as $m \to \infty$, because $u \in E(\Omega)$ and thus $\operatorname{div} u \in L^2(\Omega)$. We also obtain by equality (2.8) in Remark 2.1

$$\begin{aligned}
|(u_i, D_i w_m - D_i w)| &\le \|u_i\|_{L^2(\Omega)} \|D_i(w_m - w)\|_{L^2(\Omega)} \\
&\le \|u_i\|_{L^2(\Omega)} \|w_m - w\|_{H^1(\Omega)} \to 0,
\end{aligned}$$

as $m \to \infty$, because $u_i \in L^2(\Omega)$. Thus

$$(\operatorname{div} u, w) + (u, Dw) = 0,$$

and we have

$$(\operatorname{div} u, w_1) + (u, Dw_1) = (\operatorname{div} u, w_2) + (u, Dw_2).$$

We also know that there exists an operator $\sigma \in \mathcal{L}(H^{1/2}(\partial\Omega), H^1(\Omega))$ such that $\gamma_0 \circ \sigma$ is the identity [6, p. 9], and we choose now $w = \sigma\phi$. Then σ is a continuous linear operator for which $\|\sigma\phi\|_{H^1(\Omega)} \leq c_1\|\phi\|_{H^{1/2}(\partial\Omega)}$. We have

$$
\begin{aligned}
|L_u(\phi)| &= |(u, Dw) + (\operatorname{div} u, w)| \\
&\leq \sum_{i=1}^{n} |(u_i, D_i w)| + |(\operatorname{div} u, w)| \\
&\leq \sum_{i=1}^{n} \|u_i\|_{L^2(\Omega)} \|D_i w\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\
&\leq \sum_{i=1}^{n} \|u_i\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} \\
&= \left(\sum_{i=1}^{n} \|u_i\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} \right) \|w\|_{H^1(\Omega)} \\
&\leq c_0 \|u\|_{E(\Omega)} \|w\|_{H^1(\Omega)} \\
&= c_0 c_1 \|u\|_{E(\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)}.
\end{aligned}
$$

The mapping $\phi \mapsto L_u(\phi)$ is linear and continuous from $H^{1/2}(\partial\Omega)$ to $\mathbb{R}$. Thus there exists a linear mapping $g = g(u)$ in the dual space $H^{-1/2}(\partial\Omega)$ for which

$$\langle g, \phi \rangle = L_u(\phi).$$

The linearity of mapping $u \mapsto g(u)$ is clear, and by calculation above, we have

$$\|g\|_{H^{-1/2}(\partial\Omega)} \leq c_0 c_1 \|u\|_{E(\Omega)},$$

and thus, g is also continuous.

Define $\gamma_\nu u = g(u)$. Then equation (4.5) is satisfied. The conclusion (4.4) in Theorem 4.2 follows from Lemma 4.2 below. $\square$

Lemma 4.2. *Let $u \in \{\mathcal{D}(\overline{\Omega})\}^n$. Then*

$$\gamma_\nu u = \text{ the restriction of } u \cdot \nu \text{ to } \partial\Omega.$$

Proof. Let $u \in \{\mathcal{D}(\overline{\Omega})\}^n$ and $w \in \mathcal{C}^\infty(\overline{\Omega})$. Then by Theorem 2.12 $\gamma_0 w =$

$w|_{\partial\Omega}$, and we have

$$\begin{aligned}
L_u(\gamma_0 w) &= (u, Dw) + (\operatorname{div} u, w) \\
&= \int_\Omega \sum_{i=1}^n u_i D_i w + D_i u_i w \, dx = \sum_{i=1}^n \int_\Omega D_i(u_i w) \, dx \\
&= \sum_{i=1}^n \int_{\partial\Omega} u_i w \nu_i \, dS \\
&= \int_{\partial\Omega} (u \cdot \nu) w \, dS = \int_{\partial\Omega} (u \cdot \nu)(\gamma_0 w) \, dS
\end{aligned}$$

Thus for mapping $u \cdot \nu \in H^{-1/2}(\partial\Omega)$ we have $\langle u \cdot \nu, \phi \rangle = L_u(\phi)$ for all ϕ in $\gamma_0(\mathcal{C}^\infty(\overline{\Omega}))$. But we know that $\gamma_0(\mathcal{C}^\infty(\overline{\Omega}))$ is a dense subset of $H^{1/2}(\partial\Omega) = \gamma_0(H^1(\Omega))$. Thus there exists a sequence $\{\phi_m\}_{m=1}^\infty \subset \gamma_0(\mathcal{C}^\infty(\overline{\Omega}))$ such that $\|\phi - \phi_m\|_{H^{1/2}(\Omega)} \to 0$, as $m \to \infty$ for each $\phi \in H^{1/2}(\partial\Omega)$. By continuity of $u \cdot \nu$ and L_u we have for all $\phi \in H^{1/2}(\partial\Omega)$

$$L_u(\phi) = \lim_{m\to\infty} L_u(\phi_m) = \lim_{m\to\infty} \langle u \cdot \nu, \phi_m \rangle = \langle u \cdot \nu, \phi \rangle.$$

Because on the other hand $L_u(\phi) = \langle \gamma_\nu u, \phi \rangle$, we obtain $\gamma_\nu u = u \cdot \nu|_{\partial\Omega}$. □

Theorem 4.3. *The kernel of γ_ν is equal to $E_0(\Omega)$.*

For the proof of Theorem 4.3, we refer to [6, p. 12].

Next, we introduce some auxiliary spaces. The basic auxiliary space is

$$\mathcal{V} = \{u \in \{\mathcal{D}(\Omega)\}^n : \operatorname{div} u = 0\}.$$

The other spaces are the closures of $\mathcal{V}$ in $\{L^2(\Omega)\}^n$ and in $\{H_0^1(\Omega)\}^n$:

$$\begin{aligned}
H = \{u \in \{L^2(\Omega)\}^n :\ &\exists\, \{u_i\}_{i=1}^\infty \subset \mathcal{V} \\
&\text{such that } \|u - u_i\|_{L^2(\Omega)} \to 0 \text{ as } i \to \infty\} \\
V = \{u \in \{H_0^1(\Omega)\}^n :\ &\exists\, \{u_i\}_{i=1}^\infty \subset \mathcal{V} \\
&\text{such that } \|u - u_i\|_{H_0^1(\Omega)} \to 0 \text{ as } i \to \infty\}.
\end{aligned}$$

Let Ω be an open set in $\mathbb{R}^n$, and let $p \in \mathcal{D}'(\Omega)$. For any $v \in \mathcal{V}$, we have

$$\langle Dp, v \rangle = \sum_{i=1}^n \langle D_i p, v_i \rangle = \sum_{i=1}^n -\langle p, D_i v_i \rangle = -\langle p, \sum_{i=1}^n D_i v_i \rangle = -\langle p, \underbrace{\operatorname{div} v}_{=0} \rangle = 0.$$

We have now proved part of Theorem 4.4 below.

Theorem 4.4. *Let Ω be an open set of $\mathbb{R}^n$ and $f = (f_1, \ldots, f_n)$, $f_i \in \mathcal{D}'(\Omega)$, $i = 1, \ldots, n$. Then*

$$f = Dp \text{ for some } p \in \mathcal{D}'(\Omega)$$

in a weak sense, if and only if

$$\langle f, v \rangle = 0 \quad \forall\, v \in \mathcal{V}.$$

Theorem 4.5. *Let Ω be a bounded $\mathcal{C}^1$ smooth open set in $\mathbb{R}^n$.*

1. *If $p \in \mathcal{D}'(\Omega)$ has all its first-order weak derivatives $D_i p$, $i = 1, \ldots, n$ in $L^2(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)} \le c(\Omega)\|Dp\|_{L^2(\Omega)}.$$

2. *If $p \in \mathcal{D}'(\Omega)$ has all its first-order weak derivatives $D_i p$, $i = 1, \ldots, n$ in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)} \le c(\Omega)\|Dp\|_{H^{-1}(\Omega)}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

For the proofs of Theorems 4.4 and 4.5, we refer to [6, p. 14].

The inequalities in Theorem 4.5 are inequalities of Poincaré type in the dual space $\mathcal{D}'(\Omega)$. Here the dual space of $L^2(\Omega)$ is denoted by $L^2(\Omega)$ that makes sense by Riesz representation theorem, Theorem 2.14.

Remark 4.2.

1. *If $f \in \{H^{-1}(\Omega)\}^n$ and $\langle f, v \rangle = 0$, for all $v \in \mathcal{V}$, then $f = Dp$ in a weak sense with $p \in L^2(\Omega)$.*

2. *If $f \in \{L^2(\Omega)\}^n$ and $(f, v) = 0$, then $f = Dp$ in a weak sense with $p \in H^1(\Omega)$.*

Proof. Let $f \in \{H^{-1}(\Omega)\}^n$. Then $f \in \mathcal{D}'(\Omega)$. Because $\langle f, v \rangle = 0$ for all $v \in \mathcal{V}$, by Theorem 4.4 $f = Dp$ for some $p \in \mathcal{D}'(\Omega)$ in a weak sense that is

$$\langle f, D\phi \rangle = -\langle p, \phi \rangle \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

Because $Dp = f \in \{H^{-1}(\Omega)\}^n$, p has all its first-order weak derivatives $D_i p$, $i = 1, \ldots, n$ in $H^{-1}(\Omega)$, and thus by Theorem 4.5 $p \in L^2(\Omega)$.

In the second case, $\langle f, v \rangle = 0$ for all $v \in \mathcal{V}$, and thus by Theorem 4.4 $f = Dp$ for some $p \in \mathcal{D}'(\Omega)$ in a weak sense. Because $Dp = f \in \{L^2(\Omega)\}^n$, p has all its first-order weak derivatives $D_i p$, $i = 1, \ldots, n$ in $L^2(\Omega)$, and thus, by Theorem 4.5, $p \in L^2(\Omega)$. Since $Dp \in \{L^2(\Omega)\}^n$, we have $p \in H^1(\Omega)$. $\square$

We want to give useful characterizations for the spaces H and V we introduced before. These characterizations actually show the reasons for studying them.

Theorem 4.6. *Let Ω be a $\mathcal{C}^1$ smooth open bounded set.*

$$H^{\perp} = \{u \in \{L^2(\Omega)\}^n : u = Dp,\ p \in H^1(\Omega)\}$$
$$H = \{u \in \{L^2(\Omega)\}^n : \operatorname{div} u = 0,\ \gamma_\nu u = 0\}$$

where $u = Dp$ in the weak sense and $\operatorname{div} u = 0$ *in the weak sense.*

Proof. Let $u \in H^{\perp}$, that is $(u, v) = 0$ for each $v \in H$. Then $(u, v) = 0$ for each $v \in \mathcal{V}$. By Theorem 4.4 there exists $p \in \mathcal{D}'(\Omega)$ such that $u = Dp$ in the weak sense. But $u \in \{L^2(\Omega)\}^n$, and thus, by Theorem 4.5, p belongs to $L^2(\Omega)$ and further into $H^1(\Omega)$.

Conversely, if $p \in H^1(\Omega)$, $u = Dp \in \{L^2(\Omega)\}^n$ weakly, we have for each $v \in \mathcal{V}$

$$(u, v) = -(p, \operatorname{div} v) = 0.$$

Let then $u \in H$. By the definition of space H, $u \in \{L^2(\Omega)\}^n$, and there exists a sequence of functions $\{u_m\}_{m=1}^{\infty} \subset \mathcal{V}$ such that $\|u - u_m\|_{L^2(\Omega)} \to 0$ as $m \to \infty$. This convergence implies that, as $m \to \infty$, we have

$$(u_m, \phi) \to (u, \phi) \quad \forall\, \phi \in \{\mathcal{D}(\Omega)\}^n.$$

Because each $u_m \in \mathcal{V}$, we have $\operatorname{div} u_m = 0$ and thus

$$0 = -(\operatorname{div} u_m, \phi) = (u_m, D\phi) \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

By Hölder's inequality we have

$$|(u_m - u, D\phi)| \le \|u_m - u\|_{L^2(\Omega)}\, \|D\phi\|_{L^2(\Omega)},$$

and thus, because $D\phi \in L^2(\Omega)$, by letting $m \to \infty$, we obtain

$$(u, D\phi) = 0 \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

Thus $\operatorname{div} u = 0$ in the weak sense, and $u \in E(\Omega)$. In addition

$$\|u - u_m\|_{E(\Omega)} = \|u - u_m\|_{L^2(\Omega)},$$

and thus u_m converges to u in $E(\Omega)$. Thu,s $u \in E_0(\Omega)$. Now for the function γ_ν given by trace theorem 4.2, we have by Theorem 4.3 that $u \in \ker \gamma_\nu$. Therefore, $\gamma_\nu u = 0$.

We refer to [6, p. 16] for the rest of the proof. □

Remark 4.3. *If Ω is any open set in $\mathbb{R}^n$, then*

$$H^{\perp} = \{u \in \{L^2(\Omega)\}^n : u = Dp,\ p \in L^2_{loc}(\Omega)\}.$$

Next, we characterize the space V.

Theorem 4.7. *Let Ω be an open bounded $\mathcal{C}^1$ smooth set in $\mathbb{R}^n$. Then*

$$V = \{u \in \{H^1_0(\Omega)\}^n : \operatorname{div} u = 0\}$$

where $\operatorname{div} u = 0$ *in the weak sense.*

Proof. Let $u \in V$. Then, there exists a sequence $\{u_m\}_{m=1}^{\infty} \subset \mathcal{V}$ such that $\|u - u_m\|_{H^1_0(\Omega)} \to 0$ as $m \to 0$. Indeed, we have

$$\begin{aligned}
\|\operatorname{div} u - \operatorname{div} u_m\|_{L^2(\Omega)} &= \|\operatorname{div}(u - u_m)\|_{L^2(\Omega)} \\
&= \|\sum_{i=1}^{n} D_i(u - u_m)_i\|_{L^2(\Omega)} \\
&\leq \sum_{i=1}^{n} \|D_i(u - u_m)_i\|_{L^2(\Omega)} \\
&\leq \sum_{i=1}^{n} \|(u - u_m)_i\|_{H^1_0(\Omega)}.
\end{aligned}$$

As in the proof of Theorem 4.6, we have convergence in the weak sense. Finally, because $\operatorname{div} u_m = 0$, $\operatorname{div} u = 0$ in the weak sense.

We refer to [6, p. 18] for the rest of the proof. □

Let us introduce some embedding theorems.

Proposition 4.1. *Let Ω be $\mathcal{C}^1$ smooth and bounded, and let $u \in H^1_0(\Omega)$. Then*

1. *if $n = 2$,*

$$\|u\|_{L^q(\Omega)} \leq c(q, \Omega)\|u\|_{H^1_0(\Omega)} \quad \forall\, q, 1 \leq q < \infty.$$

2. *if $n \geq 3$,*

$$\|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq c(\Omega)\|u\|_{H^1_0(\Omega)}.$$

Theorem 4.8. *Let Ω be a $\mathcal{C}^1$ smooth bounded open set of $\mathbb{R}^n$, and let $1 \leq p < n$. Then, the embedding*

$$W^{1,p}(\Omega) \subset L^q(\Omega),$$

is compact for any q, for which $1 \leq q < p^* = \frac{np}{n-p}$. If $p \geq n$, this embedding is compact for any q, $1 \leq q < \infty$.

For the proof of Proposition 4.1, we refer to [2, p. 270] and for the proof of Theorem 4.8 to [2, p. 272].

The space V is contained in H, is dense in H and the injection is continuous [6, p. 248]. We denote the dual spaces of H and V by H' and V'. H' can be identified with a dense subspace of V' [6], and by Riesz representation theorem, we may identify H and H'.

$$V \subset H \equiv H' \subset V'$$

where each space is dense in the following one and the injections are continuous.

As a consequence of previous identifications, the scalar product in H of $f \in H$ and $u \in V$ is the same as the scalar product of f and u in the duality between V' and V:

$$\langle f, u \rangle = (f, u) \quad \forall\, f \in H,\ \forall\, u \in V. \tag{4.6}$$

For each $u \in V$, the mapping $v \mapsto ((u, v))$ is linear and continuous on V. Thus there exists $Au \in V'$ such that

$$\langle Au, v \rangle = ((u, v)) \quad \forall\, v \in V. \tag{4.7}$$

This mapping $u \mapsto Au$ is known to be linear, continuous, an isomorphism from V onto V' (Ω bounded) [6].

Let a, b be two extended real numbers, $-\infty \leq a, b \leq \infty$, X a Banach space and $1 \leq p < \infty$. We denote

$$L^p(a, b; X) = \{u : [a, b] \to X : \|u\|_{L^p(a,b;X)} < \infty\}$$

where the norm is defined by

$$\|u\|_{L^p(a,b;X)} = \Big(\int_a^b \|u(t)\|_X^p \, dt \Big)^{1/p}.$$

And we write

$$L^\infty(a,b;X) = \{u : [a,b] \to X : \|u\|_{L^\infty(a,b;X)} < \infty\},$$

with the norm

$$\|u\|_{L^\infty(a,b;X)} = \operatorname{ess\,sup}_{t\in[a,b]} \|u(t)\|_X.$$

We also have the space

$$\mathcal{C}([a,b];X) = \{u : [a,b] \to X : f \text{ continuous}, \ \sup_{t\in[a,b]} \|u(t)\|_X < \infty\}.$$

Lemma 4.3. *Let X be a Banach space with its dual space X' and let u and g be in $L^1(a,b;X)$. Then, the following are equivalent*

1. *u is almost everywhere equal to a primitive function of g*

$$u(t) = \xi + \int_0^t g(s)\,ds, \quad \xi \in X, \ \text{a.e. } t \in [a,b].$$

2. *for each test function $\phi \in \mathcal{D}(a,b)$,*

$$\int_a^b u(t)\phi'(t)\,dt = -\int_a^b g(t)\phi(t)\,dt$$

   where $\phi'(t) = \frac{d}{dt}\phi$.

3. *for each $\eta \in X'$*

$$\tfrac{d}{dt}\langle u, \eta\rangle = \langle g, \eta\rangle$$

   in the weak sense on (a,b).

If the items above are satisfied for u and g, then u is a.e. equal to a continuous function from $[a,b]$ to X.

For the proof of Lemma 4.3, we refer to [6, p. 250].

4.2 Stokes equations

The Stokes equations are the linearized stationary form of the full Navier-Stokes equations. In this subsection we consider the variational problem for the Stokes equations and sketch the proof for the existence of the weak solution.

Let us give a formulation of the Stokes problem with boundary condition $u = 0$.

Let $f \in L^2(\Omega)$ be a vector-valued function in Ω. We seek a vector-valued function $u = (u_1, \dots, u_n)$ and a scalar function p, which are defined in Ω and satisfy

$$
\begin{aligned}
-\mu\Delta u + Dp &= f \quad \text{in } \Omega, && \text{(4.8a)}\\
\operatorname{div} u &= 0 \quad \text{in } \Omega, && \text{(4.8b)}\\
u &= 0 \quad \text{on } \partial\Omega. && \text{(4.8c)}
\end{aligned}
$$

Example 4.1. *Let $f \equiv 0$. Let $u \equiv 0$ and $p =$ constant. Then, u, p solve the boundary value problem* (4.8)*.*

Remark 4.4. *The examples of the uniform rotation (Examples 3.1 and 3.2) do not give solutions to the Stokes problem. These examples show that there are very simple flows to which the Stokes equations can not be applied.*

Then we study the variational problem for the Stokes equations (4.8). The definition includes the space V we introduced in subsection 4.1.

Definition 4.1. *Let $f \in \{L^2(\Omega)\}^n$. The problem to find a function u such that*

$$
u \in V \quad \textit{and} \quad \mu((u,v)) = (f,v) \quad \forall\, v \in V, \tag{4.9}
$$

is called the variational problem *for equation* (4.8)*.*

In the following, we show that if there exists smooth functions f, u and p satisfying (4.8), then f, u and p solve (4.9). We take a scalar product of (4.8a) with a function $v \in \mathcal{V} \subset \{H_0^1(\Omega)\}^n$. We obtain

$$
(-\mu\Delta u + Dp, v) = (f, v),
$$

that is

$$
\mu(-\Delta u, v) + (Dp, v) = (f, v).
$$

By integrating by parts, we have that

$$
\begin{aligned}
(-\Delta u, v) &= \sum_{i=1}^{n} \int_\Omega -\Delta u_i v_i \, dx = \sum_{i=1}^{n} \int_\Omega -\sum_{j=1}^{n} D_j D_j u_i v_i \, dx \\
&= -\sum_{i,j=1}^{n} \left(\int_{\partial\Omega} D_j u_i \underbrace{v_i}_{=0 \text{ on } \partial\Omega} \nu_j \, dS - \int_\Omega D_j u_i D_j v_i \, dx \right) \\
&= \sum_{i,j=1}^{n} \int_\Omega D_j u_i D_j v_i \, dx = \sum_{i=1}^{n} \sum_{j=1}^{n} (D_j u_i, D_j v_i) \\
&= \sum_{i=1}^{n} ((u_i, v_i)) = ((u, v)),
\end{aligned}
$$

and that

$$
\begin{aligned}
(Dp, v) &= \sum_{i=1}^{n} \int_\Omega D_i p v_i \, dx \\
&= \sum_{i=1}^{n} \left(\int_{\partial\Omega} p \underbrace{v_i}_{=0} \nu_i \, dS - \int_\Omega p D_i v_i \, dx \right) \\
&= -\int_\Omega p \sum_{i=1}^{n} D_i v_i \, dx = -\int_\Omega p \underbrace{\operatorname{div} v}_{=0} \, dx = 0.
\end{aligned}
$$

Thus, we obtain

$$\mu((u, v)) = (f, v) \quad \forall\, v \in \mathcal{V}.$$

Recalling that V is the closure of $\mathcal{V}$ in $\{H_0^1(\Omega)\}^n$ and observing that both sides of the equation above depend linearly and continuously on v for the $\{H_0^1(\Omega)\}^n$ topology, we conclude that this equation is also valid for each $v \in V$ by continuity:

Let $v \in V$. Then, there exists a sequence $\{v_m\}_{m=1}^{\infty} \subset \mathcal{V}$ such that $\|v - v_m\| \to 0$ as $m \to \infty$. Define $L(v) = \mu((u, v)) - (f, v)$. Then by Lemma 2.1, we obtain

$$\begin{aligned}|L(v)-L(v_m)| &= |\mu((u,v))-(f,v)-\mu((u,v_m))+(f,v_m)| \\ &= |\mu((u,v-v_m))+(f,v_m-v)| \\ &\le \mu|((u,v-v_m))|+|(f,v_m-v)| \\ &\le \mu\sum_{i=1}^{n}|((u^i,v^i-v^i_m))|+\sum_{i=1}^{n}|(f^i,v^i_m-v^i)| \\ &\le \mu\sum_{i=1}^{n}c\|u^i\|\|v^i-v^i_m\|+\sum_{i=1}^{n}\|f^i\|_{L^2(\Omega)}\underbrace{\|v^i_m-v^i\|_{L^2(\Omega)}}_{\le\|v^i-v^i_m\|} \\ &\le C\|v-v_m\|\to 0,\end{aligned}$$

as $m\to 0$.

If Ω is of class $\mathcal{C}^2$, then $u\in\{H^1_0(\Omega)\}^n$ because of (4.8c). And because $\operatorname{div} u=0$ in Ω, by (4.8b), we have by Theorem 4.7, that $u\in V$. Thus, u solves the variational problem (4.9).

Definition 4.2. *The function $u\in\{H^1_0(\Omega)\}^n$ satisfies* (4.8) *in the* weak sense *if the following holds.*

$$\exists\, p\in L^2(\Omega)\ s.t.\ \mu\sum_{i,j=1}^{n}(D_ju_i,D_j\phi_i)-(p,\operatorname{div}\phi)=(f,\phi)\quad\forall\,\phi\in\{\mathcal{D}(\Omega)\}^n, \tag{4.10a}$$

$$(u,D\phi)=0 \text{ for all } \phi\in\mathcal{D}(\Omega), \tag{4.10b}$$

$$\gamma_0 u=0. \tag{4.10c}$$

Then u is a weak solution *to* (4.8).

Lemma 4.4. *Let Ω be an open bounded set of class $\mathcal{C}^2$. Then the following are equivalent:*

1. *u satisfies variational problem* (4.9).

2. *u belongs to $\{H^1_0(\Omega)\}^n$ and satisfies* (4.8) *in the weak sense, i.e. u satisfies* (4.10).

Proof. First, assume that $u \in V$ satisfies (4.9). Because $V \subset \{H_0^1(\Omega)\}^n$ each $u_i \in H_0^1(\Omega) = \ker \gamma_0$ (by Theorem 2.13). Thus, $\gamma_0 u = 0$ in $H^{1/2}(\partial\Omega)$. Because $u \in V$ by Theorem 4.7 we have $\operatorname{div} u = 0$ in the weak sense in Ω. In addition, we have $-\mu((u, v)) + (f, v) = 0$ for all $v \in V$ where the left hand side is linear and continuous for all $v \in H_0^1(\Omega)$. Thus, there exists $\mu\Delta u + f \in H^{-1}(\Omega) \in \mathcal{D}'(\Omega)$ for which

$$\langle \mu\Delta u + f, v\rangle = 0 \quad \forall\, v \in \mathcal{V} \subset V.$$

By Remark 4.2, there exists $p \in L^2(\Omega)$ such that

$$\mu\Delta u + f = Dp \quad \text{in the weak sense in } \Omega.$$

Thus, $u \in \{H_0^1(\Omega)\}^n$ and u satisfies (4.10).

Then, suppose that $u \in \{H_0^1(\Omega)\}^n$ satisfies (4.8) in the weak sense. By Theorem 4.7 and by (4.10b) $u \in V$. And (4.10a) implies by Theorem 4.4, that

$$\langle \mu\Delta u + f, v\rangle = 0 \quad \forall\, v \in \mathcal{V}.$$

We saw above that this implies that u satisfies (4.9). □

Remark 4.5. *This variational problem* (4.9) *was introduced by Jean Leray. Remarkable on it is that now we only have to find u, because the existence of p follows by Theorem 4.4.*

If Ω is not smooth, Theorem 4.7 might not hold.

Theorem 4.9. *For any $f \in \{L^2(\Omega)\}^n$, problem* (4.9) *has a unique solution u. Moreover, there exists a function $p \in L^2_{loc}(\Omega)$ such that* (4.10a) *and* (4.10b) *are satisfied. If Ω is an open bounded set of class $\mathcal{C}^2$, then $p \in L^2(\Omega)$ and* (4.10) *is satisfied by u and p.*

We prove Theorem 4.9 by the projection theorem, Theorem 4.10, below. We refer to [6, p. 24] for the proof of Theorem 4.10.

Theorem 4.10 (Projection Theorem)**.** *Let W be a separable real Hilbert space with norm $\|\cdot\|_W$ and let $a(u, v)$ be a bilinear continuous form on $W \times W$, which is coercive i.e., there exists $\alpha > 0$ such that*

$$a(u, u) \geq \alpha\|u\|_W^2 \quad \forall\, u \in W.$$

Then, for each l in W', which is the dual space of W, there exists one and only one $u \in W$ such that

$$a(u, v) = \langle l, v\rangle \quad \forall\, v \in W.$$

Proof of Theorem 4.9. V is a separable Hilbert space as a subspace of the separable metric space $\{H_0^1(\Omega)\}^n$, $\mu > 0$ and

$$\mu((u,u)) = \mu\|u\|^2 \quad \forall\, u \in V$$

where $\mu((\cdot,\cdot)) : V \times V \to R$ is a bilinear continuous form. f defines a linear and continuous form on V by $\langle f, v\rangle = (f, v)$. Thus, by the Projection theorem, Theorem 4.10, there exists a unique $u \in V$ satisfying

$$\mu((u,v)) = \langle f, v\rangle \quad \forall\, v \in V.$$

The conclusion in Theorem 4.9 follows from Lemma 4.4. $\square$

Remark 4.6. *The linear operator $v \mapsto a(u,v)$ is continuous on W. Hence, there exists an element of W' depending on u, denoted by $A(u)$, such that*

$$a(u,v) = \langle A(u), v\rangle \quad \forall\, v \in W.$$

By projection theorem, $a(u,v) = \langle l, v\rangle$ for $l \in W'$, and thus, we have

$$\langle l, v\rangle = \langle A(u), v\rangle \quad \forall\, v \in W.$$

This is equivalent to

$$A(u) = l \text{ in } W'.$$

As a consequence of Theorem 4.9, we have the existence result for the following Stokes problem. For the proof, we refer to [6, p. 31].

Theorem 4.11. *Let Ω be an open bounded set of class $\mathcal{C}^2$ in $\mathbb{R}^n$. Given $f \in \{H^{-1}(\Omega)\}^n$ and $\phi \in H^{1/2}(\partial\Omega)$. Then, there exists $u \in H^1(\Omega)$ and $p \in L^2(\Omega)$ such that the problem*

$$-\mu\Delta u + Dp = f \quad \text{in } \Omega, \tag{4.11a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{4.11b}$$

$$u = \phi \quad \text{on } \partial\Omega. \tag{4.11c}$$

is solved in the weak sense and u is unique.

4.3 Steady-state Navier-Stokes equations

We now represent the steady-state Navier-Stokes equations with the boundary value problem.

Let Ω be a $\mathcal{C}^1$ smooth, bounded open set in $\mathbb{R}^n$ with boundary $\partial\Omega$. Let $f \in \{L^2(\Omega)\}^n$. We look for functions $u = (u_1, \ldots, u_n)$ and p, which are defined in Ω and satisfy

$$-\mu\Delta u + u \cdot Du + Dp = f \text{ in } \Omega, \tag{4.12a}$$

$$\operatorname{div} u = 0 \text{ in } \Omega \tag{4.12b}$$

$$u = 0 \text{ on } \partial\Omega. \tag{4.12c}$$

As in the case of the Stokes equations, we will formulate the variational problem for the equations (4.12). But now there is the nonlinear term $u \cdot Du$ with which we have difficulties.

If f, u and p are smooth functions satisfying problem (4.12), then $u \in V$ and for each $v \in \mathcal{V}$,

$$\mu((u, v)) + b(u, u, v) = (f, v)$$

where

$$b(u, v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_\Omega u_i \, D_i v_j \, w_j \, dx.$$

But for u and v in V, the expression $b(u, u, v)$ does not necessarily make sense. In the following, we study some properties of the form b.

Definition 4.3. *Denote*

$$\tilde{V} = \textit{the closure of } \mathcal{V} \textit{ in } \{H_0^1(\Omega)\}^n \cap \{L^n(\Omega)\}^n,$$

with the norm

$$\|u\|_{H_0^1(\Omega)} + \|u\|_{L^n(\Omega)}.$$

Remark 4.7. *If Ω is bounded, we have by Proposition 4.1*

$$\|u\|_{L^2(\Omega)} \leq c\|u\|_{H_0^1(\Omega)}$$

for $n = 2$, and

$$\|u\|_{L^n(\Omega)} \leq c'\|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq c\|u\|_{H_0^1(\Omega)}$$

for $n = 3, 4$, because $\frac{2n}{n-2} \geq n$ for $n = 3, 4$. Thus, $\tilde{V} = V$ for $n = 2, 3, 4$.

The form b is trilinear*, that is linear with respect to u, v and w, on V and $\tilde{V}$.*

Lemma 4.5. *The form b is defined and trilinear continuous on*

$$\{H_0^1(\Omega)\}^n \times \{H_0^1(\Omega)\}^n \times \left(\{H_0^1(\Omega)\}^n \cap \{L^n(\Omega)\}^n\right)$$

where $\Omega \subset \mathbb{R}^n$ is any open set.

Proof. Let $u, v \in \{H_0^1(\Omega)\}^n$ and $w \in \{H_0^1(\Omega)\}^n \cap \{L^n(\Omega)\}^n$. Let $n \geq 3$. By Proposition 4.1, we have $u_i \in L^{\frac{2n}{n-2}}(\Omega)$. We also have $D_i v_j \in L^2(\Omega)$ and $w_j \in L^n(\Omega)$, $1 \leq i, j \leq n$. By Hölder's inequality in Theorem 2.8,

$$\begin{aligned}
|\int_\Omega u_i\, D_i v_j\, w_j\, dx| &\leq \int_\Omega |u_i\, D_i v_j\, w_j\, dx| \\
&\leq \left(\int_\Omega |D_i v_j|^2 dx\right)^{1/2} \\
&\qquad \left(\int_\Omega |u_i|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{2n}} \left(\int_\Omega |w_j|^n\, dx\right)^{1/n} \\
&= \|D_i v_j\|_{L^2(\Omega)} \|u_i\|_{L^{\frac{2n}{n-2}}(\Omega)} \|w_j\|_{L^n(\Omega)} \\
&\leq c(\Omega) \|v_j\|_{H_0^1(\Omega)} \|u_i\|_{H_0^1(\Omega)} \|w_j\|_{H_0^1(\Omega)\cap L^n(\Omega)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|b(u,v,w)| &= |\sum_{i,j=1}^n \int_\Omega u_i\, D_i v_j\, w_j\, dx| \\
&\leq \sum_{i,j=1}^n |\int_\Omega u_i\, D_i v_j\, w_j\, dx| \\
&\leq \sum_{i,j=1}^n c(\Omega) \|v_j\|_{H_0^1(\Omega)} \|u_i\|_{H_0^1(\Omega)} \|w_j\|_{H_0^1(\Omega)\cap L^n(\Omega)} \\
&\leq c(\Omega, n) \|v\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)\cap L^n(\Omega)}.
\end{aligned}$$

Thus, b is continuous, and well-defined. The linearity of b is trivial.

Let then $n = 2$. Now $\{H_0^1(\Omega)\}^2 \cap \{L^2(\Omega)\}^2 = \{H_0^1(\Omega)\}^2$. In this case, we have, by Proposition 4.1, $u_i \in L^4_{loc}(\Omega)$, $D_i v_j \in L^2(\Omega)$ and $w_j \in L^4_{loc}(\Omega)$, $1 \leq i, j \leq 2$. By Hölder's inequality, we have $u_i\, D_i v_j\, w_j$ in $L^1_{loc}(\Omega)$, and

$$\begin{aligned}
|\int_\Omega u_i\, D_i v_j\, w_j\, dx| &\leq \int_\Omega |u_i\, D_i v_j\, w_j\, dx| \\
&\leq \|D_i v_j\|_{L^2(\Omega)} \|u_i\|_{L^4(\Omega)} \|w_j\|_{L^4(\Omega)} \\
&\leq c(\Omega) \|v_j\|_{H_0^1(\Omega)} \|u_i\|_{H_0^1(\Omega)} \|w_j\|_{H_0^1(\Omega)}.
\end{aligned}$$

$\square$

Corollary 4.1. *The form b is defined, trilinear and continuous on $V \times V \times \tilde{V}$, when $\Omega \subset \mathbb{R}^n$ is any open set. If Ω is bounded and $n = 2, 3, 4$, b is trilinear continuous on $V \times V \times V$.*

Lemma 4.6. *For any open set Ω,*

1. *$b(u, v, v) = 0$ for all $u \in V$, $v \in \tilde{V}$.*
2. *$b(u, v, w) = -b(u, w, v)$ for all $u \in V$, $v, w \in \tilde{V}$.*

Proof. By continuity, it suffices to show the first item for $u \in \mathcal{V}$ and $v \in \mathcal{V}$, because $\mathcal{V}$ is dense in V and $\mathcal{V}$ is dense in $\tilde{V}$. For such u and v, we obtain by integrating by parts

$$\begin{aligned} \int_\Omega u_i\, D_i v_j\, v_j\, dx &= \int_\Omega u_i\, D_i(\frac{v_j^2}{2})\, dx \\ &= \int_{\partial\Omega} u_i \underbrace{\frac{v_j^2}{2}}_{=0} \nu_i\, dS - \int_\Omega D_i u_i \frac{v_j^2}{2}\, dx, \end{aligned}$$

and thus

$$\begin{aligned} b(u, v, v) &= \sum_{i,j=1}^n \big(- \int_\Omega D_i u_i \frac{v_j^2}{2}\, dx \big) \\ &= -\tfrac{1}{2} \sum_{j=1}^n \int_\Omega \sum_{i=1}^n D_i u_i\, v_j^2\, dx \\ &= -\tfrac{1}{2} \sum_{j=1}^n \int_\Omega \underbrace{\operatorname{div} u}_{=0}\, v_j^2\, dx = 0. \end{aligned}$$

The second claim follows from the first one. Let $u \in V$, $v, w \in \tilde{V}$. Then $v + w \in \tilde{V}$. We have

$$\begin{aligned} 0 = b(u, v+w, v+w) &= b(u, v, v+w) + b(u, w, v+w) \\ &= \underbrace{b(u, v, v)}_{=0} + b(u, v, w) + b(u, w, v) + \underbrace{b(u, w, w)}_{=0} \\ &= b(u, v, w) + b(u, w, v), \end{aligned}$$

and the claim follows. $\square$

Definition 4.4. *Let $f \in \{L^2(\Omega)\}^n$. The problem to find a function u such that*

$$u \in V \text{ and } \mu((u,v)) + b(u,u,v) = (f,v)\ \forall\, v \in \tilde{V}, \tag{4.13}$$

is the variational problem *for equation* (4.12)*.*

If u, p and f are smooth functions that satisfy equation (4.12), then they satisfy problem (4.13).

Conversely, if $u \in V$ satisfies (4.13), then

$$\langle -\mu\Delta u + \sum_{i=1}^{n} u_i D_i u - f, v\rangle = 0 \quad \forall\, v \in \mathcal{V}$$

where $\Delta u \in H^{-1}(\Omega)$, $f \in \{L^2(\Omega)\}^n$ and $u_i D_i u \in L^{\frac{n}{n-1}}(\Omega)$, since $u_i \in L^{\frac{2n}{n-2}}(\Omega)$ by 4.1 and $D_i u \in L^2(\Omega)$. In the case where n is arbitrary, we obtain by Theorem 4.4 and a further regularization argument that $p \in L^1_{loc}(\Omega)$ [6, p. 163], such that (4.12a) is satisfied in the weak sense. u belonging to V implies that u satisfies (4.12b) and (4.12c) in the weak and the trace theorem senses.

Definition 4.5. *The function $u \in \{H^1_0(\Omega)\}^n$ satisfies* (4.12) *in the* weak sense *if*

$$\exists\, p \in L^1_{loc}(\Omega) \text{ s.t. } \mu((u,\phi)) + (u \cdot Du, \phi) - (p, \operatorname{div}\phi) = (f,\phi)\ \forall\, \phi \in \{\mathcal{D}(\Omega)\}^n, \tag{4.14a}$$

$$(u, D\phi) = 0 \text{ for all } \phi \in \mathcal{D}(\Omega), \tag{4.14b}$$

$$\gamma_0 u = 0. \tag{4.14c}$$

Then u is a weak solution *to* (4.12)*.*

Lemma 4.7. *The following are equivalent.*

1. *u satisfies variational problem* (4.13)*.*

2. *u satisfies problem* (4.14)*.*

Proof. Let us prove the case $n \leq 4$. In these cases, we may replace requirement $p \in L^1_{loc}(\Omega)$ in (4.14a) with $p \in L^2(\Omega)$.

Because Ω is supposed to be bounded and $n \leq 4$, we have by Remark 4.7, that $V = \tilde{V}$. Thus, problem (4.13) becomes

$$u \in V \text{ and } \mu((u,v)) + b(u,u,v) = (f,v)\ \forall\, v \in V. \tag{4.15}$$

Thus, there exists $\mu\Delta u - u \cdot Du + f \in \{\mathcal{D}'(\Omega)\}^n$, such that

$$\langle \mu\Delta u - u \cdot Du + f, v\rangle = 0 \quad \forall\, v \in \mathcal{V} \subset V.$$

In the case of the Stokes problem, we already observed that $\mu\Delta u \in H^{-1}(\Omega)$ and $f \in H^{-1}(\Omega)$. Because the norms of $\{H_0^1(\Omega)\}^n$ and $\{H_0^1(\Omega)\}^n \cap \{L^n(\Omega)\}^n$ are equivalent (now when Ω is bounded and $n \leq 4$), we have by Proposition 4.1, as in the proof of Lemma 4.5

$$|b(u, v, w)| \leq c(n)\|v\|_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)}\|w\|_{H_0^1(\Omega)}.$$

And thus, $-u \cdot Du$ is linear and continuous for $H_0^1(\Omega)$ topology, and thus, it belongs to $H^{-1}(\Omega)$. By Remark 4.2, there exists $p \in L^2(\Omega)$ such that

$$\mu\Delta u - u \cdot Du + f = Dp,$$

in the weak sense, that is

$$\mu((u, \phi)) + (u \cdot Du, \phi) - (p, \operatorname{div}\phi) = (f, \phi) \quad \forall\, \phi \in \{\mathcal{D}(\Omega)\}^n,$$

or

$$\mu \sum_{i,j=1}^{n} \int_\Omega D_j u_i D_j \phi_i \, dx + \sum_{i,j=1}^{n} \int_\Omega u_j D_j u_i \phi_i \, dx - \int_\Omega p \operatorname{div}\phi \, dx = \sum_{i,j=1}^{n} \int_\Omega f_i \phi_i \, dx$$

$\forall\, \phi \in \{\mathcal{D}(\Omega)\}^n$. Equations (4.14b) and (4.14c) follow as in the case of the Stokes problem.

In the other direction, we obtain easily that

$$\langle \mu\Delta u - u \cdot Du + f, v\rangle = 0 \quad \forall\, v \in \mathcal{V}.$$

But this holds also for each $v \in V$ by a similar continuity argument as in the case of the Stokes problem. $\square$

For the solution of the variational problem for the steady-state Navier-Stokes equations (4.13), we have the following existence result.

Theorem 4.12. *Let Ω be a bounded set in $\mathbb{R}^n$ and let f be given in $H^{-1}(\Omega)$. Problem* (4.13) *has at least one solution $u \in V$ and there exists $p \in L^1_{loc}(\Omega)$ such that* (4.14) *holds.*

Proof. We prove the case $n \le 4$. In this case, we have $\tilde{V} = V$. The general result is proved in Temam [6, p. 164].

V is a separable Hilbert space as a subspace of the separable space $\{H_0^1(\Omega)\}^n$.

On the other hand, V is the closure of $\mathcal{V}$ in $\{H_0^1(\Omega)\}^n$ topology, $\overline{\langle \mathcal{V} \rangle} = V$.

There exists an independent set $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{V}$ such that $\overline{\langle \{w_k\}_{k\in\mathbb{N}} \rangle} = V$. If every independent set $\{w_i\}_{i\in J}$ of $\mathcal{V}$ with $\overline{\langle \{w_i\}_{i\in J} \rangle} = V$ were uncountable, it would contradict with the separability of V.

Define $u_m = \sum_{i=1}^m c_{i,m} w_i$, $c_{i,m} \in \mathbb{R}$ such that

$$\mu((u_m, w_k)) + b(u_m, u_m, w_k) = \langle f, w_k \rangle, \quad k = 1, \dots, m.$$

Let $X_m = \langle \{w_1, \dots, w_m\} \rangle$. Define $P = P_m$ by

$$(P_m(u), v)_{X_m} = \mu((u, v)) + b(u, u, v) - \langle f, v \rangle \quad \forall\, u, v \in X_m.$$

P_m is then continuous since everything on the right hand side is continuous for u. Furthermore

$$\begin{aligned} (P_m(u), u)_{X_m} &= \mu \|u\|^2 + \underbrace{b(u, u, u)}_{=0} - \langle f, u \rangle \\ &\ge \mu \|u\|^2 - \|f\|_{V'} \|u\| \\ &= \|u\| (\mu \|u\| - \|f\|_{V'}). \end{aligned}$$

Choosing $k > \frac{1}{\mu \|f\|_{V'}}$ when $\|f\|_{V'} > 0$ and any $k > 0$ when $\|f\|_{V'} = 0$, we have

$$(P_m(u), u)_{X_m} \ge k(\mu k - \|f\|_{V'}) > 0.$$

Thus, we may apply Lemma 4.8 below and we know that there exists $u_m \in X_m$ such that $P_m(u_m) = 0$. This implies

$$\mu((u_m, w_k)) + b(u_m, u_m, w_k) - \langle f, w_k \rangle = (P_m(u_m), v)_X = (0, v)_X = 0,$$

for each v in X_m. Thus

$$\mu((u_m, w_k)) + b(u_m, u_m, w_k) = \langle f, w_k \rangle, \quad k = 1, \dots, m. \tag{4.16}$$

Multiply (4.16) by $c_{k,m}$ and sum up the equations over k. Then, we have

$$\sum_{k=1}^m c_{k,m} [\mu((u_m, w_k)) + b(u_m, u_m, w_k) - \langle f, w_k \rangle] = 0,$$

that is

$$\mu((u_m, \sum_{k=1}^{m} c_{k,m} w_k)) + b(u_m, u_m, \sum_{k=1}^{m} c_{k,m} w_k) - \langle f, \sum_{k=1}^{m} c_{k,m} w_k \rangle = 0.$$

We defined $u_m = \sum_{k=1}^{m} c_{k,m} w_k$, and thus, we have

$$\mu \|u_m\|^2 + b(u_m, u_m, u_m) = \langle f, u_m \rangle. \tag{4.17}$$

But $b(u_m, u_m, u_m) = 0$, because $u_m \in V$, and $\langle f, u_m \rangle \leq \|f\|_{V'} \|u_m\|$. Thus, we have by (4.17)

$$\|u_m\| \leq \frac{1}{\mu} \|f\|_{V'}.$$

$\{u_m\}_m$ is therefore a bounded sequence in V which is reflexive as a Hilbert space. In a reflexive Hilbert space, every bounded sequence has a subsequence that converges weakly in the space concerned. Thus by Theorem 2.15, there exists u in V such that

$$\langle g, u_m \rangle \rightarrow \langle g, u \rangle \text{ for each } g \in V'.$$

By Theorem 4.8, the embedding $V \subset \{L^2(\Omega)\}^n$ is compact, that is $\|u\|_{L^2(\Omega)} \leq c\|u\|_V$. Furthermore, for each bounded sequence $\{u_m\}_{m=1}^{\infty} \subset V$, there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ and $u \in \{L^2(\Omega)\}^n$ such that

$$\|u_{m_j} - u\|_{L^2(\Omega)} \rightarrow 0,$$

as $j \rightarrow \infty$.

By the observations above and by Lemma 4.9 below, if we let $j \rightarrow \infty$ with the subsequence in (4.16), we find that

$$\mu((u, w_k)) + b(u, u, w_k) = \langle f, w_k \rangle, \tag{4.18}$$

for any $w_k,\ k = 1, 2, 3, \ldots$. Equation (4.18) is also true for any v that is a linear combination of w_k. These linear combinations are dense in V and the terms in equation (4.18) are continuous on V and thus (4.18) holds also for each $v \in V$. Then u is a solution of problem (4.13), and by Lemma 4.7, u satisfies (4.14). $\square$

Remark 4.8. *The procedure used in the proof of Theorem 4.12 in which we find a proper approximate solution in increasing set of finite-dimensional subspaces X_m of V and then be able to pass the limit, is called the Galerkin method. The Galerkin method approximation is also used in finite element analysis solving the equations numerically and is well adapted to finite element applications [1].*

Lemma 4.8. *Let X be finite dimensional Hilbert space with scalar product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Let P be a continuous mapping from X to itself such that*

$$(P(\xi),\xi)_X > 0 \text{ for every } \xi \text{ with } \|\xi\|_X = k > 0.$$

Then there exists $\xi_0 \in X$ such that $\|\xi_0\|_X \leq k$ and

$$P(\xi_0) = 0.$$

Proof. The proof follows from the Brouwer fixed point theorem. Suppose that P has no zero in the closed ball B of radius k and centered at origin. Define the mapping S

$$S(\xi) = -k\frac{P(\xi)}{\|P(\xi)\|_X}.$$

This S is continuous, since P is continuous, and it clearly maps B to itself, and B is convex and compact as a closed subset of a finite dimensional Banach space X. Thus, S satisfies the assumptions of the Brouwer fixed point theorem, and S has a fixed point $\xi_0 \in B$ such that

$$-k\frac{P(\xi_0)}{\|P(\xi_0)\|_X} = \xi_0,$$

for which $\|\xi_0\|_X = \|-k\frac{P(\xi_0)}{\|P(\xi_0)\|_X}\|_X = k$, and on the other hand

$$\|\xi_0\|_X^2 = (\xi_0,\xi_0)_X = (-k\frac{P(\xi_0)}{\|P(\xi_0)\|_X},\xi_0) = -k\frac{(P(\xi_0),\xi_0)}{\|P(\xi_0)\|_X},$$

and thus,

$$\underbrace{\|\xi_0\|_X^2}_{\geq 0}\underbrace{\|P(\xi_0)\|_X}_{\geq 0} = -\underbrace{k}_{>0}\underbrace{(P(\xi_0),\xi_0)}_{>0},$$

and we have a contradiction. Therefore, the proof of Lemma 4.8 is finished. $\square$

Lemma 4.9. *If u_l converges to u in $\{L^2(\Omega)\}^n$ strongly, then*

$$b(u_l, u_l, v) \to b(u, u, v) \quad \forall\, v \in \mathcal{V}. \tag{4.19}$$

Proof. By Lemma 4.6, $b(u, u, v) = -b(u, v, u)$, $u, v \in V$. Thus because $v \in \mathcal{V} \subset V$ and $u_l \in V$, we have

$$b(u_l, u_l, v) = -b(u_l, v, u_l) = -\sum_{i,j=1}^{n} u_{l,i} u_{l,j} D_i v_j \, dx.$$

Because $\|u_{l,i} - u_i\|_{L^2(\Omega)} \to 0$ by assumption and $D_i v_j \in L^\infty(\Omega)$, we have

$$\begin{aligned}
&|\int_\Omega u_{l,i} u_{l,j} D_i v_j \, dx - \int_\Omega u_i u_j D_i v_j \, dx| \\
&\qquad = |\int_\Omega (u_{l,i} u_{l,j} - u_i u_j) D_i v_j \, dx| \\
&\qquad \le \underbrace{\|D_i v_j\|_{L^\infty(\Omega)}}_{=:M<\infty} \int_\Omega |u_{l,i} u_{l,j} - u_i u_j| \, dx \\
&\qquad = M \int_\Omega |u_{l,i} u_{l,j} - u_{l,j} u_i + u_{l,j} u_i - u_i u_j| \, dx \\
&\qquad \le M \int_\Omega |u_{l,j}||u_{l,i} - u_i| + |u_i||u_{l,j} - u_j| \, dx \\
&\qquad = M \int_\Omega |u_{l,j}||u_{l,i} - u_i| \, dx + \int_\Omega |u_i||u_{l,j} - u_j| \, dx \\
&\qquad \le M\|u_{l,j}\|_{L^2(\Omega)} \|u_{l,i} - u_i\|_{L^2(\Omega)} + M\|u_i\|_{L^2(\Omega)} \|u_{l,j} - u_j\|_{L^2(\Omega)}.
\end{aligned}$$

But $u_{l,j} \in L^2(\Omega)$ and $u_i \in L^2(\Omega)$, and thus, the expression above converges to 0 as $l \to \infty$. This implies (4.19) in Lemma 4.9. $\square$

For the uniqueness of the solution of (4.13), we only have the following result [6, p. 167].

Theorem 4.13. *If $n \le 4$ and μ is sufficiently large or f sufficiently small such that*

$$\mu^2 > c(n)\|f\|_{V'}, \tag{4.20}$$

then there exists a unique solution u of (4.13). $c(n)$ is a constant from the estimate for the form b in the proof of Lemma 4.5.

Proof. In this case $n \leq 4$, we have $V = \tilde{V}$. Furthermore, if u satisfies (4.15), it satisfies (4.13). Choosing $v = u$ in (4.15), we have

$$\mu((u,u)) + \underbrace{b(u,u,u)}_{=0} = \langle f, u \rangle \leq \|f\|_{V'} \|u\|.$$

Thus, for any solution u of problem (4.15), we have

$$\|u\| \leq \frac{1}{\mu} \|f\|_{V'}.$$

Let then u_1 and u_2 be two solutions of (4.15). Define $u = u_1 - u_2$. We have

$$\mu((u_1, v)) + b(u_1, u_1, v) = \langle f, v \rangle \quad \forall\, v \in V,$$

and

$$\mu((u_2, v)) + b(u_2, u_2, v) = \langle f, v \rangle \quad \forall\, v \in V.$$

We obtain

$$\mu((u,v)) + \underbrace{b(u_1, u_1, v) + (-b(u_1, u_2, v)}_{=b(u_1,u,v)}$$
$$+ \underbrace{b(u_1, u_2, v)) - b(u_2, u_2, v)}_{=b(u,u_2,v)} = 0 \quad \forall\, v \in V,$$

and thus

$$\mu((u,v)) + b(u_1, u, v) + b(u, u_2, v) = 0 \quad \forall\, v \in V.$$

Let us choose again $v = u$. We have $b(u_1, u, u) = 0$ and obtain

$$\mu \|u\|^2 + b(u, u_2, u) = 0.$$

For $b(u, u_2, u)$, we have the estimate

$$|b(u, u_2, u)| \leq c(n) \|u_2\| \|u\|^2.$$

From the observations above, we obtain (u_2 is a solution)

$$\mu \|u\|^2 \leq c(n) \|u_2\| \|u\|^2 \leq \frac{c(n)}{\mu} \|f\|_{V'} \|u\|^2.$$

And finally,

$$(\mu - \frac{c(n)}{\mu} \|f\|_{V'}) \|u\|^2 \leq 0.$$

But by assumption (4.20), we conclude that $\|u\| = 0$. And hence, $u_1 = u_2$ a.e. $\square$

In the case of lower dimensions, we have a strong result for the regularity of the solution. For the proof of Proposition 4.2 below, we refer to [6, p. 172].

Proposition 4.2. *Let Ω be an open set of class $\mathcal{C}^\infty$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let f be given in $\mathcal{C}^\infty(\overline{\Omega})$. Then any solution $\{u, p\}$ of equation (4.12) belongs to $\mathcal{C}^\infty(\overline{\Omega}) \times \mathcal{C}^\infty(\overline{\Omega})$.*

4.4 Evolution Navier-Stokes equations

Let us first study the linear case of the evolution Navier-Stokes equations. Let Ω be an open, bounded, $\mathcal{C}^1$ smooth set in $\mathbb{R}^n$, let $T > 0$ be fixed. Denote $Q = \Omega \times (0, T)$. Let us formulate the boundary value problem:

To find a vector function $u : \Omega \times [0, T] \to \mathbb{R}^n$, a scalar function $p : \Omega \times [0, T] \to \mathbb{R}$ such that

$$D_t u - \mu \Delta u + Dp = f \text{ in } \Omega \times (0, T), \tag{4.21a}$$

$$\operatorname{div} u = 0 \text{ in } \Omega \times (0, T), \tag{4.21b}$$

$$u = 0 \text{ on } \partial\Omega \times [0, T], \tag{4.21c}$$

$$u(x, 0) = u_0(x) \text{ in } \Omega. \tag{4.21d}$$

Here, $f : \Omega \times [0, T] \to \mathbb{R}^3$ and $u_0 : \Omega \to \mathbb{R}^3$ are given.

Suppose that u and p are classical solutions of (4.21), $u \in \mathcal{C}^2(\overline{Q})$ and $p \in \mathcal{C}^1(\overline{Q})$. Let $v \in \mathcal{V}$. As in the case of the Stokes equations, we obtain by taking the scalar product of (4.21a) with v the following equality:

$$(D_t u, v) + \mu((u, v)) = (f, v) \quad \forall\, v \in \mathcal{V}.$$

By continuity, this equation holds for each $v \in V$ also.

By Lemma 3.1, we have

$$\begin{aligned} \tfrac{d}{dt}(u, v) &= \sum_{i=1}^{n} \tfrac{d}{dt} \int_\Omega u_i v_i \, dx = \sum_{i=1}^{n} \int_\Omega D_t(u_i v_i) \, dx \\ &= \sum_{i=1}^{n} \int_\Omega (D_t u_i) v_i \, dx = (D_t u, v). \end{aligned}$$

We define the following problem.

Definition 4.6. *For f and u_0 given, $f \in L^2(0,T;V')$, $u_0 \in H$, to find u satisfying*

$$u \in L^2(0,T;V), \tag{4.22a}$$
$$\tfrac{d}{dt}(u,v) + \mu((u,v)) = \langle f, v\rangle \quad \forall\, v \in V, \tag{4.22b}$$
$$u(0) = u_0. \tag{4.22c}$$

Another alternative formulation of problem (4.21) is:
For f and u_0 given, $f \in L^2(0,T;V')$, $u_0 \in H$, to find u satisfying

$$u \in L^2(0,T;V),\ u' \in L^2(0,T;V'), \tag{4.23a}$$
$$u' + \mu A u = f \text{ on } (0,T) \text{ in the weak sense}, \tag{4.23b}$$
$$u(0) = u_0. \tag{4.23c}$$

Lemma 4.10. *The following are equivalent.*

1. *u satisfies problem* (4.22)*.*
2. *u satisfies problem* (4.23)

Proof. By representations (4.6) and (4.7), we may write (4.22b) as

$$\tfrac{d}{dt}\langle u, v\rangle = \langle f - \mu A u, v\rangle \quad \forall\, v \in V. \tag{4.24}$$

Since A is linear and continuous from V to V' and $u \in L^2(0,T;V)$, the function Au belongs to $L^2(0,T;V')$. Thus, $f - \mu Au \in L^2(0,T;V')$. By (4.24) and Lemma 4.3, the weak time derivative of u, u', belongs to $L^2(0,T;V')$. Furthermore, by Lemma 4.3, the equality (4.24) implies that

$$u' = f - \mu A u$$

in the weak sense.

Conversely, if u satisfies (4.23), we see easily that u satisfies (4.22b) for all $v \in V$. □

Remark 4.9. *By Lemma 4.3, $u \in L^2(0,T;V)$ solving the weak problems is almost everywhere equal to an absolutely continuous function from $[0,T]$ to V'. Any function satisfying* (4.22a) *and* (4.22b) *is, after modification on a set measure zero, a continuous function from $[0,T]$ into V'. Therefore, condition* (4.22c) *makes sense.*

For the existence of the weak solution of problem (4.23), we have the following two theorems proved in [6].

Theorem 4.14. *For given f and u_0, $f \in L^2(0,T;V')$, $u_0 \in H$, there exists a unique function u which satisfies* (4.23). *Moreover,*

$$u \in \mathcal{C}([0,T];H).$$

The following theorem shows in which sense the solutions of equation (4.23) given by Theorem 4.14 solve equation (4.21).

Theorem 4.15. *Let the assumptions be as in Theorem 4.14. Then, there exists functions u and p defined in Q such that the function u defined by Theorem 4.14, and p satisfy* (4.21a) *in the weak sense in Q. Furthermore,* (4.21b) *is satisfied in the weak sense, and* (4.21d) *is satisfied in the sense that*

$$u(t) \to u_0 \text{ in } L^2(\Omega) \text{ as } t \to 0.$$

Consider now the full Navier-Stokes equations and the case $n \leq 4$. In its classical formulation, the initial boundary value problem of the full Navier-Stokes equations is the following:

To find a vector function $u : \Omega \times [0,T] \to \mathbb{R}^n$ and a scalar function $p : \Omega \times [0,T] \to \mathbb{R}$ such that

$$D_t u - \mu\Delta u + u \cdot Du + Dp = f \text{ in } \Omega \times (0,T), \tag{4.25a}$$

$$\operatorname{div} u = 0 \text{ in } \Omega \times (0,T), \tag{4.25b}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T), \tag{4.25c}$$

$$u(x,0) = u_0(x), \text{ in } \Omega. \tag{4.25d}$$

Functions f and u_0 are given and defined on $\Omega \times [0,T]$ and Ω respectively.

We formulate the following problem (4.25), introduced by Jean Leray.

Definition 4.7. *Let $f \in L^2(0,T;V')$ and $u_0 \in H$ be given. To find u satisfying*

$$u \in L^2(0,T;V), \tag{4.26a}$$

$$\tfrac{d}{dt}(u,v) + \mu((u,v)) + b(u,u,v) = \langle f, v\rangle \quad \forall\, v \in V, \tag{4.26b}$$

$$u(0) = u_0. \tag{4.26c}$$

In the following, we show that if u satisfies equation (4.25) then u solves problem (4.26) of it. Assume that u and p are the classical solutions of (4.25), $u \in \mathcal{C}^2(\overline{Q})$, $p \in \mathcal{C}^1(\overline{Q})$. Then $u \in L^2(0,T;V)$, and for each $v \in \mathcal{V}$, we obtain similarly to the case above that

$$\tfrac{d}{dt}(u,v) + \mu((u,v)) + b(u,u,v) = \langle f, v \rangle.$$

By continuity, this holds for each $v \in V$.

Lemma 4.11. *Assume that u belongs to $L^2(0,T;V)$ ($n \le 4$). Then, the function Bu defined by*

$$\langle Bu(t), v \rangle = b(u(t), u(t), v) \quad \forall\, v \in V, \ \textit{a.e. } t \in [0,T],$$

belongs to $L^1(0,T;V')$.

We refer to [6] for the the proof.

Problem (4.26) is also stated as follows:

Let f and u be given with $f \in L^2(0,T;V')$ and $u_0 \in H$. To find u satisfying

$$u \in L^2(0,T;V), \ u' \in L^1(0,T;V'), \tag{4.27a}$$

$$u' + \mu A u + Bu = f \text{ on } (0,T) \text{ in the weak sense}, \tag{4.27b}$$

$$u(0) = u_0. \tag{4.27c}$$

In the following, we show that if u satisfies problem (4.26) then it satisfies problem (4.27), and vice versa. Let u satisfy (4.26a) and (4.26b). Then, by (4.6) and (4.7), we may write (4.26b) as

$$\tfrac{d}{dt}\langle u, v \rangle = \langle f - \mu A u - Bu, v \rangle \quad \forall\, v \in V.$$

Since $Au \in L^2(0,T;V')$, as in the linear case, the function $f - \mu Au - Bu$ belongs to $L^1(0,T;V')$ by Lemma 4.11. Lemma 4.3 implies that

$$\begin{aligned} &u' \in L^1(0,T;V'), \\ &u' = f - \mu A u - Bu \text{ in the weak sense}, \end{aligned}$$

Then, u is almost everywhere equal to a continuous function from $[0,T]$ into V'. Thus condition (4.26c) makes sense.

The converse is seen easily.

For the full Navier-Stokes equations we have the following existence result. For the proof we refer to [6].

Theorem 4.16. *Let $n \leq 4$. Let f and u be given with $f \in L^2(0,T;V')$ and $u_0 \in H$. Then there exists at least one function u satisfying* (4.27)*. Moreover,*

$$u \in L^\infty(0,T;H)$$

and u is weakly continuous from $[0,T]$ into H, that is, $\forall\, v \in H$, $t \mapsto (u(t), v)$ is a continuous scalar function.

Remark 4.10. *Theorem 4.16 does not say anything about the existence of p.*

In the two-dimensional case, the solution u of problems (4.26) *and* (4.27) *given by Theorem 4.16 is unique. Moreover u is almost everywhere equal to a function continuous from $[0,T]$ into H and $u_0(t) \to u_0$ in H as $t \to 0$.*

In the three-dimensional case, the uniqueness of the solution u given by Theorem 4.16 is not known. The uniqueness has been shown in a class of functions where the existence is not known.

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