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# MAPPINGS OF FINITE DISTORTION: FORMATION OF CUSPS

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I dedicate this work to my loving family: Kaisa, Leo and Nooa.

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Juhani Takkinen

## List of included articles

This dissertation consists of an introductory part and the following publications:

[A] P. Koskela and J. Takkinen, *Mappings of finite distortion: formation of cusps*, Publ. Mat. **51** (2007), no. 1, 223–242.

[B] J. Takkinen, *Mappings of finite distortion: Formation of cusps II*, Conform. Geom. Dyn. **11** (2007), 207–218

[C] P. Koskela and J. Takkinen, *Mappings of finite distortion: formation of cusps III*, Preprint 351, Department of Mathematics and Statistics, University of Jyväskylä, 2007.

In this introductory part these articles will be referred to [A], [B],. . . , whereas other references will be numbered as [1], [2],. . . .

The author of this dissertation has actively taken part in research of the joint papers [A] and [C].

# Introduction

## 1. Quasiconformal mappings

Let $\Omega$ and $\Omega'$ be domains, i.e. open and connected subsets, in $\mathbb{R}^n$ and $1 \leq K < \infty$ a constant. We say that a homeomorphism $f\colon \Omega \to \Omega'$ is ($K$-)quasiconformal if the following properties are satisfied:

(i) $f \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$,
(ii) $\|Df(x)\|^n \leq K J_f(x)$ for almost every $x \in \Omega$.

Here $Df(x)$ denotes the derivative of $f$ at $x \in \Omega$, $\|\cdot\|$ is the operator norm and $J_f(x)$ is the Jacobian determinant of $f$ at $x \in \Omega$. Note that for any homeomorphism $f \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, $J_f(x)$ is locally integrable and $Df(x)$ exists almost everywhere. The inequality (ii) in the above definition also implies that $J_f(x) \geq 0$ almost everywhere in $\Omega$ and thus the mappings in question are sense preserving.

Our interest will be in the planar case ($n = 2$), where the theory of quasiconformal mappings is nowadays quite well understood [2, 13, 15]. One part of the theory is the extendability of a quasiconformal mapping defined in the unit disk to a quasiconformal mapping of the entire plane. This property is described completely by the concept of a quasidisk.

## 2. Quasidisks

A ($K$-)quasidisk is the image of the unit disk or a half plane under a ($K$-)quasiconformal mapping of the entire plane. In fact, as already suggested, it is true that each quasiconformal mapping $f\colon B \to \Omega$ extends to a quasiconformal mapping of the entire plane if and only if $\Omega$ is a quasidisk [13]. What makes this all interesting is the fact that there exist at least a dozen essentially different characterizations for quasidisks that do not utilize the concept of quasiconformality [4]. Perhaps one of the most famous of them is obtained by using the so-called Ahlfors three point property.

2.1. **Ahlfors three point property.** We say that a domain $D \subset \overline{\mathbb{R}}^2$ admits a quasiconformal reflection in its boundary $\partial D$ if there exists a homeomorphism $f$ of $\overline{D}$, whose restriction to $D$ is anti-quasiconformal, such that $f(D) = \overline{\mathbb{R}}^2 \setminus \overline{D}$ and $f$ is identity on $\partial D$. By anti-quasiconformal we mean that the mapping is sense reversing, i.e. we have $J_f(x) \leq 0$ for almost all $x \in D$, but in this case the condition (ii) in the definition of quasiconformality holds for $-J_f(x)$ instead of $J_f(x)$.

In [1], Ahlfors showed that a Jordan domain $D$ admits a quasiconformal reflection if and only if it satisfies the three point property. By using the formulation used by Gehring in [4], this property can be conveniently stated as follows: A domain $D \subset \overline{\mathbb{R}}^2$ has the three point property if it is a Jordan domain and if there exists a constant $C$ such that for each pair of finite points $P_1$, $P_2 \in \partial D$,

$$\min_{i=1,2} \operatorname{diam}(\gamma_i) \leq C|P_1 - P_2|.$$

where $\gamma_1$, $\gamma_2$ are the components of $\partial D \setminus \{P_1, P_2\}$. This condition implies that $|P_1 - P_3| \leq C|P_1 - P_2|$ for each point $P_3$ in the component of $\partial D \setminus \{P_1, P_2\}$ with the minimum diameter. The latter property is similar to the condition that Ahlfors gave in [1], from where the name "three point property" is derived.

On the other hand, a domain $D \subset \overline{\mathbb{R}}^2$ is a quasidisk if and only if it admits a quasiconformal reflection in $\partial D$ (cf. [5]). Thus by combining these two equivalences, we arrive at a characterization of quasidisks in terms of the three point property: *A Jordan domain $D$ is a quasidisk if and only if the boundary $\partial D$ satisfies the three point property.*

2.2. **Typical example of a standard cusp.** Although the boundary of a quasidisk has always zero planar measure, it can still be in some sense quite "wild". For example, the Hausdorff dimension of the boundary can be arbitrarily close to 2 (cf. [4], Example 3.2). Still, one quite easily observes that the aforementioned three point property rules out cusps, i.e. points where the boundary forms a zero angle. The domain in Figure 1 exhibits what we call a standard cusp.

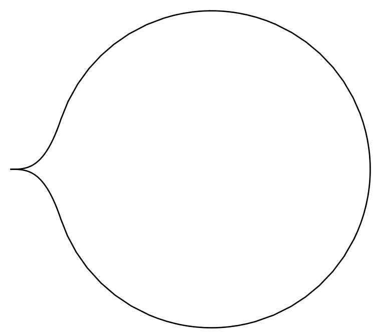

FIGURE 1. Example of a standard cusp domain.

We parametrize the degree of the cusp by $s > 0$ and define our model domains $\Omega_s \subset \mathbb{R}^2$ by setting

$$\Omega_s = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, \, |x_2| < x_1^{1+s}\} \cup B(x_s, r_s),$$

where $x_s = (s + 2, 0)$ and $r_s = \sqrt{(s + 1)^2 + 1}$. As the domains $\Omega_s$ are not quasidisks, they cannot be the images of a unit disk or half plane under quasiconformal mappings of the entire plane. One is then motivated to ask what happens when we move from quasiconformal mappings to a more general class of homeomorphisms.

## 3. Mappings of finite distortion

A mapping $f\colon \Omega \to \Omega'$ between domains $\Omega$, $\Omega' \subset \mathbb{R}^n$ is called a mapping of finite distortion, if for some measurable $K\colon \Omega \to [1,\infty]$ that is finite almost everywhere, the following properties are satisfied:

(i) $f \in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^n)$,
(ii) $\|Df(x)\|^n \le K(x)J_f(x)$ for almost every $x \in \Omega$,
(iii) $J_f(x) \in L^1_{\text{loc}}(\Omega)$.

In the recent years these mappings have gathered much interest, especially the case when one assumes that the distortion function $K(x)$ satisfies

$$\exp(\lambda K(x)) \in L^1_{\text{loc}}(\Omega) \text{ for some } \lambda > 0. \tag{1}$$

With this assumption these mappings are shown to have many good properties, e.g. they satisfy the condition N, are continuous, open and discrete, to mention some (cf. [8, 10, 11]).

As noted before, we know (precisely) which Jordan domains are the images of the unit disk under quasiconformal mappings of the entire plane. It would be of interest to to know if something similar holds for homeomorphic mappings of finite distortion. In this setting, the image of the unit disk is always a Jordan domain, but the Hausdorff dimension of $f(\partial B)$ can be 2. Towards this goal, it has been considered proper to first study some fixed example domains that are in some sense "almost" quasidisks. Hence the choice of the domains $\Omega_s$ as the preferred images.

As we have made the restriction to the planar case, i.e. $\Omega$, $\Omega' \subset \mathbb{R}^2$, we only need to assume that a homeomorphism $f\colon \Omega \to \Omega'$ satisfies

(i) $f \in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^2)$ and
(ii) $\|Df(x)\|^2 \le K(x)J_f(x)$ for almost every $x \in \Omega$,

in order for it to be a mapping of finite distortion. This is because of the facts that a planar Sobolev homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^2)$ is differentiable almost everywhere and its Jacobian determinant is locally integrable (cf. [9, 13]). Now, if we would assume $K(x)$ to be bounded, we would recover the class of quasiconformal mappings as defined in the beginning. The initiation of the study of homeomorphisms with locally exponentially integrable distortion is usually credited to David, who in [3] considered homeomorphic solutions of the Beltrami equation.

It was shown by Haïssinsky in [6] that cusps can be formed by homeomorphisms of finite distortion, even when (1) is satisfied. In the included articles we have considered homeomorphisms of finite distortion with the additional assumption that either (1) is satisfied or that the distortion function of the inverse $f^{-1}$ satisfies $K_{f^{-1}}(x) \in L^p_{\text{loc}}(\mathbb{R}^2)$ for some $p \ge 1$. We have shown that if $f(B) = \Omega_s$ for this kind of a mapping, then the degree $s$ of the cusp gives a bound for the parameters $\lambda$ and $p$, or conversely. Also, example mappings have been constructed in order to evaluate the sharpness of these bounds.

## 4. Results

In this section we gather the main theorems from the included articles [A], [B] and [C].

For a (planar) homeomorphism $f$ of finite distortion, it is convenient to denote the optimal distortion function by $K_f$. It is obtained by setting $K_f(x) = \|Df(x)\|^2/J_f(x)$ when $Df(x)$ exists and $J_f(x) > 0$ and by letting $K_f(x) = 1$ otherwise.

4.1. **Formation of cusps.** In what follows, $B$ denotes the unit disk. The main theorem of the first included article [A] deals with the case when $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of finite distortion that is quasiconformal on $B$. In this case the assumption of exponentially integrable distortion of the distortion function $K_f(x)$ leads to an essentially sharp result, stated as follows.

**Theorem 1.** *Let $f\colon B \to \Omega_s$ be a quasiconformal mapping and assume that $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphic extension of $f$. If $\hat{f}$ is a mapping of finite distortion, then*

$$\int_{2B} \exp(\lambda K_f(x))\, dx = \infty \qquad \textit{for all } \lambda > 1/s.$$

*Conversely, there exists a quasiconformal mapping $f\colon B \to \Omega_s$, which extends to a homeomorphism of finite distortion $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with*

$$\int_{2B} \exp(\lambda K_f(x))\, dx < \infty \qquad \textit{for all } \lambda < 1/s.$$

In the second included article [B], the assumption of quasiconformality is abandoned. More precisely, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is assumed only to be a homeomorphism of finite, exponentially integrable distortion.

**Theorem 2.** *For $s > 0$ given, there is no homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion with $\exp(\lambda K_f) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ so that $f(B) = \Omega_s$ when $\lambda > 4/s$. On the other hand, for $\lambda < 2/s$ such an $f$ exists.*

In this case, as seen above, the result is not as optimal as it was in the case of partial quasiconformality. The constructed example suggests that the correct border case for $\lambda$ should be $2/s$, but this remains to be proven.

Notice that the bound on $\lambda$ in Theorem 1 is $1/s$ and that the optimal bound under the setting of Theorem 2 is between $2/s$ and $4/s$. This is slightly surprising because of the results for quasiconformal mappings. Indeed, $\Omega$ is a $K$-quasidisk if and only if $\Omega$ is the image of $B$ (or a half plane) under a $K^2$-quasiconformal mapping which is conformal in $B$ (or in the half plane). This follows by combining results from [4] and [5].

In the last included article [C] we consider the distortion of the inverse function. As the inverse of a homeomorphism of finite distortion is also of finite distortion [7], it is of interest to consider how assumptions on the distortion function of $f^{-1}$ affect on the cusp formation of $f$. Again, the example domain is $\Omega_s$ and $f$ may or may not be quasiconformal on $B$. The results are stated as two theorems.

**Theorem 3.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of finite distortion such that $K_{f^{-1}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for some $1 \le p < \infty$. If $f(B) = \Omega_s$ for some $s > 0$, then necessarily $s \le 4/(p-1)$. If, in addition, $f$ is assumed to be quasiconformal on $B$, then $s \le 2/p$.*

**Theorem 4.** *For $s > 0$ given, there exists a homeomorphism of finite distortion, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, with $f(B) = \Omega_s$, such that $K_{f^{-1}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p < 2/s + 1$. If one only requires that $K_{f^{-1}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p < 2/s$, then $f$ can be made quasiconformal on $B$.*

One immediately sees that in the quasiconformal case one has an essentially sharp bound, but that the general case exhibits again some gap. Again, it seems that the correct bound for the general case in Theorem 3 should be $s \le 2/(p-1)$, but as before, this remains to be proven.

4.2. **Modulus of continuity estimates.** Here we present some new modulus of continuity results from [A] and [B] as they play a central role in the methods used to prove Theorems 1 and 3. Theorem 2 utilizes a known modulus of continuity result by Koskela, Onninen and Zhong [12, 14], which states that the sharp modulus of continuity for a mapping of finite, exponentially integrable distortion is of the type $\log^{-\lambda/n}(1/|x-y|)$, when the dimension of the space is $n$.

The first result is from [A] and it shows that if $f$ is assumed to be quasiconformal on $B$, then in the planar case instead of the bound $\log^{-\lambda/2}(1/|x-y|)$ we get with an arbitrary $\varepsilon > 0$ a modulus of continuity of the type $\log^{-\lambda/(1+\varepsilon)}(1/|x-y|)$ on the closed unit disk $\overline{B}$. Note that even if one were able to take $\varepsilon = 0$, the technique used to prove Theorem 1 would not allow one to improve on the result of Theorem 1. An example from [A] shows that we cannot allow $\varepsilon$ to be negative.

**Theorem 5.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of finite distortion such that $\exp(\lambda K_f(x)) \in L^1(2B)$ for some $\lambda > 0$. If the restriction of $f$ to the open unit disk $B$ is quasiconformal, then for any $\varepsilon > 0$ there exist positive constants $\hat{C}$ and $\tilde{C} \ge 2$ such that*

$$|f(x) - f(y)| \le \frac{\hat{C}}{\log^{\frac{\lambda}{1+\varepsilon}} \frac{\tilde{C}}{|x-y|}}, \tag{2}$$

*whenever $x, y \in \overline{B}$.*

The final theorem of this review is from [C] and it provides a modulus of continuity estimate for a homeomorphism of finite distortion in the case when one has a local $L^p$-integrability assumption on the distortion of the inverse of $f$. Simple examples show that the indicated modulus of continuity is sharp.

**Theorem 6.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ a homeomorphism of finite distortion. If $K_{f^{-1}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for some $1 \le p < \infty$, then for all $|x-y| < 1/2$*

$$|f(x) - f(y)| \le \frac{C(p, \|K_{f^{-1}}\|_{L^p(G)})}{\log^{\frac{p}{2}}(1/|x-y|)}, \tag{3}$$

*where $G = f(B(x,1))$.*

## 5. Final comments

The results of this dissertation hopefully provide a starting point for understanding the geometry of the image of the unit disk under a homeomorphism of finite distortion. A possible next step could be to try to find an optimal analog for the three point property. Also, the role of reflections could be studied.

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