ITERATED FUNCTION SYSTEMS: NATURAL MEASURE AND LOCAL STRUCTURE

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List of included articles

This thesis consists of an introductory part and the following two publications:

[**C**] A. Käenmäki, *On the geometric structure of the limit set of conformal iterated function systems*, Publ. Mat. **47** (2003), 133–141.

[**E**] A. Käenmäki, *On natural invariant measures on generalised iterated function systems*, Preprint 279, Department of Mathematics and Statistics, University of Jyväskylä, 2003.

Iteroidun funktiosysteemin rajajoukon rakenteesta ja luonnollisesta mitasta

Tässä väitöskirjatyössä, joka siis koostuu artikkeleista [**C**] ja [**E**], on tutkittu iteroidun funktiosysteemin (IFS) rajajoukkoa. Erityisen kiinnostuksen kohteena on ollut rajajoukon rakenne ja rajajoukolle luonnollisen mitan olemassaolo.

Iteroitujen funktiosysteemien historia juontaa juurensa 80–luvun alkuun, jolloin Hutchinson antoi IFS:lle muodollisen määritelmän. Tästä lähtien itse–similaarit joukot, jotka ovat ehkä tunnetuinpia esimerkkejä IFS:ien rajajoukoista, ovat olleet mielenkiinnon kohteena. Itse–similaarisuudella tarkoitetaan siis sitä, että joukko sisältää kopioita itsestään useissa eri mittakaavoissa. Ilmiönä itse–similaarisuus esiintyy luonnossa hyvin usein ja tyypillisiä esimerkkejä ovat rantaviiva, keuhkot ja saniaisen lehti. Sovelluksia tälle teorialle on löytynyt esimerkiksi kuvankäsittelystä — rasterikuvan koon suurentaminen onnistuu paremmin, jos pystytään sanomaan jotain kuvan rakenteesta. Matemaatikoille itse–similaarit joukot ovat tarjonneet suhteellisen helpon tavan muodostaa hyvin epäsäännöllisiä joukkoja. Sen lisäksi, että tämän kaltaisten ns. fraktaalijoukkojen ominaisuudet ovat jo sellaisenaankin mielenkiintoisia, ovat ne hyödyllisiä myös muilla matematiikan osa–alueilla.

Iteroitu funktiosysteemi on itse asiassa vain kokoelma vaadittavat oletukset täyttäviä kuvauksia ja rajajoukko ns. symboliavaruuden projektio. Tämä projektio muodostetaan käyttämällä hyväksi IFS:n kuvauksia. Usein onkin helpompaa tutkia symboliavaruutta pyrkien projisoimaan vastaava tilanne rajajoukolle kuin tarkastella suoraan rajajoukkoa. Itse–similaari joukko on rajajoukko iteroidusta funktiosysteemistä, jonka kuvauksiksi on valittu yksinkertaisimmat mahdolliset kuvaukset, similariteetit. Onkin mielenkiintoista tarkastella yleisempien kuvausten tuottamia rajajoukkoja. Tässä väitöskirjassa on yleistetty joitain itse–similaareille joukoille tunnettuja tuloksia itse–konformisille ja itse–affiineille joukkoille. Artikkelissa [**C**] osoitetaan, että itse–konforminen joukko joko sisältyy l–ulotteiseen sileään pintaan tai leikkaa jokaista tällaista pintaa pienessä joukossa. Artikkelilla [**E**] on ollut useita tavoitteita. Ensinnäkin siellä esitellään IFS–tyylinen merkintätapa, jota voidaan käyttää hyvin yleisten IFS:ien rajajoukkojen tutkimiseen. Käyttäen tätä merkintätapaa osoitetaan ergodisen ja invariantin tasapainomitan olemassa olo. Tämä yleistää itse–similaareille ja itse–konformisille joukoille jo tunnetun tuloksen hyvinkin yleisille rajajoukoille. Erityisesti itse–affiineille joukoille tämä on hyödyllinen tulos. Artikkelissa osoitetaankin sovelluksena, että tyypillisille itse–affiineille joukoille on olemassa ergodinen ja invariantti mitta, jolla on täysi dimensio. Tämä antaa myös osittaisen positiivisen vastauksen Kenyonin ja Peresin 90–luvun puolivälissä esittämään avoimeen ongelmaan.

1. Introduction

In 1981, Hutchinson [**5**] introduced the formal definition of iterated function systems (IFS). Some ideas in this direction have been presented also earlier, especially in early works of Cantor and also by Moran [**13**] and Mandelbrot [**7**]. Since then self–similar sets, the limit sets of the so called similitude IFS's, have aroused great interest. By self–similarity we mean that the set contains copies of itself on many different scales. As a phenomenon, self–similarity often occurs in nature; a shoreline, lungs and a fern are common examples of these kind of objects. For mathematicians, the setting of a similitude IFS has provided a sufficiently easy environment to produce highly irregular sets. Besides that these sets and their properties are interesting on their own, they are used extensively in many other areas of analysis. Since self–similar sets are widely studied and with suitable extensions to this setting we can achieve, for example, so called Julia sets, it is interesting to study limit sets of more general systems.

This thesis consists of the following two articles, [**C**] and [**E**]. The first describes the local geometric structure of the limit set of conformal IFS's, the so called self–conformal set. This continues the works of Mattila [**8**], Springer [**17**] and Mauldin, Mayer and Urbański [**9**] and [**12**]. The second article has three main goals. Besides that, an IFS–style notation is introduced for studying measures on the limit set of very general IFS's, including affine and conformal systems, among others, the existence of a natural Borel probability measure is proved in this setting. "Naturality" here means that the measure is ergodic, invariant and satisfies the so called equilibrium state. As an application, it is shown that for typical self–affine sets there exists an ergodic invariant measure having the same Hausdorff dimension as the set itself. These results continue the works of Falconer [**4**], Barreira [**1**] and Kenyon and Peres [**6**].

2. Setting

Let I be a finite set with at least two elements. Put $I^* = \bigcup_{n=1}^{\infty} I^n$ and $I^\infty = I^{\mathbb{N}} = \{(i_1, i_2, \ldots) : i_j \in I \text{ for } j \in \mathbb{N}\}$. Thus, if $\mathtt{i} \in I^*$, there is $k \in \mathbb{N}$ such that $\mathtt{i} = (i_1, \ldots, i_k)$, where $i_j \in I$ for all $j = 1, \ldots, k$. We call this k the *length* of $\mathtt{i}$ and we denote $|\mathtt{i}| = k$. If $\mathtt{j} \in I^* \cup I^\infty$, then with the notation $\mathtt{i}, \mathtt{j}$ we mean the element obtained by juxtaposing the terms of $\mathtt{i}$ and $\mathtt{j}$. If $\mathtt{i} \in I^\infty$, we denote $|\mathtt{i}| = \infty$, and for $\mathtt{i} \in I^* \cup I^\infty$ we put $\mathtt{i}|_k = (i_1, \ldots, i_k)$ whenever $1 \le k < |\mathtt{i}|$. We define $[\mathtt{i}; A] = \{\mathtt{i}, \mathtt{j} : \mathtt{j} \in A\}$ as $\mathtt{i} \in I^*$ and $A \subset I^\infty$ and we call the set $[\mathtt{i}] = [\mathtt{i}, I^\infty]$ the *cylinder set of level* $|\mathtt{i}|$. We say that two elements $\mathtt{i}, \mathtt{j} \in I^*$ are *incomparable* if $[\mathtt{i}] \cap [\mathtt{j}] = \emptyset$. Furthermore, we call a set $A \subset I^*$ incomparable if all its elements are mutually incomparable.

Define

$$|\mathtt{i} - \mathtt{j}| = \begin{cases} 2^{-\min\{k-1 \,:\, \mathtt{i}|_k \ne \mathtt{j}|_k\}}, & \mathtt{i} \ne \mathtt{j} \\ 0, & \mathtt{i} = \mathtt{j} \end{cases} \tag{2.1}$$

whenever $\mathtt{i}, \mathtt{j} \in I^\infty$. Then the couple $(I^\infty, |\cdot|)$ is a compact metric space. Let us call $(I^\infty, |\cdot|)$ a *symbol space* and an element $\mathtt{i} \in I^\infty$ a *symbol*. If there is no danger of misunderstanding, let us also call an element $\mathtt{i} \in I^*$ a symbol. Define the *left shift* $\sigma : I^\infty \to I^\infty$ by setting

$$\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots). \tag{2.2}$$

Clearly, σ is continuous and surjective. If $\mathtt{i} \in I^n$ for some $n \in \mathbb{N}$, then with the notation $\sigma(\mathtt{i})$ we mean the symbol $(i_2, \ldots, i_n) \in I^{n-1}$.

The symbol space will provide us with a simple structured environment for finding measures with desired properties. It is, however, more interesting to study geometric projections of these measures and the symbol space. Let $X \subset \mathbb{R}^d$ be a compact set with nonempty interior. Choose then a collection $\{X_\mathtt{i} : \mathtt{i} \in I^*\}$ of closed subsets of X with nonempty interior satisfying

(1) $X_{\mathtt{i},i} \subset X_\mathtt{i}$ for every $\mathtt{i} \in I^*$ and $i \in I$,
(2) $d(X_\mathtt{i}) \to 0$, as $|\mathtt{i}| \to \infty$.

Here d means the diameter of a given set. Define now a *projection mapping* $\pi : I^\infty \to X$ such that

$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} X_{\mathtt{i}|_n} \tag{2.3}$$

as $\mathtt{i} \in I^\infty$. It is clear that π is continuous. We call the compact set $E = \pi(I^\infty)$ the *limit set* of this collection, and if there is no danger of misunderstanding, we also call the projected cylinder set a cylinder set.

To be able to say something about the geometric properties of the limit set, we first have to avoid in having too much overlapping among the sets $X_{\mathtt{i}}$. In other words, we need a decent separation condition. We say that a *strong separation condition (SSC)* is satisfied if $X_{\mathtt{i}} \cap X_{\mathtt{j}} = \emptyset$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable. Assuming the SSC would be enough in many cases, but it is a rather restrictive assumption, and usually we do not need that much. We say that an *open set condition (OSC)* is satisfied if $\mathrm{int}(X_{\mathtt{i}}) \cap \mathrm{int}(X_{\mathtt{j}}) = \emptyset$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable. With the notation $\mathrm{int}(A)$ we mean the interior of a given set A. Furthermore, we say that a *weak bounded overlapping* is satisfied if the cardinality of incomparable subsets of $\{\mathtt{i} \in I^* : x \in X_{\mathtt{i}}\}$ is bounded as $x \in X$. Trivially, the SSC implies the weak bounded overlapping.

We denote the collection of all Borel regular probability measures on I^∞ with $\mathcal{M}(I^\infty)$. Denote

$$\mathcal{M}_\sigma(I^\infty) = \{\mu \in \mathcal{M}(I^\infty) : \mu \text{ is invariant}\}, \tag{2.4}$$

where the invariance of μ means that $\mu([\mathtt{i}]) = \mu\big(\sigma^{-1}([\mathtt{i}])\big)$ for every $\mathtt{i} \in I^*$. Now $\mathcal{M}_\sigma(I^\infty)$ is a nonempty closed subset of the compact set $\mathcal{M}(I^\infty)$ in the weak topology. We will show that under the assumption of the weak bounded overlapping, we can have the same structure in the limit set as in the symbol space with respect to any projected invariant measure.

Theorem A ([**E**, Theorem 3.7]). *Suppose the weak bounded overlapping is satisfied. Then for $m = \mu \circ \pi^{-1}$, where $\mu \in \mathcal{M}_\sigma(I^\infty)$, we have*

$$m(X_{\mathtt{i}} \cap X_{\mathtt{j}}) = 0 \tag{2.5}$$

whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable.

3. Thermodynamical formalism

Take $t \geq 0$ and $\mathtt{i} \in I^*$. We call a function $\psi^t_{\mathtt{i}} : I^\infty \to (0,\infty)$ a *cylinder function* if it satisfies the following three conditions:

(1) There exists $K_t \geq 1$ not depending on $\mathtt{i}$ such that

$$\psi^t_{\mathtt{i}}(\mathtt{h}) \leq K_t \psi^t_{\mathtt{i}}(\mathtt{j}) \tag{3.1}$$

for any $\mathtt{h}, \mathtt{j} \in I^\infty$.

(2) For every $\mathtt{h} \in I^\infty$ and integer $1 \leq j < |\mathtt{i}|$ we have

$$\psi^t_{\mathtt{i}}(\mathtt{h}) \leq \psi^t_{\mathtt{i}|_j}\big(\sigma^j(\mathtt{i}), \mathtt{h}\big)\psi^t_{\sigma^j(\mathtt{i})}(\mathtt{h}). \tag{3.2}$$

(3) There exist constants $0 < \underline{s}, \overline{s} < 1$ such that
$$\psi^t_{\mathtt{i}}(\mathtt{h})\underline{s}^{\delta|\mathtt{i}|} \le \psi^{t+\delta}_{\mathtt{i}}(\mathtt{h}) \le \psi^t_{\mathtt{i}}(\mathtt{h})\overline{s}^{\delta|\mathtt{i}|} \tag{3.3}$$
for every $\mathtt{h} \in I^\infty$.

Note that when we speak about one cylinder function, we always assume there is a collection of them defined as $\mathtt{i} \in I^*$ and $t > 0$. Let us comment on these conditions. The first one is called the *bounded variation principle (BVP)*, and it says that the value of $\psi^t_{\mathtt{i}}(\mathtt{h})$ cannot vary too much; roughly speaking, $\psi^t_{\mathtt{i}}$ is essentially constant. The second condition is called the *subchain rule*. If the subchain rule is satisfied with equality, we call it a *chain rule*. The third condition is there just to guarantee the nice behaviour of the cylinder function with respect to the parameter t.

For fixed $\mathtt{h} \in I^\infty$, we call the following limit
$$P(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \psi^t_{\mathtt{i}}(\mathtt{h}) \tag{3.4}$$
a *topological pressure*. From the definition of the cylinder function it follows that the topological pressure is continuous, strictly decreasing and independent of $\mathtt{h}$. Define now for each $n \in \mathbb{N}$ a Perron–Frobenius operator $\mathcal{F}_{t,n}$ by setting
$$\big(\mathcal{F}_{t,n}(f)\big)(\mathtt{h}) = \sum_{\mathtt{i}\in I^n} \psi^t_{\mathtt{i}}(\mathtt{h}) f(\mathtt{i},\mathtt{h}) \tag{3.5}$$
for every continuous function $f : I^\infty \to \mathbb{R}$. Let us then denote with $\mathcal{F}^*_{t,n}$ the dual operator of $\mathcal{F}_{t,n}$. Due to the Riesz representation theorem it acts on $\mathcal{M}(I^\infty)$. Relying now on the definition of this operator, we find a special measure using a suitable fixed point theorem. If the chain rule is satisfied, this is a known result. For example, see Bowen [**2**], Sullivan [**18**] and Mauldin and Urbański [**10**].

Theorem B ([**E**, Theorem 2.5]). *For each $n \in \mathbb{N}$ and $t \ge 0$ there exists a measure $\nu_n \in \mathcal{M}(I^\infty)$ such that*
$$\nu_n([\mathtt{i};A]) = \Pi_n^{-1} \int_A \psi^t_{\mathtt{i}}(\mathtt{h}) d\nu_n(\mathtt{h}), \tag{3.6}$$
where $\mathtt{i} \in I^n$, $A \subset I^\infty$ is a Borel set and $\lim_{n\to\infty} \frac{1}{n}\log \Pi_n = P(t)$. Furthermore, if the cylinder function satisfies the chain rule, then $\nu_n = \nu$ for every $n \in \mathbb{N}$, where
$$\nu([\mathtt{i};A]) = e^{-|\mathtt{i}|P(t)} \int_A \psi^t_{\mathtt{i}}(\mathtt{h}) d\nu(\mathtt{h}) \tag{3.7}$$
as $\mathtt{i} \in I^*$ and $A \subset I^\infty$ is a Borel set.

The measure ν above is called a *t–conformal measure*. For a given $\mu \in \mathcal{M}_\sigma(I^\infty)$ we define an *energy* $E_\mu(t)$, by setting
$$E_\mu(t) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) \tag{3.8}$$
and an *entropy* h_μ by setting
$$h_\mu = -\lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \mu([\mathtt{i}]). \tag{3.9}$$

It follows that $P(t) \geq h_\mu + E_\mu(t)$ whenever $\mu \in \mathcal{M}_\sigma(I^\infty)$. An invariant measure which satisfies this inequality with equality is called a *t–equilibrium measure*. Furthermore, a measure $\mu \in \mathcal{M}(I^\infty)$ is *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for every Borel set $A \subset I^\infty$ for which $A = \sigma^{-1}(A)$. The main result of [**E**] is the following theorem.

Theorem C ([**E**, Theorems 2.6 and 4.1]). *For each $t \geq 0$ there exists an ergodic t–equilibrium measure.*

From now on, we will always assume that an equilibrium measure is chosen to be ergodic. The relationship between the equilibrium measure and the conformal measure is, of course, an interesting question. The following theorem gives a partial answer to that. See also Mauldin and Urbański [**10**].

Theorem D ([**E**, Theorem 2.8]). *Suppose that the cylinder function satisfies the chain rule and $t \geq 0$. Then*

$$K_t^{-1}\nu(A) \leq \mu(A) \leq K_t\nu(A) \tag{3.10}$$

for every Borel set $A \subset I^\infty$, where ν is a t–conformal measure and μ is a t–equilibrium measure.

Observe that the conformal measure satisfies $\nu([\mathtt{i};A]) = \int_A \psi_\mathtt{i}^t(\mathtt{h})d\nu(\mathtt{h})$ as $P(t) = 0$ and the projected conformal measure $m = \nu \circ \pi^{-1}$ has also $m(X_\mathtt{i} \cap X_\mathtt{j}) = 0$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable and the weak bounded overlapping is satisfied.

4. Iterated function system

Take $\Omega \supset X$ to be an open subset of $\mathbb{R}^d$. Let $\{\varphi_\mathtt{i} : i \in I^*\}$ be a collection of contractive injections from Ω to Ω such that the collection $\{\varphi_\mathtt{i}(X) : \mathtt{i} \in I^*\}$ satisfies

(1) $\varphi_{\mathtt{i},i}(X) \subset \varphi_\mathtt{i}(X)$ for every $\mathtt{i} \in I^*$ and $i \in I$,
(2) $d\big(\varphi_\mathtt{i}(X)\big) \to 0$, as $|\mathtt{i}| \to \infty$.

By *contractivity* we mean that for every $\mathtt{i} \in I^*$ there exists a constant $0 < s_\mathtt{i} < 1$ such that $|\varphi_\mathtt{i}(x) - \varphi_\mathtt{i}(y)| \leq s_\mathtt{i}|x-y|$ whenever $x, y \in \Omega$. This kind of collection is called a *general iterated function system.* Furthermore, we call the collection $\{\varphi_i : i \in I\}$ of the same kind of mappings an *iterated function system (IFS).* Defining $\varphi_\mathtt{i} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_{|\mathtt{i}|}}$, as $\mathtt{i} \in I^*$, we have clearly fulfilled the assumptions of the general IFS. In fact, we have $d\big(\varphi_\mathtt{i}(X)\big) \leq (\max_{i \in I} s_i)^{|\mathtt{i}|}d(X)$.

It seems that by assuming only the mappings of a general IFS to be Lipschitz it is very difficult to get information about the geometry of the limit set. Assuming the mappings $\varphi_\mathtt{i}$ to be bi–Lipschitz, we denote the "maximal derivative" with

$$L_\mathtt{i}(x) = \limsup_{y \to x} \frac{|\varphi_\mathtt{i}(x) - \varphi_\mathtt{i}(y)|}{|x-y|} \tag{4.1}$$

and the "minimal derivative" with

$$l_\mathtt{i}(x) = \liminf_{y \to x} \frac{|\varphi_\mathtt{i}(x) - \varphi_\mathtt{i}(y)|}{|x-y|}. \tag{4.2}$$

We say that a general IFS is *bi–Lipschitz* if the mappings $\varphi_\mathtt{i}$ are bi–Lipschitz and there exist cylinder functions $\underline{\psi}_\mathtt{i}^t$ and $\overline{\psi}_\mathtt{i}^t$ satisfying the chain rule such that $\underline{\psi}_\mathtt{i}^t(\mathtt{h}) \leq l_\mathtt{i}\big(\pi(\mathtt{h})\big)^t$

and $\overline{\psi}^t_{\mathtt{i}}(\mathtt{h}) \geq L_{\mathtt{i}}\big(\pi(\mathtt{h})\big)^t$ for all $\mathtt{h} \in I^\infty$, and in both functions the parameter t is an exponent, that is, $\underline{\psi}^t_{\mathtt{i}}(\mathtt{h}) = \big(\underline{\psi}^1_{\mathtt{i}}(\mathtt{h})\big)^t$ and $\overline{\psi}^t_{\mathtt{i}}(\mathtt{h}) = \big(\overline{\psi}^1_{\mathtt{i}}(\mathtt{h})\big)^t$. We also assume that the bi–Lipschitz constants for the mappings $\varphi_{\mathtt{i}}$ are $\underline{s}_{\mathtt{i}} = \inf_{\mathtt{h}\in I^\infty} \underline{\psi}^1_{\mathtt{i}}(\mathtt{h})$ and $\overline{s}_{\mathtt{i}} = \sup_{\mathtt{h}\in I^\infty} \overline{\psi}^1_{\mathtt{i}}(\mathtt{h})$.

Observe that for IFS it suffices to require $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ for $i \neq j$ to satisfy the SSC, and, similarly, $\varphi_i\big(\mathrm{int}(X)\big) \cap \varphi_j\big(\mathrm{int}(X)\big) = \emptyset$ for $i \neq j$ to satisfy the OSC. We also say that a general IFS has *bounded overlapping* if the cardinality of the set $Z(x,r) = \{\mathtt{i} \in Z(r) : \varphi_{\mathtt{i}}(X) \cap B(x,r) \neq \emptyset\}$ is uniformly bounded as $x \in X$ and $0 < r < r_0 = r_0(x)$. Here $Z(r) = \{\mathtt{i} \in I^* : \underline{s}_{\mathtt{i}} < r \leq \underline{s}_{\mathtt{i}|_{|\mathtt{i}|-1}}\}$. The bounded overlapping is useful in studying the geometric properties of the limit set. In some cases, we have to use also the following *boundary condition*: there exists $\varrho_0 > 0$ such that

$$\inf_{x\in\partial X} \inf_{0<r<\varrho_0} \frac{\mathcal{H}^d\big(B(x,r) \cap \mathrm{int}(X)\big)}{\mathcal{H}^d\big(B(x,r)\big)} > 0, \tag{4.3}$$

where ∂X denotes the boundary of the set X.

Theorem E (**E**, Lemmas 3.3 and 3.4 and Propositions 3.5 and 3.6). *The following relationships hold:*

(1) *Suppose a general IFS has bounded overlapping. Then it has also weak bounded overlapping.*
(2) *A bi–Lipschitz IFS satisfying the SSC has bounded overlapping.*
(3) *A bi–Lipschitz general IFS satisfying the OSC and the boundary condition has weak bounded overlapping provided that $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}}$ is bounded as $\mathtt{i} \in I^*$.*
(4) *Suppose a bi–Lipschitz general IFS has weak bounded overlapping such that $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}}$ is bounded as $\mathtt{i} \in I^*$. Then it has also bounded overlapping.*

Although we develop our theory in a very general setting, we are mainly interested in the self–affine case described below. Our theory applies also on familiar self–similar and self–conformal cases. More general situations are discussed in [**E**, Examples 3.13 and 3.14].

Definition. Let the mappings of IFS be *similitudes*, that is, for each $i \in I$ there exists $0 < s_i < 1$ such that $|\varphi_i(x) - \varphi_i(y)| = s_i|x-y|$ whenever $x, y \in \Omega$. We call this kind of setting a *similitude IFS* and the corresponding limit set a *self–similar set*.

If for each $\mathtt{i} \in I^*$ we choose $\psi^t_{\mathtt{i}} \equiv s^t_{\mathtt{i}}$, where $s_{\mathtt{i}} = s_{i_1} \cdots s_{i_{|\mathtt{i}|}}$, then $\psi^t_{\mathtt{i}}$ is a constant cylinder function satisfying the chain rule. We call this choice of cylinder function in this setting a *natural cylinder function*. Observe that if $P(t) = 0$, then the projected equilibrium measure $m = \mu \circ \pi^{-1}$ has $m\big(\varphi_{\mathtt{i}}(X)\big) = s^t_{\mathtt{i}}$ for every $\mathtt{i} \in I^*$. A similitude IFS is bi–Lipschitz since mappings $\varphi_{\mathtt{i}}$ are clearly bi–Lipschitz. We also have $\underline{s}_{\mathtt{i}} = \overline{s}_{\mathtt{i}} = s_{\mathtt{i}}$. Hence also $m\big(\varphi_{\mathtt{i}}(X) \cap \varphi_{\mathtt{j}}(X)\big) = 0$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable and the weak bounded overlapping is satisfied. This setting has been studied by many authors. For example, see Hutchinson [**5**] and Schief [**16**].

Definition. Suppose $d \geq 2$. Let the mappings of IFS be C^1 and *conformal* on an open set $\Omega_0 \supset \overline{\Omega}$. Hence $|\varphi_i'|^d = |J_{\varphi_i}|$ for every $i \in I$, where J stands for the usual Jacobian and the norm on the left–hand side is just a standard "sup–norm" for linear mappings. We call this kind of setting a *conformal IFS* and the corresponding limit set a *self–conformal set*.

Observe that the conformal mapping is complex analytic in the plane and, by Liouville's theorem, a Möbius transformation in higher dimensions. So, in fact, conformal mappings are C^∞ and infinitesimally similitudes. Notice also that it is essential to use the bounded set Ω here since conformal mappings contractive in the whole $\mathbb{R}^d$ are similitudes. If for each $\mathtt{i} \in I^*$ we choose $\psi^t_{\mathtt{i}}(\mathtt{h}) = \big|\varphi'_{\mathtt{i}}\big(\pi(\mathtt{h})\big)\big|^t$, then $\psi^t_{\mathtt{i}}$ is a cylinder function satisfying the chain rule. The BVP for $\psi^t_{\mathtt{i}}$ is guaranteed by the smoothness of mappings $\varphi_{\mathtt{i}}$, [**E**, Proposition 2.1] and the chain rule. Again, we call this choice of cylinder function in this setting a *natural cylinder function*. A conformal IFS is bi–Lipschitz since mappings $\varphi_{\mathtt{i}}$ are bi–Lipschitz. We can also choose $\underline{\psi}^t_{\mathtt{i}} = \overline{\psi}^t_{\mathtt{i}} = \psi^t_{\mathtt{i}}$. Observe that if $P(t) = 0$, then the projected conformal measure $m = \nu \circ \pi^{-1}$ has $m\big(\varphi_{\mathtt{i}}(X)\big) = \int_X |\varphi'_{\mathtt{i}}(x)|^t dm(x)$ for every $\mathtt{i} \in I^*$ and also $m\big(\varphi_{\mathtt{i}}(X) \cap \varphi_{\mathtt{j}}(X)\big) = 0$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable and the weak bounded overlapping is satisfied. This setting has also been studied by many authors. For example, see Mauldin and Urbański [**10**] and Peres, Rams, Simon and Solomyak [**14**].

Definition. Let the mappings of IFS be *affine*, that is, $\varphi_i(x) = A_i x + a_i$ for every $i \in I$, where A_i is a contractive non–singular linear mapping and $a_i \in \mathbb{R}^d$. We call this kind of setting an *affine IFS* and the corresponding limit set a *self–affine set*.

Clearly, the products $A_{\mathtt{i}} = A_{i_1} \cdots A_{i_{|\mathtt{i}|}}$ are also contractive and non–singular. Singular values of a non–singular matrix are the lengths of the principle semiaxes of the image of the unit ball. On the other hand, the singular values $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d > 0$ of a contractive, non–singular matrix A are the non–negative square roots of the eigenvalues of A^*A, where A^* is the transpose of A. Define a singular value function α^t by setting $\alpha^t(A) = \alpha_1\alpha_2\cdots\alpha_{l-1}\alpha_l^{t-l+1}$, where l is the smallest integer greater than t or equal to it. For all $t > d$ we put $\alpha^t(A) = (\alpha_1 \cdots \alpha_d)^{t/d}$. It is clear that $\alpha^t(A)$ is continuous and strictly decreasing in t. If for each $\mathtt{i} \in I^*$ we choose $\psi^t_{\mathtt{i}} \equiv \alpha^t(A_{\mathtt{i}})$, then $\psi^t_{\mathtt{i}}$ is a constant cylinder function. The subchain rule for $\psi^t_{\mathtt{i}}$ is satisfied by Lemma 2.1 of Falconer [**3**]. We call this choice of a cylinder function in this setting a *natural cylinder function*. An affine IFS is bi–Lipschitz since mappings $\varphi_{\mathtt{i}}$ are bi–Lipschitz. We can also choose $\underline{\psi}^t_{\mathtt{i}} \equiv \alpha^t_d$ and $\overline{\psi}^t_{\mathtt{i}} \equiv \alpha^t_1$, where α_d (α_1) is the smallest (largest) singular value of the mapping $A_{\mathtt{i}}$. Since in this case we do not have the chain rule, it is very difficult to say anything "concrete" about the equilibrium measure. We also show ([**E**, Example 3.12]) that without assuming the SSC it is difficult to get a sufficient separation.

5. Measures with full dimension

Our aim is to study the Hausdorff dimension of measures on self–similar, self–conformal and self–affine sets. We say that the Hausdorff dimension of a given Borel probability measure m is $\dim_H(m) = \inf\{\dim_H(A) : A \text{ is a Borel set such that } m(A) = 1\}$. Checking whether $\dim_H(m) = \dim_H(E)$ is one way to examine how well a given measure m is spread out on a given set E. The following theorem is a known result. For example, see Hutchinson [**5**], Mauldin and Urbański [**10**] and [**11**] and Kenyon and Peres [**6**].

Theorem F ([**E**, Proposition 3.6 and Theorem 3.8]). *Suppose a conformal (or a similitude) IFS is equipped with the natural cylinder function and has weak bounded overlapping. Assume also that $P(t) = 0$ and m is the projected equilibrium (or conformal) measure. Then $\dim_H(m) = \dim_H(E) = t$.*

For the affine IFS, our main application, we are not able to obtain the desired information from [**E**, Theorem 3.8] because, in this case, it will give us only upper and lower bounds for the Hausdorff dimension of the equilibrium measure. We first have to study dimensions in our more general setting. For a given cylinder function and fixed $t \geq 0$, we denote with μ_t the corresponding equilibrium measure. We define for each $n \in \mathbb{N}$

$$\mathcal{G}_n^t(A) = \inf\left\{ \sum_{j=1}^\infty \int_{I^\infty} \psi_{\mathtt{i}_j}^t(\mathtt{h}) d\mu_t(\mathtt{h}) : A \subset \bigcup_{j=1}^\infty [\mathtt{i}_j],\ |\mathtt{i}_j| \geq n \right\} \tag{5.1}$$

whenever $A \subset I^\infty$. Assumptions in Carathéodory's construction are now satisfied, and we have a Borel regular measure $\mathcal{G}^t$ on I^∞ with

$$\mathcal{G}^t(A) = \lim_{n\to\infty} \mathcal{G}_n^t(A). \tag{5.2}$$

Using this measure, we define

$$\begin{aligned} \dim_\psi(A) &= \inf\{t \geq 0 : \mathcal{G}^t(A) = 0\} \\ &= \sup\{t \geq 0 : \mathcal{G}^t(A) = \infty\}, \end{aligned} \tag{5.3}$$

and we call this "critical value" the *equilibrium dimension* of the set $A \subset I^\infty$. We also define the *equilibrium dimension of a measure* $\mu \in \mathcal{M}(I^\infty)$ by setting $\dim_\psi(\mu) = \inf\{\dim_\psi(A) : A$ is a Borel set such that $\mu(A) = 1\}$.

Theorem G ([**E**, Theorems 3.2 and 4.3]). *Suppose $P(t) = 0$ and μ is the equilibrium measure. Then $\dim_\psi(\mu) = t$.*

It is also true that if $\dim_\psi(I^\infty) = t$, then $P(t) = 0$. Now the desired result follows from this theorem by applying Falconer's result for the Hausdorff dimension of self–affine sets; see [**3**].

Theorem H ([**E**, Theorem 4.5]). *Suppose an affine IFS is equipped with the natural cylinder function and the mappings are of the form $\varphi_i(x) = A_i x + a_i$, where $|A_i| < \frac{1}{2}$. We also assume that $P(t) = 0$ and m is a projected t–equilibrium measure. Then for $\mathcal{H}^{d\#I}$–almost all $a = (a_1, \ldots, a_{\#I}) \in \mathbb{R}^{d\#I}$ we have*

$$\dim_H(m) = \dim_H(E), \tag{5.4}$$

where $E = E(a)$.

6. Local geometric structure

We are particularly interested in whether the limit set is contained in an l–dimensional C^1–submanifold of $\mathbb{R}^d$ for some $0 < l < d$. The following theorem shows us that if a self–conformal set does not satisfy this property, it will be "totally spread out". For the theorem to hold, we have to make an extra assumption, namely the convexity of the set Ω, to guarantee that the diameter of each projected cylinder set is comparable to the derivative of the corresponding function.

THEOREM I ([**C**, Theorem 2.1]). *Suppose a conformal IFS is equipped with the natural cylinder function, has weak bounded overlapping, $P(t) = 0$ and $0 < l < d$. Then either $\mathcal{H}^t(E \cap M) = 0$ for every l–dimensional C^1–submanifold $M \subset \mathbb{R}^d$ or the closure of E is contained in an l–dimensional affine subspace or an l–dimensional geometric sphere whenever d exceeds 2 and an analytic curve if d equals 2.*

Using this theorem, we are able to find the minimal amount of essential directions in which the set E is spread out. It also follows that if t is an integer, then the limit set is always either t–rectifiable or purely t–unrectifiable. Observe that the theorem remains true if I is a countable set. This generalises the theorem of Mattila [**8**], which concerns the case of a finite similitude IFS. We should mention also that Springer has proved in [**17**] a similar result in the plane, and Mauldin, Mayer and Urbański have studied the same behaviour for connected self–conformal sets in [**9**] and [**12**].

Notice that under the assumptions of the theorem, $\dim_H(E) = t$ as $P(t) = 0$. Also, the projected conformal measure, which is a crucial tool in the proof, satisfies $m\big(\varphi_{\mathtt{i}}(X)\big) = \int_X |\varphi'_{\mathtt{i}}(x)|^t dm(x)$ for $\mathtt{i} \in I^*$ and $m\big(\varphi_{\mathtt{i}}(X) \cap \varphi_{\mathtt{j}}(X)\big) = 0$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable.

7. Further remarks

Below we discuss some of the questions raised during the preparation of this work.

(1) It would be interesting to know whether the result of Theorem I holds also for other limit sets. Considering affine functions and the invariance property of the self–affine set with respect to these mappings, it would be reasonable to think that a self–affine set is either contained in an l–dimensional affine subspace or "totally spread out".
(2) There might be a possibility to develop fine multifractal analysis on self–affine sets based on the main results of [**E**]. At least, in the theory, ergodic invariant equilibrium measures are needed. It also seems that the following questions are closely related to this problem: Is the topological pressure convex? If μ_t is a t–equilibrium measure, is $P'(t) = E_{\mu_t}(t)$? What is the connection between the measure $\mathcal{G}^t$ constructed in (5.2) and an equilibrium measure?
(3) The properties of an equilibrium measure are, of course, an interesting question, especially the uniqueness. Also, can we approximate the measure of infinite small cylinder sets with the cylinder function?
(4) What can we say about equilibrium measures on infinite systems?
(5) There is also a possible application in image processing: For "almost all" self–affine sets we found an invariant ergodic measure with full dimension. Checking whether a given measure has full dimension is one way to examine how well it is spread out on a given set. When we draw a limit set of iterated function systems using a computer, we actually draw the measure rather than the set.

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