DISCRETE MARTINGALES AND APPLICATIONS TO ANALYSIS

JOSÉ G. LLORENTE

The use of probabilistic techniques in Analysis has experienced a remarkable success in the last fifteen years. It turns out, for instance, that the boundary behaviour of harmonic functions and conformal mappings is better understood when rephrased in discrete terms.

In a series of influential papers, N.G. Makarov ([Mak89a], [Mak89b], [Mak85], [Mak87]) proved a number of deep results on the boundary behaviour of conformal maps, many of them being direct consequences of properties of the asymptotic behaviour of discrete martingales in the unit interval.

Still in the eighties, [CWW85], [BKM88], [BM89], [BKM90] used dyadic martingales, in more or less direct ways, to prove results on the boundary behaviour of harmonic functions in the upper half-space or in more general domains. Since then, and specially in the last ten years, dyadic martingales have shown to be an illuminating tool not only in boundary behaviour (see [BFL00], [Can98], [Don01b], [DP99], [GN01] and [Llo98]) but also in the study of Zygmund measures and Zygmund functions ([CD96], [Don01a]), regularity of measures ([GN01], [Llo98], [Llo02]) and, surprisingly, hyperbolic manifolds and Kleinian groups ([BJ95], [BJ97]).

Roughly speaking, it can be said that the main reason why the interplay "continuous–discrete" is fruitful relies on the mean value property. It is well known that harmonic functions are those continuous functions that satisfy a spherical mean value property. Now, if we consider a dyadic directed tree then, harmonic functions on the tree – that is, functions satisfying a one-sided mean value property – are canonically identified with dyadic martingales in $[0,1]$ (see Chapter 1 for details). It is then natural to expect some sort of parallelism between the boundary behaviour of harmonic functions and the asymptotic behaviour of dyadic martingales. Through Chapters 1 and 2 we will see that this is indeed the case, and sometimes the parallelism is quite direct and satisfactory.

The author was supported in part by funds of European Erasmus program and a grant of Ministerio de Educación, Spain.

The structure of the notes is the following:

Contents

In Chapter 1, I have intended to give a brief introduction to dyadic martingales, with special emphasis in their asymptotic properties. Since some of the applications in Chapter 2 and Chapter 3 depend on the upper bound of the Law of the Iterated Logarithm, I have included a detailed proof in the simplest case. Through Chapter 1, I have tried to point out the parallelisms (most of the times only heuristic but illuminating) between the discrete and the continuous setting. The material is basically self-contained, though sometimes I have preferred to skip a proof and to give instead a reference. The general approach of the notes owes much to [Mak89a]. [Shi84] is an exceptional general reference for the probabilistic part, and [Sto84] is also a good reference for martingales and limit theorems. Part of the material included in Chapters 2 and 3 can also be found in the forthcoming [GM]. [BM99] is a monograph containing advanced topics on martingales and boundary behaviour.

These notes originated in a series of lectures that I gave at the University of Jyväskylä in June 2001, as part of the Erasmus exchange program between the University of Jyväskylä and the Universitat Autònoma de Barcelona. It is a pleasure for me to thank professor Pekka Koskela for his invitation and his encouragement to prepare these notes, during and after my visit. My thanks go also to the Department of Mathematics of the University of Jyväskylä for its hospitality, to professor Stefan Geiss, who took care of the organization of the course and also to all who participated in the lectures.

The notes were written when I was visiting the University of Michigan during the academic course 2001-2002, funded by a grant of the Ministerio de Educación of Spain. The author thanks both institutions for their support.

Some comments about notation: We refer to [Shi84] for the basic facts and notation about probability spaces. It will be understood that any random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{G}, \mu)$ belongs to $L^1(\mu)$. $|\cdot|$ stands for Lebesgue measure in $[0, 1)$ or $\partial\mathbb{D}$.

1. Dyadic martingales

1.1. **Conditional expectation.** Let $(\Omega, \mathcal{G}, \mu)$ be a probability space, $X : \Omega \to \mathbb{R}$ a random variable and $\mathcal{F} \subset \mathcal{G}$ a σ-algebra. We say that the random variable Y is the *conditional expectation* of X with respect to $\mathcal{F}$ (denoted $Y = E[X/\mathcal{F}]$) if:

(i) Y is $\mathcal{F}$-measurable,
(ii) $\int_F X d\mu = \int_F Y d\mu$ for any $F \in \mathcal{F}$.

From the point of view of gambling, the whole σ-algebra $\mathcal{G}$ can be seen as the total potential information of the gambler, whereas $\mathcal{F}$ can be seen as the current real information. Then $E[X/\mathcal{F}]$ intuitively represents the "correction" of X provided the accessible information $\mathcal{F}$. The extreme cases where X is $\mathcal{F}$-measurable and X is independent of $\mathcal{F}$ (that is, X and $\mathbf{1}_F$ are independent for any $F \in \mathcal{F}$) correspond, respectively to the cases where the gambler has all information about X or has no *a priori* information about X.

We list below the basic properties of conditional expectation:

(1) If $\mu_1 = \int_F X\, d\mu$ and $\mu_2 = \mu|\mathcal{F}$, then $E[X/\mathcal{F}] = \frac{d\mu_1}{d\mu_2}$ (Radon–Nikodym derivative). In particular, the conditional expectation is well defined and unique.
(2) If X is $\mathcal{F}$-measurable (perfect information), then $E[X/\mathcal{F}] = X$.
(3) If X is independent of $\mathcal{F}$ (no a priori information), then $E[X/\mathcal{F}] = \int_\Omega X\, d\mu$.

(4) If $\{\Omega_k\}_{k=1}^N$ is a partition of Ω and $\mathcal{F}$ is the σ-algebra generated by $\{\Omega_1, \cdots, \Omega_N\}$, then
$$E[X/\mathcal{F}] = \sum_{k=1}^{N} \left(\fint_{\Omega_k} X\, d\mu \right) \mathbf{1}_{\Omega_k},$$
where $\fint_A X\, d\mu$ denotes, hereafter, $\frac{1}{\mu(A)} \int_A X\, d\mu$.

(5) If X and Z are random variables and Z is $\mathcal{F}$-measurable, then
$$E[XZ/\mathcal{F}] = Z \cdot E[X/\mathcal{F}] \quad (\text{start by } Z = \mathbf{1}_F, \text{ where } F \in \mathcal{F}).$$

A basic example.
Let $\Omega = [0,1)$, μ a Borel probability measure in [0,1) and denote by
$$\mathcal{D}_n = \left\{ \left[\frac{m-1}{2^n}, \frac{m}{2^n}\right) \ : \ m = 1, 2, \cdots, 2^n \right\}$$
the family of dyadic intervals of the generation n. Define $\mathcal{F}_n$ to be the σ-algebra generated by $\mathcal{D}_n$. We say that $\{\mathcal{F}_n\}_{n=0}^\infty$ is the *dyadic filtration*. Then, if $X : [0,1) \to \mathbb{R}$ is a random variable,
$$E[X/\mathcal{F}_n] = \sum_{I_n \in \mathcal{D}_n} \left(\fint_{I_n} X\, d\mu \right) \mathbf{1}_{I_n}.$$
This shows that, in this particular setting, conditional expectation is just obtained by averaging.

1.2. **Martingales and dyadic martingales.** An increasing sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{G}$ in a probability space $(\Omega, \mathcal{G}, \mu)$ is called a *filtration*.

Given a filtration $\{\mathcal{F}_n\}$ and a sequence of random variables $(S_n)_{n=0}^\infty$ we say that $(S_n, \mathcal{F}_n)$ is a *martingale* if, for all n:

(i) S_n is $\mathcal{F}_n$-measurable,

(ii)
$$E[S_n/\mathcal{F}_{n-1}] = S_{n-1}. \tag{1.1}$$

The definition of martingale not only includes the sequence (S_n), but also the underlying probability space $(\Omega, \mathcal{G}, \mu)$ and the filtration $\{\mathcal{F}_n\}$. Nevertheless, if there is no risk of confusion, we will often omit both $\mathcal{G}$ and $\{\mathcal{F}_n\}$ and will only refer to a martingale (S_n) in (Ω, μ).

If we think of (S_n) as the fortune of a gambler at the instant n of a game, then condition (i) above expresses the trivial fact that the result of the game totally determines the state of the fortune at any instant, while condition (ii) says that the game is "fair" in the sense that the expected fortune after any trial must be the same that the fortune before the trial.

If we replace $=$ by $\geq$ (resp. $\leq$) in (1.1) then we say that (S_n) is a *submartingale* (resp. a supermartingale). Then submartingales (resp. supermartingales) are models of favorable (resp. unfavorable) games.

Given a martingale $(S_n)_{n=0}^{\infty}$, it is often useful to introduce the *increments* $X_k = S_k - S_{k-1}$, so $S_n = S_0 + \sum_{k=1}^{n} X_k$. Now, (S_n) is a martingale iff:

(i') X_k is $\mathcal{F}_k$-measurable,
(ii') $E[X_k/\mathcal{F}_{k-1}] = 0$

for all k. Note that (ii') is expressing now that the "expected gain" at any time of the game is 0.

Basic properties of martingales.

(1) If (S_n) is a martingale (resp. submartingale), then

$E[S_k/\mathcal{F}_m] = S_m$ (resp. $E[S_k/\mathcal{F}_m] \geq S_m$) for all $k \geq m$.

In particular,

$$\int_\Omega S_n \, d\mu = \int_\Omega S_0 \, d\mu \quad \left(\text{resp. } \int_\Omega S_n \, d\mu \geq \int_\Omega S_0 \, d\mu \right).$$

(2) If $S_n = S_0 + \sum_{k=1}^{n} X_k$ is a martingale, then the increments are orthogonal in the sense that

$$\int_\Omega X_k X_m \, d\mu = 0 \quad (k \neq m).$$

In particular, if $n \geq m$,

$$\int_\Omega (S_n - S_m)^2 \, d\mu = \int_\Omega S_n^2 \, d\mu - \int_\Omega S_m^2 \, d\mu$$

and, if $S_0 \equiv 0$, then

$$\int_\Omega S_n^2 \, d\mu = \sum_{k=1}^{n} \int_\Omega X_k^2 \, d\mu.$$

(3) If (S_n) is a martingale and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, then $\varphi(S_n)$ is a submartingale (use Jensen Inequality).

Examples.

(1) If $(X_k)_{k=1}^{\infty}$ is a sequence of independent random variables with zero mean, and $\mathcal{F}_n$ is the σ-algebra generated by $X_1, \cdots, X_n$ ([Shi84]), then $S_n = \sum_{k=1}^{n} X_k$ is a martingale.
(2) (*Dyadic martingales*)
As in the basic example in Section 1.1, take $\Omega = [0,1)$, μ a Borel probability in $[0,1)$, $\mathcal{D}_n$ the family of dyadic intervals of the generation n and $\mathcal{F}_n$ the σ-algebra generated by $\mathcal{D}_n$. Then $(S_n)_{n=0}^{\infty}$ is a *dyadic martingale* in $([0,1), \mu)$ if $(S_n, \mathcal{F}_n)$ is a martingale, that is, for all n:
(i) (S_n) is constant on any $I_n \in \mathcal{D}_n$,
(ii) $S_{n-1}|I_{n-1} = ⨍_{I_{n-1}} S_n \, d\mu$ for any $I_{n-1} \in \mathcal{D}_{n-1}$.

If $I_n^1, I_n^2 \in \mathcal{D}_n$ are the dyadic "children" of I_{n-1} and $S_n^i = S_n|I_n^i$ $(i = 1, 2)$, then (ii) above is equivalent to

$$S_{n-1}|I_{n-1} = S_n^1|I_n^1 + S_n^2|I_n^2. \tag{1.2}$$

Dyadic submartingales and supermartingales are defined in the same way. In terms of the increments, then $S_n = S_0 + \sum_{k=1}^n X_k$ is a dyadic martingale iff:

(i) X_k is constant on any $I_k \in \mathcal{D}_k$,

(ii)

$$⨍_{I_{k-1}} X_k \, d\mu = 0 \quad \text{for any} \quad I_{k-1} \in \mathcal{D}_{k-1}. \tag{1.3}$$

In the special case that $\mu =$ Lebesgue measure in $[0, 1)$, then (1.2) and (1.3) reduce to

$$S_{n-1}|I_{n-1} = \frac{1}{2}(S_n^1 + S_n^2) \quad \text{and} \tag{1.4}$$

$$X_k^1 + X_k^2 = 0, \tag{1.5}$$

where $X_k^i = S_k^i - S_{k-1}$. In particular, X_k^2 is constant on any $I_{k-1} \in \mathcal{D}_{k-1}$, that is, $\mathcal{F}_{k-1}$-measurable.

(3) (*Rademacher martingale*)

Let $(X_k)_{k=1}^\infty$ be the Rademacher system in $[0, 1)$, that is, X_k alternates $+1$ and -1 on the dyadic intervals of the generation k. For instance, $X_1 = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$, $X_2 = \mathbf{1}_{[0,1/4)\cup[1/2,3/4)} - \mathbf{1}_{[1/4,1/2)\cup[3/4,1)}$ and so on. Then $(X_k)_{k=1}^\infty$ are independent, identically distributed random variables with zero mean and variance one. In particular, $S_n = \sum_{k=1}^\infty X_k$ is a dyadic martingale in $([0, 1), |\cdot|)$. We can interpret S_n as the fortune , after n trials, of a gambler who wins or loses one unit with probability $\frac{1}{2}$ at each trial. Also, S_n can be thought as the position, at the n, of a random traveller starting from the origin and moving one unit left or right with probability $\frac{1}{2}$ (usual random walk).

(4) Let $f \in L^1[0, 1)$. If $I_n \in \mathcal{D}_n$, set $S_n|I_n = ⨍_{I_n} f(x)\, dx$. Then, $(S_n)_n$ is a dyadic martingale in $([0, 1), |\cdot|)$. It is easy to check that $S_n = E[f/\mathcal{F}_n]$.

(5) The following generalization of the example above will be useful later. Let $(f_k) \subset L^1[0, 1)$ and suppose that for any $I_n \in D_n$, the limit

$$S_{I_n} = \lim_{k\to\infty} ⨍_{I_n} f_k(x)\, dx \tag{1.6}$$

exists and is finite. Then if we define $S_n|I_n = S_{I_n}$, $(S_n)_n$ is also a dyadic martingale in $([0, 1), |\cdot|)$.

Remarks.

(1) If $\mu =$ Lebesgue measure, (1.4) says that the value of S_{n-1} at any dyadic interval of the generation $n - 1$ is the arithmetic

mean of S_n at its dyadic children of the generation n. Then (1.4) must be seen as a discrete version of the mean value property of harmonic functions and it is, of course, responsible of many of the analogies between harmonic functions and martingales. Exactly in the same way, there is a natural heuristic identification between subharmonic (resp. superharmonic) functions and submartingales (resp. supermartingales).

(2) There is a graphic interpretation of dyadic martingales which is often helpful. Suppose that (S_n) is a dyadic martingale in $([0,1), |\cdot|)$. Associated to (S_n) we construct a directed tree as follows: the original node is chosen to be the point $(0, S_0)$ in the plane and, once the point $(n-1, S_{n-1})$ has been selected, then we add the points $(n, S_n^1), (n, S_n^2)$ where S_n^1, S_n^2 are as in (1.4).

In this way we produce a directed dyadic tree, the original node being $(0, S_0)$, such that the vertical coordinates of the nodes at each level n inform us about the values of S_n (see Fig. 1).

1.3. **Quadratic characteristic.** Let $S_n = \sum_{k=1}^n X_k$ be a martingale in (Ω, μ). Assume w.l.o.g. that $S_0 = 0$. Then

$$\langle S\rangle_n = \sum_{k=1}^n E[X_k^2/\mathcal{F}_{k-1}]$$

is called the *Quadratic characteristic* of (S_n). Observe that

(i) $\langle S\rangle_n$ is $\mathcal{F}_{n-1}$-measurable

(ii)
$$\int_\Omega \langle S\rangle_n \, d\mu = \int_\Omega \sum_{k=1}^n X_k^2 \, d\mu = \int_\Omega S_n^2 \, d\mu, \tag{1.7}$$

where the definition and the orthogonality property (2) in Section 1.2 have been used in (ii).

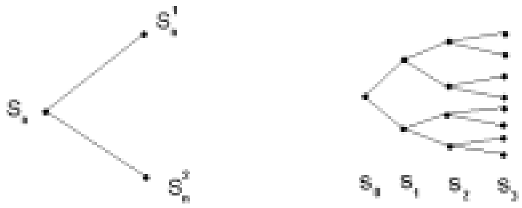

FIGURE 1

EXAMPLES.

(1) If (S_n) is dyadic martingale in $([0,1),\mu)$, then

$$E[X_k^2/\mathcal{F}_{k-1}](x) = \fint_{I_{k-1}(x)} X_k^2 \, d\mu \tag{1.8}$$

$$\langle S\rangle_n(x) = \sum_{k=1}^{n} \left[\frac{\mu(I_k^1)}{\mu(I_{k-1}(x))}(X_k^1)^2(x) + \frac{\mu(I_k^2)}{\mu(I_{k-1}(x))}(X_k^2)^2(x) \right], \tag{1.9}$$

where $\{I_j(x)\}$ are the dyadic intervals containing x, I_k^1, I_k^2 are the dyadic children of $I_{k-1}(x)$ and $X_k^i = X_k|I_k^i$. In particular, if μ is Lebesgue measure, then

$$\langle S\rangle_n = \sum_{k=1}^{n} X_k^2. \tag{1.10}$$

(2) If $(X_k)_{k=1}^{\infty}$ are independent, identically distributed random variables with zero mean, variance 1, $\mathcal{F}_k$ is the σ-algebra generated by $X_1, \dots, X_k$, and $S_n = \sum_{k=1}^{n} a_k X_k$, with $a_k \in \mathbb{R}$, then

$$\langle S\rangle_n = \sum_{k=1}^{n} a_k^2.$$

REMARKS.

(1) Sequences (S_n) such that S_n is $\mathcal{F}_{n-1}$-measurable for all n play an important role in Martingale Theory. They are usually called *predictable* sequences.

(2) Sometimes it is instructive to look at the quadratic characteristic as a sort of intrinsic random time associated to the martingale. For instance, if $|X_k| = 1$ for all k, then $\langle S\rangle_n = n$.

(3) The quadratic characteristic is a very important quantity related to a martingale, in the sense that it determines most of its asymptotic behavior, as we will see in the next sections. It can be understood as a discrete counterpart of the area function in Harmonic Analysis. If u is defined in $\mathbb{R}^n_+$ and $t > 0$, then the t-truncated area function of u at $x_0 \in \mathbb{R}^{n-1}$ is defined (in the simplest formulation) by:

$$(S_t u)(x_0) = \int_{\Gamma_t(x_0)} y^{2-n} |\nabla u|^2 \, dx_1 \dots dx_{n-1} dy,$$

where

$$\Gamma_t(x_0) = \{(x,y) \ : \ x \in \mathbb{R}^{n-1}, |x - x_0| \le y, t \le y \le 1\}.$$

Then, for small t,

$$(S_t u)(x_0) \asymp \fint_{\Gamma_t(x_0)} y^2 |\nabla u|^2 \, dx_1 \dots dx_{n-1} dy.$$

At this point, it should be mentioned that it is often useful to think in the hyperbolic gradient $y|\nabla u|$ in the upper half-space as a continuous analogue of the increments $|X_k|$, where y and k are related by $y \sim 2^{-k}$. These observations explain why it is convenient to see the area function as a continuous counterpart of the quadratic characteristic.

1.4. **Stopping times.** Let $\{\mathcal{F}_n\}$ be a filtration in the probability space $(\Omega, \mathcal{G}, \mu)$. Then

$$\tau : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$$

is called an *stopping time* if $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Example. Typical examples of stopping times are provided by barriers: let $M \in \mathbb{R}$ and (S_n) be any sequence of random variables such that S_n is $\mathcal{F}_n$-measurable for all n. Define τ by:

$$\tau(x) = n \Leftrightarrow S_0(x) \le M, \dots, S_{n-1}(x) \le M, S_n(x) > M$$
$$\tau(x) = \infty \Leftrightarrow \sup_n S_n(x) \le M.$$

Then τ is a stopping time.

Suppose now that $S_n = \sum_{k=1}^n X_k$ is any random sequence, with X_k being $\mathcal{F}_k$-measurable for all k, and τ is an stopping time. Then the stopped sequence (S_n^τ) is defined by

$$S_n^\tau(x) = S_{\min\{\tau, n\}}(x).$$

Since

$$S_k^\tau - S_{k-1}^\tau = (S_k - S_{k-1})\mathbf{1}_{\{\tau \ge k\}},$$

we get the representation

$$S_n^\tau = \sum_{k=1}^n \mathbf{1}_{\{\tau \ge k\}} X_k. \tag{1.11}$$

The following proposition says that the martingale structure is preserved by stopping times.

Proposition 1.1. *If (S_n) is a martingale and τ is any stopping time, then the stopped sequence (S_n^τ) is also a martingale.*

Proof. Note that $\{\tau \ge k\}$ is $\mathcal{F}_k$- measurable. Combine (1.11) with this observation. □

Remark. It is instructive to compare stopping time techniques with the so called "localization techniques" in Function Theory. Here is an example: suppose that u is harmonic in the unit disc $\mathbb{D}$, $M \in \mathbb{R}$ and G is a non-empty component of the set $\{u < M\}$. Then G is simply connected by the Maximum Principle and if φ is a conformal map from $\mathbb{D}$ onto G, then $v = u \circ \varphi$ is harmonic in $\mathbb{D}$ and $\sup v = M$. Thus, if (S_n) is a martingale, v can be seen as a continuous version of the stopped martingale (S_n^τ), where τ is the stopping time corresponding to the upper barrier M, as in the example above.

1.5. **The Basic Convergence Theorem.**

Theorem 1.1 (Doob)**.** *Let $1 \leq p < \infty$. If (S_n) is a martingale in (Ω, μ) such that*

$$\sup_n \int_\Omega |S_n|^p \, d\mu \ < \infty, \tag{1.12}$$

then $\lim_n S_n(x)$ exists and is finite for $\mu - a.e.\ x \in \Omega$.

Remarks.

(1) By Hölder's inequality, the result follows in the range $p \geq p_0$ provided it is proved for $p = p_0$. This shows that it is enough to take $p = 1$ in Theorem 1.1. Nevertheless, the proof given below exploits the orthogonality and works for $p = 2$ (and, therefore, for $p \geq 2$). In the general case $p = 1$, most of the proofs use the "upcrossing inequalities" technique, a combinatorial trick controlling the oscillation ([Shi84], [Sto84]).

(2) Theorem 1.1 still holds if "martingale" is replaced by "submartingale".

(3) The class of martingales satisfying (1.12) can be seen as the discrete analogue of the harmonic Hardy class H_u^p, consisting of all harmonic functions in the unit ball $\mathbb{B}$ such that

$$\sup_{0<r<1} \int_{\partial \mathbb{B}} |u(r\xi)|^p \, d\xi < \infty. \tag{1.13}$$

Therefore, Theorem 1.1 corresponds to the fact that each function in H_u^p has finite radial limits a.e. on $\partial \mathbb{B}$.

Lemma 1.1 (Kolmogorov inequality)**.** *Let $S_n = \sum_{k=1}^n X_k$ be a martingale in (Ω, μ), with $S_0 = 0$. Then, for any $\varepsilon > 0$,*

$$\mu \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \int_\Omega S_n^2 \, d\mu.$$

Proof. Define the stopping time τ by:

$$\tau(x) = n \Leftrightarrow |S_1(x)| \leq \varepsilon, \ldots, |S_{n-1}(x)| \leq \varepsilon, |S_n(x)| > \varepsilon$$
$$\tau(x) = \infty \Leftrightarrow \sup_n |S_n(x)| \leq \varepsilon.$$

Observe that

$$\left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right\} = \{\tau \leq n\}.$$

Let (S_n^τ) be the stopped martingale. Then, by (1.11) and orthogonality (Property (2), Section 1.2):

$$\int_\Omega (S_n^\tau)^2 \, d\mu = \int_\Omega \sum_{k=1}^n \mathbf{1}_{\{\tau \geq k\}} X_k^2 \, d\mu \leq \int_\Omega \sum_{k=1}^n X_k^2 \, d\mu = \int_\Omega S_n^2 \, d\mu.$$

Thus,

$$\varepsilon^2 \mu\{\tau \leq n\} \leq \int_{\{\tau = n\}} (S_n^\tau)^2 \, d\mu \leq \int_\Omega S_n^2 \, d\mu,$$

which proves the lemma. □

Proof of Theorem 1.1 (p=2).
Note, by Properties (1) and (3) in Section 1.2 that (S_n^2) is a submartingale and the sequence $\int_\Omega S_n^2\,d\mu$ is non-decreasing. Let

$$M = \lim_n \int_\Omega S_n^2\,d\mu < \infty.$$

Since

$$\left\{\nexists \lim_n S_n\right\} = \bigcup_{j=1}^{\infty}\bigcap_{m=1}^{\infty}\left\{\sup_{k\geq m}|S_k - S_m| \geq \frac{1}{j}\right\},$$

it is enough to prove that

$$\mu\left\{\sup_{k\geq m}|S_k - S_m| \geq \varepsilon\right\} \underset{m\to\infty}{\longrightarrow} 0$$

for any $\varepsilon > 0$. Set

$$A_{n,m}(\varepsilon) = \left\{\sup_{m\leq k\leq n}|S_k - S_m| \geq \varepsilon\right\}$$

and

$$A_m(\varepsilon) = \left\{\sup_{k\geq n}|S_k - S_m| \geq \varepsilon\right\}.$$

Then $A_{n,m}(\varepsilon) \uparrow A_m(\varepsilon)$ as $n \to \infty$. Let $T_k = S_{m+k} - S_m$. Since $(T_k)_{k=0}^{\infty}$ is a martingale (with respect to the filtration $\{\mathcal{F}_{m+k}\}_{k=0}^{\infty}$), then by Lemma 1.1 and Property (2), Section 1.2:

$$\begin{aligned}
\mu(A_{n,m}(\varepsilon)) &\leq \frac{1}{\varepsilon^2}\int_\Omega (S_n - S_m)^2\,d\mu \\
&= \frac{1}{\varepsilon^2}\left(\int_\Omega S_n^2\,d\mu - \int_\Omega S_m^2\,d\mu\right) \leq \frac{1}{\varepsilon^2}\left(M - \int_\Omega S_m^2\,d\mu\right),
\end{aligned}$$

which shows that

$$\mu(A_m(\varepsilon)) \leq \frac{1}{\varepsilon^2}\left(M - \int_\Omega S_m^2\,d\mu\right) \underset{m\to\infty}{\longrightarrow} 0.$$

□

Corollary 1.1 (Fatou theorem for martingales.)**.** *Let (S_n) be a martingale in (Ω, μ) and $C \in \mathbb{R}$. If $\inf S_n \geq C$ (resp. $\sup S_n \leq C$) μ-a.e. then $\lim_n S_n(x)$ exists and is finite μ-a.e. $x \in \Omega$. In particular, any non-negative martingale converges μ-a.e.*

Proof. Suppose that $C = 0$ and $S_n \geq 0$. Then $\int_\Omega |S_n|\,d\mu = \int_\Omega S_n\,d\mu$, which is independent of n by property (1), section 1.2. This shows that Corollary 1.1 follows from Theorem 1.1 with $p = 1$. □

Corollary 1.1 suggests the following definition. If (S_n) is a martingale in (Ω,μ) we define its *Fatou set* $F(S)$ as:

$$F(S)=\left\{x\in\Omega\ :\ \lim_n S_n \text{ exists and is finite }\right\}.$$

The following definition will also be useful in these notes. We say that the martingale $S_n=S_0+\sum_{k=1}^n X_k$ has *uniformly bounded increments* if $\sup_k |X_k|\le C\ <\infty$.

Theorem 1.2. *Let (S_n) be a martingale in (Ω,μ), with uniformly bounded increments. Then*

$$F(S)=\left\{\overline{\lim_n} S_n<+\infty\right\}=\left\{\underline{\lim_n}\, S_n>-\infty\right\},$$

where the identities must be understood μ-a.e.

Proof. Fix $M>0$. It is enough to show that $\{\sup_n S_n\le M\}\subset F(S)$ μ-a.e. Define the stopping time τ by

$$\begin{aligned}\tau(x)=n &\Leftrightarrow S_1(x)\le M,\dots,S_{n-1}(x)\le M, S_n(x)>M\\ \tau(x)=\infty &\Leftrightarrow \sup_n S_n(x)\le M\end{aligned}$$

and let (S_n^τ) be the stopped martingale. Then, by construction, $S_n^\tau=S_n$ on $\{\sup_n S_n\le M\}$. Since $S_n^\tau\le M+C$ (here is where the assumption on the increments is used), then by Corollary 1.1, (S_n^τ) converges μ-a.e. and in particular (S_n) converges μ-a.e. on the set $\{\sup_n S_n\le M\}$. $\square$

Corollary 1.2. *Let (S_n) be a martingale in (Ω,μ) with uniformly bounded increments. Then the following dichotomy result holds:*

$$\Omega=F(S)\cup\left\{\overline{\lim_n} S_n=+\infty,\ \underline{\lim_n}\, S_n=-\infty\right\}\cup N,$$

where $\mu(N)=0$. In particular,

$$\begin{aligned}\mu\left\{\lim_n S_n=+\infty\right\}&=\mu\left\{\lim_n S_n=-\infty\right\}\\ &=\mu\left(\left\{\overline{\lim_n}|S_n|<\infty\right\}\setminus F(S)\right)=0.\end{aligned}$$

Remark.

Corollary 1.1 is the martingale version of the classical Fatou theorem on the existence of radial limits for non-negative harmonic functions in the unit disc or the unit ball. Corollary 1.2 as well is the martingale counterpart of the "local Fatou-type" results for harmonic functions in the ball or the upper half-space. If u is harmonic in $\mathbb{R}^n_+$ and we denote by $F(u)$ the *Fatou set* of u, that is, the set of all $x\in\mathbb{R}^{n-1}$ such that u has finite non-tangential limit at x, then ([Ste70]):

$$\mathbb{R}^{n-1}=F(u)\cup\left\{x\in\mathbb{R}^{n-1}\ :\ \overline{\lim_{z\lhd x}}u(z)=+\infty,\underline{\lim_{z\lhd x}}\,u(z)=-\infty\right\}\cup N,$$

where $z \lhd x$ denotes non-tangential approach and N has $(n-1)$-Lebesgue measure zero.

Corollary 1.2 has also nice probabilistic interpretations: when seeing (S_n) as the partial states of a gambler's fortune, it says that, with probability 1, the asymptotic behaviour of the fortune is, either very regular or completely oscillatory, provided that the bets are uniformly bounded. In terms of one-dimensional random walks with uniformly bounded steps then, either the asymptotic position of the random traveller is stationary (convergent behaviour) or totally oscillatory (recurrent behaviour).

1.6. **Limit Theorems vs. Quadratic Characteristic.** Let $S_n = S_0 + \sum_{k=1}^n X_k$ be a martingale in (Ω, μ) and $\langle S\rangle_n = \sum_{k=1}^n E[X_k^2/\mathcal{F}_{k-1}]$ its quadratic characteristic. Since $\langle S\rangle_n$ is non-decreasing, we will denote $\langle S\rangle_\infty = \lim_n \langle S\rangle_n$. In this section we will show that the asymptotic behaviour of a martingale is closely related to its quadratic characteristic.

Theorem 1.3. *Let (S_n) be a martingale in (Ω, μ). Then*

(a) $\{\langle S_n\rangle_\infty < \infty\} \subset F(S)$ *μ-a.e.*

(b) *If, in addition, (S_n) has uniformly bounded increments, then* $\{\langle S_n\rangle_\infty < \infty\} = F(S)$ *μ-a.e.*

Proof. (a) Fix $M > 0$. It is enough to prove $\{\langle S\rangle_\infty \le M\} \subset F(S)$ μ-a.e. Assume w.l.o.g. that $S_0 = 0$. We define the following stopping time τ by:

$$\tau(x) = n \Leftrightarrow \langle S\rangle_1(x) \le M, \dots, \langle S\rangle_n(x) \le M, \langle S\rangle_{n+1}(x) > M$$
$$\tau(x) = \infty \Leftrightarrow \langle S\rangle_\infty(x) \le M.$$

Note that, since $\langle S\rangle_n$ is predictable, then τ is an stopping time. The important fact is that if (S_n^τ) is the stopped martingale, then $\langle S^\tau\rangle_n \le M$. Now, by (1.7),

$$\int_\Omega (S_n^\tau)^2 \, d\mu = \int_\Omega \langle S^\tau\rangle_n \, d\mu \le M,$$

so by Theorem 1.1, (S_n^τ) converges μ-a.e. In particular, $\lim_n S_n = \lim_n S_n^\tau$ exists on $\{\tau = \infty\} = \{\langle S\rangle_\infty \le M\}$. This proves (a).

(b) Analogously, it is enough to show that

$$\left\{\sup_n |S_n| \le a\right\} \subset \{\langle S\rangle_\infty < \infty\} \ \mu\text{-a.e.}$$

for fixed $a > 0$, by Corollary 1.2. Now, consider the stopping time

$$\tau(x) = n \Leftrightarrow |S_1| \le a, \dots, |S_{n-1}| \le a, |S_n| > a$$
$$\tau(x) = \infty \Leftrightarrow \sup_n |S_n| \le a.$$

Then $|S_n^\tau| \leq a + C$, provided $\sup_k |S_k - S_{k-1}| \leq C$, where (S_n^τ) is the stopped martingale. By (1.7),

$$\int_\Omega \langle S^\tau \rangle_n \, d\mu = \int_\Omega (S_n^\tau)^2 \, d\mu \leq (a+c)^2,$$

so by monotone convergence

$$\int_\Omega \langle S^\tau \rangle_\infty \, d\mu \leq (a+b)^2 \text{ and } \langle S^\tau \rangle_\infty < \infty \ \mu\text{-a.e.}$$

Since $\langle S^\tau \rangle_\infty = \langle S \rangle_\infty$ on

$$\{\tau = \infty\} = \left\{ \sup_n |S_n| \leq a \right\},$$

it follows that

$$\left\{ \sup_n |S_n| \leq a \right\} \subset \{\langle S \rangle_\infty < \infty\}$$

μ-a.e., which proves (b). □

Corollary 1.3. *Let (S_n) be a martingale in (Ω, μ) with uniformly bounded increments. Then*

$$\Omega = \{\langle S \rangle_\infty < \infty\} \cup \left\{ \overline{\lim_n} S_n = +\infty \, , \, \underline{\lim}_n S_n = -\infty \right\} \cup N,$$

where $\mu(N) = 0$.

From Theorem 1.3, Corollary 1.2 and Example (2) in Section 1.3, we get:

Corollary 1.4 (Kolmogorov, Khinchin)**.** *Let $(X_k)_{k=1}^\infty$ be independent, identically distributed random variables with zero mean and finite variance. Let $(a_k)_{k=1}^\infty$ be a bounded real sequence. Then, if $S_n = \sum_{k=1}^n a_k X_k$,*

$$\sum_{k=1}^\infty a_k^2 < \infty \Leftrightarrow (S_n) \text{ converges a.e.},$$

$$\sum_{k=1}^\infty a_k^2 = \infty \Leftrightarrow \overline{\lim_n} S_n = +\infty, \quad \underline{\lim}_n S_n = -\infty \text{ a.e.}$$

Remark.

Theorem 1.3 and its consequences are the martingale counterpart of a group of deep results in Harmonic Analysis connecting the boundary behaviour of harmonic functions to their area function ([Cal50b], [Cal50a], [MZ38], [Spe43], [Ste61]). Let u be harmonic in $\mathbb{R}_+^n$ and let $F(u)$ be its Fatou set, as in remark in Section 1.5. Then (see [Ste70]):

$$F(u) = \{S_0 u < \infty\} \text{ a.e.},$$

where $(S_0 u)(x) = \lim_{t \to 0} (S_t u)(x)$, $S_t u$ being the truncated area function introduced at Remark (3) in Section 1.3.

1.7. **The Law of the Iterated Logarithm (LIL).** We saw in Section 1.6 that a martingale (S_n) behaves asymptotically well on the set $\{\langle S\rangle_\infty < \infty\}$ while, in the presence of boundedness restrictions on the increments, its behaviour is quite pathological on the set $\{\langle S\rangle_\infty = \infty\}$; in particular is unbounded μ-a.e. on this set. But, what can be said about the size of $|S_n|$ on $\{\langle S\rangle_\infty = \infty\}$? The precise answer is given by *the Law of the Iterated Logarithm.* In order to give a more accessible approach, we will only prove the upper bound, in the case where $\Omega = [0,1)$ and $\mu =$ Lebesgue measure.

Theorem 1.4 (Law of the iterated logarithm)**.** *Let $(S)_n$ be a dyadic martingale in $([0,1), |\cdot|)$. Then*

(a) *(Upper bound)*

$$\overline{\lim_n} \frac{|S_n|}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} \leq 1$$

a.e. on $\{\langle S\rangle_\infty = \infty\}$.

(b) *(Lower bound) If, in addition, (S_n) has uniformly bounded increments,*

$$\underline{\lim_n} \frac{|S_n|}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} \geq 1$$

a.e. on $\{\langle S\rangle_\infty = \infty\}$.

Corollary 1.5. *If $S_n = S_0 + \sum_{k=1}^n X_k$ is a dyadic martingale in $([0,1), |\cdot|)$ and $\sup_k |X_k| \leq C < \infty$, then*

$$\overline{\lim_n} \frac{|S_n|}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} = 1$$

a.e. on $\{\langle S\rangle_\infty = \infty\}$. In particular,

$$\overline{\lim_n} \frac{|S_n|}{\sqrt{n\log\log n}} \leq \sqrt{2}C$$

a.e. in $[0,1)$.

We will only give the proof of (a) here. We have mostly followed [Mak89a]. See also [Mak89a] for the proof of (b). The key property is the following:

Proposition 1.2. *Let (S_n) be a dyadic martingale in $([0,1), |\cdot|)$, with $S_0 = 0$. Then*

$$\int_0^1 e^{S_n - \frac{1}{2}\langle S\rangle_n}\, dx \leq 1.$$

Proof. By (1.10), $\langle S\rangle_n = \sum_{k=1}^{n} X_k^2$. Fix $I_{n-1} \in \mathcal{D}_{n-1}$. Then

$$\int_{I_{n-1}} e^{S_n - \frac{1}{2}\langle S\rangle_n}\,dx = e^{S_{n-1}-\frac{1}{2}\langle S\rangle_n}|I_{n-1}\int_{I_{n-1}} e^{X_n}\,dx$$
$$\leq (e^{S_{n-1}-\frac{1}{2}\langle S\rangle_n}|I_{n-1})(e^{\frac{1}{2}X_n^2}|I_{n-1}|) = \int_{I_{n-1}} e^{S_{n-1}-\frac{1}{2}\langle S\rangle_{n-1}}\,dx$$

the inequality being a consequence of the inequality $\frac{1}{2}(e^x + e^{-x}) \leq e^{\frac{1}{2}x^2}$, which holds fo any $x \in \mathbb{R}$. Now, adding up over all $I_{n-1} \in \mathcal{D}_{n-1}$ we get:

$$\int_0^1 e^{S_n - \frac{1}{2}\langle S\rangle_n}\,dx \leq \int_0^1 e^{S_{n-1}-\frac{1}{2}\langle S\rangle_{n-1}}\,dx.$$

So, iterating:

$$\int_0^1 e^{S_n - \frac{1}{2}\langle S\rangle_n}\,dx \leq \int_0^1 e^{S_0}\,dx = 1.$$

□

Lemma 1.2. *Let (S_n) be a dyadic martingale in $([0,1), |\cdot|)$, with $S_0 = 0$. Then, for $M, N > 0$,*

$$|\{x \in [0,1) \,:\, \exists n \in \mathbb{N},\ S_n(x) > M,\ \langle S\rangle_n(x) \leq N\}| \leq e^{-\frac{M^2}{2N}}.$$

Proof. Let

$$A = \{x \in [0,1) \,:\, \exists n \in \mathbb{N},\ S_n(x) > M,\ \langle S\rangle_n(x) \leq N\}.$$

We define the stopping time τ by,

$$\tau(x) = n \Leftrightarrow S_1 \leq M, \ldots, S_{n-1} \leq M, \quad S_n > M$$
$$\tau(x) = \infty \Leftrightarrow \sup_n S_n \leq M.$$

Observe that if $x \in A$ and $n_0(x) = \tau(x) < \infty$, then $S_{n_0}(x) > M$, and $\langle S\rangle_{n_0}(x) \leq N$. In particular

$$S_n^\tau(x) > M, \text{ and } \langle S^\tau\rangle_n(x) = \langle S^\tau\rangle_{n_0}(x) \leq \langle S\rangle_n(x) \leq N$$

for all $n \geq n_0$.

Now, fix $t > 0$ and apply Proposition 1.2 to the martingale (tS_n^τ). Then:

$$1 \geq \int_0^1 e^{tS_n^\tau - \frac{t^2}{2}\langle S_n^\tau\rangle_n}\,dx \geq \int_A e^{tS_n^\tau - \frac{t^2}{2}\langle S_n^\tau\rangle_n}\,dx.$$

By the preceding comments,

$$\varliminf_n e^{tS_n^\tau - \frac{t^2}{2}\langle S_n^\tau\rangle_n(x)} \geq e^{tM - \frac{t^2}{2}N}$$

whenever $x \in A$ so, by Fatou's lemma:

$$1 \geq e^{tM - \frac{t^2}{2}N}|A|$$

and the result follows by choosing $t = \frac{M}{N}$. □

Now, the upper bound of the LIL easily follows from Lemma 1.2:

Proof of theorem 1.4(a).

Fix $\varepsilon > 0$ and set

$$A_k = \{x : \exists n,\ (1+\varepsilon)^k \le \langle S\rangle_n(x) < (1+\varepsilon)^{k+1},$$
$$S_n(x) > (1+\varepsilon)\sqrt{2\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)}\,\}.$$

It is enough to show that

$$|\{x : x \in A_k \text{ for infinitely many } k\text{'s}\}| = 0.$$

From Lemma 1.2 with

$$M = (1+\varepsilon)\sqrt{2(1+\varepsilon)^k \log\log(1+\varepsilon)^k} \text{ and } N = (1+\varepsilon)^{k+1},$$

we get $|A_k| \le \frac{C(\varepsilon)}{K^{1+\varepsilon}}$ so the result follows from the standard Borel–Cantelli argument. □

REMARKS.

(1) The proof of Proposition 1.2 and Lemma 1.2 show that only the fact, that $e^{tS_n - ct^2\langle S\rangle_n}$ is a supermartingale, for $t > 0$ and some fixed constant $c > 0$, is needed to get an upper bound of the LIL.
(2) Lemma 1.2 is the martingale version of the so called "good-λ" inequalities in Harmonic Analysis.
(3) Theorem 1.4 is valid for abstract martingales with uniformly bounded increments ([Sto64], [Sto84]).

Theorem 1.5. *Let (S_n) be a dyadic martingale in (Ω,μ) with uniformly bounded increments. Then*

$$\overline{\lim_n} \frac{|S_n|}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} = 1$$

μ-a.e. on the set $\{\langle S\rangle_\infty = \infty\}$.

In general, the proof of the LIL in the abstract setting also uses exponential inequalities but requires restrictions on the size of the increments, stronger in the case of the lower bound ([Sto64], [Sto84]). In some specific situations, such restrictions can be dropped from the upper bound (Theorem 1.4(a) is such an example; we will see another one at the end of Chapter 3).

Note that Corollary 1.5 gives a substantial improvement of the trivial global bound $S_n = O(n)$.

2. Dyadic martingales, Bloch functions and conformal mappings

2.1. **Bloch functions and conformal mappings.** Let f be analytic in the unit disc $\mathbb{D}$ (resp. the upper half-plane $\mathbb{R}^2_+$). We say that f is *a*

Bloch function ($f \in \mathcal{B}(\mathbb{D})$, resp. $f \in \mathcal{B}(\mathbb{R}^2_+)$) if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| = \|f\|_B < \infty$$

$$(\text{resp.} \ \sup_{z\in\mathbb{R}^2_+} \operatorname{Im} z|f'(z)| = \|f\|_B < \infty \ ).$$

Analogously, a harmonic function u in $\mathbb{D}$ (resp. $\mathbb{R}^2_+$) is said to be *Bloch* (u$\in \mathcal{B}_h(\mathbb{D})$ or $u \in \mathcal{B}_h(\mathbb{R}^2_+)$) if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\nabla u(z)| = \|u\|_B < \infty$$

$$(\text{resp.} \ \sup_{z\in\mathbb{R}^2_+} \operatorname{Im} z|\nabla u(z)| = \|u\|_B < \infty \ ).$$

Since $|f'| = |\nabla u|$ for any analytic function f such that $\operatorname{Re} f = u$ or $\operatorname{Im} f = u$, it follows that f is Bloch iff $\operatorname{Re} f, \operatorname{Im} f$ are Bloch and all of them have the same norm.

It is easy to see that the definitions in $\mathbb{D}$ and $\mathbb{R}^2_+$ are equivalent via Möbius transformation. Since computations are often easier in the upper half-plane, in most of this section we will work in the upper half plane and, occasionally, the corresponding results will be also stated in the unit disc.

The main properties of harmonic Bloch functions are summarized in the following proposition. We recall that the hyperbolic metric in $\mathbb{R}^2_+$ is the metric with length element $ds = \frac{|ds|}{\operatorname{Im} z}$.

Proposition 2.1. *Let $u \in \mathcal{B}_h(\mathbb{R}^2_+)$. Then*

(a) *For any Möbius transformation $g \in \operatorname{Aut}(\mathbb{R}^2_+)$, $u \circ g \in \mathcal{B}_h(\mathbb{R}^2_+)$ and $\|u \circ g\|_B = \|u\|_B$, that is, B_h is conformally invariant.*

(b) *For any z, $w \in \mathbb{R}^2_+$,*

$$|u(z) - u(w)| \leq \|u\|_B d_h(z, w),$$

where d_h denotes hyperbolic metric in $\mathbb{R}^2_+$, that is, $u : (\mathbb{R}^2_+, d_h) \to \mathbb{R}$ is Lipschitz. In particular, if $R \subset \mathbb{R}^2_+$ has hyperbolic diameter at most d, then $\underset{R}{osc}\ u \leq \|u\|_B d$.

(c) $|u(x, y)| \leq |u(x, 1)| + \|u\|_B \log \frac{1}{y}$.

(d) *If $I \subset \mathbb{R}$ is any interval and*

$$T^+(I) = \left\{(x, y) \ : \ x \in I, \ \frac{|I|}{2} \leq y \leq |I|\right\},$$

then $\underset{T^+(I)}{osc}\ u \leq C\|u\|_B$ for some absolute constant C.

(e) *If $x_0 \in \mathbb{R}$, $0 < \alpha < \pi/2$ and*

$$C_{n,\alpha} = \left\{(x, y) \ : \ |x - x_0| \leq \ ytg\ \alpha, \ 2^{-(n+1)} \leq y \leq 2^{-n}\right\},$$

then $\underset{C_{n,\alpha}}{osc}\ u \leq C\|u\|_B$ for some absolute constant C.

Proof. (a) If f is analytic in $\mathbb{R}^2_+$, with $\operatorname{Re} f = u$, then

$$\begin{aligned}|\nabla(u\circ g)(z)| &= |(f\circ g)'(z)| = |f'(g(z))||g'(z)|\\ &= |\nabla u(g(z))||g'(z)|\end{aligned}$$

and the result follows from the hypothesis together with the identity $|g'(z)|\operatorname{Im} z = \operatorname{Im} g(z)$ which holds for any $g \in \operatorname{Aut}(\mathbb{R}^2_+)$.

(b) From (a) and the fact that Möbius transformations are isometries in the hyperbolic metric ([Pom92], chap. 1, section 1.2), we can assume that $z = i, w = iy, y > 0$. The result then follows by integration.

(c) immediate from integration.

(d), (e) follow from (b) and the fact that $T^+(I)$ and $C_{n,\alpha}$ have hyperbolic diameter bounded by some fixed absolute constant. $\square$

There is a close connection between Bloch functions and conformal mappings.

Theorem 2.1. *Let $f : \mathbb{R}^2_+ \to \mathbb{C}$ be conformal. Then*

$$\operatorname{Im} z \left|\frac{f''(z)}{f'(z)}\right| \le 6.$$

In particular, $\log f' \in \mathcal{B}(\mathbb{R}^2_+)$, $\log|f'|, \operatorname{Arg} f' \in \mathcal{B}(\mathbb{R}^2_+)$ and $\|\log f'\|_B \le 6$. Conversely, if $b \in \mathcal{B}(\mathbb{R}^2_+)$ and $\|b\|_B \le 1$, then there is a conformal map $f : \mathbb{R}^2_+ \to \mathbb{C}$ such that $b = \log f'$. Both constants 6 and 1 are best possible. (See [Pom92], *Proposition 4.1 for the proof).*

Remark. Note that by (e) in Proposition 2.1, the non-tangential boundary behaviour of any Bloch function is the same, up to a bounded error, than its vertical boundary behaviour.

2.2. **Applications to the boundary behaviour of Bloch functions and conformal mappings.** Suppose that $u \in \mathcal{B}_h(\mathbb{R}^2_+)$ and we fix an interval in $\mathbb{R}$, that will be assumed to be the unit interval $[0, 1]$. We will see in this section that it is possible to associate to u a dyadic martingale in $([0, 1), |\cdot|)$ in such a way that the boundary behaviour of u turns out to be essentially equivalent to the asymptotic behaviour of the martingale, so that many questions on the boundary behaviour of u can be directly reformulated in terms of martingale. The key property of Bloch functions needed for such reduction is given by the following.

Proposition 2.2. *Let $u \in \mathcal{B}(\mathbb{R}^2_+)$, $I = [a, b]$, $l = b - a$. Then*

(a) $\lim_{\varepsilon\to 0} \fint_I u(x, \varepsilon)\, dx = S_I$ *exists and is finite.*

(b) *There are absolute constants $C_1, C > 0$ such that*

$$\left|\fint_I (u(x,\varepsilon) - u(x,l))\, dx\right| \le C_1\|u\|_B \tag{2.1}$$

and, in particular, $|S_I - u(x, l)| \le C\|u\|_B$ for each $x \in I$.

Proof. Fix ε, $0<\varepsilon<l$ and let $R=I\times[\varepsilon,l]$. By Green's theorem

$$\int_{\partial R}\frac{\partial u}{\partial n}=0,$$

so

$$\begin{aligned}\left|\int_I\frac{\partial u}{\partial y}(x,\varepsilon)\,dx-\int_I\frac{\partial u}{\partial y}(x,l)\,dx\right|&=\left|\int_\varepsilon^l\frac{\partial u}{\partial x}(b,y)\,dy-\int_\varepsilon^l\frac{\partial u}{\partial x}(a,y)\,dy\right|\\&\le 2\|u\|_B\int_\varepsilon^y\frac{dy}{y}=2\|u\|_B\log\frac{l}{\varepsilon}.\end{aligned}$$

Therefore, if $\varphi(y)=\int_I u(x,y)\,dx$, last inequality shows that

$$|\varphi'(y)|\le|\varphi'(l)|+2\|u\|_B\log\frac{l}{y},$$

thus $\varphi'\in L^1(0,l)$, which implies (a).

To prove (b), note that it is enough to prove (2.1), by proposition 2.1 (d). But, by (a):

$$\begin{aligned}|\varphi(\varepsilon)-\varphi(l)|=\left|\int_\varepsilon^l\varphi'(y)\,dy\right|&\le\int_\varepsilon^l\left(|\varphi'(l)|+2\|u\|_B\log\frac{l}{y}\right)dy\\&\le|\varphi'(l)|l+2\|u\|_B\int_0^l\log\frac{l}{y}\,dy\\&\le\left|\int_I\frac{\partial u}{\partial y}(x,l)\,dx\right|l+2l\|u\|_B\int_0^1 log\frac{1}{t}\,dt\\&\le\left(1+2\int_0^1\log\frac{1}{t}\,dt\right)\|u\|_Bl.\end{aligned}$$

□

REMARK 1.

Inequality (2.1) can be seen as a weak BMO-type inequality for Bloch functions. It should be pointed out that Bloch functions do not have, in general, radial limits.

Suppose now that we restrict our attention to a fixed interval, say $[0,1]$. Then, by example (5) in Section 1.2, the assignment

$$S_{I_n}=\lim_{\varepsilon\to0}⨍_{I_n}u(x,\varepsilon)\,dx$$

defines a dyadic martingale (S_n) in $([0,1),|\cdot|)$.

Proposition 2.3. *If $u\in\mathcal{B}_h(\mathbb{R}^2_+)$, there is a dyadic martingale (S_n) in $([0,1),|\cdot|)$ such that*

(a) $|S_n(x)-u(x,y)|\le C\|u\|_B$, *for any $x\in[0,1)$ and $2^{-(n+1)}\le y\le 2^{-n}$.*

(b) $\sup_n|S_n-S_{n-1}|\le 2C\|u\|_B$, *that is, (S_n) has uniformly bounded increments.*

Here C is some absolute constant.

Proof. (a) follows from Proposition 2.2 and Proposition 2.1 (d). To prove (b), note that, by (a)

$$\begin{aligned}|S_n(x) - S_{n-1}(x)| &\leq |S_n(x) - u(x, 2^{-n})| + |u(x, 2^{-n}) - S_{n-1}(x)| \\ &\leq 2C\|u\|_B.\end{aligned}$$

□

REMARK 2.

If $u, (S_n)$ are as in Proposition 2.3, then (S_n) pointwise approximates the vertical behaviour of u (thus also its non-tangential behaviour, by the remark at the end of section 2.1) up to a bounded error term. Therefore, any statement on the vertical (or non-tangential) boundary behaviour of u stable up to a bounded error is equivalent to the corresponding statement for the boundary behaviour of the martingale. In particular,

$$\left\{x \in [0,1) \,:\, \overline{\lim_{z \lhd x}} |u(z)| < \infty\right\} = \left\{x \in [0,1) \,:\, \overline{\lim_{z \lhd x}} |S_n(x)| < \infty\right\},$$

$$\left\{x \in [0,1) \,:\, \lim_{z \lhd x} u(z) = \infty\right\} = \left\{x \in [0,1) \,:\, \lim_{z \lhd x} S_n(x) = \infty\right\}.$$

If we apply the LIL to the martingale (S_n) and use Proposition 2.3 we get the following substantial improvement of the pointwise growth estimate given by Proposition 2.1, (c).

Theorem 2.2 (Law of the Iterated Logarithm for Bloch functions)**.** *If $u \in \mathcal{B}_h(\mathbb{R}^2_+)$, then*

$$\overline{\lim} \frac{|u(x,y)|}{\sqrt{\log \frac{1}{y} \log\log\log \frac{1}{y}}} \leq C\|u\|_B$$

for a.e. $x \in \mathbb{R}$, where C is some absolute constant.

Corollary 2.1 (Law of the iterated logarithm for conformal mappings)**.** *If $f : \mathbb{R}^2_+ \to \mathbb{C}$ is conformal, then*

$$\overline{\lim_{y \to 0}} \frac{|\log|f'(x+iy)||}{\sqrt{\log \frac{1}{y} \log\log\log \frac{1}{y}}} \leq C$$

for a.e. $x \in \mathbb{R}$, where C is some absolute constant.

Theorem 2.3. *Let $u \in B_h(\mathbb{D})$. Then*

$$\overline{\lim_{r \to 1}} \frac{|u(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \leq C\|u\|_B$$

for a.e. $e^{i\theta} \in \partial\mathbb{D}$.

Corollary 2.2. *If $f : \mathbb{D} \to \mathbb{C}$ is conformal,*

$$\overline{\lim_{r\to 1}} \frac{|\log|f'(re^{i\theta})||}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \leq C$$

for a.e. $e^{i\theta} \in \partial\mathbb{D}$.

Remark 3.
Theorem 2.3 is due to Makarov ([Mak85]), see also [Mak89a], [GM], [Lyo90] and [Pom92]. An extension of inequality (2.1) in Lipschitz domains with the corresponding applications to boundary behaviour, also from a martingale point of view, can be found in [Llo98]. [BKM88], [BKM90], [BM89] and [BM99] contain additional material on the Law of the Iterated Logarithm. See [BFL00], [GK01] and [GKLN01] for more recent related results.

3. Exceptional sets, martingales and measures

3.1. **Hausdorff measures.** Hausdoff measures and contents are often used in Analysis to distinguish the size of small sets (of Lebesgue measure zero).

Let $\Psi : [0, +\infty) \to [0, +\infty)$ be continuous, increasing, such that $\Psi(0) = 0$. We call such functions *measure functions.*
Let $E \subset \mathbb{R}^N$, and Ψ a measure function. The *Hausdorff Ψ-content* of E is defined by

$$M_\Psi(E) = \inf \sum_k \Psi(\text{diam } Q_k),$$

where the infimum is taken over the coverings $\{Q_k\}$ of E by cubes. The *Hausdorff Ψ-measure* of E is

$$\Lambda_\Psi = \sup_{\delta>0} \inf \sum_k \Psi(\text{diam } Q_k),$$

where now the infimum is taken over all coverings $\{Q_k\}$ of E by cubes with diam $Q_k \leq \delta$.

Specially important are the choices $\Psi(t) = t^\alpha$, $0 < \alpha \leq N$, in which case we just write $M_\alpha(E)$ and $\Lambda_\alpha(E)$. Observe that M_N is just Lebesgue outer content.

The *Hausdorff dimension* of E is defined by

$$\dim E = \inf \{\alpha \in [0, N] \, : \, M_\alpha(E) = 0\}.$$

As an example, the usual Cantor set has dimension $\frac{\log 2}{\log 3}$ and the snowflake curve has dimension $\frac{\log 4}{\log 3}$ (see [Fal85]).

Proposition 3.1. (a) *$M_\Psi(E) = 0$ iff $\Lambda_\Psi(E) = 0$.*

(b) *If $\Psi_1 \leq \Psi_2$, then $M_{\Psi_1} \leq M_{\Psi_2}$. If $\overline{\lim}_{t\to 0} \frac{\Psi_1(t)}{\Psi_2(t)} < \infty$, then $M_{\Psi_2}(E) > 0$ whenever $M_\Psi(E) > 0$.*

(c) *If* $\overline{\lim}_{t\to 0}\frac{\Psi(t)}{t^\alpha} < \infty$ *and* $M_\Psi(E) > 0$, *then* $\dim E \geq \alpha$.

Since, by (a) M_Ψ, Λ are null simultaneously, M_Ψ is more used in Analysis because it has the advantage that it is finite on bounded sets.

Hausdorff measures will appear in the rest of the chapter in two situations: to obtain fine estimates on the size of some exceptional sets of Lebesgue measure zero arising from the boundary behaviour of Bloch functions and also when comparing a given measure to Hausdorff measures.

3.2. **Exceptional sets.** We saw in Section 1.5 (see Corollary 1.2) that martingales with uniformly bounded increments exhibit a dichotomy-type behaviour implying that sets like

$$\left\{\lim_n S_n = \pm\infty\right\}, \left\{\overline{\lim_n}|S_n| < \infty\right\} \setminus F(S)$$

have measure zero. Now, we can ask about the size of these sets, in terms of Hausdorff dimension. The following theorems ([Mak89a], [Mak89b]) give an extaordinarily precise answer to that question. We refer to [Mak89a], [GM] and [Llo98] for the proofs.

Theorem 3.1. *Let* (S_n) *be a dyadic martingale in* $([0,1), |\cdot|)$, *with uniformly bounded increments. Then, for any interval* $I \subset [0,1)$, *either*

$$|F(S) \cap I| > 0 \text{ or } M_\Psi\left(\left\{\lim_n S_n = \pm\infty\right\} \cap I\right) > 0,$$

where $\Psi(t) = t\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. *If this second possibility occurs then, in particular,* $\dim\left(\{\lim_n S_n = \pm\infty\} \cap I\right) = 1$. *Furthermore, the measure function* Ψ *is sharp.*

Theorem 3.2. *Let* (S_n) *be a dyadic martingale in* $([0,1), |\cdot|)$ *with uniformly bounded increments. Then for any interval* $I \subset [0,1)$, *either*

$$|F(S) \cap I| > 0 \text{ or } \dim\left(\left\{\sup_n |S_n| < \infty\right\} \cap I\right) = 1.$$

The result is best possible in the sense that "dimension one" cannot be replaced by $M_\Psi > 0$, *where* $\Psi(t) = o(t^\alpha)$ *for any* $0 < \alpha < 1$.

The corresponding translations in terms of Bloch functions follow from Proposition 2.3. We recall that $F(u)$ denotes the Fatou set of u.

Theorem 3.3. *Let* $u \in \mathcal{B}_h(\mathbb{R}^2_+)$. *Then, for any interval* $I \subset \mathbb{R}$, *either*

$$|F(u) \cap I| > 0 \text{ or } M_\Psi\left(\left\{x \in \mathbb{R} : \lim_{z \lhd x} u(z) = \pm\infty\right\} \cap I\right) > 0,$$

where Ψ *is as in Theorem 3.1. In particular, if the second possibility occurs, then*

$$\dim\left(\left\{x \in \mathbb{R} : \lim_{z \lhd x} u(z) = \pm\infty\right\} \cap I\right) = 1.$$

Theorem 3.4. *Let $u \in \mathcal{B}_h(\mathbb{R}^2_+)$. Then, for any interval $I \subset \mathbb{R}$, either*

$$|F(u) \cap I| > 0 \text{ or } \dim\left(\left\{x \in \mathbb{R} : \overline{\lim_{z \lhd x}} |u(z)| < \infty\right\} \cap I\right) = 1.$$

Again, counterexamples showing that that Theorems 3.3, 3.4 are sharp can be obtained from the martingale counterexamples.

Theorem 3.3 solved a problem that had challenged many analysts for a long time: *the angular derivative problem.* Let $f : \mathbb{R}^2_+ \to \mathbb{C}$ be conformal. Suppose that f has a finite non-tangential limit $f(x)$ at $x \in \mathbb{R}$. Then we say that f has an angular derivative at x if the limit

$$\lim_{z \lhd x} \frac{f(z) - f(x)}{z - x}$$

exists (possibly ∞). By [Pom92], Proposition 4.7, f has a finite angular derivative at x iff f' has the same non-tangential limit at x.

From Theorem 3.3 we get ([Mak89b]):

Theorem 3.5. *Let $f : \mathbb{R}^2_+ \to \mathbb{C}$ be conformal. Then f has a finite angular derivative on a set $E \subset \mathbb{R}$ with $M_\Psi(E) > 0$, where Ψ is as in Theorem 3.1.*

Proof. We restrict our attention to $[0, 1)$. Approximate $u = \log |f'|$ by a dyadic martingale (S_n) in $[0, 1)$, as in Section 2.2. If $|F(S)| > 0$, then by Proposition 2.3,

$$\left|\left\{x : \overline{\lim_{z \lhd x}} |u(z)| < \infty\right\}\right| > 0$$

so by the local Fatou theorem for analytic functions (see [Koo98]) it follows that f' has finite non-tangential limit, hence f has finite angular derivative on a set of positive measure.

If $|F(S)| = 0$ then by Theorem 3.1,

$$M_\Psi \left\{\lim_n S_n = -\infty\right\} > 0,$$

so, by Proposition 2.3,

$$M_\Psi \left\{x\,;\, \lim_{z \lhd x} u(z) = -\infty\right\} > 0,$$

thus

$$M_\Psi \left\{x : f' \text{ has non-tangential limit } 0\right\} > 0.$$

Since f has a finite non-tangential limit at any point at which f' does, the result follows. ☐

Remarks.

(1) Theorems 3.3 and 3.4 have been extended to the case of Bloch harmonic functions in Lipschitz domains ([Llo98]). The proof also uses dyadic martingales, the key for the reduction being an appropriate version on inequality (2.1) in terms of harmonic measure.

(2) Makarov actually proved Theorem 3.4 for analytic Bloch functions, which is an stronger result. See [Don01b] and [Roh93] for further refinements in the analytic case.

3.3. **Comparison of measures.** Many problems in Analysis lead to the following situation: we are given a set E with $\mu(E) > 0$, where μ is some measure intrinsically associated to the problem and we ask how large E is, in terms of the ambient space, for instance what can be said about its Hausdorff dimension. Then one would like to compare μ with a Hausdorff measure.

Let μ be a positive Borel measure in $\mathbb{R}^N$ and Ψ a measure function. We say that μ is *absolutely continuous* with respect to Hausdorff Ψ-measure (denoted $\mu \ll \Lambda_\Psi$) if

$$\mu(E) > 0 \Rightarrow M_\Psi(E) > 0. \tag{3.1}$$

To check (3.1) always requires a control of the μ-measures of cubes in terms of their diameters. If, for instance, we have the following global control

$$\mu(Q) \le C\Psi(\operatorname{diam} Q)$$

for any cube Q then $\mu \ll \Lambda_\Psi$ is automatically satisfied. However, a local control is enough to get the same conclusion. As in the one-dimensional case, let

$$\mathcal{D}_n = \left\{ \prod_{i=1}^{N} \left[\frac{m_i - 1}{2^n}, \frac{m_i}{2^n} \right) \,:\, m_i \in \mathbb{Z},\, i = 1, \dots, N \right\}$$

be the family of dyadic cubes of the generation n in $\mathbb{R}^N$. For $x \in \mathbb{R}^N$, let $Q_n(x)$ be the only cube in $\mathcal{D}_n$ containing x.

Proposition 3.2. *If*

$$\overline{\lim_n} \frac{\mu(Q_n(x))}{\Psi(\operatorname{diam} Q_n(x))} < \infty$$

for μ-a.e. x, then $\mu \ll \Lambda_\Psi$.

Proof. Choose E with $\mu(E) > 0$. It is enough to assume that

$$\mu(Q_n(x)) \le M\Psi(\operatorname{diam} Q_n(x))$$

for all $x \in E$, all n and some $M > 0$. Suppose that $E \subset \cup_k Q_k$ where Q_k are cubes. Fix such a Q_k and let $n \in \mathbb{N}$ be such that

$$2^{-n} \le \operatorname{diam} Q_k < 2^{-(n-1)}.$$

Then Q_k is covered by, at most, 2^N cubes $Q_k^1, \cdots, Q_k^{2^n} \in \mathcal{D}_n$ and

$$\left\{ Q_k^j \,:\, j = 1, \dots, 2^n,\, k = 1, 2, \dots \right\}$$

is a covering of E by dyadic cubes. It can also be assumed that $Q_k^j \cap E \neq \varnothing$ for all k, j so

$$\Psi(\operatorname{diam} Q_k^j) \ge M^{-1}\mu(Q_k^j).$$

Then,

$$\sum_{k=1}^{\infty} \Psi(\operatorname{diam} Q_k) \geq 2^{-n} \sum_{k=1}^{\infty} \sum_{j=1}^{2^n} \Psi(\operatorname{diam} Q_k^j) \geq M^{-1} 2^{-N} \mu(E)$$

which shows that $M_\Psi(E) > 0$. □

We list now some remarkable situations where comparison of measures arises.

EXAMPLES.

(1) (Zygmund measures).
A positive measure in $\mathbb{R}$ is a *Zygmund measure* if there is $C > 0$ such that

$$|\mu(I) - \mu(I')| \leq C|I| \tag{3.2}$$

for any two adjacent intervals $I, I' \subset \mathbb{R}$ with the same length. From (3.2) it is easy to get the global estimate $\mu(I) \leq C_1 |I| \log \frac{1}{|I|}$ which implies $\mu \ll \Lambda_{\Psi_1}$, where $\Psi_1(t) = t \log \frac{1}{t}$. However, the optimal result ([Mak89b]) asserts that $\mu \ll \Lambda_\Psi$, where $\Psi(t) = t\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}$ and this measure function is sharp. The idea behind this result can be described briefly as follows. Let us restrict to the unit interval. Then $S_I = \frac{\mu(I)}{|I|}$ defines a dyadic martingale in $([0,1), |\cdot|)$ with uniformly bounded increments so, by Corollary 1.5, $S_n = O(\sqrt{n \log\log n})$ a.e. Actually, the stronger estimate $S_n = O(\sqrt{n \log\log n})$ μ-a.e. also holds and this (which is the difficult part of the proof) implies the result.

(2) (Harmonic measure).
Let $\Omega \subset \mathbb{C}$ be a Jordan domain and $f : \mathbb{D} \to \Omega$ a Riemann mapping, with $f(0) = z_0 \in \Omega$. By Caratheodory's Theorem ([Pom92], Theorem 2.6), f extends to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. If $E \subset \partial\Omega$, we define the *harmonic measure* of E, in Ω, from z by $\omega(E, z_0, \Omega) = \frac{|f^{-1}(E)|}{2\pi}$, that is, the value at z_0 of the solution to the Dirichlet problem in Ω with boundary values $\mathbf{1}_E$ on $\partial\Omega$. Then $\omega(\cdot, z_0, \Omega)$ is a Borel probability measure in $\partial\Omega$, called the *harmonic measure* with base point z_0. A lot of remarkable results in classical Complex Analysis rely on harmonic measure estimates on special domains (see [Fuc67], [Nev70], [Bae88]).

One of the most challenging problems in Geometric Function Theory during the last thirty years has been to try to figure out the geometric structure of harmonic measure in plane or higher dimensional domains. In other words, how big are sets of positive harmonic measure in terms of Hausdorff measures? Or, in the language introduced above; for which Ψ's is it true that $\omega \ll \Lambda_\Psi$? We discuss this problem in the next section.

(3) (Distortion of homeomorpisms of the real line).

Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism and μ_g the Lebesgue-Stieltjes measure associated to g, that is, $\mu_g(E) = |g(E)|$. Increasing version of the Cantor function show that g can be singular, that is, can map a set of zero length onto a set of positive length. Therefore this shows that, in general, $\mu_g \not\ll \Lambda_1$. The situation can be worse. We say that g is *quasi-symmetric* if there is $M > 0$ such that

$$M^{-1} \leq \frac{g(x+t) - g(x)}{g(x) - g(x-t)} \leq M$$

for all $x \in \mathbb{R}$, and $t > 0$. In [BA56] it was shown that quasi-symmetric homeomorphisms are exactly those which can be quasi-conformally extended to $\mathbb{R}^2_+$. It was also proved there that there are singular quasi-symmetric homeomorphisms, so it is still possible that $\mu_g \not\ll \Lambda_1$ even if g is quasi-symmetric. This suggests the question of how much can g expand, that is, how large must $E \subset \mathbb{R}$ be if we know that $|g(E)| > 0$? In terms of Hausdorff measures, this is equivalent to ask whether $\mu_g \ll \Lambda_\Psi$ for some measure function Ψ. We will come back to this problem in Section 3.5.

3.4. **On the size of harmonic measure.** As in example (2) in Section 3.3, let Ω be a Jordan domain and suppose that one seeks estimates of the form $\omega \ll \Lambda_\Psi$ where ω is harmonic measure in Ω from some fixed base point in Ω.

If $\partial\Omega$ is rectifiable, then by a result of Riesz ([Pom92], Theorem 6.8), $f' \in H^1(\mathbb{D})$, where $f : \mathbb{D} \to \Omega$ is a Riemann mapping and ω, $\Lambda_1|\partial\Omega$ are mutually absolutely continuous, that is, $\omega(E) > 0$ iff $M_1(E) > 0$.

However, if $\partial\Omega$ is non-rectifiable, it is not necessarily true that $\omega \ll \Lambda_1|\partial\Omega$ ([Lav63]). It may even happen that $\omega(E) = 1$ but $M_1(E) = 0$ for some $E \subset \partial\Omega$, so the conformal mapping can compress a set of full length in $\partial\mathbb{D}$ into a set of zero length ([MP73], see also [Pom92], Section 6.5). In fact, this pathological behaviour is typical from domains with fractal boundary.

Therefore, the next step is to ask whether a conformal mapping can compress much further, or, for which α's, with $0 < \alpha < 1$, it is true that $\omega \ll \Lambda_\alpha$?

Suppose that $\Omega \subset \mathbb{C}$ is a Jordan domain, $z_0 \in \Omega$, $d = \operatorname{dist}(z_0, \partial\Omega)$ and $0 < r < \frac{d}{2}$. Let $E \subset \partial\Omega$ with $\operatorname{diam} E = r$. Assume that $0 \in E$, $z_0 \in \mathbb{R}$, $d \leq z_0$. Then $E \subset D = \{z : |z| < r\}$ and if Ω_0 is the component of $\Omega \setminus \overline{D}$ containing z_0 then, by the Maximum Principle, $\omega(E, z_0, \Omega) \leq \omega(\partial D, z_0, \Omega_0)$. Again by the Maximum Principle we can assume that $\partial D \cap \partial\Omega_0$ consists on one single arc J. If $T(z) = \frac{r}{z}$ and $T(z_0) = z_0'$, $T(\Omega_0) = \Omega'$, $T(\partial\Omega_0 \setminus \partial D) = \gamma'$ and $T(J) = J'$ then $\Omega' \subset \mathbb{D}$, γ' is some arc in $\mathbb{D}$ joining two points of $\partial\mathbb{D}$ (possibly the same) and separating z_0' from 0, and $J' = \partial\Omega' \setminus \gamma' = \partial\Omega' \cap \partial\mathbb{D}$. By conformal invariance, $\omega(\partial D, z_0, \Omega_0) = \omega(J', z_0', \Omega')$.

From the Beurling–Carleman–Milloux estimate ([Nev70]), the last harmonic measure does not exceed

$$\omega([-1,0], z_0', \mathbb{D} \setminus [-1,0]) = \frac{4}{\pi} \arctan \sqrt{z_0'} \leq \frac{4}{\pi} \arctan \sqrt{\frac{r}{d}}.$$

Therefore, we have proved the global estimate

$$\omega(E, z_0, \Omega) \leq C(z_0, \Omega)\sqrt{\operatorname{diam} E}$$

which implies $\omega \ll \Lambda_{\frac{1}{2}}$.

An important advance was obtained by Carleson [Car73], who proved that $\omega \ll \Lambda_{\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$. Kaufman and Wu ([KW82]) showed that there are Jordan domains such that $\omega \not\ll \Lambda_\Psi$, where $\Psi(t) = t \exp\{C\sqrt{\log \frac{1}{t}}\}$ for any $C > 0$. The final achievement is due to Makarov ([Mak85]), who proved the following astonishingly precise result:

Theorem 3.6. *There is some absolute constant $C > 0$ such that $\omega \ll \Lambda_\Psi$, where*

$$\Psi(t) = t \exp\left\{ C\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}} \right\}.$$

Furthermore, Ψ is sharp, up to the value of C.

We will give the proof of the first statement in Theorem 3.6 in the specific case where $\partial\Omega$ is a quasicircle. We say that a Jordan curve J is a *quasicircle* if there is a constant $M \geq 1$ such that if $a, b \in J$, then $\operatorname{diam} J(a,b) \leq M|a-b|$ where $J(a,b)$ is the subarc of J with smallest diameter joining a, b. A Jordan domain bounded by a quasicircle is called a *quasidisc*.

Proof of Theorem 3.6 for quasidiscs.

Let $f : \mathbb{D} \to \Omega$ be conformal. Let $A \subset \mathbb{D}$, with $|A| > 0$. By Corollary 2.2,

$$\overline{\lim_{r\to 1}} \frac{|\log|f'(re^{i\theta})||}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \leq C < \infty$$

for a.e. $e^{i\theta} \in \partial\mathbb{D}$ and some absolute constant $C \geq 1$.

Choose $A' \subset A$, with $|A'| > 0$ and r_0, $0 < r_0 < 1$ such that

$$|f'(re^{i\theta})| \geq \exp\left\{ -2C\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}} \right\} \tag{3.3}$$

whenever $e^{i\theta} \in A'$, and $r_0 \leq r < 1$.

By [Pom92], Proposition 4.19 or Theorem 4.20, there is an absolute constant $K \geq 1$ such that if $I \subset \partial\mathbb{D}$ is any arc and

$$z \in T(I) = \left\{ \left(1 - \frac{|I|}{2\pi}\right) e^{i\theta} \,:\, e^{i\theta} \in I \right\},$$

then

$$|I||f'(z)| \leq K \operatorname{diam} f(I) \tag{3.4}$$

(note that the quasidisc property is not needed here).

Now, let $\delta_0 = (KM)^{-1} 2\pi(1-r_0) \min_{|z|\leq r_0} |f'(z)|$, where M is the quasicircle constant. Cover $f(A')$ by open squares $\{Q_k\}$ of diameters $\{\delta_k\}$, with $\delta_k \leq \delta_0$, and such that $Q_k \cap f(A') \neq \varnothing$ for all k. Denote by J_k the maximal subarc of $\partial\Omega$ intersecting D_k, and set $I_k = f^{-1}(J_k)$. Then, by construction, $A' \subset \cup_k I_k$ and $A' \cap I_k \neq \varnothing$ for all k. If $e^{i\theta_k} \in A' \cap I_k$, let

$$z_k = \left(1 - \frac{I_k}{2\pi}\right) e^{i\theta_k} \in T(I_k).$$

By the quasicircle property and (3.4) we get:

$$|I_k||f'(z_k)| \leq K \operatorname{diam} J_k \leq KM\delta_k \leq 2\pi(1-r_0) \min_{|z|\leq r_0} |f'(z)| \tag{3.5}$$

which implies that $|I_k| \leq 2\pi(1-r_0)$ and $|z_k| \geq r_0$.

Now, if Ψ is any measure function, we get from (3.3) and (3.5):

$$\begin{aligned} \sum_k \Psi(\delta_k) &\geq \sum_k \Psi\left((KM)^{-1}|I_k||f'(z_k)|\right) \\ &\geq \sum_k \Psi\left((KM)^{-1}|I_k| \exp\left\{-2C\sqrt{\log\frac{2\pi}{|I_k|} \log\log\log\frac{2\pi}{|I_k|}}\right\}\right) \end{aligned} \tag{3.6}$$

and if we choose $\Psi(t) = t\exp\left\{2C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right\}$, it is easily shown that the right term in (3.6) is bounded below by $C(M)\sum_k |I_k|$ for some $C(M) > 0$. Then

$$\sum_k \Psi(\delta_k) \geq C(M) \sum_k |I_k| \geq C(M)|A'| > 0,$$

which proves that $\Lambda_\Psi(f(A)) \geq \Lambda_\Psi(f(A')) > 0$. $\square$

Remarks.

(1) Quasidiscs with fractal boundary actually provide examples for the sharpness part in Theorem 3.6. See [Mak89a] for further details.
(2) Theorem 3.6 is a compression result in the sense that ω must "live" on sets of Hausdorff dimension at least 1, so the conformal mapping cannot compress more than this. In fact, the corresponding expansion result, that is, that ω must live on

subsets of dimension at most 1 is also true, even in non-simply connected plane domains ([JW88]).

3.5. **Regularity of measures in terms of their doubling properties.** This section describes some recent results that show how dyadic martingales can be applied to the study of regularity of measures in terms of their multiplicative properties.

Let μ be a positive, finite measure in $[0,1)$. Then $S_I = \frac{\mu(I)}{|I|}$ defines a dyadic martingale in $([0,1), |\cdot|)$ such that

$$S_n - S_{n-1} = \frac{\mu(I_n) - \mu(I'_n)}{2|I_n|},$$

where I_n, I'_n are sibling dyadic intervals of the generation n. As it was pointed out in example (1) in Section 3.3, this explains why dyadic martingales are well suited to study additive properties of measures.

On the other hand, in the rest of the section we will be concerned with the *multiplicative* or *doubling* properties of a measure.

If μ is a positive, locally finite Borel measure in $\mathbb{R}$ we say that μ is *doubling* if there is some constant $M \geq 1$ such that

$$M^{-1} \leq \frac{\mu(I)}{\mu(I')} \leq M$$

for any two adjacent intervals of the same length. This is equivalent to say that there is δ, $0 < \delta \leq \frac{1}{2}$, such that

$$\frac{\mu(I)}{\mu(J)} \geq \delta,$$

where $J \subset \mathbb{R}$ is any interval and $I \subset J$ is either one of the left or right half-intervals in which J is divided.

Now, for a finite positive Borel measure μ in $[0,1)$, we say that μ is *dyadic doubling* if there is δ, $0 < \delta \leq \frac{1}{2}$ such that

$$\frac{\mu(I)}{\mu(J)} \geq \delta$$

whenever $J \in \mathcal{D}_{n-1}$ and $I \subset J$, $I \in \mathcal{D}_n$.

Note that "doubling" is stronger than "dyadic doubling". Examples of dyadic doubling measures which are not doubling will appear later in this section.

The application of dyadic martingales to the study of multiplicative properties of measures started in [GN01] with the following observation: since the multiplicative properties of a given measure μ are better captured by $Z_I = \log \frac{\mu(I)}{|I|}$, which is no longer a dyadic martingale, one could try to approximate Z_I by a dyadic martingale and hope that this approximation gives useful information about Z_I.

From this principle, the authors proved in [GN01] a number of remarkable results on the regularity of doubling measures in $\mathbb{R}^N$, with applications to the boundary behaviour of harmonic and analytic functions. The approximation procedure in the one-dimensional case can be briefly described as follows (Lemma 2.1 in [GN01]). Let μ be a positive, finite measure in $[0,1)$. Then there is a dyadic martingale (S_n) in $([0,1), |\cdot|)$ and a non-negative, non-decreasing sequence of functions (P_n) in $[0,1)$, which is predictable (each P_n is constant on any $I_{n-1} \in \mathcal{D}_{n-1}$) such that

$$\log \frac{\mu(I_n)}{|I_n|} = S_n - P_n$$

for any $I_n \in \mathcal{D}_n$. Furthermore, if μ is dyadic doubling,

$$\langle S \rangle_n(x) \asymp A_n^2(\mu)(x),$$
$$P_n(x) \asymp A_n^2(\mu)(x),$$

where

$$A_n^2(\mu)(x) = \sum_{k=1}^{n} \left(\frac{\mu(I_k(x))}{\mu(I_{k-1}(x))} - \frac{1}{2} \right)^2,$$

$\{I_k(x)\}$ are the dyadic intervals containing x and the comparison constants only depend on the doubling constant of μ. Then $(S_n(x))$ converges (and so does $P_n(x)$) a.e. x on $\{A_\infty^2(\mu) < \infty\}$ by Theorem 1.3 and, on the other hand, by the LIL (Theorem 1.4) applied to (S_n),

$$S_n = O(\sqrt{\langle S \rangle_n \log \log \langle S \rangle_n}) = o(P_n)$$

a.e. on $\{A_\infty^2(\mu) = \infty\}$.

The following is a one-dimensional restatement of some results in [GN01]:

Theorem 3.7 ([GN01], Theorem 1.1)**.** *Let μ be a positive dyadic doubling measure in $[0,1)$, and let $\mu = f\,dx + \mu_S$ be its decomposition into absolutely continuous and singular part. Then*

(a) *The sets $\{x : f(x) > 0\}$ and $\{x : A_\infty^2(\mu)(x) < \infty\}$ can only differ in a set of Lebesgue measure zero*
(b) *μ is singular iff $A_\infty^2(\mu)(x) = \infty$ for a.e. (Lebesgue) $x \in [0,1)$.*
(c) *There is an absolute constant $C > 0$ such that $\exp\left(CA_\infty^2(\mu)\right) \in L^1[0,1]$ implies $\mu \ll \Lambda_1$.*

After that, [GN01] moves into a group of deep results relating the boundary behaviour of the Poisson integral of a positive measure in the upper half-space to the so called multiplicative area function which is the usual area function of $\log u$, instead of u.

The rest of the section is devoted to describe some recent results on the regularity of measures contained in [LN02]. Let μ be a positive, finite Borel measure in $[0,1)$, not necessarily dyadic doubling. For each $k \in \mathbb{N}$, let

$$\delta_k = \inf \frac{\mu(I_k)}{\mu(I_{k-1})}, \tag{3.7}$$

where the infimum runs over all dyadic intervals $I_{k-1} \in \mathcal{D}_{k-1}$ and $I_k \subset I_{k-1}$, $I_k \in \mathcal{D}_k$. Then $0 \le \delta_k \le \frac{1}{2}$. δ_k informs about the dyadic doubling properties of μ at scale k so it is natural to expect that the smaller the $\delta_k' s$ are, the more singular μ is. In particular, μ is dyadic doubling iff $\inf_k \delta_k > 0$.

A natural question is:

> Given the sequence (δ_k), how can we get comparison estimates like $\mu \ll \Lambda_\Psi$ for some measure function depending on the sequence (δ_k)?

The starting point in [LN02] is, somehow, dual to the followed in [GN01] for now one would like to approximate $\log \frac{\mu(I)}{|I|}$ by a dyadic martingale with respect to the measure μ itself.

An important role in the explanations which follow will be played by the entropy. We recall that if $0 \le \lambda \le 1$, then the *entropy* of the distribution $\{\lambda, 1-\lambda\}$ is defined by $h(\lambda) = \lambda \log \frac{1}{\lambda} + (1-\lambda)\log \frac{1}{1-\lambda}$. Then $0 \le h(\lambda) \le \log 2$, with $h(\lambda) = 0$ iff $\lambda = 0$ or 1 and $h(\lambda) = \log 2$ iff $\lambda = \frac{1}{2}$.

Entropy has a nice interpretation in Information Theory; it measures the expected information (in terms of surprise) associated to the probability distribution $\{\lambda, 1-\lambda\}$. The maximal entropy corresponds to the case $\lambda = \frac{1}{2}$ (maximal uncertainty) and the minimal entropy corresponds to $\lambda = 0$ or $\lambda = 1$ (no, surprise, deterministic). See [App96] for more details and interesting historical comments.

Let us come back now to the measure μ. We will write, hereafter,

$$\lambda_k(x) = \frac{\mu(I_k(x))}{\mu(I_{k-1}(x))}.$$

The following is Proposition 3.1. in [LN02].

Proposition 3.3. *Let μ be a Borel probability measure in $[0,1)$. Then there is a dyadic martingale (S_n) in $([0,1),\mu)$ and a non-negative, non-decreasing predictable sequence (P_n) such that*

$$\log \frac{\mu(I_n)}{|I_n|} = S_n + P_n.$$

In fact,

$$S_n(x) = \sum_{k=1}^{n} (1-\lambda_k(x)) \log \frac{\lambda_k(x)}{1-\lambda_k(x)},$$
$$P_n(x) = n\log 2 - H_n(x),$$

where $H_n(x) = \sum_{k=1}^{n} h(\lambda_k(x))$, $h(\lambda_k)$ being the entropy associated to λ_k.

Corollary 3.1. *If μ, (S_n), H_n are as above, then*

$$\mu(I_n(x)) = \exp\{S_n(x) - H_n(x)\}. \tag{3.8}$$

Corollary 3.2. *The quadratic characteristic of* (S_n) *is given by*

$$\langle S\rangle_n = \sum_{k=1}^{n} \lambda_k(x)(1-\lambda_k(x)) \log^2 \frac{\lambda_k(x)}{1-\lambda_k(x)}.$$

Hereafter, we will refer to H_n as the entropy term in (3.8). Note that the better the doubling properties of μ are (that is, the bigger the δ_k's are), the bigger the entropy is. This is confirmed by the following result, which is a counterpart of statements (a), (b) in Theorem 3.7.

Theorem 3.8.

(a) μ *is singular iff* $\lim_n(n\log 2 - H_n(x)) = \infty$ μ*-a.e.,*

(b) μ *is absolutely continuous iff* $\lim_n(n\log 2 - H_n(x)) < \infty$ μ*-a.e.*

Before continuing with the general case, we will describe an important class of dyadic doubling measures that are extremal for the type of problems we are interested in. See [Mak86], [Bou93] for applications of the construction in Geometric Function Theory.

Example. Fix $0 < \delta \le \frac{1}{2}$. Suppose that we give mass $1-\delta$ to the interval $[0,\frac{1}{2})$ and mass δ to the interval $[\frac{1}{2},1)$. Iterating this pattern, we can construct a probability measure μ_δ in $[0,1)$ such that, if $I_{n-1}\in\mathcal{D}_{n-1}$, $I_{n-1} = I_n^- \cup I_n^+$, where $I_n^-, I_n^+ \in \mathcal{D}_n$ are the left and right dyadic children on I_{n-1}, then

$$\mu(I_n^-) = (1-\delta)\mu(I_{n-1}),$$
$$\mu(I_n^+) = \delta\mu(I_{n-1}) \quad \text{(see [Shi84], page 152)}.$$

We can explicitly compute $\langle S\rangle_n$ and H_n:

$$\langle S\rangle_n = n\delta(1-\delta)\log^2\frac{1-\delta}{\delta},$$
$$H_n = n\left(\delta\log\frac{1}{\delta} + (1-\delta)\log\frac{1}{1-\delta}\right) = nh(\delta).$$

Note also that, by Proposition 3.3, $\sup_n |S_n - S_{n-1}| \le C(\delta) < \infty$ so (S_n) ha uniformly bounded increments. By Theorem 1.5, we get:

$$\overline{\lim}\frac{|S_n|}{\sqrt{n\log\log n}} \le C < \infty \tag{3.9}$$

μ_δ-a.e. Now, from Proposition 3.3, (3.8) and (3.9) we can derive some interesting consequences. Let $\alpha(\delta) = \frac{h(\delta)}{\log 2}$. Then:

(i)

$$\frac{\mu_\delta(I_n(x))}{|I_n(x)|} = \exp\{S_n(x) - n(1-\alpha(\delta))\} \xrightarrow{n\to\infty} \infty \quad \mu\text{-a.e. } x,$$

as soon as $\delta \ne \frac{1}{2}$. This shows that μ_δ is singular if $0<\delta<\frac{1}{2}$. (Of course, $\mu_{\frac{1}{2}}$ = Lebesgue measure.)

(ii)
$$\overline{\lim_n} \frac{\mu_\delta(I_n(x))}{2^{-n\alpha(\delta)} \exp\left\{C\sqrt{n \log\log n}\right\}} < \infty \ \mu\text{-a.e. } x.$$
Therefore, by Proposition 3.2, $\mu_\delta \ll \Lambda_{\Psi_\delta}$, where
$$\Psi_\delta(t) = t^{\alpha(\delta)} \exp\left\{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right\}. \tag{3.10}$$
In particular, $\mu_\delta \ll \Lambda_{\alpha(\delta)-\varepsilon}$ for any $\varepsilon > 0$. Note also that μ_δ is not doubling.

Essentially the same procedure works if μ is dyadic doubling for in this case, there is δ, $0 < \delta \leq \frac{1}{2}$ such that $\lambda_k(x) \geq \delta_k \geq \delta$, all k, all $x \in [0,1)$ and
$$\langle S\rangle_n = O(n),\ H_n \geq nh(\delta),\ |S_n - S_{n-1}| = O(1),$$
so, using the LIL (Theorem 1.5) in the same way we get:

Corollary 3.3. *Let μ be a dyadic doubling measure in $[0,1)$, with $\inf_k \delta_k \geq \delta > 0$, for some $0 < \delta < \frac{1}{2}$. Then $\mu \ll \Lambda_{\Psi_\delta}$, where Ψ_δ is given by (3.10). In particular, $\mu \ll \Lambda_{\alpha(\delta)-\varepsilon}$ for any $\varepsilon > 0$. Here $\alpha(\delta) = \frac{h(\alpha)}{\log 2}$ is the normalized entropy associated to δ.*

The assertion $\mu \ll \Lambda_{\alpha(\delta)-\varepsilon}$ in Corollary 3.3 had already been proved by Heurteaux ([Heu95], [Heu98]).

The general (non-doubling) situation requires more work. When trying to run the argument above, two different questions arise:

(a) Is there an upper bound of the LIL for the martingale (S_n) in (3.8)?
(b) Assuming that (a) has been achieved, under what conditions is it true that $\sqrt{\langle S\rangle_n \log\log\langle S\rangle_n} = o(H_n)$?

First, it should be pointed out why question (a) is meaningful. Recall from remark (3) at Section 1.7 that, for general martingales, the LIL (even the upper bound) requires some restrictions on the increments. But, unless μ is dyadic doubling, the martingale (S_n) in (3.8) does not have uniformly bounded increments and the growth restrictions needed for applying the general LIL would be quite unpleasant here. It turns out that, if the martingale (S_n) is related to μ as in (3.8), it is possible to drop any assumption on the increments in order to get a one-sided upper bound.

Theorem 3.9. *Let μ be a probability measure in $[0,1)$ and (S_n) the martingale associated to μ as in (3.8). Then*
$$\overline{\lim_n} \frac{S_n}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} \leq 1$$
μ-a.e. on $\{\langle S\rangle_n = \infty\}$.

To prove Theorem 3.9 it is enough to check that $e^{tS_n - \frac{1}{2}t^2\langle S\rangle_n}$ is a supermartingale in $([0,1),\mu)$ (see Remark (1) in Section 1.7). This is seen to be equivalent to the following elementary inequality

$$\lambda e^{t(1-\lambda)\log\frac{\lambda}{1-\lambda}} + (1-\lambda)e^{t\lambda\log\frac{1-\lambda}{\lambda}} \le \exp\left\{\frac{1}{2}t^2\lambda(1-\lambda)\log^2\frac{1-\lambda}{\lambda}\right\},$$

where $0 \le \lambda \le 1$, $t > 0$. In the higher dimensional case, we were able to prove an analogous, though more involved, inequality, with $\frac{1}{2}$ replaced by some dimensional constant (Lemma 3.3 in [LN02]).

Regarding question (b), it should also be remarked that *some* restriction on the sequence (δ_k) must be required in order to get $\mu \ll \Lambda_\Psi$ for some measure function Ψ. For instance, this is impossible for any Ψ if μ has atoms, which can happen if $\sum_k \delta_k < \infty$ (imitate the construction of μ_δ in the example with variable δ_k's).

According to Theorem 3.9, to control (S_n) by H_n requires first to control $\langle S\rangle_n$ by H_n.

Lemma 3.1.

$$\langle S\rangle_n \le CH_n \log\left(\frac{n\log 2}{H_n}\right)$$

for some absolute constant C.

Now, Theorem 3.9, Theorem 1.3 and Lemma 3.1 give:

Corollary 3.4. *There is an absolute constant $C > 0$ such that*

$$\varlimsup_n \frac{S_n}{\sqrt{H_n\log\left(\frac{2n\log 2}{H_n}\right)\log\log\left(H_n\log\left(\frac{2n\log 2}{H_n}\right)\right)}} \le C$$

μ-a.e. on $\{\lim_n H_n = \infty\}$.

Now the final step is to show that the square root in Corollary 3.4 is small compared to H_n as soon as H_n is big enough (which can be accomplished if the δ_k's are not too small). This is also elementary.

Lemma 3.2. *Assume that*

$$\lim_n \frac{H_n}{(\log n)(\log\log\log n)} = \infty.$$

Then

$$\sqrt{H_n\log\left(\frac{2n\log 2}{H_n}\right)\log\log\left(H_n\log\left(\frac{2n\log 2}{H_n}\right)\right)} = o(H_n)$$

as $n \to \infty$.

Now we are ready to state the main results in [LN02]:

Theorem 3.10. *Suppose that*

$$\lim_n \frac{\sum_{k=1}^n \delta_k \log \frac{1}{\delta_k}}{(\log n)(\log\log\log n)} = \infty.$$

Then $\mu \ll \Lambda_\Psi$ for any measure function Ψ such that

$$\Psi(2^{-n}) =$$
$$= \exp\left\{ C\sqrt{A_n \log\left(\frac{2n\log 2}{A_n}\right)\log\log\left(A_n \log\left(\frac{2n\log 2}{A_n}\right)\right)} - A_n \right\},$$

where $A_n = \sum_{k=1}^n h(\delta_k)$ and C is some absolute constant.

Note that if $\delta_k \geq \delta$, then $A_n \geq nh(\delta)$ and we recover Corollary 3.3. In fact, only a control of δ_k at many scales is needed.

Corollary 3.5. *Suppose that $\delta_k \geq \delta > 0$ for all $k \in A \subset \mathbb{N}$. Then*

(a) *If A has density 1, that is, $\lim_n \frac{\#(A\cap[1,n])}{n} = 1$, then $\mu \ll \Lambda_{\alpha(\delta)-\varepsilon}$ for any $\varepsilon > 0$.*

(b) *If A has positive lower density, that is, $\varliminf_n \frac{\#(A\cap[1,n])}{n} > 0$, then $\mu \ll \Lambda_\beta$ for some $\beta > 0$.*

While one of the main motivations in [LN02] was to obtain regularity estimates for measures which are not necessarily doubling, it turns out that the same techniques give also new results for measures with good doubling behaviour. We say that μ is *dyadic symmetric* if $\delta_k \to \frac{1}{2}$ as $k \to \infty$. Symmetric measures (which can be seen as multiplicative counterparts of Zygmund measures) have received special attention recently ([AAN79], [Can98], [DN02]) because of their interesting connections with Function Theory. The first regularity result on symmetric measures goes back to Carleson [Car67] who proved (with our notation), that if $\sum_k \left(\frac{1}{2} - \delta_k\right)^2 < \infty$, then $\mu \ll \Lambda_1$. Note that Theorem 3.7 (c) provides a substantial improvement.

In the dyadic symmetric case we get from Proposition 3.3, Corollary 3.2:

$$\langle S\rangle_n \asymp \sum_{k=1}^n \left(\frac{1}{2} - \delta_k\right)^2,$$

$$H_n \asymp n\log 2 - \mathcal{O}\left(\sum_{k=1}^n \left(\frac{1}{2} - \delta_k\right)^2\right).$$

Theorem 3.11. *If μ is dyadic symmetric then $\mu \ll \Lambda_\Psi$, where Ψ is any measure function such that*

$$\Psi(2^{-n}) = 2^{-n}\exp\left\{ C\sum_{k=1}^n \left(\frac{1}{2} - \delta_k\right)^2 \right\}$$

for some absolute constant $C > 0$.

Now all these results can be applied to the distortion of homeomorphisms on the real line. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphisms we denote by μ_g its Lebesgue–Stieltjes measure. Note that μ_g is doubling iff g is quasi-symmetric. In general, suppose that

$$(M(t))^{-1} \le \frac{g(x+t)-g(x)}{g(x)-g(x-t)} \le M(t)$$

for some $M(t) \ge 1$ which is assumed to be monotone (the case $M(t) \uparrow \infty$ as $t \downarrow 0$ is allowed). Then:

Corollary 3.6. *There is an absolute constant $C > 0$ such that*

(a) *If $M(t) \le M$ then $\mu_g \ll \Lambda_{\Psi_M}$, where*

$$\Psi_M(t) = t^{\alpha(M)} \exp\left\{ C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}} \right\}$$

and $\alpha(M) = \frac{h\left(\frac{1}{M+1}\right)}{\log 2}$.

(b) *If*

$$\lim_{t\to 0} \frac{\int_t^1 \frac{\log M(S)}{M(S)}\frac{dS}{S}}{(\log\log\frac{1}{t})(\log\log\log\log\frac{1}{t})} = \infty,$$

then $\mu_g \ll \Lambda_\Psi$, where

$$\Psi(t) = \exp\left\{ -C\int_t^1 \frac{\log M(S)}{M(S)}\frac{dS}{S} \right\}.$$

(c) *If $M(t) \downarrow 1$ as $t \uparrow 0$, (μ_g is symmetric) then $\mu_g \ll \Lambda_\Psi$, where*

$$\Psi(t) = t\exp\left\{ C\int_t^1 (M(S)-1)^2\frac{dS}{S} \right\}.$$

Remarks.

(1) Corollary 3.6 (b) can be rephrased in terms of harmonic measure of the operator obtained as a pull back of the Laplacian with the Beurling–Ahlfors extension of g. The lack of general existence and uniqueness results for such degenerate operators is the responsible of the one-sided formulation.

(2) The use of martingales in the study of fine properties of measures has been recently used in [Llo02], where some questions in Geometric Function Theory motivated the construction of a special class of singular doubling measures in $\mathbb{R}^N$.

References

[AAN79] A.B. Aleksandrov, J.M. Anderson, and A. Nicolau, *Inner functions, Bloch spaces and symmetric measures*, Proc. London Math. Soc.(3) **79** (1979), 318–352.

[App96] D. Applebaum, *Probability and Information*, Cambridge University Press, 1996.

[BA56] A. Beurling and L.V. Ahlfors, *The boundary correspodence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.

[Bae88] Baernstein A. II, *Alfors and conformal invariants*, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), no. 3, 289–312.

[BFL00] J. Betancor, J.C. Fariña, and J.G. Llorente, *Axiomatic potential theory, asymptotic values and elliptic differential operators*, Potential Analysis **12** (2000), no. 2, 115–146.

[BJ95] C.J. Bishop and P.W. Jones, *The law of the iterated logarithm for Kleinian groups*, Lipa's Legacy, Amer. Math. Soc. Providence, RI. (1995), 17–50.

[BJ97] ———, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997), no. 1, 1–39.

[BKM88] R. Bañuelos, I. Klemeš, and C.N. Moore, *An analogue for harmonic functions of Kolmogorovs law of the iterated logarithm*, Duke Math. J. **57** (1988), 37–68.

[BKM90] R. Bañuelos, I. Klemeš, and C.N Moore, *The lower bound in the law of the iterated logarithm for harmonic functions*, Duke Math. J. **60** (1990), no. 3, 689–715.

[BM89] R. Bañuelos and C.N Moore, *Sharp estimates for the non-tangential maximal function and the Lusin area function in Lipschitz domains*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 641–662.

[BM99] ———, *Probabilistic Behaviour of Harmonic Functions*, Progress in Math. Birkhauser, Basel, 1999.

[Bou93] J. Bourgain, *On the radial variation of bounded analytic functions on the disc*, Duke Math. J. **69** (1993), no. 3, 671–682.

[Cal50a] A.P. Calderón, *On a Theorem of Marcinkiewicz and Zygmund*, Trans. Amer. Math. Soc. **68** (1950), 47–54.

[Cal50b] ———, *On the behaviour of harmonic functions near the boundary*, Trans. Amer. Math. Soc. **68** (1950), 47–54.

[Can98] A. Cantón, *Singular measures and the little Bloch space*, Publ. Mat. **42** (1998), 211–222.

[Car67] L. Carleson, *On mappings, conformal at the boundary*, Journal d'Anal. Math. **19** (1967), 1–13.

[Car73] ———, *On the distortion of sets on a jordan curve under conformal mappings*, Duke Math. J. **40** (1973), 547–559.

[CD96] J.J. Carmona and J.J. Donaire, *On removable singularities for the analytic Zygmund class*, Michigan Math. J. **43** (1996), no. 1, 51–65.

[CWW85] S.Y. Chang, J.M. Wilson, and T.H. Wolff, *Some weighted norm inequalities for the schrödinger operator*, Comment Math. Helv. **60** (1985), 217–246.

[DN02] E. Doubtsov and A. Nicolau, *Symmetric and Zygmund measures in several variables*, Ann. Inst. Fourier **52** (2002), no. 1, 153–177.

[Don01a] J.J. Donaire, *Porosity of sets and the Zygmund class*, to appear in Bull. London Math. Soc.

[Don01b] ———, *Radial behaviour of inner functions in $\mathcal{B}_0$*, J. London Math. Soc. **2** (2001), no. 1, 141–158.

[DP99] J.J. Donaire and Ch. Pommerenke, *On radial behaviour and balanced Bloch functions*, Rev. Mat. Iber. **15** (1999), no. 3, 429–449.

[Fal85] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[Fuc67] W.H.J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Van Nostrand, 1967.

[Gar86] J.B Garnett, *Applications of Harmonic Measure*, Wiley&Sons, 1986.

[GK01] M.J. Gonzáles and P. Koskela, *Radical growth of solutions to the Poisson equation*, Complex Variables **46** (2001), no. 1, 59–72.

[GKLN01] M.J. Gonzáles, P. Koskela, J.G. Llorente, and A Nicolau, *Distributional inequalities for non-harmonic functions*, Preprint, 2001.

[GM] J.B. Garnett and D. Marshall, *Harmonic Measure*, In Press.

[GN01] M.J. Gonzáles and A. Nicolau, *Hyperbolic area function*, Preprint, 2001.

[Heu95] Y. Heurteaux, *Sur la comparison des measures avec les measures de Hausdorff*, C.R. Acad. Sci. Paris Ser. 1 Math. **321** (1995), 61–65.

[Heu98] ______, *Estimations de la dimension infériure et de la dimension supérieure des measures*, Ann. Inst. H. Poincaré Probab. Stat. **34** (1998), no. 3, 309–338.

[JW88] P.W. Jones and T.H. Wolff, *Hausdorff dimension of harmonic measures in the plane*, Acta Math. **161** (1988), 131–144.

[Koo98] P. Koosis, *Introduction to H^p Spaces*, Cambridge Univ. Press, 1998.

[KW82] R. Kaufman and J.M. Wu, *Distortion of the boundary under conformal mapping*, Mich. Math. J. **29** (1982), 267–280.

[Lav63] M.A. Lavrentiev, *Boundary problems on the Theory of Univalent functions*, Amer. Math. Soc. Transl. **32** (1963), 1–35.

[Llo98] J.G. Llorente, *Boundary values os harmonic Bloch functions in Lipschitz domains: a martingale approach*, Potential Analysis **9** (1998), 229–260.

[Llo02] ______, *On the Gehring-Hayman property, Riesz Theorem and a result of Semmes*, Preprint, 2002.

[LN02] J.G. Llorente and A. Nicolau, *Non doubling measures, entropy and the Law of the Iterated logarithm*, Preprint, 2002.

[Lyo90] T. Lyons, *A synthetic proof of Makarov's Law of the iterated logarithm*, Bull. London Math. Soc **22** (1990), no. 2, 159–162.

[Mak85] N.G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. **51** (1985), no. 3, 369–384.

[Mak86] ______, *A note on the integral means of the derivative in conformal mapping*, Proc. Amer. Math. Soc. **96** (1986), 233–236.

[Mak87] ______, *Conformal mapping and Hausdorff measures*, Arkiv Mat. **25** (1987), 41–89.

[Mak89a] ______, *Probability methods in the theory of conformal mapping*, Algebra i Analiz (1989), 3–59 (Russian), [English transl. : Leningrad Math. J. 1 (1990),1-56].

[Mak89b] ______, *Smooth measures and the law of iterated logarithm*, Izv. Akad. Navk SSSR Ser. Mat. **53** (1989), 439–446 (Russian), [English transl. : Math. USSR Izv. 34 (1990), 455-463].

[MP73] J.E. McMillan and G. Piranian, *Compression and expansion of boundary sets*, Duke Math. J. **40** (1973), 599–605.

[MZ38] J. Marcinkiewicz and A. Zygmund, *A Theorem of Lusin*, Duke Math. J. **4** (1938), 473–485.

[Nev70] R. Nevanlinna, *Analytic Functions*, Springer, 1970.

[Pom92] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer Verlag, 1992.

[Roh93] S. Rohde, *The Boundary Behaviour of Bloch Functions*, J. London Math. Soc. **48** (1993), no. 2, 488–499.

[Shi84] A.N. Shiryayev, *Probability*, Springer Verlag, 1984.

[Spe43] D.C Spencer, *A function theoretic identity*, Amer. J. Math. **65** (1943), 143–160.

[Ste61] E.M Stein, *On the theory of harmonic functions of several variables ii*, Acta Math. **106** (1961), 137–174.

[Ste70] ———, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[Sto64] W.F. Stout, *A martingale analogue of Kolmogorov's law of the iterated logarithm*, Z. Wahrs. Verw. Geb. **3** (1964), 211–226.

[Sto84] ———, *Almost Sure Convergence*, Academic Press, 1984.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain