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DEPARTMENT OF MATHEMATICS
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REPORT 114

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REGULARITY PROPERTIES OF MAXIMAL OPERATORS

HANNES LUIRO

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HANNES LUIRO

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Jyväskylä, July 2008

Hannes Luiro

List of included articles

This dissertation consists of an introductory part and the following publications:

[A] H. Luiro, Continuity of the maximal operator in Sobolev spaces. Proc. of AMS., 135 (2007), no.1, 243-251.

[B] H. Luiro, The regularity of the Hardy-Littlewood maximal operator on subdomains of $\mathbb{R}^n$. Proceedings of the Edinburgh Mathematical Society, to appear.

[C] H. Luiro, On the continuous and discontinuous maximal operators.

INTRODUCTION

1. The Hardy-Littlewood maximal operator

The Hardy-Littlewood maximal operator has long been a central tool in mathematical analysis. The operator first appeared in the one-dimensional case in a paper of the famous British mathematicians, G. H. Hardy and J. E. Littlewood [HL]. The n-dimensional analogue was soon after studied by N. Wiener [W].

The Hardy-Littlewood maximal operator M is defined

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy \tag{1}$$

for locally integrable function $f : \mathbb{R}^n \mapsto \mathbb{R}$. Informally, the value of the maximal function of f at x is the largest average value of f on any ball centered at x.

There are various ways to define a maximal operator. For example, we may replace the x-centered balls in the definition by balls that contain x (the *non-centered* maximal operator), or use cubes instead of balls. In most situations these definitions turn out to be essentially equivalent.

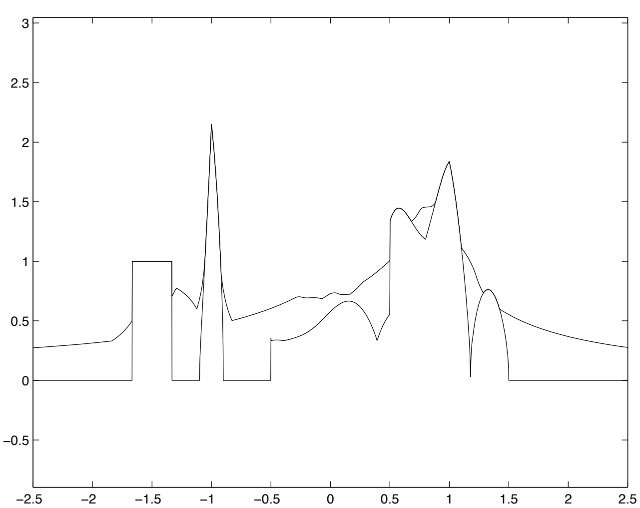

Figure 1. 1-dimensional example of a function (the lower graph) and its maximal function (the upper graph)

2. Basic properties of the maximal operator

The Hardy-Littlewood maximal operator and its variants play a key role in differentation theory and in the theory of singular integrals. One often needs to estimate some quantities depending on a given function f. That can be shown to be dominated by Mf. Hence, a natural question arises: how 'big' can the maximal function of a given function be? Pointwise inequalities are not possible, but the bounds in the L^p-sense can be given. The fundamental result of Hardy-Littlewood-Wiener asserts that the maximal operator is bounded on $L^p(\mathbb{R}^n)$ when $1 < p \leq \infty$ [St]. This means that there is a constant $C(n,p) > 0$ such that

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)}\,, \tag{2}$$

for all $f \in L^p(\mathbb{R}^n)$. When $p = 1$, this result fails. However there is a constant $C(n) > 0$ such that for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$ it holds that

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq C(n)\frac{\|f\|_1}{\lambda}\,. \tag{3}$$

Thus, M is said to be of "weak type (1,1)", and it is a bounded operator from L^1 to the weak Lebesque space L^1_w. The inequality (3) follows from a standard covering theorem, and (2) follows from (3) by interpolation between L^1_w and L^∞.

3. Mapping properties of the maximal operator

While the size of the maximal function is of principal interest, it is also useful and interesting to study how the maximal operator preserves the regularity properties of functions. A simple observation is that the set

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \tag{4}$$

is open for arbitrary $f \in L^1_{loc}$ and $\lambda > 0$. In the other words, Mf is always lower semi-continuous. Moreover, it is easy to see that if f is continuous, then so is Mf as well (if $Mf \not\equiv \infty$). What about the smoothness properties of the maximal function? In 1997, Kinnunen [K] observed that M is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$. Moreover, he showed the pointwise inequality

$$|D_i Mf(x)| \leq M(D_i f)(x) \tag{5}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$ and a.e $x \in \mathbb{R}^n$ Here $D_i f$ denotes the weak partial derivative of f in the direction e_i. Kinnunen applied this result to the study Lebesgue points of Sobolev functions.

The boundedness of M in Sobolev spaces is basically implied by two facts that also will be important for further results. First, the maximal operator commutes with translations. Denote for $h \in \mathbb{R}^n$ that $f_h(x) = f(x+h)$. Then it holds that $M(f_h)(x) = (Mf)_h(x)$ for every $x \in \mathbb{R}^n$. Second, M is sublinear, which means that given locally integrable functions f and g, it holds that $|Mf(x) - Mg(x)| \leq M(f-g)(x)$ a.e. From these facts we see that

$$\frac{1}{|h|}|Mf(x+h) - Mf(x)| \leq M\left(\frac{f_h - f}{|h|}\right)(x) \text{ a.e.} \tag{6}$$

The boundedness of M on $W^{1,p}(\mathbb{R}^n)$ and the pointwise inequality (5) are easily implied by this inequality.

After [K] several articles devoted to the same topic have appeared. We mention here [AP],[Bu],[HO],[KL],[KS], [Ko] and [Ta].

4. Different maximal operators

In mathematical analysis, a number of different maximal operators have been used for various purposes. Accordingly, one natural way to extend the result of Kinnunen is to study the regularity of some other classical maximal operators.

We now give the definitions of the local and fractional maximal operators. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary subdomain (i.e. an open and connected subset) of $\mathbb{R}^n$. The *local* Hardy-Littlewood maximal operator $M_\Omega f$, where $f \in L^1_{loc}(\Omega)$, is defined by

$$M_\Omega f(x) = \sup_{0<r<d(x,\Omega^c)} \fint_{B(x,r)} |f(y)|\, dy\,. \tag{7}$$

Here $d(x, A)$ denotes the Euclidean distance from x to set A.

The *fractional* maximal operator M^α, where $0 < \alpha < n$, has applications in potential theory and partial differential equations. It is defined by

$$M^\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\, dy \tag{8}$$

for $x \in \mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n)$.

Again, as it was in the case of the original maximal operator, we could define the local or the fractional maximal operator also by using cubes or non-centered balls.

A result of Kinnunen and P. Lindqvist states that M_Ω is bounded on $W^{1,p}(\Omega)$ [KL]. This local case is more difficult to treat because M_Ω does not commute with translations. The proof is quite involved, and it does not generalize to the case where the maximal operator is defined by cubes. A simpler proof that also applies to the cube-based operator was obtained by P. Hajlasz and J. Onninen [HO].

The fractional maximal operator was studied by Kinnunen and E. Saksman [KS] who proved that M^α is smoothing in the sense that it maps L^p-spaces boundedly to certain first order Sobolev spaces.

5. Continuity of the Hardy-Littlewood maximal operator

As M is a sublinear operator, the boundedness of the maximal operator on $L^p(\mathbb{R}^n)$ implies its continuity on $L^p(\mathbb{R}^n)$. The result of Kinnunen now leads us to another question: is the maximal operator continuous on $W^{1,p}(\mathbb{R}^n)$? This question was posed in [HO] where it was attributed to T. Iwaniec. In general, boundedness does not need imply continuity. An important example is given in [AL], where it is shown that the symmetric decreasing rearrangement, defined for measurable $u : \mathbb{R}^n \mapsto [0, \infty)$ as the mapping $u \mapsto u^*$, where

$$u^*(x) := \sup\{t \geq 0 : |\{x \in \mathbb{R}^n : u(x) > t\}| \geq |B(0, |x|)|\}\,, \tag{9}$$

is bounded on $W^{1,p}(\mathbb{R}^n)$ but not continuous when $1 < p < n$. We will later see that this phenomenon surprisingly takes place for a natural class of maximal operators, as well.

A main result of this thesis is the positive answer [A, Theorem 4.1] to the question of Iwaniec:

Theorem 5.1. *M is continuous $W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$, when $1 < p < \infty$.*

An important role in the proof is played by the following new concept:

$$\mathcal{R}f(x) = \{r \geq 0 : Mf(x) = \fint_{B(x,r)} |f(y)|\,dy\}. \tag{10}$$

Here we make the convention that $\fint_{B(x,0)} |f| = |f(x)|$. In other words, $\mathcal{R}f(x)$ is the set containing the radii for which the maximum average is achieved. We call this set the set of "best radii" at the point x. These sets turn out to be non-empty and closed for almost every x. A major part of the proof of Theorem 5.1 is based on the following results, which can be found in [A, Lemma 2.1] and [A, Theorem 3.1]. We denote for $A \subset \mathbb{R}^n$ and $\lambda > 0$, $A_{(\lambda)} = \{x : d(x, A) \leq \lambda\}$.

Lemma 5.2. *Let $1 \leq p < \infty$, and suppose $f_j \to f$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Then for all $R > 0$ and $\lambda > 0$ it holds that*

$$m(\{x \in B(0,R) : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) \to 0 \quad \textit{as } j \to \infty. \tag{11}$$

Theorem 5.3. *Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Then for almost all $x \in \mathbb{R}^n$,*

$$(1)\ D_i Mf(x) = \fint_{B(x,r)} D_i|f|(y)\,dy \quad \textit{for all } 0 < r \in \mathcal{R}f(x),$$

$$(2)\ D_i Mf(x) = D_i|f|(x) \quad \textit{if } 0 \in \mathcal{R}f(x).$$

Lemma 5.2 reveals that the sets of best radii of the functions f and g are typically "near" to each other if $\|f - g\|_p$ is small. This is used to prove Theorem 5.3, which gives a somewhat surprising formula for the derivative of the maximal function.

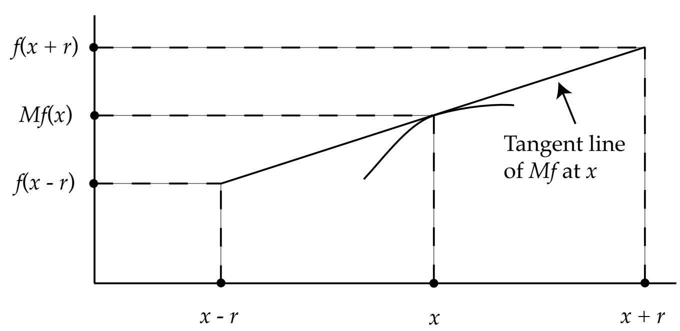

FIGURE 2. tangent of the maximal function at point x

Figure 2 depicts the 1-dimensional situation. In this case Theorem 5.3 implies that

$$(Mf)'(x) = \frac{1}{2r}(f(x+r) - f(x-r)) \text{ for } r \in \mathcal{R}f(x)$$

and where it also holds that

$$\frac{1}{2}(f(x-r)+f(x+r)) = Mf(x) \text{ for } r \in \mathcal{R}f(x).$$

6. Continuity of the local maximal operator

The second major result of this thesis is the continuity of the local maximal operator on $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a subdomain and $1 < p < \infty$ [B, Theorem 2.11].

Theorem 6.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is subdomain and $1 < p < \infty$. Then $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is continuous.*

The proof of the local case is somewhat similar to the global case. We extend the concept of the best radii and analogous results from the global case to the local case. In the local setting some of the auxiliary results are slightly modified and technically complicated to prove, although the philosophy is the same. However, the continuity itself does not seem to easily follow from the continuity in the global case.

In the local case, the formula for the derivative of the maximal function (Theorem 5.3) is stated in the following way [B, Theorem 2.4].

Theorem 6.2. *Assume that $p > 1$ and $f \in W^{1,p}(\Omega)$. Then for almost all $x \in \Omega$ it holds that*

(1) $D_i M_\Omega f(x) = \fint_{B(x,r)} D_i|f|(y)\,dy$ *for all $r \in \mathcal{R}f(x)$, $0 < r < \delta(x)$ and*

(2) $D_i M_\Omega f(x) = D_i|f|(x)$ *if $0 \in \mathcal{R}f(x)$.*

The difficulty in the local case lies at the points where the maximum average is achieved with the largest ball contained in Ω. On these points, the straightforward extension of Theorem 5.3 would say that

$$D_i M_\Omega f(x) = \fint_{B(x,\delta(x))} D_i|f|(y)\,dy\,. \tag{12}$$

Unfortunately, this can *not* hold. A formula for the derivative of the maximal function at these points must depend on the derivative of the function $\delta(x)$. The most interesting and difficult case occurs when, in considering a sequence $f_j \to f \in W^{1,p}(\Omega)$, there are sets S_j with $|S_j| > \lambda$ uniformly for some $\lambda > 0$ and such that for each $x \in S_j$ only $Mf(x)$ is achieved on $B(x, \delta(x))$ but $Mf_j(x)$ in slightly smaller ball. For the above reasons, in the local case we have to prove several technical lemmata before attacking the continuity.

The proof does not use the special geometric properties of balls. In particular, the *spherical* maximal operator is not used. Nor do we assume any smoothness for Ω. The proof also applies, for example, to the local maximal operator defined by cubes.

7. Maximal operator on Triebel-Lizorkin spaces

Another way to extend the regularity theory of maximal operators is to study its behaviour on different function spaces. As we know the boundedness in the spaces $L^p(\mathbb{R}^n) = W^{0,p}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, the next question is about the boundedness on the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, when $0 < s < 1$. The work of S. Korry [Ko] also covers this case. Actually, Korry proves that the maximal operator is bounded on the Triebel-Lizorkin spaces $F^p_{s,q}(\mathbb{R}^n)$ when $1 < p, q < \infty$ and $0 < s < 1$, and it is known that $F^p_{s,2}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. Let $1 \le r < \infty$ be fixed. The definition of the Triebel-Lizorkin spaces is often given in terms of the Fourier transform, but an equivalent definition can also be given in the following way [Tr2, p.194]. For the above range of indices, the space $F^p_{s,q}(\mathbb{R}^n)$ consists of those measurable functions on $\mathbb{R}^n$ for which the norm

(13)

$$\|f\|_{F^p_{s,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\int_0^1\left(\int_{B(0,1)}\left|\frac{f(x+th)-f(x)}{t^{s+\frac{1}{q}}}\right|^r dh\right)^{\frac{q}{r}} dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

is finite. Different values of r lead to equivalent norms.

Korry's proof was based on iterating the result of Benedek-Calderon-Panzone concerning linear operators acting on Banach-space valued functions [BCP], combined with the fundamental fact (6).

In the following theorem [B, Theorems 3.2 and 4.2] we extend the result of Korry to the local case of M_Ω on $F^p_{s,q}(\Omega)$.

Theorem 7.1. *M_Ω is bounded and continuous on $F^p_{s,q}(\Omega)$, when $1 < p, q < \infty$.*

The local Triebel-Lizorkin spaces are defined by Triebel [Tr1, 3.1] as a restrictions of functions in $F^p_{s,q}(\mathbb{R}^n)$:

$$F^p_{s,q}(\Omega) = \{f_{|\Omega} : f \in F^p_{s,q}(\mathbb{R}^n)\}. \tag{14}$$

When considering the local case, we are again, as it is in the case of M_Ω on $W^{1,p}(\Omega)$), faced with the problem that M_Ω does not commute with translations. We find a suitable property for M_Ω replacing commutativity and allowing us to use the BCP result in roughly the same way as Korry did.

The proof of the continuity of M_Ω is easier in the case of Triebel-Lizorkin spaces (when $s < 1$) than in the case of Sobolev spaces ($s = 1$).

8. General results for a wide class of maximal operators on different function spaces

The results above are proven for the global or local centered Hardy-Littlewood maximal operator, defined using balls. However, it is easy to observe that all of these proofs are valid, for example, in the case where balls are replaced by cubes. This raises two interesting questions:

(Q1) For which maximal operators can the techniques used above also be used to imply continuity from boundedness?

(Q2) Are there any natural maximal operators for which continuity is not implied by boundedness?

Our last main results deal with these questions.

Let us first discuss (Q1). We will consider the class of maximal operators which are determined by an *admissible* family of operators $\{T_r\}_{r\in I}$. This means that $\{T_r\}_{r\in I}$ has some natural properties which are given in (A1)-(A7) in [C, Section 2], and a maximal operator T^* can be defined by

$$T^*f(x) = \sup_{r\in I} |T_r f(x)| . \tag{15}$$

We study T^* acting on functions on normed space $\mathcal{F}_1 \subset L^p(\mathbb{R}^n)$ of a natural type, where $1 < p < \infty$. Typically $\mathcal{F}_1$ can be thought to be a certain Lebesgue space or Sobolev space. In this general setting we will prove results that are counterparts to Theorems 5.1 and 5.3 and Lemma 5.2.

We show [C, Lemma 2.5 and Theorem 2.9] that if $\{T_r\}_{r\in I}$ is admissible then for every $f \in \mathcal{F}_1$,

$$D_i T^* f \in L^q(\mathbb{R}^n) \text{ and } \|D_i T^* f\|_q \le C\|f\|_{\mathcal{F}_1}, \tag{16}$$

The numbers $1 < q, s < \infty$ above come from (A1)-(A7). Moreover, for a sequence $(h_k)_{k=1}^{\infty}$ such that $h_k > 0$ and $h_k \to 0$, for a.e. $x \in \mathbb{R}^n$ it holds that

$$D_i T^* f(x) = \lim_{k\to\infty} \frac{|T_r f(x + h_k e_i)| - |T_r f(x)|}{h_k} \text{ for all } r \in \mathcal{R}f(x) . \tag{17}$$

The main general result is the following [C, Theorem 2.10].

Theorem 8.1. *If $\{T_r\}_{r\in I}$ is admissible and $f, f_j \in \mathcal{F}_1$ so that $\|f_j - f\|_{\mathcal{F}_1} \to 0$, then*

$$\|D_i T^* f_j - D_i T^* f\|_s \to 0 \text{ as } j \to \infty . \tag{18}$$

Equation (17) above and Theorem 8.1 generalize Theorem 5.3 and Theorem 5.1.

One of the assumptions (A1)-(A7) is that for almost every x the function $r \mapsto T_r f(x)$ is continuous on $I \subset \mathbb{R}$. More precisely, we find representatives of the functions $T_r f$ so that this holds. This makes it possible to generalize the definition of the best radii simply by setting

$$\mathcal{R}f(x) = \{r \in I : T^*f(x) = |T_r f(x)|\} . \tag{19}$$

Concerning the derivatives of the maximal operator, we recall that $f \in \mathcal{F}_1$ does not need to have functions as weak partial derivatives, for example in the case where $\mathcal{F}_1$ is some L^p-space or fractional Sobolev-space with smoothness index less than 1. However, in many cases it still holds that T^* maps $\mathcal{F}_1$ boundedly into a certain Sobolev-space. We show that if assumptions (A1)-(A7) hold, then this occurs.

The assumption (A2) is that there exists a Sobolev-space $W^{1,p}(\mathbb{R}^n)$ such that for every *fixed* $r \in I$ it holds that $T_r f \in W^{1,p}(\mathbb{R}^n)$. This enables us to define the maximal operator R^* controlling the derivative of $T_r f$ by setting

$$R^*f(x) = \sup_{r\in I_0, 1\le i\le n} |D_i(T_r f)(x)| , \tag{20}$$

where I_0 is some countable and dense subset of I . The property that is most closely connected to the continuity of T^* between $\mathcal{F}_1$ and Sobolev spaces is (A7), which requires that R^* is bounded $\mathcal{F}_1 \to L^s(\mathbb{R}^n)$ for some $s > 1$.

9. Some applications of the general results

The classical *spherical* maximal operator S is defined for $f \in L^1_{loc}(\mathbb{R}^n)$, $n \geq 2$, by

$$Sf(x) = \sup_{r>0} \fint_{\partial B(x,r)} |f(y)|\, d\sigma(y)\,. \tag{21}$$

Above σ denotes the $n-1$-dimensional Hausdorff measure on $\mathbb{R}^n$. It is a deep result of Stein [St] ($n \geq 3$) and Bourgain [Bo] ($n = 2$) that S is bounded on $L^p(\mathbb{R}^n)$ when $p > \frac{n}{n-1}$. Knowing this, it is easy to observe that S is also bounded on $W^{1,p}(\mathbb{R}^n)$ when $p > \frac{n}{n-1}$ [HO]. The continuity is more involved but it turns out that the results in the previous chapter can be applied to S [C, Section 3], yielding:

Theorem 9.1. *S is continuous $W^{1,p}(\mathbb{R}^n) \mapsto W^{1,p}(\mathbb{R}^n)$, when $\frac{n}{n-1} < p < \infty$.*

Our results can be applied also to maximal singular integral operators. As an example, we verify that the maximal Hilbert transform operator H^*, defined by

$$H^*f(x) = \sup_{0<r<1} \Big| \int_{\{|x-y|>r\}} \frac{f(y)}{x-y}\, dy \Big|\,, \tag{22}$$

is continuous on $W^{1,p}(\mathbb{R})$, when $1 < p < \infty$ [C, Section 3].

As we mentioned earlier, it was shown in [KS] that the fractional maximal operator M^α, $1 \leq \alpha < \frac{n}{p}$, has the following property: if $f \in L^p(\mathbb{R}^n)$, $1 < p < n$, then there exists $C = C(n,p,\alpha)$ so that

$$\|D_i M^\alpha f\|_q \leq C\|f\|_p\,, \tag{23}$$

where $q = \frac{np}{n-(\alpha-1)p}$. We extend this result by proving [C, Section 3] that if $f_j \to f$ in $L^p(\mathbb{R}^n)$ and p, α, q are as above, then

$$\|D_i M^\alpha f_j - D_i M^\alpha f\|_q \to 0\,. \tag{24}$$

10. Bounded but discontinuous maximal operator

We now discuss the question (Q2). This is related to the necessity of the properties of R^* for results (16), (17) and, especially, for Theorem 8.1.

Let us define the maximal operator which in some sense is at the core of this question. The *restricted* Hardy-Littlewood maximal operator M^Q_λ, $\lambda \geq 0$, is given by

$$M^Q_\lambda f(x) = \sup_{r\geq\lambda} \fint_{Q(x,r)} |f(y)|\, dy\,, \tag{25}$$

for each $f \in L^1_{loc}(\mathbb{R}^n)$. Here $Q(x,r)$ denotes the cube centered at x with side length $2r$. We also denote by M^B_λ the maximal operator where the cubes $Q(x,r)$ in (25) are replaced by balls $B(x,r)$.

For $\lambda > 0$, rather elementary properties imply that M^Q_λ and M^B_λ are bounded from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ for every $1 < p < \infty$ with a constant comparable to $\frac{1}{\lambda}$ [C, Theorem 4.1]. However, we observe that the R^* corresponding to M^Q_λ is basically defined by taking the supremum of averages of a function f on sets $\partial Q(x,r)$, $r \geq \lambda$. In this case R^* is definitely unbounded. From this we conclude that the boundedness (or other similar property) of R^* is not necessary for (16).

The question about the necessity of the properties of R^* for (17) is more delicate. Actually, we do not know if even (17) holds for M_λ^Q, but we guess that it may at least be proved in some modified form. However, the non-boundedness of R^* is in accordance with the following result:

Theorem 10.1. *M_λ^Q is discontinuous from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ when $1 < p < \infty$ and $\lambda \geq 0$.*

This result is based on the construction in [C, Theorem 4.4]. There we find a sequence of functions $f_j \in W^{1,p}(\mathbb{R})$, which are even smooth except at a finite set of points so that each f_j is supported in $(-10,-9) \cup (9,10)$, $-1 \leq f_j \leq 1$ everywhere, $\|f_j\|_1 \to 0$, and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |D(M(\chi_{(-10,10)} + f_j))(x)|\, dx \geq C \text{ for all } j \in \mathbb{N}. \tag{26}$$

This construction is essentially based on three facts. The first necessary condition is that we are able to find a set $A \subset \mathbb{R}$ with measure zero such that for almost every $x \in [-\frac{1}{2}, \frac{1}{2}]$ there is $r > 0$ so that *both* $x-r$ and $x+r$ are in A. This set can be found rather easily by using Cantor sets. In fact, we will use the set $(C-10) \cup (C+9)$, where C is the standard Cantor $\frac{1}{3}$-set.

The second fact we need is that in the middle of two suitable bumps with integral zero, we obtain a peak in the graph of the maximal function. The crucial point is that the sharpness of this peak does not essentially change if we scale the bumps.

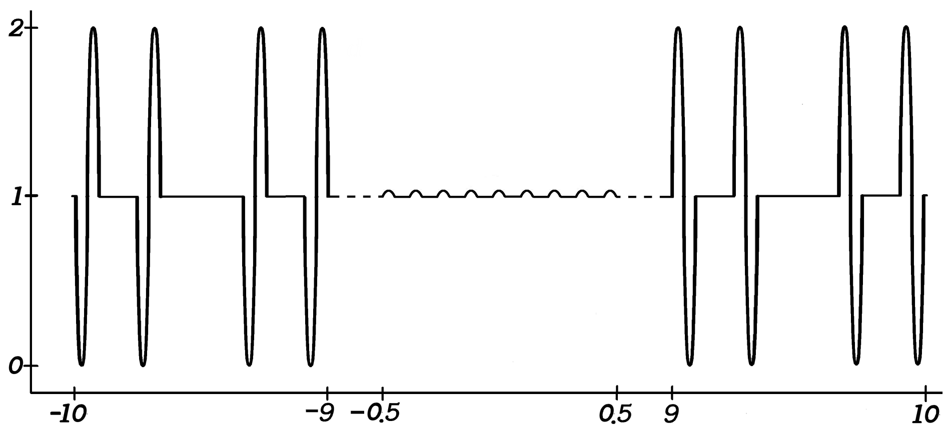

FIGURE 3. Function $\chi_{-\frac{1}{2},\frac{1}{2}} + f_2$ from (26) and in the middle its maximal function on interval $[-\frac{1}{2}, \frac{1}{2}]$ (the amplitude of the fluctuation of the maximal function is magnified).

Finally, it is possible to combine the above facts. The way we do this is described in Figure 3. Let us denote by C_-^j and C_+^j the j:th generation of intervals in the construction of $(C-10)$ and $(C+9)$. The width of these intervals is 3^{-j}. We add to the characteristic function of an interval

$(-10, 10)$ the bumps like on the left in the picture to every interval of C_-^j and bumps like on the right on every interval of C_+^j. The outcome will be that the maximal function of this function fluctuates on the interval $(-\frac{1}{2}, \frac{1}{2})$ with frequency 3^{-j} and amplitude greater than $c3^{-j}$, where c depends only on the shape of the original bump and the distance between the intervals of C_-^j and C_+^j. For that, the decisive fact is that for every $j \in \mathbb{N}$, there exists a set of points $D_j \subset (-\frac{1}{2}, \frac{1}{2})$ with density 3^{-j} and such that for any $x \in D^j$ there does *not* exist $r > 0$ such that $x - r \in C_-^j$ and $x + r \in C_+^j$ (here C_+^j and C_-^j are assumed to be open). This reveals how the Cantor set used satisfies surprisingly well the requirements for the construction.

Since we know that M_λ^Q is bounded but not continuous from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$, let us consider the operator M_λ^B . It turns out that the operator R^* corresponding to this operator is controlled by $\frac{1}{\lambda}S$, which is bounded on $L^p(\mathbb{R}^n)$ when $\frac{n}{n-1} < p < \infty$. Then it is easy to observe that (17) and (8.1) hold for M_λ^B. Hence M_λ^B is continuous from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ when $\frac{n}{n-1} < p < \infty$.

The above results clarify the picture on the regularity properties of the maximal operators. Here it is observed for the first time the real difference between continuity and boundedness of a classical maximal operators and, at the same time, how the regularity properties of the classical maximal operators may change if they are defined by cubes or by balls.

11. Questions

During our research, several natural questions have arisen related to the regularity theory of different maximal operators. We mention here those that are most closely related to the results presented in this thesis.

Question 1. What is the infimum of those $s > 1$ for which M does not preserve the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$? Or, what is the supremum of $s > 1$ so that M is bounded (or continous) on $W^{s,p}(\mathbb{R}^n)$? It has been showed by Korry [Ko] that $s \leq 1 + \frac{1}{p}$.

Question 2. Suppose that the maximal operator T^* corresponding to the family $\{T_r\}$, as in [C, Section 2], is bounded on $W^{1,p}(\mathbb{R}^n)$. What are the minimal assumptions guaranteeing that the formula for the derivative of the maximal function is valid, i.e

$$D_i T^* f(x) = T_r(D_i f)(x) \quad \textit{for all } r \in \mathcal{R}f(x) \ ?$$

Question 3. Let us denote by M_Q^α the fractional maximal operator defined by using cubes instead of balls. Are the regularity properties of M_Q^α and M^α similar? In particular, does (24) hold for M_Q^α? The example given by operators M_λ^Q and M_λ seems to suggest that the answer may be negative. However, there are real difficulties coming in when trying to modify the argument of Theorem 10.1.

Question 4. We proved the continuity of the derivatives of M^α in the case where $p > \frac{n}{n-1}$, see (24). To prove the boundedness of the operator R^* corresponding to M^α we used the spherical maximal operator, which is bounded only when $p > \frac{n}{n-1}$. However, M^α is bounded for any $p > 1$. Accordingly, it is reasonable to ask if the assumption $p > \frac{n}{n-1}$ is necessary for the result in (24).

References

[AP] J.M. Aldaz and J. Perez Lazaro, Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities, Trans. Amer. Math. Soc. 359 (2007), no. 5, 2443-2461 (electronic).

[BCP] A. Benedek, A.P. Calderón and R. Panzone, Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci. 48 (1962), 356-365.

[AL] F.J. Almgren and E.H. Lieb, Symmetric decreasing rearrangement is sometimes continuous. J. Amer. Math. Soc., 2 (1989), 683-773.

[Bo] J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Anal. Math. 47 (1986), 69-85.

[Bu] S.M. Buckley, Is the maximal function of a Lipschitz function continuous?, Ann. Acad. Sci. Fenn. Math. 24 (1999), 519-528.

[HL] G.H. Hardy and J.E. Littlewood, A maximal theorem with function-theoretic applications, Acta Math. 54 (1930), 81-116.

[HO] P. Hajłasz and J. Onninen, On boundedness of maximal functions in Sobolev spaces, Annales Academiae Scientiarum Fennicae Mathematica 29 (2004), 167-176.

[K] J. Kinnunen, The Hardy-Littlewood maximal function of a Sobolev-function, Israel J.Math. 100 (1997), 117-124.

[KL] J. Kinnunen and V. Latvala, Lebesque points for Sobolev functions on metric spaces, Rev.Mat.Iberoamericana 18 (2002), 685-700.

[KS] J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, Bull. London Math. Soc. 35 (2003), no.4, 529-535.

[Ko] S. Korry, Boundedness of Hardy-Littlewood maximal operator in the framework of Lizorkin-Triebel spaces, Rev. Mat. Univ. Complut. Madrid 15 (2002), 401-416.

[SS] W. Schlag and C. Sogge, Local smoothing estimates related to the circular maximal theorem, Math. Res. Lett. 4 (1997), 1-15.

[St] E.M. Stein, Maximal functions. I. Spherical means, Proc. Nat. Acad. Sci. U.S.A 73, 1976, 2174-2175.

[St2] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

[Ta] H. Tanaka, A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function, Bull. Austral. Math. Soc. 65 (2002), 253-258.

[Tr1] H. Triebel, Theory of function spaces, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag (1983).

[Tr2] H. Triebel, Theory of function spaces II, Monographs in Mathematics, Vol. 84, Birkhäuser Verlag (1992).

[W] N. Wiener, The ergodic theorem, Duke Math. J. 5 (1939) 1-18.

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