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KARI NISSINEN

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Abstract

The key question of small area estimation is how to obtain reliable regional statistics when the sample data contain too few observations for statistical inference of adequate precision. A common procedure is to employ statistical models, which make it possible to "borrow strength" for the estimation by utilizing data from similar or neighbouring areas or from earlier similar surveys. This work concentrates on the latter alternative, which is often called borrowing strength over time.

We consider here small area estimation from longitudinal unit-level survey data collected with a panel or a rotating panel design, where the sampled units are observed several times according to a specified scheme. We apply a 3-level variance component model to such data and under it we derive formulas for an estimator of small area total and its mean squared error within the empirical best linear unbiased prediction (EBLUP) framework.

Through a simulation study we show that in small area estimation it is extremely useful to utilize rotating panel data instead of cross-sectional data or complete panel data when (cross-sectional) small area totals are to be estimated. Increasing the effective sample size by using rotating panel data reduces relative estimation errors, leading to smaller bias and more accurate point estimates. Also the mean squared errors of the estimates are reduced, which leads to narrower but still valid confidence intervals. In addition, utilizing rotating panel data provides protection for possible bias caused by misspecified model. These merits appear particularly in the smallest areas.

Keywords: small area estimation, linear mixed model, empirical best unbiased prediction (EBLUP), rotating panel data, borrowing strength over time

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I am most indebted to Professor Jukka Nyblom for helping me to get this work to the end. His extensive knowledge of mathematical statistics and matrix algebra guided me over quite a few tricky issues. His comments and suggestions sharpened my writing and the structure of my thesis remarkably.

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My special thanks go to all the wonderful colleagues in the department. I have enormously enjoyed the years with you. Since it seems now that our ways do part, you are to know that I am going to miss you much.

Outside the academic world I have had a great pleasure to maintain friendships with several good people, who all would deserve to be acknowledged here. At this very moment, however, I want specifically mention my mates in the band. Thanks for being around for 25 years and happy rock'n'roll!

Throughout my life my parents have supported me to such extent that cannot be underestimated. I dedicate this thesis to them. Equally, I dedicate this thesis to my wife Eila and lovely daughters Veera and Ella. My deepest thanks go to you for your patience and making my life so rich for so long time.

Finally, I thank Ellen and Artturi Nyyssönen Foundation for financial support for this work.

Mattilanniemi, 1st December 2008
Kari Nissinen

Selected list of notation

Some frequently used notation, which appears systematically throughout the monograph, is introduced here. The list is not complete, symbols with minor use are left out. The adopted notation follows the established conventions of statistical literature as far as possible. Unfortunately the notational conventions of survey sampling are very different from those of mixed model theory, for example. Thus, some symbols can have different meanings in different contexts. The most regularly used meanings are given in the list.

In the monograph the notation is defined in each context as clearly as possible, and it is hoped that the double use of certain symbols will not cause any confusion.

General notation

i	area
j	unit (within area)
t	time, occasion
m	number of areas
T	number of occasions
y	study variable, response
Y	total of y
$\mathbf{x}$	vector of auxiliary variables $x_1, \ldots, x_p$
U	population
s	sample
r	non-sample, remainder, $U - s$
N	population size
n	sample size
M	number of units in longitudinal data
$\mathbf{I}$	identity matrix
$\mathbf{J}$	unity matrix
$\mathbf{1}$	unity vector
K	number of replications in simulation study

Specific notation for mixed models

$\mathbf{y}$	vector of y observations
$\mathbf{X}$	model matrix of the fixed part
$\boldsymbol{\beta}$	vector of fixed effects
$\mathbf{Z}$	model matrix of the random part
$\mathbf{u}$	vector of random effects
$\mathbf{e}$	vector of error terms
p	number of fixed effects
q	number of random effects
u_i	random effect of area i
v_{ij}	random effect of unit j in area i
$\mathbf{v}_i$	vector of unit effects v_{ij}

$\mathbf{u}_i$	vector of both area effect u_i and unit effects v_{ij}
$\mathbf{G}$	covariance matrix of $\mathbf{u}$
$\mathbf{R}$	covariance matrix of $\mathbf{e}$
$\mathbf{V}$	covariance matrix of $\mathbf{y}$
$\boldsymbol{\sigma}$	vector of variance parameters in $\mathbf{V}$ (and $\mathbf{G}$ and $\mathbf{R}$)
σ_u^2	variance of area effects
σ_v^2	variance of unit effects
σ_e^2	variance of error terms
$\tilde{\mathbf{u}}$	BLUP of $\mathbf{u}$
$\widehat{\mathbf{u}}$	empirical BLUP of $\mathbf{u}$
$\mathbf{l}$	coefficient vector of fixed effects in linear combination $\mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$
$\mathbf{m}$	coefficient vector of random effects in linear combination $\mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$

Specific notation for panel data and rotating panel data

U_{it}	population of area i at time t
U_t	overall population at time t $(U_{1t} \cup \cdots \cup U_{mt})$
U_i^*	panel population of area i $(U_{i1} \cup \cdots \cup U_{iT})$
U^*	overall panel population $(U_1^* \cup \cdots \cup U_m^*$ or $U_1 \cup \cdots \cup U_T)$
N_{it}	size of U_{it}
N_t	size of U_t $(N_{1t} + \cdots + N_{mt})$
N_i^*	size of U_i^* $(N_{i1} + \cdots + N_{iT})$
N^*	size of U^* $(N_1^* + \cdots + N_m^*$ or $N_1 + \cdots + N_T)$
s_{it}	sample from area i at time t
s_t	overall sample at time t $(s_{1t} \cup \cdots \cup s_{mt})$
s_i^*	panel sample data of area i $(s_{i1} \cup \cdots \cup s_{iT})$
s^*	overall panel sample data $(s_1^* \cup \cdots \cup s_m^*$ or $s_1 \cup \cdots \cup s_T)$
n_{it}	size of s_{it}
n_t	size of s_t $(n_{1t} + \cdots + n_{mt})$
n_i^*	size of s_i^* $(n_{i1} + \cdots + n_{iT})$
n^*	size of s^* $(n_1^* + \cdots + n_m^*$ or $n_1 + \cdots + n_T)$
T_{ij}	number of repeated observations on unit j in area i
M_i	number of separate units j in panel population U_i^* or (rotating) panel sample s_i^*
M	total number of separate units j in panel population U^* or (rotating) panel sample s^*
r_{it}	non-sample of area i at time t $(U_{it} - s_{it})$
r_t	overall non-sample at time t $(U_t - s_t)$
$r1_{it}$	subset of r_{it} containing units, which are sampled at some other time than t, i.e. appear in the rotating panel data s_i^*, but not in s_{it}
$r2_{it}$	subset of r_{it} containing the units, which have never been sampled, i.e. do not appear in s_i^* $(r_{it} - r1_{it})$
$N_{r_{it}}$	size of r_{it} $(N_{it} - n_{it})$
N_{r_t}	size of r_t $(N_t - n_t)$
$N_{r1_{it}}$	size of $r1_{it}$
$N_{r2_{it}}$	size of $r2_{it}$

Contents

1 Introduction

In the sample survey methodology the small area estimation has become a field of active research. The small area problem essentially concerns obtaining reliable estimates of quantities of interest — totals or means of study variables, for example — for geographical regions, when the regional sample sizes are small in the survey data set. If the regional estimates are to be obtained by the traditional direct survey estimators, based only on the sample data from the area of interest itself, small sample sizes lead to undesirably large standard errors for them. For instance, due to their low precision the estimates might not satisfy the generally accepted publishing criteria in official statistics. It may even happen that there are no sample members at all from some areas, making the direct estimation impossible. All this gives rise to the need of special small area estimation methodology.

An important motive for the research of small area estimation methods comes from the increasing demand of regional or subgroup statistics both in administration and scientific research (Ghosh and Rao 1994, Mukhopadhyay 1998, Rao 1999, Pfeffermann 2002, Rao 2003). Examples of disciplines and topics where the regional statistics can play an important role are social policy, research on living conditions, environmental sciences, education, regional economics, unemployment and public health. The governmental decisions on e.g. regional distribution of national funds are largely based on regional statistics and therefore ask for highly reliable estimates.

The small areas need not to be understood as only geographically defined regions like provinces, municipalities or health service districts. They can be as well demographically, economically or socially defined subpopulations or domains like age-sex groups or groups of labor workers in different industries. As subpopulations, such domains are not necessarily small, but they may consist of thousands of units, for example. In the context of small area estimation an area or domain becomes small when its sample size is too small for direct estimation of adequate precision (Rao 2003). This happens frequently because many surveys conducted e.g. by national statistical offices are (and often have to be) designed principally for national or international purposes and their sampling designs might not be tuned for obtaining estimates for regions of "subnational" level. For instance, a stratification, which is relevant for national needs, does not necessarily coincide with the regions, if these are not of primary interest. As a result, the obtained sample does not allow acceptable regional inference.

The opinion polls carried out before elections are an illustrative example. The typical sample size of opinion polls in Finland is around 1000 people (from approximately 4 million people entitled to vote), which gives good estimates at the national level. But if there is a need to get estimates for districts like the 416 municipalities or the 15 constituencies in Finland, it is evident that splitting a sample of 1000 people into 416 or even 15 subsets makes the number of respondents small at least in some districts. Increasing the sample size to make the direct estimates reliable for all the districts of interest would demand too much resources in practice.

The research on small area estimation deals with the problem of getting good estimates for small areas or domains, when the survey data contain few, if any, observations from these areas. The way to improve estimation is to "borrow strength" from outside the area of interest, that is, to make use of information about the study variable coming from sources that are external to the sample data from that area. Typically such sources are data from neighbouring or otherwise similar areas and/or data from earlier time periods. The first case is often called borrowing strength over space and the second borrowing strength over time (EURAREA Consortium 2004). By borrowing strength it is generally meant increasing the "effective" sample size for estimation (Rao 2003), which is done by using appropriate models to link the external data to the sample data. The resulting estimators are called indirect. The increased effective sample size is then expected to turn to decreased mean squared error (MSE) of the small area estimator. The MSE consists of two components: the variance and the squared bias of the estimator. Thus, improving an estimator means reducing its variance or bias or both, if possible. Usually it is found that indirect estimators introduce some bias, but at the same time their variance is so small that the overall reduction in MSE is substantial.

This study concerns model-based small area estimation, in which strength is borrowed over time by using longitudinal data sets at unit level, collected by a panel design or a rotating panel design, and applying a linear mixed model to them. It is shown that the rotating panel design is particularly powerful here, because the composition of the sample changes in every survey occasion so that the data sets of different occasions consist of (at least partially) different units. Supplementing the current, possibly small, regional data with earlier data from the same areas by a plausible model brings then a lot of "new" units into the estimation, which substantially increases the effective sample size. The rotating panel design is widely recognized and applied in official statistics (Feder, Nathan and Pfeffermann 2000; Singh, Kennedy and Wu 2001), but before this work it is seldom, if ever, considered in the context of small area estimation.

In general, longitudinal unit level data sets collected either by a panel design or by some variation of it are rare. However, in some European countries, especially in the Nordic countries, which have a long tradition of official statistics and register data sets on individuals, such data are fairly common not only in the official statistics but also in the econometric research or the research on public health. Longitudinal unit level data sets are rich in information, but so far there has not been available small area methodology, by which this richness could be fully utilized. For example, the official monthly unemployment rate in Finland is estimated only by cross-sectional methods despite the fact that the data are collected under rotating panel design. The research at hand is aimed to help in filling this gap.

The longitudinal data sets also make it possible to successfully estimate change between the survey occasions. This also requires moving away from the cross-sectional models and starting to apply models and estimators, which are appropriately tuned for the (rotating) panel data. Estimation of change in small area totals or means using linear mixed models is a relatively straightforward extension of the "cross-sectional" model-based small area estimation, but it will not be considered in this paper. Instead, it will be left as the target of future research.

Outside some unit level models meant for non-panel data from repeated surveys (EURAREA Consortium 2004) the research on utilizing longitudinal information in small area estimation only considers area level models. The current study is then arguably the first one dealing with unit level models.

This report considers small area estimation from panel data and rotating panel data by a linear mixed model within the best linear unbiased prediction (BLUP) framework and it is organized in the following way. First, in Chapter 2 we review the literature on small area estimation especially from the model-based point of view, which gives the background of the study. In Chapter 3 a general review of the theory on linear mixed models and their estimation and inference is given. Chapter 4 introduces the general principles and problems in applying linear mixed models to small area estimation. The unit level nested error regression model for cross-sectional data is used as an example with which the needed concepts are illustrated. Chapter 5 introduces the linear mixed model to be applied to panel and rotating data. In Chapter 6 the empirical BLUP (EBLUP) estimation of small area totals is presented for the panel data and in Chapter 7 the corresponding presentation is given for the rotating panel data. A Monte Carlo simulation study for evaluating the performance of the derived estimators and their MSE estimators under various conditions are presented and discussed in Chapters 8–13.

2 Approaches to small area estimation

2.1 Direct and indirect estimators

Following the definition given by Rao (1999, 2003), a small area estimator is direct when it uses the sample values of study variable y (say) from the specified area only. A simple example is the expansion estimator

$$\widehat{Y}_i = \sum_{j \in s_i} w_{ij} y_{ij} \tag{2.1}$$

of area total

$$Y_i = \sum_{j \in U_i} y_{ij}.$$

Here U_i is the population of area i, s_i is a random sample from it, y_{ij} is the y value of unit j in area i and w_{ij} is a weight. When $w_{ij} = 1/\pi_{ij}$, where π_{ij} is the inclusion probability of unit ij in the chosen design, the estimator (2.1) is the well known design-based Horvitz-Thompson estimator. The variance of $\widehat{Y}_i$ is of order $O(n_i^{-1})$, where n_i is the sample size of area i. If n_i is small, the variance can be large.

The efficiency of estimation can be increased by auxiliary variables $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, which correlate with the study variable y and can be introduced as covariates e.g. in a linear regression model for y. This leads to model-based estimation or design-based model-assisted estimation, depending on the chosen framework. The generalized regression estimation (GREG) (Särndal et al. 1992) is a design-based model-assisted approach with numerous applications to domain estimation and it is sometimes used also in small area estimation. A GREG estimator of a small area total Y_i is

$$\widehat{Y}_{i,GREG} = \widehat{Y}_i + (\mathbf{X}_i - \widehat{\mathbf{X}}_i)'\widehat{\mathbf{b}}_i, \tag{2.2}$$

where

$$\mathbf{X}_i = \sum_{j \in U_i} \mathbf{x}_{ij}$$

is the vector of area population totals of x variables and

$$\widehat{\mathbf{X}}_i = \sum_{j \in s_i} w_{ij} \mathbf{x}_{ij}$$

the direct estimate of it. The estimator is a sum of the Horvitz-Thompson estimator and a regression-based adjustment term. The vector $\widehat{\mathbf{b}}_i$ of regression coefficients of area i is obtained by solving the sample weighted least squares equations (Särndal et al. 1992; Rao 2003 p. 13). The GREG estimator is model-assisted in the sense that regressing y on $\mathbf{x}$ is done only for removing the unexplained variation from y to increase estimation accuracy. Unlike in the model-based approach, any probability model determining the relation between y and $\mathbf{x}$ is not assumed. It is only required that nonzero correlations exist between y and $\mathbf{x}$. The variance of the GREG estimator is calculated from the residuals of the regression fit within the design-based framework.

The estimator (2.2) is completely area-specific. It employs area-specific auxiliary totals $\mathbf{X}_i$ and also the coefficient $\widehat{\mathbf{b}}_i$ is calculated from area-specific data. It is clearly a direct estimator. Another GREG estimator could be

$$\widehat{Y}_{i,GREG} = \widehat{Y}_i + (\mathbf{X}_i - \widehat{\mathbf{X}}_i)'\widehat{\mathbf{b}}, \tag{2.3}$$

where the regression coefficient is now estimated jointly from all the areas, or

$$\widehat{Y}_{i,GREG} = \widehat{Y}_i + (\mathbf{X} - \widehat{\mathbf{X}})'\widehat{\mathbf{b}}, \tag{2.4}$$

where not only $\widehat{\mathbf{b}}$ but also the auxiliary totals are obtained from the whole data. The estimator (2.3) is called survey regression predictor by Battese, Harter and Fuller (1988).

Särndal (2001) considers an estimator direct if it uses y values only from the area of interest, even if observations on auxiliary variables from other areas were employed. In this sense (2.3) and (2.4) are not direct, because they utilize the observed y values from all areas in estimating the regression coefficient. On the other hand, Rao (2003, Ch. 2.5) argues that unlike "true" indirect estimators, the estimators (2.3) and (2.4) fail to increase the effective sample size since the order of their variance is $O(n_i^{-1})$, the same as that of the Horvitz-Thompson estimator (2.1). Therefore Rao calls them modified direct estimators, which do not genuinely borrow strength. According to Rao, a small area estimator is indirect only if it reduces the order of variance (or MSE) compared to the usual direct estimators. This characterization defines the model-assisted design-based estimation direct. Myrskylä (2007, Ch. 2.1.3) discusses the terminology and suggests that "borrowing information" would be a more pertinent term than "borrowing strength".

All model-based small area estimation is indirect. An appropriate model is used here in predicting the unobserved y values (or their sum) from area i to get an estimate of area total or mean. The model links the small areas to each other in some way and it is estimated from all the observed data. Rao (2003, Ch. 4) mentions also indirect estimators that are not based on explicit use of models. Among these are the synthetic estimators, which rely on implicit assumptions like that small area characteristics are approximately similar to those of the overall population, the composite estimators, which combine direct and synthetic estimators, and the James-Stein estimators.

2.2 Design-based approach

The design-based approach to small area estimation or, more generally, to domain estimation is based on the traditional probability sampling theory (e.g. Cochran 1977, Cassel et al. 1977), which rests on the assumption of finite and fixed population and drawing random samples from it with selection probabilities defined by the sampling design. In this approach the survey population U is a collection of N units and with each unit j is associated a value y_j of the study variable y. The y_j is not treated as a random variable, but rather a known (if observed) or unknown (if unobserved) constant. No model or distributional assumptions are specified for y. As fixed constants, the population values of y can be regarded as a vector-valued parameter of very large dimension.

In the sense of statistical modelling, the unknown parameters to be estimated e.g. in small area problems are not model parameters, but some functions $h(y_1, y_2, \ldots, y_N)$ of the y values in the population. To make a distinction to model parameters Pfeffermann (1993) calls these functions as "descriptive population quantities" (DPQ). A simple example of DPQ is the total $Y = \sum_{j=1}^{N} y_j$ of the study variable y.

The model-based inference refers to the probability distribution of the random variable y. In the design-based inference the only random element is the composition of sample s (of size n), selected with probability $p(s)$. The statistical properties of an estimator are evaluated with respect to the sampling distribution determined by the design. That is, the bias and variance of an estimator $\widehat{Y}$ of Y (say) are calculated over all possible samples under the specified design and they are therefore called the design bias and the design variance.

The design expectation of an estimator $\widehat{Y}$ of population total is

$$E_p(\widehat{Y}) = \sum p(s)\widehat{Y}_s,$$

where the summation is over all possible samples s and $\widehat{Y}_s$ is the value of $\widehat{Y}$ in the sample s. If $E_p(\widehat{Y}) = Y$, the estimator is design unbiased. The design variance of $\widehat{Y}$ is

$$Var_p(\widehat{Y}) = E_p\left[\widehat{Y} - E_p(\widehat{Y})\right]^2.$$

An estimator $\widehat{Y}$ is design consistent if its design bias and design variance tend to zero as the sample size reaches the population size. If both $\widehat{Y}$ and its variance estimator $\widehat{Var}(\widehat{Y})$ are design consistent, the pivot

$$(\widehat{Y} - Y)/\sqrt{\widehat{Var}(\widehat{Y})}$$

converges in distribution to $N(0,1)$, making e.g. the inference by confidence intervals possible (Rao 2003, p. 11).

As such, an estimator is not necessarily purely design based or purely model based. At least in some simple cases the design-based estimator and model-based estimator may coincide. For instance, under the simple random sampling or stratified sampling with proportional allocation the ordinary sample mean $\bar{y}$ is a design-based estimator of the finite population mean $\bar{Y}$. On the other hand, if the study variable y is regarded as a random variable with a probability distribution, the sample mean is a model-based estimator of the expected value μ of this distribution. Thus, in the first place the choice between design-based and model-based approaches is not a matter of selecting the estimator, but selecting the framework for inference, i.e. the probability distribution characterizing the random variation of the estimator. Pfeffermann (1993) and Valliant et al. (2000, Ch. 1) discuss the relation between design-based and model-based inference in more detail.

It is possible to evaluate model-based estimators from a design-based standpoint by examining their sampling distributions, and this is particularly useful in the survey methodology, where various estimators are often compared under finite population sampling. On the other hand, it is also possible to study model-based properties of a design-based estimator, but this happens less commonly.

In general, the design-based estimators make explicit use of sampling weights, which are inverses of the inclusion probabilities of sampling units. A simple design-based estimator is the Horvitz-Thompson estimator (2.1) of area total Y_i. It is design unbiased and has the design variance

$$Var_p(\widehat{Y}_i) = \sum_{j \in U_i} \sum_{k \in U_i} \left(\frac{\pi_{ijk}}{\pi_{ij}\pi_{ik}} - 1 \right) y_{ij} y_{ik},$$

where π_{ijk} is the joint inclusion probability of units j and k in area i (Särndal et al. 1992, Ch. 2.8). An unbiased estimator of this variance is given in Särndal et al. (1992, Ch. 2.8) and also in Rao (2003, p. 12).

The GREG estimators (2.2)–(2.4) are design consistent and approximately design unbiased. The bias, which often is negligible, arises from the fact that $\widehat{\mathbf{b}}$ obtained from the sample weighted least squares equations is not a design-unbiased estimator of the population regression coefficient $\mathbf{b}$ (Särndal et al. 1992, Ch. 6.; Valliant et al. 2000, Ch. 2.7). It is worth noting that $\mathbf{b}$, as a characteristic of a finite population, can be considered a DPQ in the sense of Pfeffermann (1993).

The approximate design variance, obtained from a Taylor series expansion, of the GREG estimators (2.2)–(2.4) is of form

$$Var_p(\widehat{Y}_{i,GREG}) = \sum_{j \in U_i} \sum_{k \in U_i} \left(\frac{\pi_{ijk}}{\pi_{ij}\pi_{ik}} - 1 \right) e_{ij} e_{ik},$$

where $e_{ij} = y_{ij} - \mathbf{x}'_{ij}\mathbf{b}_i$ (for (2.2)) or $e_{ij} = y_{ij} - \mathbf{x}'_{ij}\mathbf{b}$ (for (2.3) and (2.4)) is the residual from the population regression fit for unit j in area i. This variance is of the same order $O(n_i^{-1})$ as the variance of the Horvitz-Thompson estimator, but in practice one can anticipate that the regression adjustment makes the actual variance of GREG estimators remarkably lower (Särndal et al. 1992, p. 226, 238). A variance estimator based on weighted residuals from the sample regression fit is given in Särndal et al. (1992, Ch. 6.6) and also in Rao (2003, p. 15) and Estevao et al. (1995).

2.3 Synthetic estimator

An alternative expression for the GREG estimator (2.2) is

$$\widehat{Y}_{i,GREG} = \mathbf{X}_i'\widehat{\mathbf{b}}_i + (\widehat{Y}_i - \widehat{\mathbf{X}}_i'\widehat{\mathbf{b}}_i), \tag{2.5}$$

which is a sum of the area total predicted by the fitted regression model and the sum of sample residuals. The similar expression for the survey regression predictor (2.3) is

$$\widehat{Y}_{i,GREG} = \mathbf{X}_i'\widehat{\mathbf{b}} + (\widehat{Y}_i - \widehat{\mathbf{X}}_i'\widehat{\mathbf{b}}). \tag{2.6}$$

The first term in these expressions can be considered as an estimator of a small area total in its own right and it is called the synthetic estimator (Särndal et al. 1992, p. 399; Rao 2003, Ch. 4.2). Hence the GREG estimator can be viewed as a synthetic estimator with a correction term for the design bias. The most commonly used synthetic estimator comes from (2.6), where the regression coefficient is obtained from the whole sample data to yield

$$\widehat{Y}_{i,SYN} = \mathbf{X}_i'\widehat{\mathbf{b}}. \tag{2.7}$$

This can be calculated even if there are no sample members from the area i. Note that in such case the bias correction term of the estimator (2.6) goes to zero making the GREG and synthetic estimators equal.

The synthetic estimator (2.7) is an example of an estimator, which can be considered either model-based or design-based model-assisted. In the both cases the specified linear relationship between y and the auxiliary variables, described with the parameter $\mathbf{b}$, plays an important role. In the design-based approach we assume no explicit model, but the more correlated y is with the auxiliaries the more efficient is the estimator. If we adopt the model-based approach, we assume that the linear model

$$y_{ij} = \mathbf{x}_{ij}'\mathbf{b} + e_{ij}, \tag{2.8}$$

where the e_{ij}'s are independent identically distributed random variables with some probability distribution f, holds for every y_{ij} in the population. If $\widehat{\mathbf{b}}$ is the unbiased OLS estimator of $\mathbf{b}$, then $\widehat{Y}_{i,SYN}$ is unbiased with respect to the model (using the sample weighted least squares would introduce some model bias). The model-based variance

$$Var_f(\widehat{Y}_{i,SYN}) = \mathbf{X}_i'Cov(\widehat{\mathbf{b}})\mathbf{X}_i$$

of $\widehat{Y}_{i,SYN}$ comes immediately from the covariance matrix of $\widehat{\mathbf{b}}$. Since $\mathbf{b}$ is estimated from the sample data from all areas (of total size of n), the variance of $\widehat{Y}_{i,SYN}$ is $O(n^{-1})$, being considerable lower than in the Horvitz-Thompson or the GREG estimation. Hence the synthetic estimator has the ability to borrow strength and is clearly indirect.

On the other hand, we may regard the synthetic estimator as a special case of the GREG estimator (with no correction for design bias), and as such it is a model-assisted estimator, whose inference can be carried out in the design-based framework. Then it is not explicitly required that the assisting model (2.8) holds, but it appears that the estimator performs well only if the true regression coefficient, which is now a DPQ, happens to be close to **b** in all considered areas. Otherwise the synthetic estimator can be highly biased in some areas. The assumption that the area-specific regression coefficients are approximately equal is an example of what Rao (2003) calls implicit modelling.

The design variance of the synthetic estimator will be small (order n^{-1}), but the bias can make the mean squared error large (see, for example, the empirical findings in the EURAREA Consortium (2004) research report). Mukhopadhyay (1998, Ch. 3.3) and Rao (2003, Ch. 4.2.4) consider the estimation of design MSE of $\widehat{Y}_{i,SYN}$. One approximate MSE estimator is

$$\widehat{MSE}_p(\widehat{Y}_{i,SYN}) \approx (\widehat{Y}_{i,SYN} - \widehat{Y}_i)^2 - \widehat{Var}_p(\widehat{Y}_i),$$

where $\widehat{Y}_i$ is a design-unbiased direct estimator and $\widehat{Var}_p(\widehat{Y}_i)$ its estimated design variance.

An area-specific synthetic estimator could be obtained by using the first term in (2.5). This estimator would be design unbiased, but the design variance would be large instead.

2.4 Model-based approach

2.4.1 Comparisons with design-based approach

In the practice of survey statistics the direct design-based estimators have been highly appreciated, because they are at least approximately design unbiased even in complex sampling designs. They are also robust to the possible misspecification of the assisting model so that the estimator is design unbiased, whether the model is correct or not. The model-based estimators, which rest on the sampling from an infinite hypothetical population (superpopulation) characterized by a stochastic model, can suffer from severe bias if the model is not correct. In addition, they may be biased and inconsistent with respect to the chosen design.

In the small area estimation the question is often about the tradeoff between variance and bias. With small sample sizes the unbiasedness (and robustness) of the direct estimators may be of no practical value due to the large variance of the estimator. The point estimates itself can be "bad" or the standard errors (or coefficients of variation) can be so large that e.g. the accuracy criteria set for the publication of official statistics are not fulfilled. The model-based estimators are prone to bias, but they have the advantage of small variances (also in the design-based sense) compared to the design-based counterparts, especially in the small area context. In the GREG estimator (2.6) the synthetic part $\mathbf{X}_i'\widehat{\mathbf{b}}$ has a low design-based variance, of order $O(n^{-1})$, where n is the total sample size, but the bias can be considerable. Introducing the design-based regression adjustment term removes the bias, but at the same time the variance of the estimator becomes of order $O(n_i^{-1})$, where n_i is the area sample size. If n_i is small, the increase of variance is large. Usually some bias is tolerated, if the mutual reduction of variance ends up in reduced MSE.

There is evidence that the indirect model-based small area estimators outperform the direct estimators with respect to the estimation accuracy measured with the MSE or absolute relative errors (Torabi and Rao 2008 and the large Monte Carlo studies of EURAREA Consortium (2004)). This is probably why the model-based approach is widely accepted as the prime framework for the small area estimation. Only a small portion of the papers published on the small area estimation seem to deal with the design-based approach (Särndal and Hidiroglou 1989 and Särndal 2001, for example).

2.4.2 Ignorable sample selection

A simple example of a model-based estimator is the synthetic estimator (2.7) under the model (2.8). All properties of the estimator are derived from the postulated probability model for the random variable y_{ij} and they are valid if the realized y_{ij}'s in the sample obey this model. Unfortunately, this cannot always be guaranteed. The condition, which is required, is that the sample selection mechanism is ignorable or, synonymously, noninformative (Valliant et al. 2000, Ch. 2.6; Pfeffermann 1993).

Let $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_N)$ be the sample inclusion vector of a population of size N. That is, $\delta_i = 1$ when the unit i is in the sample, otherwise $\delta_i = 0$. The vector $\boldsymbol{\delta}$ is a random variable of which each sample selection produces a realization. Denote the distribution of $\boldsymbol{\delta}$ by $f_{\boldsymbol{\delta}}(\boldsymbol{\delta}; \boldsymbol{\phi})$, where $\boldsymbol{\phi}$ is an unknown parameter, on which the selection mechanism may depend. This distribution gives the sample probabilities, which are commonly denoted by $p(s)$ in the design-based literature (also in Section 2.2).

Let then $\mathbf{z}$ denote a vector of variables, which may affect the sample selection. For instance, $\mathbf{z}$ may contain variables that define the strata, the probabilities in the PPS sampling or the clusters. A sample selection mechanism is ignorable if it does not depend on the study variable y given z or, more formally, if

$$f_{\boldsymbol{\delta}|y,\mathbf{z}}(\boldsymbol{\delta}|y, \mathbf{z}; \boldsymbol{\phi}) = f_{\boldsymbol{\delta}|\mathbf{z}}(\boldsymbol{\delta}|\mathbf{z}; \boldsymbol{\phi}). \tag{2.9}$$

Under ignorable sampling design the probability model of y is a justified basis for the inference. Examples of ignorable sampling schemes are the simple random sampling and the stratified sampling with proportional allocation. In general, all selection mechanisms, which yield self-weighting samples, are ignorable. If the sampling is not self-weighting and depends on $\mathbf{z}$, the proper inference is obtained by putting the $\mathbf{z}$ variables into the model as covariates (Valliant et al. 2000, p. 39; Rao 2003, p. 79).

In the small area estimation with the basic area level models (Fay and Herriot 1979; Ghosh and Rao 1994; Rao 1999, 2003) the sampling design is taken implicitly into account as the models operate with area level quantities, which are computed by "aggregating out" the design on individual units. For the small area estimation with unit level models in non-ignorable designs Prasad and Rao (1999) and You and Rao (2002, 2003) have proposed approaches, where the survey weights are incorporated in an originally model-based inference to produce model-assisted design-consistent estimators.

2.4.3 General prediction theorem

The model-based approach to finite population theory treats the population vector $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ as a realization of a random variable $\mathbf{Y}$. The probability distribution of $\mathbf{Y}$ is characterized by a model ξ. The aim is to estimate the value of a descriptive population quantity, which is a function $h(\mathbf{y})$ of $\mathbf{y}$, typically a linear combination $\mathbf{c}'\mathbf{y}$. If $\mathbf{c} = \mathbf{1}$, where $\mathbf{1}$ is an unity vector, then $h(\mathbf{y})$ is the population total, and if $\mathbf{c} = \mathbf{1}/N$, then $h(\mathbf{y})$ is the population mean.

Let s denote a sample of size n from a finite population U and let r denote the non-sampled remainder of U so that $U = s \cup r$. Correspondingly, let the vector $\mathbf{y}_s$ contain the y observations in the sample and $\mathbf{y}_r$ the rest of $\mathbf{y}$. It is advisable to order the population vector so that

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_s \\ \mathbf{y}_r \end{bmatrix}$$

and respectively

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_s \\ \mathbf{c}_r \end{bmatrix}.$$

The population quantity to be estimated is now

$$h(\mathbf{y}) = \mathbf{c}_s'\mathbf{y}_s + \mathbf{c}_r'\mathbf{y}_r,$$

a realization of the random variable

$$h(\mathbf{Y}) = \mathbf{c}_s'\mathbf{Y}_s + \mathbf{c}_r'\mathbf{Y}_r.$$

The first term $\mathbf{c}_s'\mathbf{y}_s$ is observed from the sample, whereas the second term must be estimated (or predicted, in the frequentist terminology, since it is a function of random variables, not a fixed parameter). Thus, estimating $h(\mathbf{y})$, or predicting $h(\mathbf{Y})$, is essentially predicting the value $\mathbf{c}_r'\mathbf{y}_r$ of the unobserved random variable $\mathbf{c}_r'\mathbf{Y}_r$. The information needed in the prediction will come from the sample vector $\mathbf{Y}_s$, and the predictor (or estimator) of $h(\mathbf{Y})$ can be written

$$\widehat{h}(\mathbf{Y}) = \mathbf{c}_s'\mathbf{Y}_s + \mathbf{a}'\mathbf{Y}_s, \tag{2.10}$$

where $\mathbf{a}$ is some $n \times 1$ vector defining the predictor $\mathbf{a}'\mathbf{Y}_s$ of $\mathbf{c}_r'\mathbf{Y}_r$.

The estimator $\widehat{h}(\mathbf{Y})$ is model unbiased, if

$$E_\xi[\widehat{h}(\mathbf{Y}) - h(\mathbf{Y})] = 0.$$

The model-based error variance of $\widehat{h}(\mathbf{Y})$ is

$$Var_\xi[\widehat{h}(\mathbf{Y}) - h(\mathbf{Y})] = E_\xi[\widehat{h}(\mathbf{Y}) - h(\mathbf{Y})]^2.$$

The general prediction theorem by Royall (1976, see also Valliant et al. 2000) gives the best linear unbiased predictor (BLUP) of $\mathbf{c}'\mathbf{Y}$ as well as its error variance under the general linear model, in the case of finite population. The best linear unbiased predictor means here a model-unbiased predictor, which is linear in $\mathbf{Y}_s$ and has the minimum model-based error variance among all linear unbiased predictors. The theorem serves as a general basis of the BLUP approach to small area estimation with unit level models. Either we may consider only the population U_i of an area i, which leads to the direct estimation, or we may consider the overall population U (and a sample from it), but define the coefficient vector $\mathbf{c}$ so that it picks only those elements of $\mathbf{y}$, which come from the area i. This leads to the indirect estimation with ability to borrow strength. At area level it is more straightforward to develop the BLUP approach within the standard theory of linear mixed models.

Define the general linear model ξ with

$$E_\xi(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$$

and

$$Var_\xi(\mathbf{Y}) = \mathbf{V},$$

where $\mathbf{X}$ contains auxiliary variables, $\boldsymbol{\beta}$ is a vector of unknown parameters and $\mathbf{V}$ is an arbitrary positive definite covariance matrix. The model ξ covers a variety of special cases, including the linear mixed models. The theorem does not require normality.

In accordance with the partition $U = s \cup r$ we can arrange $\mathbf{X}$ and $\mathbf{V}$ so that

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{bmatrix}$$

and

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_s & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_r \end{bmatrix}.$$

The general prediction theorem. Under the model ξ for a finite population U the best linear model-unbiased predictor of $h(\mathbf{Y}) = \mathbf{c}'\mathbf{Y}$ is

$$BLUP(\mathbf{c}'\mathbf{Y}) = \mathbf{c}_s'\mathbf{Y}_s + \mathbf{c}_r'[\mathbf{X}_r\widehat{\boldsymbol{\beta}} + \mathbf{V}_{rs}\mathbf{V}_s^{-1}(\mathbf{Y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}})], \tag{2.11}$$

where

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{Y}_s \tag{2.12}$$

is the general least squares (GLS) estimator of $\boldsymbol{\beta}$. The GLS estimator is also the best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$, that is, it has the minimum variance among linear unbiased estimtors (e.g. Silvey 1975, McCulloch and Searle 2001). The error variance is

$$
\begin{aligned}
Var_\xi[BLUP(\mathbf{c}'\mathbf{Y}) - \mathbf{c}'\mathbf{Y}] = &\ \mathbf{c}_r'(\mathbf{V}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{V}_{sr})\mathbf{c}_r \\
&+ \mathbf{c}_r'(\mathbf{X}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{X}_s)(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{X}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{X}_s)'\mathbf{c}_r.
\end{aligned}
$$

Proof. The proof is sketched here by following the presentation of Valliant et al. (2000, Ch. 2.2). The result (2.11) is obtained by minimizing

$$
\begin{aligned}
E_\xi(\mathbf{a}'\mathbf{Y}_s - \mathbf{c}_r'\mathbf{Y}_r)^2 &= Var(\mathbf{a}'\mathbf{Y}_s - \mathbf{c}_r'\mathbf{Y}_r) + [E_\xi(\mathbf{a}'\mathbf{Y}_s - \mathbf{c}_r'\mathbf{Y}_r)]^2 \\
&= \mathbf{a}'\mathbf{V}_s\mathbf{a} + \mathbf{c}_r'\mathbf{V}_r\mathbf{c}_r - 2\mathbf{a}'\mathbf{V}_{sr}\mathbf{c}_r + [(\mathbf{a}'\mathbf{X}_s - \mathbf{c}_r'\mathbf{X}_r)\boldsymbol{\beta}]^2 \qquad (2.13)
\end{aligned}
$$

with respect to $\mathbf{a}$ under the model unbiasedness constraint

$$E_\xi(\mathbf{a}'\mathbf{Y}_s - \mathbf{c}_r'\mathbf{Y}_r) = (\mathbf{a}'\mathbf{X}_s - \mathbf{c}_r'\mathbf{X}_r)\boldsymbol{\beta} = 0$$

for all $\boldsymbol{\beta}$, which is equivalent to

$$\mathbf{a}'\mathbf{X}_s - \mathbf{c}_r'\mathbf{X}_r = \mathbf{0}.$$

The minimization can be done using the Lagrange method. The function to be minimized is

$$H(\mathbf{a}, \boldsymbol{\lambda}) = \mathbf{a}'\mathbf{V}_s\mathbf{a} - 2\mathbf{a}'\mathbf{V}_{sr}\mathbf{c}_r + 2(\mathbf{a}'\mathbf{X}_s - \mathbf{c}_r'\mathbf{X}_r)\boldsymbol{\lambda},$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. Setting the partial derivative

$$\frac{\partial H(\mathbf{a}, \boldsymbol{\lambda})}{\partial \mathbf{a}} = 2\mathbf{V}_s\mathbf{a} - 2\mathbf{V}_{sr}\mathbf{c}_r + 2\mathbf{X}_s\boldsymbol{\lambda}$$

equal to zero yields

$$\mathbf{X}_s\boldsymbol{\lambda} = \mathbf{V}_{sr}\mathbf{c}_r - \mathbf{V}_s\mathbf{a} \qquad (2.14)$$

and further

$$\mathbf{a} = \mathbf{V}_s^{-1}(\mathbf{V}_{sr}\mathbf{c}_r - \mathbf{X}_s\boldsymbol{\lambda}). \qquad (2.15)$$

Multiplying (2.14) on the left by $\mathbf{X}_s'\mathbf{V}_s^{-1}$, using the unbiasedness constraint and solving for $\boldsymbol{\lambda}$ yields

$$\boldsymbol{\lambda} = (\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{V}_{sr} - \mathbf{X}_r')\mathbf{c}_r$$

and substituting this into (2.15) gives the expression

$$\mathbf{a} = \mathbf{V}_s^{-1}[\mathbf{V}_{sr} - \mathbf{X}_s(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{V}_{sr} - \mathbf{X}_r')]\mathbf{c}_r. \qquad (2.16)$$

The result (2.11) is obtained after straightforward calculations by inserting (2.16) into (2.10) and using (2.12). The expression of error variance is obtained similarly by inserting (2.16) into (2.13). □

The inference concerning the BLUP of $\mathbf{c}'\mathbf{y}$ usually appeals to the normal distribution. Valliant et al. (2000, Ch. 2.5) give three fairly reasonable conditions, under which the BLUP is asymptotically normal.

2.5 Models for small area estimation

2.5.1 Random area effects

The model-based small area estimation largely employs linear mixed models involving random area effects. The auxiliary variables $\mathbf{x}$ are introduced in the fixed part of the model as covariates. Usually the random terms in the model, and the target variable y with them, are assumed normal.

In area level models the observational units for both the target variable and auxiliaries are the areas. The random area effect arises then "naturally" as the residual of a linear model specified for the target variable.

In unit level models the observational units are the individuals in the areas. The areas have often a clustering effect making the individual observations within an area closer together than the observations overall. As a result, the observations are not independent but have a positive "intra-area" correlation, which must be taken into account in a proper statistical inference. A convenient way to proceed is to model the clustering by the random area effect.

The area effect could also be treated as fixed, especially when the number of areas is small. Then it is assumed that area affects only on the mean level of the target variable and no within-area clustering is present. The assumption of correlated observations is often more realistic, however.

From the perspective of small area estimation there is also another advantage in modelling the area effects as random. When the area effects are considered random variables with a common distribution, samples from all areas give information from this distribution and as a result each area contributes to the estimation of the other area effects. This leads to a phenomenon, which is known as shrinkage and appears even if the areas are independent of each other. When a sample from an area is small, there is little information on the effect of that area. Then, through the assumed common distribution, the weak information is supplemented with samples from other areas to yield an area effect estimate, which is "shrunken" (biased) towards the overall mean of area effects, but is more stable than an estimate based on the data from that area alone. In this way the shrinkage is closely related to the concept of borrowing strength.

The predicted values of the random area effects are needed in many applications, the small area estimation being one of them. The best linear unbiased predictors (BLUP), which can be derived within either frequentist or Bayesian approach (see Robinson 1991), are usually employed here.

If the target variable y is discrete, e.g. binary, it can be modelled with a generalized linear mixed model (Zeger and Karim 1991, Goldstein 1995, Ghosh et al. 1998), where an appropriate link function for y's expected value and an error distribution for the observations would be specified. The random area effects are usually assumed normal also here.

The basic area level model for small area estimation is often called as Fay-Herriot model according to the seminal paper by Fay and Herriot (1979). The basic unit level model is the nested error regression model, a two-level variance component model introduced by Battese et al. (1988) for small area estimation purposes.

Many extensions of the basic models are proposed in the literature on small area estimation. In spatial models (Cressie 1993) the area effects are allowed to correlate spatially, e.g. as a function of the distance. With these models the strength for estimation is borrowed "over space" from the neighbouring or similar areas (EURAREA Consortium 2004). In another class of models strength is borrowed "over time" using temporal data in form of area-level time series or unit-level data from previous surveys. In these models random time effects are sometimes considered (e.g. Datta et al. 1994, Ghosh et al. 1996). Also the state-space models (Durbin and Koopman 2001) are frequently applied to small area problems with time series data (e.g. Binder and Dick 1989, Pfeffermann and Burck 1990, Singh et al. 1994, Feder et al. 2000).

2.5.2 Basic area level model

Denote the finite population of area i by U_i, $i = 1, 2, \ldots, m$, and the unknown population mean of a target variable y in area i by $\bar{Y}_i$. Assume that auxiliary data are available at area level, that is, we have area-specific data vectors $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{pi})$ with known values for each area. Let $\theta_i = h(\bar{Y}_i)$ denote some function of the unknown area mean, it is typically the area mean itself or the area total. The target is in the estimation of the value of θ_i.

In the basic area level model it is assumed that θ_i is related to the auxiliary data through a linear regression model

$$\theta_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i, \qquad (2.17)$$

where the error terms u_i are uncorrelated with mean zero and variance σ_u^2. Normality of u_i's is often assumed. We define $\widehat{\theta}_i = h(\widehat{\bar{Y}}_i)$, where $\widehat{\bar{Y}}_i$ is the direct design-based estimator of $\bar{Y}_i$. We can specify a "sampling model"

$$\widehat{\theta}_i = \theta_i + e_i, \qquad (2.18)$$

where the e_i's are independent sampling errors with zero mean (meaning that $\widehat{\theta}_i$ is design-unbiased for θ_i) and known variances ψ_i. Combining models (2.17) and (2.18) yields the area level linear mixed model

$$\widehat{\theta}_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i + e_i, \qquad (2.19)$$

introduced by Fay and Herriot (1979). The model errors u_i and sampling errors e_i are assumed mutually independent. When θ_i is a linear function of $\bar{Y}_i$, it is often safe to assume normality of $\widehat{\theta}_i$'s due to the central limit theorem.

The model takes the sampling design into account implicitly through the survey weights in the direct estimator $\widehat{\bar{Y}}_i$. The assumption of known ψ_i's may cause problems. Rao (1999, 2003) suggests using smoothed estimates obtained from unit-level sample data. When θ_i is a nonlinear function of $\bar{Y}_i$, the sampling model (2.18) may not be valid. As a result, the estimator $\widehat{\theta}_i$ may be biased, especially with small sample sizes n_i, even if the direct estimator $\widehat{\bar{Y}}_i$ is design-unbiased. In this case the standard linear model theory does not apply and alternative methods are needed (Rao 1999).

The natural way to estimate $\theta_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$ is to replace $\boldsymbol{\beta}$ and u_i with the respective estimator $\widehat{\boldsymbol{\beta}}$ and predictor $\tilde{u}_i$. The model-based estimation with the Fay-Herriot model (2.19) is usually carried out within the frequentist BLUP approach or within the empirical (EB) or hierarchical (HB) Bayes approaches. Reviews of these are given by Rao (1999, 2003) and Ghosh and Rao (1994). Under (2.19) the BLUP estimator of θ_i can be expressed as a weighted combination of the direct estimator $\widehat{\theta}_i$ and the regression-synthetic estimator $\mathbf{x}_i'\widehat{\boldsymbol{\beta}}$, where $\widehat{\boldsymbol{\beta}}$ is the BLUE of $\boldsymbol{\beta}$, obtained by the GLS method, for example. The estimator is

$$\begin{aligned}\tilde{\theta}_{i,BLUP} &= \mathbf{x}_i'\tilde{\boldsymbol{\beta}} + \tilde{u}_i \\ &= \mathbf{x}_i'\tilde{\boldsymbol{\beta}} + \gamma_i(\widehat{\theta}_i - \mathbf{x}_i'\tilde{\boldsymbol{\beta}}) \\ &= \gamma_i\widehat{\theta}_i + (1-\gamma_i)\mathbf{x}_i'\tilde{\boldsymbol{\beta}}\end{aligned}$$

with

$$\gamma_i = \frac{\sigma_u^2}{\sigma_u^2 + \psi_i}.$$

The term $\tilde{u}_i = \gamma_i(\widehat{\theta}_i - \mathbf{x}_i'\tilde{\boldsymbol{\beta}})$ is the best linear unbiased predictor of the model error u_i and it is obtained as

$$\tilde{u}_i = E(u_i|\widehat{\theta}_i) = E(u_i) + Cov(\widehat{\theta}_i, u_i)[Cov(\widehat{\theta}_i)]^{-1}[\widehat{\theta}_i - E(\widehat{\theta}_i)]$$

(McCulloch and Searle 2001, Ch. 9) under model (2.19).

Since the Fay-Herriot model deals with the area level quantities θ_i and not with the individual observations, the BLUP estimator is valid for general sampling designs (Ghosh and Rao 1994). It is also design-consistent since $\gamma_i \to 1$ as $\psi_i \to 0$. If the direct estimator is not available, e.g. the sample size of area i is zero, then the BLUP estimator reduces to the regression-synthetic estimator.

The variance component σ_u^2 can be estimated by the usual methods associated with the linear mixed models, e.g. by ML, REML or the method of moments. Replacing σ_u^2 with its estimate in γ_i yields the empirical BLUP (EBLUP) estimator of θ_i. Under normal errors this is also the empirical Bayes estimator of θ_i (Ghosh and Rao 1994).

2.5.3 Basic unit level model

When the auxiliary data are available at the individual level, the basic model is so-called nested error regression model. It is a two-level variance component model and in the context of small area estimation it was introduced by Battese, Harter and Fuller (1988).

The nested error regression model takes the form

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + u_i + e_{ij}, \tag{2.20}$$

where y_{ij} is the response of individual (or unit) j, $j = 1, 2, \dots, N_i$, in area i, $\mathbf{x}_{ij}$ is the vector of auxiliary variables, $\boldsymbol{\beta}$ is the vector of fixed parameters, u_i is the random effect of area i and e_{ij} the random individual error term. The area effects u_i are assumed independent with zero mean and variance σ_u^2. Similarly, the errors e_{ij} are independent with zero mean and variance σ_e^2. In addition, the u_i's and the e_{ij}'s are assumed mutually independent. Normality of u_i's and e_{ij}'s is often imposed to obtain tractable MSE approximations for the small area estimates. The model (2.20) is assumed to hold in the population U as well as in the sample s. This requires ignorable sampling design.

The population quantity of interest is usually the total Y_i or the mean $\bar{Y}_i$ of area i. The estimation is carried out either in the frequentist (BLUP) framework or in the Bayesian (EB, HB) framework. A review and examples of these approaches are given in Ghosh and Rao (1994) and Rao (2003).

The general prediction theorem gives the basis for the BLUP estimation in the unit-level case. Let s_i denote a sample from the finite population U_i of area i and $r_i = U_i - s_i$ the remainder. Consider the estimation of total

$$Y_i = \sum_{j \in U_i} y_{ij} = \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} y_{ij}.$$

The sample sum is observed so that the estimation of Y_i reduces to prediction of the unobserved remainder sum with the model (2.20). The best predictor of y_{ij} is

$$\tilde{y}_{ij} = \mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + \tilde{u}_i,$$

where $\widehat{\boldsymbol{\beta}}$ is the BLUE of $\boldsymbol{\beta}$ and $\tilde{u}_i$ is the BLUP of u_i. These are calculated from the overall sample data $s = \cup_{i=1}^m s_i$. The BLUP estimator of Y_i is then

$$\begin{aligned} \widehat{Y}_{i,BLUP} &= \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} \tilde{y}_{ij} \\ &= \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} \mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + (N_i - n_i)\tilde{u}_i, \end{aligned} \tag{2.21}$$

where n_i is the size of sample s_i. As the employed model is linear, for the BLUP estimation it suffices to know the area population sum $\sum_{j \in U_i} \mathbf{x}_{ij}$ of the auxiliaries, the individual values are not needed. The remainder sum of $\mathbf{x}_{ij}$'s is simply

$$\sum_{j \in r_i} \mathbf{x}_{ij} = \sum_{j \in U_i} \mathbf{x}_{ij} - \sum_{j \in s_i} \mathbf{x}_{ij},$$

the last sum being obtained from the sample. It is not difficult to show that $\widehat{Y}_{i,BLUP}$ (2.21) agrees completely with the general prediction theorem (2.11) under the model (2.20).

Usually the variance components σ_u^2 and σ_e^2 are unknown. When their estimates are used in the BLUP approach, the resulting small area estimator is the empirical BLUP estimator.

For categorical or count variables the normality assumption of individual errors is not valid. Then the generalized linear mixed models with random area effects u_i have been applied (e.g. MacGibbon and Tomberlin 1989; Farrell, MacGibbon and Tomberlin 1997; Ghosh, Natarajan, Stroud and Carlin 1998).

2.6 Extensions that borrow strength over time

2.6.1 Repeated surveys

Many surveys are regularly repeated in time. In fact, almost all surveys conducted by national statistical agencies are carried out periodically (EURAREA Consortium 2004) so that longitudinal data are available at least at the area level and sometimes at the unit level also, depending on the adopted design.

Some surveys are repeated so that on each occasion a completely new sample is drawn. Then the resulting data set is not strictly longitudinal, but a series of cross-sectional data sets. The triennial comprehensive school achievement studies are a Finnish example of this. The schools keep the same over time, but the pupils are changed. The joint data set contains then time series information at school level, but not at pupil level.

Sometimes the data are collected with panel designs. In these a selected sample of units is surveyed repeatedly over the desired period to get a complete (excluding possible drop-outs) longitudinal unit-level data set. The European Community Household Panel is an example of this kind of survey. A special target of such surveys is to measure change in the study variables. The rotating panel design is a special form of panel design, where the sample changes on each survey occasion according to a preplanned scheme so that some units enter the sample for a certain time period, replacing other units which drop out. The merits of rotating panel designs are the decrease of response burden per sampling unit and keeping the current sample always up-to-date. For instance, the continuing monthly labour force survey in Finland is implemented in this way.

The idea of looking for gains in the survey estimation by utilizing previously collected data is not new. Early papers on the analysis of repeated surveys were given by Scott and Smith (1974), employing standard time series methods, and Jones (1980), considering the best linear unbiased estimation with stochastic least squares. These authors, however, did not consider small area estimation. The potential of utilizing time series models in obtaining small area estimates does not seem to have drawn any remarkable attention before late 1980's. Feder et al. (2000) and Singh et al. (2001) have considered survey estimation in rotating panel designs, but not in the context of small area estimation.

This far, the majority of the publications on small area estimation from repeated surveys deal with area level models. The possible reason for this is that in many countries, and especially in the United States, the infrastructure of the official statistics does not support longitudinal data sets at individual level. On the other hand, research on small area estimation from unit level panel data is clearly needed, because aggregating individual level data to adapt for area level models may cause unnecessary loss of information.

2.6.2 Area level models

The work on combining longitudinal and cross-sectional data for purposes of small area estimation in the mixed model framework is concentrated mostly on the area level models. In an important paper, Rao and Yu (1994) discuss various models that combine time series with the cross-sectional data to exploit the information in the data from previous time points. They propose the following extension of the basic area level model.

Let $\widehat{\theta}_{it}$ be the direct estimator of θ_{it}, where $i = 1, 2, \ldots, m$ refers to the area and $t = 1, 2, \ldots, T$ to the time, and

$$\widehat{\theta}_{it} = \theta_{it} + e_{it}. \tag{2.22}$$

The vector $\mathbf{e}_i = (e_{i1}, e_{i2}, \ldots, e_{iT})'$ of sampling errors for area i is assumed to follow $N(\mathbf{0}, \mathbf{\Psi}_i)$, where the covariance matrix $\mathbf{\Psi}_i$ is known. The linking model for the parameter of interest is

$$\theta_{it} = \mathbf{x}_{it}'\boldsymbol{\beta} + u_i + v_{it}, \tag{2.23}$$

where u_i is a random area effect and v_{it} is a random area by time effect following an AR(1) process:

$$v_{it} = \rho v_{i,t-1} + \varepsilon_{it}, \quad |\rho| < 1. \tag{2.24}$$

The u_i's and ε_{it}'s are assumed independent and distributed as $N(0, \sigma_u^2)$ and $N(0, \sigma^2)$, respectively. Rao and Yu note that more complex models instead of the AR(1) process could be formulated, but they feel that resulted gains are unlikely to be significant.

Under model (2.22)–(2.24) Rao and Yu give a two-stage estimator for θ_{it}, when this is a small area mean. Extending the work of Prasad and Rao (1990), they derive a second-order correct estimator of the MSE of their two-stage estimator. The HB approach, with diffuse priors, is also considered by Rao and Yu. The computations are performed by direct numerical integration or by Gibbs sampling.

Datta, Lahiri and Maiti (1999) and You (1999) used the model (2.22)–(2.24), but replaced the AR(1) process by a random walk model with $\rho = 1$. They used the EBLUP and EB approaches in the estimation and obtained also an estimator of the MSE for the estimator of the small area mean. Datta, Lahiri and Maiti estimated the model parameters by ML and REML, while You used the method of moments estimators.

Ghosh, Nangia and Kim (1996) used the sampling model (2.22) with a linking model with no area-specific random effects, but with time-specific random effects instead. In univariate form, the model can be written as

$$\theta_{it} = \mathbf{x}_{it}'\boldsymbol{\beta} + \mathbf{z}_{it}'\mathbf{b}_t + v_{it}, \tag{2.25}$$

where $v_{it} \sim N(0, \sigma_v^2)$. For the time-specific effects a random walk model

$$\mathbf{b}_t = \mathbf{b}_{t-1} + \boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, was assumed. For the estimation Ghosh, Nangia and Kim adopted the HB approach with Gibbs sampling with proper normal and inverse Wishart priors for the model parameters. Rao (1999) criticizes the model for the lack of random area effects, which tends to produce oversmooth estimates.

Datta, Lahiri and Lu (1994) used the model (2.25) with an additional area-specific random effect u_i and allowed random slopes $\boldsymbol{\beta}_i$ vary over areas. Their linking model was

$$\theta_{it} = \mathbf{x}_{it}'\boldsymbol{\beta}_i + u_i + \mathbf{z}_{it}'\mathbf{b}_t + v_{it}$$

with normality assumptions for all the random parameters $\boldsymbol{\beta}_i$, u_i and v_{it}. A random walk model for $\mathbf{b}_t$'s was assumed.

Datta, Lahiri, Maiti and Lu (1999) employed a linking model with additional terms for seasonal variation. They carried the estimation out by the HB approach with Gibbs sampling.

2.6.3 Unit level models

In the EU-funded research project EURAREA for small area estimation some extensions of the unit level nested error regression model were proposed (EURAREA Consortium 2004, Sec. C4). These apply to a series of independent cross-sectional samples, were the data come from repeated surveys for the same areas, but the sampled units change at every occasion. A unit level model with random cross-classified area and time effects is

$$y_{ijt} = \mathbf{x}_{ijt}'\boldsymbol{\beta}_t + u_i + v_t + e_{ijt}, \tag{2.26}$$

where y_{ijt} is response of the unit j in area i at time point t, $\mathbf{x}_{ijt}$ is the corresponding auxiliary data vector, $\boldsymbol{\beta}_t$ is a possibly time-dependent fixed parameter, u_i is a random area effect, v_t a random time effect and e_{ijt} the random error. The random terms are assumed mutually independent and normal with zero means and variances σ_u^2, σ_v^2 and

σ_e^2 respectively. The estimation of small area totals and means under model (2.26) was carried out within the EBLUP framework (EURAREA Consortium 2004).

Various covariance structures could be specified for the random effects in (2.26). A simple but well-performing structure is obtained by assuming that both the area effects and the time effects are independent and identically distributed. Then the covariance matrix of the y_{ijt}'s has a variance component structure, which implies that the correlations between observations in the same area at different time points are constant. This structure is also known as compound symmetry. It can be thought unrealistic sometimes, but it has an important merit being linear with respect the unknown variance components. The linearity enables a tractable and well-performing Taylor series approximation for the MSE estimator of EBLUP estimated small area totals and means (Datta and Lahiri 2000) as well as for the covariance matrix of fixed effect estimates (Kenward and Roger 1997).

One alternative to the variance component model is to assume that the time effects in (2.26) follow an AR(1) process with autocorrelation parameter ρ. This makes the covariance structure more complex and maybe more realistic, but the linearity is lost so that the traditional results on the MSE estimation do not apply. In the Monte Carlo simulations with reality-based data sets, executed in the EURAREA project (2004), it also turned out that the ML or REML estimation of ρ is very problematic in many practical situations, typically showing no convergence. Assumably this was due to the fact that considered data sets did not contain enough time points for reliable autocorrelation estimation.

To overcome the autocorrelation estimation problems with model (2.26) a model with time-varying area-effects was also suggested. The model is

$$y_{ijt} = \mathbf{x}'_{ijt}\boldsymbol{\beta}_t + u_{it} + e_{ijt}, \tag{2.27}$$

where the area by time random effects u_{it} follow independent AR(1) processes with the same parameter ρ in each area $i = 1, 2, \ldots, m$. Then each area provides an independent replication of the process, which would lead to more successful estimation of ρ. An interesting property of the model (2.27) is that the cross-sectional differences of area effects do not keep constant over time, unlike in the random main effects model (2.26). In practical experiments it appeared that the autocorrelation can be estimated efficiently under the time-varying area effects model, but, unfortunately, the error variance of predicted u_{it}'s can be high leading to increased MSE's of the small area estimates.

In general, the international Monte Carlo studies carried out by the EURAREA project (2004) showed that substantial gains can be achieved by using models for repeated survey data instead of models for just a single time point data. Among the repeated survey models the one with a simple variance component covariance structure performed always well. The estimation of the AR(1) model seldom succeeded, but when that happened, the no gains were found over the simple variance component model. However, a model with fixed time effect and a random area effect perfomed practically equally well as the models with random time effect. In fact, the fixed time approach is much more convenient to implement than the approaches with random time effect, and by virtue of this convenience it is the preferable model choice especially when the number of time points is not large.

2.6.4 Model for panel data and rotating panel data

In the panel design the samples at different time points contain exactly the same units, and if there are no drop-outs, a complete panel data set, where all the sampled units are observed at every time point, is obtained. The panel design is schematically illustrated in the Table 2.1, where we have a sample of three units observed on five occasions.

Table 2.1. A panel design of $T = 5$ time points with sample size $n = 3$. The "X" marks an observation.

	time					
unit	1	2	3	4	5	
1	X	X	X	X	X	the sampled
2	X	X	X	X	X	units
3	X	X	X	X	X	(s_{iT})
4						the non-sampled
5						units
6						(r_{iT})
7						
8						

An appropriate model for the panel data takes the covariance of the repeated observations from the same unit into account. Within the family of linear mixed models it is possible to fit a variety of covariance structures to the panel data. However, for the purposes of small area estimation by the empirical BLUP approach it is reasonable to stick to covariance structures, which are linear with respect to the unknown parameters. Such structures are given e.g. by multilevel variance component models or random coefficient regression models (Goldstein 1995, Longford 1993).

One simple model, yielding the compound symmetry covariance structure for the repeated observations of an unit, is the three-level variance component model

$$y_{ijt} = \mathbf{x}'_{ijt}\boldsymbol{\beta} + u_i + v_{ij} + e_{ijt}, \tag{2.28}$$

where y_{ijt} is response of the unit j in area i at time point t, $\mathbf{x}_{ijt}$ is the auxiliary data vector, $\boldsymbol{\beta}$ is the fixed regression parameter, u_i is a random area effect, v_{ij} a random effect of an unit j within area i and e_{ijt} the observation-specific random error within unit ij. Essentially the same model is used by Stukel and Rao (1999) for small area estimation under two-stage cluster sampling. Stukel and Rao call this model two-fold nested error regression model. In the multilevel hierarchy areas correspond to level 3, units within areas to level 2 and observations within units to level 1. The time effect is modelled as fixed by means of indicator variables for each survey occasion, and these are included in the vector $\mathbf{x}_{ijt}$. The random terms u_i, v_{ij} and e_{ijt} are assumed mutually independent and normal with zero means and variances σ_u^2, σ_v^2 and σ_e^2 respectively.

The presence of the random area effect induces a constant intra-area covariance σ_u^2 for units in same area and, additionally, the presence of the random unit effect induces a constant intra-unit covariance $\sigma_u^2 + \sigma_v^2$ for the repeated observations on the same unit.

The BLUP estimation of small area total is again based on the general prediction theorem. Let $U_{iT} = s_{iT} \cup r_{iT}$ denote a finite population of area i at "current" time T. The corresponding sample and the remainder are s_{iT} and r_{iT}, respectively. The total of variable y in area i at time T is

$$Y_{iT} = \sum_{s_{iT}} y_{ijT} + \sum_{r_{iT}} y_{ijT},$$

where the latter sum needs to be predicted by the model.

Under (2.28) the best predictor of y_{ijt} is

$$\tilde{y}_{ijt} = \mathbf{x}'_{ijt}\widehat{\boldsymbol{\beta}} + \tilde{u}_i + \tilde{v}_{ij},$$

where again $\widehat{\boldsymbol{\beta}}$ is the BLUE of $\boldsymbol{\beta}$ and $\tilde{u}_i$ and $\tilde{v}_{ij}$ are the BLUP's of u_i and v_{ij}. For the units ij belonging to the remainder r_{iT}, however, the BLUP of v_{ij} is not available. Then, for any $y_{ijT} \in r_{iT}$ the best predictor is

$$\tilde{y}_{ijT} = \mathbf{x}'_{ijT}\widehat{\boldsymbol{\beta}} + \tilde{u}_i, \tag{2.29}$$

which leads to the estimator

$$\begin{aligned} \widehat{Y}_{iT} &= \sum_{s_{iT}} y_{ijT} + \sum_{r_{iT}} \tilde{y}_{ijT} \\ &= \sum_{s_{iT}} y_{ijT} + \sum_{r_{iT}} \mathbf{x}'_{ijT}\widehat{\boldsymbol{\beta}} + (N_{iT} - n_{iT})\tilde{u}_i, \end{aligned}$$

where N_{iT} is the size of U_{iT} and n_{iT} is the size of s_{iT}.

The estimator is practically of the same form as the estimator based on the cross-sectional nested error model since the predicted effects $\tilde{v}_{ij}$ of the observed units cannot play any role in predicting the sum of the unobserved units. However, in estimating temporal changes in small area totals the panel model is anticipated to be more useful, because the $\tilde{v}_{ij}$'s bear the information on the temporal correlations.

The model (2.28) applies to the rotating panel data as well, and then more gains are expected than in the panel case, since the information on unit effects v_{ij} can now be incorporated into estimation. An example of rotation scheme is given in the Table 2.2. There one unit enters the sample on every occasion, stays for three occasions and then drops out. At each time point there is a cross-sectional sample of size $n = 3$, but the sample composition is partially changed between time points. Here the overlap in consecutive samples is two thirds.

Table 2.2. A rotating panel design of $T = 5$ time points, where units are observed on three consecutive occasions. The "X" marks an observation.

	time					
unit	1	2	3	4	5	
1					X	the units in
2				X	X	the current
3			X	X	X	sample (s_{iT})
4		X	X	X		the units not in
5	X	X	X			the current sample,
6	X	X				but in previous
7	X					samples ($r1_{iT}$)
8						the units never
9						sampled ($r2_{iT}$)

Another example of a rotation scheme is given in the Table 2.3, which shows a fraction of the real-life design of the monthly Labour Force Survey by Statistics Finland (in reality, every unit in the figure corresponds a sampled panel of over 2000 respondents and a monthly sample consists of five such panels). In this design each unit is observed five times: first three times at intervals of three months, then there is a half-year break and the final two times again at intervals of three months. For example, the unit number 8 in the figure enters the survey in month 2, is reobserved in months 5 and 8 and will be observed again in month 2 and 8 in the next year. After this it is dropped out. In this design there is no overlap in consecutive months, but there is 60 % quarterly overlap and 40 % overlap at interval of one year (Salonen 1997).

Table 2.3. The rotating panel design of the Finnish Labour Force Survey.

	time											
unit	1	2	3	4	5	6	7	8	9	10	11	12
1	X						X			X		
2		X						X			X	
3			X						X			X
4	X			X						X		
5		X			X						X	
6			X			X						X
7	X			X			X					
8		X			X			X				
9			X			X			X			
⋮												

It can be seen from the Tables 2.2 and 2.3 that a rotation scheme divides the population U_{iT} (say) into three parts: s_{iT} = the units in the current sample, $r1_{iT}$ = the units which are not in the current sample, but have included in earlier samples, and $r2_{iT}$ = the units, which have never been sampled. It is also important to note that in principle the incompleteness of the sample panel data is now determined only by the design, making the selection mechanism fully ignorable. In practice it is of course possible to have non-random drop-outs.

The total of variable y in area i at time T is now

$$Y_{iT} = \sum_{s_{iT}} y_{ijT} + \sum_{r1_{iT}} y_{ijT} + \sum_{r2_{iT}} y_{ijT}$$

and its BLUP estimator is

$$\widehat{Y}_{iT,BLUP} = \sum_{s_{iT}} y_{ijT} + \sum_{r1_{iT}} \tilde{y}_{ijT} + \sum_{r2_{iT}} \tilde{y}_{ijT}.$$

For those units, which are included in $r1_{iT}$, there is sample data available from earlier occasions, providing information for predicting unit effects v_{ij}. Hence, the best predictor of $y_{ijT} \in r1_{iT}$ is

$$\tilde{y}_{ijT} = \mathbf{x}'_{ijT}\widehat{\boldsymbol{\beta}} + \tilde{u}_i + \tilde{v}_{ij},$$

which is more accurate than the predictor (2.29) in the panel case. The y_{ijT} values in $r2_{iT}$ must still be predicted with (2.29) since their unit effects cannot be predicted from the sample data. By virtue of the rotation, however, the number of units $y_{ijT} \in r2_{iT}$ is reduced remarkably compared to cross-sectional or panel data. The BLUP estimator $\widehat{Y}_{iT,BLUP}$ can now be written as

$$\widehat{Y}_{iT,BLUP} = \sum_{s_{iT}} y_{ijT} + \sum_{j \in r_{iT}} \mathbf{x}'_{ijT}\widehat{\boldsymbol{\beta}} + (N_{iT} - n_{iT})\tilde{u}_i + \sum_{j \in r1_{it}} \tilde{v}_{ij},$$

where $r_{iT} = r1_{iT} \cup r2_{iT}$.

Compared to panel data or cross-sectional data, the rotating panel should also increase the prediction accuracy of the area effects u_i, because the non-overlap of repeated samples increases the amount of area-specific information considerably.

3 Review of mixed model theory

3.1 Linear mixed model

The ordinary fixed effects linear model is usually written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}, \tag{3.1}$$

where $\mathbf{y}$ is an $n \times 1$ random vector of response data, $\boldsymbol{\beta}$ is a $p \times 1$ vector of fixed effects parameters, $\mathbf{X}$ is a known $n \times p$ model matrix comprising the values of explanatory variables and/or zeros and ones corresponding with the considered design and $\mathbf{e}$ is a vector of random errors. For $\mathbf{e}$ it is assumed that $E(\mathbf{e}) = \mathbf{0}$ and $Cov(\mathbf{e}) = \sigma_e^2\mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Also the normal distribution is often assigned to $\mathbf{e}$ to make

$$\mathbf{y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma_e^2\mathbf{I}_n).$$

Hence the vector $\mathbf{y}$ contains random variables that are independent with equal variability.

The linear mixed model is obtained by incorporating a $q \times 1$ vector $\mathbf{u}$ of random effects, i.e. effects that are considered random variables instead of fixed constants, with an appropriate model matrix $\mathbf{Z}$ into the fixed effects model (3.1):

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}. \tag{3.2}$$

The model matrix $\mathbf{Z}$ is often an incidence matrix (design matrix) of zeros and ones only, but it may also contain explanatory variables (that usually are present also in $\mathbf{X}$). In the latter case the model (3.2) is often called random coefficient regression model. If $\mathbf{Z}$ is an incidence matrix and the random effects in $\mathbf{u}$ are uncorrelated, we have a special case called variance component model.

For the random vectors $\mathbf{u}$ and $\mathbf{e}$ we make the following assumptions:

$$\begin{aligned} &E(\mathbf{u}) = \mathbf{0} \qquad && Cov(\mathbf{u}) = \mathbf{G} \\ &E(\mathbf{e}) = \mathbf{0} && Cov(\mathbf{e}) = \mathbf{R} \\ &Cov(\mathbf{u}, \mathbf{e}) = \mathbf{0}, && \end{aligned}$$

where, in principle, $\mathbf{G}$ and $\mathbf{R}$ can be arbitrary positive definite covariance matrices. In variance component models the matrices $\mathbf{G}$ and $\mathbf{R}$ are diagonal.

Under these assumptions the expected value of $\mathbf{y}$ is

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta},$$

and if $\mathbf{u}$ is given,

$$E(\mathbf{y}|\mathbf{u}) = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}.$$

The covariance matrix of $\mathbf{y}$ is

$$Cov(\mathbf{y}) = \mathbf{V} = \mathbf{ZGZ}' + \mathbf{R},$$

and if $\mathbf{u}$ is given,

$$Cov(\mathbf{y}|\mathbf{u}) = \mathbf{R}.$$

We see that the fixed part $\mathbf{X}\boldsymbol{\beta}$ of model (3.2) defines the mean structure of $\mathbf{y}$, whereas the random part $\mathbf{Zu}+\mathbf{e}$ defines the covariance structure. By an appropriate specification of the random part and covariance matrices $\mathbf{G}$ and $\mathbf{R}$ we can define a variety of covariance structures. Hence, both the assumption of uncorrelated observations and the assumption of homoscedastic observations, which are typical of traditional linear models, can be relaxed. This is reasonable especially when longitudinal, spatial or otherwise hierarchical data sets are considered and makes mixed models a powerful framework for statistical modelling of data.

The normal distribution is usually assigned to both random terms $\mathbf{u}$ and $\mathbf{e}$ so that

$$\mathbf{u} \sim N_q(\mathbf{0}, \mathbf{G}) \text{ and } \mathbf{e} \sim N_n(\mathbf{0}, \mathbf{R})$$

independently. Normality is often a feasible characterization of the behaviour of the random terms (which are essentially latent) and also makes applying the likelihood methods, which are the principal approach to the mixed model estimation, convenient. Other distributions could also be considered, but it seems that the theory and estimation methods under them are not yet well-established.

Under the normality assumptions the distribution of $\mathbf{y}$ given $\mathbf{u}$ is $N(\mathbf{X}\boldsymbol{\beta}+\mathbf{Zu}, \mathbf{R})$, which is analogous with the linear fixed effects model, where $\mathbf{u}$ is treated fixed, with possibly correlated errors. The distributions of $\mathbf{u}$ and $\mathbf{y}$ given $\mathbf{u}$ imply that

$$\mathbf{y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \mathbf{ZGZ}' + \mathbf{R}). \tag{3.3}$$

This is called the marginal formulation of linear mixed model (Verbeke and Molenberghs 1997, 2000). Note that although the marginal model (3.3) follows from the linear mixed model (3.2), the models are not equivalent, because the marginal model does not explicitly define the random effect structure in (3.2). The marginal model, however, gives the basis of the maximum likelihood estimation of the model parameters.

The covariance matrices $\mathbf{G}$ and $\mathbf{R}$ are functions of a set of scalar parameters. Here we call them variance parameters and denote the $r \times 1$ (say) vector of them by $\boldsymbol{\sigma}$. The estimation of $\mathbf{G}$ and $\mathbf{R}$ turns now to estimation of $\boldsymbol{\sigma}$. Occasionally, depending on the context, we shall write $\mathbf{G} = \mathbf{G}(\boldsymbol{\sigma})$, $\mathbf{R} = \mathbf{R}(\boldsymbol{\sigma})$ and $\mathbf{V} = \mathbf{V}(\boldsymbol{\sigma})$. The (fixed) model parameters to be estimated are vectors $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. Often the parameters in $\boldsymbol{\sigma}$ are simply variances and covariances, but sometimes they may have other interpretations like the autoregressive parameter and moving average component in time series applications (e.g. Box and Jenkins 1970) or the sill and range in geostatistical applications (e.g. Cressie 1991).

3.2 Estimation of model parameters

3.2.1 Traditional approaches

For variance component models, where all the parameters in $\boldsymbol{\sigma}$ are variances, the classical method for estimating $\boldsymbol{\sigma}$ is so-called ANOVA estimation, based on equating the mean squares of the analysis of variance to their expected values and solving the estimates from the resulting equations. The ANOVA estimation, originally meant for balanced data, was adapted to unbalanced data by Henderson in 1950's. The Henderson 3 method, also known as method of fitting constants, has still fairly recently been suggested to be used with small area models (Prasad and Rao 1990, Morales 2002). ANOVA methods are non-iterative and therefore easy to implement, give unbiased variances estimates (which sometimes appear negative, though) and they require no normality of random effects in the model. Their major drawback is that they only apply to a limited choice of models.

The regression coefficients $\boldsymbol{\beta}$ can be estimated by the ordinary least squares:

$$\widehat{\boldsymbol{\beta}}_{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}, \tag{3.4}$$

or, more preferably, by the generalized least squares:

$$\widehat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}. \tag{3.5}$$

It is worth noting that both the OLS and GLS estimators of $\boldsymbol{\beta}$ are unbiased. The estimator $\widehat{\boldsymbol{\beta}}_{GLS}$ is generally the best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$. When $Cov(\mathbf{e}) = \mathbf{V} = \sigma_e^2\mathbf{I}_n$, like in standard linear models, $\widehat{\boldsymbol{\beta}}_{OLS}$ is the BLUE of $\boldsymbol{\beta}$.

If the covariance matrix $\mathbf{V} = \mathbf{Z}\mathbf{G}(\boldsymbol{\sigma})\mathbf{Z}' + \mathbf{R}(\boldsymbol{\sigma})$ is unknown, it will be replaced with its estimate $\widehat{\mathbf{V}} = \mathbf{Z}\mathbf{G}(\widehat{\boldsymbol{\sigma}})\mathbf{Z}' + \mathbf{R}(\widehat{\boldsymbol{\sigma}})$, where $\widehat{\boldsymbol{\sigma}}$ is obtained by the ANOVA method, for example. Kackar and Harville (1981) have shown that the resulting estimator of $\boldsymbol{\beta}$ is unbiased if the distribution of $\mathbf{y}$ is symmetric about its mean and $\widehat{\boldsymbol{\sigma}}$ is an even and translation-invariant function of the data vector $\mathbf{y}$. That is, $\widehat{\boldsymbol{\sigma}}(\mathbf{y}) = \widehat{\boldsymbol{\sigma}}(-\mathbf{y})$ (even) and $\widehat{\boldsymbol{\sigma}}(\mathbf{y}+\mathbf{X}\boldsymbol{\beta}) = \widehat{\boldsymbol{\sigma}}(\mathbf{y})$ for all $\mathbf{y}$ and $\boldsymbol{\beta}$ (translation invariant). Even and translation-invariant variance estimators are obtained by the ANOVA method or, under normality, by the maximum likelihood (ML) or residual ML (REML) methods, for example.

In the above formulas (3.4) and (3.5), as well as later in this presentation, we assume for simplicity that the model matrix $\mathbf{X}$ is of full rank. If this does not hold, we introduce appropriate constraints or replace the matrix inverses $(\mathbf{X}'\mathbf{X})^{-1}$ and $(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$ with the generalized inverses $(\mathbf{X}'\mathbf{X})^{-}$ and $(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}$, respectively. In the latter case, to be precise, the vector $\boldsymbol{\beta}$ is not estimable since the $\widehat{\boldsymbol{\beta}}_{OLS}$ and $\widehat{\boldsymbol{\beta}}_{GLS}$ are not invariant to the choice of $(\mathbf{X}'\mathbf{X})^{-}$ and $(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}$. The invariance holds for $\mathbf{X}\widehat{\boldsymbol{\beta}}_{OLS}$ and $\mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS}$ instead, and therefore it is often defined that $\mathbf{X}\widehat{\boldsymbol{\beta}}_{OLS} = BLUE(\mathbf{X}\boldsymbol{\beta})$ (for $Cov(\mathbf{e}) = \sigma_e^2\mathbf{I}_n$) and $\mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS} = BLUE(\mathbf{X}\boldsymbol{\beta})$, respectively. The problem of less-than-full-rank matrices is met with the usual analysis of variance models. Detailed discussion of it can be found e.g. in McCulloch and Searle (2001) and Searle, Casella and McCulloch (1992).

Besides ANOVA estimation there are also other non-iterative methods, which do not always require normality, for estimating variance components. They are based on minimizing certain optimality criteria and typically have acronyms like MINQUE (minimum norm quadratic unbiased estimation) or MIVQUE (minimum variance quadratic unbiased estimation). Searle, Casella and McCulloch (1992) give a detailed presentation of all these "historical" methods in the context of variance component estimation. The MIVQUE estimation is also considered by Frees (2004).

Here we concentrate on the ML and restricted (or residual) maximum likelihood (REML) estimation methods, which have superseded the earlier methods by virtue of their applicability to more general models and attractive properties like consistency, efficiency and asymptotic normality of the estimators (see Searle, Casella and McCulloch 1992, Ch. 6.8). We construct the likelihood function under the standard assumption of normal random effects.

3.2.2 Maximum likelihood estimation (ML)

The likelihood function to be maximized for the estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ comes from the marginal model (3.3), where the covariance matrix $\mathbf{V} = \mathbf{ZGZ}' + \mathbf{R}$ is a function $\mathbf{V}(\boldsymbol{\sigma})$ of variance parameters $\boldsymbol{\sigma}$. The density of $\mathbf{y} \sim N_n(\boldsymbol{\beta}, \mathbf{V})$ is

$$f(\mathbf{y}; \boldsymbol{\beta}, \mathbf{V}) = (2\pi)^{-\frac{n}{2}} |\mathbf{V}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right\} \tag{3.6}$$

leading to the log likelihood function

$$\log L(\boldsymbol{\beta}, \boldsymbol{\sigma}) = \text{const} - \frac{1}{2}\log|\mathbf{V}(\boldsymbol{\sigma})| - \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\boldsymbol{\sigma})^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}), \tag{3.7}$$

which is then maximized with respect of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. The parameter space of $\boldsymbol{\beta}$ is $\Theta_\beta = \mathbb{R}^p$. For $\boldsymbol{\sigma}$, the parameter space Θ_σ is a subset of $\mathbb{R}^r$ such that $\mathbf{G}$ and $\mathbf{R}$ are non-negative definite. For variance component models this means that all the variance components are non-negative. However, if a variance component lies on the boundary of Θ_σ, i.e. is zero, it causes problems for the inference.

The partial derivative of the log likelihood with respect to $\boldsymbol{\beta}$ is

$$\frac{\partial \log L(\boldsymbol{\beta}, \boldsymbol{\sigma})}{\partial \boldsymbol{\beta}} = \mathbf{X}'\mathbf{V}(\boldsymbol{\sigma})^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}),$$

and setting this to zero leads to the GLS solution (3.5). For given $\mathbf{V}$ this is also the maximum likelihood estimator $\widehat{\boldsymbol{\beta}}_{ML}$ of $\boldsymbol{\beta}$. For unknown $\mathbf{V}$ the ML estimator of $\boldsymbol{\beta}$ is

$$\widehat{\boldsymbol{\beta}}_{ML} = (\mathbf{X}'\widehat{\mathbf{V}}_{ML}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\mathbf{V}}_{ML}^{-1}\mathbf{y},$$

where $\widehat{\mathbf{V}}_{ML} = \mathbf{V}(\widehat{\boldsymbol{\sigma}}_{ML})$ is the ML estimator of $\mathbf{V}$. The estimator $\widehat{\boldsymbol{\beta}}_{ML}$ is unbiased for $\boldsymbol{\beta}$ under normality of $\mathbf{y}$ (Kackar and Harville 1981, 1984).

To estimate $\mathbf{V}(\boldsymbol{\sigma})$, we substitute the GLS solution (3.5) into (3.7) to obtain the profile log likelihood

$$\log L(\boldsymbol{\sigma}) = \text{const} - \frac{1}{2}\log|\mathbf{V}(\boldsymbol{\sigma})| - \frac{1}{2}\mathbf{r}(\boldsymbol{\sigma})'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{r}(\boldsymbol{\sigma}), \tag{3.8}$$

where $\mathbf{r}(\boldsymbol{\sigma}) = \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{y}$. The profile log likelihood (3.8) is then maximized to find $\widehat{\boldsymbol{\sigma}}_{ML}$. Searle et al. (1992, Ch. 6.2) provide the ML estimation equations for variance component models. In general the maximization requires numerical optimization techniques like Newton-Raphson, Fisher scoring or EM algorithms. These algorithms are discussed e.g. by Lindstrom and Bates (1988), Longford (1993, 2005) and Wolfinger et al. (1994) and they have been implemented in the procedures for mixed models in many popular packages like R, SAS and SPSS. It is also worth noting that the iterative generalized least squares (IGLS) algorithm of Goldstein (1986, 1995) leads under normality to maximum likelihood estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. The IGLS algorithm is specially implemented in the MLwiN software.

In estimation of variance parameters, however, the ML method suffers from some well-known problems. First, it tends to give variance estimates that are biased downwards. The familiar example is the estimation of variance of a univariate normal distribution $N(\mu, \sigma^2)$, when the mean μ is unknown, from sample observations $y_1, y_2, \ldots, y_n$. The ML estimator of σ^2 is

$$\widehat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2,$$

where $\bar{y}$ is the sample mean. The unbiased estimator is the sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \tag{3.9}$$

with denominator $n-1$ instead of n. The bias of the ML estimator, due to the estimation of μ, is then $-\sigma^2/n$. The bias is larger with linear regression models with several explanatory variables. In the regression model (3.1) we are to estimate a $p \times 1$ vector $\boldsymbol{\beta}$ of fixed parameters instead of just one μ. The ML estimator of the error variance σ^2 is now

$$\widehat{\sigma}^2_{ML} = \frac{SS_E}{n},$$

where SS_E denotes the observed error sum of squares of the model. Again, the unbiased estimator

$$\widehat{\sigma}^2 = \frac{SS_E}{n-p}. \tag{3.10}$$

of σ^2 differs from the ML estimator in its denominator, now containing the correct number $n-p$ of degrees of freedom. We see that the bias of ML estimator can be most serious if the number p of covariates is large compared to the sample size n.

Another problem of ML estimation is, as Diggle et al. (2002, p. 66) point out, that being essentially model-dependent it can be highly sensitive to a misspecified model matrix $\mathbf{X}$. It may happen that the ML estimators of the variance parameters are not even consistent. One strategy to fight against this inconsistency is to overly increase the number p of columns (i.e. explanatory variables) in $\mathbf{X}$ for "safety reasons", but then this would make the bias problem worse.

At least to some extent, these problems of ML estimation can be avoided by adopting another likelihood-based procedure called residual ML or restricted ML (REML).

3.2.3 Restricted maximum likelihood estimation (REML)

The REML method for variance component estimation was first introduced by Patterson and Thompson (1971) and further developed by Harville (1974, 1977). It is based on such linear transformation of the data $\mathbf{y}$ that the resulting distribution does not depend on the fixed effects parameter vector $\boldsymbol{\beta}$. Hence $\boldsymbol{\beta}$ is eliminated from the log likelihood, but at the same time the loss of degrees of freedom involved in estimating $\boldsymbol{\beta}$ is taken into account in the estimation of $\mathbf{V}(\boldsymbol{\sigma})$. In a way REML corrects the denominator of variance estimators to reduce the bias in variance parameter estimates. In some simple cases like balanced ANOVA models it gives variance component estimates that are unbiased and agree with ANOVA or MIVQUE estimates, provided that the non-negativity constraint of variance parameters does not come into play (Harville 1977). In the two examples in the preceding section the unbiased variance estimators (3.9) and (3.10) are also REML estimators.

The REML method is based on the likelihood principle and has the same merits, like consistency, efficiency and asymptotic normality, as the ML method. Since the REML estimators produce unbiased or nearly unbiased variance estimates, have the same desirable properties as the ML estimators and do not require computations that are essentially more complex than those needed in ML estimation, the REML method is now a widely preferred approach to estimate variance parameters in mixed models (Searle et al. 1992, Pinheiro and Bates 2000, Verbeke and Molenberghs 2000, Diggle et al. 2002).

Introduce a linear transformation $\mathbf{z} = \mathbf{K}'\mathbf{y}$ of the normal response vector $\mathbf{y}$, where $\mathbf{K}$ is a $n \times (n-p)$ matrix of full rank, for which $\mathbf{K}'\mathbf{X} = \mathbf{0}$. The distribution of $\mathbf{z}$ is then $N_{n-p}(\mathbf{0}, \mathbf{K}'\mathbf{V}\mathbf{K})$, which does not depend on $\boldsymbol{\beta}$. The elements of $\mathbf{z}$ are sometimes referred as error contrasts (e.g. Harville 1974, 1977; Verbeke and Molenberghs 2000). The REML estimators of the variance parameters $\boldsymbol{\sigma}$ are obtained by maximizing the likelihood function associated with the error contrasts $\mathbf{z}$ instead of the original data $\mathbf{y}$. The fixed parameter vector $\boldsymbol{\beta}$ is then estimated by applying the GLS formula (3.5), the covariance matrix $\mathbf{V}$ being replaced with its estimate $\widehat{\mathbf{V}}_{REML} = \mathbf{V}(\widehat{\boldsymbol{\sigma}}_{REML})$.

An appropriate $\mathbf{K}$ is found by selecting $n-p$ columns from the projection matrix

$$\mathbf{Q} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}',$$

which transforms $\mathbf{y}$ to the usual OLS residuals (the term "residual maximum likelihood" originates from the idea to base the likelihood on residuals of the OLS fit to the data).

However, the resulting likelihood function and inference do not depend on which columns are used, and nor even the choice of $\mathbf{K}$, as Harville (1974) has shown (see also Diggle et al. 2002, Frees 2004). Instead, any full-rank $n \times (n-p)$ matrix $\mathbf{K}$ giving the property $E(\mathbf{z}) = \mathbf{0}$ for all $\boldsymbol{\beta}$ will do.

To find a convenient expression for the REML likelihood function, define a non-zero $n \times (n-p)$ matrix $\mathbf{A}$ such that $\mathbf{AA}' = \mathbf{Q}$ and $\mathbf{A}'\mathbf{A} = \mathbf{I}$. The matrix $\mathbf{A}$ is also an appropriate choice for $\mathbf{K}$ and we define $\mathbf{z} = \mathbf{A}'\mathbf{y}$. The expected value of $\mathbf{z}$ is $\mathbf{0}$ and the covariance matrix is $\mathbf{A}'\mathbf{VA}$. Because $\mathbf{A}'\mathbf{X} = \mathbf{0}$, we can also write $\mathbf{z} = \mathbf{A}'\mathbf{y} = \mathbf{A}'(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})$, where $\widehat{\boldsymbol{\beta}}_{GLS}$ is given in (3.5). The density of $\mathbf{z}$ is then

$$
\begin{aligned}
f_{\mathbf{z}}(\mathbf{z}) &= (2\pi)^{-\frac{n-p}{2}} |\mathbf{A}'\mathbf{VA}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mathbf{z}'(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{z}\right\} \\
&= (2\pi)^{-\frac{n-p}{2}} |\mathbf{A}'\mathbf{VA}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})'\mathbf{A}'(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})\right\}.
\end{aligned}
$$

Applying the result 33 in Rao (1973, p. 77) yields

$$\mathbf{A}'(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1},$$

which leads straightforwardly to

$$(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})'\mathbf{A}'(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS}) = (\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS}). \tag{3.11}$$

To find an expression for the determinant $|\mathbf{A}'\mathbf{VA}|$ we define $\mathbf{H} = [\mathbf{A} \quad \mathbf{G}]$, where $\mathbf{G} = \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$ (note that $\widehat{\boldsymbol{\beta}}_{GLS} = \mathbf{G}'\mathbf{y}$). Then

$$|\mathbf{H}'\mathbf{VH}| = \begin{vmatrix} \mathbf{A}'\mathbf{VA} & \mathbf{A}'\mathbf{VG} \\ \mathbf{G}'\mathbf{VA} & \mathbf{G}'\mathbf{VG} \end{vmatrix} = \begin{vmatrix} \mathbf{A}'\mathbf{VA} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}'\mathbf{VG} \end{vmatrix} = |\mathbf{A}'\mathbf{VA}||\mathbf{G}'\mathbf{VG}|$$

so that

$$|\mathbf{A}'\mathbf{VA}| = |\mathbf{H}'\mathbf{VH}||\mathbf{G}'\mathbf{VG}|^{-1}. \tag{3.12}$$

By a straightforward calculation

$$|\mathbf{G}'\mathbf{VG}| = |\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|^{-1}. \tag{3.13}$$

For $|\mathbf{H}'\mathbf{VH}|$ we note that

$$|\mathbf{H}'\mathbf{VH}| = |\mathbf{H}'||\mathbf{V}||\mathbf{H}| = |\mathbf{V}||\mathbf{H}'\mathbf{H}|, \tag{3.14}$$

because $\mathbf{H}$ is a square matrix. By using $\mathbf{AA}' = \mathbf{Q}$ and $\mathbf{A}'\mathbf{A} = \mathbf{I}$ and the standard result for block determinants we get

$$|\mathbf{H}'\mathbf{H}| = \begin{vmatrix} \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{G} \\ \mathbf{G}'\mathbf{A} & \mathbf{G}'\mathbf{G} \end{vmatrix} = |\mathbf{I}||\mathbf{G}'\mathbf{G} - \mathbf{G}'\mathbf{AA}'\mathbf{G}| = |\mathbf{G}'\mathbf{G} - \mathbf{G}'\mathbf{QG}| = |\mathbf{X}'\mathbf{X}|^{-1}. \tag{3.15}$$

Collecting (3.11)–(3.15) together leads to the following expression of the density of the error contrasts $\mathbf{z} = \mathbf{A}'\mathbf{y}$:

$$f_{\mathbf{z}}(\mathbf{z}) = (2\pi)^{-\frac{n-p}{2}} |\mathbf{X}'\mathbf{X}|^{\frac{1}{2}} |\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|^{-\frac{1}{2}} |\mathbf{V}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS}) \right\}.$$

Now it is seen that the matrix $\mathbf{A}$ defining the error contrasts $\mathbf{z}$ does not explicitly appear in the density. It is implicitly related to the matrix $\mathbf{X}'\mathbf{X}$, but this does not depend on $\mathbf{V}$ and therefore does not affect the maximization of the likelihood. The consequence is that the REML estimation does not depend on the choice of $n-p$ error contrasts.

The density of $\mathbf{z}$ leads to the restricted or residual log likelihood, which is written here using the notation of (3.8):

$$\text{(3.16)} \quad \log L_{REML}(\boldsymbol{\sigma}) = \text{const} - \frac{1}{2}\log|\mathbf{X}'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{X}| - \frac{1}{2}\log|\mathbf{V}(\boldsymbol{\sigma})| - \frac{1}{2}\mathbf{r}(\boldsymbol{\sigma})'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{r}(\boldsymbol{\sigma}).$$

Maximizing (3.16) produces the REML estimate $\widehat{\boldsymbol{\sigma}}_{REML}$. Searle et al. (1992, Ch. 6.6) provide the REML estimation equations for variance component models. In general the maximization requires numerical methods. If we compare the ML log likelihood (3.8) and the REML log likelihood (3.16), we note that their only difference is the penalty term $-\frac{1}{2}\log|\mathbf{X}'\mathbf{V}(\boldsymbol{\sigma})^{-1}\mathbf{X}|$ in (3.16). Thus, the same algorithms as in the ML estimation (e.g. Newton-Raphson) can be used here, with only minor adaptations (Longford 2003).

Substituting the estimator $\widehat{\mathbf{V}}_{REML} = \mathbf{V}(\widehat{\boldsymbol{\sigma}}_{REML})$ into the GLS formula (3.5) gives the "REML estimator"

$$\widehat{\boldsymbol{\beta}}_{REML} = (\mathbf{X}'\widehat{\mathbf{V}}_{REML}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\mathbf{V}}_{REML}^{-1}\mathbf{y}$$

of $\boldsymbol{\beta}$. This is not identical to $\widehat{\boldsymbol{\beta}}_{ML}$. However, if $\mathbf{y}$ has a symmetric (e.g. normal) distribution, the both estimators are unbiased (Kackar and Harville 1981, 1984).

The often cited justification of the REML approach has been given by Patterson and Thompson (1971), who maintain that in the absence of information on fixed $\boldsymbol{\beta}$ no information on $\boldsymbol{\sigma}$ is lost if the inference is based on the error contrasts $\mathbf{z}$ instead of the data $\mathbf{y}$. Harville (1974) showed from a Bayesian point of view that using only error contrasts in inferences on $\boldsymbol{\sigma}$ is equivalent to using all the data to make these inferences, but ignoring any prior information on $\boldsymbol{\beta}$ (Verbeke and Molenberghs 2000, p. 46).

3.3 Prediction of random effects

3.3.1 Best linear unbiased predictor (BLUP)

Technically speaking, the random effects $\mathbf{u}$ in model (3.2) are not model parameters like $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. However, as Pinheiro and Bates (2000) point out, in a way they behave like parameters and since they are unobservable, there often is interest in obtaining estimates of their values. For example, estimates of random area effects are needed in the estimation of small area means. In the frequentist theory the concept of estimation is usually reserved only for the fixed parameters, and since the vector $\mathbf{u}$ contains random variables, not unknown constants, we say that we do not estimate but predict their values (for opposite points of view, see Robinson 1991).

It can be shown that the best predictor BP of $\mathbf{u}$, in the sense that it minimizes the mean squared prediction error, is the conditional mean

$$\tilde{\mathbf{u}} = BP(\mathbf{u}) = E(\mathbf{u}|\mathbf{y}).$$

The normality assumptions for model (3.2) imply that $\mathbf{u}$ and $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$ have a joint multivariate normal distribution

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix} \sim N_{q+n}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{X}\boldsymbol{\beta} \end{bmatrix}, \begin{bmatrix} \mathbf{G} & \mathbf{G}\mathbf{Z}' \\ \mathbf{Z}\mathbf{G} & \mathbf{V} \end{bmatrix} \right),$$

and under the normal theory the mean of $\mathbf{u}$ given $\mathbf{y}$ is

$$\begin{aligned} E(\mathbf{u}|\mathbf{y}) &= E(\mathbf{u}) + Cov(\mathbf{u}, \mathbf{y})[Cov(\mathbf{y})]^{-1}(\mathbf{y} - E(\mathbf{y})) \\ &= \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \end{aligned} \tag{3.17}$$

This is the best predictor of $\mathbf{u}$, and being a linear function of $\mathbf{y}$ it also is the best linear predictor (BLP) of $\mathbf{u}$.

In practice the unknown $\boldsymbol{\beta}$ in (3.17) is replaced with its estimator $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}_{GLS}$, which is the BLUE of $\boldsymbol{\beta}$, yielding the best linear unbiased predictor (BLUP) of $\mathbf{u}$

$$\tilde{\mathbf{u}} = BLUP(\mathbf{u}) = \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}). \tag{3.18}$$

The unbiasedness means here that both the random variable $\mathbf{u}$ and its predictor have the same expected value

$$E(\tilde{\mathbf{u}}) = E(\tilde{\mathbf{u}} - \mathbf{u}) = E(\mathbf{u}) = \mathbf{0}.$$

In the sense of the "usual" point estimation the unbiasedness of $\tilde{\mathbf{u}}$ does not hold since

$$\begin{aligned} E(\tilde{\mathbf{u}}|\mathbf{u}) &= \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}[E(\mathbf{y}|\mathbf{u}) - \mathbf{X}E(\widehat{\boldsymbol{\beta}}|\mathbf{u})] \\ &= \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}]\mathbf{Z}\mathbf{u}, \end{aligned}$$

which, in general, is not equal to $\mathbf{u}$. Thus the use of the term "unbiased" in this context is sometimes criticized, e.g. by Robinson in his important review paper (1991). In fact, the components $\tilde{u}_i$ of $\tilde{\mathbf{u}}$ have a tendency to be "biased" towards zero, the common expected value of the all u_i's. In a way, each $E(\tilde{u}_i|u_i)$ can be considered a linear combination of the "global" mean zero and u_i. This phenomenon, which also can be described by the fact that $\tilde{u}_i$'s have less spread than u_i's (or the GLS estimates of u_i's, which would be used if the u_i's were regarded as fixed effects), is called shrinkage. This is seen from

$$\begin{aligned} Var(u_i) &= Var(E(u_i|\mathbf{y})) + E(Var(u_i|\mathbf{y})) \\ &= Var(\tilde{u}_i) + \text{a non-negative value.} \end{aligned}$$

Hence the BLUP $\tilde{\mathbf{u}}$ in (3.18) is sometimes called a shrinkage estimator of $\mathbf{u}$ (Robinson 1991).

In the context of small area estimation, the shrinkage property of the BLU predictors is often desirable. For instance, in estimating small area means the BLUP approach produces composite estimators, which are linear combinations of the local area sample means and the global overall sample mean and thus shrink the local means towards the global mean. Since the local area sample mean is an unbiased estimator of the area mean, combining the global sample mean with it results in a biased estimator. On the other hand, the presence of the global sample mean in the composite estimator (or shrinkage estimator) gives some stability for it, thus increasing its efficiency, mostly when there are little data from the area. In the small area estimation the question is often about the trade-off between bias and efficiency.

The concept of shrinkage is closely related to the concept of borrowing strength (Longford 2005). When the data set from a region is small, giving weak information on that region, the regional estimate is strengthened by supplementing the regional data with global data. The smaller are the regional data, the more weight the global information gets in the estimation. This is equivalent to strong shrinkage towards the global estimate. When the regional data are large, the local information receives more weight compared to the global information, and there will be less shrinkage in the regional estimate. The shrinkage also gives some protection for extreme estimates, which may occur by chance, especially when the regional data are small.

3.3.2 Mixed model equations

Henderson (in Henderson et al. 1959) introduced a set of equations, solutions of which give simultaneously the GLS estimator of $\boldsymbol{\beta}$ and the BLUP of $\mathbf{u}$. The equations are derived by maximizing the joint density of $\mathbf{y}$ and $\mathbf{u}$ with respect to $\boldsymbol{\beta}$ and $\mathbf{u}$.

Since $\mathbf{y}|\mathbf{u} \sim N_n(\mathbf{X}\boldsymbol{\beta} + \mathbf{Zu}, \mathbf{R})$ and $\mathbf{u} \sim N_q(\mathbf{0}, \mathbf{G})$, the joint density of $\mathbf{y}$ and $\mathbf{u}$ is

$$\begin{aligned} f(\mathbf{y}, \mathbf{u}) &= f(\mathbf{y}|\mathbf{u}) f(\mathbf{u}) \\ &= (2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp\{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu})'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu})\} \\ &\quad \times (2\pi)^{-q/2} |\mathbf{G}|^{-1/2} \exp\{-\frac{1}{2}\mathbf{u}'\mathbf{G}^{-1}\mathbf{u}\} \\ &= \frac{\exp\{-\frac{1}{2}[(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu})'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu}) + \mathbf{u}'\mathbf{G}^{-1}\mathbf{u}]\}}{(2\pi)^{(n+q)/2} |\mathbf{R}|^{1/2} |\mathbf{G}|^{1/2}}. \end{aligned}$$

To maximize the density $f(\mathbf{y}, \mathbf{u})$ calculate the partial derivatives of

$$\log f(\mathbf{y}, \mathbf{u}) = -\frac{n+q}{2}\log(2\pi) - \frac{1}{2}\log|\mathbf{R}| - \frac{1}{2}\log|\mathbf{G}| - \frac{1}{2}[(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u})'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}) + \mathbf{u}'\mathbf{G}^{-1}\mathbf{u}]$$

with respect to $\boldsymbol{\beta}$ and $\mathbf{u}$:

$$\frac{\partial \log f(\mathbf{y}, \mathbf{u})}{\partial \boldsymbol{\beta}} = \mathbf{X}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u})$$
$$\frac{\partial \log f(\mathbf{y}, \mathbf{u})}{\partial \mathbf{u}} = \mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}) - \mathbf{G}^{-1}\mathbf{u}.$$

Setting these to zero yields equations

$$\begin{cases} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X}\boldsymbol{\beta} + \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z}\mathbf{u} = \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}\mathbf{u} + \mathbf{G}^{-1}\mathbf{u} = \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y}, \end{cases}$$

which are written in matrix form as

$$\begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{bmatrix}. \tag{3.19}$$

These are Henderson's mixed model equations. Solving them produces the $\widehat{\boldsymbol{\beta}}_{GLS}$ in (3.5) and the $\tilde{\mathbf{u}}$ in (3.18). Note that if $\mathbf{Z} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$, i.e. we have only fixed effects in the model, the mixed model equations reduce to the normal equations related to the generalized least squares. Also note that if $\mathbf{G}$ is "very large", i.e. $\mathbf{G}^{-1} \approx \mathbf{0}$, meaning that the variances of u_i's are very large, the BLU predictors of u_i's are close to the GLS estimates (obtained as if the u_i's were fixed effects), implying weak shrinkage. On the other hand, if $\mathbf{G}$ is close to $\mathbf{0}$, its inverse is large and dominates the equations. The result is that the shrinkage becomes strong making $\tilde{\mathbf{u}}$ close to zero.

A practical merit of the mixed model equations is their computational convenience, because there, unlike in (3.18), is no need for inverting the $n \times n$ covariance matrix $\mathbf{V}$. The inverses of $q \times q$ matrix $\mathbf{G}$ and $n \times n$ matrix $\mathbf{R}$ are needed instead, but they are often easy to compute: q is usually not that large and $\mathbf{R}$ is usually diagonal.

3.3.3 Joint covariance matrix of estimation and prediction errors

It follows from the properties of model (3.2) that the covariance matrix of $\widehat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$, where $\mathbf{V}$ is known, is

$$Cov(\widehat{\boldsymbol{\beta}}_{GLS}) = Cov(\widehat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta}) = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}.$$

To obtain a convenient expression for the covariance matrix of the BLU predictor $\tilde{\mathbf{u}} = \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{GLS})$ of $\mathbf{u}$ we define

$$\mathbf{P} = \mathbf{V}^{-1}[\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}]$$

and write

$$\tilde{\mathbf{u}} = \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{y}.$$

Under (3.2) its covariance matrix is

$$Cov(\tilde{\mathbf{u}}) = \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{Z}\mathbf{G},$$

by $\mathbf{PVP} = \mathbf{P}$. Since the purpose is to predict $\mathbf{u}$, not its expected value zero, it is more pertinent to consider the covariance matrix of prediction errors $\tilde{\mathbf{u}} - \mathbf{u}$. This is

$$\begin{aligned} Cov(\tilde{\mathbf{u}} - \mathbf{u}) &= Cov(\tilde{\mathbf{u}}) + Cov(\mathbf{u}) - 2Cov(\tilde{\mathbf{u}}, \mathbf{u}) \\ &= \mathbf{G} - \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{Z}\mathbf{G}. \end{aligned}$$

In addition,

$$Cov(\widehat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta}, \tilde{\mathbf{u}} - \mathbf{u}) = -(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Z}\mathbf{G}.$$

Collecting these results together yields the joint covariance matrix

$$(3.20) \quad Cov\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta} \\ \tilde{\mathbf{u}} - \mathbf{u} \end{bmatrix} = \begin{bmatrix} (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} & -(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Z}\mathbf{G} \\ -\mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} & \mathbf{G} - \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{Z}\mathbf{G} \end{bmatrix}.$$

An alternative expression for (3.20) has been presented by Henderson (1975), who showed that the covariance matrix can be obtained as the (generalized) inverse of the left-hand-side matrix in (3.19), that is,

$$(3.21) \quad Cov\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta} \\ \tilde{\mathbf{u}} - \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} \end{bmatrix}^{-1}.$$

The equivalence of (3.20) and (3.21) is easy to verify by a straightforward calculation. The expression (3.21) is computationally economical, because the inverse of $\mathbf{V}$ is not needed.

3.3.4 Empirical best linear unbiased predictor (EBLUP)

Usually the covariance matrices $\mathbf{V}$, $\mathbf{G}$ and $\mathbf{R}$ are unknown. Then, in predicting $\mathbf{u}$ by the BLUP formula (3.18) they will be replaced with their REML or ML estimates to yield

$$(3.22) \quad \hat{\mathbf{u}} = EBLUP(\mathbf{u}) = \widehat{\mathbf{G}}\mathbf{Z}'\widehat{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}).$$

The predictor $\hat{\mathbf{u}}$ is called empirical best linear unbiased predictor (EBLUP) of $\mathbf{u}$, the word "empirical" referring to the fact that the values of $\mathbf{G}$ and $\mathbf{V}$ have been obtained from the observed data (cf. empirical Bayes). The estimator $\widehat{\boldsymbol{\beta}}$ is now the GLS estimator (3.5), where $\mathbf{V}$ is replaced with its estimate (i.e. $\widehat{\boldsymbol{\beta}}$ is typically $\widehat{\boldsymbol{\beta}}_{REML}$ or $\widehat{\boldsymbol{\beta}}_{ML}$), and it is sometimes called empirical BLUE of $\boldsymbol{\beta}$.

Both $\widehat{\boldsymbol{\beta}}$ and $\hat{\mathbf{u}}$ can be obtained by solving the mixed model equations (3.19) where $\mathbf{V}$, $\mathbf{G}$ and $\mathbf{R}$ are substituted by the corresponding estimates.

In small area estimation the approach, where e.g. small area totals are estimated by utilizing the empirical BLU predictors of random area effects, is often referred as EBLUP method or approach (Ghosh and Rao 1994, Rao 1999, Pfeffermann 2002, Rao 2003).

3.4 Statistical inference

3.4.1 Fixed effects

When the covariance matrix $\mathbf{V}$ is known, the fixed parameter vector $\boldsymbol{\beta}$ is unbiasedly estimated by the GLS solution (3.5)

$$\widehat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}.$$

Its covariance matrix is

$$Cov(\widehat{\boldsymbol{\beta}}_{GLS}) = \boldsymbol{\Phi} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}. \tag{3.23}$$

In testing a linear null hypothesis $H_0 : \mathbf{L}'\boldsymbol{\beta} = \mathbf{c}$, where $\mathbf{L}$ and $\mathbf{c}$ are given constants, we can use the Wald statistic

$$X^2 = (\mathbf{L}'\widehat{\boldsymbol{\beta}}_{GLS} - \mathbf{c})'(\mathbf{L}'\boldsymbol{\Phi}\mathbf{L})^{-1}(\mathbf{L}'\widehat{\boldsymbol{\beta}}_{GLS} - \mathbf{c}). \tag{3.24}$$

Under H_0 and normality the Wald statistic follows a central χ^2 distribution with $r_{\mathbf{L}}$ degrees of freedom, where $r_{\mathbf{L}}$ is the rank of the coefficient matrix $\mathbf{L}$. We can also construct confidence intervals for linear combinations of form $\mathbf{l}'\boldsymbol{\beta}$, where $\mathbf{l}$ is a given vector, as

$$\mathbf{l}'\widehat{\boldsymbol{\beta}}_{GLS} \pm z_\alpha s.e.(\mathbf{l}'\widehat{\boldsymbol{\beta}}_{GLS}), \tag{3.25}$$

where z_α is the desired critical value of the standard normal distribution and

$$s.e.(\mathbf{l}'\widehat{\boldsymbol{\beta}}_{GLS}) = \sqrt{\mathbf{l}'\boldsymbol{\Phi}\mathbf{l}}$$

is the standard error of $\mathbf{l}'\widehat{\boldsymbol{\beta}}_{GLS}$.

When $\mathbf{V} = \mathbf{V}(\boldsymbol{\sigma})$ is unknown, the inference becomes more complicated due to the uncertainty about $\boldsymbol{\sigma}$. Let $\widehat{\boldsymbol{\sigma}}$ be some estimator of the variance parameter vector $\boldsymbol{\sigma}$ and $\widehat{\mathbf{V}} = \mathbf{V}(\widehat{\boldsymbol{\sigma}})$ the corresponding estimator of $\mathbf{V}$. Kackar and Harville (1981) have shown that if (i) the data vector $\mathbf{y}$ is symmetrically distributed about its expected value and (ii) the estimator of $\boldsymbol{\sigma}$ is even and translation-invariant, then the estimator

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\widehat{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\mathbf{V}}^{-1}\mathbf{y}$$

of GLS form is unbiased for $\boldsymbol{\beta}$. This result holds especially for the ML and REML estimation of variance components under normal distribution. The formula (3.23) gives now the asymptotic covariance matrix of $\widehat{\boldsymbol{\beta}}$. Following (3.23), the covariance matrix of $\widehat{\boldsymbol{\beta}}$ could be estimated simply with

$$\widehat{\boldsymbol{\Phi}} = (\mathbf{X}'\widehat{\mathbf{V}}^{-1}\mathbf{X})^{-1},$$

which is asymptotically valid, but in finite samples underestimates the true variability of $\widehat{\boldsymbol{\beta}}$ since it implicitly treats $\widehat{\mathbf{V}}$ as known, without taking the variation in $\widehat{\mathbf{V}}$ into account. In fact, as McCulloch and Searle (2001) point out, $\widehat{\boldsymbol{\Phi}}$ is an estimate of $Cov(\widehat{\boldsymbol{\beta}}_{GLS}) = \boldsymbol{\Phi} = Cov((\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y})$. It is not an estimate of $Cov(\widehat{\boldsymbol{\beta}}) = Cov((\mathbf{X}'\widehat{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\mathbf{V}}^{-1}\mathbf{y})$ (see also Searle, Casella and McCulloch 1992, Ch. 9.1).

The problem of valid estimation of $Cov(\widehat{\boldsymbol{\beta}})$ (and, in general, $Cov(\mathbf{l}'\widehat{\boldsymbol{\beta}} + \mathbf{m}'\widehat{\mathbf{u}})$, where $\mathbf{l}$ and $\mathbf{m}$ are given coefficient vectors) in the context of linear mixed models has been considered by Kackar and Harville (1984), Harville and Jeske (1992) and further, under the REML estimation, by Kenward and Roger (1997). Under $\mathbf{m} = \mathbf{0}$ the result of Kackar and Harville (1984) for translation-invariant estimators of $\boldsymbol{\sigma}$ give

$$\begin{aligned} Cov(\widehat{\boldsymbol{\beta}}) &= Cov(\widehat{\boldsymbol{\beta}}_{GLS}) + Cov(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_{GLS}) \\ &= \boldsymbol{\Phi} + \boldsymbol{\Lambda} \end{aligned} \tag{3.26}$$

for the fixed effects estimator $\widehat{\boldsymbol{\beta}}$. The matrix $\boldsymbol{\Lambda} = Cov(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_{GLS})$ measures the amount by which the asymptotic covariance matrix underestimates $Cov(\widehat{\boldsymbol{\beta}})$. Kackar and Harville (1984) derive a Taylor series approximation for $\boldsymbol{\Lambda}$, written by Kenward and Roger (1997) as

$$\boldsymbol{\Lambda} \approx \boldsymbol{\Phi} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} [Cov(\widehat{\boldsymbol{\sigma}})]_{ij} (\mathbf{Q}_{ij} - \mathbf{P}_i \boldsymbol{\Phi} \mathbf{P}_j) \right\} \boldsymbol{\Phi}, \tag{3.27}$$

where

$$\mathbf{P}_i = \mathbf{X}' \frac{\partial \mathbf{V}^{-1}}{\partial \sigma_i} \mathbf{X},$$

$$\mathbf{Q}_{ij} = \mathbf{X}' \frac{\partial \mathbf{V}^{-1}}{\partial \sigma_i} \mathbf{V} \frac{\partial \mathbf{V}^{-1}}{\partial \sigma_j} \mathbf{X}$$

and $Cov(\widehat{\boldsymbol{\sigma}})$ is the covariance matrix of the estimator $\widehat{\boldsymbol{\sigma}}$. As Kenward and Roger employ the REML estimation, they obtain $Cov(\widehat{\boldsymbol{\sigma}})$ as the inverse of the corresponding Fisher information matrix. An estimator $\widehat{\boldsymbol{\Lambda}}$ of $\boldsymbol{\Lambda}$ is obtained by substituting $\mathbf{V}$ with $\widehat{\mathbf{V}}$ in the approximation (3.27).

Kenward and Roger consider the bias of $\widehat{\boldsymbol{\Phi}}$ in estimating $\boldsymbol{\Phi}$. Using a Taylor series expansion they derive a general, but rather complicated expression for the bias. However, if the covariance matrix $\mathbf{V}$ has a linear structure, i.e. it is a linear function

$$\mathbf{V} = \sum_{i=1}^{r} \sigma_i \mathbf{K}_i \tag{3.28}$$

of the parameters σ_i in $\boldsymbol{\sigma}$, where each $\mathbf{K}_i$ is a known matrix, the second derivatives in the expansion vanish. Kenward and Roger show that this, along with the approximate unbiasedness of the REML estimator of $\mathbf{V}$, leads to a convenient result

$$E(\widehat{\boldsymbol{\Phi}}) \approx \boldsymbol{\Phi} - \boldsymbol{\Lambda},$$

which suggests together with (3.26) an adjusted "Kenward-Roger estimator"

$$\widehat{\mathbf{\Phi}}_{KR} = \widehat{\mathbf{\Phi}} + 2\widehat{\mathbf{\Lambda}} \tag{3.29}$$

of $Cov(\widehat{\boldsymbol{\beta}})$. This bias-corrected estimator is found to perform well in small samples.

It appears that under a linear covariance structure a bias correction of similar form to (3.29) can be applied also in estimating the variance of prediction error, when the value of a linear combination $\mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$ of fixed and random effects is to be predicted. Such prediction is an essential part of model-based small area estimation.

Linear covariance structures arise from a variety of linear mixed models. In particular, the variance component models yield covariance matrices $\mathbf{V}$ with a linear structure. The small area models considered in this report (in later chapters) also possess this property.

Under ML estimation we can test the linear hypothesis $H_0 : \mathbf{L}'\boldsymbol{\beta} = \mathbf{c}$ with the likelihood ratio

$$\lambda = \frac{L(\widehat{\boldsymbol{\beta}}_{H_0}, \widehat{\boldsymbol{\sigma}}_{H_0})}{L(\widehat{\boldsymbol{\beta}}_{ML}, \widehat{\boldsymbol{\sigma}}_{ML})},$$

where $\widehat{\boldsymbol{\beta}}_{ML}$ and $\widehat{\boldsymbol{\sigma}}_{ML}$ are the ML estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\beta}}_{H_0}$ and $\widehat{\boldsymbol{\sigma}}_{H_0}$ are the respective estimates under H_0. The test statistic $-2\log\lambda$ is asymptotically $\chi^2(r_{\mathbf{L}})$. For REML estimation this test is not valid, because the error contrasts used for eliminating the fixed parameters from the REML log likelihood are not the same under the null hypothesis and the alternative hypothesis (Verbeke and Molenberghs 2000). Then the log likelihoods are not comparable any more.

It is often more convenient to apply the Wald statistic (3.24), where $\mathbf{\Phi}$ has been replaced with its estimate, $\widehat{\mathbf{\Phi}}$ or $\widehat{\mathbf{\Phi}}_{KR}$. Under H_0 the statistic X^2 follows the $\chi^2(r_{\mathbf{L}})$ distribution asymptotically. An often preferred alternative (e.g. Pinheiro and Bates 2000, Brown and Prescott 1999) is to use the Wald F statistic

$$F = X^2 / r_{\mathbf{L}}, \tag{3.30}$$

which follows under H_0 the F distribution with degrees of freedom $r_{\mathbf{L}}$ and $DF2$ asymptotically. The calculation of the denominator degrees of freedom $DF2$ is usually complicated. One way to obtain $DF2$ is the well-known Satterthwaite approximation (1946, see also Verbeke and Molenberghs 1997). Kenward and Roger (1997) derived a scaled Wald F statistic to be used in small samples with the adjusted estimator $\widehat{\mathbf{\Phi}}_{KR}$ and an approximation of $DF2$, which essentially is the Satterthwaite approximation applied to $\widehat{\mathbf{\Phi}}_{KR}$ instead of $\widehat{\mathbf{\Phi}}$.

In constructing confidence intervals for single linear combinations $\mathbf{l}'\boldsymbol{\beta}$ under estimated $\mathbf{\Phi}$ the critical value z_α in (3.25) should be replaced with the corresponding value from t distribution with $DF2$ degrees of freedom.

Finally we remark that the inference on $\boldsymbol{\beta}$ is sometimes based on a so-called robust or empirical estimator of $Cov(\widehat{\boldsymbol{\beta}})$ (Liang and Zeger 1986, Verbeke and Molenberghs 2000, Diggle et al. 2002). The covariance matrix of the GLS estimator

$$\widehat{\boldsymbol{\beta}}_W = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{y}$$

with some symmetric weight matrix $\mathbf{W}$ can be estimated with a sandwich estimator

$$[(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}]\tilde{\mathbf{V}}[\mathbf{W}\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}],$$

where $\tilde{\mathbf{V}}$ is a consistent estimator of $\mathbf{V}$ regardless of the true covariance structure. The sandwich estimator is consistent for $Cov(\widehat{\boldsymbol{\beta}}_W)$ and gives a robust inference on $\boldsymbol{\beta}$ with respect to the model misspecification of the covariance structure. It may also help alleviate some of the small sample bias (Brown and Prescott 1999, p. 74).

We may choose $\mathbf{W} = \widehat{\mathbf{V}}^{-1}$, where $\widehat{\mathbf{V}}$ is e.g. the "model-based" REML estimator. If we also choose $\tilde{\mathbf{V}} = \widehat{\mathbf{V}}$, the sandwich estimator reduces to $\widehat{\boldsymbol{\Phi}}$, which is consistent only if the specified covariance structure is correct. In the context of longitudinal data analysis some feasible choices for $\tilde{\mathbf{V}}$ are discussed by Diggle et al. (2002, Sec. 4.6). An apparently "terrible" choice for $\tilde{\mathbf{V}}$, which still may yield an adequate sandwich estimator, is the singular $(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_W)(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_W)'$ (see Liang and Zeger 1986; Diggle et al. 2002, p. 72–73)

3.4.2 Variance parameters

The statistical inference on the parameter vector $\boldsymbol{\sigma}$ determining the covariance matrix $\mathbf{V} = \mathbf{V}(\boldsymbol{\sigma})$ is based on the classical theory of maximum likelihood estimation. It appears that under some regularity conditions the distribution of the ML (as well as REML) estimators $\widehat{\boldsymbol{\sigma}}$ is approximately normal with mean vector $\boldsymbol{\sigma}$ and a covariance matrix $Cov(\widehat{\boldsymbol{\sigma}})$, which is the inverse of the matrix of the Fisher information matrix (Verbeke and Molenberghs 2000). Recall that under the regularity conditions the Fisher information matrix is the negative of the expected matrix of the second derivatives of the log likelihood with respect to the model parameters. Searle, Casella and McCulloch (1992) give an expression of the information matrix for variance component models, both in the ML estimation (Ch. 6.3) and the REML estimation (Ch. 6.6). For more general models with non-linear covariance structures it is difficult to give any general expression, because the second derivatives with respect to the variance parameters often become complicated.

Both the approximate confidence intervals for $\boldsymbol{\sigma}$ and the related Wald χ^2 significance tests can be based on the asymptotic normality of $\widehat{\boldsymbol{\sigma}}$. However, it is known that the validity of the normal approximation strongly depends on the true value of $\boldsymbol{\sigma}$: the closer the true value of the parameter is to the boundary of the parameter space, the larger sample size is needed for a reasonable approximation, and if the true value is on the boundary, the standard asymptotic theory does not hold (e.g. Verbeke and Molenberghs 2000). An important case where the boundary problem is met is testing the null hypothesis that the variance of a random effect is zero. For covariances or correlations the value zero is in the interior of the parameter space and causes no problems.

The Wald statistic for a linear null hypothesis $H_0 : \mathbf{L}'\boldsymbol{\sigma} = \mathbf{c}$ has the same form as the statistic (3.24) for the fixed parameters. The parameter $\boldsymbol{\beta}_0$ in (3.24) will be replaced with $\widehat{\boldsymbol{\sigma}}$, which is either the ML or REML estimator of $\boldsymbol{\sigma}$ and $\boldsymbol{\Phi}$ will be replaced with the asymptotic covariance matrix of $\widehat{\boldsymbol{\sigma}}$. Similar hypotheses can also be tested by the likelihood ratio statistic

$$\lambda = \frac{L(\widehat{\boldsymbol{\beta}}_{H_0}, \widehat{\boldsymbol{\sigma}}_{H_0})}{L(\widehat{\boldsymbol{\beta}}_{ML}, \widehat{\boldsymbol{\sigma}}_{ML})}$$

for the ML estimation and

$$\lambda = \frac{L_{REML}(\widehat{\boldsymbol{\sigma}}_{H_0})}{L_{REML}(\widehat{\boldsymbol{\sigma}}_{REML})}$$

for the REML estimation. The subscript H_0 refers here to estimates obtained by the chosen method (ML or REML) under the constraints in the null hypothesis. The test statistic $-2\log\lambda$ (as well as the Wald statistic) is asymptotically distributed as $\chi^2(s)$, where s is the number of constraints on $\boldsymbol{\sigma}$ defined in H_0. The likelihood ratio test is often considered more reliable than the Wald test due to its more rapid asymptotics. In testing hypotheses that lie on the boundary, however, the likelihood ratio test suffers from the same problems as the Wald test.

Stram and Lee (1994) have examined the behaviour of the likelihood ratio test when the null hypothesis lies on the boundary of the parameter space and noticed that the usual chi-square approximation produces a conservative test, which wrongly favors the null hypothesis even asymptotically. They argue, however, that for simple hypotheses, e.g. for just one variance component, the error in significance level is typically small. Stram and Lee also show that in tests for nonzero variance components the distribution of the likelihood ratio statistic can be approximated well with a 50:50 mixture of distributions $\chi^2(s)$ and $\chi^2(s-1)$.

For testing a hypothesis $H_0 : \sigma^2 = 0$ on a single nonzero variance component McCulloch (2005) suggests a modified likelihood ratio test, where the p value obtained from $\chi^2(1)$ distribution is cut in half. This is because the alternative hypothesis is now one-sided $H_1 : \sigma^2 > 0$, but the likelihood ratio test is inherently two-sided. The Wald test is sometimes modified in the same way, i.e. in the MIXED procedure of the SAS package, by taking the square root of the $\chi^2(1)$ distributed statistic X^2 for a single variance component and calculating the one-sided p value from the $N(0,1)$ distribution.

3.4.3 Random effects

Recall first that when the covariance matrix $Cov(\mathbf{y}) = \mathbf{V}$ is known (that is, the parameter vector $\boldsymbol{\sigma}$ determining $\mathbf{V} = \mathbf{V}(\boldsymbol{\sigma})$ is known), the BLUP of the random effects vector $\mathbf{u}$ is

$$\tilde{\mathbf{u}} = BLUP(\mathbf{u}) = \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) = \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{y}$$

with $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$ and $\mathbf{P} = \mathbf{V}^{-1}[\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}]$. The covariance matrices of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}} - \mathbf{u}$ are

$$Cov(\tilde{\mathbf{u}}) = \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{Z}\mathbf{G}$$

and

$$Cov(\tilde{\mathbf{u}} - \mathbf{u}) = \mathbf{G} - \mathbf{G}\mathbf{Z}'\mathbf{P}\mathbf{Z}\mathbf{G}, \tag{3.31}$$

respectively. When $\mathbf{V}$ (actually $\boldsymbol{\sigma}$) is estimated, we have the empirical BLUP

$$\widehat{\mathbf{u}} = EBLUP(\mathbf{u}) = \widehat{\mathbf{G}}\mathbf{Z}'\widehat{\mathbf{P}}\mathbf{y}$$

obtained by substituting $\mathbf{G}$ and $\mathbf{V}$ (and the resulting $\mathbf{P}$) with their estimates. Now the uncertainty about $\mathbf{V}$ causes that a straightforward application of (3.31), which is valid for known $\mathbf{V}$, leads to underestimated variances of the prediction errors. This is similar to what happens also for the variances of the fixed effect estimators.

Kackar and Harville (1984), followed by Prasad and Rao (1990) and Harville and Jeske (1992), have considered the mean squared error of the predictor of a linear combination

$$\tau = \mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$$

of fixed effects $\boldsymbol{\beta}$ and random effects $\mathbf{u}$, when $\mathbf{V}$ is unknown. Their results have important applications to small area estimation by linear mixed models. Recall also that the paper of Kackar and Harville (1984) provided the starting point for the work of Kenward and Roger (1997) on the inference for fixed effects. We consider here the MSE of the EBLUP of the random part $\mathbf{m}'\mathbf{u}$.

The mean squared error of $BLUP(\mathbf{m}'\mathbf{u}) = \mathbf{m}'\tilde{\mathbf{u}}$ is simply

$$\begin{aligned} MSE(\mathbf{m}'\tilde{\mathbf{u}}) &= E[\mathbf{m}'\tilde{\mathbf{u}} - \mathbf{m}'\mathbf{u}]^2 \\ &= \mathbf{m}'Cov(\tilde{\mathbf{u}} - \mathbf{u})\mathbf{m} \end{aligned}$$

and its estimate is

$$\widehat{MSE}(\mathbf{m}'\tilde{\mathbf{u}}) = \mathbf{m}'\widehat{Cov}(\tilde{\mathbf{u}} - \mathbf{u})\mathbf{m},$$

where

$$\widehat{Cov}(\tilde{\mathbf{u}} - \mathbf{u}) = \widehat{\mathbf{G}} - \widehat{\mathbf{G}}\mathbf{Z}'\widehat{\mathbf{P}}\mathbf{Z}\widehat{\mathbf{G}}.$$

For $EBLUP(\mathbf{m}'\mathbf{u}) = \mathbf{m}'\widehat{\mathbf{u}}$ we note that

$$\begin{aligned} MSE(\mathbf{m}'\widehat{\mathbf{u}}) &= E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}} + \mathbf{m}'\tilde{\mathbf{u}} - \mathbf{m}'\mathbf{u}]^2 \\ &= E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}}]^2 + E[\mathbf{m}'\tilde{\mathbf{u}} - \mathbf{m}'\mathbf{u}]^2 + 2E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}}][\mathbf{m}'\tilde{\mathbf{u}} - \mathbf{m}'\mathbf{u}]. \end{aligned}$$

Kackar and Harville (1984) show that

$$E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}}][\mathbf{m}'\tilde{\mathbf{u}} - \mathbf{m}'\mathbf{u}] = 0$$

provided that the random effects and error terms in the mixed model (3.2) are normal and the estimator $\widehat{\boldsymbol{\sigma}}$ of $\boldsymbol{\sigma}$ is translation invariant. The ML and REML estimators of $\boldsymbol{\sigma}$ are translation invariant (Kackar and Harville 1981). It follows that

$$MSE(\mathbf{m}'\widehat{\mathbf{u}}) = MSE(\mathbf{m}'\tilde{\mathbf{u}}) + E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}}]^2, \tag{3.32}$$

i.e. the MSE of EBLUP is the MSE of BLUP plus a non-negative correction term accounting for the uncertainty about $\mathbf{V}$.

Since $\mathbf{m}'\tilde{\mathbf{u}}$ is a function of $\boldsymbol{\sigma}$ and $\mathbf{m}'\widehat{\mathbf{u}}$ a similar function of $\widehat{\boldsymbol{\sigma}}$, we can write a Taylor series approximation

$$\mathbf{m}'\widehat{\mathbf{u}} \approx \mathbf{m}'\tilde{\mathbf{u}} + \left(\frac{\partial \mathbf{m}'\tilde{\mathbf{u}}}{\partial \boldsymbol{\sigma}}\right)'(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}),$$

which gives

$$E[\mathbf{m}'\widehat{\mathbf{u}} - \mathbf{m}'\tilde{\mathbf{u}}]^2 \approx E\left[\left(\frac{\partial \mathbf{m}'\tilde{\mathbf{u}}}{\partial \boldsymbol{\sigma}}\right)'(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\right]^2. \tag{3.33}$$

Kackar and Harville (1984) give conditions under which the equation

$$E\left[\left(\frac{\partial \mathbf{m}'\tilde{\mathbf{u}}}{\partial \boldsymbol{\sigma}}\right)'(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\right]^2 = tr[A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})],$$

where

$$A(\boldsymbol{\sigma}) = Cov\left(\frac{\partial \mathbf{m}'\tilde{\mathbf{u}}}{\partial \boldsymbol{\sigma}}\right)$$

and

$$B(\boldsymbol{\sigma}) = E[(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})'],$$

holds exactly. This leads them to suggest the "Kackar-Harville approximation"

$$MSE(\mathbf{m}'\widehat{\mathbf{u}}) \approx MSE(\mathbf{m}'\tilde{\mathbf{u}}) + tr[A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})]. \tag{3.34}$$

In practice the MSE matrix $B(\boldsymbol{\sigma})$ is often replaced by the asymptotic covariance matrix of $\widehat{\boldsymbol{\sigma}}$.

Prasad and Rao (1990) derived a further, computationally convenient approximation for $tr[A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})]$ in (3.34). Define $\mathbf{b}' = \mathbf{m}'\mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}$ so that $\mathbf{m}'\tilde{\mathbf{u}} = \mathbf{b}'(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}})$. The general form of the "Prasad-Rao approximation" is then

$$g_3(\boldsymbol{\sigma}) = tr[(\partial\mathbf{b}'/\partial\boldsymbol{\sigma})'\mathbf{V}(\partial\mathbf{b}'/\partial\boldsymbol{\sigma})B(\boldsymbol{\sigma})] \approx tr[A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})]. \tag{3.35}$$

They explicitly worked out the approximation (3.35) for three simple small area models, including the Fay-Herriot model (2.19) and nested error regression model (2.20), using Henderson's method 3 for the variance component estimation. They also showed that their approximation

$$MSE(\mathbf{m}'\widehat{\mathbf{u}}) \approx MSE(\mathbf{m}'\tilde{\mathbf{u}}) + g_3(\boldsymbol{\sigma}) \tag{3.36}$$

can be estimated with

$$\widehat{MSE}(\mathbf{m}'\widehat{\mathbf{u}}) \approx \widehat{MSE}(\mathbf{m}'\tilde{\mathbf{u}}) + 2g_3(\widehat{\boldsymbol{\sigma}}), \tag{3.37}$$

where $g_3(\widehat{\boldsymbol{\sigma}})$ is obtained by substituting $\boldsymbol{\sigma}$ with its estimate in $g_3(\boldsymbol{\sigma})$. The estimator (3.37) is approximately unbiased. We note that it has actually the same form (with the bias correction factor of two) as the result (3.29) of Kenward and Roger for fixed effects, which results from assuming that $\mathbf{V}$ has a linear structure. Prasad and Rao (1990) also give the regularity conditions under which the neglected terms in their approximation, as well as in the Kackar-Harville approximation, are $o(m^{-1})$ for large m, where m is the number of small areas in the considered models.

Datta and Lahiri (2000) showed later that the Prasad-Rao approximation (3.36) is valid for general linear mixed models if the REML estimation is employed. For the ML estimation they give an additive adjustment term to account for the bias in the variance estimates. Similarly, they showed that the estimator (3.37) is approximately unbiased if the covariance structure is linear. For the ML estimation, however, an additional bias correction term is again needed.

Finally we note that Das, Jiang and Rao (2004) have thoroughly studied the MSE approximation for the EBLUP of $\tau = \mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$ under general linear mixed models, when REML or ML estimation is employed. They essentially give rigorous proofs for the earlier results by Prasad and Rao and Datta and Lahiri. In a highly technical paper they also specify the precise order of the neglected terms in the MSE approximation and its estimator and present the regularity conditions, which are to be satisfied.

4 Small area estimation with unit level mixed model for cross-sectional data

4.1 Preliminaries

In this chapter we apply the theory on linear mixed models and the related EBLUP theory to the estimation of a small area total. The application is presented for the basic unit level model for cross-sectional data, i.e. the nested error regression model. In the following chapters we show how the application will be extended for panel data and rotating panel data.

Consider a finite population U of size N, which divides into m disjoint areas so that

$$U = U_1 \cup U_2 \cup \ldots \cup U_i \cup \ldots \cup U_m$$

and

$$N = N_1 + N_2 + \ldots + N_i + \ldots + N_m,$$

where N_i is the size of area population U_i, $i = 1, 2, \ldots, m$. Assume then that a random sample s of size n is drawn from U and

$$s = s_1 \cup s_2 \cup \ldots \cup s_i \cup \ldots \cup s_m$$

with

$$n = n_1 + n_2 + \ldots + n_i + \ldots + n_m.$$

Furthermore, each area population U_i divides into the sample s_i and the remainder $r_i = U_i - s_i$. It possible that for some areas the sample size n_i is zero so that $r_i = U_i$.

The total of target variable y in area i is

$$Y_i = \sum_{j \in U_i} y_{ij} = \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} y_{ij},$$

where $j = 1, 2, \ldots, N_i$ denotes the unit. The sample sum $\sum_{j \in s_i} y_{ij}$ is observed, which makes the estimation of Y_i reduce to prediction of the unobserved remainder sum $\sum_{j \in r_i} y_{ij}$. This is done by using an appropriate model for y_{ij}'s.

4.2 Nested error regression model

We consider here the nested error regression model

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + u_i + e_{ij},$$

which has already been introduced in Section 2.5.3 as model (2.20). Here y_{ij} is the response of unit j in area i, $\mathbf{x}_{ij}$ is the corresponding vector of auxiliary variables, $\boldsymbol{\beta}$

is the vector of fixed parameters, u_i is the random effect of area i and e_{ij} the random individual error term. The area effects u_i are assumed independent with zero mean and variance σ_u^2. Similarly, the errors e_{ij} are independent with zero mean and variance σ_e^2. In addition, the u_i's and the e_{ij}'s are assumed mutually independent. Under these assumptions

$$E(y_{ij}) = \mathbf{x}'_{ij}\boldsymbol{\beta}$$

and

$$Var(y_{ij}) = \sigma_u^2 + \sigma_e^2.$$

We also make the usual assumption that both u_i and e_{ij} are normally distributed.

It is advisable to write the model in the matrix form. With the notation of (3.2) the model equation of the $N_i \times 1$ vector

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iN_i} \end{bmatrix}$$

of the population U_i of area i is

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{u}_i + \mathbf{e}_i,$$

where

$$\mathbf{X}_i = \begin{bmatrix} \mathbf{x}'_{i1} \\ \mathbf{x}'_{i2} \\ \vdots \\ \mathbf{x}'_{iN_i} \end{bmatrix}, \qquad (N_i \times p)$$

$\mathbf{Z}_i = \mathbf{1}_{N_i}$ (the unity vector with N_i elements), $\mathbf{u}_i = u_i$ (scalar) and

$$\mathbf{e}_{ij} = \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ e_{iN_i} \end{bmatrix} \qquad (N_i \times 1).$$

With this notation

$$E(\mathbf{y}_i) = \mathbf{X}_i\boldsymbol{\beta}$$

and

$$Cov(\mathbf{y}_i) = \mathbf{V}_i = \mathbf{J}_{N_i}\sigma_u^2 + \mathbf{I}_{N_i}\sigma_e^2,$$

where $\mathbf{J}_{N_i} = \mathbf{1}_{N_i}\mathbf{1}'_{N_i}$ is the square matrix of ones. Using the usual mixed model notation we can write

$$\mathbf{V}_i = \mathbf{Z}_i\mathbf{G}_i\mathbf{Z}'_i + \mathbf{R}_i,$$

where $\mathbf{G}_i = \sigma_u^2$ and $\mathbf{R}_i = \sigma_e^2\mathbf{I}_{N_i}$. Under the given assumptions the responses y_{ij} and $y_{ij'}$ of two units j and j' ($j' \neq j$) coming from the same area i have a common covariance

$$Cov(y_{ij}, y_{ij'}) = \sigma_u^2.$$

The corresponding intra-area correlation is then

$$\rho = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2}.$$

If we further stack the area vectors $\mathbf{y}_i$ into one response vector

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_m \end{bmatrix} \qquad (N \times 1),$$

we have the model in the form (3.2) of the general linear mixed model. The model is now

(4.1) $$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$$

with

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{bmatrix}, \qquad (N \times p)$$

$$\mathbf{Z} = \oplus_{i=1}^m \mathbf{Z}_i = \begin{bmatrix} \mathbf{1}_{N_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_2} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{N_m} \end{bmatrix}, \qquad (N \times m)$$

$$\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \qquad (m \times 1)$$

and

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix}. \qquad (N \times 1)$$

Units (and observations) coming from different areas i and i' are not correlated. The covariance matrix $\mathbf{V}$ of the response vector $\mathbf{y}$ is block-diagonal

$$Cov(\mathbf{y}) = \mathbf{V} = \oplus_{i=1}^m \mathbf{V}_i = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_m \end{bmatrix} \qquad (N \times N),$$

or

$$\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R},$$

where $\mathbf{G} = \sigma_u^2 \mathbf{I}_m$ and $\mathbf{R} = \sigma_e^2 \mathbf{I}_N$.

4.3 BLUP and EBLUP estimation of area total

The target is to estimate the area total

$$Y_i = \sum_{j \in U_i} y_{ij} = \mathbf{c}_i'\mathbf{y},$$

where $\mathbf{c}_i$ is an $N \times 1$ vector of N_i ones and $N - N_i$ zeros such that the ones correspond to those y_{ij}'s of $\mathbf{y}$, which belong to area i.

Let the vectors $\mathbf{y}$ and $\mathbf{e}$ as well as the matrices $\mathbf{X}$ and $\mathbf{Z}$ in (4.1) be partitioned into sample parts $\mathbf{y}_s$, $\mathbf{e}_s$, $\mathbf{X}_s$ and $\mathbf{Z}_s$ (of n rows) and remainder parts $\mathbf{y}_r$, $\mathbf{e}_r$, $\mathbf{X}_r$ and $\mathbf{Z}_r$ (of $N - n$ rows). Then the nested error regression model takes the form

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_s \\ \mathbf{y}_r \end{bmatrix} = \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \mathbf{Z}_s \\ \mathbf{Z}_r \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{e}_s \\ \mathbf{e}_r \end{bmatrix}.$$

The corresponding partition of the covariance matrix of $\mathbf{y}$ is

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_s & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_r \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_s\mathbf{G}\mathbf{Z}_s' & \mathbf{Z}_s\mathbf{G}\mathbf{Z}_r' \\ \mathbf{Z}_r\mathbf{G}\mathbf{Z}_s' & \mathbf{Z}_r\mathbf{G}\mathbf{Z}_r' \end{bmatrix} + \begin{bmatrix} \sigma_e^2\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \sigma_e^2\mathbf{I}_{N-n} \end{bmatrix}.$$

Applying the similar partition to $\mathbf{c}_i$ yields

$$\mathbf{c}_i = \begin{bmatrix} \mathbf{c}_{is} \\ \mathbf{c}_{ir} \end{bmatrix},$$

where $\mathbf{c}_{is}$ picks the units in the sample s_i from area i and $\mathbf{c}_{ir}$ picks those in the remainder r_i. Now the area total to be estimated is

$$Y_i = \mathbf{c}_{is}'\mathbf{y}_s + \mathbf{c}_{ir}'\mathbf{y}_r$$

and the general prediction theorem (2.11) can be applied directly. This gives

$$\widehat{Y}_{i,BLUP} = \mathbf{c}_{is}'\mathbf{y}_s + \mathbf{c}_{ir}'[\mathbf{X}_r\widehat{\boldsymbol{\beta}} + \mathbf{V}_{rs}\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}})],$$

where the GLS estimate

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{y}_s$$

is calculated over the whole sample data s. The covariance matrix $\mathbf{V}$ is assumed known here. By $\mathbf{V}_{rs} = \mathbf{Z}_r\mathbf{G}\mathbf{Z}_s'$ we can write

$$\begin{aligned} \widehat{Y}_{i,BLUP} &= \mathbf{c}_{is}'\mathbf{y}_s + \mathbf{c}_{ir}'\mathbf{X}_r\widehat{\boldsymbol{\beta}} + \mathbf{c}_{ir}'\mathbf{Z}_r[\mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}})] \\ &= \mathbf{c}_{is}'\mathbf{y}_s + (\mathbf{l}_i'\widehat{\boldsymbol{\beta}} + \mathbf{m}_i'\tilde{\mathbf{u}}), \end{aligned}$$

where $\mathbf{l}_i' = \mathbf{c}_{ir}'\mathbf{X}_r$, $\mathbf{m}_i' = \mathbf{c}_{ir}'\mathbf{Z}_r$ and $\tilde{\mathbf{u}}$ is the BLUP of $\mathbf{u}$ by (3.18). Note that $\mathbf{l}_i'\widehat{\boldsymbol{\beta}} + \mathbf{m}_i'\tilde{\mathbf{u}}$ is the BLUP of linear combination $\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u}$.

By observing that

$$\mathbf{c}'_{is}\mathbf{y}_s = \sum_{j\in s_i} y_{ij}, \quad \mathbf{l}'_i = \mathbf{c}'_{ir}\mathbf{X}_r = \sum_{j\in r_i} \mathbf{x}'_{ij}$$

and (under the nested error regression model) $\mathbf{m}'_i = \mathbf{c}'_{ir}\mathbf{Z}_r$ is a row vector with $N_i - n_i$ in the ith entry and zeros elsewhere, we obtain the BLUP estimator

$$\widehat{Y}_{i,BLUP} = \sum_{j\in s_i} y_{ij} + (\sum_{j\in r_i} \mathbf{x}'_{ij})\widehat{\boldsymbol\beta} + (N_i - n_i)\tilde{u}_i \tag{4.2}$$

seen already in (2.21).

In practice the covariance matrix $\mathbf{V}$ is unknown and needs to be estimated. In the estimation we prefer the REML method. The "REML estimator" $\widehat{\boldsymbol\beta}_{REML}$ of $\boldsymbol\beta$ is then obtained by replacing the $\mathbf{V}_s$ in the GLS formula with its REML estimate $\widehat{\mathbf{V}}_{s,REML}$. The estimator $\widehat{\boldsymbol\beta}_{REML}$ is unbiased under mild conditions (Kackar and Harville 1981). The empirical BLUP of Y_i under REML estimation is

$$\begin{aligned}\widehat{Y}_{i,EBLUP} &= \mathbf{c}'_{is}\mathbf{y}_s + \mathbf{c}'_{ir}[\mathbf{X}_r\widehat{\boldsymbol\beta}_{REML} + \widehat{\mathbf{V}}_{rs,REML}\widehat{\mathbf{V}}^{-1}_{s,REML}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol\beta}_{REML})] \\ &= \sum_{j\in s_i} y_{ij} + (\sum_{j\in r_i} \mathbf{x}'_{ij})\widehat{\boldsymbol\beta}_{REML} + (N_i - n_i)\widehat{u}_i, \end{aligned} \tag{4.3}$$

where $\widehat{u}_i$ is the empirical BLUP of u_i obtained from

$$\widehat{\mathbf{u}} = \widehat{\mathbf{G}}_{REML}\mathbf{Z}'_s\widehat{\mathbf{V}}^{-1}_{s,REML}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol\beta}_{REML}),$$

the EBLUP of $\mathbf{u}$ under REML estimation.

4.4 Mean squared error

Consider first the BLUP estimator (4.2), where $\mathbf{V}$ is assumed known. The estimation error is

$$\begin{aligned}\widehat{Y}_{i,BLUP} - Y_i &= \mathbf{c}'_{is}\mathbf{y}_s + \mathbf{c}'_{ir}\mathbf{X}_r\widehat{\boldsymbol\beta} + \mathbf{c}'_{ir}\mathbf{Z}_r\tilde{\mathbf{u}} - \mathbf{c}'_{is}\mathbf{y}_s - \mathbf{c}'_{ir}\mathbf{y}_r \\ &= (\sum_{j\in r_i} \mathbf{x}'_{ij})\widehat{\boldsymbol\beta} + (N_i - n_i)\tilde{u}_i - \sum_{j\in r_i}(\mathbf{x}'_{ij}\boldsymbol\beta + u_i + e_{ij}) \\ &= (\sum_{j\in r_i} \mathbf{x}'_{ij})(\widehat{\boldsymbol\beta} - \boldsymbol\beta) + (N_i - n_i)(\tilde{u}_i - u_i) - \sum_{j\in r_i} e_{ij}.\end{aligned}$$

The BLUP estimator is model unbiased since the model-based expected value of the estimation error is $E(\widehat{Y}_{i,BLUP} - Y_i) = 0$. Hence the model-based mean squared error of the estimator is

$$\begin{aligned}MSE(\widehat{Y}_{i,BLUP}) &= E(\widehat{Y}_{i,BLUP} - Y_i)^2 = Var(\widehat{Y}_{i,BLUP} - Y_i) \\ &= \left[\sum_{j\in r_i} \mathbf{x}'_{ij} \quad N_i - n_i\right] Cov\begin{bmatrix}\widehat{\boldsymbol\beta} - \boldsymbol\beta \\ \tilde{u}_i - u_i\end{bmatrix}\begin{bmatrix}\sum_{j\in r_i} \mathbf{x}'_{ij} \\ N_i - n_i\end{bmatrix} + Var(\sum_{j\in r_i} e_{ij}) \\ &= \left[\mathbf{l}'_i \quad \mathbf{m}'_i\right] Cov\begin{bmatrix}\widehat{\boldsymbol\beta} - \boldsymbol\beta \\ \tilde{\mathbf{u}} - \mathbf{u}\end{bmatrix}\begin{bmatrix}\mathbf{l}_i \\ \mathbf{m}_i\end{bmatrix} + (N_i - n_i)\sigma_e^2,\end{aligned} \tag{4.4}$$

where the joint covariance matrix of $\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}$ and $\tilde{\mathbf{u}} - \mathbf{u}$ is given in (3.20) or (3.21).

The first term in (4.4) is the variance of prediction error $(\mathbf{l}_i'\widehat{\boldsymbol{\beta}} + \mathbf{m}_i'\tilde{\mathbf{u}}) - (\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u})$. It is a common practice to write this variance as a decomposition

$$\begin{bmatrix}\mathbf{l}_i' & \mathbf{m}_i'\end{bmatrix} Cov \begin{bmatrix}\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \tilde{\mathbf{u}} - \mathbf{u}\end{bmatrix} \begin{bmatrix}\mathbf{l}_i \\ \mathbf{m}_i\end{bmatrix} = g_{1i}(\boldsymbol{\sigma}) + g_{2i}(\boldsymbol{\sigma}),$$

where

$$g_{1i}(\boldsymbol{\sigma}) = \mathbf{m}_i'(\mathbf{G} - \mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}\mathbf{Z}_s\mathbf{G})\mathbf{m}_i$$

and

$$g_{2i}(\boldsymbol{\sigma}) = (\mathbf{l}_i' - \mathbf{m}_i'\mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{l}_i - \mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{Z}_s\mathbf{G}\mathbf{m}_i)$$

(recall that both $\mathbf{V}_s$ and $\mathbf{G}$ are functions of variance parameters $\boldsymbol{\sigma}$, here $\boldsymbol{\sigma} = (\sigma_u^2, \sigma_e^2)$). The decomposition is obtained by a straightforward matrix algebra and it has a nice interpretation as seen below. We use the subscript "s" with matrices $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{V}$ to emphasize that they are now associated with the observed sample data.

Firstly, we note that if both $\boldsymbol{\beta}$ and $\mathbf{V}$ were known, the best predictor of $\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u}$ is $\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u}_0$, where $\mathbf{u}_0 = \mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\boldsymbol{\beta})$ is the best predictor of the unobserved $\mathbf{u}$. We find that in this case the variance of prediction error $(\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u}_0) - (\mathbf{l}_i'\boldsymbol{\beta} + \mathbf{m}_i'\mathbf{u}) = \mathbf{m}_i'\mathbf{u}_0 - \mathbf{m}_i'\mathbf{u}$ is equal to $g_{1i}(\boldsymbol{\sigma})$ and it arises from the uncertainty about $\mathbf{u}$.

Secondly, we note that

$$g_{2i}(\boldsymbol{\sigma}) = Var[(\mathbf{l}_i' - \mathbf{m}_i'\mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})]$$

since $Cov(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = (\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}$. Thus, $g_{2i}(\boldsymbol{\sigma})$ arises from the uncertainty about $\boldsymbol{\beta}$.

If we define

$$g_{4i}(\boldsymbol{\sigma}) = (N_i - n_i)\sigma_e^2,$$

we have the decomposition

$$MSE(\widehat{Y}_{i,BLUP}) = g_{1i}(\boldsymbol{\sigma}) + g_{2i}(\boldsymbol{\sigma}) + g_{4i}(\boldsymbol{\sigma}) \qquad (4.5)$$

for the BLUP estimator of Y_i. It shows that the model-based MSE of BLUP estimator based on linear mixed model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$ emerges from three additive sources:

(i) $g_{1i}(\boldsymbol{\sigma})$, the uncertainty about random effects $\mathbf{u}$

(ii) $g_{2i}(\boldsymbol{\sigma})$, the uncertainty about fixed effects $\boldsymbol{\beta}$

(iii) $g_{4i}(\boldsymbol{\sigma})$, the uncertainty about the individual y_{ij}'s in r_i

The term $g_{4i}(\boldsymbol{\sigma})$ measures the unit-level error variation in predicting the y_{ij}'s in r_i. This random variation would be present even if $\boldsymbol{\beta}$, $\mathbf{V}$ and $\mathbf{u}$ were all known.

Consider then the EBLUP estimator (4.3). It is of the same form as the BLUP estimator (4.2), but the unknown covariance matrices $\mathbf{V} = \mathbf{V}(\boldsymbol{\sigma})$ and $\mathbf{G} = \mathbf{G}(\boldsymbol{\sigma})$ have been replaced with their REML (say) estimates $\widehat{\mathbf{V}} = \mathbf{V}(\widehat{\boldsymbol{\sigma}})$ and $\widehat{\mathbf{G}} = \mathbf{G}(\widehat{\boldsymbol{\sigma}})$. Also the EBLUP estimator is model unbiased.

According to the results of Kackar and Harville (1984), introduced already in Section 3.4.3,

$$MSE(\widehat{Y}_{i,EBLUP}) = MSE(\widehat{Y}_{i,BLUP}) + E(\widehat{Y}_{i,EBLUP} - \widehat{Y}_{i,BLUP})^2$$

holds under normality of $\mathbf{y}$ for translation-invariant (e.g. REML and ML) estimators of $\boldsymbol{\sigma}$. The expectation $E(\widehat{Y}_{i,EBLUP} - \widehat{Y}_{i,BLUP})^2$ is generally intractable, but it can be approximated with

$$E(\widehat{Y}_{i,EBLUP} - \widehat{Y}_{i,BLUP})^2 \approx tr\left[Cov\left(\frac{\partial \widehat{Y}_{i,BLUP}}{\partial \boldsymbol{\sigma}}\right) Cov(\widehat{\boldsymbol{\sigma}})\right]$$

(cf. (3.34)). A commonly used further approximation of $E(\widehat{Y}_{i,EBLUP} - \widehat{Y}_{i,BLUP})^2$ is the Prasad-Rao approximation

$$g_{3i}(\boldsymbol{\sigma}) = tr[(\partial \mathbf{m}_i' \mathbf{G} \mathbf{Z}_s' \mathbf{V}_s^{-1} / \partial \boldsymbol{\sigma})' \mathbf{V}_s (\partial \mathbf{m}_i' \mathbf{G} \mathbf{Z}_s' \mathbf{V}_s^{-1} / \partial \boldsymbol{\sigma}) Cov(\widehat{\boldsymbol{\sigma}})],$$

which is justified by noting that $\widehat{\boldsymbol{\beta}}$ can be regarded as fixed for the order of the considered approximation (Prasad and Rao 1990; Singh, Stukel and Pfeffermann 1998).

The approximate MSE of the EBLUP estimator $\widehat{Y}_{i,EBLUP}$ can now be expressed as the four-term decomposition

$$MSE(\widehat{Y}_{i,EBLUP}) \approx g_{1i}(\boldsymbol{\sigma}) + g_{2i}(\boldsymbol{\sigma}) + g_{3i}(\boldsymbol{\sigma}) + g_{4i}(\boldsymbol{\sigma}). \tag{4.6}$$

The additional term $g_{3i}(\boldsymbol{\sigma})$ arises from the uncertainty about variance parameters $\boldsymbol{\sigma}$, which is not present in the BLUP estimation.

The estimation of the approximate MSE (4.6) has been studied e.g. by Prasad and Rao (1990), Harville and Jeske (1992) and Datta and Lahiri (2000). Datta and Lahiri consider the ML and REML methods in estimating $\boldsymbol{\sigma}$ and show that under a general normal linear mixed model (3.2) with linear covariance structure (3.28) and some regularity conditions

$$\begin{aligned} E(g_{1i}(\widehat{\boldsymbol{\sigma}})) &\approx g_{1i}(\boldsymbol{\sigma}) - g_{3i}(\boldsymbol{\sigma}) \\ E(g_{2i}(\widehat{\boldsymbol{\sigma}})) &\approx g_{2i}(\boldsymbol{\sigma}) \\ E(g_{3i}(\widehat{\boldsymbol{\sigma}})) &\approx g_{3i}(\boldsymbol{\sigma}), \end{aligned}$$

when $\widehat{\boldsymbol{\sigma}}$ is the approximately unbiased REML estimator of $\boldsymbol{\sigma}$. If the ML method is employed, a correction term accounting for the bias in estimates of $\boldsymbol{\sigma}$ should be added to $g_{1i}(\widehat{\boldsymbol{\sigma}})$. The neglected terms in the approximate expected values are all of order $o(m^{-1})$, where m is the number of areas (Datta and Lahiri 2000).

An estimator of the approximate MSE of the EBLUP estimator is then

$$\widehat{MSE}(\widehat{Y}_{i,EBLUP}) = g_{1i}(\widehat{\boldsymbol{\sigma}}) + g_{2i}(\widehat{\boldsymbol{\sigma}}) + 2g_{3i}(\widehat{\boldsymbol{\sigma}}) + g_{4i}(\widehat{\boldsymbol{\sigma}}). \tag{4.7}$$

It is approximately unbiased under normal linear mixed models with a linear covariance structure, if the variance parameters are estimated by REML. The restriction to linear covariance structures can be problematic in longitudinal or spatial applications. Singh et al. (2005) have considered the MSE estimation with area level models, where this assumption is relaxed.

Simulation studies for exploring the performance of the estimator (4.7) have been carried out e.g. by Prasad and Rao (1990), Datta and Ghosh (1991), Hulting and Harville (1991) and Singh et al. (1998). According to these authors the estimator (4.7) performs mainly well. Considerable positive bias may occur, when m is small and the area variance σ_u^2 is very small compared to the unit-level error variance σ_e^2 (Singh et al. 1998).

According to the simulation results of Moura and Holt (1999) and the empirical results of EURAREA Consortium (2004) (with various unit-level models) the g_{1i} term contributes generally over 90 % of the estimated MSE. The simulation study of our research (Chapters 8–12) supports these findings. The proportion of g_{1i} is typically at least 85 %, often over 95 %, depending on data, model and area size. This suggests that the variation of small area estimates is mostly related to the uncertainty about the latent area effects (and other possible random effects in the model). On the other hand, the term that contributes least to the MSE is g_{4i}, the share of which is hardly ever over 1 %. In our study the g_{3i} term associated with the estimation of covariance parameters contributes typically no more than 2 %. The contribution of the g_{2i} term, associated with fixed effects, seems to depend much on situation. With cross-sectional or complete panel data the contribution is typically only 1–4 %. In rotating panel applications it is usually between 5 and 14 %, and increases with area size. For large areas it may appear as high as 50 %, reducing the share of the g_{1i} term correspondingly.

5 Mixed model for unit level panel data

5.1 Notation for finite panel population

Consider areas $i = 1, 2, \dots, m$, and a series of time points $t = 1, 2, \dots, T$. Let U_{it} denote the finite population of area i at time point t consisting of units $j = 1, 2, \dots, N_{it}$. We define the longitudinal population of area i as

$$U_i^* = U_{i1} \cup U_{i2} \cup \cdots \cup U_{iT}$$

and call it panel population. It contains a total number of

$$N_i^* = \sum_{t=1}^{T} N_{it}$$

"observational elements", in which we count the individual repeated observations on unit j in area i over the T time points.

The adopted notation system, which is rather complicated, is as follows. When we consider a cross-sectional population (with no repeated measurements), we denote it by U, and when we consider a panel population (with T repeated measurements), we denote it by U^*. Correspondingly, when we denote the cross-sectional number of separate units j, we use N, and when we denote the number of observational elements in a panel population, we use N^*. In addition, we will use the letter M to denote the total number of units in panel populations.

For each time point t we define the joint cross-sectional population as

$$U_t = U_{1t} \cup U_{2t} \cup \cdots \cup U_{mt},$$

the size of which is

$$N_t = \sum_{i=1}^{m} N_{it}.$$

The overall panel population is the union

$$\begin{aligned} U^* &= U_1^* \cup U_2^* \cup \cdots \cup U_m^* \\ &= U_1 \cup U_2 \cup \cdots \cup U_T \end{aligned}$$

and it consists of

$$N^* = \sum_{i=1}^{m} N_i^* = \sum_{t=1}^{T} N_t$$

observational elements. A tabular summary of this notation is given in Table 5.1.

Table 5.1. The notation for population sizes.

area	time 1	2	...	T	total
1	N_{11}	N_{12}	...	N_{1T}	N_1^*
2	N_{21}	N_{22}	...	N_{2T}	N_2^*
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
m	N_{m1}	N_{m2}	...	N_{mT}	N_m^*
total	N_1	N_2	...	N_T	N^*

In the course of time some units usually leave the population and other units enter it, which implies that the populations U_{it} and their sizes N_{it} do not keep the same over the time points and the number of observational elements is not the same for all units in the area. We denote the number of observational elements (i.e. repeated observations) on unit j from area i by T_{ij}, where $1 \leq T_{ij} \leq T$.

The total "cross-sectional" number of separate units j, which appear in the panel population U_i^* (meaning that they appear in U^* on at least one occasion t) is denoted by M_i. Thus, in introducing the mixed model for the panel data we will generally have $j = 1, 2, \ldots, M_i$. This notation is illustrated with the example in Table 5.2. We also note that

$$N_i^* = \sum_{j=1}^{N_i} T_{ij}.$$

Table 5.2. A panel population U_i^* of area i with $M_i = 7$ units and $T = 5$ time points. The "X" marks an observational element.

unit	time 1	2	3	4	5	
1	X	X				$T_{i1} = 2$
2	X	X	X	X		$T_{i2} = 4$
3	X	X	X	X	X	$T_{i3} = 5$
4	X	X	X	X	X	$T_{i4} = 5$
5	X	X	X	X	X	$T_{i5} = 5$
6		X	X	X	X	$T_{i6} = 4$
7				X	X	$T_{i7} = 2$
	$N_{i1} = 5$	$N_{i2} = 6$	$N_{i3} = 5$	$N_{i4} = 6$	$N_{i5} = 5$	$N_i^* = 27$

We denote the number of separate units in the overall panel population U^* by

$$M = \sum_{i=1}^{m} M_i.$$

5.2 Mixed model formulation

Let y denote the response (or target variable) and consider p covariates (or auxiliary variables) $\mathbf{x} = (x_1, x_2, \ldots, x_p)$.

Define the following linear mixed model for the random observation y_{ijt} of unit j in area i at time t:

$$y_{ijt} = \mu_t + \mathbf{x}'_{ijt}\boldsymbol{\beta} + u_i + v_{ij} + e_{ijt}, \tag{5.1}$$

where u_i is the random area effect, v_{ij} is the random unit effect and e_{ijt} is the error term. We recall that (5.1) is essentially similar to the two-fold nested error regression model of Stukel and Rao (1999). In the model equation we distinguish the $p \times 1$ vector $\boldsymbol{\beta}$ of fixed regression coefficients of $\mathbf{x}$ variables and the fixed time-specific intercept μ_t. In this model we assume that $\boldsymbol{\beta}$ keeps constant over time. Instead, the intercept term is allowed to vary due to the time effect. This is equivalent to introducing a fixed categorical time factor with T levels, which in turn is equivalent to introducing orthogonal polynomials for trends up to order $T-1$. We prefer the categorical time effect (to the linear trend, for instance), because it gives a "saturated" fit for response means at different time points, thus helping us to avoid unnecessary bias in time-specific small area estimates.

Later in this text, however, we do not separate μ_t's from the regression coefficients, except when necessary. Instead, we regard μ_t's ($t = 1, 2, \ldots, T$) as elements of the fixed parameter vector $\boldsymbol{\beta}$ and let the vector $\mathbf{x}$ contain also the corresponding dummy variables, which pick the specific time point t out of the time points $1, 2, \ldots, T$.

From the perspective of multilevel models (Goldstein 1995) the model (5.1) is a three-level model with residual e_{ijt} at level 1 (observation or measurement level), unit effect v_{ij} at level 2 (individual level) and area effect u_i at level 3 (area level). The random effects u_i, v_{ij} and e_{ijt} are assumed mutually independent with distributions

$$\begin{aligned} u_i &\overset{i.i.d.}{\sim} N(0, \sigma_u^2) \\ v_{ij} &\overset{i.i.d.}{\sim} N(0, \sigma_v^2) \\ e_{ijt} &\overset{i.i.d.}{\sim} N(0, \sigma_e^2). \end{aligned}$$

The random effects define the covariance structure of variables y_{ijt}. The considered model is a variance component model with variance components σ_u^2, σ_v^2 and σ_e^2, yielding

$$Var(y_{ijt}) = \sigma_u^2 + \sigma_v^2 + \sigma_e^2.$$

The random area and unit effects are needed also in making the model to allow for the intra-area and intra-unit correlation among y_{ijt}'s. This is advisable since the units coming from the same area often tend to be homogeneous and there usually exists a strong covariance between the repeated observations from the same unit. However, in certain cases, especially when the number of areas is small, the area effect could also be considered fixed. This is equivalent to assuming no intra-area correlation.

In some applications the fixed time effect has been replaced with a random one. Then the observations coming from the same time point would be correlated (see Section 2.6).

In the matrix formulation of the model we define the longitudinal response vector of unit j in area i, where $j = 1, 2, \ldots, M_i$, as

$$\mathbf{y}_{ij} = \begin{bmatrix} y_{ij1} \\ y_{ij2} \\ \vdots \\ y_{ijT_{ij}} \end{bmatrix} \qquad (T_{ij} \times 1).$$

The model equation for $\mathbf{y}_{ij}$ takes the form

$$\mathbf{y}_{ij} = \mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{1}_{T_{ij}}u_i + \mathbf{1}_{T_{ij}}v_{ij} + \mathbf{e}_{ij}, \tag{5.2}$$

where

$$\mathbf{X}_{ij} = \begin{bmatrix} \mathbf{x}'_{ij1} \\ \mathbf{x}'_{ij2} \\ \vdots \\ \mathbf{x}'_{ijT_{ij}} \end{bmatrix} \qquad (T_{ij} \times p)$$

and

$$\mathbf{e}_{ij} = \begin{bmatrix} e_{ij1} \\ e_{ij2} \\ \vdots \\ e_{ijT_{ij}} \end{bmatrix} \qquad (T_{ij} \times 1).$$

The vector $\mathbf{1}_{T_{ij}}$ is the unity vector of T_{ij} elements.

Collecting the vectors $\mathbf{y}_{ij}$ into one panel population vector of area i gives

$$\mathbf{y}_{i} = \begin{bmatrix} \mathbf{y}_{i1} \\ \mathbf{y}_{i2} \\ \vdots \\ \mathbf{y}_{iM_i} \end{bmatrix} \qquad (N_i^* \times 1),$$

and the model equation for $\mathbf{y}_i$ is

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_{1i}u_i + \mathbf{Z}_{2i}\mathbf{v}_i + \mathbf{e}_i, \tag{5.3}$$

where

$$\mathbf{X}_{i} = \begin{bmatrix} \mathbf{X}_{i1} \\ \mathbf{X}_{i2} \\ \vdots \\ \mathbf{X}_{iM_i} \end{bmatrix}, \qquad (N_i^* \times p)$$

$$\mathbf{Z}_{1i} = \mathbf{1}_{N_i^*} = \begin{bmatrix} \mathbf{1}_{T_{i1}} \\ \mathbf{1}_{T_{i2}} \\ \vdots \\ \mathbf{1}_{T_{iM_i}} \end{bmatrix}, \qquad (N_i^* \times 1)$$

$$\mathbf{Z}_{2i} = \oplus_{j=1}^{M_i} \mathbf{1}_{T_{ij}} = \begin{bmatrix} \mathbf{1}_{T_{i1}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{T_{i2}} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{T_{iM_i}} \end{bmatrix}, \qquad (N_i^* \times M_i)$$

$$\mathbf{v}_i = \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iM_i} \end{bmatrix} \qquad (M_i \times 1)$$

and

$$\mathbf{e}_i = \begin{bmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \\ \vdots \\ \mathbf{e}_{iM_i} \end{bmatrix}. \qquad (N_i^* \times 1)$$

By uniting the u_i and $\mathbf{v}_i$ into one random effect vector $\mathbf{u}_i$ of area i we can write the model equation (5.3) as

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{u}_i + \mathbf{e}_i \tag{5.4}$$

with

$$\mathbf{Z}_i = [\mathbf{Z}_{1i} \quad \mathbf{Z}_{2i}] = \begin{bmatrix} \mathbf{1}_{T_{i1}} & \mathbf{1}_{T_{i1}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{1}_{T_{i2}} & \mathbf{0} & \mathbf{1}_{T_{i2}} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{1}_{T_{iM_i}} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{T_{iM_i}} \end{bmatrix} \qquad (N_i^* \times (1 + M_i))$$

and

$$\mathbf{u}_i = \begin{bmatrix} u_i \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_{i1} \\ v_{i2} \\ \vdots \\ v_{iM_i} \end{bmatrix}. \qquad ((1 + M_i) \times 1)$$

Finally, let

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_m \end{bmatrix} \qquad (N^* \times 1)$$

be the response vector of the overall panel population and write

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e} \tag{5.5}$$

with

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{bmatrix}, \qquad (N^* \times p)$$

$$\mathbf{Z} = \oplus_{i=1}^{m} \mathbf{Z}_i = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{Z}_m \end{bmatrix} \qquad (N^* \times (m+M))$$

and

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} u_1 \\ \mathbf{v}_1 \\ u_2 \\ \mathbf{v}_2 \\ \vdots \\ u_m \\ \mathbf{v}_m \end{bmatrix} \qquad ((m+M) \times 1).$$

5.3 Covariance structure

The model (5.1) is a variance component model with variance components $Var(u_i) = \sigma_u^2$, $Var(v_{ij}) = \sigma_v^2$ and $Var(e_{ijt}) = \sigma_e^2$. It induces a relatively simple covariance structure for the observations y_{ijt}.

The variance of observation y_{ijt} is

$$Var(y_{ijt}) = \sigma_u^2 + \sigma_v^2 + \sigma_e^2.$$

This is constant over measurements $t = 1, 2, \dots, T$; an assumption which is not always realistic, however.

The covariance of two observations y_{ijt} and $y_{ijt'}$ from the same unit ij at different time points $(t \neq t')$ is

$$Cov(y_{ijt}, y_{ijt'}) = \sigma_u^2 + \sigma_v^2.$$

The corresponding intra-unit correlation or autocorrelation between two arbitrary measurements for unit ij, is then a constant

$$\rho_{ij} = \frac{\sigma_u^2 + \sigma_v^2}{\sigma_u^2 + \sigma_v^2 + \sigma_e^2}. \tag{5.6}$$

Due to the random area effect, the observations of two different units ij and ij' $(j \neq j')$ from the same area i also have the covariance

$$Cov(y_{ijt}, y_{ij't'}) = \sigma_u^2,$$

no matter if $t = t'$ or $t \neq t'$. The intra-area correlation, due to the clustering effect of area, is now

$$\rho_i = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2 + \sigma_e^2}. \tag{5.7}$$

Units (and observations) coming from different areas i and i' are not correlated.

According to the model the correlation between two repeated measurements on the same unit has a constant value. The covariance structure of uniform correlations is known as compound symmetry. In terms of the above variance components the covariance matrix of the y observations of unit ij is

$$Cov(\mathbf{y}_{ij}) = \mathbf{V}_{ij} = \begin{bmatrix} \sigma_u^2 + \sigma_v^2 + \sigma_e^2 & \sigma_u^2 + \sigma_v^2 & \dots & \sigma_u^2 + \sigma_v^2 \\ \sigma_u^2 + \sigma_v^2 & \sigma_u^2 + \sigma_v^2 + \sigma_e^2 & \dots & \sigma_u^2 + \sigma_v^2 \\ \vdots & \vdots & & \vdots \\ \sigma_u^2 + \sigma_v^2 & \sigma_u^2 + \sigma_v^2 & \dots & \sigma_u^2 + \sigma_v^2 + \sigma_e^2 \end{bmatrix} \quad (T_{ij} \times T_{ij}).$$

Other covariance structures for the repeated measurements could also be considered. One popular alternative is the first-order autoregressive structure AR(1), which lets the covariance decay as the distance between measurements increases. AR(1) is often found realistic in repeated measurement studies and is not difficult to specify within the framework of linear mixed models. For instance, we can introduce a model

$$y_{ijt} = \mu_t + \mathbf{x}'_{ijt}\boldsymbol{\beta} + u_i + e_{ijt}, \tag{5.8}$$

which differs from the model (5.1) by the omission of unit effect v_{ij} and letting the error terms e_{ijt} from the same unit ij correlate according to the AR(1) structure. This means that the vectors

$$\mathbf{e}_{ij} = \begin{bmatrix} e_{ij1} & e_{ij2} & \dots & e_{ijT_{ij}} \end{bmatrix}'$$

of errors follow now independently a multivariate normal distribution $N_{T_{ij}}(\mathbf{0}, \mathbf{R})$, where the covariance matrix $\mathbf{R}$ has the AR(1) form

$$\mathbf{R} = \sigma_e^2 \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{T_{ij}-1} \\ \rho & 1 & \rho & \dots & \rho^{T_{ij}-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{T_{ij}-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho^{T_{ij}-1} & \rho^{T_{ij}-2} & \rho^{T_{ij}-3} & \dots & 1 \end{bmatrix} \quad (T_{ij} \times T_{ij}).$$

The non-zero autocorrelation parameter ρ compensates for the omission of unit effects. For $\mathbf{y}_{ij}$ the model (5.8) gives a covariance matrix

$$Cov(\mathbf{y}_{ij}) = \mathbf{V}_{ij} = \begin{bmatrix} \sigma_u^2 + \sigma_e^2 & \sigma_u^2 + \rho\sigma_e^2 & \dots & \sigma_u^2 + \rho^{T_{ij}-1}\sigma_e^2 \\ \sigma_u^2 + \rho\sigma_e^2 & \sigma_u^2 + \sigma_e^2 & \dots & \sigma_u^2 + \rho^{T_{ij}-2}\sigma_e^2 \\ \vdots & \vdots & & \vdots \\ \sigma_u^2 + \rho^{T_{ij}-1}\sigma_e^2 & \sigma_u^2 + \rho^{T_{ij}-2}\sigma_e^2 & \dots & \sigma_u^2 + \sigma_e^2 \end{bmatrix} \qquad (T_{ij} \times T_{ij}).$$

However, we do not apply the AR(1) model here, in spite of some appealing properties of it. The reason is that in estimation of small area totals or means by linear mixed models, the existing estimators of the approximate MSE (originating from seminal papers of Kackar and Harville (1984) and Prasad and Rao (1990)) meet problems. To be precise, their bias correction terms are valid only for covariance structures of linear form (3.28), which make the second derivatives in the Taylor series expansion vanish (Datta and Lahiri 2000, Rao 2003 p. 109, also Kenward and Roger 1997). The AR(1) structure does not satisfy this requirement because of the powers of ρ.

Going back to the three-level variance component model (5.1) we note that since the area effects u_i are assumed independent, the covariance matrix $\mathbf{V}$ of the overall population vector $\mathbf{y}$ is block-diagonal:

$$Cov(\mathbf{y}) = \mathbf{V} = \oplus_{i=1}^{m}\mathbf{V}_i = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_m \end{bmatrix} \qquad (N^* \times N^*),$$

where $\mathbf{V}_i$ is the covariance matrix of data vector $\mathbf{y}_i$ of area i. Using the notation of models (5.3) and (5.4) the $N_i^* \times N_i^*$ covariance matrix $\mathbf{V}_i$ is

$$\begin{aligned} Cov(\mathbf{y}_i) = \mathbf{V}_i &= \mathbf{J}_{N_i^*}\sigma_u^2 + \mathbf{Z}_{2i}\mathbf{Z}_{2i}'\sigma_v^2 + \mathbf{I}_{N_i^*}\sigma_e^2 \\ &= \mathbf{Z}_i\mathbf{G}_i\mathbf{Z}_i' + \mathbf{R}_i, \end{aligned}$$

where $\mathbf{J}_{N_i^*}$ is the $N_i^* \times N_i^*$ matrix of ones, $\mathbf{Z}_i$ is given with (5.4),

$$\mathbf{G}_i = Cov(\mathbf{u}_i) = \begin{bmatrix} \sigma_u^2 & \mathbf{0}' \\ \mathbf{0} & \sigma_v^2\mathbf{I}_{M_i} \end{bmatrix} \qquad ((1 + M_i) \times (1 + M_i))$$

and

$$\mathbf{R}_i = \sigma_e^2\mathbf{I}_{N_i^*}.$$

We note that the covariance matrix $\mathbf{V}_{ij}$ of data vector $\mathbf{y}_{ij}$ can be written as

$$Cov(\mathbf{y}_{ij}) = \mathbf{J}_{T_{ij}}\sigma_u^2 + \mathbf{J}_{T_{ij}}\sigma_v^2 + \mathbf{I}_{T_{ij}}\sigma_e^2.$$

We also note that the covariance matrix of vectors $\mathbf{y}_{ij}$ and $\mathbf{y}_{ij'}$, where $j \neq j'$, is

$$Cov(\mathbf{y}_{ij}, \mathbf{y}_{ij'}) = \mathbf{1}_{T_{ij}}\mathbf{1}_{T_{ij'}}'\sigma_u^2.$$

Thus, the area covariance matrix $\mathbf{V}_i$ is not block-diagonal.

6 Small area estimation with mixed model for panel data

6.1 Notation for sample data

Assume then that at each time point $t = 1, 2, \ldots, T$ a random sample s_t is drawn from the cross-sectional population U_t and all the resulting samples $s_1, s_2, \ldots, s_T$ contain the same units, thus forming a complete panel sample data

$$s^* = s_1 \cup s_2 \cup \cdots \cup s_T$$

with T repeated observations for every unit in the data. Each cross-sectional sample s_t divides into m areas so that

$$s_t = s_{1t} \cup s_{2t} \cup \cdots \cup s_{mt},$$

where s_{it} is the sample data set of area i at time t. On the other hand, the panel data s^* can be regarded as the union

$$s^* = s_1^* \cup s_2^* \cup \cdots \cup s_m^*$$

of the regional panel sample data sets

$$s_i^* = s_{i1} \cup s_{i2} \cup \cdots \cup s_{iT}, \quad i = 1, 2, \ldots, m.$$

A key point is here that the sampling design is typically specified for national purposes and does not necessarily correspond well to the regional division in the population. For instance, the strata can be constructed with no respect to small areas, which may appear to be of interest. The sizes of regional panel data sets $s_1^*, \ldots, s_m^*$ are then random, and as a consequence they can become very small, which gives rise to the small area estimation problem.

Note that we assume here that the sampled units stay in their population and do not drop out during the period when the observations are collected. This assumption pertains only to the sample data, we do not require that the populations U_{it} or their sizes N_{it} should keep constant over time. Also the assumption of constant samples will be explicitly relaxed later in the context of rotating panel data.

We may denote the size of the regional sample s_{it} by n_{it}. However, in complete panel data the sample size n_{it} keeps the same for every time point t, which allows us to write $n_{it} = n_i$ for every i and t. We denote the size of the aggregated sample s_t at time t by

$$n = \sum_{i=1}^{m} n_i$$

for every $t = 1, 2, \ldots, T$.

The total number of observations in panel data s_i^* of area i is

$$n_i^* = \sum_{t=1}^{T} n_{it} = n_i T$$

and in the whole sample data s^*

$$n^* = nT.$$

A summary of this sample size notation is given in Table 6.1.

Table 6.1. The notation for sample sizes in the case of complete panel data.

	time point				
area	1	2	...	T	total
1	n_1	n_1	...	n_1	$n_1 T$
2	n_2	n_2	...	n_2	$n_2 T$
⋮	⋮	⋮		⋮	⋮
m	n_m	n_m	...	n_m	$n_m T$
total	n	n	...	n	nT

Denote the set of non-sampled units (remainder) of U_{it} by r_{it}, that is, $U_{it} = s_{it} \cup r_{it}$. The non-sample size of area i at time t is then

$$N_{r_{it}} = N_{it} - n_{it} = N_{it} - n_i$$

for every i and t. Correspondingly, $U_t = s_t \cup r_t$ and the size of r_t is

$$N_{r_t} = N_t - n$$

for every t.

6.2 BLUP and EBLUP estimation of area total

First we arrange and partition the panel population U^* into the sample s^* and the non-sample r^* in the same way as in the case of the cross-sectional model (Section 4.3). For brevity, we use in the following formulas subscripts s and r instead of s^* and r^*, respectively, even though they refer to panel data sets. Now the sample parts of the response vector $\mathbf{y}$ and the other data matrices and vectors in model (5.5) have nT rows and the non-sample parts respectively $N^* - nT$ rows. The model equation is

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_s \\ \mathbf{y}_r \end{bmatrix} = \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \mathbf{Z}_s \\ \mathbf{Z}_r \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{e}_s \\ \mathbf{e}_r \end{bmatrix}.$$

The corresponding partition of the covariance matrix $Cov(\mathbf{y}) = \mathbf{V}$ is

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_s & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_r \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_s\mathbf{G}\mathbf{Z}_s' & \mathbf{Z}_s\mathbf{G}\mathbf{Z}_r' \\ \mathbf{Z}_r\mathbf{G}\mathbf{Z}_s' & \mathbf{Z}_r\mathbf{G}\mathbf{Z}_r' \end{bmatrix} + \begin{bmatrix} \sigma_e^2\mathbf{I}_{nT} & \mathbf{0} \\ \mathbf{0} & \sigma_e^2\mathbf{I}_{N^*-nT} \end{bmatrix},$$

where

$$\mathbf{G} = Cov(\mathbf{u}) = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{G}_m \end{bmatrix} \qquad ((M+m) \times (M+m)).$$

The target is to estimate the total

$$Y_{it} = \sum_{j \in U_{it}} y_{ijt} = \mathbf{c}_{it}'\mathbf{y},$$

where the vector $\mathbf{c}_{it}$ of ones and zeros is such that the ones pick the y_{ijt}'s, which belong to area i at time t, out of $\mathbf{y}$.

Applying the partition to $\mathbf{c}_{it}$ yields

$$\mathbf{c}_{it} = \begin{bmatrix} \mathbf{c}_{its} \\ \mathbf{c}_{itr} \end{bmatrix},$$

where $\mathbf{c}_{its}$ picks the units in the sample s_{it} from area i and $\mathbf{c}_{itr}$ picks those in the non-sample r_{it}, and we write the area total to be estimated as

$$Y_{it} = \mathbf{c}_{its}'\mathbf{y}_s + \mathbf{c}_{itr}'\mathbf{y}_r.$$

Again, the general prediction theorem gives the BLUP estimator

$$\widehat{Y}_{it,BLUP} = \mathbf{c}_{its}'\mathbf{y}_s + \mathbf{c}_{itr}'[\mathbf{X}_r\widehat{\boldsymbol{\beta}} + \mathbf{V}_{rs}\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}})]$$

with the GLS estimator $\widehat{\boldsymbol{\beta}} = (\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{y}_s$. The formula for BLUP estimator simplifies to

$$\widehat{Y}_{it,BLUP} = \sum_{j \in s_{it}} y_{ijt} + (\sum_{j \in r_{it}} \mathbf{x}_{ijt}')\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i + \sum_{j \in r_{it}} \tilde{v}_{ij}, \tag{6.1}$$

where $\tilde{u}_i$ and $\tilde{v}_{ij}$ are the BLUPs of the area effect u_i and the unit effect v_{ij}, respectively, obtained from the BLUP

$$\tilde{\mathbf{u}} = \mathbf{G}\mathbf{Z}_s'\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}})$$

of $\mathbf{u}$. The sample data do not provide information on the non-sampled units, though. Therefore the effects of the units, which belong to r_{it}, are predicted with their expected value zero and the BLUP estimator reduces to

$$\widehat{Y}_{it,BLUP} = \sum_{j \in s_{it}} y_{ijt} + (\sum_{j \in r_{it}} \mathbf{x}_{ijt}')\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i, \tag{6.2}$$

which, in fact, is similar to the BLUP estimator (4.2) of the cross-sectional case.

When the covariance matrix $\mathbf{V}$ is unknown, we again estimate it by the REML method. The empirical BLUP of Y_{it} under REML estimation is now

$$\widehat{Y}_{it,EBLUP} = \sum_{j \in s_{it}} y_{ijt} + (\sum_{j \in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}}_{REML} + N_{r_{it}}\widehat{u}_i, \tag{6.3}$$

where $\widehat{\boldsymbol{\beta}}_{REML}$ and the EBLUP $\widehat{u}_i$ are obtained by substituting the known covariance matrices with their REML estimates in the formulas of $\widehat{\boldsymbol{\beta}}$ and $\tilde{\mathbf{u}}$.

The population sums of auxiliary variables $\mathbf{x}$ in area i at time t must be known here. The non-sample sum $\sum_{j \in r_{it}} \mathbf{x}'_{ijt}$ is then obtained by subtracting the corresponding sample sum from the population sum.

From the computational point of view it is worth emphasizing that when the areas are independent, like here, the covariance matrix $\mathbf{V}$ has a block-diagonal structure

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1\mathbf{G}_1\mathbf{Z}'_1 + \mathbf{R}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2\mathbf{G}_2\mathbf{Z}'_2 + \mathbf{R}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{Z}_m\mathbf{G}_m\mathbf{Z}'_m + \mathbf{R}_m \end{bmatrix}$$

(note that this block-diagonality applies to $\mathbf{G}$ and $\mathbf{Z}$ as well). Then it is practical to compute the BLUP of each area-specific random effect vector

$$\mathbf{u}_i = \begin{bmatrix} u_i \\ \mathbf{v}_i \end{bmatrix}, \qquad i = 1, 2, \dots, m,$$

separately with

$$\tilde{\mathbf{u}}_i = \mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}(\mathbf{y}_{is} - \mathbf{X}_{is}\widehat{\boldsymbol{\beta}}),$$

where $\mathbf{y}_{is}, \mathbf{X}_{is}, \mathbf{Z}_{is}$ and $\mathbf{V}_{is}$ all correspond to the area-specific subset s_i^* of the observed panel data s^*. In the EBLUP case the unknown covariance matrices are of course replaced with their estimates.

6.3 Mean squared error

Consider first the BLUP estimator (6.2), where $\mathbf{V}$ is assumed known. The estimation error is

$$\begin{aligned} \widehat{Y}_{it,BLUP} - Y_{it} &= (\sum_{j \in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i - \sum_{j \in r_{it}} y_{ijt} \\ &= (\sum_{j \in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i - \sum_{j \in r_{it}} (\mathbf{x}'_{ijt}\boldsymbol{\beta} + u_i + v_{ij} + e_{ijt}) \\ &= (\sum_{j \in r_{it}} \mathbf{x}'_{ijt})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + N_{r_{it}}(\tilde{u}_i - u_i) - \sum_{j \in r_{it}} (v_{ij} + e_{ijt}). \end{aligned}$$

Again, the BLUP estimator is model unbiased and the model-based mean squared error of the estimator is

$$
\begin{aligned}
MSE(\widehat{Y}_{it,BLUP}) &= Var(\widehat{Y}_{it,BLUP} - Y_{it}) \\
&= \left[\sum_{j\in r_{it}} \mathbf{x}'_{ijt} \quad N_{r_{it}}\right] Cov \begin{bmatrix} \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \tilde{u}_i - u_i \end{bmatrix} \begin{bmatrix} \sum_{j\in r_{it}} \mathbf{x}'_{ijt} \\ N_{r_{it}} \end{bmatrix} \\
&+ Var\left[\sum_{j\in r_{it}} (v_{ij} + e_{ijt})\right] \\
&= \left[\mathbf{l}'_{it} \quad \mathbf{m}'_{it}\right] Cov \begin{bmatrix} \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \tilde{\mathbf{u}} - \mathbf{u} \end{bmatrix} \begin{bmatrix} \mathbf{l}_{it} \\ \mathbf{m}_{it} \end{bmatrix} + N_{r_{it}}(\sigma_v^2 + \sigma_e^2),
\end{aligned}
$$

where $\mathbf{l}'_{it} = \mathbf{c}'_{itr}\mathbf{X}_r = \sum_{j\in r_{it}} \mathbf{x}'_{ijt}$ and $\mathbf{m}'_{it} = \mathbf{c}'_{itr}\mathbf{Z}_r$, which reduces to a row vector with $N_{r_{it}}$ in the entry corresponding to u_i and zeros elsewhere. Recall that the joint covariance matrix of $\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}$ and $\tilde{\mathbf{u}} - \mathbf{u}$ is given in (3.20) or, equivalently, in (3.21).

As with the cross-sectional model in Section 4.4, we can decompose the MSE as

$$MSE(\widehat{Y}_{it,BLUP}) = g_{1it}(\boldsymbol{\sigma}) + g_{2it}(\boldsymbol{\sigma}) + g_{4it}(\boldsymbol{\sigma}),$$

where $g_{1it}(\boldsymbol{\sigma})$ measures the uncertainty about random effects $\mathbf{u}_i$, $g_{2it}(\boldsymbol{\sigma})$ measures the uncertainty from estimating $\boldsymbol{\beta}$ and $g_{4it}(\boldsymbol{\sigma})$ measures the uncertainty about the individual y_{ijt}'s in r_{it}.

When the variance parameters $\boldsymbol{\sigma}$ are unknown we estimate them. When the employed estimators are at least approximately unbiased (e.g. REML estimator, ANOVA estimator), the approximate MSE of the resulting EBLUP estimator of area total Y_{it} has the form

$$MSE(\widehat{Y}_{it,EBLUP}) \approx g_{1it}(\boldsymbol{\sigma}) + g_{2it}(\boldsymbol{\sigma}) + g_{3it}(\boldsymbol{\sigma}) + g_{4it}(\boldsymbol{\sigma})$$

seen already in (4.6), but the additional term $g_{3it}(\boldsymbol{\sigma})$ measures now the uncertainty about $\boldsymbol{\sigma} = (\sigma_u^2, \sigma_v^2, \sigma_e^2)$. The explicit formulas needed for computing $g_{1it}(\boldsymbol{\sigma})$, $g_{2it}(\boldsymbol{\sigma})$, $g_{3it}(\boldsymbol{\sigma})$ and $g_{4it}(\boldsymbol{\sigma})$ will be given in (6.6)–(6.10).

Estimators of the "g terms" are obtained by replacing $\boldsymbol{\sigma}$ with its (approximately unbiased) REML estimate $\hat{\boldsymbol{\sigma}}$ in above formulas. Recall that under the REML estimation

$$
\begin{aligned}
E(g_{1it}(\hat{\boldsymbol{\sigma}})) &\approx g_{1it}(\boldsymbol{\sigma}) - g_{3it}(\boldsymbol{\sigma}) \\
E(g_{2it}(\hat{\boldsymbol{\sigma}})) &\approx g_{2it}(\boldsymbol{\sigma}) \\
E(g_{3it}(\hat{\boldsymbol{\sigma}})) &\approx g_{3it}(\boldsymbol{\sigma}) \\
E(g_{4it}(\hat{\boldsymbol{\sigma}})) &\approx g_{4it}(\boldsymbol{\sigma}).
\end{aligned}
$$

Thus, an approximately unbiased estimator of the approximate MSE is still

$$\widehat{MSE}(\widehat{Y}_{it,EBLUP}) = g_{1it}(\widehat{\boldsymbol{\sigma}}) + g_{2it}(\widehat{\boldsymbol{\sigma}}) + 2g_{3it}(\widehat{\boldsymbol{\sigma}}) + g_{4it}(\widehat{\boldsymbol{\sigma}}),$$

which is valid under regularity conditions and when covariance matrix $\mathbf{V}(\boldsymbol{\sigma})$ has the linear structure (3.28).

6.4 Computational formulas for MSE

6.4.1 Term g_{1it}

The general formula of $g_{1it}(\boldsymbol{\sigma})$ is

$$g_{1it}(\boldsymbol{\sigma}) = \mathbf{m}'_{it}(\mathbf{G} - \mathbf{G}\mathbf{Z}'_s\mathbf{V}_s^{-1}\mathbf{Z}_s\mathbf{G})\mathbf{m}_{it},$$

where $\mathbf{m}_{it}$ is a $(m+M)\times 1$ vector with $N_{r_{it}}$ in the entry corresponding to u_i and zeros elsewhere. Recall that m is the number of areas and $M = M_1 + M_2 + \cdots + M_m$ is the number of units in these areas. To utilize the block-diagonality of $\mathbf{G}$, $\mathbf{Z}$ and $\mathbf{V}$ we redefine $\mathbf{m}_{it}$ as a $(1+M_i)\times 1$ vector

$$\mathbf{m}_{it} = \begin{bmatrix} N_{r_{it}} & \mathbf{0}' \end{bmatrix}'$$

to get

$$g_{1it}(\boldsymbol{\sigma}) = \mathbf{m}'_{it}(\mathbf{G}_i - \mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}\mathbf{Z}_{is}\mathbf{G}_i)\mathbf{m}_{it},$$

where $\mathbf{G}_i$, $\mathbf{Z}_{is}$ and $\mathbf{V}_{is}$ are the area-specific blocks. Now straightforward calculation gives

$$g_{1it}(\boldsymbol{\sigma}) = N^2_{r_{it}}(\sigma_u^2 - \sigma_u^2\mathbf{1}'_{n_iT}\mathbf{V}_{is}^{-1}\mathbf{1}_{n_iT}\sigma_u^2).$$

Define then

$$\gamma = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2/T}.$$

To find the inverse of covariance matrix $\mathbf{V}_{is}$ we write

$$\begin{aligned} \mathbf{V}_{is} &= \sigma_u^2\mathbf{J}_{n_iT} + \sigma_v^2\left[\oplus_{j=1}^{n_i}\mathbf{J}_T\right] + \sigma_e^2\mathbf{I}_{n_iT} \\ &= \mathbf{A}_i + \sigma_u^2\mathbf{1}_{n_iT}\mathbf{1}'_{n_iT}, \end{aligned} \tag{6.4}$$

where

$$\mathbf{A}_i = \sigma_v^2\left[\oplus_{j=1}^{n_i}\mathbf{J}_T\right] + \sigma_e^2\mathbf{I}_{n_iT} = \oplus_{j=1}^{n_i}\mathbf{A}_{ij} \qquad (n_iT\times n_iT)$$

with

$$\mathbf{A}_{ij} = \sigma_v^2\mathbf{J}_T + \sigma_e^2\mathbf{I}_T. \tag{6.5}$$

By applying a well-known result on matrix inverses (see e.g. Rao 1973, p. 33) to (6.4) we obtain

$$\mathbf{V}_{is}^{-1} = \mathbf{A}_i^{-1} - \mathbf{B}_i,$$

where

$$\mathbf{B}_i = \frac{\sigma_u^2(1-\gamma)^2\mathbf{J}_{n_iT}}{\sigma_e^4 + n_iT\sigma_u^2\sigma_e^2(1-\gamma)}$$

and

$$\mathbf{A}_i^{-1} = \oplus_{j=1}^{n_i}\mathbf{A}_{ij}^{-1},$$

since $\mathbf{A}_i$ is block-diagonal. Further, the inverse of (6.5) is

$$\mathbf{A}_{ij}^{-1} = \frac{1}{\sigma_e^2}\left(\mathbf{I}_T - \frac{\gamma}{T}\mathbf{J}_T\right) \qquad (T \times T).$$

With these results we end up to

$$\mathbf{1}'_{n_iT}\mathbf{V}_{is}^{-1}\mathbf{1}_{n_iT} = \frac{n_iT(1-\gamma)}{\sigma_e^2 + n_iT(1-\gamma)\sigma_u^2},$$

which leads to a computationally convenient formula

$$g_{1it}(\boldsymbol{\sigma}) = N_{r_{it}}^2\sigma_u^2\left[\frac{\sigma_e^2}{\sigma_e^2 + n_iT(1-\gamma)\sigma_u^2}\right]. \tag{6.6}$$

6.4.2 Term g_{2it}

The general formula of $g_{2it}(\boldsymbol{\sigma})$ is

$$g_{2it}(\boldsymbol{\sigma}) = (\mathbf{l}'_{it} - \mathbf{m}'_{it}\mathbf{G}\mathbf{Z}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{l}_{it} - \mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{Z}_s\mathbf{G}\mathbf{m}_{it}),$$

and by the block-diagonality of $\mathbf{G}$, $\mathbf{Z}$ and $\mathbf{V}$ it receives the area-specific form

$$g_{2it}(\boldsymbol{\sigma}) = (\mathbf{l}'_{it} - \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}\mathbf{X}_{is})(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}(\mathbf{l}_{it} - \mathbf{X}'_{is}\mathbf{V}_{is}^{-1}\mathbf{Z}_{is}\mathbf{G}_i\mathbf{m}_{it}),$$

with $\mathbf{m}_{it}$ redefined again as

$$\mathbf{m}_{it} = \begin{bmatrix} N_{r_{it}} & \mathbf{0}' \end{bmatrix}'.$$

Note that $(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}$ is the covariance matrix (3.23) of the GLS estimator $\widehat{\boldsymbol{\beta}}$ obtained from the overall sample data under known $\mathbf{V}_s$.

Writing

$$\mathbf{b}'_{it} = \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1} \qquad (1 \times n_iT)$$

and

$$\mathbf{d}'_{it} = \mathbf{l}'_{it} - \mathbf{b}'_{it}\mathbf{X}_{is} \qquad (1 \times p)$$

we have

$$g_{2it}(\boldsymbol{\sigma}) = Cov(\mathbf{d}'_{it}\widehat{\boldsymbol{\beta}}) = \mathbf{d}'_{it}(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{d}_{it}. \tag{6.7}$$

For $\mathbf{b}'_{it}$ we get the expression

$$\mathbf{b}'_{it} = N_{r_{it}}\sigma_u^2\mathbf{1}'_{n_iT}\mathbf{V}_{is}^{-1} = N_{r_{it}}\sigma_u^2\left[\frac{1-\gamma}{\sigma_e^2 + n_iT(1-\gamma)\sigma_u^2}\right]\mathbf{1}'_{n_iT}$$

and finally

$$\mathbf{d}'_{it} = \sum_{j\in r_{it}}\mathbf{x}'_{ijt} - \left[\frac{N_{r_{it}}(1-\gamma)\sigma_u^2}{\sigma_e^2 + n_iT(1-\gamma)\sigma_u^2}\right]\sum_{jt\in s_i^*}\mathbf{x}'_{ijt}, \tag{6.8}$$

where the latter summation over the sample panel data s_i^* means that the $\mathbf{x}'_{ijt}$ vectors are summed over all n_iT observations in s_i^*. That is, the sum is calculated over all occasions $t = 1, 2, \ldots, T$ and all sampled units $j = 1, 2, \ldots, n_i$ in area i.

6.4.3 Term g_{3it}

This term is generally intractable and we employ the Prasad-Rao approximation

$$g_{3it}(\boldsymbol{\sigma}) \approx tr\left[\left(\frac{\partial \mathbf{b}'_{it}}{\partial \boldsymbol{\sigma}}\right)\mathbf{V}_{is}\left(\frac{\partial \mathbf{b}'_{it}}{\partial \boldsymbol{\sigma}}\right)' Cov(\hat{\boldsymbol{\sigma}})\right],$$

where $Cov(\hat{\boldsymbol{\sigma}})$ is the asymptotic covariance matrix of estimates of $\boldsymbol{\sigma} = (\sigma_u^2, \sigma_v^2, \sigma_e^2)$. Here we use the REML estimation and take $Cov(\hat{\boldsymbol{\sigma}})$ as the inverse of the information matrix obtained from the REML log likelihood.

We obtain the derivative $\partial \mathbf{b}'_{it}/\partial \boldsymbol{\sigma}$ from the expression

$$\mathbf{b}'_{it} = \left[\frac{\sigma_u^2}{n_i T\sigma_u^2 + T\sigma_v^2 + \sigma_e^2}\right] N_{r_{it}} \mathbf{1}'_{n_i T}.$$

Then

$$\frac{\partial \mathbf{b}'_{it}}{\partial \boldsymbol{\sigma}} = \frac{N_{r_{it}}}{(n_i T\sigma_u^2 + T\sigma_v^2 + \sigma_e^2)^2}\begin{bmatrix}(T\sigma_v^2 + \sigma_e^2)\mathbf{1}'_{n_i T}\\ -T\sigma_u^2 \mathbf{1}'_{n_i T}\\ -\sigma_u^2 \mathbf{1}'_{n_i T}\end{bmatrix}.$$

Noting that

$$\mathbf{1}'_{n_i T}\mathbf{V}_{is}\mathbf{1}_{n_i T} = (n_i T)^2\sigma_v^2 + n_i T^2\sigma_u^2 + n_i T\sigma_e^2$$

the formula of g_{3it} can be developed into

$$g_{3it}(\boldsymbol{\sigma}) = \frac{N_{r_{it}}^2 n_i T}{(n_i T\sigma_u^2 + T\sigma_v^2 + \sigma_e^2)^3}\begin{bmatrix}(T\sigma_v^2 + \sigma_e^2)\\ -T\sigma_u^2\\ -\sigma_u^2\end{bmatrix}' Cov(\hat{\boldsymbol{\sigma}})\begin{bmatrix}(T\sigma_v^2 + \sigma_e^2)\\ -T\sigma_u^2\\ -\sigma_u^2\end{bmatrix}. \tag{6.9}$$

6.4.4 Term g_{4it}

This term, arising from the errors in predicting individual y_{ijt}'s in r_{it}, is simply

$$g_{4it}(\boldsymbol{\sigma}) = (\sigma_u^2 + \sigma_e^2)N_{r_{it}}. \tag{6.10}$$

7 Small area estimation with mixed model for rotating panel data

7.1 Notation for sample data

We turn to the case where data are collected with a rotating panel design. Then at each time point $t = 1, 2, \ldots, T$ a random sample s_t is taken from population U_t, and the resulting cross-sectional samples $s_1, s_2, \ldots, s_T$ have some overlap according to a scheme specified in advance. Each cross-sectional sample s_t further divides into m areas so that

$$s_t = s_{1t} \cup s_{2t} \cup \cdots \cup s_{mt},$$

where s_{it} is the sample from U_{it}, the population of area i at time t. We denote the size of s_{it} by n_{it} and the total sample size at time t by

$$n_t = \sum_{i=1}^{m} n_{it}.$$

The cross-sectional samples $s_1, s_2, \ldots, s_T$ compose a longitudinal data set

$$s^* = s_1 \cup s_2 \cup \cdots \cup s_T$$

with T_{ij} $(1 \leq T_{ij} \leq T)$ repeated observations for every unit j from area i. The data s^* can be also regarded as the union

$$s^* = s_1^* \cup s_2^* \cup \cdots \cup s_m^*$$

of the regional longitudinal sample data sets

$$s_i^* = s_{i1} \cup s_{i2} \cup \cdots \cup s_{iT}, \quad i = 1, 2, \ldots, m.$$

The total number of observations in data s_i^* is

$$n_i^* = \sum_{t=1}^{T} n_{it}$$

and in the whole longitudinal data s^* correspondingly

$$n^* = \sum_{i=1}^{m} n_i^* = \sum_{t=1}^{T} n_t.$$

A summary of this sample size notation is given in Table 7.1.

In addition, we define M_i as the number of separate units j from the area i, which appear in the sample data s_i^* (that is, they appear in s_i^* at least on one occasion t). In fact, we used the same symbol already with the notation for population data in the same purpose. However, we believe that this does not cause any confusion. We note now that

$$n_i^* = \sum_{j=1}^{M_i} T_{ij}.$$

An example of using this notation is given in Table 7.2. We denote the number of separate units in the overall longitudinal data s^* by

$$M = \sum_{i=1}^{m} M_i.$$

Table 7.1. The notation for sample sizes in the case of rotating panel data.

	time				
area	1	2	...	T	total
1	n_{11}	n_{12}	...	n_{1T}	n_1^*
2	n_{21}	n_{22}	...	n_{2T}	n_2^*
⋮	⋮	⋮		⋮	⋮
m	n_{m1}	n_{m2}	...	n_{mT}	n_m^*
total	n_1	n_2	...	n_T	n^*

Table 7.2. A longitudinal data s_i^* of area i with $M_i = 7$ units and $T = 5$ time points. The "X" marks an observational element.

	time					
unit	1	2	3	4	5	
1	X					$T_{i1} = 1$
2	X	X				$T_{i2} = 2$
3	X	X	X			$T_{i3} = 3$
4	X	X	X	X		$T_{i4} = 4$
5	X	X	X	X	X	$T_{i5} = 5$
6		X	X	X	X	$T_{i6} = 4$
7			X	X	X	$T_{i7} = 3$
	$n_{i1} = 5$	$n_{i2} = 5$	$n_{i3} = 5$	$n_{i4} = 4$	$n_{i5} = 3$	$n_i^* = 22$

Again, we denote the remainder or the set of non-sampled units in U_{it} by r_{it}. The non-sample size of area i at time t is then

$$N_{r_{it}} = N_{it} - n_{it}$$

for every i and t. Correspondingly, the remainder of the overall cross-sectional population U_t is r_t, the size of which is

$$N_{r_t} = N_t - n_t$$

for every t.

In a rotating panel design the sampled units are observed at specified time points within a period and then they are dropped out. At every time point a specific subset of units, which have been sampled earlier, leaves the data and a new subset is sampled for a replacement. This kind of subset is often called a panel or a wave. The rest of the earlier-sampled data stays at least for the next time point. As a result, the composition of the sample data partially changes between time points and there is certain overlap between the consecutive samples. This procedure is called rotation and it follows a specific design defining the scheme, by which the sampled units stay in the sample, and the degree of overlap between occasions. A variety of continuing surveys are implemented in this way. It is important to note that in principle the incompleteness of the panel data is now determined only by the design, making the selection mechanism fully ignorable. In practice it is of course possible to have non-random drop-outs.

A fragmentary example of continuing rotation scheme is given in Table 7.3. There a wave of two units enters the data at each time point, stays for two occasions, skips an occasion, comes back for one occasion and finally drops out. For each time point t there is a cross-sectional sample data set of three waves and $n_t = 6$ units. For instance, at time point 1 the data consist of waves 1, 3 and 4. Of these, wave 1 appears for the last time before dropping out, wave 3 appears for the second time and wave 4 has just entered the data. This scheme yields a 1/3 overlap between any two samples which are not more than three occasions apart from each other. Otherwise the overlap is zero. For example, the overlaps in "occasion pairs" $(1, 2)$, $(1, 3)$ and $(1, 4)$ are all 1/3, whereas the overlap in $(1, 5)$ is zero.

Table 7.3. A fragment of rotating panel design with $T = 5$ time points. The "X" marks an observation.

		time				
wave	unit	1	2	3	4	5
1	1	X				
	2	X				
2	3		X			
	4		X			
3	5	X		X		
	6	X		X		
4	7	X	X		X	
	8	X	X		X	
5	9		X	X		X
	10		X	X		X
6	11			X	X	
	12			X	X	
7	13				X	X
	14				X	X
8	15					X
	16					X

7.2 BLUP and EBLUP estimation of small area total

The estimation of area total from rotating panel data follows the same outlines as the estimation from complete panel data. The target is to estimate the total

$$Y_{it} = \sum_{j \in U_{it}} y_{ijt} = \sum_{j \in s_{it}} y_{ijt} + \sum_{j \in r_{it}} y_{ijt},$$

where s_{it} and r_{it} again denote the sample and the remainder of area i at time t, respectively. The general prediction theorem leads to the BLUP estimator

$$\widehat{Y}_{it,BLUP} = \sum_{j \in s_{it}} y_{ijt} + (\sum_{j \in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i + \sum_{j \in r_{it}} \tilde{v}_{ij}, \tag{7.1}$$

where $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{y}_s$ and $\tilde{u}_i$ and $\tilde{v}_{ij}$ are the BLUP's of the area effect u_i and the unit effect v_{ij}, obtained from the vector

$$\tilde{\mathbf{u}} = \mathbf{G}\mathbf{Z}'_s\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\widehat{\boldsymbol{\beta}}).$$

Note again that the subscript s in $\mathbf{y}_s, \mathbf{X}_s, \mathbf{V}_s$ and $\mathbf{Z}_s$ actually refers to longitudinal sample data set s^* so that all the observed temporal data is used in the estimation. The subscript s is used instead of s^* for convenience.

The estimator (7.1) has exactly the same form as the estimator (6.1) in the panel case. However, under the panel design we noted that

$$\sum_{j \in r_{it}} \tilde{v}_{ij} = 0,$$

because the sample data do not provide information on v_{ij}'s in r_{it}. The case of rotating panel design is different. At each time point t the rotation scheme partitions the remainder r_{it} as

$$r_{it} = r1_{it} \cup r2_{it},$$

where the set $r1_{it}$ contains those units from area i, which do not appear the sample at time t, but do appear in the longitudinal data s_i^* at some other time, and the set $r2_{it}$ contains those units from area i, which do not appear in the longitudinal data s_i^* (i.e. they have never been sampled). We denote the sizes of $r1_{it}$ and $r2_{it}$ by $N_{r1_{it}}$ and $N_{r2_{it}}$, respectively. Then

$$N_{r_{it}} = N_{r1_{it}} + N_{r2_{it}}$$

and

$$M_i = n_{it} + N_{r1_{it}},$$

where M_i denotes the total number of separate units in the data s_i^*.

The partition can be illustrated with Table 7.3 (or with Tables 2.2 and 2.3 shown earlier). If we consider Table 7.3 as an example of s_i^*, we see that for $t = 5$, for instance, the sample s_{i5} consists of units 9, 10 and 13–16. The first "remainder set" $r1_{i5}$ consists of units 1–8, 11 and 12 and the second set $r2_{i5}$ consists of the rest of U_{i5}, which does not appear in the table. We note also that $M_i = 16$, $n_{i5} = 6$ and $N_{r1_{i5}} = M_i - n_{i5} = 10$.

Now, due to the rotation, information on the v_{ij}'s in $r1_{it}$ is available in the data. Thus, they can be predicted and, unlike in the panel case, the predictors contribute to the estimator (7.1). The BLUP estimator $\widehat{Y}_{it,BLUP}$ for the rotating panel design is then

$$\widehat{Y}_{it,BLUP} = \sum_{j\in s_{it}} y_{ijt} + (\sum_{j\in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i + \sum_{j\in r1_{it}} \tilde{v}_{ij}. \tag{7.2}$$

It is anticipated that the contribution from the predicted v_{ij}'s improves accuracy of the BLUP estimator.

The EBLUP estimator is correspondingly

$$\widehat{Y}_{it,EBLUP} = \sum_{j\in s_{it}} y_{ijt} + (\sum_{j\in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\widehat{u}_i + \sum_{j\in r1_{it}} \widehat{v}_{ij}, \tag{7.3}$$

where the estimator $\widehat{\boldsymbol{\beta}}$ and predictors $\widehat{u}_i$ and $\widehat{v}_{ij}$ are obtained using the REML estimates of variance components $\boldsymbol{\sigma}$.

7.3 Mean squared error

Again, consider first the BLUP estimator (7.2). The estimation error is

$$\begin{aligned}\widehat{Y}_{it,BLUP} - Y_{it} &= (\sum_{j\in r_{it}} \mathbf{x}'_{ijt})\widehat{\boldsymbol{\beta}} + N_{r_{it}}\tilde{u}_i + \sum_{j\in r1_{it}} \tilde{v}_{ij} - \sum_{j\in r_{it}}(\mathbf{x}'_{ijt}\boldsymbol{\beta} + u_i + v_{ij} + e_{ijt}) \\ &= (\sum_{j\in r_{it}} \mathbf{x}'_{ijt})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + N_{r_{it}}(\tilde{u}_i - u_i) + \sum_{j\in r1_{it}} (\tilde{v}_{ij} - v_{ij}) - \sum_{j\in r2_{it}} v_{ij} - \sum_{j\in r_{it}} e_{ijt}.\end{aligned}$$

The BLUP estimator is model unbiased.

We let $\mathbf{v}_{r1_{it}}$ denote the $N_{r1_{it}} \times 1$ vector of those v_{ij}'s, which belong to the units in the set $r1_{it}$. Now

$$\sum_{j\in r1_{it}} (\tilde{v}_{ij} - v_{ij}) = \mathbf{1}'_{N_{r1_{it}}}(\tilde{\mathbf{v}}_{r1_{it}} - \mathbf{v}_{r1_{it}})$$

and the model-based mean squared error of the BLUP estimator can be written as

$$\begin{aligned}MSE(\widehat{Y}_{it,BLUP}) &= Var(\widehat{Y}_{it,BLUP} - Y_{it}) \\ &= \begin{bmatrix}\sum_{j\in r_{it}} \mathbf{x}'_{ijt} & N_{r_{it}} & \mathbf{1}'_{N_{r1_{it}}}\end{bmatrix} Cov \begin{bmatrix}\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \tilde{u}_i - u_i \\ \tilde{\mathbf{v}}_{r1_{it}} - \mathbf{v}_{r1_{it}}\end{bmatrix} \begin{bmatrix}\sum_{j\in r_{it}} \mathbf{x}'_{ijt} \\ N_{r_{it}} \\ \mathbf{1}_{N_{r1_{it}}}\end{bmatrix} \\ &+ Var\left[\sum_{j\in r2_{it}} v_{ij} + \sum_{j\in r_{it}} e_{ijt}\right] \\ &= \begin{bmatrix}\mathbf{l}'_{it} & \mathbf{m}'_{it}\end{bmatrix} Cov \begin{bmatrix}\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \tilde{\mathbf{u}}_i - \mathbf{u}_i\end{bmatrix} \begin{bmatrix}\mathbf{l}_{it} \\ \mathbf{m}_{it}\end{bmatrix} + N_{r2_{it}}\sigma_v^2 + N_{r_{it}}\sigma_e^2,\end{aligned}$$

where

$$\mathbf{u}_i = \begin{bmatrix} u_i \\ \mathbf{v}_i \end{bmatrix}, \quad ((1+M_i)\times 1)$$

$$\mathbf{l}'_{it} = \sum_{j\in r_{it}} \mathbf{x}'_{ijt}$$

and

$$\mathbf{m}'_{it} = \begin{bmatrix} N_{r_{it}} & \delta_{i1} & \delta_{i2} & \dots & \delta_{iM_i} \end{bmatrix}, \quad (1\times(1+M_i))$$

where

$$\delta_{ij} = \begin{cases} 0, & \text{if unit } ij \text{ is in the sample data } s_{it} \\ 1, & \text{if unit } ij \text{ is in the remainder part } r1_{it} \end{cases}$$

for each $j = 1, 2, \dots, M_i$. Thus, the vector $\mathbf{m}'_{it}$ contains $N_{r1_{it}}$ ones and $n_{it} = M_i - N_{r1_{it}}$ zeros.

In the computations we again utilize the decomposition

$$MSE(\widehat{Y}_{it,BLUP}) = g_{1it}(\boldsymbol{\sigma}) + g_{2it}(\boldsymbol{\sigma}) + g_{4it}(\boldsymbol{\sigma}),$$

where $g_{1it}(\boldsymbol{\sigma})$ measures the uncertainty about random effects, $g_{2it}(\boldsymbol{\sigma})$ measures the uncertainty about $\boldsymbol{\beta}$ and $g_{4it}(\boldsymbol{\sigma})$ measures the uncertainty about the individual y_{ijt}'s in r_{it}. The approximate MSE of the EBLUP estimator is correspondingly

$$MSE(\widehat{Y}_{it,EBLUP}) \approx g_{1it}(\boldsymbol{\sigma}) + g_{2it}(\boldsymbol{\sigma}) + g_{3it}(\boldsymbol{\sigma}) + g_{4it}(\boldsymbol{\sigma}),$$

where the additional term $g_{3it}(\boldsymbol{\sigma})$ again measures the uncertainty about $\boldsymbol{\sigma} = (\sigma_u^2, \sigma_v^2, \sigma_e^2)$. The explicit formulas needed for computing $g_{1it}(\boldsymbol{\sigma})$, $g_{2it}(\boldsymbol{\sigma})$, $g_{3it}(\boldsymbol{\sigma})$ and $g_{4it}(\boldsymbol{\sigma})$ are given in the next section. The approximately unbiased estimator of the approximate MSE is still

$$\widehat{MSE}(\widehat{Y}_{it,EBLUP}) = g_{1it}(\widehat{\boldsymbol{\sigma}}) + g_{2it}(\widehat{\boldsymbol{\sigma}}) + 2g_{3it}(\widehat{\boldsymbol{\sigma}}) + g_{4it}(\widehat{\boldsymbol{\sigma}})$$

and it is valid especially with REML estimation under regularity conditions and when covariance matrix $\mathbf{V}(\boldsymbol{\sigma})$ has the linear structure (3.28).

7.4 Computational formulas for MSE

7.4.1 Term g_{1it}

The basic formula of $g_{1it}(\boldsymbol{\sigma})$ is

$$\begin{aligned} g_{1it}(\boldsymbol{\sigma}) &= \mathbf{m}'_{it}(\mathbf{G}_i - \mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}\mathbf{Z}_{is}\mathbf{G}_i)\mathbf{m}_{it} \\ &= \mathbf{m}'_{it}\mathbf{G}_i\mathbf{m}_{it} - \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}\mathbf{Z}_{is}\mathbf{G}_i\mathbf{m}_{it}, \end{aligned} \tag{7.4}$$

where $\mathbf{G}_i$, $\mathbf{Z}_{is}$ and $\mathbf{V}_{is}$ are the blocks of $\mathbf{G}_s$, $\mathbf{Z}_s$ and $\mathbf{V}_s$ specific to area i. However, the formula (7.4) is not feasible for straightforward computing, because it requires inverting the often large $\mathbf{V}_{is}$. By utilizing the relatively simple structure of $\mathbf{V}_{is}$ we work out an alternative formula for $g_{1it}(\boldsymbol{\sigma})$, where the computation of $\mathbf{V}_{is}^{-1}$ is avoided.

Deriving the inverse of $\mathbf{V}_{is}$ follows here the same guidelines as in the panel data case, but the practice is somewhat more complicated due to unequal T_{ij}'s. We first write the covariance matrix $\mathbf{V}_{is}$ as

$$\begin{aligned}\mathbf{V}_{is} &= \sigma_u^2 \mathbf{J}_{n_i^*} + \sigma_v^2 \left[\oplus_{j=1}^{M_i} \mathbf{J}_{T_{ij}}\right] + \sigma_e^2 \mathbf{I}_{n_i^*} \qquad (n_i^* \times n_i^*) \\ &= \sigma_u^2 \mathbf{1}_{n_i^*} \mathbf{1}'_{n_i^*} + \mathbf{A}_i,\end{aligned}$$

where

$$\mathbf{A}_i = \oplus_{j=1}^{M_i} \left(\sigma_v^2 \mathbf{J}_{T_{ij}} + \sigma_e^2 \mathbf{I}_{T_{ij}}\right).$$

Then we apply the result of Rao (1973, p. 33) to obtain

$$\mathbf{V}_{is}^{-1} = \mathbf{A}_i^{-1} - \frac{\mathbf{A}_i^{-1} \mathbf{1}_{n_i^*} \mathbf{1}'_{n_i^*} \mathbf{A}_i^{-1} \sigma_u^2}{1 + \mathbf{1}'_{n_i^*} \mathbf{A}_i^{-1} \mathbf{1}_{n_i^*} \sigma_u^2}, \tag{7.5}$$

where

$$\mathbf{A}_i^{-1} = \oplus_{j=1}^{M_i} \left[\frac{1}{\sigma_e^2} \left(\mathbf{I}_{T_{ij}} - \frac{\gamma_{ij}}{T_{ij}} \mathbf{J}_{T_{ij}}\right)\right],$$

by e.g. McCulloch and Searle (2001, p. 292), with

$$\gamma_{ij} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2 / T_{ij}}.$$

Substituting (7.5) into (7.4) yields the expression

$$\begin{aligned}g_{1it}(\boldsymbol{\sigma}) =& \mathbf{m}'_{it} \mathbf{G}_i \mathbf{m}_{it} - \mathbf{m}'_{it} \mathbf{G}_i \mathbf{Z}'_{is} \mathbf{A}_i^{-1} \mathbf{Z}_{is} \mathbf{G}_i \mathbf{m}_{it} \\ &+ \frac{\mathbf{m}'_{it} \mathbf{G}_i \mathbf{Z}'_{is} \mathbf{A}_i^{-1} \mathbf{1}_{n_i^*} \mathbf{1}'_{n_i^*} \mathbf{A}_i^{-1} \mathbf{Z}_{is} \mathbf{G}_i \mathbf{m}_{it} \sigma_u^2}{1 + \mathbf{1}'_{n_i^*} \mathbf{A}_i^{-1} \mathbf{1}_{n_i^*} \sigma_u^2},\end{aligned} \tag{7.6}$$

which is computationally more convenient than (7.4). The computing of $g_{1it}(\boldsymbol{\sigma})$ using (7.6) can be further simplified by utilizing some additional results, which are presented in the following.

First we note that

$$\mathbf{m}'_{it} \mathbf{G}_i = \begin{bmatrix} N_{r_{it}} \sigma_u^2 & \delta_{i1} \sigma_v^2 & \delta_{i2} \sigma_v^2 & \ldots & \delta_{iM_i} \sigma_v^2 \end{bmatrix} \quad (1 \times (1 + M_i)) \tag{7.7}$$

so that the n_{it} entries corresponding to the units in s_{it} are equal to zero and the $N_{r1_{it}}$ entries corresponding to the units in $r1_{it}$ are equal to σ_v^2. The first term of the right-hand-side of (7.6) becomes then

$$\mathbf{m}'_{it} \mathbf{G}_i \mathbf{m}_{it} = N_{r_{it}}^2 \sigma_u^2 + N_{r1_{it}} \sigma_v^2. \tag{7.8}$$

Further,

$$\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*} = \frac{1}{\sigma_e^2}\begin{bmatrix}(1-\gamma_{i1})\mathbf{1}_{T_{i1}} \\ \vdots \\ (1-\gamma_{iM_i})\mathbf{1}_{T_{iM_i}}\end{bmatrix} \tag{7.9}$$

yielding

$$\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*} = \frac{1}{\sigma_e^2}\sum_{j=1}^{M_i} T_{ij}(1-\gamma_{ij}) = \sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}, \tag{7.10}$$

$$\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*} = \frac{1}{\sigma_e^2}\begin{bmatrix}\sum_{j=1}^{M_i} T_{ij}(1-\gamma_{ij}) \\ T_{i1}(1-\gamma_{i1}) \\ \vdots \\ T_{iM_i}(1-\gamma_{iM_i})\end{bmatrix} = \begin{bmatrix}\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1} \\ (\sigma_v^2+\sigma_e^2/T_{i1})^{-1} \\ \vdots \\ (\sigma_v^2+\sigma_e^2/T_{iM_i})^{-1}\end{bmatrix} \tag{7.11}$$

and

$$\begin{aligned}&\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{Z}_{is} \\ &= \begin{bmatrix}\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1} & (\sigma_v^2+\sigma_e^2/T_{i1})^{-1} & \dots & (\sigma_v^2+\sigma_e^2/T_{iM_i})^{-1} \\ (\sigma_v^2+\sigma_e^2/T_{i1})^{-1} & (\sigma_v^2+\sigma_e^2/T_{i1})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_v^2+\sigma_e^2/T_{iM_i})^{-1} & 0 & \dots & (\sigma_v^2+\sigma_e^2/T_{iM_i})^{-1}\end{bmatrix}.\end{aligned} \tag{7.12}$$

It can be helpful to note that the summing over $j = 1, 2, \dots, M_i$ is equivalent to summing over the set $s_{it} \cup r1_{it}$ of $M_i = n_{it} + N_{r1_{it}}$ units j.

By (7.7) and (7.12) we arrive at the expression

$$\begin{aligned}&\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{Z}_{is}\mathbf{G}_i\mathbf{m}_{it} \\ &= N_{r_{it}}^2\sigma_u^4\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1} + (2N_{r_{it}}\sigma_u^2\sigma_v^2+\sigma_v^4)\sum_{j\in r1_{it}}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}\end{aligned} \tag{7.13}$$

for the second term of the right-hand-side of (7.6).

For the last term of the right-hand-side of (7.6) we get

$$\begin{aligned}&\frac{\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{Z}_{is}\mathbf{G}_i\mathbf{m}_{it}\sigma_u^2}{1+\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\sigma_u^2} \\ &= \frac{\left[N_{r_{it}}\sigma_u^2\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}+\sigma_v^2\sum_{j\in r1_{it}}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}\right]^2\sigma_u^2}{1+\sigma_u^2\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}}\end{aligned} \tag{7.14}$$

by (7.7), (7.10) and (7.11).

From (7.8), (7.13) and (7.14) we obtain an expression for (7.6) in scalar form.

7.4.2 Term g_{2it}

The general formula for $g_{2it}(\boldsymbol{\sigma})$ is

$$g_{2it}(\boldsymbol{\sigma}) = Cov(\mathbf{d}'_{it}\widehat{\boldsymbol{\beta}}) = \mathbf{d}'_{it}(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{d}_{it}, \tag{7.15}$$

where

$$\mathbf{d}'_{it} = \mathbf{l}'_{it} - \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1}\mathbf{X}_{is}. \qquad (1 \times p)$$

Recall that the vectors $\mathbf{l}'_{it}$ and $\mathbf{m}'_{it}$ are

$$\mathbf{l}'_{it} = \sum_{j \in r_{it}} \mathbf{x}'_{ijt}$$

and

$$\mathbf{m}'_{it} = \begin{bmatrix} N_{r_{it}} & \delta_{i1} & \delta_{i2} & \dots & \delta_{iM_i} \end{bmatrix}.$$

The covariance matrix $Cov(\widehat{\boldsymbol{\beta}}) = (\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}$ can be obtained e.g. from the mixed model equations.

The straight inversion of $\mathbf{V}_{is}$ in $\mathbf{d}'_{it}$ can again be avoided by applying (7.5). Then we get the expression

$$\mathbf{d}'_{it} = \mathbf{l}'_{it} - \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{X}_{is} + \frac{\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{X}_{is}\sigma_u^2}{1 + \mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\sigma_u^2} \tag{7.16}$$

for $\mathbf{d}'_{it}$. The computing of (7.16) can be further simplified by making use of (7.7), (7.10), (7.11) and additional results given below.

First,

$$\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{X}_{is} = \frac{1}{\sigma_e^2}\sum_{j=1}^{M_i}[(1-\gamma_{ij})\sum_t \mathbf{x}'_{ijt}] \tag{7.17}$$

and

$$\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{X}_{is} = \frac{1}{\sigma_e^2}\begin{bmatrix} \sum_{j=1}^{M_i}[(1-\gamma_{ij})\sum_t \mathbf{x}'_{ijt}] \\ (1-\gamma_{i1})\sum_t \mathbf{x}'_{i1t} \\ \vdots \\ (1-\gamma_{iM_i})\sum_t \mathbf{x}'_{iM_it} \end{bmatrix}. \tag{7.18}$$

The summation $\sum_t \mathbf{x}'_{ijt}$ in (7.17) and (7.18) gives a vector where the auxiliary data vectors $\mathbf{x}'_{ijt}$ of unit j in area i are summed over those time points $t \in \{1, 2, \dots, T\}$ at which the unit is observed. Due to the rotating panel design, these time points are not the same for all units.

By (7.7) and (7.18) we get

$$\begin{aligned}
&\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{X}_{is} \\
(7.19) \qquad &= \frac{1}{\sigma_e^2}\left\{N_{r_{it}}\sigma_u^2\sum_{j=1}^{M_i}[(1-\gamma_{ij})\sum_t \mathbf{x}'_{ijt}] + \sigma_v^2\sum_{j\in r1_{it}}[(1-\gamma_{ij})\sum_t \mathbf{x}'_{ijt}]\right\}
\end{aligned}$$

for the second term of the right-hand-side of (7.16). For the third term of the right-hand-side of (7.16) we get

$$\begin{aligned}
&\frac{\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{X}_{is}\sigma_u^2}{1+\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\sigma_u^2} \\
(7.20) \qquad &= \frac{N_{r_{it}}\sigma_u^2\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1} + \sigma_v^2\sum_{j\in r1_{it}}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}}{1+\sigma_u^2\sum_{j=1}^{M_i}(\sigma_v^2+\sigma_e^2/T_{ij})^{-1}} \\
&\quad\times\frac{\sigma_u^2}{\sigma_e^2}\sum_{j=1}^{M_i}[(1-\gamma_{ij})\sum_t \mathbf{x}'_{ijt}]
\end{aligned}$$

by (7.7), (7.10), (7.11) and (7.17). Substituting (7.19) and (7.20) into (7.16) leads to an alternative computational formula for $\mathbf{d}'_{it}$.

7.4.3 Term g_{3it}

The Prasad-Rao approximation of $g_{3it}(\boldsymbol{\sigma})$ has the general form

$$g_{3it}(\boldsymbol{\sigma}) \approx tr\left[\left(\frac{\partial\mathbf{b}'_{it}}{\partial\boldsymbol{\sigma}}\right)\mathbf{V}_{is}\left(\frac{\partial\mathbf{b}'_{it}}{\partial\boldsymbol{\sigma}}\right)' Cov(\hat{\boldsymbol{\sigma}})\right],$$

where $Cov(\hat{\boldsymbol{\sigma}})$ is the asymptotic covariance matrix of (e.g. REML) estimates of $\boldsymbol{\sigma} = (\sigma_u^2, \sigma_v^2, \sigma_e^2)$ and

$$\mathbf{b}'_{it} = \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{V}_{is}^{-1} \qquad (1\times n_i^*).$$

By applying (7.5) the vector $\mathbf{b}'_{it}$ can be written as

$$(7.21) \qquad \mathbf{b}'_{it} = \mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1} - a_{it}\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\sigma_u^2,$$

where

$$(7.22) \qquad a_{it} = a_{it}(\boldsymbol{\sigma}) = \frac{\mathbf{m}'_{it}\mathbf{G}_i\mathbf{Z}'_{is}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}}{1+\mathbf{1}'_{n_i^*}\mathbf{A}_i^{-1}\mathbf{1}_{n_i^*}\sigma_u^2} = \frac{\sum_{j=1}^{M_i} T_{ij}\frac{N_{r_{it}}\sigma_u^2+\delta_{ij}\sigma_v^2}{T_{ij}\sigma_v^2+\sigma_e^2}}{1+\sigma_u^2\sum_{j=1}^{M_i}\frac{T_{ij}}{T_{ij}\sigma_v^2+\sigma_e^2}}.$$

In addition, we have

$$
\begin{aligned}
\mathbf{m}_{it}'\mathbf{G}_i\mathbf{Z}_{is}'\mathbf{A}_i^{-1} &= \begin{bmatrix} \frac{N_{r_{it}}\sigma_u^2+\delta_{i1}\sigma_v^2}{T_{i1}\sigma_v^2+\sigma_e^2}\mathbf{1}_{T_{i1}} \\ \vdots \\ \frac{N_{r_{it}}\sigma_u^2+\delta_{iM_i}\sigma_v^2}{T_{iM_i}\sigma_v^2+\sigma_e^2}\mathbf{1}_{T_{iM_i}} \end{bmatrix}' \\
&= \begin{bmatrix} \frac{N_{r_{it}}\sigma_u^2+\delta_{i1}\sigma_v^2}{T_{i1}\sigma_v^2+\sigma_e^2} \\ \vdots \\ \frac{N_{r_{it}}\sigma_u^2+\delta_{iM_i}\sigma_v^2}{T_{iM_i}\sigma_v^2+\sigma_e^2} \end{bmatrix}' \begin{bmatrix} \mathbf{1}_{T_{i1}}' & \cdots & \mathbf{0}' \\ \vdots & \ddots & \vdots \\ \mathbf{0}' & \cdots & \mathbf{1}_{T_{iM_i}}' \end{bmatrix}
\end{aligned} \tag{7.23}
$$

and

$$
\mathbf{1}_{n_i^*}'\mathbf{A}_i^{-1}\sigma_u^2 = \begin{bmatrix} \frac{\sigma_u^2}{T_{i1}\sigma_v^2+\sigma_e^2} \\ \vdots \\ \frac{\sigma_u^2}{T_{iM_i}\sigma_v^2+\sigma_e^2} \end{bmatrix}' \begin{bmatrix} \mathbf{1}_{T_{i1}}' & \cdots & \mathbf{0}' \\ \vdots & \ddots & \vdots \\ \mathbf{0}' & \cdots & \mathbf{1}_{T_{iM_i}}' \end{bmatrix}. \tag{7.24}
$$

Substituting (7.23) and (7.24) into (7.21) leads to the expression

$$
\mathbf{b}_{it}' = \mathbf{c}_{it}' \begin{bmatrix} \mathbf{1}_{T_{i1}}' & \cdots & \mathbf{0}' \\ \vdots & \ddots & \vdots \\ \mathbf{0}' & \cdots & \mathbf{1}_{T_{iM_i}}' \end{bmatrix},
$$

where

$$
\mathbf{c}_{it} = \begin{bmatrix} \frac{(N_{r_{it}}-a_{it})\sigma_u^2+\delta_{i1}\sigma_v^2}{T_{i1}\sigma_v^2+\sigma_e^2} \\ \vdots \\ \frac{(N_{r_{it}}-a_{it})\sigma_u^2+\delta_{iM_i}\sigma_v^2}{T_{iM_i}\sigma_v^2+\sigma_e^2} \end{bmatrix},
$$

and to the derivative

$$
\frac{\partial \mathbf{b}_{it}'}{\partial \boldsymbol{\sigma}} = \left(\frac{\partial \mathbf{c}_{it}'}{\partial \boldsymbol{\sigma}}\right) \begin{bmatrix} \mathbf{1}_{T_{i1}}' & \cdots & \mathbf{0}' \\ \vdots & \ddots & \vdots \\ \mathbf{0}' & \cdots & \mathbf{1}_{T_{iM_i}}' \end{bmatrix}. \tag{7.25}
$$

For the computation it suffices to find the derivative of the jth element

$$
c_{itj} = \frac{(N_{r_{it}} - a_{it}(\boldsymbol{\sigma}))\sigma_u^2 + \delta_{ij}\sigma_v^2}{T_{ij}\sigma_v^2 + \sigma_e^2}, \quad j = 1, 2, \ldots, M_i, \tag{7.26}
$$

of $\mathbf{c}_{it}$. The notation $a_{it}(\boldsymbol{\sigma})$ emphasizes here that a_{it} is a function of the variance components σ_u^2, σ_v^2 and σ_e^2. Since

$$
\frac{\partial \mathbf{c}_{it}'}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \partial \mathbf{c}_{it}'/\partial\sigma_u^2 \\ \partial \mathbf{c}_{it}'/\partial\sigma_v^2 \\ \partial \mathbf{c}_{it}'/\partial\sigma_e^2 \end{bmatrix}, \qquad (3 \times M_i)
$$

we shall explicitly work out the partial derivatives of (7.26) with respect to each variance component. These partial derivatives, given in (7.27)–(7.29), are then used in constructing the matrix (7.25) needed in the approximation of g_{3it}.

First, for σ_u^2 we get

$$\frac{\partial c_{itj}}{\partial \sigma_u^2} = \frac{N_{r_{it}} - a_{it} - \sigma_u^2(\partial a_{it}/\partial \sigma_u^2)}{T_{ij}\sigma_v^2 + \sigma_e^2}, \tag{7.27}$$

where

$$\frac{\partial a_{it}}{\partial \sigma_u^2} = \left(\sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)\left(N_{r_{it}} - \sigma_v^2 \sum_{j=1}^{M_i} \frac{\delta_{ij}T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right) \Bigg/ \left(1 + \sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)^2.$$

Then, for σ_v^2 we get

$$\frac{\partial c_{itj}}{\partial \sigma_v^2} = \frac{-(N_{r_{it}} - a_{it})T_{ij}\sigma_u^2 - (T_{ij}\sigma_v^2 + \sigma_e^2)\sigma_u^2(\partial a_{it}/\partial \sigma_v^2) + \delta_{ij}\sigma_e^2}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}, \tag{7.28}$$

where

$$\begin{aligned}\frac{\partial a_{it}}{\partial \sigma_v^2} = &\left(\sum_{j=1}^{M_i} \frac{\delta_{ij}T_{ij}\sigma_e^2 - N_{r_{it}}T_{ij}^2\sigma_u^2}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}\right) \Bigg/ \left(1 + \sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right) \\ &+ \left(\sum_{j=1}^{M_i} \frac{N_{r_{it}}T_{ij}\sigma_u^2 - \delta_{ij}T_{ij}\sigma_v^2}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)\left(\sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}^2}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}\right) \Bigg/ \left(1 + \sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)^2.\end{aligned}$$

Finally, for σ_e^2 we get

$$\frac{\partial c_{itj}}{\partial \sigma_e^2} = \frac{-(N_{r_{it}} - a_{it})\sigma_u^2 - (T_{ij}\sigma_v^2 + \sigma_e^2)\sigma_u^2(\partial a_{it}/\partial \sigma_e^2) - \delta_{ij}\sigma_v^2}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}, \tag{7.29}$$

where

$$\begin{aligned}\frac{\partial a_{it}}{\partial \sigma_e^2} = &\left(-\sum_{j=1}^{M_i} \frac{N_{r_{it}}T_{ij}\sigma_u^2 + \delta_{ij}T_{ij}\sigma_v^2}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}\right) \Bigg/ \left(1 + \sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right) \\ &+ \left(\sum_{j=1}^{M_i} \frac{N_{r_{it}}T_{ij}\sigma_u^2 + \delta_{ij}T_{ij}\sigma_v^2}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)\left(\sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{(T_{ij}\sigma_v^2 + \sigma_e^2)^2}\right) \Bigg/ \left(1 + \sigma_u^2 \sum_{j=1}^{M_i} \frac{T_{ij}}{T_{ij}\sigma_v^2 + \sigma_e^2}\right)^2.\end{aligned}$$

7.4.4 Term g_{4it}

This term is simply

$$g_{4it}(\boldsymbol{\sigma}) = N_{r2_{it}}\sigma_v^2 + N_{r_{it}}\sigma_e^2. \tag{7.30}$$

8 Description of simulation study

8.1 Overview

We carry out a Monte Carlo simulation study to investigate the performance of the introduced EBLUP estimators based on the three-level model (5.1) for panel data and rotating panel data. For comparison, we also consider the EBLUP estimators based on the nested error regression model (2.20) for cross-sectional data and on the random time model (2.26) for repeated survey data, which is close to rotating panel data, if the overlap in temporal samples is mild. Thus, in total four "competing" models and related EBLUP estimators are considered and compared: (1) three-level model under panel data, (2) three-level model under rotating panel data, (3) nested error regression model for cross-sectional data and (4) random time model for repeated survey data. We use hereafter the following abbreviations for the models: NESTED for the nested error regression model, PANEL for the three-level model under panel data, ROTPANEL for the three-level model under rotating panel data and RANTIME for the random time model.

The task is to estimate the area totals of the response variable at the last ("current") time point in the considered panel or rotating panel data with each of the above models. The auxiliary $\mathbf{x}$ variables used in the estimation are the same for every model. The factors, whose effects on the estimation performance are studied and which define the design of the simulation experiment, are as follows:

A. The magnitude of the *intra-area correlation* (5.7)
B. The magnitude of the *intra-unit correlation* (5.6)
C. The *length of the panel* (for estimation from panel or rotating panel data)
D. Using *a correct or an incorrect model* in the estimation.

The magnitude of the intra-area correlation of the units measures how homogeneous are the units, which belong to the same area. This factor is given two levels: (1) high correlation and (2) low correlation. In the first case the area variance is high with respect to the unit and residual variances, and the second case is the opposite.

The magnitude of the intra-unit correlation or (in a way) autocorrelation measures how highly the repeated observations of a unit are correlated. Also this factor is given two levels: (1) high correlation and (2) low correlation, which are related to the magnitude of both the unit variance and the area variance with respect to the residual variance.

By the length of the panel we mean the number of time points (or lags) in the data used for the estimation. Again for this factor we define two levels: (1) 10 time points, i.e. the current occasion and 9 preceding occasions, and (2) 5 time points. The purpose is to assess the possible loss of efficiency when shorter panel data are used. The NESTED model is of course applied to the cross-sectional data of the last survey occasion.

By the correct model we mean here that all the relevant auxiliary variables are included in the fixed part of the model. In other words, the employed model contains exactly those $\mathbf{x}$ variables, which have appeared in generating the population. In an incorrect model some important variables are missing from $\mathbf{x}$. In reality, a statistical model can of course be incorrect in a variety of ways. For instance, the covariance structure can be misspecified or the true model might not be linear at all. However, we concentrate here only on linear mixed models with misspecified fixed part. In addition, we consider briefly a case, where model has not only a misspecified fixed part but also a misspecified covariance structure, which ignores nonconstant intra-unit correlations and unequal variances over time. We believe that these cases illustrate reasonably well the robustness of the EBLUP estimation when the data come from a population, where the employed model does not hold.

The Monte Carlo study is design-based in the sense that we first create a synthetic finite longitudinal population of $m = 30$ areas and $T = 10$ time points using a specified statistical model and then repeatedly draw probability samples from it. The properties of the estimators are evaluated with respect to this finite population sampling. The two-level factors A (intra-area correlation) and B (intra-unit correlation) define $2 \times 2 = 4$ different covariance structures for the statistical model, acting as "cells" in the simulation design: (1) high intra-area correlation and high intra-unit correlation, (2) high intra-area correlation and low intra-unit correlation, (3) low intra-area correlation and high intra-unit correlation and (4) low intra-area correlation and low intra-unit correlation. A separate fixed population will be generated under each covariance structure. We name the resulting populations POP1, POP2, POP3 and POP4, in the respective order (see Table 8.1).

Table 8.1. The four covariance structures ("cells") and corresponding populations.

A. intra-area correlation	B. intra-unit correlation: high	B. intra-unit correlation: low
high	POP1	POP2
low	POP3	POP4

From each longitudinal population we draw both panel samples and rotating panel samples $K = 1000$ times. In other words, there are $K = 1000$ replicates in each cell. For comparability, the samples are matched so that in each of the K replicates the panel sample and the rotating panel sample agree fully at the last survey occasion, whose regional totals are to be estimated. No drop-outs or non-response are introduced. The sampling method is the simple random sampling. Hence the regional sample sizes are not controlled in any way and they merely reflect the regional population sizes. The overall sample size per time point is set $n = 1000$. Now and then the sample sizes of the smallest regions appear to be zero (see Table 8.6). The adopted rotation scheme is the one used in the Finnish Labour Force Survey, described in Table 2.3 and in more detail in Table 8.7.

In each replicate the models ROTPANEL and RANTIME are applied to the rotating panel sample and the models PANEL and NESTED are applied to the panel sample (the NESTED model of course uses the last occasion data only). Recall that the RANTIME model is valid for neither panel nor rotating panel data since it does not allow for any overlap in the repeated survey data. However, we estimate it from the rotating panel data to see how its performance is affected if some overlap is present.

The parameters of the mixed models are estimated from the sample data by the REML and GLS methods. The EBLUP estimates of the area totals at the last occasion are then calculated under each competing model.

The factors C (length of panel data in use) and D (correct/incorrect model) are related to the model choice in the estimation stage. We deal with these factors so that in each replication, for every competing model, the EBLUP estimates are obtained both with the correctly specified model and with an incorrectly specified model and under different numbers of occasions in the sample data sets (where appropriate). This leads to altogether $7 \times 2 = 14$ estimation tasks per replication. This scheme is presented in Table 8.2 and it is replicated $K = 1000$ times for each of the four populations.

Table 8.2. The scheme of models used in the simulation study.

C. number of occasions	D. model	
	correct	incorrect
1	NESTEDc	NESTEDi
5	PANEL5c	PANEL5i
	ROTPANEL5c	ROTPANEL5i
	RANTIME5c	RANTIME5i
10	PANEL10c	PANEL10i
	ROTPANEL10c	ROTPANEL10i
	RANTIME10c	RANTIME10i

In the above table PANEL5 denotes the case where the PANEL model is used with data of 5 time points and PANEL10 denotes the case where the panel data consist of 10 time points. Similar notation is used for ROTPANEL and RANTIME. In addition, the letter 'c' refers to correct model and the letter 'i' to incorrect model. We emphasize that the models PANEL5, ROTPANEL5 and RANTIME5 are applied to the same samples than PANEL10, ROTPANEL10 and RANTIME10, but they only use the data of 5 last survey occasions. The NESTED model naturally uses only the last survey occasion.

The empirical distributions of the EBLUP estimates and their MSE estimates obtained by K replications are examined to assess and compare the performance, e.g. bias and accuracy, of the considered estimators.

Outside the systematical examination of the factors A, B, C and D we have also made some separate experiments to assess how the overlap degree of the rotation scheme affects to the estimation. In certain cases we have also considered the effect of specifying time as a fixed effect instead of random (or vice versa). We also have generated and sampled the population POP1 twice with different seed numbers to see if the choice of seed has any effect to the results. A brief summary of the findings of these experiments will be reported.

The SAS software was used in all computations as well as in generating the populations for simulation. The needed programs were all written by the author as SAS macros, with the exception of EBLUP estimation with the RANTIME model, which was executed by a SAS macro written by Dr. Ari Veijanen from Statistics Finland for the EURAREA Consortium (2004). The SAS procedure MIXED was utilized in estimating the mixed model parameters by the REML method. The computations of the EBLUP estimation were programmed in SAS/IML language. The SURVEYSELECT procedure (along with the random number generator implemented in SAS) was used in selecting the random samples in the simulation studies.

8.2 Generation of populations

The basis for the generation of populations POP1–POP4 is a real data set provided by Statistics Finland originally for purposes of the EURAREA research project (2004). It contains a complete panel population of $N = 1084764$ 10–77-year-old people from $m = 30$ NUTS4 regions in the province of Western Finland. NUTS, i.e. Nomenclature of Statistical Territorial Units, is the standard coding system for the administrative regional divisions within the European Union. In Finland the NUTS4 regions are groups of municipalities (the Finnish expression is "seutukunta"). The data cover $T = 10$ consecutive months from March 1997 to December 1997, and the following variables, which will serve as auxiliary variables in the EBLUP estimation, are recorded for every individual:

AGE = age in years
JS = jobseeker status (1 = jobseeker, 0 = not)
SEX = sex (1 = male, 0 = female)

Those people who have died or moved away during the 10 months have not been included in the data so that the population sizes of the areas are the same in every month. These are shown along with the regional means of the auxiliary variables in Table 8.3.

For each panel population POP1–POP4 a normal response variable Y was created using AGE, JS and SEX as covariates. The model equation used in generating the 10847640

Y values for the 10-month panel data of 1084764 individuals (or units) was in all cases

$$(8.1)\quad Y_{ijt} = 10 + 0.1AGE_{ijt} + 1.5JS_{ijt} + 2SEX_{ijt} + MONTH_t + REGION_i + IND_{ij} + \epsilon_{ijt},$$

where $MONTH_t$, $REGION_i$ and IND_{ij} are the effects of survey month t, NUTS4 region i and unit ij, respectively, end ϵ_{ijt} is the error term. The values of AGE, JS and SEX were taken from the basis data set as such and are considered fixed in the employed models. The 10 values of $MONTH_t$ are also considered fixed, although they were created by adding a $N(0,1)$ distributed random term to the systematical linear trend $-9/6, -7/6, -5/6, \ldots, 5/6, 7/6, 9/6$. The seed given here to the SAS random number generator was 97312.

Table 8.3. Regional (cross-sectional) population sizes N_i and means of the auxiliary variables AGE, JS and SEX.

region i	N_i	mean(AGE)	mean(JS)	mean(SEX)
1	16026	43.1	0.17	0.54
2	12402	43.9	0.16	0.53
3	19956	44.9	0.16	0.53
4	52441	41.1	0.15	0.50
5	8728	45.0	0.13	0.52
6	27537	42.1	0.15	0.53
7	53025	42.4	0.15	0.51
8	14155	42.1	0.14	0.53
9	28719	44.7	0.14	0.53
10	47415	42.6	0.14	0.50
11	29056	45.3	0.14	0.52
12	18538	45.4	0.13	0.53
13	55575	43.7	0.13	0.52
14	7859	45.7	0.14	0.52
15	90468	41.6	0.14	0.49
16	11351	44.4	0.14	0.51
17	75975	41.8	0.14	0.50
18	28835	43.9	0.13	0.51
19	10008	45.2	0.13	0.52
20	18614	44.5	0.14	0.52
21	34838	44.1	0.13	0.50
22	15491	46.1	0.12	0.52
23	5475	46.8	0.13	0.52
24	58600	43.9	0.12	0.50
25	34754	45.2	0.13	0.51
26	16267	45.4	0.12	0.53
27	57440	43.0	0.12	0.50
28	73616	44.5	0.12	0.51
29	18841	42.4	0.13	0.50
30	142759	39.8	0.14	0.50
total	1084764			

The random effects, i.e. the 30 values of $REGION_i$, 1084764 values of IND_{ij} and 10847640 values of ϵ_{ijt} were independently generated from normal distributions. The different covariance structures of the generated populations POP1–POP4 were caused by using different variances in generating the random effects. The target was to create intra-area correlations (5.7) and intra-unit correlations (5.6), the magnitudes of which could be met in real-life applications. The variances and the resulting (expected) correlations are shown in Table 8.4. The seed numbers for generating $REGION_i$, IND_{ij} and ϵ_{ijt} were 97130, 9716026 and 971084764, respectively. It should be noted that the variance of Y is constant over time.

Table 8.4. The variances of the random effects and the resulting intra-area and intra-unit correlations in the four generated populations.

	POP1	POP2	POP3	POP4
area variance	3	3	1	1
individual variance	4	1	5	1
error variance	1	4	2	6
intra-area corr.	0.375	0.375	0.125	0.125
intra-unit corr.	0.875	0.5	0.75	0.25

After the generation of the populations we computed the realized raw correlations of Y, AGE, JS and SEX to illustrate the relevance of AGE, JS and SEX as auxiliary variables in small area estimation. The correlations were practically the same in all populations, and we present in Table 8.5 the last (10th) month's correlation matrix of POP1.

Table 8.5. The raw correlations of Y and the auxiliary variables AGE, JS and SEX in the generated population POP1, computed from the last month's data.

	Y	AGE	JS	SEX
Y	1	0.52	0.12	0.27
AGE	0.52	1	−0.03	−0.03
JS	0.12	−0.03	1	−0.07
SEX	0.27	−0.03	−0.07	1

In the simulation study the correct model refers to a model, which contains (in addition to the relevant random effects) all three variables AGE, JS and SEX as covariates. In an incorrect model only JS, the weakest of the three, is present. For the PANEL and ROTPANEL models, for instance, this means that the correct model takes the form

$$Y_{ijt} = \mu_t + \beta_1 AGE_{ijt} + \beta_2 JS_{ijt} + \beta_3 SEX_{ijt} + v_i + u_{ij} + e_{ijt}, \tag{8.2}$$

where i is region, j is unit and t is month, and region and unit effects v_i and u_{ij} are random. Note that the possible time trend is accounted by a fixed month-specific intercept μ_t to avoid unnecessary bias when estimating small area totals in the last month. The incorrect model is correspondingly

$$Y_{ijt} = \mu_t + \beta_1 JS_{ijt} + v_i + u_{ij} + e_{ijt}. \tag{8.3}$$

8.3 Performance criteria

In this simulation study the performances of the EBLUP estimators of small area totals under various models are examined from two general standpoints: the accuracy of the point estimates and the bias of the MSE estimates. The former is considered through the relative errors and absolute relative errors of the EBLUP estimates, and the latter through the realized coverage rates of 95 % confidence intervals and by comparing the MSE estimates with the realized variation of the EBLUP estimates.

8.3.1 Relative errors and bias

The bias of each EBLUP estimator is measured through the mean relative error. The relative error of the estimate $\widehat{Y}_{it}$ of the true total Y_{it} of area i at time t is

$$RE(\widehat{Y}_{it}) = \frac{\widehat{Y}_{it} - Y_{it}}{Y_{it}}.$$

The relative (design) bias of the estimator $\widehat{Y}_{it}$ is the (design) expectation of $RE(\widehat{Y}_{it})$, where Y_{it} is regarded as fixed, and it is estimated by the observed mean

$$MRE(\widehat{Y}_{it}) = \frac{1}{K}\sum_{k=1}^{K} RE(\widehat{Y}_{itk}) \tag{8.4}$$

of the relative errors $RE(\widehat{Y}_{itk})$, where $\widehat{Y}_{itk}$ is the estimate of Y_{it} obtained from the kth replication ($k = 1, 2, \ldots, K$) in the simulations. The mean relative error is calculated for every area i under each competing estimator and model.

8.3.2 Absolute relative errors

The accuracy of the EBLUP estimates is measured similarly through the absolute relative errors. These are defined as

$$ARE(\widehat{Y}_{it}) = \frac{|\widehat{Y}_{it} - Y_{it}|}{Y_{it}}.$$

A practical measure of the accuracy of $\widehat{Y}_{it}$ is the observed mean absolute relative error

$$MARE(\widehat{Y}_{it}) = \frac{1}{K}\sum_{k=1}^{K} ARE(\widehat{Y}_{itk}). \tag{8.5}$$

Also this is calculated for every area under each competing estimator and model.

8.3.3 Coverage rates

In determining confidence intervals of Y_{it} we appeal to the asymptotic normality of the EBLUP estimators. The asymptotic 95 % confidence interval is

$$\widehat{Y}_{it} \pm 1.96\sqrt{\widehat{MSE}(\widehat{Y}_{it})}.$$

The coverage rate is the observed percentage of confidence intervals covering the true Y_{it} in K replications and it should be close to 0.95. It mainly measures the validity of the estimated MSE and to some extent also the possible bias of $\widehat{Y}_{it}$. Also the coverage rates are calculated separately for every area under each competing estimator and model.

8.3.4 Bias of MSE estimators

In sampling from a large finite population the true MSE of an EBLUP estimator $\widehat{Y}_{it}$ is unknown, but it can be approximated by the empirical mean squared error

$$EMSE(\widehat{Y}_{it}) = \frac{1}{K}\sum_{k=1}^{K}(\widehat{Y}_{itk} - Y_{it})^2 \tag{8.6}$$

obtained by simulation. Now the validity or bias of a MSE estimator can be assessed by comparing the observed mean

$$M\widehat{MSE}(\widehat{Y}_{it}) = \frac{1}{K}\sum_{k=1}^{K}\widehat{MSE}(\widehat{Y}_{itk})$$

of the MSE estimates with $EMSE(\widehat{Y}_{it})$. Since the MSE of a total estimate is often very large, we prefer to operate with RMSE, the square root of the MSE, which is used in computing confidence intervals. We define the approximate relative error of an estimated RMSE as

$$RE[\widehat{RMSE}(\widehat{Y}_{it})] = \frac{M\widehat{RMSE}(\widehat{Y}_{it}) - \sqrt{EMSE(\widehat{Y}_{it})}}{\sqrt{EMSE(\widehat{Y}_{it})}}, \tag{8.7}$$

where

$$M\widehat{RMSE}(\widehat{Y}_{it}) = \frac{1}{K}\sum_{k=1}^{K}\sqrt{\widehat{MSE}(\widehat{Y}_{itk})}. \tag{8.8}$$

The relative error (8.7) can interpreted as a measure of bias of the RMSE estimator. It is calculated for every area i under each competing estimator and model.

8.4 Repeated sampling

Recall first that target was to estimate the regional population totals of the study variable Y in December 1997, the last of the 10 time points included in the synthetic population data sets POP1–POP4. From each of these populations the repeated sampling of the simulation study was carried out in similar way. The number of repeated samples, or replications, was $K = 1000$.

Table 8.6. Regional (cross-sectional) population sizes and minimum, maximum and mean sample sizes of the last month's data in the 1000 replications of the simulation study.

region i	N_i	$\min(n_i)$	$\text{mean}(n_i)$	$\max(n_i)$
1	16026	4	14.9	29
2	12402	3	11.4	23
3	19956	5	18.3	35
4	52441	30	48.4	70
5	8728	1	8.1	18
6	27537	11	25.4	41
7	53025	27	48.9	72
8	14155	4	13.0	25
9	28719	14	26.4	40
10	47415	22	43.8	69
11	29056	12	26.7	45
12	18538	4	17.1	30
13	55575	33	51.4	76
14	7859	0	7.3	16
15	90468	60	83.0	113
16	11351	1	10.4	23
17	75975	44	70.1	97
18	28835	10	26.7	43
19	10008	1	9.2	20
20	18614	6	17.1	30
21	34838	17	32.1	52
22	15491	4	14.3	31
23	5475	0	4.9	12
24	58600	30	54.1	79
25	34754	15	32.0	49
26	16267	5	14.9	28
27	57440	34	53.1	80
28	73616	45	67.7	89
29	18841	7	17.6	31
30	142759	97	131.6	165
total	1084764		1000	

In each replication an SRS sample of $n = 1000$ units was first drawn from the December 1997 population data of $N = 1084764$ units. The EBLUP estimation with NESTED model was done from this cross-sectional sample data. The seed number given to the SAS procedure SURVEYSELECT, by which the sampling was carried out, was $6121959 + k$, where $k = 1, 2, \ldots, 1000$ is the number of replication. Some statistics of the realized regional sample sizes n_i are shown in Table 8.6.

The complete panel sample of 10 time points was then constituted by adding the similar samples from the 9 preceding months to the December sample data. The estimation with PANEL model was done from this data of $10 \times 1000 = 10000$ observations.

The design for obtaining rotating panel data follows the one used in the monthly Finnish Labour Force Survey of Statistics Finland, which has already been illustrated in Table 2.3. Table 8.7 presents the design employed in the simulations.

Table 8.7. The rotating panel design of the simulation study.

	time point									
panel	1	2	3	4	5	6	7	8	9	10
1	X									
2		X								
3			X							
4	X			X						
5		X			X					
6			X			X				
7				X			X			
8					X			X		
9						X			X	
10	X						X			X
11		X						X		
12			X						X	
13	X			X						X
14		X			X					
15			X			X				
16	X			X			X			
17		X			X			X		
18			X			X			X	
19				X			X			X
20					X			X		
21						X			X	
22							X			X
23								X		
24									X	
25										X

The rotating panel sample data consist of 25 panels of 200 units. Each panel is surveyed at time points indicated by 'X' in Table 8.7. For instance, panel 4 provides observations from time points 1 and 4 (March and June 1997), in total $200 + 200 = 400$ observations for the sample data set. The sample data of each month consist of 5 panels and 1000 observations. It is seen that the data of adjacent months have no overlap. Instead, there is 60 % overlap (3 panels out of 5) at interval of three months. Altogether, the data contain $10 \times 1000 = 10000$ observations from $25 \times 200 = 5000$ units.

In each replication the rotating panel data was constituted in the following way. First, the previously drawn December sample data was randomly split into five equal-sized panels (numbers 10, 13, 19, 22 and 25 in Table 8.7). In this way the rotating panel sample matches completely with the panel sample and the cross-sectional sample at the last survey month. Then, another cross-sectional SRS sample of 4000 units was drawn from the December population so that it does not contain any already sampled units. This sample was again randomly split into the rest 20 panels. The eventual 10-month sample data was then composed by gathering up the monthly observations for each panel according to the design shown in Table 8.7.

This data set was used in the EBLUP estimation with ROTPANEL and RANTIME models. The seed number was $20903 + k$ in drawing the sample of 4000 units and $321 + k$ in the panel splitting, where k again denotes the replication number.

9 Simulation study with correct model: results for longitudinal data of 10 occasions

In the estimation of area totals of the last (Tth) month with PANEL and ROTPANEL approaches the employed model was shown in (8.2). The model for NESTED approach is

$$Y_{ijT} = \mu + \beta_1 AGE_{ijT} + \beta_2 JS_{ijT} + \beta_3 SEX_{ijT} + v_i + e_{ijT}, \tag{9.1}$$

which only uses the cross-sectional data of the last month T, and the model for RANTIME approach is

$$Y_{ijt} = \mu + \beta_1 AGE_{ijt} + \beta_2 JS_{ijt} + \beta_3 SEX_{ijt} + v_i + u_t + e_{ijt}, \tag{9.2}$$

where u_t is the normal random effect of month $t = 1, 2, \ldots, T$. This model assumes no panel design, that is, the individuals at different times are independent and no individual random effect is present. The model (9.2) was also considered with time effects fixed, but the observed estimator performance was virtually unchanged.

In this chapter we consider the results of using the longitudinal data of 10 months in the estimation with PANEL, ROTPANEL and RANTIME approaches and compare them with those of the NESTED approach. That is, we examine the results obtained under the correct models PANEL10c, ROTPANEL10c, RANTIME10c and NESTEDc, cf. Table 8.2. The results for the populations POP1–POP4 are presented together to explore the effect of covariance structure (characterized by the intra-area and intra-unit correlations, i.e. factors A and B in Section 8.1) on the estimator performance. The results for 5-month data are presented and compared with these results in Chapter 11.

9.1 Bias

The bias of the region total estimates is measured by the mean of the observed relative estimation errors. The results for the populations POP1–POP4 are given in Appendix A in Tables A.1, A.6, A.11 and A.16 (respectively). The tables present the observed bias under models NESTEDc, PANEL10c, ROTPANEL10c and RANTIME10c together with the mean cross-sectional sample size of the last month in 1000 simulations. The corresponding plots of the bias against the last month's mean sample size are presented in Figures 9.1, 9.2, 9.3 and 9.4.

We emphasize that the interest is here in the design bias calculated over the repeated samples from the fixed populations. All the EBLUP estimates are here model-unbiased since the employed models agree with the one used in generating populations. However, because each fixed population is just one specific realization of the correct model, it may happen that the model does not hold well in all areas. Consequently the model-based

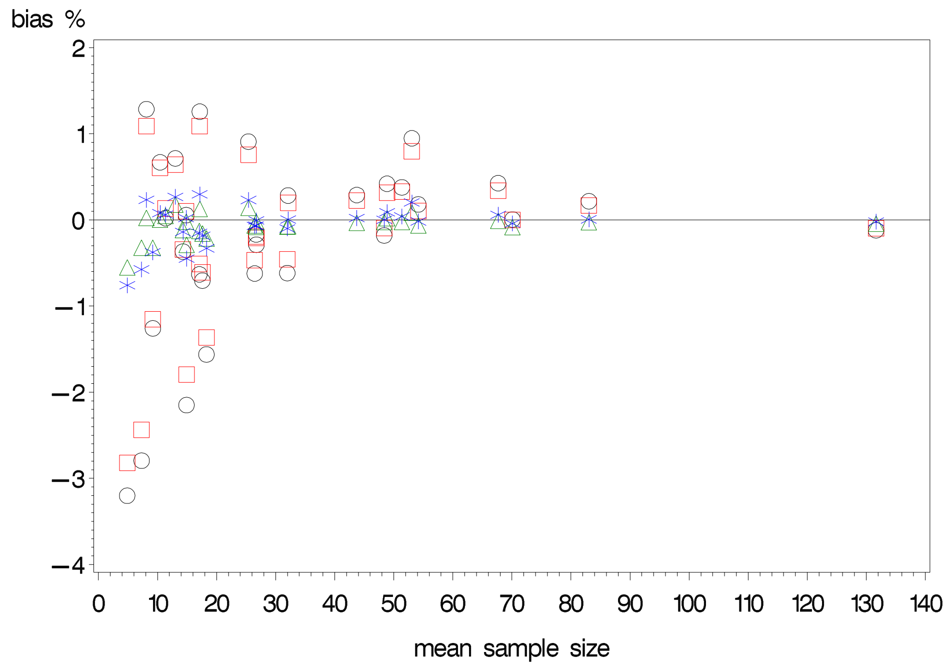

Figure 9.1: Relative bias plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

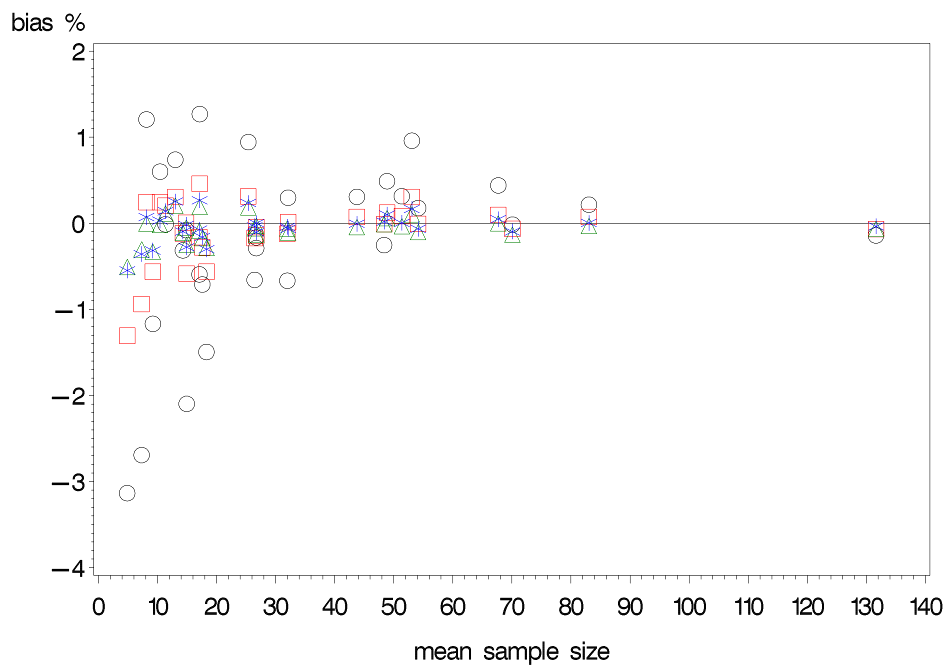

Figure 9.2: Relative bias plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

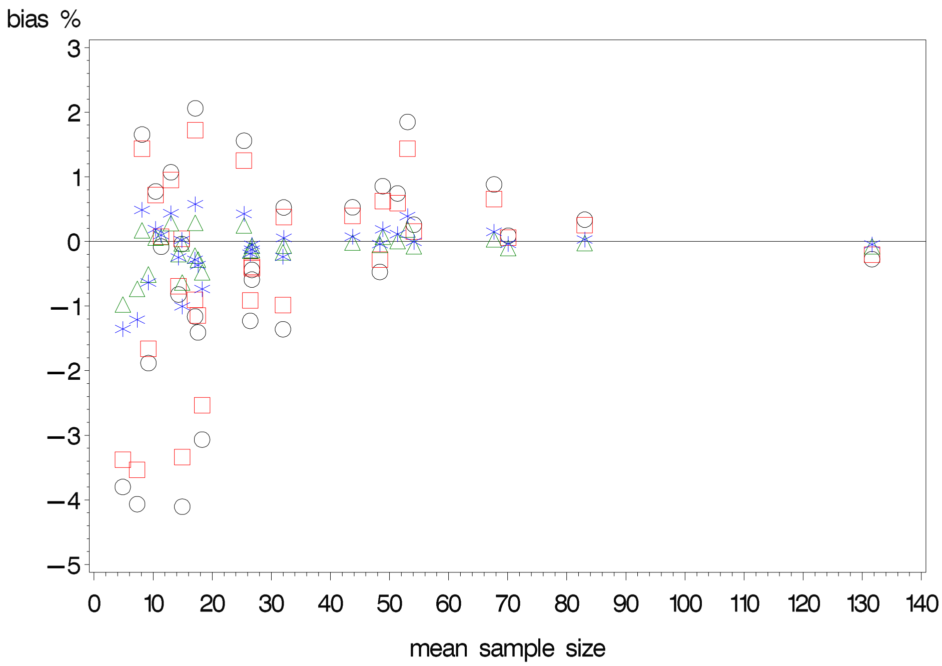

Figure 9.3: Relative bias plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

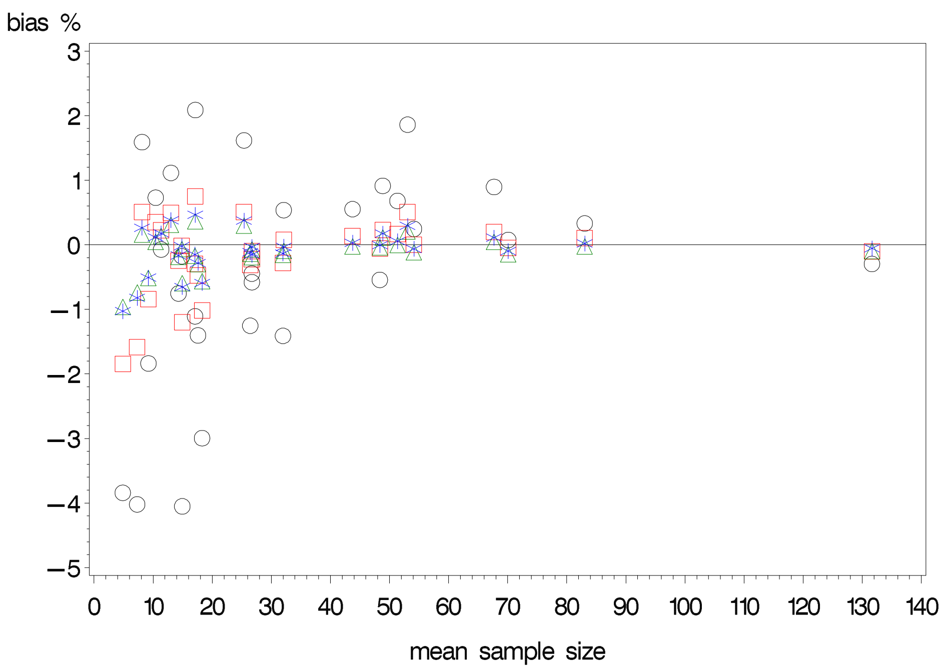

Figure 9.4: Relative bias plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

estimators may have some design bias over the repeated samples, even with the SRS sampling.

In general, the design biases are not serious and in the largest areas the models give almost equally unbiased estimates. In smaller areas, however, it is seen that especially the estimates obtained from the cross-sectional data with the NESTED model are prone to some design bias, and when the intra-unit correlation is high (Figures 9.1 and 9.3), the complete panel data (PANEL model) gives only slightly better results. In this case the observations from previous time points offer only little additional information over the cross-sectional information at the current time point. As a result the effective sample size does not increase much along with the increased number of survey occasions.

When the "longitudinal" intra-unit correlation is low (Figures 9.2 and 9.4), the observed biases with the PANEL model reduce towards the small biases obtained with the ROTPANEL and RANTIME models. Now the previous observations differ more from the current observations, thus increasing the effective sample size more than under high intra-unit correlation.

The ROTPANEL and RANTIME models, which use the rotating panel data, correct for the bias significantly. Overall, the RANTIME model seems to perform best, but the difference between it and ROTPANEL is negligible.

On the whole, the observed biases seem to increase as the within-area heterogeneity increases, or the intra-area correlation decreases (see Figures 9.3 and 9.4). The estimators from cross-sectional or complete panel data show notable biases in several areas, which are not necessarily small, while the estimators from rotating panel data perform well. One explanation is that under large within-area variance the probability to get small cross-sectional samples, which do not represent well the population, is increased, and for some reason these "bad" samples have not cancelled each other out in the simulation. Another possibility is that the large unit variance employed in generating the fixed population have increased the chance to get such realizations of area populations, which do not obey well the original model. A natural result from this is that the model-based estimates for those areas are not always good.

Although the population-specific results do not differ much from each other, comparisons between Figures 9.1–9.4 suggest that biases are smallest when the intra-area correlation is high and intra-unit correlation low (POP2) and largest when the circumstances are opposite (POP3). Thus, as far as bias is concerned, ideal data for small area estimation would be longitudinal data, in which the units within areas are fairly homogeneous so that even a small cross-sectional sample is representative, but at the same time there is enough variation over time so that the effective sample size is genuinely increased by the observations from previous occasions.

We take then a further look at the bias occurring in the smallest areas and use the results for region 23 in the simulations from POP1 as an example. Figure 9.5 shows the 1000 relative estimation errors obtained by the models NESTED and ROTPANEL

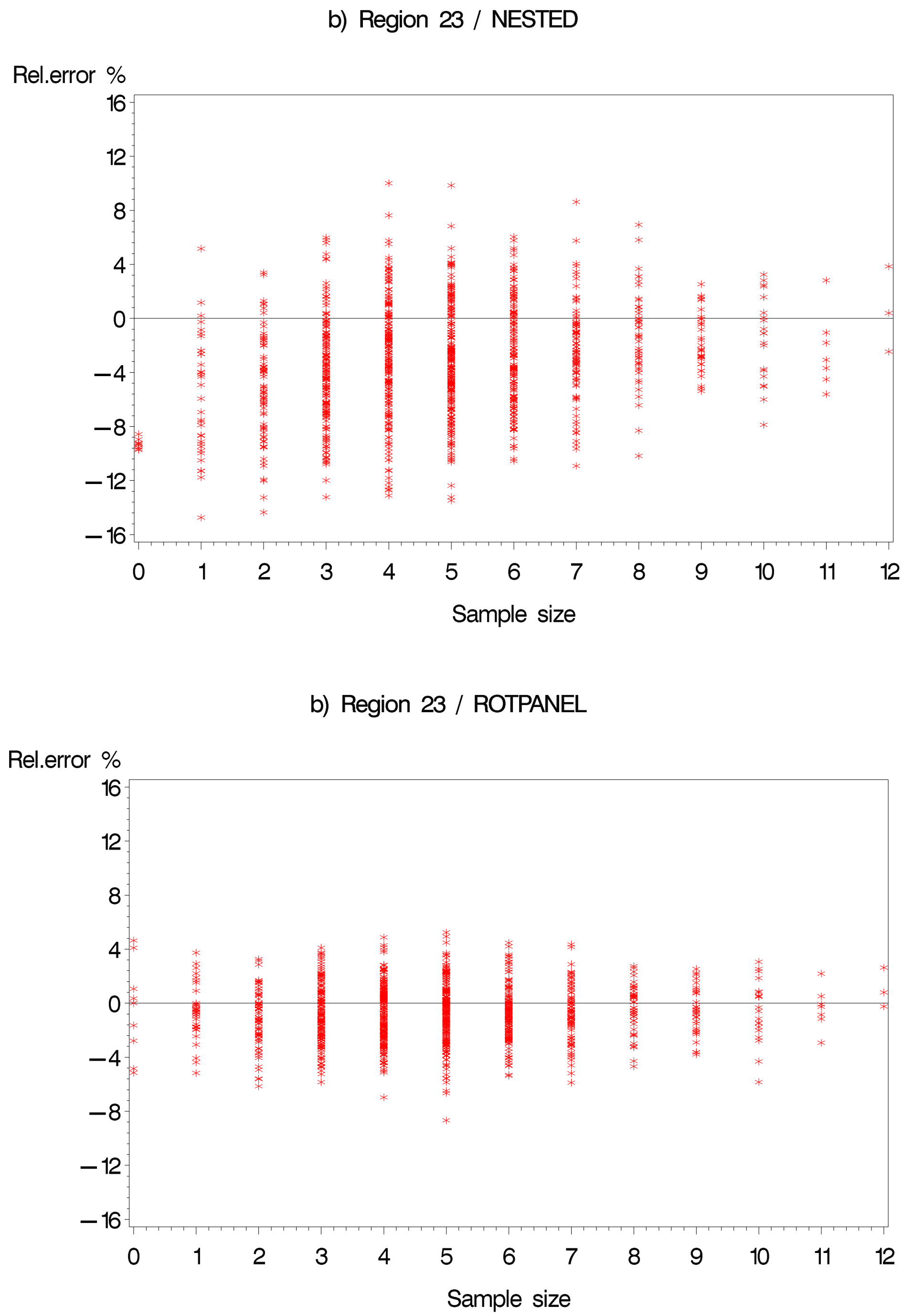

Figure 9.5: Relative errors of EBLUP estimates in region 23 under a) NESTED model b) ROTPANEL model in simulations from POP1.

in this region. In this case the NESTED model represents the PANEL model and the ROTPANEL model represents the RANTIME model well so that we present figures for the two models only.

For the NESTED model (upper part of Figure 9.5) we see that the sample size is often very small, sometimes zero, and when sample size increases, the bias reduces. When the sample size is zero, the EBLUP estimate comes only from the fixed part of the model (i.e. from auxiliary variables), since the predicted area effect has shrunken to zero because of missing information. When sample size is close to zero, the predicted area effect is close to zero due to the weakness of the information about that area and the fixed part dominates in the estimate. Understandably, the panel design does not help very much here.

Now we know that the true effect of area 23 in POP1 is clearly positive so that the shrinkage towards zero makes the EBLUP estimates overly negative in the smallest samples. The same happens in area 14. Then, in area 5, for example, we could see the opposite: the true area effect is negative, and the shrinkage in predicting it causes positive bias in the EBLUP estimation. As the sample size increases, the shrinkage decreases, leading to better estimates.

The small sample bias is largely avoided by using the rotating panel data (lower part of Figure 9.5, note that sample size refers to the cross-sectional sample size of the last month). Then the data provide additional observations from earlier occasions, which come to strengthen especially the prediction of area effects in small regions. This again decreases the shrinkage and the EBLUP estimates become more accurate.

9.2 Absolute relative errors

The accuracy of a point estimate of a small area total is measured by the mean absolute relative error (MARE). The smaller is the MARE, the higher is the accuracy of the estimates. The results for the populations POP1–POP4 are given in Appendix A in Tables A.2, A.7, A.12 and A.17 (respectively). The corresponding plots are presented in Figures 9.6–9.9.

From these figures we see immediately the effect of sample size to the accuracy. The rate of the absolute decrease in MARE, as the sample size increases, is fastest for the models NESTED and PANEL, but the rate of the relative decrease is almost the same for every model. For the considered models, the MARE of the total estimate of the largest area (area 30 with mean cross-sectional sample size over 130) is 20–30 % of that of the smallest area (area 23 with mean cross-sectional sample size 5), and the MARE is approximately halved as the cross-sectional sample size increases from 5 to 25.

The Figures 9.6–9.9 also show the superior performance of the models ROTPANEL and RANTIME. Using rotating panel data with the ROTPANEL model reduces the MARE

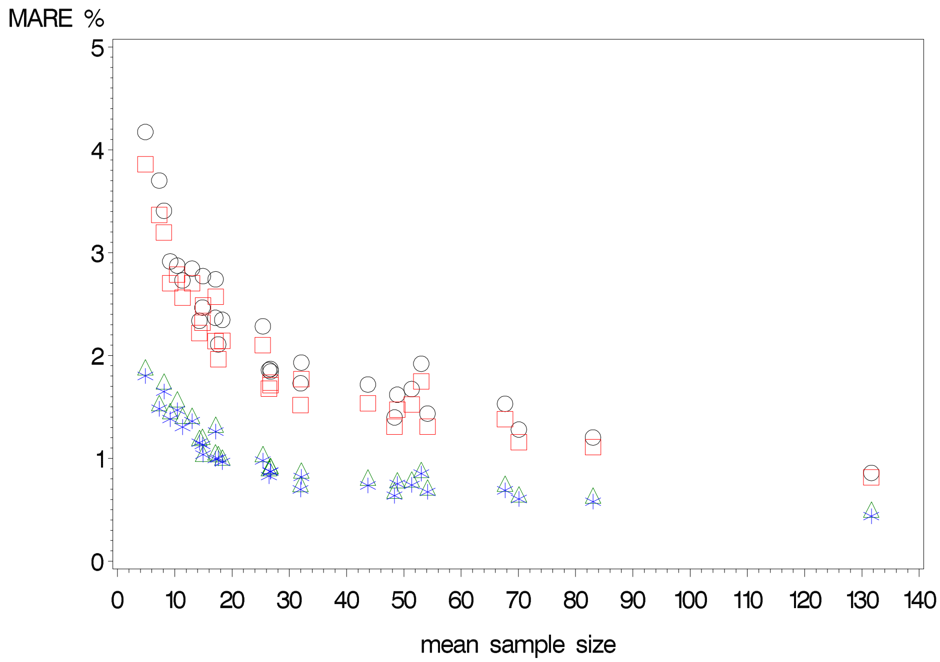

Figure 9.6: MARE plotted against last month's mean sample size of region in simulations from POP1. Symbols used are ○ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

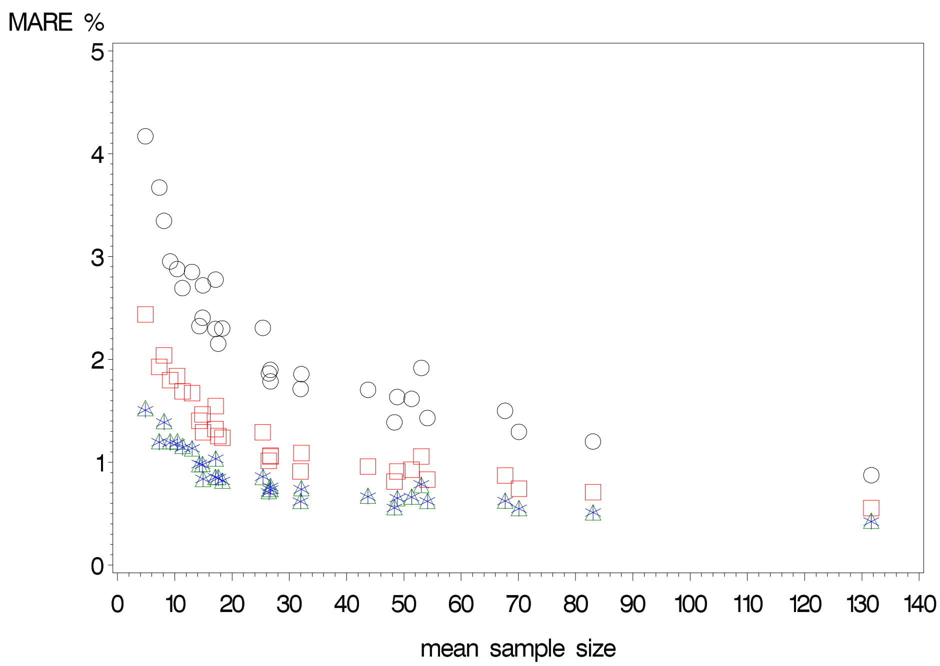

Figure 9.7: MARE plotted against last month's mean sample size of region in simulations from POP2. Symbols used are ○ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

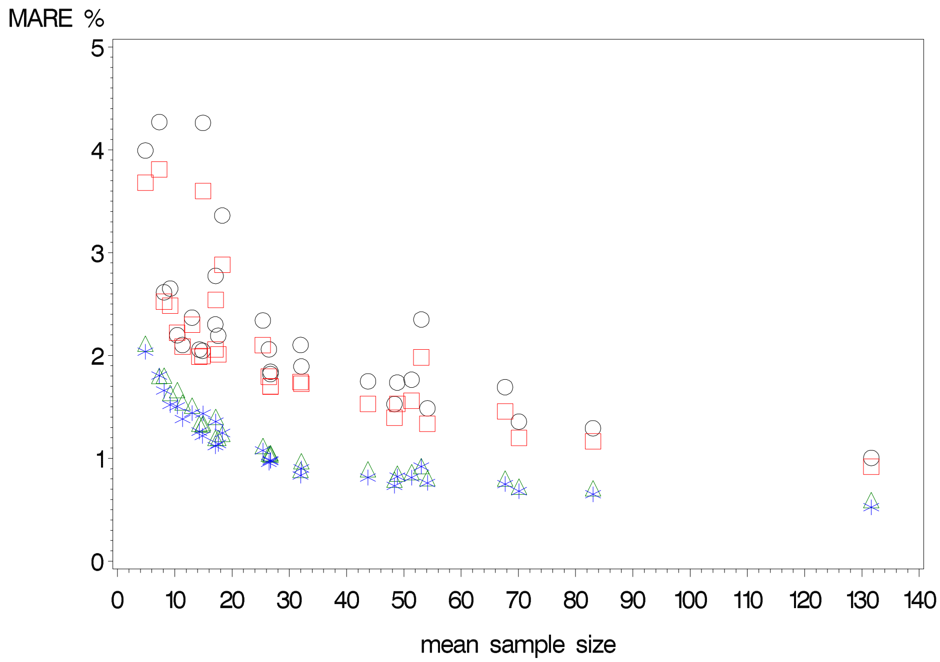

Figure 9.8: MARE plotted against last month's mean sample size of region in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

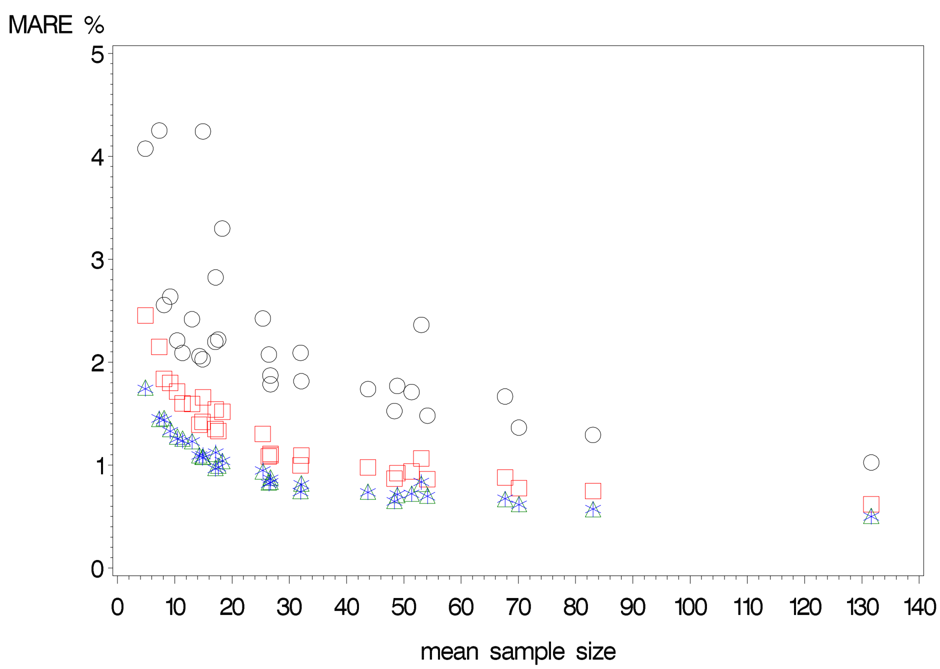

Figure 9.9: MARE plotted against last month's mean sample size of region in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

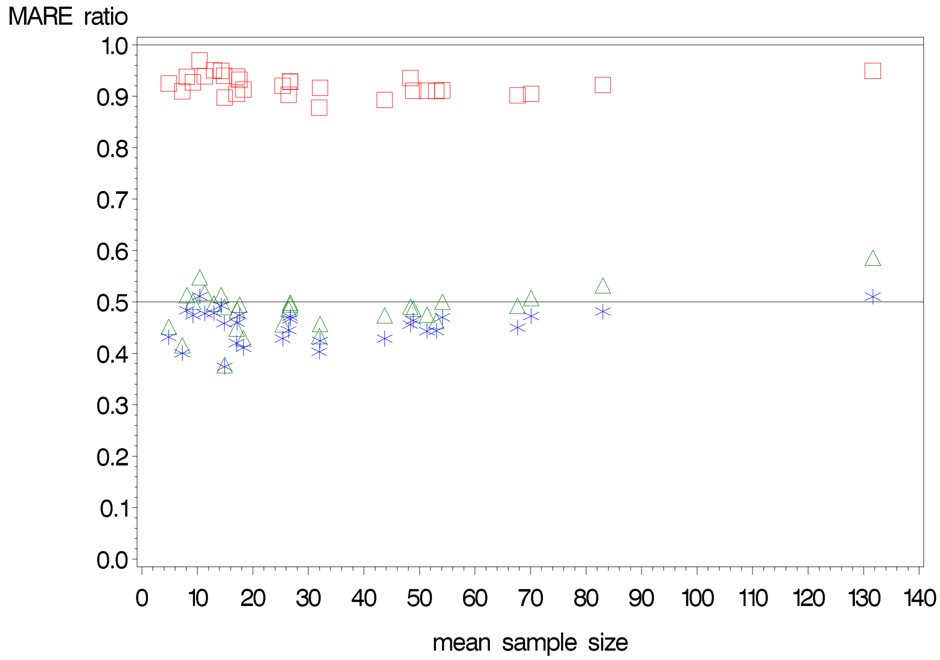

Figure 9.10: MARE reduction in terms of MARE ratios plotted against last month's mean sample size in simulations from POP1. Symbols used are □ = PANEL10c/NESTEDc, ∗ = ROTPANEL10c/NESTEDc and △ = RANTIME10c/NESTEDc.

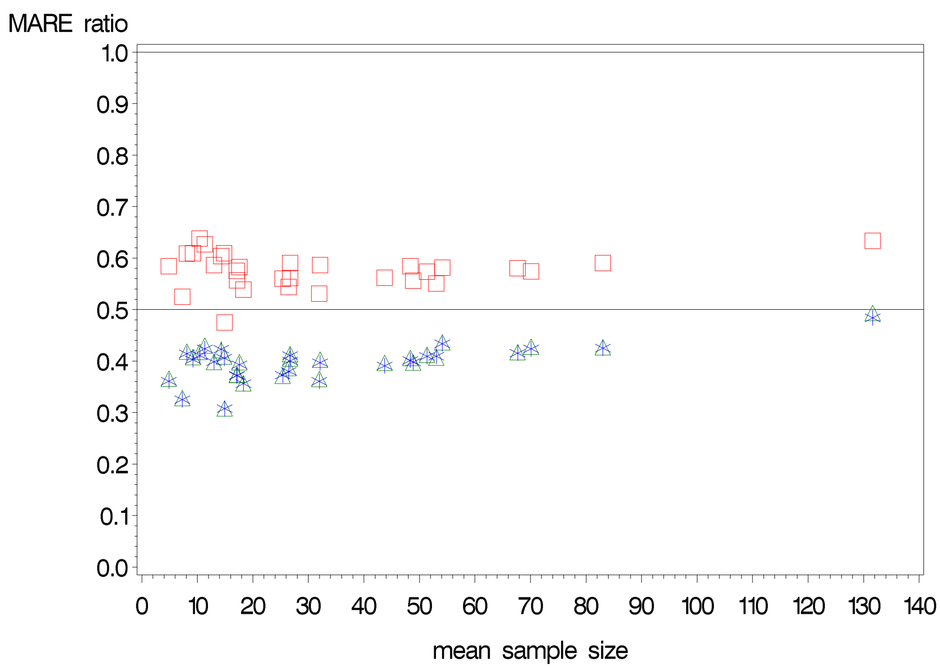

Figure 9.11: MARE reduction in terms of MARE ratios plotted against last month's mean sample size in simulations from POP2. Symbols used are □ = PANEL10c/NESTEDc, ∗ = ROTPANEL10c/NESTEDc and △ = RANTIME10c/NESTEDc.

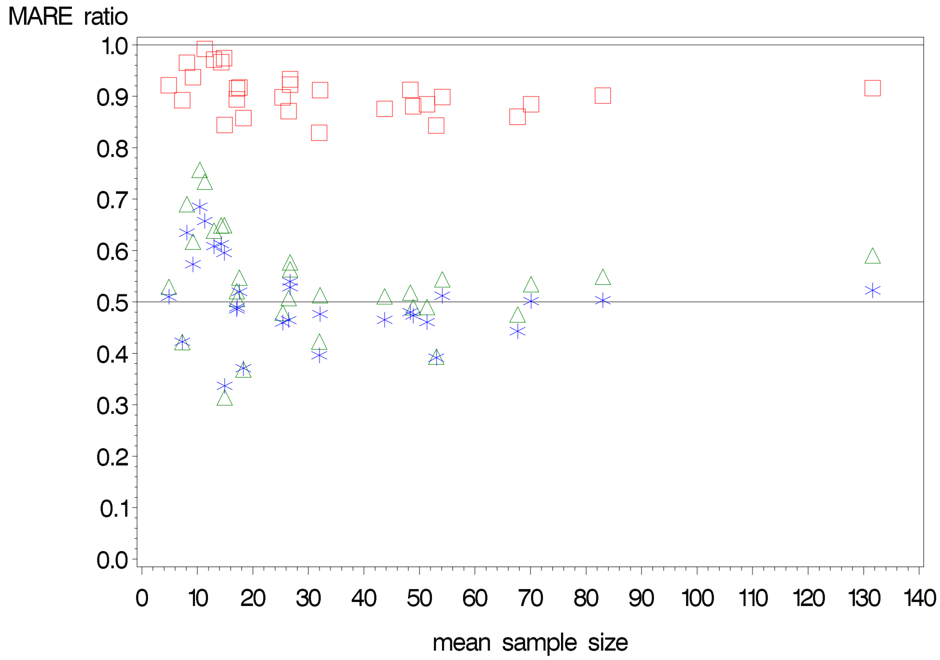

Figure 9.12: MARE reduction in terms of MARE ratios plotted against last month's mean sample size in simulations from POP3. Symbols used are □ = PANEL10c/NESTEDc, ∗ = ROTPANEL10c/NESTEDc and △ = RANTIME10c/NESTEDc.

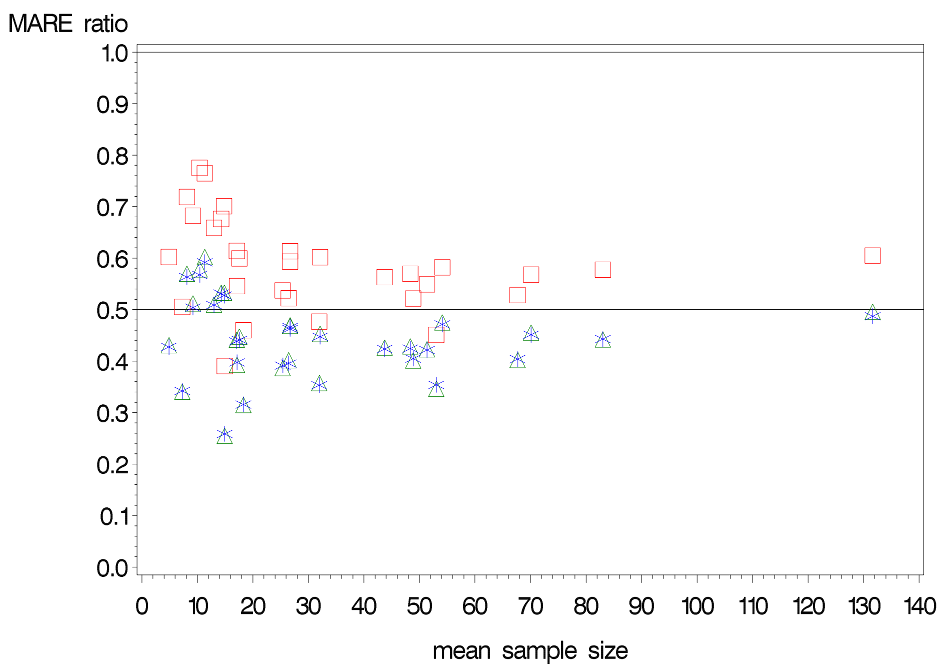

Figure 9.13: MARE reduction in terms of MARE ratios plotted against last month's mean sample size in simulations from POP4. Symbols used are □ = PANEL10c/NESTEDc, ∗ = ROTPANEL10c/NESTEDc and △ = RANTIME10c/NESTEDc.

on average 50-60 % compared to the cross-sectional case (model NESTED), depending on the population. The reduction is largest with the population POP2 of high within-area homogeneity and low longitudinal intra-unit correlation (Figure 9.11). The results for the RANTIME model are almost similar, the average MARE reduction being 46–60 %. In the point estimation the RANTIME model seems to perform almost as well as the ROTPANEL model even if it wrongly assumes that the longitudinal unit-level observations are not correlated.

The performance of the PANEL model depends essentially on the magnitude of the intra-unit correlation. When the correlation is high (POP1 and POP3), the average gain of using panel data instead of cross-sectional data is only 9 % in terms of MARE reduction. When the correlation is low (POP2 and POP4), the average reduction is 42 %. We see again how the high intra-unit correlation inhibits the increase of effective sample size in the case of complete panel data. A similar conclusion was made already with the bias considerations.

The MARE reductions are illustrated in Figures 9.10–9.13, where the MARE ratios, i.e. the ratios of the PANEL model MARE, the ROTPANEL model MARE and the RANTIME model MARE to the NESTED model MARE, are plotted against the mean sample size. It seems that the gains of using rotating panel data are often biggest in medium-sized areas, although the differences are small.

The certain large biases in the estimates under models NESTED and PANEL, which came up with the reduced within-area homogeneity (Figures 9.3 and 9.4), carry over into the average MARE also. This shows in Figures 9.8 and 9.9 as irregularities in the MARE decrease with the sample size increase. Similar irregularities were not met with the populations POP1 and POP2 of higher within-area homogeneity.

Besides the mean absolute relative errors it is interesting to examine the maximum absolute relative errors, which reflect the "worst possible" performance of the estimators. These are presented in Figures 9.14–9.17. The behaviour of the maximum absolute relative errors is in much similar to that of the mean absolute relative errors. The best results are obtained with the ROTPANEL and RANTIME models, and the performance of the PANEL model improves as the intra-unit correlation decreases. Overall, utilizing the rotating panel data either by ROTPANEL model or the RANTIME model does not only improve the average performance of the estimators but also protects from very bad estimates.

If we compare the Figures 9.14–9.17 with each other, we note that the lower intra-unit correlation (POP2 and POP4) turns to stronger protection by rotating panel data (and to lesser extent by panel data). On the other hand, we observe that the performance of NESTED is correspondingly deteriorated. We conclude that the estimation from panel or rotating panel data benefits from the lowered longitudinal correlation through the increased effective sample size, but the estimation from cross-sectional data only suffers from the increased error variance.

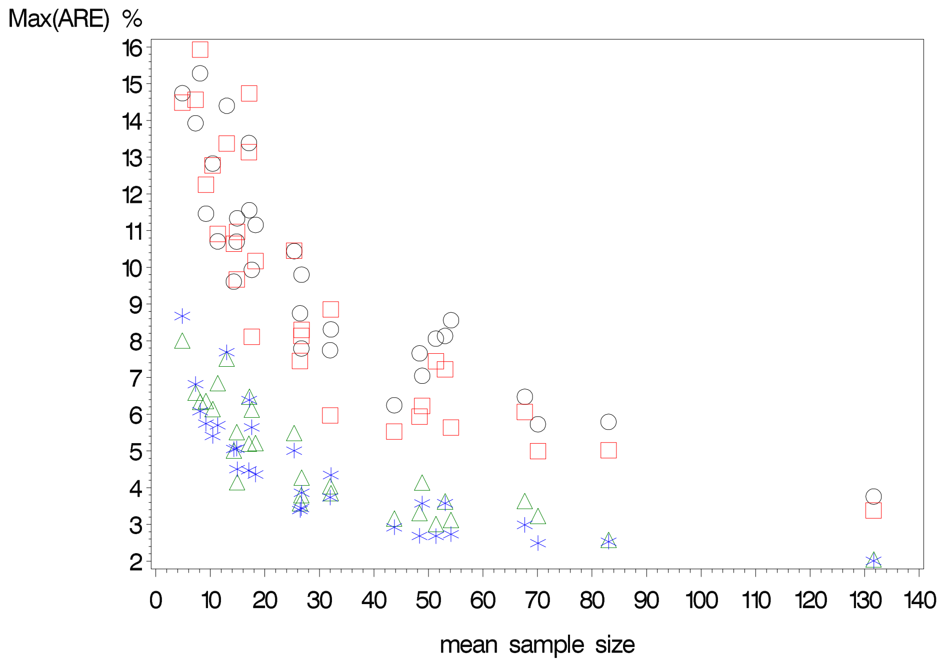

Figure 9.14: Maximum absolute relative errors plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

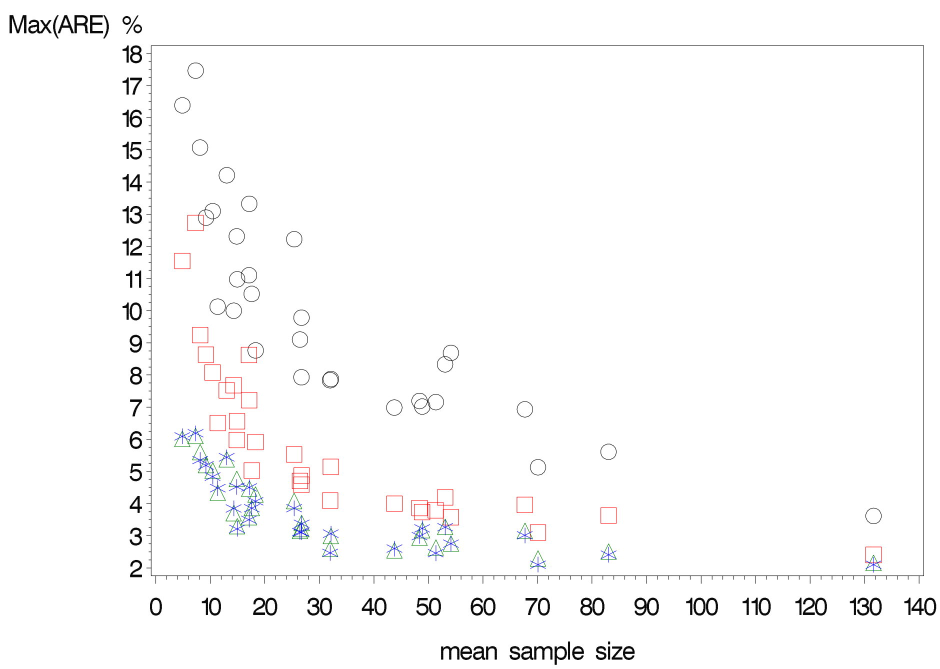

Figure 9.15: Maximum absolute relative errors plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

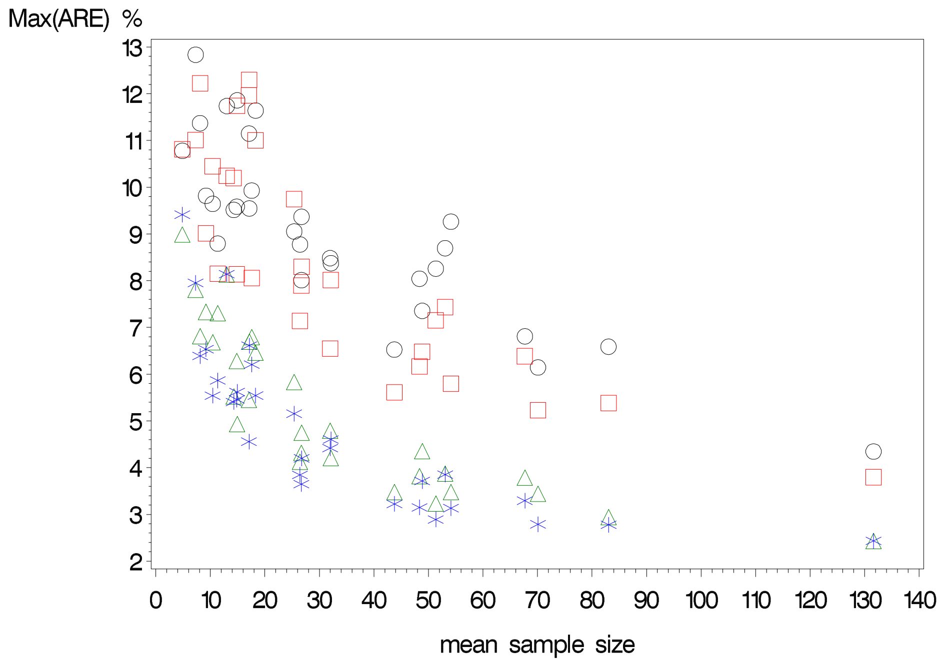

Figure 9.16: Maximum absolute relative errors plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

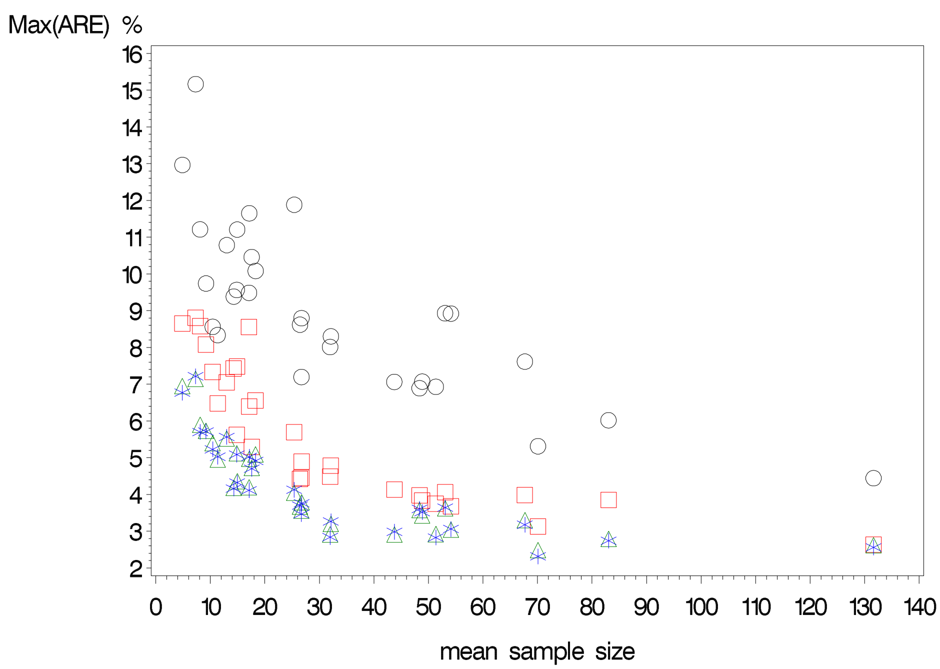

Figure 9.17: Maximum absolute relative errors plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

9.3 Coverage of confidence intervals

The observed coverage rates of the nominal 95 % confidence intervals calculated from the EBLUP point estimates and their estimated mean squared errors for the populations POP1–POP4 are shown in Tables A.3, A.8, A.13 and A.18 of the appendix A. Correspondingly, the Figures 9.18–9.21 show the coverage rates plotted against the mean sample sizes of regions in the last month. In these figures the nominal 95 % level is indicated by the horizontal line. In this simulation study of 1000 replications we tolerate a deviation of 1–2 percentage units from the nominal level.

We note first that the RANTIME model performs poorly, giving too low rates especially when the intra-unit correlation is strong (Figures 9.18 and 9.20). The reason is that it does not recognize the panel design and treats the repeated measurements erroneously as independent replications, which leads to underestimated MSE's. The underestimation reduces slightly as the sample size increases. When the intra-unit correlation is not strong (Figures 9.19 and 9.21), the RANTIME model still gives too low rates, but this "bias" is not so striking any more (average rate is over 0.93). The relatively low correlation between the longitudinal observations takes here the independence assumption of the RANTIME model closer to the truth.

We look then at the populations POP1 and POP2 of small within-area heterogeneity (Figures 9.18 and 9.19). In general, the other models give here acceptable rates, which suggests that both the region totals and the MSE's are adequately estimated. Particularly this holds for the ROTPANEL model. It is also seen, however, that the deviations from the nominal rate are slightly larger in smaller areas, and in certain areas especially the NESTED model gives too low coverage rates.

The area 26 seems especially problematic, and we examine its properties to find an explanation for the low coverage. It appears that among all the considered areas the area effect, and consequently the mean of response Y, is highest in area 26. The shrinkage in predicting the area effect brings then some negative design bias (see Tables A.1 and A.6) into the EBLUP estimate of the area total. Since the observed variance of the EBLUP estimates of area 26 is not that large, we conclude that the squared bias raises the empirical MSE to a level, which the MSE estimates cannot fully reach (cf. Figures 9.26 and 9.27).

The performance of the PANEL model in the problematic areas depends again on the magnitude of the intra-area correlation. When the correlation is high (Figure 9.18), the PANEL model does not perform much better than the NESTED model, but when the correlation is low (Figure 9.19), the performance improves due to the reduced bias of the EBLUP estimates.

As for the results for the populations POP3 and POP4 of large within-area heterogeneity, the ROTPANEL model again performs adequately. The average coverage rates, computed over all 30 areas, of the NESTED and PANEL models are also adequate. However, the Figures 9.20 and 9.21 show that in smaller areas the NESTED and PANEL

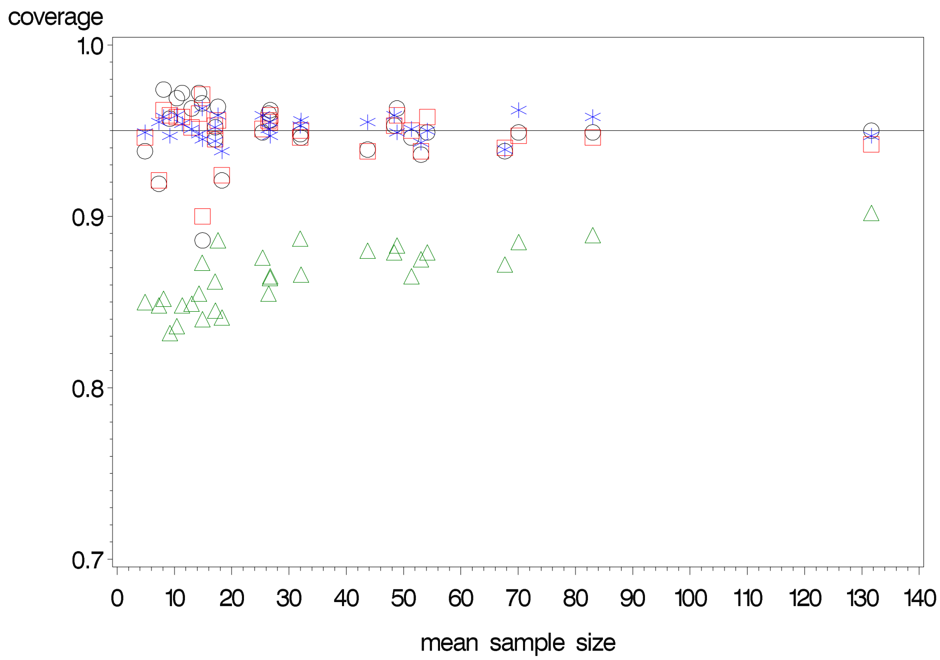

Figure 9.18: Coverage of asymptotic 95 % confidence intervals plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

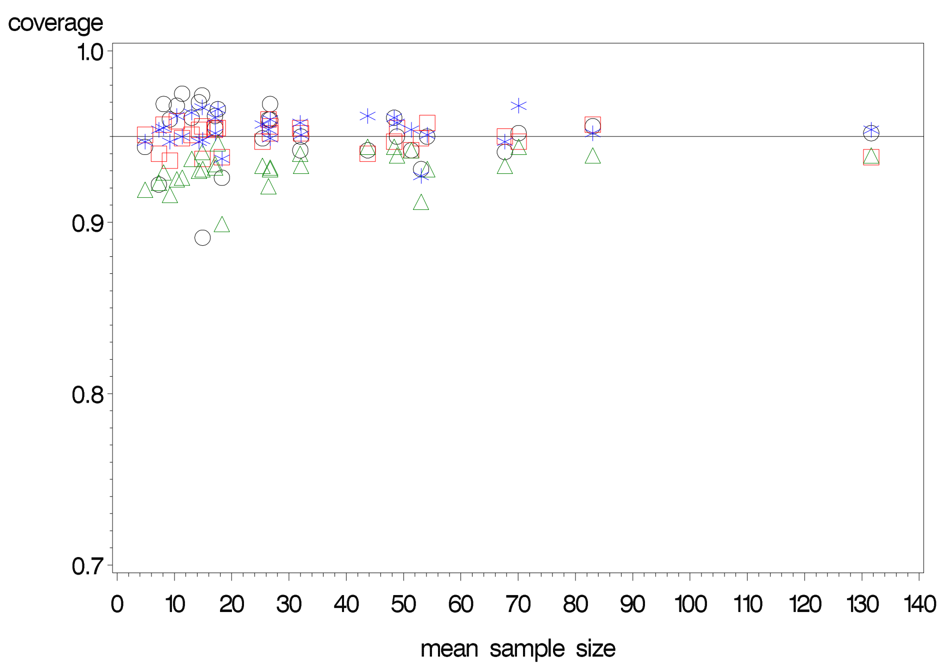

Figure 9.19: Coverage of asymptotic 95 % confidence intervals plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

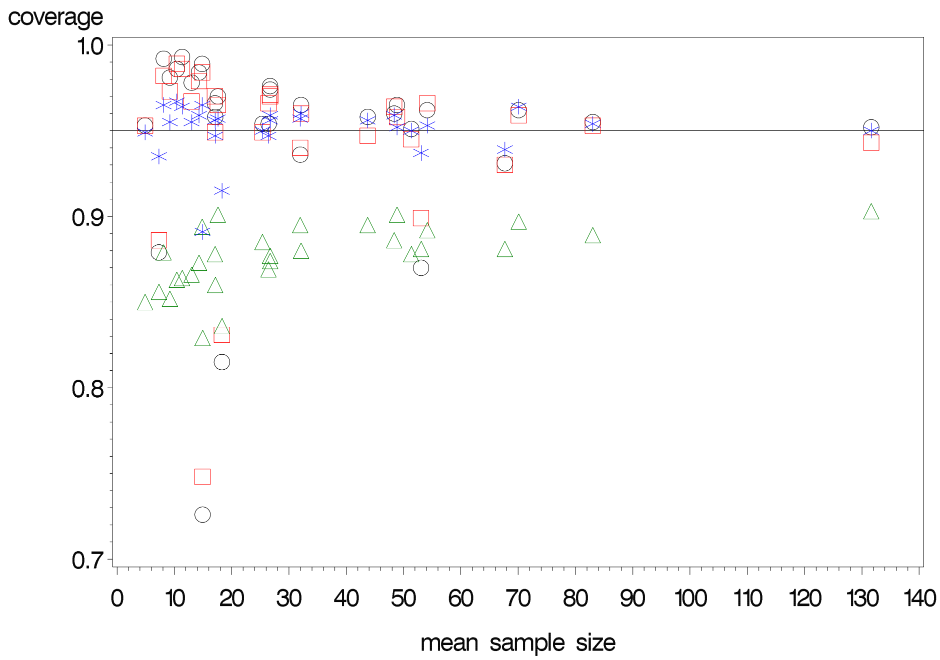

Figure 9.20: Coverage of asymptotic 95 % confidence intervals plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

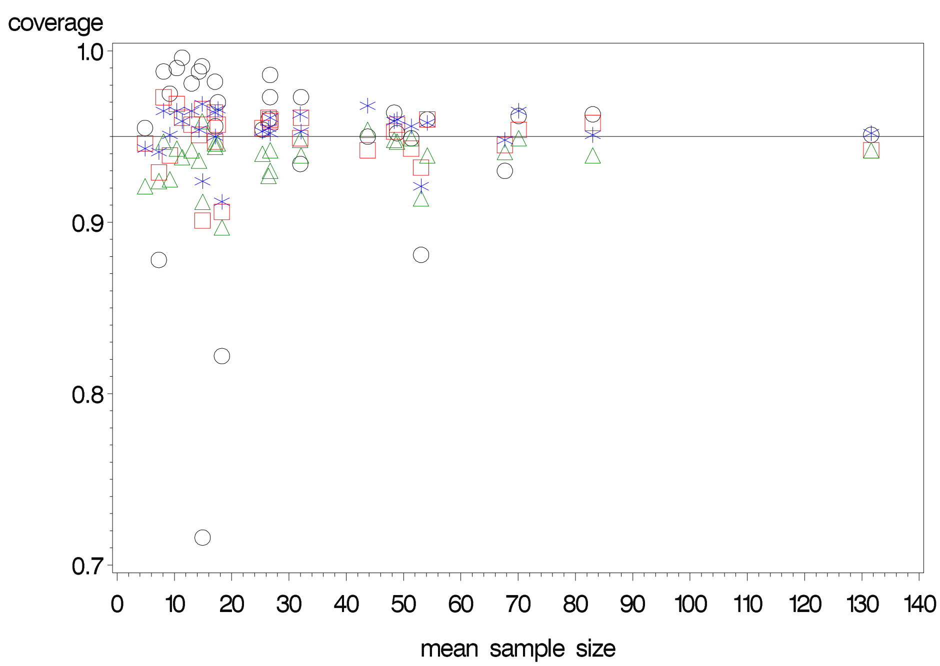

Figure 9.21: Coverage of asymptotic 95 % confidence intervals plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

models yield now severe deviations from the nominal coverage rate (the performance of the PANEL improves as the intra-unit correlation goes down, though). In most areas, especially in the smallish ones, the empirical coverage rate is overly high, sometimes very close to 1. On the other hand, in some areas, especially in areas 26 and 3, which already appeared problematic, the rate is unacceptably low. The problematic coverage rates indicate problems in the MSE estimation. These can stem from problems in estimating the design variance of the EBLUP estimators or possible design biases in the EBLUP estimates (cf. Figures 9.3 and 9.4). The reasons for the latter under large within-area heterogeneity were briefly discussed in Section 9.1. Anyway, using rotating panel data with the ROTPANEL model makes the coverage rates settle reasonably close to the nominal 95 % level in every area.

9.4 MSE estimation

In this section we evaluate the performance of the estimators of the mean squared errors of the area total estimates. For convenience, instead of MSE estimates we consider their square roots, i.e. RMSE estimates, which are proportional to the lengths of confidence intervals.

The bias of the RMSE estimates is examined by comparing the mean (8.8) of RMSE estimates to corresponding empirical root mean squared error (ERMSE), i.e. the square root of (8.6), which is regarded as an estimate of the unknown true RMSE. The approximate relative error (8.7) is used as the measure of bias.

For each population POP1–POP4, the regional ERMSE's and the observed means of regional RMSE estimates are shown with the approximate relative error in Tables A.4 and A.5 (for POP1), A.9 and A.10 (for POP2), A.14 and A.15 (for POP3) and A.19 and A.20 (for POP4) of the appendix. The observed means of the estimated RMSE's for the populations POP1–POP4 are shown in Figures 9.22–9.25, where they are plotted against the mean sample sizes of the last month.

As expected, the cross-sectional NESTED model always yields the largest RMSE estimates. When the intra-unit correlation is high (POP1 and POP3), indicating high stability of longitudinal observations, the PANEL model reduces the estimates only slightly compared to the NESTED model. The average RMSE reduction is only about 10 % here. When the intra-unit correlation is low (POP2 and POP4), the PANEL model improves considerably, yielding 43 % average reduction. This corresponds the decreased similarity of the observations made over time, which increases the effective sample size and makes borrowing strength over time genuinely possible even when the very same units are observed.

The gains of using rotating panel data are notably larger. Under high intra-unit correlation (POP1 and POP3) the average RMSE reduction is 53 % with ROTPANEL model and 60 % with RANTIME model. Under high intra-unit correlation (POP2 and

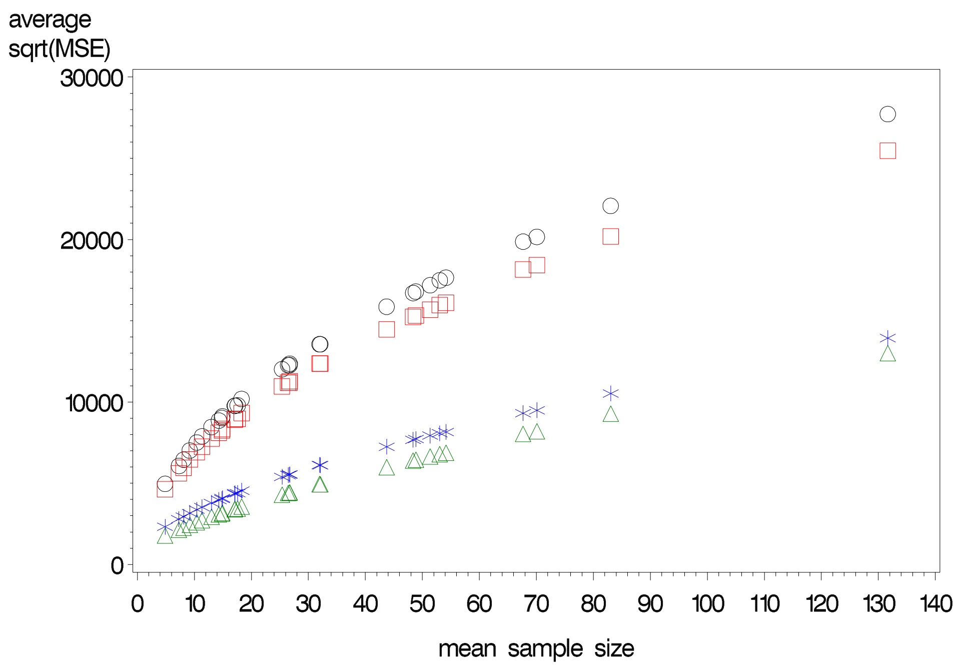

Figure 9.22: Mean estimated RMSE plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

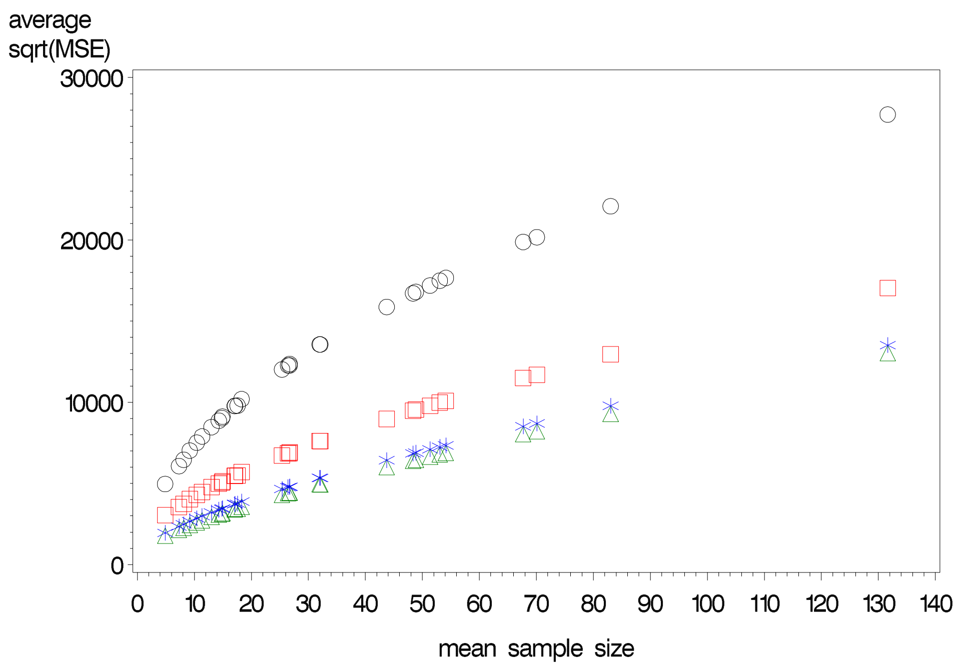

Figure 9.23: Mean estimated RMSE plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

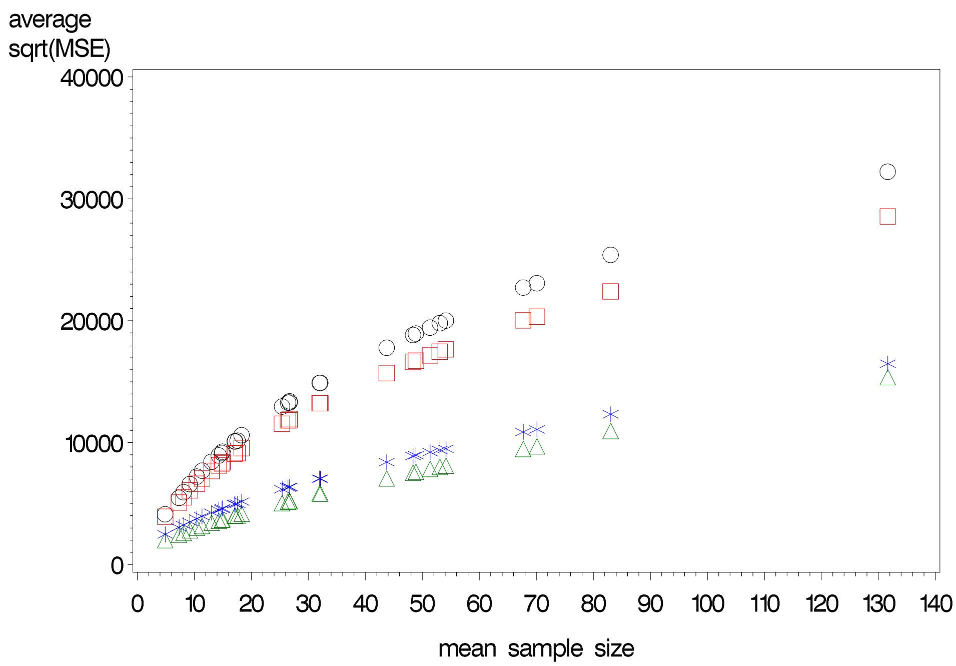

Figure 9.24: Mean estimated RMSE plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

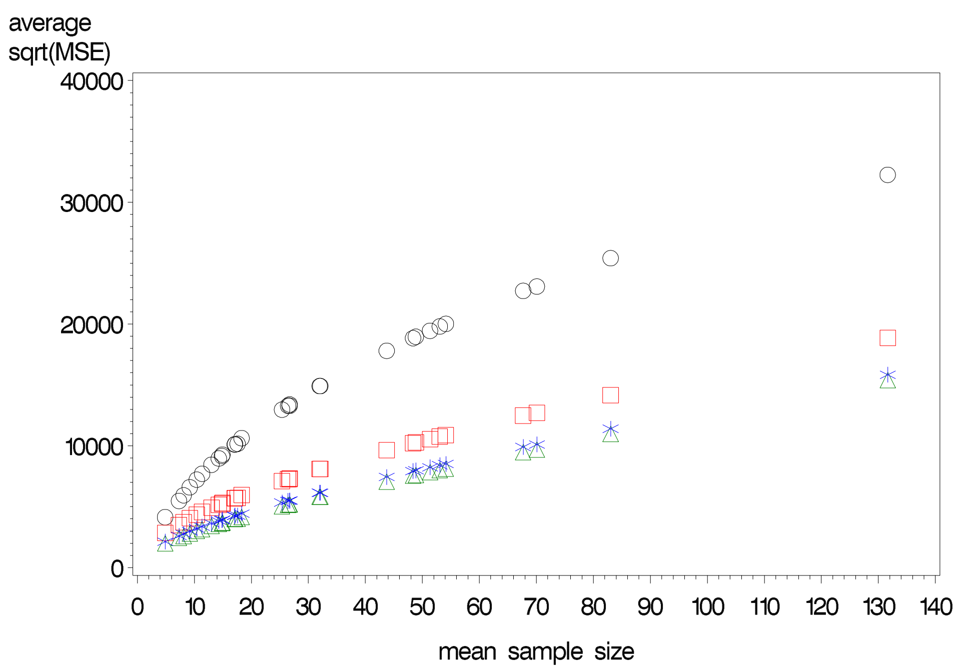

Figure 9.25: Mean estimated RMSE plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

POP4) the average RMSE reduction is 58 % with ROTPANEL model and 60 % with RANTIME model. However, the RMSE reduction by RANTIME model is spurious, since the RMSE estimates under it are biased downwards, especially under high intra-unit correlation (Figures 9.26 and 9.28).

The RMSE reduction does not vary much over the considered areas. However, the reductions are largest in smallish or medium-sized areas with mean cross-sectional sample size between 10 and 30 and smallest in either smallest or largest areas, depending on the population.

All these observations are very similar to those made on the point estimation accuracy (MARE considerations in Section 9.2.). The reduction in RMSE is often close to the reduction in MARE. Under strong intra-unit correlations it is difficult to borrow much strength by using complete panel data instead of cross-sectional data.

Figures 9.22–9.25 are much alike, but it is seen that the overall level of the RMSE estimates is higher in the populations POP3 and POP4 (Figures 9.24 and 9.25). Hence the increased within-area heterogeneity (or lowered intra-area correlation) causes increases in RMSE. The average increase is here 8 % for the NESTED model, 6 % for the PANEL model, 15 % for the ROTPANEL model and 17 % for the RANTIME model. On the other hand, increase of the within-unit heterogeneity (i.e. decrease in the stability of longitudinal observations) reduces the estimated RMSE not only for the PANEL model but also for the ROTPANEL model. The average reduction is here 37 % for the PANEL model and 11 % for the ROTPANEL model. Since the NESTED and RANTIME models do not account for the intra-unit correlation, this does not affect their performance in the MSE estimation.

The biases of the RMSE estimates are illustrated in Figures 9.26–9.29, which present the approximate relative estimation errors plotted against the mean sample sizes of the regions. They give virtually the same information as Figures 9.18–9.21 about the confidence interval coverage rates.

It is seen that the ROTPANEL model gives the most valid RMSE estimates in every case. The increased within-area heterogeneity (Figures 9.28 and 9.29) seems to increase the relative errors, though. The RMSE underestimation by the RANTIME model is particularly considerable when the intra-unit correlation is high (Figures 9.26 and 9.28).

The RMSE estimates under the NESTED model are unsatisfactory in many small areas, which was already seen with the problems with confidence interval coverage rates. The relative errors can be large especially when the within-area heterogeneity is large (POP3 and POP4). When the intra-unit correlation is low (POP2 and POP4), moving to the PANEL model offers some correction, but usually not sufficiently. When the intra-unit correlation is high, also the PANEL model performs badly. Similar bias problems were met already with EBLUP estimates from cross-sectional or complete panel data, when the within-area heterogeneity is large (cf. Section 9.1.).

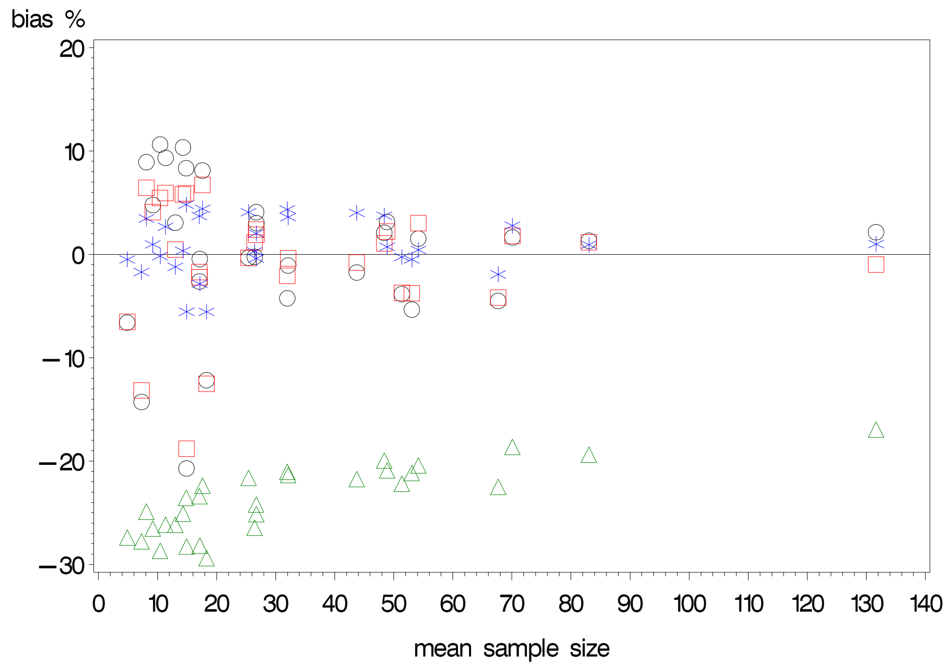

Figure 9.26: Approximate bias of RMSE estimates plotted against last month's mean sample size in simulations from POP1. Symbols used are ○ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

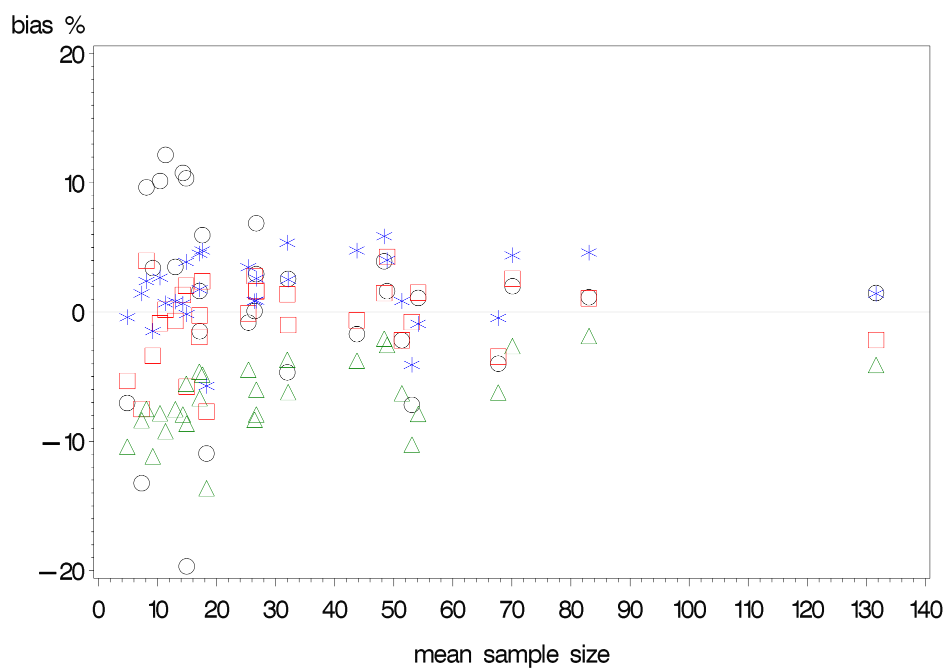

Figure 9.27: Approximate bias of RMSE estimates plotted against last month's mean sample size in simulations from POP2. Symbols used are ○ = NESTEDc, □ = PANEL10c, ∗ = ROTPANEL10c and △ = RANTIME10c.

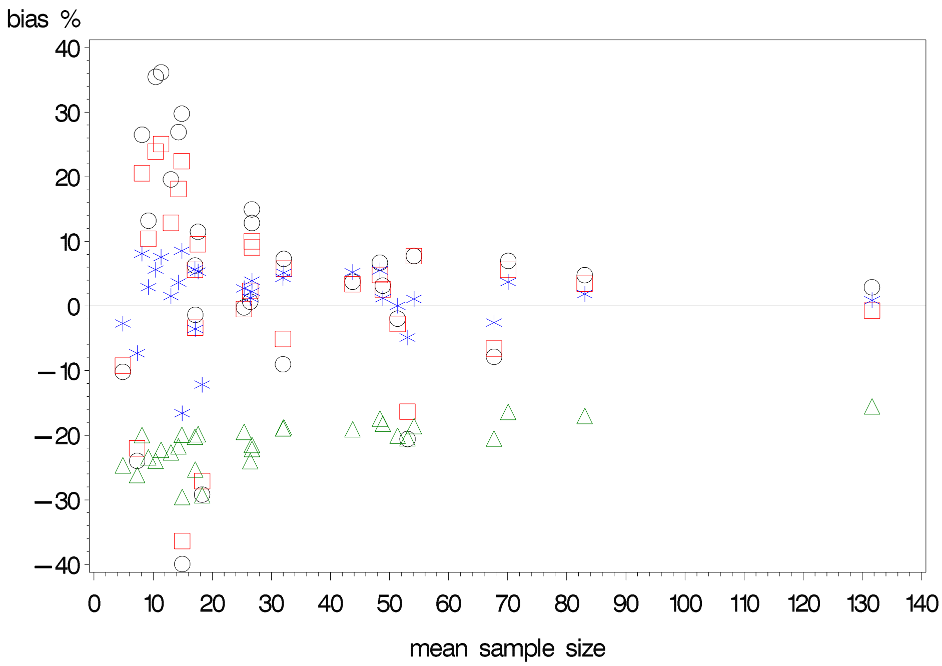

Figure 9.28: Approximate bias of RMSE estimates plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

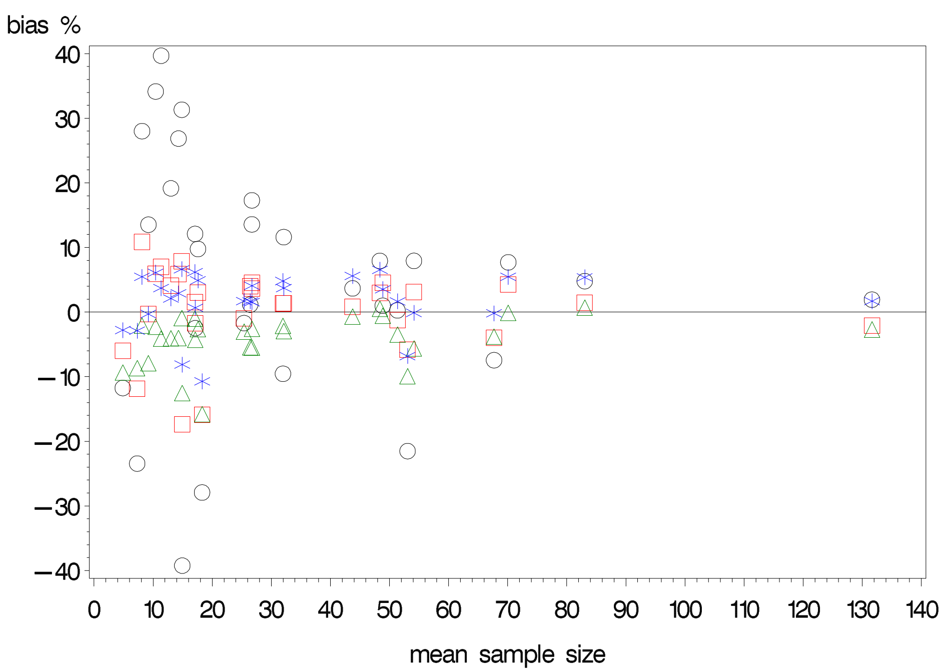

Figure 9.29: Approximate bias of RMSE estimates plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDc, □ = PANEL10c, $*$ = ROTPANEL10c and △ = RANTIME10c.

We believe that the explanation of these problems is that the large unit variance used in generating the populations POP3 and POP4 produced some area populations, which do not obey well the actual model. However, we note again that using the ROTPANEL model helps to correct for these problems.

We give one more illustration of the performance of the RMSE estimators under the competing models. Figure 9.30 contains four plots, where the square root of empirical MSE is compared with the estimated one. The plots are drawn from the population POP3 of high intra-unit correlation and large within-area heterogeneity, which gave the most problematic estimator performance.

We note first that for the ROTPANEL model the empirical and estimated RMSE agree well. The RMSE underestimation by the RANTIME model is also seen immediately. The plots of NESTED and PANEL models are practically similar. These models perform adequately for many areas, but we note a few circles, which stand out in the plots. In the corresponding areas the absolute estimation errors are large, presumably due to the badly fitting model. When the within-area heterogeneity decreases (POP1 and POP2), these errors become smaller, and when the intra-unit correlation decreases (POP2 and POP4), the bias of the RANTIME model reduces essentially.

9.5 Summary

The simulation study was carried out with four fixed populations, which differ from each other in the magnitude of heterogeneity within areas and the correlation of the repeated observations. The performances of the EBLUP estimators under the competing models were assessed in terms of relative bias, estimation accuracy measured with absolute relative errors and validity of estimated mean squared errors in correspondence with coverage rates of the asymptotic confidence intervals.

In general, it was found that utilizing the rotating panel data with the three-level model (5.1) called ROTPANEL, which adequately reflects the properties of the design, outperforms the other approaches in every way. The simple explanation is the borrowing strength in form of increased amount of longitudinal data made available for the estimation. Under the ROTPANEL model the observed biases of EBLUP estimates were negligible, the mean absolute relative errors were smaller than under the other models and also the MSE estimates were valid, producing short confidence intervals with good coverage properties.

For comparison, we also fitted another model, the RANTIME model (2.26) with independent random time effects, to the rotating panel data. Using the same data, it yielded EBLUP estimates of approximately the same (un)bias and accuracy as the ROTPANEL model, but its ignorance about the longitudinal correlations in the data caused downward biases in the MSE estimation, the severity of which increase with the correlations.

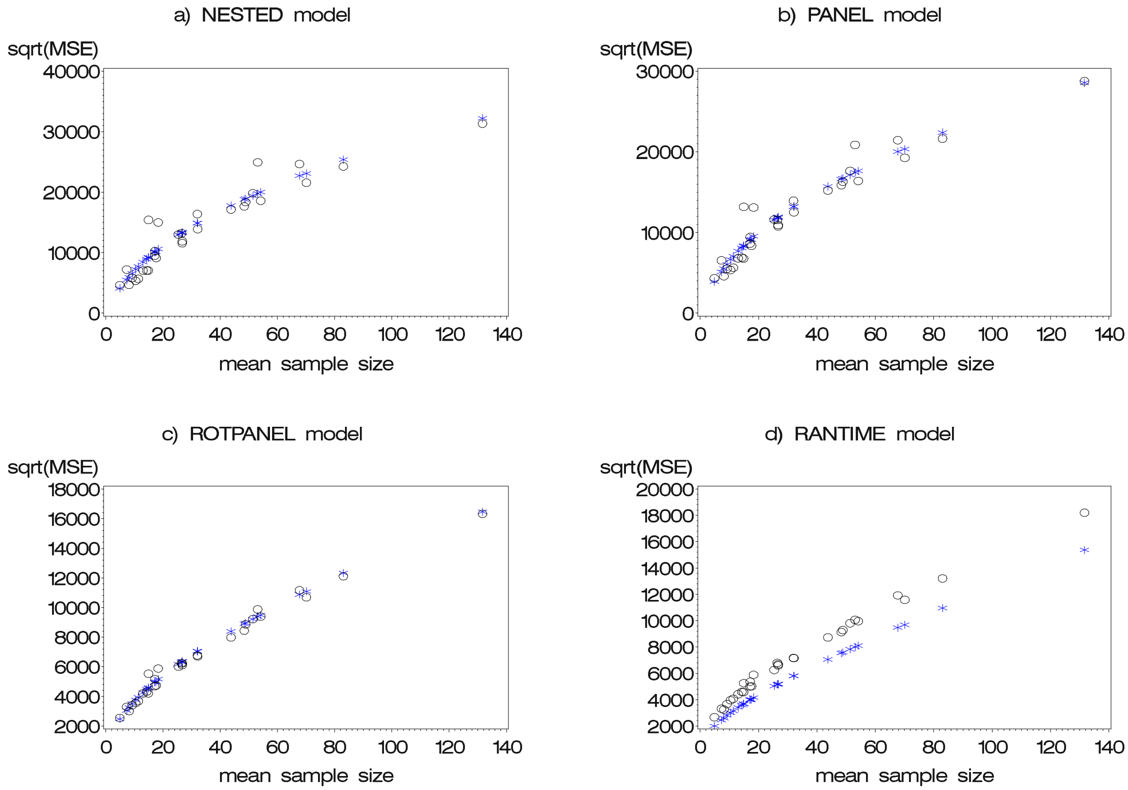

Figure 9.30: Square root of EMSE and mean estimated RMSE plotted against last month's mean sample size for a) NESTEDc, b) PANEL10c, c) ROTPANEL10c and d) RANTIME10c models in simulations from POP3. Symbols used are ◯ = root EMSE, $*$ = mean estimated RMSE.

Using only cross-sectional data and the model (2.20) called NESTED makes the EBLUP estimates of the smallest areas prone to remarkable biases, which in turn, along with smaller effective sample, lead to inferior accuracy and MSE estimates. The performance of the NESTED model gets slightly better as the within-area variance decreases.

The utility of using complete panel data (with the model (5.1)) depends on the magnitude of the correlation of repeated observations. If this correlation is high, the estimates from panel data are not much better than those from cross-sectional data. This is because the data sets from earlier occasions are then so similar to the current one that they do not truly increase the amount of information. If the correlations are weaker, the PANEL model gives better results, but still it cannot compete successfully with the ROTPANEL model.

We point out, however, that these remarks concern only the estimation of a "cross-sectional" area total at some specific occasion. If we are interested in estimating change, the complete panel data might be of essentially greater value.

If we compare the four different populations in light of the obtained results, we note that the differences are small. However, the best overall estimation accuracy (in terms of MARE and valid MSE) was observed with the population POP2, which had high intra-area and low intra-unit correlation. The population with the opposite properties was POP3, and with it the observed estimation accuracy was worst. This suggests that ideal data for small area estimation would be longitudinal data, in which the units within areas are fairly homogeneous, but at the same time there is enough variation over time so that the effective sample size is genuinely increased by the observations from previous occasions.

All these conclusions are made assuming that the model employed in estimation is correct. In what follows, we research the performance of the estimators under incorrect models.

10 Simulation study with incorrect model: results for longitudinal data of 10 occasions

We consider next the performance of the models NESTED, PANEL, ROTPANEL and RANTIME, when the only auxiliary variable (or covariate) in the fixed part of the model is JS, which is weakly correlated with the target variable Y. This incorrect model (in the sense that important covariates are missing from the model) for the PANEL and ROTPANEL approaches was shown in (8.3). For the NESTED approach, the model equation is

$$Y_{ijT} = \mu + \beta_1 JS_{ijT} + v_i + e_{ijT} \tag{10.1}$$

and for the RANTIME approach it is correspondingly

$$Y_{ijt} = \mu + \beta_1 JS_{ijt} + v_i + u_t + e_{ijt}. \tag{10.2}$$

In this chapter we present selected results concerning the EBLUP estimation of area totals under the incorrect models (i.e. models NESTEDi, PANEL10i, ROTPANEL10i and RANTIME10i, cf. Table 8.2). The discussion focuses on how moving from the correct model to an incorrect model affects the estimator performance in different situations.

All simulation results on biases, absolute relative errors and coverage rates are shown for every population (POP1–POP4) in Tables B.1–B.20 in Appendix B. Here we shall present only a few figures selected to illustrate the main findings with incorrect models.

10.1 Bias

The biases, measured with the mean relative errors, of the region total estimates for populations POP1–POP4 are shown in Tables B.1, B.6, B.11 and B.16, respectively. The effect of inferior modelling is almost similar in every population. In general, using a worsened model increases (the small) bias in all areas except in the largest, where the bias is practically zero regardless of the selected model.

We let the results for POP2 and POP3 to represent all results. They are displayed in Figures 10.1 and 10.2 for comparison with the corresponding Figures 9.2 and 9.3.

The median percentual increases in bias (over all areas) for different models and populations are shown in Table 10.1. Many of them seem remarkable. In general, the increases are largest, when the intra-area correlations are high (the populations POP1 and POP2). It is also seen that the performance of the PANEL model has strongly suffered under low intra-unit correlation (POP2 and POP4). As opposed to the results with correct models, the panel data does not give here much gains over the cross-sectional data.

It should be noted, however, that although the percentual increases in bias are substantial, the biases are still small, less than 6 % at the maximum. The rotating panel data outperforms again the complete panel and cross-sectional data.

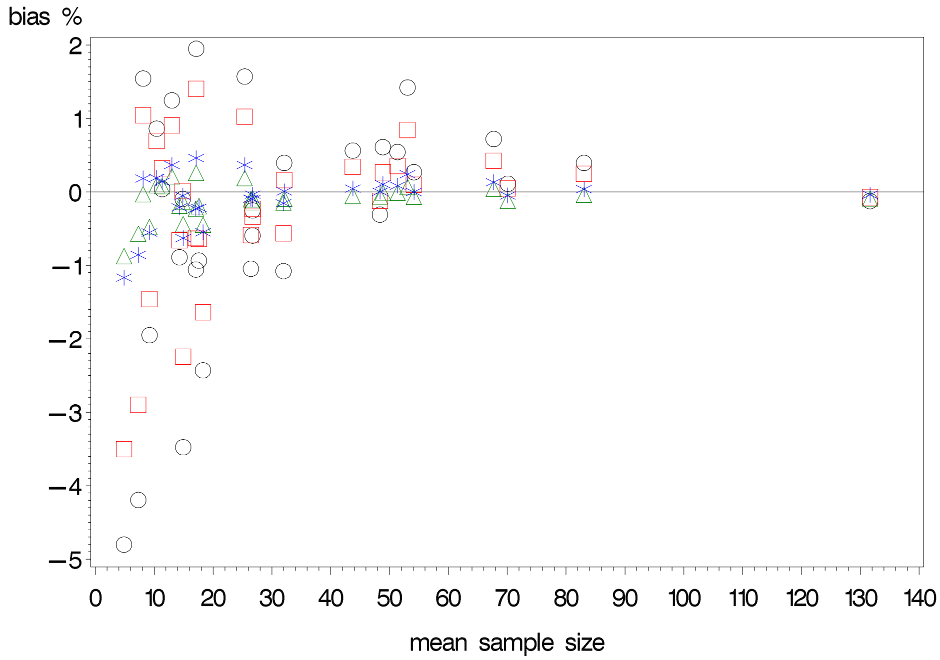

Figure 10.1: Relative bias under incorrect model plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

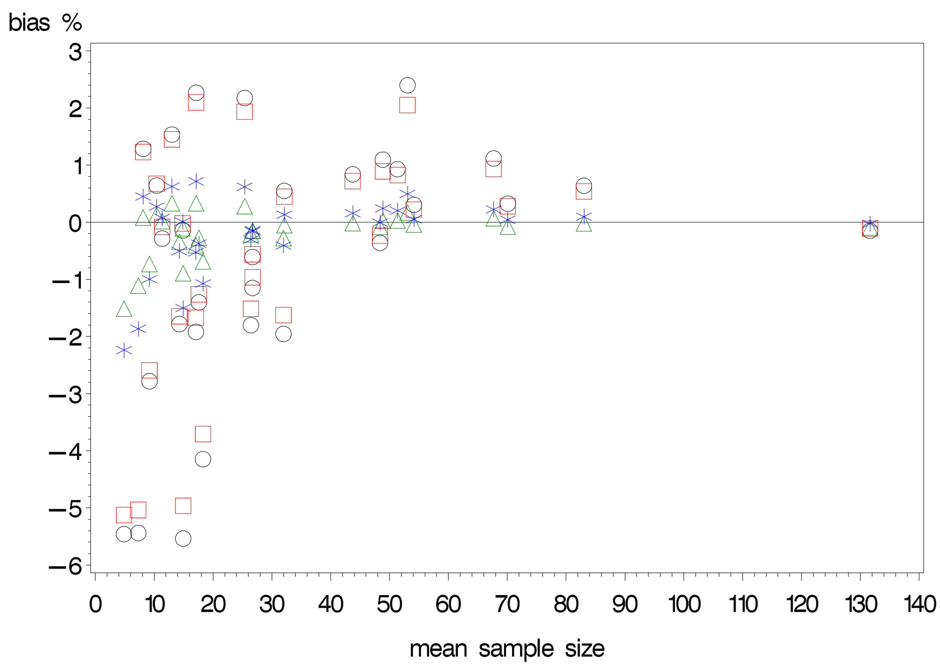

Figure 10.2: Relative bias under incorrect model plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

Table 10.1. The median percentual increase in bias of EBLUP total estimates caused by incorrect modelling.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	60	71	76	57
POP2	61	219	98	44
POP3	37	43	47	31
POP4	37	165	69	27

10.2 Absolute relative errors

The mean absolute relative errors of the EBLUP estimates are shown in Tables B.2, B.7, B.12 and B.17. In all cases the inferior model increases the MARE. On average, the relative increases are approximately the same regardless of the area size, although they have more variation in smaller areas. The rotating panel data outperforms again the complete panel and cross-sectional data.

The results for POP2 and POP4 are chosen to represent all results. They are presented in Figures 10.3 and 10.4. Comparing them with the Figures 9.7 and 9.9 for correct models shows the increase in MARE. Otherwise the plots are similar.

The median percentual increases in MARE (over all areas) for different models and populations are shown in Table 10.2. They lead to conclusions, which are similar to those made for bias. The increases due to bad modelling are largest in the populations POP1 and POP2 of high intra-area correlations. The accuracy of the estimates from panel data has suffered remarkably in the populations of low intra-unit correlation. Thus, the gains of using panel data instead of cross-sectional data reduce if the model is not correct. To illustrate this we show plots of MARE ratios in Figures 10.5 and 10.6, which can be compared with Figures 9.11 and 9.13, where the employed model is correct.

Table 10.2. The median percentual increase in MARE of EBLUP total estimates caused by incorrect modelling.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	32	37	39	39
POP2	33	86	50	55
POP3	23	29	29	30
POP4	22	68	39	44

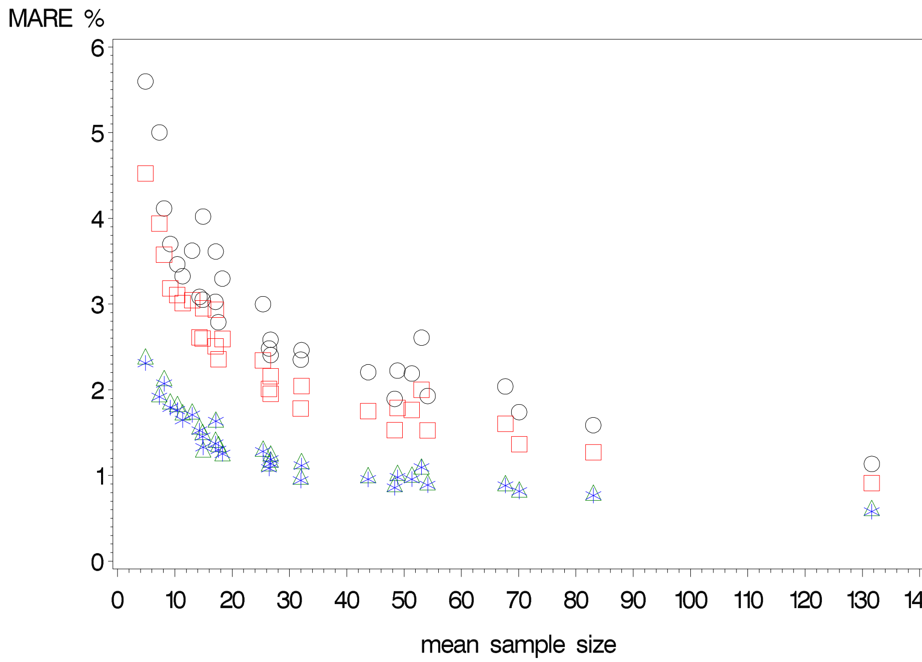

Figure 10.3: MARE under incorrect model plotted against last month's mean sample size of region in simulations from POP2. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

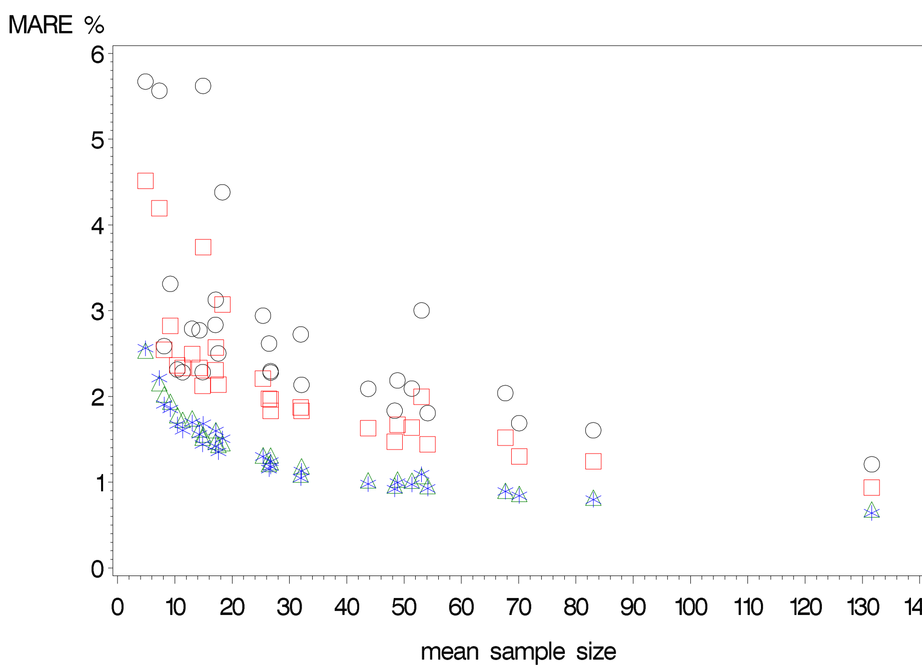

Figure 10.4: MARE under incorrect model plotted against last month's mean sample size of region in simulations from POP4. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

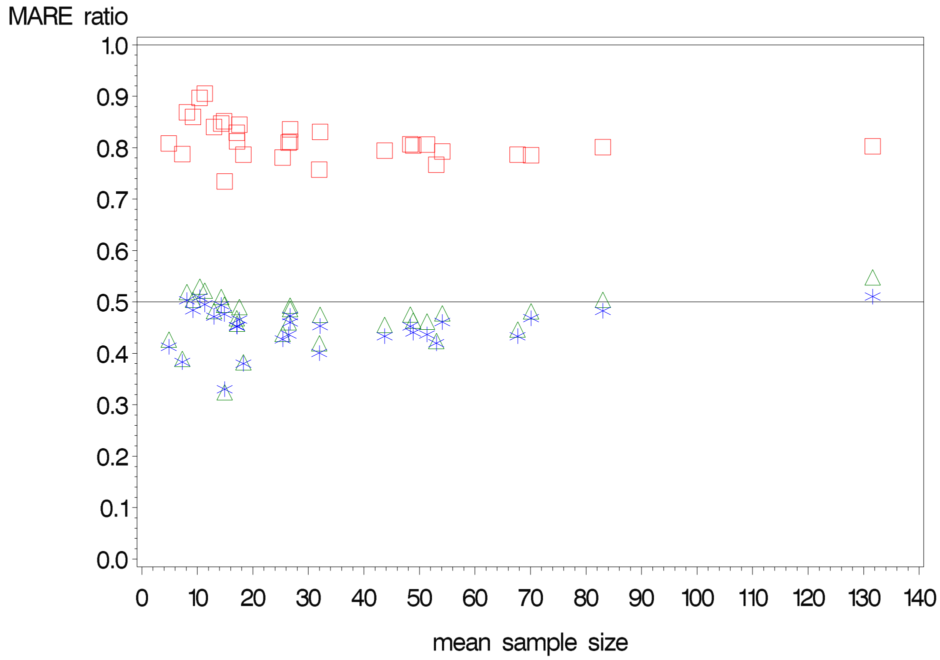

Figure 10.5: MARE reduction under incorrect model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP2. Symbols used are □ = PANEL10i/NESTEDi, ∗ = ROTPANEL10i/NESTEDi and △ = RANTIME10i/NESTEDi.

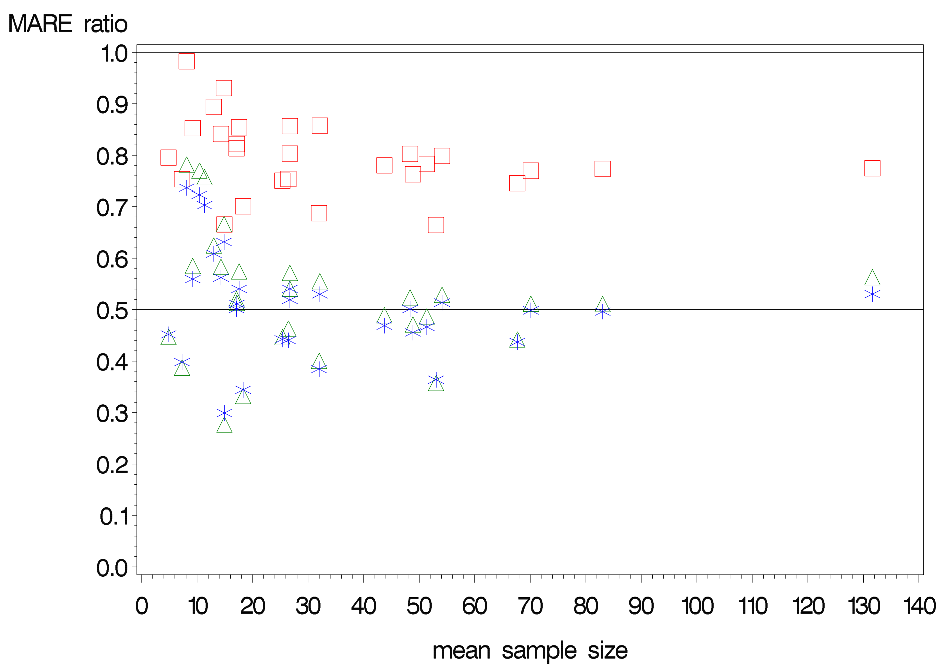

Figure 10.6: MARE reduction under incorrect model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP4. Symbols used are □ = PANEL10i/NESTEDi, ∗ = ROTPANEL10i/NESTEDi and △ = RANTIME10i/NESTEDi.

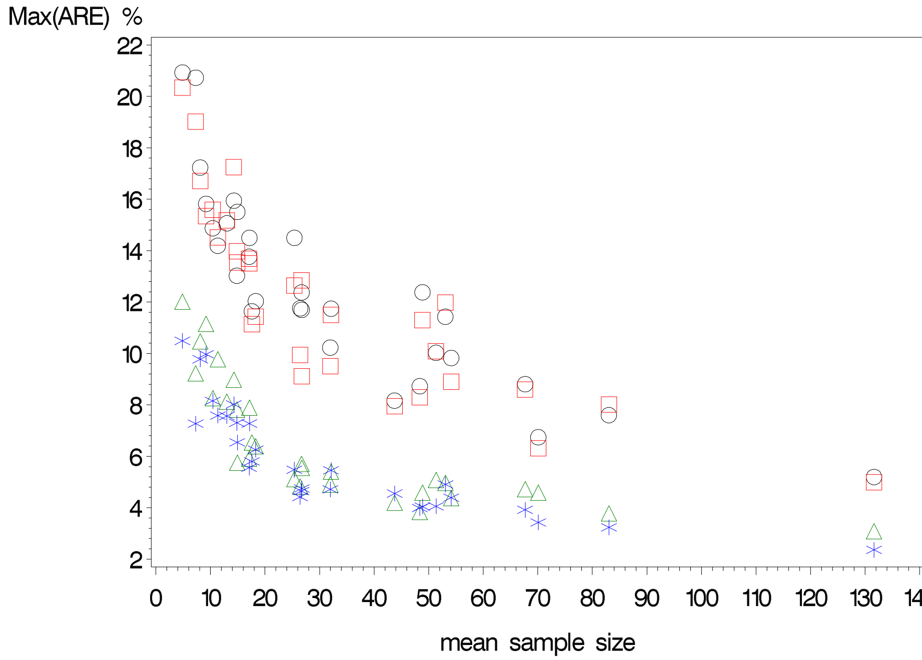

Figure 10.7: Maximum absolute relative errors under incorrect model plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIMEi.

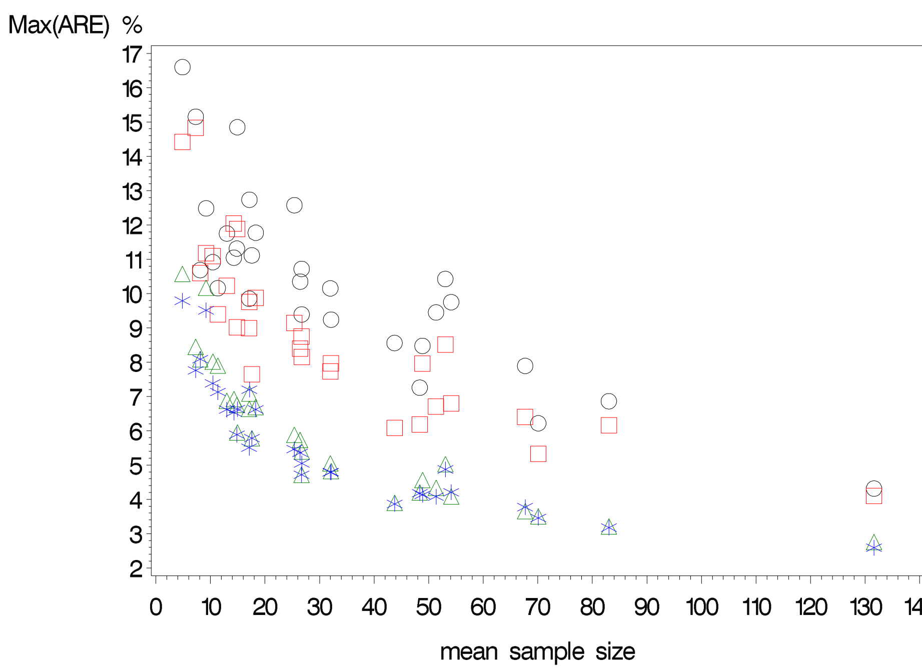

Figure 10.8: Maximum absolute relative errors under incorrect model plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

We show the maximum absolute relative errors in simulations from POP1 and POP4 in Figures 10.7 and 10.8. We notice that in the smallest areas the NESTED and PANEL models can yield remarkably worse EBLUP estimates than the ROTPANEL and RANTIME models, and they are worse under incorrect model than under correct model especially when the within-area homogeneity is strong (POP1 and the not displayed POP2) (cf. Figures 9.14 and 9.17).

10.3 Coverage of confidence intervals

The observed coverage rates of the nominal 95 % confidence intervals calculated from the EBLUP point estimates and their estimated mean squared errors are shown in Tables B.3, B.8, B.13 and B.18 of Appendix B. We illustrate the results by the plot of coverage rates for the population POP4, presented in Figure 10.9. The corresponding plot for the correct model is in Figure 9.21. In the both figures the areas 26 and 3 again stick out from the rest as especially problematic cases. Their EBLUP estimates were already found to have large biases.

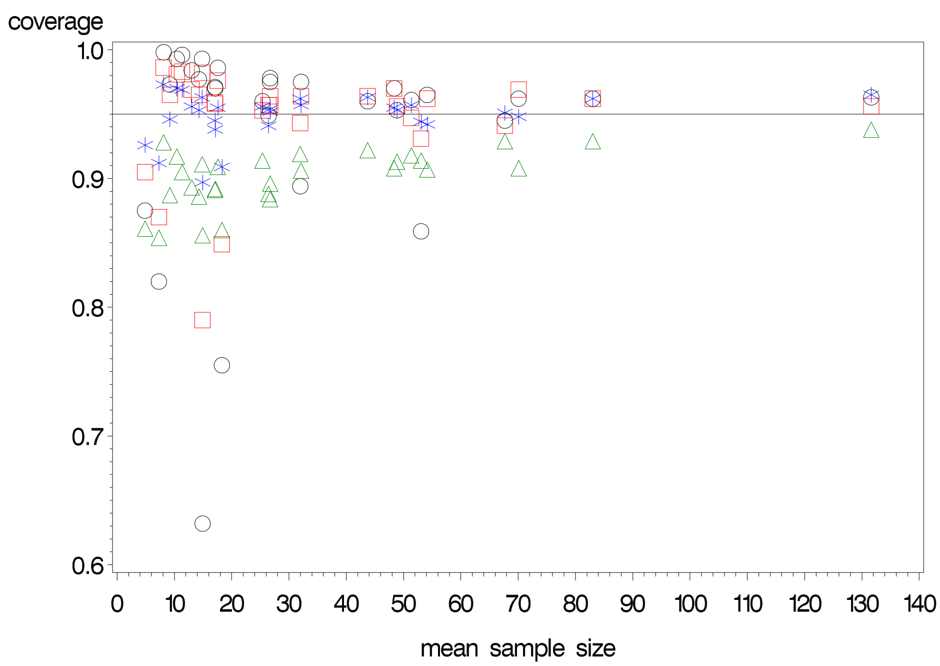

Figure 10.9: Coverage of asymptotic 95 % confidence intervals under incorrect model plotted against last month's mean sample size in simulations from POP4. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

In general, the incorrect model makes those confidence intervals, which already had poor (too high or too low) coverage rates under the correct model, even poorer. This concerns especially the performances of NESTED and PANEL models in small regions and the RANTIME model overall. Also the ROTPANEL model suffers from an incorrect covariate choice, but its performance is still acceptable.

10.4 MSE estimation

Using an incorrect model increases also the MSE estimates of the EBLUP estimates. The median percentual increases in mean estimated RMSE (over all areas) for different models and populations are shown in Table 10.3. The table looks much like Table 10.2 for the mean absolute relative errors. The increases are here highest with the PANEL model.

Table 10.3. The median percentual increase in mean RMSE of EBLUP total estimates caused by incorrect modelling.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	32	38	36	35
POP2	32	88	48	35
POP3	21	28	27	26
POP4	21	70	37	25

Depending on the model and population, the relative increase is lowest either in the smallest or in the largest areas. We illustrate this with Figure 10.10, which displays the ratios $M\widehat{RMSE}_i/M\widehat{RMSE}_c$, where c denotes correct model and i incorrect model, of the mean RMSE's (8.8) plotted against mean sample size in POP2. For the NESTED and PANEL models the lowest increase is always met with the smallest areas.

To examine the bias in MSE estimation we compare the average of RMSE estimates with the corresponding empirical root mean squared error (ERMSE) and calculate the approximate relative estimation error. These comparisons (for the four populations) are shown in Tables B.4, B.5, B.9, B.10, B.14, B.15, B.19 and B.20 of Appendix B. For illustration, we display the approximate relative errors for POP2 and POP3 in Figures 10.11 and 10.12. The corresponding plots obtained under the correct models are in Figures 9.27 and 9.28.

We learn that the biases in MSE estimation, which occur especially with smallest areas, are increased if the employed model is incorrect. Again, the ROTPANEL model gives the best results. The increases in bias under it are very moderate compared to the NESTED and PANEL models, which suffered from remarkable biases in some areas even if the model had correct covariates.

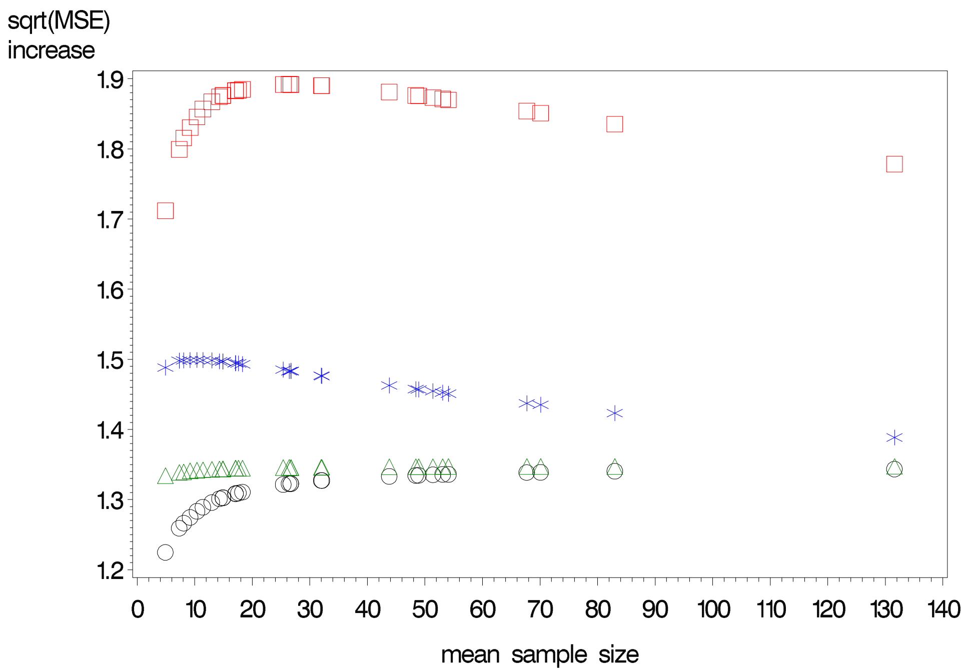

Figure 10.10: Relative increase of RMSE due to incorrect model plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDi, □ = PANEL10i, $*$ = ROTPANEL10i and △ = RANTIME10i.

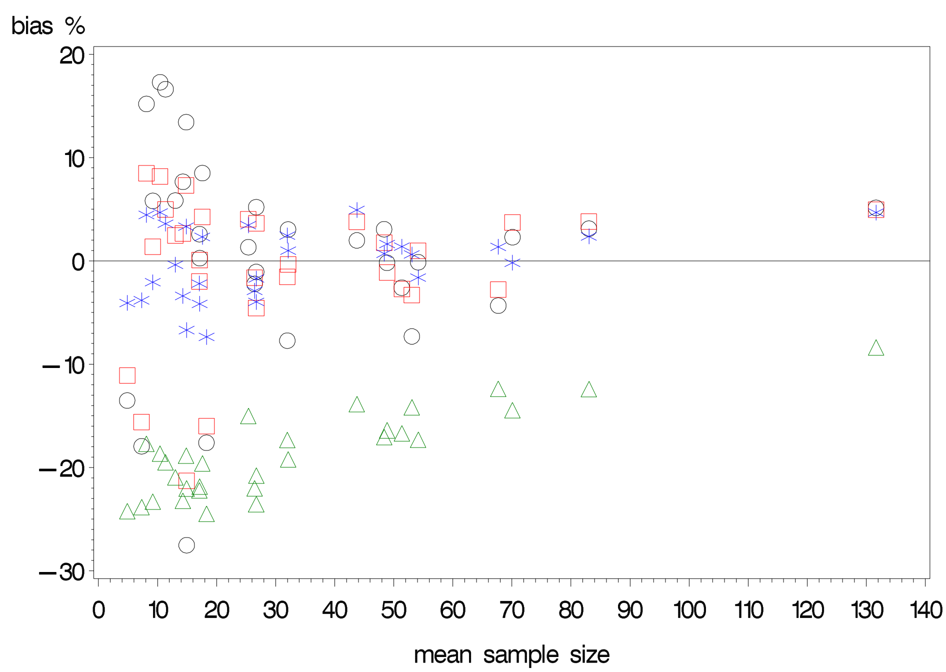

Figure 10.11: Approximate bias of RMSE estimates under incorrect model plotted against last month's mean sample size in simulations from POP2. Symbols used are ◯ = NESTEDi, □ = PANEL10i, $*$ = ROTPANEL10i and △ = RANTIME10i.

The RANTIME model performs poorly in all four populations. It performed poorly in the populations POP1 and POP3, which have strong longitudinal correlations, even under the correct model. In these populations the incorrect model increases the already large biases only slightly. In the other two populations POP2 and POP4 of low longitudinal correlation the correct RANTIME model performs somewhat better, but then the incorrect model raises the biases to almost as high level as in POP1 and POP3.

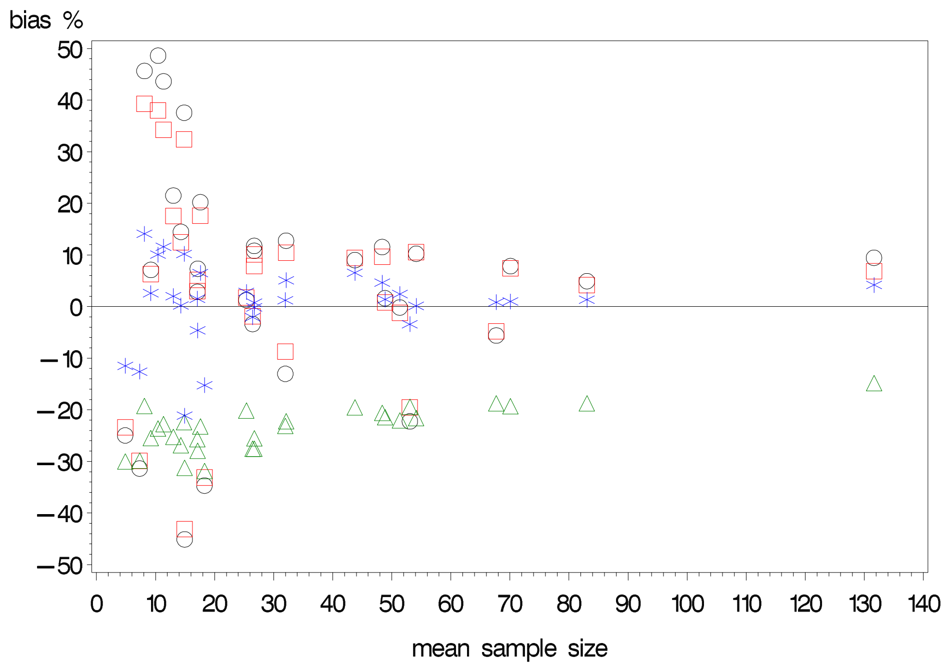

Figure 10.12: Approximate bias of RMSE estimates under incorrect model plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDi, □ = PANEL10i, ∗ = ROTPANEL10i and △ = RANTIME10i.

10.5 Summary

The behaviour of the competing models NESTED, PANEL, ROTPANEL and RANTIME does not change much when the employed model differs from the so-called correct one with respect to the included auxiliary variables. Certainly the wrong model makes the possible biases in EBLUP estimates as well as in their MSE estimates more severe and decreases the estimation accuracy and the validity of asymptotic confidence intervals.

Utilizing the rotating panel data by the ROTPANEL model yields superior results in every way. It successfully protects from very poor point estimates, and gives in most cases small standard errors and adequate confidence intervals.

With the correct auxiliary variables the use of complete panel data is gainful in some circumstances, but when the model is not correct, these gains are practically lost.

11 Simulation study with correct model: results for longitudinal data of 5 occasions

In this chapter we examine how the performances of the EBLUP estimators under the PANEL, ROTPANEL and RANTIME models change when the length of panel, i.e. the number of survey occasions in the data, is reduced from 10 to 5. The employed models are correct, including all the necessary auxiliary variables. Thus, we consider models PANEL5c, ROTPANEL5c and RANTIME5c. The cross-sectional model NESTEDc serves as a benchmark in the performance comparisons.

All the numerical results on biases, absolute relative errors and coverage rates are shown for every population (POP1–POP4) in Tables C.1–C.20 of Appendix C. Only few selected figures are presented here to illustrate the main findings.

11.1 Bias

The biases of the region total estimates calculated from the data of 5 survey occasions are shown in Tables C.1, C.6, C.11 and C.16 for the populations POP1–POP4, respectively. The reduced number of occasions in the panel data causes only slight increases in bias. Thus, the large percentual increases presented in Table 11.1 are somewhat misleading. We do not display any figures here since they would be almost identical with those obtained from the data of 10 occasions and presented in Chapter 9.

The estimates from rotating panel data, obtained under the models ROTPANEL and RANTIME, are still virtually unbiased even in the smallest areas. The estimates from panel data showed some small biases in a few small or smallish areas. These biases are now larger than before, but the increase is negligible especially when the longitudinal correlation within units is high (the populations POP1 and POP3).

Table 11.1. The median percentual increase in bias of EBLUP total estimates caused by shorter panel data.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	3	37	75
POP2	29	68	55
POP3	3	40	78
POP4	33	67	73

11.2 Absolute relative errors

The mean absolute relative errors of the EBLUP estimates are shown in Tables C.2, C.7, C.12 and C.17. Halving the number of survey occasions does not yield large increases in the mean or maximum absolute relative error. The median percentual increases in MARE for different models and populations are shown in Table 11.2. Again, under high longitudinal correlation (POP1 and POP3 the length of panel has virtually no effect on the performance of the PANEL model. The relative increases are approximately the same regardless of the area size.

Table 11.2. The median percentual increase in MARE of EBLUP total estimates caused by shorter panel data.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	1	18	16
POP2	12	27	27
POP3	1	18	16
POP4	13	27	27

Reducing the number of time points in data does not affect the comparison of the various approaches. The graphical representations of MARE would be almost identical with those obtained from the data of 10 occasions (see Chapter 9) and are omitted.

The rotating panel data outperforms the complete panel and cross-sectional data. However, the gains of using panel or rotating panel data are somewhat smaller with data of 5 occasions than with data of 10 occasions. To illustrate this we show two plots of MARE ratios in Figures 11.1 and 11.2, which can be compared with Figures 9.11 and 9.13 to see the difference.

11.3 Coverage of confidence intervals

The observed coverage rates of the nominal 95 % confidence intervals calculated from the EBLUP point estimates and their estimated mean squared errors are shown in Tables C.3, C.8, C.13 and C.18. The results are essentially similar to those obtained with data sets of 10 time points. The only exception is in the performance of the RANTIME model for rotating panel data. This is illustrated in the plot of coverage rates for the population POP1 in Figure 11.3. The corresponding plot in the case of 10 time points is in Figure 9.18. With the shorter data the observed rates have moved closer to the nominal rates, though they still show too low coverage. The change is due to the fact that in shorter data the number of units, which are mistakenly assumed independent, is reduced.

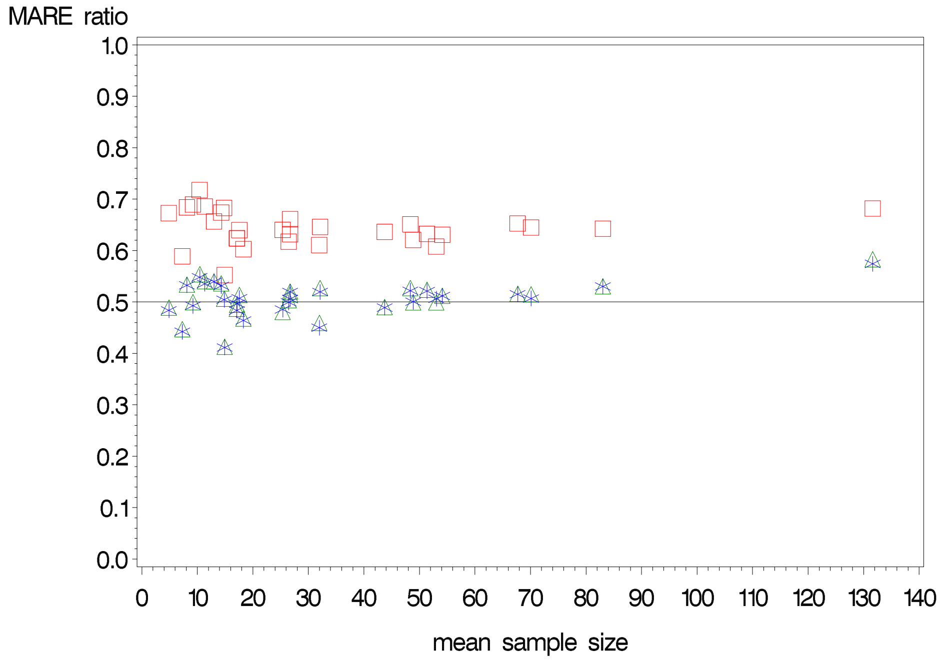

Figure 11.1: MARE reduction under correct model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP2. Symbols used are □ = PANEL5c/NESTEDc, ∗ = ROTPANEL5c/NESTEDc and △ = RANTIME5c/NESTEDc.

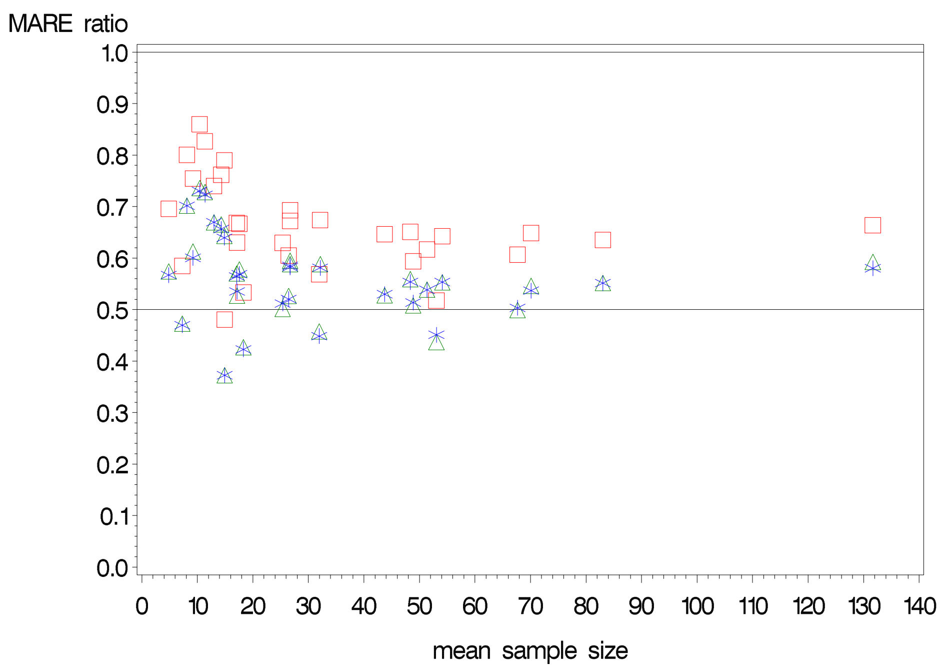

Figure 11.2: MARE reduction under correct model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP4. Symbols used are □ = PANEL5c/NESTEDc, ∗ = ROTPANEL5c/NESTEDc and △ = RANTIME5c/NESTEDc.

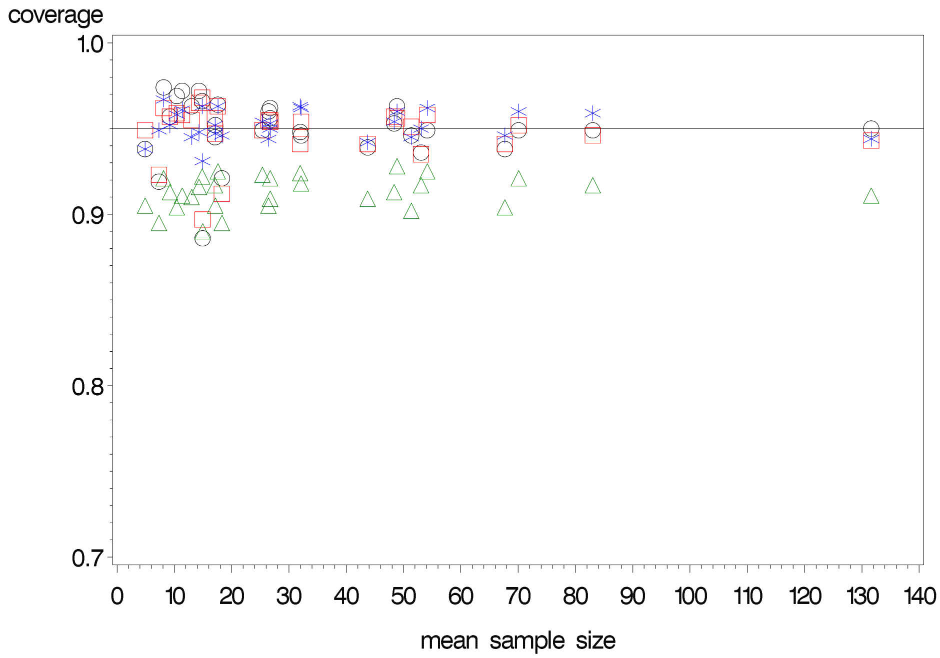

Figure 11.3: Coverage of asymptotic 95 % confidence intervals under correct model plotted against last month's mean sample size in simulations from POP1. Symbols used are ◯ = NESTEDc, □ = PANEL5c, $*$ = ROTPANEL5c and △ = RANTIME5c.

11.4 MSE estimation

Using the reduced number of survey occasions induces some increase also in the MSE estimates of the EBLUP estimates. The graphical representations are again omitted, but the median percentual increases in mean estimated RMSE (over all areas) for the considered models and populations are shown in Table 11.3. The table looks much like Table 11.2 for the mean absolute relative errors.

Table 11.3. The median percentual increase in mean RMSE of EBLUP total estimates caused by shorter panel data.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	1	17	33
POP2	12	27	33
POP3	1	17	32
POP4	14	27	32

For the PANEL model, the increases are relatively small (especially in the populations POP1 and POP3, where the intra-unit correlation is high) and do not vary much with the area size. This indicates that under high longitudinal correlation increasing "lags" in complete panel data does not result in essentially reduced variation of estimates. Under low longitudinal correlation (POP2 and POP4) the earlier observations contain more additional information and losing them causes more loss of estimation efficiency.

The ROTPANEL model and especially the RANTIME model suffer more from throwing longitudinal data away, because the number of unobserved units increases. When the number of time points is reduced from 10 to 5, the RANTIME model (erroneously) sees this as halving the number of independent observations, and while it does not recognize the intra-unit correlation, the loss of efficiency is almost the same in every population. The ROTPANEL model accounts for the magnitude of the correlation, and for this reason (as with the PANEL model) the MSE increases more in the populations of low intra-unit correlation.

For the rotating panel data, especially with the ROTPANEL model, the relative increases are usually largest in the smallest areas. This is understandable since in the biggest areas even the cross-sectional sample size is often large enough to give accurate estimates.

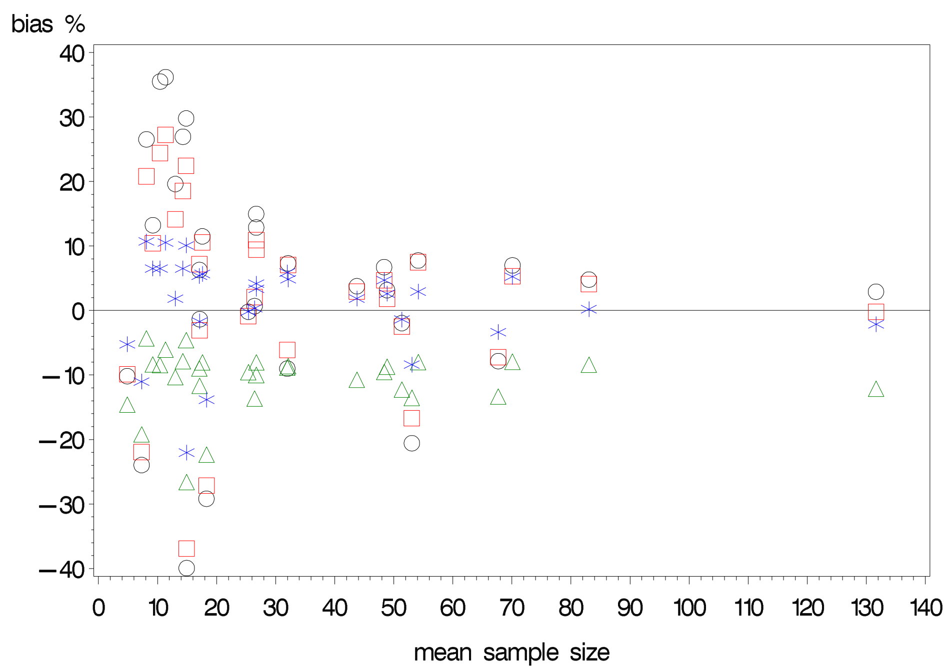

Figure 11.4: Approximate bias of RMSE estimates under correct model plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDc, □ = PANEL5c, ∗ = ROTPANEL5c and △ = RANTIME5c.

As for the bias in MSE estimation, the results are the same as for the confidence interval coverage rates. The effect of smaller number of survey occasions is seen mainly in the reduced bias of the MSE estimates under the RANTIME model. Only few small increases of bias are observed with the other models. The numerical results for the four populations POP1–POP4 are shown in respective Tables C.4 and C.5, C.9 and C.10, C.14 and C.15, C.19 and C.20. For illustration, we display the approximate relative errors of RMSE estimates for POP3 in Figure 11.4. The corresponding plot for the 10-occasion data is in Figure 9.28, and comparison of these two figures shows the changed performance of RANTIME model and the almost unchanged performances of the other models.

11.5 Summary

When the employed model is correct, reducing the number of survey occasions by half does not have much effect on the performance of models PANEL, ROTPANEL and RANTIME. Small increases in (absolute) relative errors and in mean squared errors are observed. Often the increases are highest in the smallest areas. Again, utilizing the rotating panel data by the ROTPANEL model yields the most reliable results.

12 Simulation study with incorrect model: results for longitudinal data of 5 occasions

In this chapter we present selected results concerning the EBLUP estimation of area totals from sample data sets of 5 time points under the incorrect models. The considered models are PANEL5i, ROTPANEL5i and RANTIME5i and NESTEDi as a benchmark (cf. Table 8.2). The model equations have been shown in (8.3), (10.1) and (10.2).

We have two general interests here. First, do the drawbacks of using an incorrect model change when the number of time points in sample data is reduced. Second, are the effects of using shorter panel data the same under incorrect models as under correct models.

All the numerical results on biases, absolute relative errors and coverage rates are shown for every population (POP1–POP4) in Tables D.1–D.20 in Appendix D. We present here only few figures selected to illustrate the main findings.

12.1 Bias

The observed biases of the region total estimates in the case of short panel and incorrect model are shown in Tables D.1, D.6, D.11 and D.16. As an example, the results for the population POP3 are displayed in Figure 12.1. It looks much similar to Figure 10.2 drawn from the corresponding results for data of 10 time points. By careful comparison it can be seen that the biases under the rotating panel data models ROTPANEL and RANTIME are slightly increased in the smallest areas due to the shorter data. This increase is negligible under the PANEL model.

The relative increases of bias due to the shorter data are presented in Table 12.1. However, it should be kept in mind that both the actual biases and their increases are in general very small. Thus, as for bias under incorrect models there is not much difference if the longitudinal data contains 5 or 10 time points.

Table 12.1. The median percentual increase in bias of EBLUP total estimates due to shorter panel data under incorrect models.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	1	37	79
POP2	7	46	64
POP3	2	33	84
POP4	8	49	76

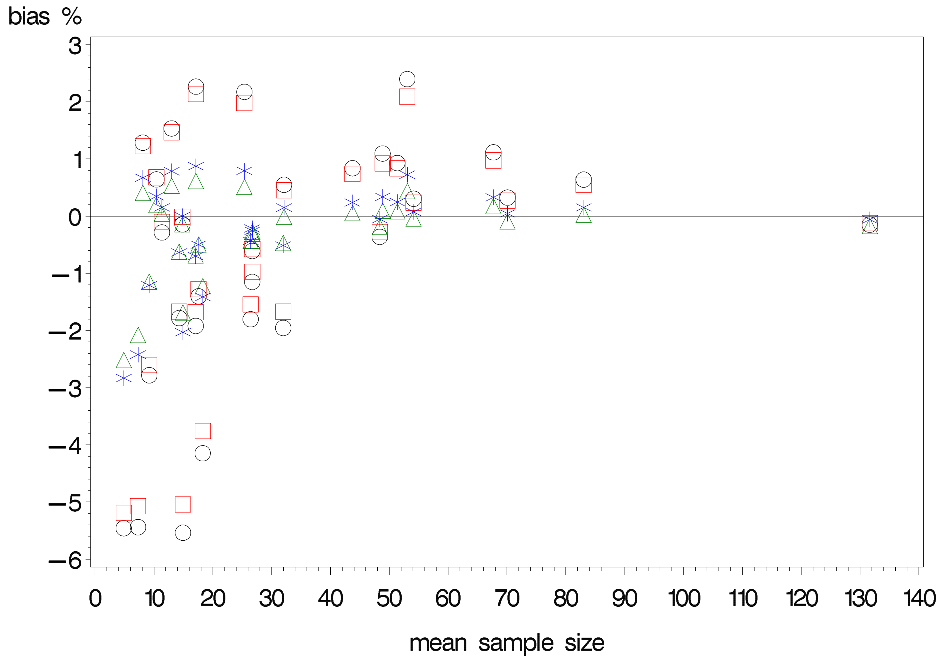

Figure 12.1: Relative bias under incorrect model plotted against last month's mean sample size in simulations from POP3. Symbols used are ◯ = NESTEDi, □ = PANEL5i, ∗ = ROTPANEL5i and △ = RANTIME5i.

On the other hand, comparing Table 12.1 with Table 11.1 suggests that the increase in bias due to shorter data can be smaller under an incorrect model than under correct model. This holds especially for the PANEL and ROTPANEL model in the populations POP2 and POP4 of low intra-unit correlation.

The relative increase of bias due to the incorrect modelling in the case of shorter panel data is shown in Table 12.2. This is similar to Table 10.1, suggesting that the number of survey occasions is not much related to the effect of correct/incorrect modelling on bias.

Table 12.2. The median percentual increase in bias of EBLUP total estimates caused by incorrect modelling in data of 5 time points.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	60	69	72	65
POP2	61	163	78	59
POP3	37	43	43	39
POP4	37	118	54	39

12.2 Absolute relative errors

The mean absolute relative errors of the EBLUP estimates are shown in Tables D.2, D.7, D.12 and D.17. As expected, the inferior model and the reduced number of time points both increase the MARE. The effect of inferior model is the stronger of these two and it does not change much with the number of time points in data. On the other hand, we can state that under an incorrect model the effect of shorter data on MARE is smaller than under the correct model. We come to these conclusions by comparing Table 12.4 with Table 10.2 and Table 12.3 with Table 11.2. They are valid also when the maximum absolute relative errors are concerned.

Table 12.3. The median percentual increase in MARE of EBLUP total estimates due to shorter panel data under incorrect models.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	1	15	14
POP2	3	20	19
POP3	1	15	13
POP4	4	19	18

Table 12.4. The median percentual increase in MARE of EBLUP total estimates caused by incorrect modelling in data of 5 time points.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	32	37	34	37
POP2	33	72	42	45
POP3	23	28	25	27
POP4	33	55	31	35

In the case of 5 occasions the gains of panel data or rotating panel data over the cross-sectional data are only slightly smaller if the model is incorrect. The populations of low intra-unit correlation make an exception in the sense that an incorrect model reduces the gains of PANEL model considerably more. This is illustrated in Figures 12.2 and 12.3, which can be compared to Figures 11.1 and 11.2, where the employed model is correct. The reduction in the gains of the ROTPANEL and RANTIME models is smaller than that of the PANEL model.

12.3 Coverage of confidence intervals

The coverage rates of the nominal 95 % confidence intervals are shown in Tables D.3, D.8, D.13 and D.18. The tendency of incorrect models to yield inferior confidence intervals

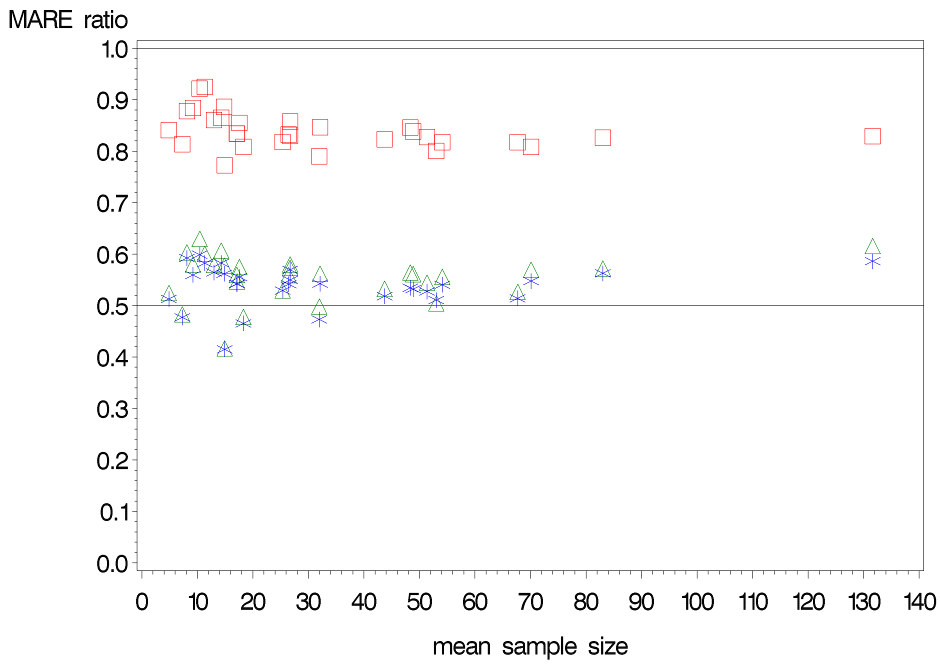

Figure 12.2: MARE reduction under incorrect model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP2. Symbols used are □ = PANEL5i/NESTEDi, $*$ = ROTPANEL5i/NESTEDi and △ = RANTIME5i/NESTEDi.

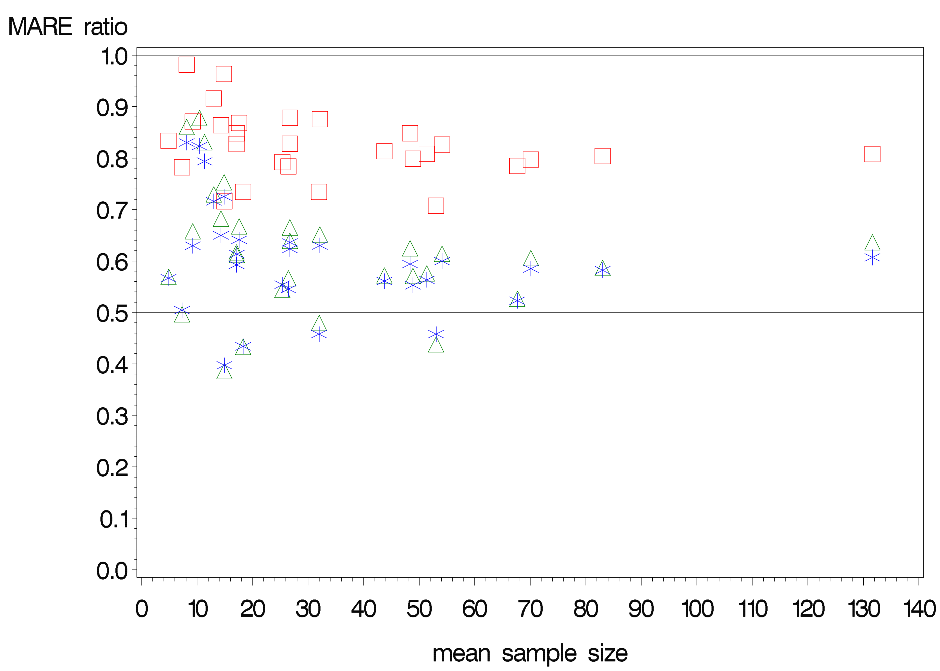

Figure 12.3: MARE reduction under incorrect model in terms of MARE ratios plotted against last month's mean sample size in simulations from POP4. Symbols used are □ = PANEL5i/NESTEDi, $*$ = ROTPANEL5i/NESTEDi and △ = RANTIME5i/NESTEDi.

occurs also with the shorter data sets. When the model is incorrect, the number of occasions has only a minor effect. Once again the ROTPANEL model performs best among the compared models.

12.4 MSE estimation

The median percentual increases in mean estimated RMSE (over all areas), caused by the shorter sample data in the case of incorrect model, are displayed in Table 12.5. The increases caused by incorrect models in the case of shorter data are in Table 12.6. These tables look much like the corresponding tables about the mean absolute relative errors and the same conclusions can be made. The increases in the estimated RMSE under the RANTIME models are not really comparable with the others, because the RMSE estimates are biased and this bias reduces as the number of occasions goes down.

Table 12.5. The median percentual increase in RMSE of EBLUP total estimates due to shorter panel data under incorrect models.

	model		
population	PANEL	ROTPANEL	RANTIME
POP1	1	16	33
POP2	3	20	33
POP3	1	15	31
POP4	4	20	31

Table 12.5 is very similar to Table 11.3 in Section 11.4, and the discussion presented there is valid also here. However, the RMSE performance of PANEL model makes some difference. We note that under the incorrect model the effect of the length of the panel on the performance is negligible in every population. Similar behaviour was noticed also with the MARE (Table 12.3).

Table 12.6. The median percentual increase in RMSE of EBLUP total estimates caused by incorrect modelling in data of 5 time points.

	model			
population	NESTED	PANEL	ROTPANEL	RANTIME
POP1	32	38	35	34
POP2	32	73	40	34
POP3	21	27	25	25
POP4	21	56	29	25

The effect of incorrect modelling on the level of MSE is slightly reduced, when there is 5 time points in the data instead of 10 (cf. Table 10.3).

About the bias in MSE estimation nothing really new can be said. The RMSE bias results are shown in Appendix D in Tables D.4 and D.5 for POP1, D.9 and D.10 for POP2, D.14 and D.15 for POP3 and D.19 and D.20 for POP4. With 5 occasions data the incorrect models yield larger biases in MSE estimates than correct models in the same way as with 10 occasions data. If the original model is incorrect, shortening the panel data yield only minor increases in bias.

12.5 Summary

It is not surprising that the small area estimation results are worst when the model is incorrect and there is less longitudinal data available. However, the effect of bad modelling on the estimator performance is found stronger than the length of the panel data. When the model is wrong, the length of the panel does not affect much. When the model is correct, long panel data give somewhat greater gains than short panel data. These gains are seen in the accuracy of both EBLUP estimates and their MSE estimates. The rotating panel data modelled by the ROTPANEL model yields again the best results.

13 Miscellaneous findings

In addition to the Monte Carlo studies reported above, we carried out a few simulation experiments mainly for preliminary purposes. These raised a couple of issues, which are worth discussing.

First, we experimented with various rotation schemes. It was soon noticed that as the overlap between survey occasions increases, the results with rotating panel data tend to those obtained with panel data. The more overlap, the less good properties are available in estimating the cross-sectional area totals at the last survey occasion. The rotation plan used in the Finnish Labour Force Survey and also in this work seems very reasonable here. The situation is of course different, when the estimation problem concerns changes in totals. It is not within the scope of this thesis, however.

Second, there was a little worry about the incorrect models applied in the simulation study since they, while obeying the postulated covariance structure and only two fixed covariates missing, might still be too good to offer a real alternative to the so-called correct models. Therefore we generated one more fixed population POP5 by a model, which was made more complicated by adding two strong interactions, a fixed interaction between month and JS and a random interaction between month and unit, into the model (8.2) used before. The former interaction introduces temporal variation in the regression coefficient of JS and the latter makes the response variances grow over time. The already familiar simple incorrect models (8.3), (10.1) and (10.2) (with the variance equality assumption) were fitted to samples from this population to obtain the EBLUP estimates.

However, this additional experiment did not offer any completely new findings. Deteriorated performances were found especially with the EBLUP estimators from cross-sectional data (NESTED) and panel data (PANEL) in the smallest areas. They were manifested in larger errors in both the EBLUP estimates and their MSE estimates, yielding larger MARE's and poorer (mainly too high) coverage rates of confidence intervals. The results for the largest areas were mostly adequate. The ROTPANEL model gave a valid performance with small EBLUP and MSE estimation errors also in this situation. The results for EBLUP bias, MARE and RMSE bias, obtained from panel data of 10 time points, are presented in Figures 13.1, 13.2 and 13.3. It is seen that the high intra-unit correlation makes again the RMSE estimates under the RANTIME model severely biased. In some areas using the data of only 5 time points deteriorates the model fit, yielding increase in bias and poor coverage rates.

One interesting issue, which was not fully met with the earlier experiments, but appeared here, was an outlying area. In the generation of POP5 area 15 happened to receive an unusually large negative random effect, which stuck out of the other area effects with its standardized value -2.6. As the area also happened to be the second largest, it caused difficulties to the area total estimation. Especially the estimates from cross-sectional data and complete panel data suffered from large positive bias (Figure 13.1) leading to large MARE (Figure 13.2) and bad MSE estimates (Figure 13.3).

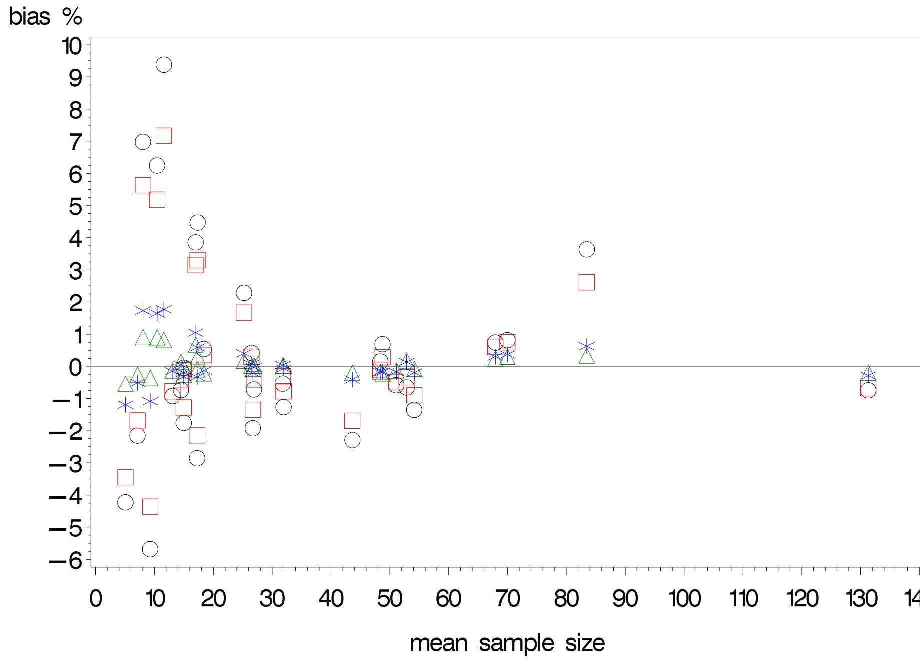

Figure 13.1: Relative bias under incorrect model plotted against last month's mean sample size in simulations from POP5. Symbols used are ◯ = NESTEDi, □ = PANEL10i, $*$ = ROTPANEL10i and △ = RANTIME10i.

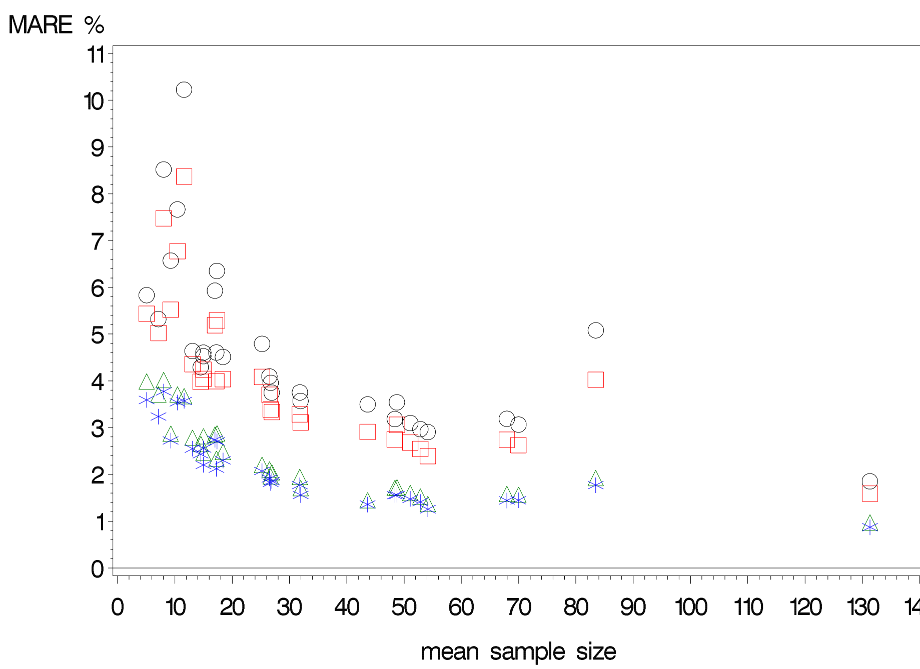

Figure 13.2: MARE under incorrect model plotted against last month's mean sample size in simulations from POP5. Symbols used are ◯ = NESTEDi, □ = PANEL10i, $*$ = ROTPANEL10i and △ = RANTIME10i.

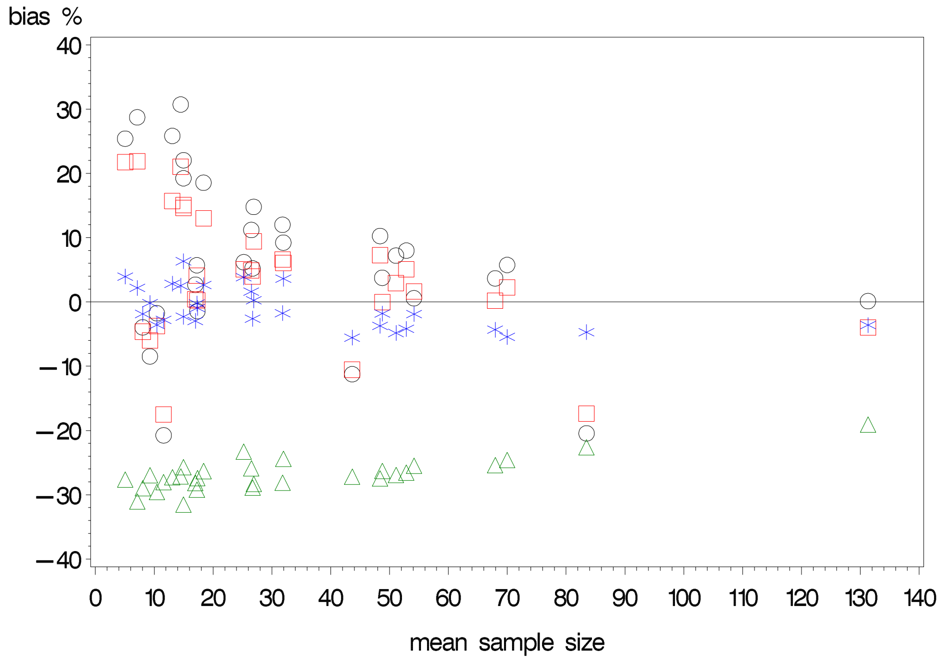

Figure 13.3: Approximate bias of RMSE estimates under incorrect model plotted against last month's mean sample size in simulations from POP2. Symbols used are $\bigcirc$ = NESTEDi, $\square$ = PANEL10i, $*$ = ROTPANEL10i and $\triangle$ = RANTIME10i.

Again, it appears that using rotating panel data offers a remarkable correction for the bias, whether data of 5 or 10 occasions are available. This in turn transfers to greater accuracy and better MSE estimates. Thus, our limited experiment suggests that when the amount of data available from an area is increased enough, which in this case means utilizing the rotating panel, even an inferior model can capture the essential properties of the area, leading to adequate small area estimates.

14 Discussion

One can hardly claim that the idea of compensating the weakness of the sample information by introducing external data is fresh. Virtually all literature on the small area estimation is about borrowing strength or increasing the effective sample size, either spatially from neighbouring or otherwise similar areas or temporally from earlier surveys.

In this work we show in the framework of best linear unbiased prediction (BLUP) how to borrow strength at unit level using longitudinal data sets collected by panel design or rotating panel design. The EBLUP estimator of area totals and the estimator of its approximate mean squared error are derived under a three-level variance component model, which is relatively simple and easy to estimate, but still accounts for both the correlations between units in an area and the correlations of the repeated observations of an unit with a reasonable accuracy.

The simulation studies showed clearly the superiority of using rotating panel data in estimating small area totals at a certain (usually the last) time point. The increase of the available information due to fully utilizing the rotating data yields great improvements in the estimation accuracy, reducing the relative errors of the EBLUP estimates as well as their mean squared errors, while still keeping the MSE estimates and the resulting confidence intervals valid.

However, we are aware of the limitations of this study. So far we have stuck to the framework of linear mixed models with normal random terms. Although it is recognized as a flexible modelling environment, which can provide pertinent results even for non-continuous and non-normally distributed responses, there are often practical situations, e.g. with a multinomial response, where one might look for a more adequate model. Some progress is made in small area estimation with generalized linear mixed models (e.g. EURAREA 2004), but the methodology seems not fully established yet. The small area estimation from unit-level longitudinal data with generalized linear or non-linear mixed models is one possible direction of future research.

The theory of generalized linear mixed models also relies largely on the normality of random effects (McCulloch and Searle 2001). In real life, however, the normality of area effects is not always justified. There are some research activities on the EBLUP estimation under non-normal area effects, but as far as we know, any papers on the topic are not yet published. Thus, apparently there is a lot of work to be done concerning the robustness of EBLUP estimators with respect the non-normality and the possibility to apply other distributions than normal.

In this thesis we have only considered the estimation of small area totals at a certain time point. In official statistics these are often of the most interest and there is a desire to estimate these as precisely as possible. Yet, longitudinal data sets are usually collected with the purpose of getting information about changes in the course of time. It is known that in estimating change a strong correlation between the data sets of different time points increases the accuracy, just contrary to the cross-sectional estimation problem. One could expect that the change in small area totals is estimated better from complete panel data than from rotating panel data, but this may depend on the distributional properties of the study variable together with the properties of the rotation design. It is possible that a rotation design optimal for estimating cross-sectional totals and their change could be found if some properties of the study variable were known. The problem of estimating small area changes is third direction of future research and it is probably the one, where the next step will be taken.

It is relatively simple to carry out the estimation of change under the linear mixed model adopted in this thesis. The estimation under generalized linear mixed models is arguably more demanding.

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A Appendix: Simulation study with correct model. Result tables for longitudinal data of 10 occasions

Table A.1. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP1.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL10c	bias ROTPANEL10c	bias RANTIME10c
1	14.9	0.06	0.10	0.02	-0.02
2	11.4	0.02	0.13	0.06	0.05
3	18.3	-1.56	-1.37	-0.33	-0.22
4	48.4	-0.18	-0.09	0.00	-0.02
5	8.1	1.29	1.09	0.23	0.02
6	25.4	0.91	0.76	0.23	0.14
7	48.9	0.42	0.32	0.09	0.02
8	13.0	0.71	0.64	0.27	0.18
9	26.4	-0.62	-0.47	-0.07	-0.06
10	43.8	0.29	0.22	0.02	-0.03
11	26.7	-0.29	-0.21	-0.01	-0.03
12	17.1	-0.63	-0.51	-0.15	-0.13
13	51.4	0.38	0.33	0.04	-0.02
14	7.3	-2.79	-2.44	-0.58	-0.32
15	83.0	0.22	0.17	0.01	-0.03
16	10.4	0.67	0.60	0.08	0.01
17	70.1	0.00	0.00	-0.04	-0.08
18	26.7	-0.17	-0.19	-0.07	-0.08
19	9.2	-1.26	-1.15	-0.38	-0.32
20	17.1	1.26	1.09	-0.30	0.13
21	32.1	0.28	0.20	0.01	-0.08
22	14.3	-0.37	-0.34	-0.14	-0.12
23	4.9	-3.20	-2.82	-0.76	-0.55
24	54.1	0.18	0.11	-0.02	-0.06
25	32.0	-0.62	-0.46	-0.10	-0.08
26	14.9	-2.15	-1.80	-0.45	-0.29
27	53.1	0.95	0.80	0.20	0.08
28	67.7	0.43	0.34	0.06	-0.01
29	17.6	-0.70	-0.61	-0.19	-0.16
30	131.6	-0.12	-0.10	-0.02	-0.04
average	33.3	-0.22	-0.19	-0.06	-0.07

Table A.2. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP1.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL10c	MARE ROTPANEL10c	MARE RANTIME10c
1	14.9	2.47	2.32	1.13	1.21
2	11.4	2.73	2.56	1.30	1.42
3	18.3	2.35	2.14	0.96	1.01
4	48.4	1.40	1.31	0.64	0.69
5	8.1	3.41	3.20	1.65	1.75
6	25.4	2.28	2.10	0.98	1.04
7	48.9	1.62	1.48	0.75	0.79
8	13.0	2.85	2.70	1.36	1.41
9	26.4	1.86	1.68	0.83	0.90
10	43.8	1.72	1.53	0.74	0.81
11	26.7	1.85	1.71	0.87	0.92
12	17.1	2.37	2.14	1.00	1.06
13	51.4	1.67	1.52	0.74	0.80
14	7.3	3.70	3.37	1.48	1.54
15	83.0	1.20	1.11	0.58	0.64
16	10.4	2.87	2.79	1.47	1.57
17	70.1	1.28	1.16	0.61	0.65
18	26.7	1.87	1.74	0.87	0.93
19	9.2	2.92	2.70	1.38	1.46
20	17.1	2.74	2.57	1.26	1.33
21	32.1	1.93	1.77	0.82	0.88
22	14.3	2.34	2.22	1.15	1.20
23	4.9	4.17	3.86	1.80	1.89
24	54.1	1.44	1.31	0.67	0.72
25	32.0	1.73	1.52	0.70	0.75
26	14.9	2.77	2.49	1.04	1.05
27	53.1	1.92	1.75	0.85	0.89
28	67.7	1.53	1.38	0.69	0.75
29	17.6	2.11	1.97	1.00	1.04
30	131.6	0.86	0.82	0.44	0.50
average	33.3	2.20	2.03	0.99	1.05

Table A.3. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP1.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL10c	coverage ROTPANEL10c	coverage RANTIME10c
1	14.9	0.966	0.971	0.963	0.873
2	11.4	0.972	0.958	0.954	0.848
3	18.3	0.921	0.924	0.938	0.841
4	48.4	0.953	0.953	0.959	0.879
5	8.1	0.974	0.962	0.958	0.852
6	25.4	0.949	0.951	0.959	0.876
7	48.9	0.963	0.959	0.949	0.883
8	13.0	0.963	0.952	0.951	0.849
9	26.4	0.960	0.956	0.951	0.855
10	43.8	0.939	0.938	0.955	0.880
11	26.7	0.962	0.959	0.947	0.865
12	17.1	0.945	0.953	0.952	0.862
13	51.4	0.946	0.950	0.951	0.865
14	7.3	0.919	0.921	0.955	0.848
15	83.0	0.949	0.946	0.958	0.889
16	10.4	0.969	0.958	0.959	0.836
17	70.1	0.949	0.947	0.962	0.885
18	26.7	0.956	0.955	0.955	0.864
19	9.2	0.957	0.959	0.947	0.832
20	17.1	0.952	0.945	0.944	0.845
21	32.1	0.946	0.950	0.956	0.866
22	14.3	0.972	0.960	0.948	0.855
23	4.9	0.938	0.946	0.949	0.850
24	54.1	0.949	0.958	0.950	0.879
25	32.0	0.948	0.946	0.953	0.887
26	14.9	0.886	0.900	0.945	0.840
27	53.1	0.936	0.938	0.943	0.875
28	67.7	0.938	0.940	0.939	0.872
29	17.6	0.964	0.956	0.959	0.886
30	131.6	0.950	0.942	0.947	0.902
average	33.3	0.950	0.952	0.952	0.865

Table A.4. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under correct model in simulations from POP1. Results for models NESTEDc and PANEL10c.

		NESTEDc			PANEL10c		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	9019	8325	8.34	8258	7796	5.93
2	11.4	7894	7220	9.34	7247	6841	5.93
3	18.3	10192	11605	-12.18	9317	10650	-12.52
4	48.4	16707	16364	2.10	15243	15081	1.07
5	8.1	6453	5925	8.91	5955	5594	6.45
6	25.4	12071	12056	-0.32	10969	11005	-0.33
7	48.9	16801	16289	3.14	15330	14996	2.23
8	13.0	8464	8212	3.07	7759	7721	0.49
9	26.4	12265	12294	-0.24	11193	11065	1.16
10	43.8	15864	16147	-1.75	14473	14589	-0.80
11	26.7	12355	11993	3.02	11274	11063	1.91
12	17.1	9767	10030	-2.62	8932	9087	-1.71
13	51.4	17192	17878	-3.84	15688	16298	-3.74
14	7.3	6065	7075	-14.28	5608	6458	-13.16
15	83.0	22076	21789	1.32	20193	19956	1.19
16	10.4	7513	6791	10.63	6908	6549	5.48
17	70.1	20160	19843	1.64	18421	18097	1.79
18	26.7	12274	11792	4.09	11201	10937	2.41
19	9.2	7020	6699	4.79	6467	6212	4.10
20	17.1	9775	9819	-0.45	8939	9140	-2.20
21	32.1	13559	13707	-1.08	12370	12417	-0.38
22	14.3	8860	8029	10.35	8114	7669	5.80
23	4.9	4965	5316	-6.60	4635	4958	-6.51
24	54.1	17660	17395	1.52	16119	15644	3.04
25	32.0	13562	14163	-4.24	12372	12636	-2.09
26	14.9	9120	11501	-20.70	8350	10281	-18.78
27	53.1	17489	18476	-5.34	15691	16586	-3.77
28	67.7	19871	20808	-4.50	18155	18943	-4.16
29	17.6	9798	9063	8.11	8961	8397	6.72
30	131.6	27724	27140	2.15	25476	25729	-0.98
average	33.3	12749	12791	0.15	11663	11747	-0.51

Table A.5. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP1. Results for models ROTPANEL10c and RANTIME10c.

		ROTPANEL10c			RANTIME10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	4036	3850	4.83	3139	4107	-23.57
2	11.4	3531	3440	2.65	2727	3693	-26.16
3	18.3	4544	4811	-5.55	3564	5047	-29.38
4	48.4	7675	7397	3.76	6399	7993	-19.94
5	8.1	2950	2851	3.47	2263	3013	-24.89
6	25.4	5393	5182	4.07	4294	5479	-21.63
7	48.9	7723	7667	0.73	6444	8145	-20.88
8	13.0	3794	3840	-1.20	2938	3979	-26.16
9	26.4	5508	5491	0.31	4399	5980	-26.44
10	43.8	7251	6972	4.00	5995	7661	-21.75
11	26.7	5548	5570	-0.39	4433	5922	-25.14
12	17.1	4364	4208	3.71	3411	4453	-23.40
13	51.4	7934	7954	-0.25	6647	8542	-22.18
14	7.3	2795	2843	-1.69	2138	2959	-27.75
15	83.0	10526	10429	0.93	9280	11513	-19.40
16	10.4	3373	3378	-0.15	2600	3646	-28.69
17	70.1	9493	9236	2.78	8207	10084	-18.61
18	26.7	5531	5421	2.03	4415	5826	-24.22
19	9.2	3168	3138	0.96	2437	3317	-26.53
20	17.1	4388	4517	-2.86	3431	4777	-28.18
21	32.1	6113	5900	3.61	4942	6283	-21.34
22	14.3	3973	3960	0.33	3089	4123	-25.08
23	4.9	2324	2335	-0.47	1783	2455	-27.37
24	54.1	8160	8125	0.43	6878	8643	-20.42
25	32.0	6125	5869	4.36	4947	6265	-21.04
26	14.9	4084	4324	-5.55	3175	4426	-28.26
27	53.1	8081	8121	-0.49	6793	8615	-21.15
28	67.7	9319	9502	-1.93	8031	10361	-22.49
29	17.6	4398	4212	4.42	3440	4432	-22.38
30	131.6	13939	13801	1.00	13011	15668	-16.96
average	33.3	5868	5811	0.93	4842	6247	-23.71

Table A.6. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP2.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL10c	bias ROTPANEL10c	bias RANTIME10c
1	14.9	-0.08	0.01	-0.02	-0.05
2	11.4	-0.01	0.20	0.14	0.11
3	18.3	-1.50	-0.56	-0.30	-0.29
4	48.4	-0.25	-0.01	0.02	-0.01
5	8.1	1.21	0.25	0.07	-0.01
6	25.4	0.94	0.31	0.24	0.19
7	48.9	0.49	0.13	0.10	0.06
8	13.0	0.74	0.31	0.25	0.21
9	26.4	-0.66	-0.17	-0.04	-0.06
10	43.8	0.31	0.07	-0.01	-0.04
11	26.7	-0.29	-0.05	0.00	-0.03
12	17.1	-0.59	-0.16	-0.07	-0.09
13	51.4	0.32	0.08	0.01	-0.03
14	7.3	-2.69	-0.94	-0.36	-0.30
15	83.0	0.22	0.08	0.00	-0.03
16	10.4	0.60	0.25	0.03	-0.02
17	70.1	-0.02	-0.06	-0.10	-0.13
18	26.7	-0.17	-0.14	-0.11	-0.13
19	9.2	-1.17	-0.56	-0.32	-0.33
20	17.1	1.27	0.46	0.27	0.19
21	32.1	0.30	0.01	-0.06	-0.11
22	14.3	-0.32	-0.12	-0.10	-0.11
23	4.9	-3.14	-1.31	-0.55	-0.51
24	54.1	0.18	-0.01	-0.06	-0.10
25	32.0	-0.67	-0.12	-0.04	-0.06
26	14.9	-2.10	-0.59	-0.28	-0.25
27	53.1	0.96	0.31	0.17	0.10
28	67.7	0.44	0.10	0.05	0.00
29	17.6	-0.71	-0.28	-0.16	-0.17
30	131.6	-0.14	-0.07	-0.03	-0.06
average	33.3	-0.22	-0.09	-0.04	-0.07

Table A.7. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP2.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL10c	MARE ROTPANEL10c	MARE RANTIME10c
1	14.9	2.41	1.47	0.98	0.98
2	11.4	2.69	1.69	1.14	1.16
3	18.3	2.30	1.24	0.82	0.82
4	48.4	1.39	0.81	0.56	0.56
5	8.1	3.35	2.04	1.39	1.40
6	25.4	2.31	1.29	0.86	0.86
7	48.9	1.64	0.91	0.65	0.65
8	13.0	2.85	1.67	1.14	1.14
9	26.4	1.86	1.01	0.71	0.72
10	43.8	1.70	0.96	0.67	0.68
11	26.7	1.79	1.06	0.73	0.74
12	17.1	2.30	1.32	0.85	0.86
13	51.4	1.62	0.93	0.66	0.67
14	7.3	3.67	1.93	1.19	1.21
15	83.0	1.20	0.71	0.51	0.51
16	10.4	2.88	1.84	1.18	1.20
17	70.1	1.30	0.74	0.55	0.56
18	26.7	1.90	1.07	0.76	0.77
19	9.2	2.95	1.80	1.20	1.20
20	17.1	2.78	1.55	1.03	1.04
21	32.1	1.86	1.09	0.74	0.75
22	14.3	2.32	1.40	0.98	0.98
23	4.9	4.17	2.44	1.51	1.53
24	54.1	1.43	0.83	0.62	0.62
25	32.0	1.71	0.91	0.62	0.63
26	14.9	2.72	1.29	0.84	0.84
27	53.1	1.92	1.06	0.79	0.78
28	67.7	1.50	0.87	0.62	0.63
29	17.6	2.15	1.25	0.85	0.86
30	131.6	0.87	0.55	0.42	0.43
average	33.3	2.18	1.26	0.85	0.86

Table A.8. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP2.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL10c	coverage ROTPANEL10c	coverage RANTIME10c
1	14.9	0.974	0.956	0.967	0.941
2	11.4	0.975	0.949	0.950	0.926
3	18.3	0.926	0.938	0.937	0.899
4	48.4	0.961	0.947	0.961	0.944
5	8.1	0.969	0.957	0.955	0.929
6	25.4	0.949	0.947	0.957	0.933
7	48.9	0.950	0.955	0.958	0.939
8	13.0	0.961	0.951	0.964	0.937
9	26.4	0.960	0.960	0.954	0.921
10	43.8	0.942	0.940	0.962	0.944
11	26.7	0.969	0.956	0.960	0.932
12	17.1	0.962	0.954	0.961	0.934
13	51.4	0.942	0.942	0.954	0.942
14	7.3	0.922	0.940	0.954	0.923
15	83.0	0.956	0.957	0.952	0.939
16	10.4	0.968	0.959	0.962	0.925
17	70.1	0.952	0.947	0.968	0.944
18	26.7	0.960	0.952	0.949	0.931
19	9.2	0.960	0.936	0.947	0.916
20	17.1	0.955	0.955	0.952	0.932
21	32.1	0.950	0.952	0.951	0.933
22	14.3	0.970	0.954	0.947	0.930
23	4.9	0.944	0.951	0.947	0.919
24	54.1	0.950	0.958	0.951	0.931
25	32.0	0.942	0.955	0.958	0.940
26	14.9	0.891	0.937	0.948	0.931
27	53.1	0.931	0.949	0.927	0.912
28	67.7	0.941	0.950	0.947	0.933
29	17.6	0.966	0.955	0.966	0.946
30	131.6	0.952	0.938	0.954	0.939
average	33.3	0.952	0.950	0.954	0.932

Table A.9. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP2. Results for models NESTEDc and PANEL10c.

		NESTEDc			PANEL10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9022	8175	10.36	5070	4968	2.05
2	11.4	7897	7041	12.16	4478	4470	0.18
3	18.3	10194	11447	-10.95	5706	6181	-7.68
4	48.4	16709	16076	3.94	9488	9351	1.47
5	8.1	6456	5888	9.65	3738	3595	3.98
6	25.4	12019	12118	-0.82	6720	6727	-0.10
7	48.9	16804	16535	1.63	9547	9155	4.28
8	13.0	8468	8182	3.50	4777	4810	-0.69
9	26.4	12268	12258	0.08	6860	6673	2.80
10	43.8	15867	16142	-1.70	8975	9032	-0.63
11	26.7	12358	11563	6.88	6911	6802	1.60
12	17.1	9770	9613	1.63	5473	5579	-1.90
13	51.4	17195	17577	-2.17	9790	10009	-2.19
14	7.3	6068	6994	-13.24	3547	3834	-7.49
15	83.0	22078	21826	1.15	12958	12823	1.05
16	10.4	7515	6823	10.14	4288	4326	-0.88
17	70.1	20162	19766	2.00	11686	11391	2.59
18	26.7	12276	11924	2.95	6866	6755	1.64
19	9.2	7024	6793	3.40	4038	4179	-3.37
20	17.1	9779	9927	-1.49	5476	5491	-0.27
21	32.1	13562	13222	2.57	7605	7681	-0.99
22	14.3	8863	8001	10.77	4984	4919	1.32
23	4.9	4970	5346	-7.03	3048	3219	-5.31
24	54.1	17661	17468	1.10	10083	9932	1.52
25	32.0	13563	14227	-4.67	7606	7503	1.37
26	14.9	9123	11357	-19.67	5125	5439	-5.77
27	53.1	17490	18842	-7.18	9975	10053	-0.78
28	67.7	19874	20699	-3.99	11492	11902	-3.44
29	17.6	9800	9249	5.96	5490	5362	2.39
30	131.6	27727	27322	1.48	17045	17424	-2.18
average	33.3	12752	12747	0.62	7295	7320	-0.51

Table A.10. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP2. Results for models ROTPANEL10c and RANTIME10c.

		ROTPANEL10c			RANTIME10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	3432	3304	3.87	3143	3327	-5.53
2	11.4	2989	2968	0.71	2730	3007	-9.21
3	18.3	3886	4122	-5.73	3567	4130	-13.63
4	48.4	6852	6472	5.87	6405	6539	-2.05
5	8.1	2485	2427	2.39	2265	2449	-7.51
6	25.4	4661	4505	3.46	4299	4498	-4.42
7	48.9	6899	6632	4.03	6451	6619	-2.54
8	13.0	3217	3191	0.81	2941	3180	-7.52
9	26.4	4771	4730	0.87	4403	4804	-8.35
10	43.8	6435	6142	4.77	6001	6236	-3.77
11	26.7	4807	4685	2.60	4438	4720	-5.97
12	17.1	3724	3563	4.52	3414	3579	-4.61
13	51.4	7108	7047	0.87	6653	7100	-6.30
14	7.3	2351	2317	1.47	2140	2336	-8.39
15	83.0	9786	9354	4.62	9289	9464	-1.85
16	10.4	2851	2777	2.66	2602	2824	-7.86
17	70.1	8700	8333	4.40	8215	8436	-2.62
18	26.7	4789	4747	0.88	4420	4800	-7.92
19	9.2	2674	2714	-1.47	2439	2746	-11.18
20	17.1	3746	3681	1.77	3434	3680	-6.68
21	32.1	5340	5208	2.53	4947	5273	-6.18
22	14.3	3378	3356	0.66	3092	3358	-7.92
23	4.9	1958	1966	-0.41	1784	1992	-10.44
24	54.1	7342	7409	-0.90	6884	7473	-7.88
25	32.0	5347	5074	5.38	4952	5142	-3.70
26	14.9	3472	3478	-0.17	3178	3478	-8.63
27	53.1	7257	7565	-4.07	6799	7576	-10.26
28	67.7	8520	8559	-0.46	8039	8571	-6.21
29	17.6	3755	3584	4.77	3443	3618	-4.84
30	131.6	13523	13333	1.43	13024	13579	-4.09
average	33.3	5202	5108	1.74	4846	5151	-6.60

Table A.11. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP3.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL10c	bias ROTPANEL10c	bias RANTIME10c
1	14.9	-0.04	0.04	0.02	-0.03
2	11.4	-0.08	0.07	0.10	0.07
3	18.3	-3.07	-2.54	-0.74	-0.48
4	48.4	-0.47	-0.28	-0.04	-0.05
5	8.1	1.66	1.43	0.49	0.17
6	25.4	1.56	1.25	0.43	0.25
7	48.9	0.86	0.63	0.19	0.07
8	13.0	1.07	0.95	0.44	0.28
9	26.4	-1.23	-0.91	-0.19	-0.14
10	43.8	0.53	0.40	0.07	-0.02
11	26.7	-0.59	-0.42	-0.05	-0.06
12	17.1	-1.16	-0.91	-0.28	-0.22
13	51.4	0.74	0.59	0.11	0.01
14	7.3	-4.07	-3.54	-1.21	-0.74
15	83.0	0.34	0.25	0.03	-0.03
16	10.4	0.77	0.72	0.19	0.06
17	70.1	0.09	0.06	-0.04	-0.10
18	26.7	-0.44	-0.39	-0.13	-0.13
19	9.2	-1.88	-1.66	-0.64	-0.51
20	17.1	2.06	1.72	0.58	0.29
21	32.1	0.53	0.38	0.06	-0.06
22	14.3	-0.82	-0.69	-0.25	-0.20
23	4.9	-3.80	-3.38	-1.36	-0.98
24	54.1	0.26	0.16	-0.01	-0.08
25	32.0	-1.36	-0.98	-0.24	-0.17
26	14.9	-4.10	-3.34	-1.01	-0.63
27	53.1	1.85	1.44	0.39	0.18
28	67.7	0.88	0.66	0.15	0.03
29	17.6	-1.41	-1.14	-0.38	-0.28
30	131.6	-0.28	-0.20	-0.06	-0.07
average	33.3	-0.39	-0.32	-0.11	-0.12

Table A.12. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP3.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL10c	MARE ROTPANEL10c	MARE RANTIME10c
1	14.9	2.05	1.99	1.22	1.33
2	11.4	2.10	2.09	1.38	1.54
3	18.3	3.36	2.88	1.25	1.24
4	48.4	1.53	1.39	0.73	0.79
5	8.1	2.61	2.52	1.66	1.81
6	25.4	2.34	2.10	1.08	1.12
7	48.9	1.74	1.53	0.82	0.85
8	13.0	2.37	2.30	1.44	1.51
9	26.4	2.06	1.79	0.96	1.05
10	43.8	1.75	1.53	0.81	0.89
11	26.7	1.84	1.70	0.97	1.04
12	17.1	2.30	2.06	1.12	1.20
13	51.4	1.77	1.56	0.81	0.87
14	7.3	4.27	3.81	1.81	1.80
15	83.0	1.29	1.17	0.65	0.71
16	10.4	2.20	2.22	1.51	1.66
17	70.1	1.36	1.20	0.68	0.73
18	26.7	1.82	1.70	0.98	1.05
19	9.2	2.65	2.49	1.52	1.64
20	17.1	2.77	2.54	1.36	1.40
21	32.1	1.89	1.73	0.90	0.97
22	14.3	2.06	1.99	1.26	1.34
23	4.9	3.99	3.68	2.04	2.11
24	54.1	1.49	1.34	0.76	0.81
25	32.0	2.10	1.74	0.83	0.89
26	14.9	4.26	3.60	1.44	1.34
27	53.1	2.35	1.98	0.92	0.93
28	67.7	1.69	1.46	0.75	0.81
29	17.6	2.19	2.01	1.14	1.20
30	131.6	1.00	0.92	0.53	0.59
average	33.3	2.24	2.03	1.11	1.17

Table A.13. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP3.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL10c	coverage ROTPANEL10c	coverage RANTIME10c
1	14.9	0.989	0.984	0.965	0.894
2	11.4	0.993	0.986	0.964	0.864
3	18.3	0.815	0.831	0.915	0.836
4	48.4	0.960	0.964	0.959	0.886
5	8.1	0.992	0.982	0.965	0.879
6	25.4	0.954	0.949	0.950	0.885
7	48.9	0.965	0.958	0.952	0.901
8	13.0	0.978	0.967	0.955	0.866
9	26.4	0.954	0.966	0.947	0.869
10	43.8	0.958	0.947	0.956	0.895
11	26.7	0.974	0.971	0.956	0.874
12	17.1	0.966	0.970	0.956	0.878
13	51.4	0.951	0.945	0.950	0.878
14	7.3	0.879	0.886	0.935	0.856
15	83.0	0.955	0.953	0.954	0.889
16	10.4	0.986	0.989	0.967	0.863
17	70.1	0.962	0.959	0.964	0.897
18	26.7	0.976	0.970	0.959	0.877
19	9.2	0.981	0.973	0.955	0.852
20	17.1	0.958	0.949	0.947	0.860
21	32.1	0.965	0.960	0.960	0.880
22	14.3	0.984	0.979	0.959	0.873
23	4.9	0.953	0.953	0.949	0.850
24	54.1	0.962	0.966	0.953	0.892
25	32.0	0.936	0.940	0.957	0.895
26	14.9	0.726	0.748	0.891	0.829
27	53.1	0.870	0.899	0.937	0.881
28	67.7	0.931	0.930	0.939	0.881
29	17.6	0.970	0.965	0.957	0.901
30	131.6	0.952	0.943	0.950	0.903
average	33.3	0.947	0.946	0.951	0.876

Table A.14. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP3. Results for models NESTEDc and PANEL10c.

		NESTEDc			PANEL10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9150	7051	29.77	8292	6774	22.41
2	11.4	7701	5656	36.16	7049	5636	25.07
3	18.3	10617	14990	-29.17	9553	13103	-27.09
4	48.4	18835	17651	6.71	16626	15864	4.80
5	8.1	5938	4693	26.53	5510	4571	20.54
6	25.4	12964	12990	-0.20	11559	11615	-0.48
7	48.9	18953	18373	3.16	16728	16309	2.57
8	13.0	8432	7050	19.60	7675	6799	12.88
9	26.4	13283	13193	0.68	11832	11561	2.34
10	43.8	17792	17146	3.77	15719	15206	3.37
11	26.7	13393	11867	12.86	11928	10934	9.09
12	17.1	10099	9503	6.27	9105	8621	5.61
13	51.4	19438	19824	-1.95	17151	17637	-2.76
14	7.3	5467	7190	-23.96	5096	6536	-22.03
15	83.0	25408	24248	4.78	22408	21656	3.47
16	10.4	7216	5327	35.46	6630	5350	23.93
17	70.1	23079	21568	7.01	20346	19259	5.64
18	26.7	13293	11564	14.95	11842	10765	10.00
19	9.2	6587	5818	13.22	6083	5508	10.44
20	17.1	10115	10255	-1.37	9116	9431	-3.34
21	32.1	14905	13889	7.32	13229	12503	5.81
22	14.3	8951	7052	26.93	8119	6874	18.11
23	4.9	4130	4598	-10.18	3915	4311	-9.19
24	54.1	20016	18574	7.76	17656	16390	7.72
25	32.0	14902	16377	-9.01	13227	13932	-5.06
26	14.9	9260	15414	-39.92	8388	13184	-36.38
27	53.1	19799	24930	-20.58	17468	20876	-16.32
28	67.7	22722	24656	-7.84	20032	21444	-6.58
29	17.6	10160	9113	11.49	9157	8359	9.55
30	131.6	32241	31332	2.90	28570	28770	-0.70
average	33.3	13828	13730	4.44	12334	12326	2.45

Table A.15. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP3. Results for models ROTPANEL10c and RANTIME10c.

		ROTPANEL10c			RANTIME10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	4559	4200	8.55	3660	4569	-19.89
2	11.4	3952	3673	7.60	3163	4068	-22.25
3	18.3	5166	5881	-12.16	4167	5892	-29.28
4	48.4	8908	8441	5.53	7544	9134	-17.41
5	8.1	3245	3001	8.13	2600	3249	-19.98
6	25.4	6182	6017	2.74	5040	6261	-19.50
7	48.9	8966	8859	1.21	7598	9289	-18.20
8	13.0	4266	4201	1.55	3417	4418	-22.66
9	26.4	6320	6242	1.25	5165	6795	-23.99
10	43.8	8402	7982	5.26	7064	8730	-19.08
11	26.7	6368	6228	2.25	5206	6684	-22.11
12	17.1	4951	4684	5.70	3985	4996	-20.24
13	51.4	9219	9215	0.04	7838	9805	-20.06
14	7.3	3055	3296	-7.31	2448	3315	-26.15
15	83.0	12336	12111	1.86	10961	13211	-17.03
16	10.4	3760	3559	5.65	3010	3958	-23.95
17	70.1	11091	10693	3.72	9689	11584	-16.36
18	26.7	6346	6106	3.93	5185	6605	-21.50
19	9.2	3510	3410	2.93	2811	3671	-23.43
20	17.1	4979	5161	-3.53	4009	5365	-25.27
21	32.1	7042	6698	5.14	5812	7155	-18.77
22	14.3	4483	4325	3.65	3598	4597	-21.73
23	4.9	2474	2543	-2.71	2008	2665	-24.65
24	54.1	9492	9389	1.10	8112	9963	-18.58
25	32.0	7055	6760	4.36	5817	7174	-18.92
26	14.9	4613	5530	-16.58	3702	5254	-29.54
27	53.1	9394	9874	-4.86	8012	10071	-20.44
28	67.7	10881	11165	-2.54	9481	11924	-20.49
29	17.6	4992	4739	5.34	4020	5015	-19.84
30	131.6	16472	16324	0.91	15380	18205	-15.52
average	33.3	6749	6677	1.29	5683	7121	-21.23

Table A.16. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP4.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL10c	bias ROTPANEL10c	bias RANTIME10c
1	14.9	-0.18	-0.01	-0.02	-0.07
2	11.4	-0.07	0.23	0.18	0.14
3	18.3	-3.00	-1.02	-0.59	-0.56
4	48.4	-0.54	-0.06	0.00	-0.04
5	8.1	1.59	0.51	0.27	0.16
6	25.4	1.62	0.51	0.38	0.30
7	48.9	0.91	0.23	0.17	0.11
8	13.0	1.11	0.50	0.38	0.31
9	26.4	-1.25	-0.31	-0.11	-0.14
10	43.8	0.55	0.14	0.03	-0.03
11	26.7	-0.58	-0.10	-0.03	-0.07
12	17.1	-1.11	-0.29	-0.15	-0.17
13	51.4	0.68	0.17	0.06	0.00
14	7.3	-4.02	-1.59	-0.82	-0.74
15	83.0	0.33	0.11	0.02	-0.03
16	10.4	0.73	0.35	0.12	0.05
17	70.1	0.07	-0.05	-0.10	-0.15
18	26.7	-0.45	-0.22	-0.15	-0.19
19	9.2	-1.84	-0.84	-0.51	-0.51
20	17.1	2.09	0.75	0.47	0.36
21	32.1	0.54	0.08	-0.03	-0.09
22	14.3	-0.75	-0.24	-0.16	-0.19
23	4.9	-3.84	-1.85	-1.03	-0.96
24	54.1	0.25	0.00	-0.06	-0.11
25	32.0	-1.41	-0.28	-0.14	-0.15
26	14.9	-4.05	-1.20	-0.65	-0.60
27	53.1	1.86	0.51	0.30	0.21
28	67.7	0.90	0.20	0.11	0.05
29	17.6	-1.40	-0.48	-0.28	-0.30
30	131.6	-0.30	-0.10	-0.06	-0.10
average	33.3	-0.39	-0.15	-0.08	-0.12

Table A.17. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP4.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL10c	MARE ROTPANEL10c	MARE RANTIME10c
1	14.9	2.03	1.42	1.07	1.08
2	11.4	2.09	1.60	1.24	1.26
3	18.3	3.30	1.52	1.04	1.04
4	48.4	1.53	0.87	0.65	0.65
5	8.1	2.56	1.84	1.44	1.46
6	25.4	2.43	1.30	0.95	0.94
7	48.9	1.77	0.92	0.72	0.71
8	13.0	2.42	1.59	1.23	1.23
9	26.4	2.07	1.08	0.82	0.83
10	43.8	1.74	0.98	0.74	0.74
11	26.7	1.79	1.10	0.83	0.84
12	17.1	2.20	1.35	0.96	0.97
13	51.4	1.71	0.94	0.72	0.72
14	7.3	4.25	2.15	1.45	1.45
15	83.0	1.29	0.75	0.57	0.57
16	10.4	2.21	1.71	1.25	1.28
17	70.1	1.36	0.78	0.62	0.62
18	26.7	1.87	1.11	0.87	0.88
19	9.2	2.64	1.80	1.33	1.35
20	17.1	2.82	1.54	1.12	1.11
21	32.1	1.82	1.09	0.81	0.82
22	14.3	2.06	1.39	1.09	1.10
23	4.9	4.08	2.45	1.74	1.76
24	54.1	1.48	0.86	0.70	0.70
25	32.0	2.09	1.00	0.74	0.75
26	14.9	4.24	1.66	1.10	1.08
27	53.1	2.36	1.07	0.84	0.82
28	67.7	1.67	0.88	0.67	0.67
29	17.6	2.22	1.33	0.98	0.99
30	131.6	1.03	0.62	0.50	0.51
average	33.3	2.24	1.29	0.96	0.96

Table A.18. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP4.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL10c	coverage ROTPANEL10c	coverage RANTIME10c
1	14.9	0.991	0.966	0.969	0.959
2	11.4	0.996	0.961	0.959	0.938
3	18.3	0.822	0.906	0.912	0.897
4	48.4	0.964	0.953	0.959	0.948
5	8.1	0.988	0.973	0.965	0.947
6	25.4	0.954	0.955	0.953	0.940
7	48.9	0.952	0.957	0.960	0.947
8	13.0	0.981	0.957	0.965	0.942
9	26.4	0.960	0.961	0.955	0.927
10	43.8	0.950	0.942	0.968	0.954
11	26.7	0.986	0.959	0.961	0.942
12	17.1	0.982	0.964	0.964	0.946
13	51.4	0.949	0.943	0.956	0.949
14	7.3	0.878	0.929	0.941	0.924
15	83.0	0.963	0.958	0.951	0.939
16	10.4	0.990	0.969	0.965	0.943
17	70.1	0.962	0.954	0.965	0.949
18	26.7	0.973	0.960	0.952	0.930
19	9.2	0.975	0.939	0.951	0.925
20	17.1	0.956	0.947	0.950	0.944
21	32.1	0.973	0.961	0.953	0.939
22	14.3	0.988	0.951	0.954	0.936
23	4.9	0.955	0.946	0.943	0.921
24	54.1	0.960	0.960	0.958	0.939
25	32.0	0.934	0.949	0.963	0.948
26	14.9	0.716	0.901	0.924	0.912
27	53.1	0.881	0.932	0.921	0.914
28	67.7	0.930	0.945	0.948	0.941
29	17.6	0.970	0.957	0.966	0.946
30	131.6	0.951	0.942	0.952	0.942
average	33.3	0.948	0.950	0.953	0.938

Table A.19. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP4. Results for models NESTEDc and PANEL10c.

		NESTEDc			PANEL10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9164	6979	31.31	5253	4870	7.86
2	11.4	7711	5521	39.67	4576	4277	6.99
3	18.3	10628	14743	-27.91	5965	7091	-15.88
4	48.4	18847	17464	7.92	10221	9926	2.97
5	8.1	5944	4644	27.99	3726	3361	10.86
6	25.4	12974	13196	-1.68	7109	7177	-0.95
7	48.9	18964	18777	1.00	10287	9841	4.53
8	13.0	8447	7089	19.16	4918	4723	4.13
9	26.4	13297	13145	1.16	7267	6987	4.01
10	43.8	17805	17172	3.69	9642	9562	0.84
11	26.7	13403	11427	17.29	7323	7005	4.54
12	17.1	10112	9020	12.11	5707	5621	1.53
13	51.4	19449	19395	0.28	10562	10694	-1.23
14	7.3	5475	7152	-23.45	3500	3970	-11.84
15	83.0	25419	24257	4.79	14160	13958	1.45
16	10.4	7225	5387	34.12	4351	4106	5.97
17	70.1	23089	21439	7.70	12712	12187	4.31
18	26.7	13305	11716	13.56	7275	7017	3.68
19	9.2	6603	5815	13.55	4059	4072	-0.32
20	17.1	10130	10388	-2.48	5713	5812	-1.70
21	32.1	14917	13363	11.63	8102	7992	1.38
22	14.3	8966	7067	26.87	5157	4871	5.87
23	4.9	4139	4690	-11.75	2861	3043	-5.98
24	54.1	20026	18553	7.94	10893	10566	3.09
25	32.0	14914	16486	-9.54	8101	7994	1.34
26	14.9	9276	15265	-39.23	5313	6430	-17.37
27	53.1	19809	25240	-21.52	10771	11435	-5.81
28	67.7	22732	24559	-7.44	12489	13002	-3.95
29	17.6	10171	9265	9.78	5731	5562	3.04
30	131.6	32253	31639	1.94	18880	19280	-2.07
average	33.3	13840	13695	4.95	7754	7748	0.38

Table A.20. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP4. Results for models ROTPANEL10c and RANTIME10c.

		ROTPANEL10c			RANTIME10c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	3899	3656	6.65	3661	3695	-0.92
2	11.4	3373	3252	3.72	3164	3300	-4.12
3	18.3	4435	4967	-10.71	4170	4953	-15.81
4	48.4	7935	7447	6.55	7550	7510	0.53
5	8.1	2771	2628	5.44	2600	2655	-2.07
6	25.4	5350	5259	1.73	5044	5200	-3.00
7	48.9	7991	7718	3.54	7604	7649	-0.59
8	13.0	3643	3567	2.13	3419	3563	-4.04
9	26.4	5479	5390	1.65	5169	5467	-5.45
10	43.8	7443	7049	5.59	7070	7119	-0.69
11	26.7	5522	5306	4.07	5210	5347	-2.56
12	17.1	4243	3996	6.18	3988	4028	-0.99
13	51.4	8238	8102	1.68	7845	8129	-3.49
14	7.3	2610	2687	-2.87	2448	2681	-8.69
15	83.0	11408	10825	5.39	10971	10894	0.71
16	10.4	3209	3024	6.12	3010	3081	-2.30
17	70.1	10121	9597	5.46	9698	9710	-0.12
18	26.7	5500	5416	1.55	5188	5490	-5.50
19	9.2	2997	3005	-0.27	2812	3054	-7.92
20	17.1	4268	4239	0.68	4011	4190	-4.27
21	32.1	6151	5930	3.73	5817	5993	-2.94
22	14.3	3834	3726	2.90	3600	3751	-4.03
23	4.9	2133	2194	-2.78	2008	2215	-9.35
24	54.1	8515	8524	-0.11	8119	8603	-5.63
25	32.0	6158	5875	4.82	5822	5948	-2.12
26	14.9	3944	4293	-8.13	3704	4233	-12.50
27	53.1	8414	9028	-6.80	8019	8903	-9.93
28	67.7	9909	9927	-0.18	9489	9864	-3.80
29	17.6	4280	4080	4.90	4022	4131	-2.64
30	131.6	15846	15582	1.69	15395	15820	-2.69
average	33.3	5987	5876	1.81	5688	5906	-4.23

B Appendix: Simulation study with incorrect model. Result tables for longitudinal data of 10 occasions

Table B.1. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP1.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL10i	bias ROTPANEL10i	bias RANTIME10i
1	14.9	0.04	0.10	0.00	-0.11
2	11.4	0.06	0.20	0.08	0.02
3	18.3	-2.50	-2.33	-0.55	-0.38
4	48.4	-0.24	-0.17	0.00	-0.08
5	8.1	1.63	1.52	0.31	-0.01
6	25.4	1.53	1.41	0.38	0.14
7	48.9	0.55	0.45	0.11	-0.05
8	13.0	1.22	1.16	0.39	0.19
9	26.4	-1.02	-0.87	-0.12	-0.11
10	43.8	0.54	0.48	0.08	-0.05
11	26.7	-0.60	-0.52	-0.04	-0.08
12	17.1	-1.10	-0.98	-0.29	-0.27
13	51.4	0.60	0.57	0.12	0.00
14	7.3	-4.28	-4.00	-1.02	-0.59
15	83.0	0.40	0.35	0.05	-0.04
16	10.4	0.92	0.90	0.22	0.11
17	70.1	0.13	0.12	0.01	-0.08
18	26.7	-0.26	-0.28	-0.07	-0.09
19	9.2	-2.03	-1.94	-0.61	-0.48
20	17.1	1.94	1.84	0.47	0.19
21	32.1	0.38	0.31	0.08	-0.06
22	14.3	-0.95	-0.92	-0.26	-0.19
23	4.9	-4.84	-4.55	-1.36	-0.91
24	54.1	0.27	0.20	0.04	-0.03
25	32.0	-1.03	-0.88	-0.21	-0.16
26	14.9	-3.54	-3.25	-0.78	-0.48
27	53.1	1.41	1.28	0.26	0.05
28	67.7	0.71	0.62	0.13	0.03
29	17.6	-0.93	-0.88	-0.24	-0.18
30	131.6	-0.10	-0.08	-0.02	-0.07
average	33.3	-0.37	-0.34	-0.09	-0.13

Table B.2. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP1.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL10i	MARE ROTPANEL10i	MARE RANTIME10i
1	14.9	3.07	2.97	1.54	1.65
2	11.4	3.37	3.31	1.74	1.91
3	18.3	3.35	3.21	1.36	1.40
4	48.4	1.91	1.83	0.88	0.96
5	8.1	4.00	3.92	2.18	2.32
6	25.4	2.93	2.77	1.37	1.44
7	48.9	2.21	2.13	1.03	1.13
8	13.0	3.64	3.54	1.84	1.94
9	26.4	2.49	2.36	1.18	1.28
10	43.8	2.21	2.10	1.00	1.10
11	26.7	2.42	2.31	1.22	1.32
12	17.1	3.10	2.94	1.44	1.54
13	51.4	2.23	2.12	0.99	1.11
14	7.3	5.06	4.84	2.07	2.15
15	83.0	1.60	1.52	0.81	0.89
16	10.4	3.49	3.49	1.92	2.07
17	70.1	1.74	1.64	0.85	0.92
18	26.7	2.63	2.55	1.26	1.38
19	9.2	3.71	3.62	1.85	1.99
20	17.1	3.57	3.47	1.77	1.85
21	32.1	2.53	2.42	1.15	1.25
22	14.3	3.05	2.98	1.60	1.71
23	4.9	5.59	5.36	2.50	2.66
24	54.1	1.90	1.78	0.93	1.00
25	32.0	2.35	2.17	0.99	1.08
26	14.9	4.06	3.84	1.45	1.42
27	53.1	2.63	2.50	1.13	1.18
28	67.7	2.04	1.94	0.90	0.99
29	17.6	2.80	2.71	1.39	1.48
30	131.6	1.12	1.08	0.58	0.67
average	33.3	2.89	2.78	1.36	1.46

Table B.3. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP1.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL10i	coverage ROTPANEL10i	coverage RANTIME10i
1	14.9	0.975	0.974	0.961	0.849
2	11.4	0.970	0.967	0.961	0.842
3	18.3	0.895	0.892	0.938	0.834
4	48.4	0.952	0.961	0.955	0.873
5	8.1	0.973	0.966	0.966	0.864
6	25.4	0.959	0.959	0.961	0.877
7	48.9	0.950	0.952	0.951	0.870
8	13.0	0.968	0.964	0.956	0.843
9	26.4	0.945	0.947	0.937	0.821
10	43.8	0.962	0.964	0.959	0.883
11	26.7	0.963	0.960	0.941	0.861
12	17.1	0.962	0.958	0.954	0.820
13	51.4	0.948	0.948	0.957	0.868
14	7.3	0.896	0.892	0.935	0.815
15	83.0	0.950	0.961	0.961	0.882
16	10.4	0.970	0.969	0.957	0.845
17	70.1	0.951	0.950	0.954	0.871
18	26.7	0.951	0.948	0.949	0.835
19	9.2	0.967	0.959	0.949	0.840
20	17.1	0.959	0.957	0.925	0.822
21	32.1	0.956	0.949	0.956	0.856
22	14.3	0.963	0.963	0.944	0.834
23	4.9	0.923	0.920	0.950	0.815
24	54.1	0.955	0.960	0.949	0.882
25	32.0	0.947	0.957	0.956	0.859
26	14.9	0.836	0.856	0.918	0.823
27	53.1	0.944	0.937	0.951	0.875
28	67.7	0.948	0.948	0.953	0.891
29	17.6	0.975	0.973	0.955	0.853
30	131.6	0.964	0.955	0.958	0.899
average	33.3	0.949	0.949	0.951	0.853

Table B.4. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP1. Results for models NESTEDi and PANEL10i.

		NESTEDi			PANEL10i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	11751	10444	12.51	11257	10162	10.78
2	11.4	10177	8919	14.10	9770	8829	10.66
3	18.3	13362	16432	-18.68	12782	15755	-18.87
4	48.4	22303	21718	2.69	21279	20768	2.46
5	8.1	8174	7066	15.68	7876	6923	13.77
6	25.4	15886	15415	3.06	15174	14658	3.52
7	48.9	22432	22457	-0.11	21402	21608	-0.95
8	13.0	10970	10355	5.94	10520	10072	4.45
9	26.4	16229	16573	-2.08	15499	15691	-1.22
10	43.8	21156	20485	3.28	20187	19358	4.28
11	26.7	16348	15645	4.49	15613	14931	4.57
12	17.1	12780	12746	0.27	12231	12237	-0.05
13	51.4	22967	23956	-4.13	21912	22858	-4.14
14	7.3	7637	9432	-19.03	7370	9068	-18.73
15	83.0	29603	28925	2.34	28267	27614	2.36
16	10.4	9642	8219	17.31	9267	8191	13.14
17	70.1	27001	26467	2.02	25773	25069	2.81
18	26.7	16241	16459	-1.32	15511	15877	-2.31
19	9.2	8948	8401	6.51	8612	8171	5.40
20	17.1	12796	12553	1.94	12244	12148	0.79
21	32.1	18001	17863	0.77	17183	17061	0.72
22	14.3	11530	10592	8.86	11047	10323	7.01
23	4.9	6081	7002	-13.15	5901	6760	-12.71
24	54.1	23602	23222	1.64	22521	21808	3.27
25	32.0	18005	19080	-5.63	17187	17652	-2.63
26	14.9	11880	16510	-28.04	11380	15619	-27.14
27	53.1	23370	25182	-7.20	22299	23849	-6.50
28	67.7	26610	27741	-4.08	25399	26320	-3.50
29	17.6	12832	11654	10.11	12280	11303	8.64
30	131.6	37262	35140	6.04	35656	34109	4.54
average	33.3	16853	16888	0.54	16113	16160	0.15

Table B.5. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP1. Results for models ROTPANEL10i and RANTIME10i.

		ROTPANEL10i			RANTIME10i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	5503	5237	5.08	4222	5707	-26.02
2	11.4	4809	4587	4.84	3664	5006	-26.81
3	18.3	6200	6720	-7.74	4795	7004	-31.54
4	48.4	10445	10136	3.05	8625	11141	-22.58
5	8.1	4003	3786	5.73	3034	4023	-24.58
6	25.4	7360	7048	4.43	5783	7425	-22.11
7	48.9	10510	10399	1.07	8686	11374	-23.63
8	13.0	5170	5210	-0.77	3949	5521	-28.47
9	26.4	7515	7799	-3.64	5924	8438	-29.79
10	43.8	9875	9387	5.20	8080	10304	-21.58
11	26.7	7571	7710	-1.80	5971	8294	-28.01
12	17.1	5954	5946	0.13	4589	6387	-28.15
13	51.4	10795	10521	2.60	8959	11747	-23.73
14	7.3	3788	3955	-4.22	2864	4131	-30.67
15	83.0	14251	14178	0.51	12512	15769	-20.65
16	10.4	4590	4454	3.05	3491	4844	-27.93
17	70.1	12878	12907	-0.22	11065	14050	-21.25
18	26.7	7547	7775	-2.93	5946	8511	-30.14
19	9.2	4306	4227	1.87	3270	4545	-28.05
20	17.1	5986	6362	-5.91	4616	6668	-30.77
21	32.1	8337	8120	2.67	6658	8866	-24.90
22	14.3	5418	5548	-2.34	4154	5914	-29.76
23	4.9	3129	3281	-4.63	2380	3464	-31.29
24	54.1	11097	11201	-0.93	9271	12081	-23.26
25	32.0	8354	8181	2.11	6664	8865	-24.83
26	14.9	5569	6125	-9.08	4270	6039	-29.29
27	53.1	10990	10897	0.85	9157	11450	-20.03
28	67.7	12645	12425	1.77	10828	13513	-19.87
29	17.6	6000	5813	3.22	4628	6265	-26.13
30	131.6	18745	18042	3.90	17545	20863	-15.90
average	33.3	7978	7933	0.26	6520	8607	-25.73

Table B.6. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP2.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL10i	bias ROTPANEL10i	bias RANTIME10i
1	14.9	-0.10	0.01	-0.05	-0.14
2	11.4	0.04	0.32	0.14	0.08
3	18.3	-2.43	-1.64	-0.55	-0.45
4	48.4	-0.31	-0.12	0.00	-0.06
5	8.1	1.54	1.04	0.18	-0.03
6	25.4	1.57	1.02	0.37	0.19
7	48.9	0.61	0.27	0.11	-0.01
8	13.0	1.25	0.91	0.37	0.21
9	26.4	-1.05	-0.59	-0.09	-0.10
10	43.8	0.56	0.34	0.05	-0.06
11	26.7	-0.60	-0.34	-0.04	-0.09
12	17.1	-1.06	-0.63	-0.23	-0.23
13	51.4	0.54	0.35	0.08	-0.01
14	7.3	-4.19	-2.90	-0.86	-0.57
15	83.0	0.39	0.25	0.04	-0.04
16	10.4	0.86	0.69	0.19	0.09
17	70.1	0.11	0.05	-0.05	-0.12
18	26.7	-0.25	-0.23	-0.10	-0.14
19	9.2	-1.95	-1.46	-0.55	-0.48
20	17.1	1.95	1.40	0.46	0.26
21	32.1	0.40	0.16	0.01	-0.10
22	14.3	-0.89	-0.66	-0.21	-0.19
23	4.9	-4.80	-3.50	-1.17	-0.88
24	54.1	0.27	0.10	0.00	-0.07
25	32.0	-1.08	-0.56	-0.15	-0.15
26	14.9	-3.48	-2.24	-0.63	-0.44
27	53.1	1.42	0.84	0.24	0.07
28	67.7	0.72	0.42	0.13	0.04
29	17.6	-0.94	-0.64	-0.22	-0.19
30	131.6	-0.13	-0.07	-0.04	-0.09
average	33.3	-0.37	-0.25	-0.09	-0.12

Table B.7. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP2.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL10i	MARE ROTPANEL10i	MARE RANTIME10i
1	14.9	3.05	2.60	1.45	1.51
2	11.4	3.33	3.01	1.65	1.74
3	18.3	3.30	2.59	1.25	1.26
4	48.4	1.89	1.53	0.86	0.90
5	8.1	4.12	3.58	2.07	2.14
6	25.4	3.00	2.34	1.28	1.31
7	48.9	2.22	1.79	0.98	1.03
8	13.0	3.62	3.05	1.71	1.75
9	26.4	2.48	2.01	1.08	1.14
10	43.8	2.21	1.75	0.96	1.01
11	26.7	2.41	1.95	1.14	1.19
12	17.1	3.03	2.51	1.37	1.42
13	51.4	2.19	1.77	0.96	1.01
14	7.3	5.00	3.94	1.92	1.95
15	83.0	1.59	1.27	0.77	0.80
16	10.4	3.46	3.11	1.76	1.84
17	70.1	1.74	1.37	0.82	0.84
18	26.7	2.58	2.16	1.19	1.26
19	9.2	3.70	3.18	1.79	1.87
20	17.1	3.61	2.94	1.63	1.65
21	32.1	2.46	2.05	1.12	1.17
22	14.3	3.08	2.61	1.52	1.57
23	4.9	5.60	4.53	2.31	2.39
24	54.1	1.93	1.53	0.89	0.92
25	32.0	2.35	1.78	0.94	0.99
26	14.9	4.02	2.95	1.33	1.30
27	53.1	2.61	2.00	1.10	1.11
28	67.7	2.04	1.60	0.88	0.91
29	17.6	2.79	2.36	1.30	1.37
30	131.6	1.14	0.91	0.58	0.62
average	33.3	2.88	2.36	1.29	1.33

Table B.8. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP2.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL10i	coverage ROTPANEL10i	coverage RANTIME10i
1	14.9	0.973	0.967	0.954	0.886
2	11.4	0.981	0.960	0.959	0.888
3	18.3	0.892	0.909	0.935	0.865
4	48.4	0.959	0.963	0.952	0.898
5	8.1	0.979	0.962	0.963	0.895
6	25.4	0.958	0.960	0.962	0.903
7	48.9	0.952	0.952	0.952	0.903
8	13.0	0.964	0.953	0.952	0.874
9	26.4	0.945	0.955	0.935	0.870
10	43.8	0.953	0.955	0.958	0.911
11	26.7	0.968	0.956	0.944	0.883
12	17.1	0.957	0.952	0.947	0.868
13	51.4	0.949	0.947	0.963	0.898
14	7.3	0.908	0.911	0.937	0.849
15	83.0	0.960	0.961	0.958	0.910
16	10.4	0.971	0.955	0.966	0.892
17	70.1	0.957	0.962	0.947	0.906
18	26.7	0.948	0.941	0.952	0.865
19	9.2	0.963	0.958	0.947	0.869
20	17.1	0.954	0.953	0.932	0.876
21	32.1	0.959	0.948	0.953	0.886
22	14.3	0.958	0.955	0.940	0.868
23	4.9	0.917	0.925	0.942	0.861
24	54.1	0.947	0.955	0.944	0.896
25	32.0	0.930	0.953	0.962	0.906
26	14.9	0.832	0.892	0.936	0.862
27	53.1	0.927	0.956	0.951	0.913
28	67.7	0.944	0.943	0.948	0.920
29	17.6	0.969	0.963	0.953	0.889
30	131.6	0.958	0.952	0.961	0.930
average	33.3	0.948	0.949	0.950	0.888

Table B.9. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP2. Results for models NESTEDi and PANEL10i.

		NESTEDi			PANEL10i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	11752	10360	13.44	9512	8865	7.30
2	11.4	10180	8729	16.62	8315	7919	5.00
3	18.3	13361	16215	-17.60	10755	12802	-15.99
4	48.4	22298	21636	3.06	17803	17496	1.75
5	8.1	8176	7097	15.20	6787	6256	8.49
6	25.4	15884	15678	1.31	12713	12221	4.03
7	48.9	22427	22467	-0.18	17906	18108	-1.12
8	13.0	10973	10366	5.86	8919	8706	2.45
9	26.4	16227	16595	-2.22	12981	13194	-1.61
10	43.8	21152	20740	1.99	16884	16270	3.77
11	26.7	16346	15538	5.20	13075	12617	3.63
12	17.1	12782	12460	2.58	10304	10512	-1.98
13	51.4	22962	23574	-2.60	18338	18853	-2.73
14	7.3	7641	9313	-17.95	6382	7562	-15.60
15	83.0	29596	28700	3.12	23782	22910	3.81
16	10.4	9644	8223	17.28	7914	7316	8.17
17	70.1	26994	26389	2.29	21630	20852	3.73
18	26.7	16238	16413	-1.07	12989	13611	-4.57
19	9.2	8953	8461	5.81	7391	7291	1.37
20	17.1	12798	12763	0.27	10314	10306	0.08
21	32.1	18000	17468	3.05	14375	14423	-0.33
22	14.3	11533	10712	7.66	9342	9099	2.67
23	4.9	6087	7037	-13.50	5218	5870	-11.11
24	54.1	23595	23627	-0.14	18854	18670	0.99
25	32.0	18002	19508	-7.72	14378	14602	-1.53
26	14.9	11884	16396	-27.52	9616	12217	-21.29
27	53.1	23365	25207	-7.31	18666	19303	-3.30
28	67.7	26604	27805	-4.32	21307	21912	-2.76
29	17.6	12832	11826	8.51	10341	9920	4.24
30	131.6	37251	35427	5.15	30314	28885	4.95
average	33.3	16851	16891	0.54	13570	13619	-0.58

Table B.10. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP2. Results for models ROTPANEL10i and RANTIME10i.

		ROTPANEL10i			RANTIME10i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	5137	4972	3.32	4223	5204	-18.85
2	11.4	4480	4324	3.61	3665	4552	-19.49
3	18.3	5802	6263	-7.36	4797	6351	-24.47
4	48.4	9987	9918	0.70	8627	10400	-17.05
5	8.1	3724	3565	4.46	3034	3687	-17.71
6	25.4	6923	6691	3.47	5785	6806	-15.00
7	48.9	10052	9890	1.64	8689	10393	-16.40
8	13.0	4821	4839	-0.37	3950	4998	-20.97
9	26.4	7076	7287	-2.90	5926	7598	-22.01
10	43.8	9413	8971	4.93	8082	9382	-13.86
11	26.7	7129	7253	-1.71	5972	7539	-20.79
12	17.1	5566	5691	-2.20	4591	5902	-22.21
13	51.4	10340	10199	1.38	8962	10758	-16.69
14	7.3	3522	3662	-3.82	2865	3761	-23.82
15	83.0	13928	13606	2.37	12516	14289	-12.41
16	10.4	4274	4083	4.68	3492	4294	-18.68
17	70.1	12487	12509	-0.18	11068	12938	-14.45
18	26.7	7105	7398	-3.96	5948	7782	-23.57
19	9.2	4008	4092	-2.05	3271	4264	-23.29
20	17.1	5598	5840	-4.14	4618	5908	-21.83
21	32.1	7883	7805	1.00	6660	8244	-19.21
22	14.3	5057	5234	-3.38	4155	5411	-23.21
23	4.9	2914	3038	-4.08	2380	3141	-24.23
24	54.1	10653	10826	-1.60	9274	11214	-17.30
25	32.0	7896	7707	2.45	6666	8064	-17.34
26	14.9	5198	5571	-6.70	4271	5478	-22.03
27	53.1	10541	10475	0.63	9159	10673	-14.19
28	67.7	12245	12080	1.37	10831	12363	-12.39
29	17.6	5611	5483	2.33	4630	5758	-19.59
30	131.6	18778	17937	4.69	17550	19154	-8.37
average	33.3	7605	7574	-0.05	6522	7877	-18.71

Table B.11. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP3.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL10i	bias ROTPANEL10i	bias RANTIME10i
1	14.9	-0.14	-0.02	0.01	-0.11
2	11.4	-0.28	-0.08	0.08	0.02
3	18.3	-4.15	-3.71	-1.07	-0.69
4	48.4	-0.36	-0.23	-0.01	-0.08
5	8.1	1.29	1.23	0.45	0.08
6	25.4	2.17	1.94	0.62	0.28
7	48.9	1.10	0.89	0.24	0.03
8	13.0	1.53	1.45	0.63	0.33
9	26.4	-1.80	-1.51	-0.30	-0.21
10	43.8	0.84	0.72	0.16	-0.01
11	26.7	-1.15	-0.96	-0.16	-0.15
12	17.1	-1.92	-1.67	-0.52	-0.40
13	51.4	0.93	0.83	0.20	0.03
14	7.3	-5.44	-5.03	-1.86	-1.11
15	83.0	0.64	0.54	0.10	-0.02
16	10.4	0.65	0.67	0.28	0.13
17	70.1	0.33	0.28	0.05	-0.08
18	26.7	-0.61	-0.56	-0.14	-0.15
19	9.2	-2.78	-2.59	-1.00	-0.73
20	17.1	2.27	2.10	0.72	0.33
21	32.1	0.55	0.45	0.13	-0.05
22	14.3	-1.78	-1.64	-0.51	-0.34
23	4.9	-5.46	-5.12	-2.24	-1.52
24	54.1	0.31	0.22	0.05	-0.04
25	32.0	-1.96	-1.62	-0.41	-0.28
26	14.9	-5.54	-4.96	-1.50	-0.89
27	53.1	2.40	2.05	0.50	0.17
28	67.7	1.12	0.93	0.23	0.07
29	17.6	-1.40	-1.26	-0.37	-0.27
30	131.6	-0.15	-0.11	-0.03	-0.09
average	33.3	-0.63	-0.56	-0.19	-0.19

Table B.12. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP3.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL10i	MARE ROTPANEL10i	MARE RANTIME10i
1	14.9	2.27	2.24	1.52	1.69
2	11.4	2.31	2.34	1.69	1.94
3	18.3	4.44	4.07	1.67	1.62
4	48.4	1.83	1.75	0.95	1.04
5	8.1	2.53	2.56	1.98	2.22
6	25.4	2.85	2.65	1.40	1.45
7	48.9	2.16	2.02	1.05	1.14
8	13.0	2.79	2.75	1.81	1.95
9	26.4	2.60	2.40	1.26	1.38
10	43.8	2.08	1.95	1.03	1.13
11	26.7	2.31	2.18	1.27	1.39
12	17.1	2.93	2.72	1.50	1.63
13	51.4	2.14	2.00	1.02	1.13
14	7.3	5.59	5.23	2.45	2.37
15	83.0	1.61	1.51	0.85	0.93
16	10.4	2.31	2.40	1.81	2.04
17	70.1	1.69	1.57	0.89	0.96
18	26.7	2.31	2.25	1.30	1.45
19	9.2	3.32	3.21	1.91	2.08
20	17.1	3.07	2.97	1.74	1.82
21	32.1	2.20	2.12	1.17	1.28
22	14.3	2.75	2.64	1.65	1.78
23	4.9	5.58	5.27	2.79	2.80
24	54.1	1.79	1.66	0.97	1.05
25	32.0	2.71	2.40	1.11	1.20
26	14.9	5.66	5.13	1.91	1.70
27	53.1	3.00	2.69	1.14	1.15
28	67.7	2.06	1.89	0.92	1.00
29	17.6	2.49	2.40	1.44	1.58
30	131.6	1.19	1.13	0.65	0.74
average	33.3	2.75	2.60	1.43	1.52

Table B.13. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP3.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL10i	coverage ROTPANEL10i	coverage RANTIME10i
1	14.9	0.993	0.992	0.973	0.866
2	11.4	0.996	0.988	0.970	0.868
3	18.3	0.762	0.784	0.910	0.826
4	48.4	0.970	0.972	0.953	0.878
5	8.1	0.996	0.989	0.978	0.890
6	25.4	0.962	0.951	0.961	0.889
7	48.9	0.960	0.957	0.952	0.881
8	13.0	0.988	0.985	0.960	0.863
9	26.4	0.946	0.946	0.941	0.841
10	43.8	0.971	0.971	0.962	0.888
11	26.7	0.975	0.970	0.944	0.872
12	17.1	0.968	0.965	0.948	0.842
13	51.4	0.958	0.955	0.950	0.873
14	7.3	0.799	0.823	0.917	0.824
15	83.0	0.962	0.960	0.959	0.889
16	10.4	0.995	0.991	0.971	0.873
17	70.1	0.963	0.961	0.956	0.880
18	26.7	0.974	0.968	0.953	0.852
19	9.2	0.977	0.972	0.952	0.853
20	17.1	0.973	0.966	0.934	0.841
21	32.1	0.969	0.965	0.961	0.875
22	14.3	0.979	0.973	0.949	0.847
23	4.9	0.881	0.887	0.931	0.831
24	54.1	0.962	0.971	0.947	0.886
25	32.0	0.914	0.928	0.951	0.870
26	14.9	0.633	0.681	0.870	0.806
27	53.1	0.868	0.892	0.945	0.879
28	67.7	0.940	0.943	0.946	0.892
29	17.6	0.986	0.984	0.960	0.870
30	131.6	0.968	0.956	0.957	0.902
average	33.3	0.940	0.942	0.949	0.865

Table B.14. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP3. Results for models NESTEDi and PANEL10i.

		NESTEDi			PANEL10i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	10837	7878	37.56	10319	7795	32.38
2	11.4	9010	6273	43.63	8635	6431	34.27
3	18.3	12681	19417	-34.69	12010	17954	-33.11
4	48.4	23240	20834	11.55	21651	19741	9.68
5	8.1	6829	4689	45.64	6602	4740	39.28
6	25.4	15693	15496	1.27	14757	14502	1.76
7	48.9	23390	23015	1.63	21788	21608	0.83
8	13.0	9921	8163	21.54	9473	8059	17.55
9	26.4	16105	16669	-3.38	15133	15420	-1.86
10	43.8	21902	20092	9.01	20426	18666	9.43
11	26.7	16243	14656	10.83	15261	13861	10.10
12	17.1	12028	11693	2.86	11409	11078	2.99
13	51.4	24018	24061	-0.18	22365	22639	-1.21
14	7.3	6264	9128	-31.38	6072	8660	-29.88
15	83.0	31649	30166	4.92	29411	28228	4.19
16	10.4	8401	5653	48.61	8072	5849	38.01
17	70.1	28675	26590	7.84	26657	24813	7.43
18	26.7	16126	14428	11.77	15153	14046	7.88
19	9.2	7625	7118	7.12	7351	6916	6.29
20	17.1	12056	11233	7.33	11430	10866	5.19
21	32.1	18191	16132	12.76	17038	15423	10.47
22	14.3	10584	9245	14.48	10080	8965	12.44
23	4.9	4648	6193	-24.95	4546	5936	-23.42
24	54.1	24754	22454	10.24	23042	20843	10.55
25	32.0	18188	20911	-13.02	17035	18661	-8.71
26	14.9	10963	19971	-45.11	10431	18326	-43.08
27	53.1	24479	31471	-22.22	22792	28315	-19.51
28	67.7	28220	29890	-5.59	26238	27565	-4.81
29	17.6	12116	10078	20.22	11488	9768	17.61
30	131.6	40347	36862	9.45	37561	35153	6.85
average	33.3	16839	16682	5.33	15808	15694	3.99

Table B.15. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP3. Results for models ROTPANEL10i and RANTIME10i.

		ROTPANEL10i			RANTIME10i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	5778	5243	10.20	4580	5901	-22.39
2	11.4	4992	4473	11.60	3951	5115	-22.76
3	18.3	6561	7739	-15.22	5222	7663	-31.85
4	48.4	11322	10820	4.64	9483	11942	-20.59
5	8.1	4069	3566	14.11	3235	4006	-19.25
6	25.4	7865	7648	2.84	6325	7919	-20.13
7	48.9	11395	11238	1.40	9551	12156	-21.43
8	13.0	5399	5292	2.02	4272	5715	-25.25
9	26.4	8041	8208	-2.03	6483	8944	-27.52
10	43.8	10684	10028	6.54	8879	11029	-19.49
11	26.7	8102	8041	0.76	6534	8775	-25.54
12	17.1	6285	6190	1.53	4992	6716	-25.67
13	51.4	11714	11435	2.44	9854	12645	-22.07
14	7.3	3822	4373	-12.60	3042	4335	-29.83
15	83.0	15600	15395	1.33	13789	16969	-18.74
16	10.4	4742	4308	10.07	3756	4918	-23.63
17	70.1	14054	13916	0.99	12186	15102	-19.31
18	26.7	8075	8086	-0.14	6507	8982	-27.56
19	9.2	4415	4303	2.60	3503	4699	-25.45
20	17.1	6319	6624	-4.60	5020	6968	-27.96
21	32.1	8962	8527	5.10	7299	9382	-22.20
22	14.3	5680	5667	0.23	4502	6156	-26.87
23	4.9	3059	3456	-11.49	2480	3543	-30.00
24	54.1	12055	12039	0.13	10200	13011	-21.60
25	32.0	8978	8868	1.24	7306	9500	-23.09
26	14.9	5848	7414	-21.12	4633	6738	-31.24
27	53.1	11934	12353	-3.39	10073	12495	-19.38
28	67.7	13792	13676	0.85	11924	14673	-18.74
29	17.6	6338	5951	6.50	5035	6556	-23.20
30	131.6	20685	19852	4.20	19354	22717	-14.80
average	33.3	8552	8491	0.69	7132	9176	-23.58

Table B.16. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP4.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL10i	bias ROTPANEL10i	bias RANTIME10i
1	14.9	-0.28	-0.05	-0.05	-0.15
2	11.4	-0.26	0.17	0.15	0.09
3	18.3	-4.07	-2.57	-0.98	-0.77
4	48.4	-0.43	-0.14	0.00	-0.07
5	8.1	1.21	0.92	0.31	0.07
6	25.4	2.24	1.40	0.57	0.33
7	48.9	1.14	0.54	0.22	0.07
8	13.0	1.58	1.19	0.57	0.36
9	26.4	-1.82	-0.98	-0.24	-0.21
10	43.8	0.87	0.50	0.11	-0.02
11	26.7	-1.13	-0.59	-0.13	-0.16
12	17.1	-1.87	-1.07	-0.40	-0.36
13	51.4	0.86	0.51	0.15	0.02
14	7.3	-5.41	-3.83	-1.57	-1.11
15	83.0	0.63	0.36	0.08	-0.02
16	10.4	0.61	0.60	0.24	0.12
17	70.1	0.31	0.15	-0.03	-0.12
18	26.7	-0.62	-0.40	-0.17	-0.20
19	9.2	-2.75	-1.98	-0.87	-0.73
20	17.1	2.30	1.63	0.66	0.41
21	32.1	0.56	0.26	0.05	-0.08
22	14.3	-1.70	-1.13	-0.40	-0.33
23	4.9	-5.53	-4.21	-1.96	-1.52
24	54.1	0.29	0.12	0.01	-0.08
25	32.0	-2.00	-1.01	-0.31	-0.26
26	14.9	-5.47	-3.45	-1.21	-0.86
27	53.1	2.41	1.29	0.43	0.19
28	67.7	1.13	0.60	0.21	0.09
29	17.6	-1.38	-0.86	-0.32	-0.29
30	131.6	-0.17	-0.08	-0.04	-0.11
average	33.3	-0.62	-0.40	-0.16	-0.19

Table B.17. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP4.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL10i	MARE ROTPANEL10i	MARE RANTIME10i
1	14.9	2.28	2.12	1.44	1.52
2	11.4	2.28	2.33	1.61	1.73
3	18.3	4.38	3.07	1.51	1.46
4	48.4	1.83	1.47	0.92	0.96
5	8.1	2.59	2.54	1.91	2.02
6	25.4	2.94	2.21	1.30	1.31
7	48.9	2.19	1.67	1.00	1.03
8	13.0	2.79	2.49	1.70	1.74
9	26.4	2.62	1.97	1.15	1.21
10	43.8	2.09	1.63	0.98	1.02
11	26.7	2.28	1.83	1.18	1.23
12	17.1	2.84	2.31	1.42	1.48
13	51.4	2.09	1.64	0.98	1.02
14	7.3	5.56	4.19	2.22	2.16
15	83.0	1.61	1.24	0.80	0.82
16	10.4	2.32	2.36	1.67	1.78
17	70.1	1.69	1.30	0.84	0.86
18	26.7	2.30	1.97	1.24	1.31
19	9.2	3.31	2.83	1.85	1.94
20	17.1	3.13	2.57	1.60	1.61
21	32.1	2.14	1.83	1.13	1.19
22	14.3	2.77	2.33	1.56	1.62
23	4.9	5.67	4.51	2.56	2.54
24	54.1	1.81	1.44	0.93	0.96
25	32.0	2.72	1.87	1.05	1.09
26	14.9	5.62	3.74	1.68	1.56
27	53.1	3.00	2.00	1.09	1.07
28	67.7	2.04	1.52	0.89	0.90
29	17.6	2.50	2.14	1.35	1.44
30	131.6	1.21	0.94	0.64	0.68
average	33.3	2.75	2.20	1.34	1.37

Table B.18. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP4.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL10i	coverage ROTPANEL10i	coverage RANTIME10i
1	14.9	0.993	0.982	0.963	0.911
2	11.4	0.996	0.983	0.990	0.905
3	18.3	0.755	0.849	0.909	0.860
4	48.4	0.970	0.970	0.955	0.908
5	8.1	0.998	0.986	0.973	0.928
6	25.4	0.960	0.953	0.956	0.914
7	48.9	0.953	0.956	0.953	0.913
8	13.0	0.984	0.969	0.956	0.893
9	26.4	0.949	0.957	0.941	0.888
10	43.8	0.960	0.964	0.963	0.922
11	26.7	0.978	0.964	0.952	0.896
12	17.1	0.971	0.959	0.945	0.891
13	51.4	0.961	0.947	0.957	0.918
14	7.3	0.820	0.870	0.912	0.854
15	83.0	0.962	0.962	0.962	0.929
16	10.4	0.993	0.981	0.970	0.917
17	70.1	0.962	0.969	0.948	0.908
18	26.7	0.975	0.957	0.954	0.884
19	9.2	0.973	0.965	0.946	0.887
20	17.1	0.970	0.958	0.938	0.892
21	32.1	0.975	0.964	0.957	0.906
22	14.3	0.977	0.965	0.953	0.886
23	4.9	0.875	0.905	0.926	0.861
24	54.1	0.965	0.962	0.942	0.907
25	32.0	0.894	0.943	0.962	0.919
26	14.9	0.632	0.790	0.897	0.856
27	53.1	0.859	0.931	0.944	0.914
28	67.7	0.945	0.941	0.951	0.929
29	17.6	0.986	0.976	0.955	0.909
30	131.6	0.963	0.956	0.965	0.938
average	33.3	0.938	0.948	0.950	0.901

Table B.19. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP4. Results for models NESTEDi and PANEL10i.

		NESTEDi			PANEL10i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	10846	7909	37.13	8835	7320	20.70
2	11.4	9020	6110	47.63	7519	6238	20.54
3	18.3	12691	19167	-33.79	10163	13932	-27.05
4	48.4	23246	20791	11.81	17747	16741	6.01
5	8.1	6836	4663	46.60	5883	4644	26.68
6	25.4	15699	15848	-0.94	12294	12094	1.65
7	48.9	23394	23206	0.81	17858	17790	0.38
8	13.0	9936	8226	20.79	8178	7379	10.83
9	26.4	16115	16689	-3.44	12587	12735	-1.16
10	43.8	21910	20451	7.13	16768	15761	6.39
11	26.7	16246	14453	12.41	12685	11737	8.08
12	17.1	12039	11377	5.82	9686	9549	1.43
13	51.4	24022	23585	1.85	18322	18451	-0.70
14	7.3	6273	9081	-30.92	5456	7235	-24.59
15	83.0	31649	29899	5.85	24115	23054	4.60
16	10.4	8412	5726	46.91	7076	5723	23.64
17	70.1	28675	26459	8.38	21825	20492	6.50
18	26.7	16133	14480	11.42	12601	12426	1.41
19	9.2	7640	7148	6.88	6504	6259	3.91
20	17.1	12069	11459	5.32	9703	9509	2.04
21	32.1	18198	15682	16.04	14083	13333	5.63
22	14.3	10601	9325	13.68	8650	8015	7.92
23	4.9	4657	6299	-26.07	4199	5264	-20.23
24	54.1	24757	22767	8.74	18869	18011	4.76
25	32.0	18194	21313	-14.63	14082	14782	-4.74
26	14.9	10981	19873	-44.74	8932	13887	-35.68
27	53.1	24481	31522	-22.34	18668	21290	-12.32
28	67.7	28220	29842	-5.44	21478	22299	-3.68
29	17.6	12128	10208	18.81	9743	8791	10.83
30	131.6	40343	37250	8.30	31117	29328	6.10
average	33.3	16847	16694	5.33	13188	13136	1.66

Table B.20. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP4. Results for models ROTPANEL10i and RANTIME10i.

		ROTPANEL10i			RANTIME10i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	5366	4995	7.43	4579	5297	-13.55
2	11.4	4634	4259	8.80	3950	4572	-13.60
3	18.3	6102	7099	-14.04	5222	6952	-24.88
4	48.4	10744	10547	1.87	9486	11015	-13.88
5	8.1	3784	3414	10.84	3234	3629	-10.88
6	25.4	7344	7227	1.62	6326	7203	-12.18
7	48.9	10817	10658	1.49	9554	11014	-13.26
8	13.0	5011	4933	1.58	4272	5111	-16.42
9	26.4	7516	7612	-1.26	6484	7935	-18.29
10	43.8	10107	9543	5.91	8881	9907	-10.36
11	26.7	7574	7551	0.30	6535	7871	-16.97
12	17.1	5842	5879	-0.63	4992	6119	-18.42
13	51.4	11136	10953	1.67	9857	11405	-13.57
14	7.3	3556	3989	-10.85	3041	3937	-22.76
15	83.0	15143	14660	3.29	13793	15191	-9.20
16	10.4	4403	3999	10.10	3755	4295	-12.57
17	70.1	13530	13349	1.36	12190	13729	-11.21
18	26.7	7546	7705	-2.06	6508	8121	-19.86
19	9.2	4103	4163	-1.44	3502	4363	-19.73
20	17.1	5874	6053	-2.96	5021	6091	-17.57
21	32.1	8410	8170	2.94	7301	8608	-15.18
22	14.3	5275	5328	-0.99	4501	5553	-18.94
23	4.9	2865	3210	-10.75	2478	3224	-23.14
24	54.1	11487	11551	-0.55	10203	11928	-14.46
25	32.0	8422	8273	1.80	7307	8571	-14.75
26	14.9	5430	6510	-16.59	4632	6086	-23.89
27	53.1	11360	11724	-3.10	10076	11577	-12.97
28	67.7	13261	13142	0.91	11928	13249	-9.97
29	17.6	5892	5628	4.69	5036	5950	-15.36
30	131.6	20598	19631	4.93	19361	20667	-6.32
average	33.3	8104	8059	0.21	7134	8306	-15.47

C Appendix: Simulation study with correct model. Result tables for longitudinal data of 5 occasions

Table C.1. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP1.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL5c	bias ROTPANEL5c	bias RANTIME5c
1	14.9	0.06	0.11	0.08	0.03
2	11.4	0.02	0.13	0.13	0.07
3	18.3	-1.56	-1.38	-0.41	-0.36
4	48.4	-0.18	-0.13	-0.02	-0.06
5	8.1	1.29	1.10	0.34	0.19
6	25.4	0.91	0.79	0.29	0.20
7	48.9	0.42	0.33	0.14	0.05
8	13.0	0.71	0.66	0.34	0.26
9	26.4	-0.62	-0.48	-0.09	-0.11
10	43.8	0.29	0.24	0.07	0.00
11	26.7	-0.29	-0.21	-0.02	-0.05
12	17.1	-0.63	-0.50	-0.17	-0.20
13	51.4	0.38	0.34	0.05	-0.01
14	7.3	-2.79	-2.47	-0.79	-0.69
15	83.0	0.22	0.17	0.02	-0.03
16	10.4	0.67	0.62	0.13	0.04
17	70.1	0.00	-0.01	-0.06	-0.11
18	26.7	-0.17	-0.20	-0.12	-0.16
19	9.2	-1.26	-1.15	-0.45	-0.49
20	17.1	1.26	1.12	0.35	0.23
21	32.1	0.28	0.20	0.00	-0.07
22	14.3	-0.37	-0.36	-0.14	-0.20
23	4.9	-3.20	-2.90	-1.07	-1.01
24	54.1	0.18	0.12	-0.01	-0.06
25	32.0	-0.62	-0.48	-0.12	-0.14
26	14.9	-2.15	-1.85	-0.64	-0.55
27	53.1	0.95	0.81	0.31	0.19
28	67.7	0.43	0.36	0.10	0.02
29	17.6	-0.70	-0.61	-0.24	-0.26
30	131.6	-0.12	-0.11	-0.03	-0.08
average	33.3	-0.22	-0.19	-0.07	-0.11

Table C.2. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP1.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL5c	MARE ROTPANEL5c	MARE RANTIME5c
1	14.9	2.47	2.34	1.33	1.40
2	11.4	2.73	2.57	1.53	1.64
3	18.3	2.35	2.16	1.14	1.19
4	48.4	1.40	1.32	0.75	0.81
5	8.1	3.41	3.23	1.91	2.02
6	25.4	2.28	2.13	1.19	1.23
7	48.9	1.62	1.49	0.86	0.92
8	13.0	2.85	2.71	1.61	1.66
9	26.4	1.86	1.70	0.98	1.06
10	43.8	1.72	1.56	0.90	0.96
11	26.7	1.85	1.73	1.01	1.03
12	17.1	2.37	2.15	1.18	1.25
13	51.4	1.67	1.54	0.88	0.93
14	7.3	3.70	3.39	1.77	1.85
15	83.0	1.20	1.12	0.69	0.72
16	10.4	2.87	2.81	1.73	1.81
17	70.1	1.28	1.18	0.69	0.75
18	26.7	1.87	1.76	1.02	1.08
19	9.2	2.92	2.74	1.59	1.68
20	17.1	2.74	2.59	1.47	1.50
21	32.1	1.93	1.78	0.99	1.04
22	14.3	2.34	2.24	1.33	1.41
23	4.9	4.17	3.91	2.13	2.22
24	54.1	1.44	1.32	0.77	0.81
25	32.0	1.73	1.55	0.81	0.87
26	14.9	2.77	2.52	1.25	1.27
27	53.1	1.92	1.77	1.01	1.03
28	67.7	1.53	1.40	0.82	0.87
29	17.6	2.11	1.96	1.15	1.21
30	131.6	0.86	0.81	0.52	0.56
average	33.3	2.20	2.05	1.17	1.23

Table C.3. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP1.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL5c	coverage ROTPANEL5c	coverage RANTIME5c
1	14.9	0.966	0.968	0.963	0.922
2	11.4	0.972	0.958	0.961	0.911
3	18.3	0.921	0.912	0.946	0.895
4	48.4	0.953	0.957	0.954	0.913
5	8.1	0.974	0.962	0.967	0.921
6	25.4	0.949	0.949	0.954	0.923
7	48.9	0.963	0.956	0.960	0.928
8	13.0	0.963	0.955	0.945	0.910
9	26.4	0.960	0.955	0.944	0.905
10	43.8	0.939	0.941	0.942	0.909
11	26.7	0.962	0.955	0.953	0.921
12	17.1	0.945	0.956	0.952	0.917
13	51.4	0.946	0.951	0.945	0.902
14	7.3	0.919	0.923	0.949	0.895
15	83.0	0.949	0.946	0.959	0.917
16	10.4	0.969	0.959	0.958	0.904
17	70.1	0.949	0.952	0.960	0.921
18	26.7	0.956	0.954	0.950	0.909
19	9.2	0.957	0.957	0.952	0.913
20	17.1	0.952	0.947	0.947	0.905
21	32.1	0.946	0.954	0.962	0.918
22	14.3	0.972	0.965	0.948	0.916
23	4.9	0.938	0.949	0.938	0.905
24	54.1	0.949	0.958	0.962	0.925
25	32.0	0.948	0.941	0.963	0.924
26	14.9	0.886	0.897	0.931	0.890
27	53.1	0.936	0.935	0.950	0.917
28	67.7	0.938	0.941	0.946	0.904
29	17.6	0.964	0.963	0.963	0.925
30	131.6	0.950	0.943	0.944	0.911
average	33.3	0.950	0.949	0.952	0.913

Table C.4. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP1. Results for models NESTEDc and PANEL5c.

		NESTEDc			PANEL5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9019	8325	8.34	8348	7900	5.67
2	11.4	7894	7220	9.34	7325	6859	6.79
3	18.3	10192	11605	-12.18	9420	10762	-12.47
4	48.4	16707	16364	2.10	15415	15237	1.17
5	8.1	6453	5925	8.91	6015	5658	6.31
6	25.4	12071	12056	-0.32	11093	11159	-0.59
7	48.9	16801	16289	3.14	15503	15252	1.65
8	13.0	8464	8212	3.07	7843	7746	1.25
9	26.4	12265	12294	-0.24	11320	11223	0.86
10	43.8	15864	16147	-1.75	14637	14832	-1.31
11	26.7	12355	11993	3.02	11402	11183	1.96
12	17.1	9767	10030	-2.62	9031	9093	-0.68
13	51.4	17192	17878	-3.84	15866	16461	-3.61
14	7.3	6065	7075	-14.28	5663	6511	-13.02
15	83.0	22076	21789	1.32	20414	20104	1.54
16	10.4	7513	6791	10.63	6980	6627	5.33
17	70.1	20160	19843	1.64	18626	18381	1.33
18	26.7	12274	11792	4.09	11328	11026	2.74
19	9.2	7020	6699	4.79	6534	6304	3.65
20	17.1	9775	9819	-0.45	9038	9207	-1.84
21	32.1	13559	13707	-1.08	12511	12464	0.38
22	14.3	8860	8029	10.35	8203	7738	6.01
23	4.9	4965	5316	-6.60	4675	5033	-7.11
24	54.1	17660	17395	1.52	16301	15887	2.61
25	32.0	13562	14163	-4.24	12513	12862	-2.71
26	14.9	9120	11501	-20.70	8441	10433	-19.09
27	53.1	17489	18476	-5.34	16141	16780	-3.81
28	67.7	19871	20808	-4.50	18357	19242	-4.60
29	17.6	9798	9063	8.11	9061	8428	7.51
30	131.6	27724	27140	2.15	25740	25871	-0.51
average	33.3	12749	12791	0.15	11791	11875	-0.49

Table C.5. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under correct model in simulations from POP1. Results for models ROTPANEL5c and RANTIME5c.

		ROTPANEL5c			RANTIME5c		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	4738	4550	4.13	4275	4763	-10.25
2	11.4	4139	3989	3.76	3731	4252	-12.25
3	18.3	5326	5642	-5.60	4818	5942	-18.92
4	48.4	8937	8656	3.25	8242	9318	-11.55
5	8.1	3456	3307	4.51	3113	3506	-11.21
6	25.4	6310	6234	1.22	5731	6441	-11.02
7	48.9	8994	8750	2.79	8292	9295	-10.79
8	13.0	4446	4526	-1.77	4010	4699	-14.66
9	26.4	6447	6505	-0.89	5863	6976	-15.95
10	43.8	8458	8448	0.12	7774	8992	-13.55
11	26.7	6489	6484	0.08	5902	6658	-11.35
12	17.1	5112	5004	2.16	4620	5279	-12.48
13	51.4	9229	9414	-1.97	8524	9944	-14.28
14	7.3	3265	3399	-3.94	2940	3548	-17.14
15	83.0	12167	12272	-0.86	11450	12787	-10.46
16	10.4	3948	3984	-0.90	3560	4189	-15.02
17	70.1	11006	10628	3.56	10281	11453	-10.23
18	26.7	6470	6372	1.54	5876	6737	-12.78
19	9.2	3720	3623	2.68	3349	3832	-12.60
20	17.1	5151	5182	-0.60	4652	5377	-13.48
21	32.1	7150	6953	2.83	6525	7357	-11.31
22	14.3	4666	4601	1.41	4216	4818	-12.49
23	4.9	2723	2792	-2.47	2463	2899	-15.04
24	54.1	9492	9352	1.50	8787	9785	-10.20
25	32.0	7157	6712	6.63	6532	7234	-9.70
26	14.9	4792	5238	-8.51	4329	5362	-19.27
27	53.1	9387	9575	-1.96	8681	9834	-11.72
28	67.7	10809	11043	-2.12	10086	11809	-14.59
29	17.6	5150	4923	4.61	4654	5171	-10.00
30	131.6	15975	16389	-2.53	15385	17695	-13.05
average	33.3	6837	6818	0.42	6289	7198	-12.91

Table C.6. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP2.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL5c	bias ROTPANEL5c	bias RANTIME5c
1	14.9	-0.08	0.03	0.05	0.00
2	11.4	-0.01	0.20	0.21	0.15
3	18.3	-1.50	-0.66	-0.42	-0.43
4	48.4	-0.25	-0.08	-0.02	-0.07
5	8.1	1.21	0.36	0.24	0.16
6	25.4	0.94	0.42	0.34	0.27
7	48.9	0.49	0.18	0.17	0.11
8	13.0	0.74	0.38	0.32	0.25
9	26.4	-0.66	-0.22	-0.06	-0.10
10	43.8	0.31	0.12	0.06	-0.01
11	26.7	-0.29	-0.05	-0.04	-0.09
12	17.1	-0.59	-0.16	-0.08	-0.12
13	51.4	0.32	0.11	0.02	-0.04
14	7.3	-2.69	-1.13	-0.61	-0.60
15	83.0	0.22	0.09	0.02	-0.04
16	10.4	0.60	0.32	0.09	0.03
17	70.1	-0.02	-0.08	-0.11	-0.16
18	26.7	-0.17	-0.16	-0.16	-0.21
19	9.2	-1.17	-0.60	-0.40	-0.44
20	17.1	1.27	0.59	0.35	0.28
21	32.1	0.30	0.03	-0.04	-0.10
22	14.3	-0.32	-0.18	-0.12	-0.17
23	4.9	-3.14	-1.65	-0.98	-0.98
24	54.1	0.18	0.03	-0.03	-0.08
25	32.0	-0.67	-0.20	-0.11	-0.15
26	14.9	-2.10	-0.79	-0.52	-0.52
27	53.1	0.96	0.38	0.31	0.22
28	67.7	0.44	0.17	0.11	0.05
29	17.6	-0.71	-0.31	-0.25	-0.29
30	131.6	-0.14	-0.09	-0.06	-0.11
average	33.3	-0.22	-0.10	-0.06	-0.11

Table C.7. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP2.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL5c	MARE ROTPANEL5c	MARE RANTIME5c
1	14.9	2.41	1.64	1.21	1.22
2	11.4	2.69	1.85	1.44	1.46
3	18.3	2.30	1.39	1.07	1.08
4	48.4	1.39	0.90	0.72	0.73
5	8.1	3.35	2.29	1.78	1.79
6	25.4	2.31	1.48	1.12	1.11
7	48.9	1.64	1.02	0.82	0.82
8	13.0	2.85	1.87	1.54	1.54
9	26.4	1.86	1.15	0.93	0.94
10	43.8	1.70	1.09	0.83	0.84
11	26.7	1.79	1.18	0.93	0.93
12	17.1	2.30	1.43	1.11	1.12
13	51.4	1.62	1.02	0.84	0.85
14	7.3	3.67	2.16	1.62	1.64
15	83.0	1.20	0.77	0.64	0.64
16	10.4	2.88	2.07	1.58	1.59
17	70.1	1.30	0.84	0.66	0.67
18	26.7	1.90	1.20	0.96	0.98
19	9.2	2.95	2.04	1.45	1.48
20	17.1	2.78	1.73	1.38	1.37
21	32.1	1.86	1.20	0.97	0.98
22	14.3	2.32	1.57	1.23	1.25
23	4.9	4.17	2.81	2.02	2.04
24	54.1	1.43	0.90	0.73	0.73
25	32.0	1.71	1.05	0.77	0.79
26	14.9	2.72	1.50	1.12	1.12
27	53.1	1.92	1.17	0.97	0.96
28	67.7	1.50	0.98	0.77	0.77
29	17.6	2.15	1.38	1.09	1.11
30	131.6	0.87	0.60	0.50	0.51
average	33.3	2.18	1.41	1.09	1.10

Table C.8. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP2.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL5c	coverage ROTPANEL5c	coverage RANTIME5c
1	14.9	0.974	0.954	0.962	0.954
2	11.4	0.975	0.952	0.952	0.940
3	18.3	0.926	0.936	0.941	0.924
4	48.4	0.961	0.941	0.957	0.947
5	8.1	0.969	0.960	0.965	0.958
6	25.4	0.949	0.949	0.948	0.942
7	48.9	0.950	0.949	0.953	0.950
8	13.0	0.961	0.954	0.957	0.945
9	26.4	0.960	0.955	0.956	0.946
10	43.8	0.942	0.934	0.955	0.945
11	26.7	0.969	0.949	0.960	0.949
12	17.1	0.962	0.954	0.966	0.958
13	51.4	0.942	0.942	0.953	0.943
14	7.3	0.922	0.936	0.949	0.938
15	83.0	0.956	0.953	0.953	0.947
16	10.4	0.968	0.952	0.961	0.944
17	70.1	0.952	0.957	0.973	0.958
18	26.7	0.960	0.957	0.948	0.942
19	9.2	0.960	0.941	0.956	0.944
20	17.1	0.955	0.958	0.946	0.941
21	32.1	0.950	0.952	0.951	0.943
22	14.3	0.970	0.949	0.962	0.954
23	4.9	0.944	0.942	0.941	0.935
24	54.1	0.950	0.959	0.961	0.953
25	32.0	0.942	0.948	0.967	0.949
26	14.9	0.891	0.931	0.941	0.928
27	53.1	0.931	0.943	0.934	0.930
28	67.7	0.941	0.939	0.950	0.949
29	17.6	0.966	0.961	0.959	0.951
30	131.6	0.952	0.941	0.949	0.941
average	33.3	0.952	0.948	0.954	0.945

Table C.9. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under correct model in simulations from POP2. Results for models NESTEDc and PANEL5c.

		NESTEDc			PANEL5c		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	9022	8175	10.36	5690	5610	1.43
2	11.4	7897	7041	12.16	5024	4862	3.33
3	18.3	10194	11447	-10.95	6401	6960	-8.03
4	48.4	16709	16076	3.94	10561	10451	1.05
5	8.1	6456	5888	9.65	4185	4076	2.67
6	25.4	12019	12118	-0.82	7529	7666	-1.79
7	48.9	16804	16535	1.63	10625	10453	1.65
8	13.0	8468	8182	3.50	5361	5357	0.07
9	26.4	12268	12258	0.08	7684	7590	1.24
10	43.8	15867	16142	-1.70	10003	10192	-1.85
11	26.7	12358	11563	6.88	7740	7616	1.63
12	17.1	9770	9613	1.63	6142	6017	2.08
13	51.4	17195	17577	-2.17	10888	10992	-0.95
14	7.3	6068	6994	-13.24	3966	4304	-7.85
15	83.0	22078	21826	1.15	14280	13912	2.65
16	10.4	7515	6823	10.14	4808	4880	-1.48
17	70.1	20162	19766	2.00	12925	12794	1.02
18	26.7	12276	11924	2.95	7691	7515	2.34
19	9.2	7024	6793	3.40	4525	4683	-3.37
20	17.1	9779	9927	-1.49	6145	6128	0.28
21	32.1	13562	13222	2.57	8505	8343	1.94
22	14.3	8863	8001	10.77	5594	5510	1.52
23	4.9	4970	5346	-7.03	3383	3685	-8.20
24	54.1	17661	17468	1.10	11204	10981	2.03
25	32.0	13563	14227	-4.67	8507	8625	-1.37
26	14.9	9123	11357	-19.67	5752	6277	-8.36
27	53.1	17490	18842	-7.18	11088	11209	-1.08
28	67.7	19874	20699	-3.99	12720	13190	-3.56
29	17.6	9800	9249	5.96	6159	5911	4.20
30	131.6	27727	27322	1.48	18554	18835	-1.49
average	33.3	12752	12747	0.62	8121	8154	-0.61

Table C.10. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP2. Results for models ROTPANEL5c and RANTIME5c.

		ROTPANEL5c			RANTIME5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	4438	4115	7.85	4277	4127	3.63
2	11.4	3873	3782	2.41	3733	3803	-1.84
3	18.3	4998	5304	-5.77	4820	5362	-10.11
4	48.4	8512	8332	2.16	8246	8424	-2.11
5	8.1	3231	3094	4.43	3114	3108	0.19
6	25.4	5940	5875	1.11	5734	5813	-1.36
7	48.9	8565	8368	2.35	8296	8355	-0.71
8	13.0	4163	4264	-2.37	4012	4261	-5.84
9	26.4	6075	6076	-0.02	5866	6122	-4.18
10	43.8	8036	7761	3.54	7778	7818	-0.51
11	26.7	6115	5924	3.22	5905	5939	-0.57
12	17.1	4794	4629	3.56	4622	4653	-0.67
13	51.4	8800	8890	-1.01	8528	8940	-4.61
14	7.3	3052	3110	-1.86	2941	3135	-6.19
15	83.0	11770	11479	2.54	11456	11462	-0.05
16	10.4	3695	3655	1.09	3561	3691	-3.52
17	70.1	10586	10015	5.70	10286	10222	0.63
18	26.7	6091	6004	1.45	5879	6086	-3.40
19	9.2	3477	3341	4.07	3350	3378	-0.83
20	17.1	4828	4884	-1.15	4654	4866	-4.36
21	32.1	6756	6728	0.42	6528	6795	-3.93
22	14.3	4374	4244	3.06	4218	4277	-1.38
23	4.9	2552	2649	-3.66	2464	2671	-7.75
24	54.1	9067	8887	2.03	8791	8912	-1.36
25	32.0	6763	6373	6.12	6535	6485	0.77
26	14.9	4492	4729	-5.01	4331	4758	-8.97
27	53.1	8960	9405	-4.73	8686	9283	-6.43
28	67.7	10388	10525	-1.30	10091	10503	-3.92
29	17.6	4829	4675	3.29	4656	4727	-1.50
30	131.6	15732	15731	0.01	15393	16001	-3.80
average	33.3	6498	6428	1.12	6292	6466	-2.82

Table C.11. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP3.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL5c	bias ROTPANEL5c	bias RANTIME5c
1	14.9	-0.04	0.05	0.09	0.02
2	11.4	-0.08	0.06	0.17	0.09
3	18.3	-3.07	-2.60	-0.97	-0.83
4	48.4	-0.47	-0.33	-0.08	-0.13
5	8.1	1.66	1.46	0.69	0.46
6	25.4	1.56	1.31	0.56	0.40
7	48.9	0.86	0.66	0.28	0.15
8	13.0	1.07	0.97	0.55	0.42
9	26.4	-1.23	-0.95	-0.26	-0.26
10	43.8	0.53	0.42	0.15	0.05
11	26.7	-0.59	-0.42	-0.10	-0.14
12	17.1	-1.16	-0.91	-0.35	-0.36
13	51.4	0.74	0.61	0.16	0.06
14	7.3	-4.07	-3.59	-1.62	-1.41
15	83.0	0.34	0.26	0.06	-0.02
16	10.4	0.77	0.74	0.28	0.14
17	70.1	0.09	0.05	-0.04	-0.13
18	26.7	-0.44	-0.40	-0.20	-0.26
19	9.2	-1.88	-1.68	-0.78	-0.79
20	17.1	2.06	1.78	0.72	0.52
21	32.1	0.53	0.39	0.08	-0.02
22	14.3	-0.82	-0.72	-0.30	-0.35
23	4.9	-3.80	-3.46	-1.82	-1.69
24	54.1	0.26	0.18	0.01	-0.06
25	32.0	-1.36	-1.04	-0.33	-0.32
26	14.9	-4.10	-3.44	-1.41	-1.20
27	53.1	1.85	1.48	0.59	0.39
28	67.7	0.88	0.70	0.23	0.11
29	17.6	-1.41	-1.17	-0.50	-0.49
30	131.6	-0.28	-0.22	-0.07	-0.13
average	33.3	-0.39	-0.33	-0.14	-0.19

Table C.12. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP3.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL5c	MARE ROTPANEL5c	MARE RANTIME5c
1	14.9	2.05	2.00	1.40	1.49
2	11.4	2.10	2.08	1.58	1.72
3	18.3	3.36	2.93	1.51	1.51
4	48.4	1.53	1.41	0.87	0.94
5	8.1	2.61	2.54	1.89	2.01
6	25.4	2.34	2.14	1.31	1.33
7	48.9	1.74	1.55	0.95	1.00
8	13.0	2.37	2.30	1.69	1.75
9	26.4	2.06	1.82	1.13	1.23
10	43.8	1.75	1.57	0.99	1.04
11	26.7	1.84	1.71	1.12	1.15
12	17.1	2.30	2.06	1.31	1.40
13	51.4	1.77	1.59	0.97	1.01
14	7.3	4.27	3.84	2.21	2.22
15	83.0	1.29	1.18	0.78	0.80
16	10.4	2.20	2.24	1.74	1.85
17	70.1	1.36	1.23	0.78	0.83
18	26.7	1.82	1.73	1.14	1.21
19	9.2	2.65	2.50	1.70	1.82
20	17.1	2.77	2.57	1.59	1.60
21	32.1	1.89	1.74	1.09	1.15
22	14.3	2.06	2.01	1.43	1.53
23	4.9	3.99	3.75	2.41	2.47
24	54.1	1.49	1.35	0.87	0.91
25	32.0	2.10	1.80	0.97	1.04
26	14.9	4.26	3.69	1.81	1.73
27	53.1	2.35	2.01	1.13	1.11
28	67.7	1.69	1.49	0.90	0.94
29	17.6	2.19	2.01	1.32	1.39
30	131.6	1.00	0.92	0.62	0.66
average	33.3	2.24	2.06	1.31	1.36

Table C.13. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP3.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL5c	coverage ROTPANEL5c	coverage RANTIME5c
1	14.9	0.989	0.983	0.969	0.935
2	11.4	0.993	0.986	0.976	0.929
3	18.3	0.815	0.833	0.918	0.882
4	48.4	0.960	0.967	0.954	0.920
5	8.1	0.992	0.983	0.976	0.948
6	25.4	0.954	0.949	0.955	0.929
7	48.9	0.965	0.957	0.957	0.931
8	13.0	0.978	0.972	0.959	0.928
9	26.4	0.954	0.958	0.946	0.909
10	43.8	0.958	0.950	0.949	0.916
11	26.7	0.974	0.971	0.957	0.931
12	17.1	0.966	0.972	0.964	0.931
13	51.4	0.951	0.953	0.951	0.919
14	7.3	0.879	0.884	0.921	0.883
15	83.0	0.955	0.951	0.956	0.923
16	10.4	0.986	0.987	0.972	0.935
17	70.1	0.962	0.962	0.965	0.932
18	26.7	0.976	0.968	0.958	0.923
19	9.2	0.981	0.974	0.959	0.928
20	17.1	0.958	0.947	0.939	0.915
21	32.1	0.965	0.964	0.962	0.920
22	14.3	0.984	0.979	0.956	0.930
23	4.9	0.953	0.953	0.940	0.907
24	54.1	0.962	0.966	0.962	0.934
25	32.0	0.936	0.938	0.968	0.925
26	14.9	0.726	0.749	0.877	0.860
27	53.1	0.870	0.893	0.933	0.908
28	67.7	0.931	0.922	0.936	0.914
29	17.6	0.970	0.969	0.964	0.929
30	131.6	0.952	0.941	0.944	0.913
average	33.3	0.947	0.946	0.951	0.920

Table C.14. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP3. Results for models NESTEDc and PANEL5c.

		NESTEDc			PANEL5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9150	7051	29.77	8403	6863	22.44
2	11.4	7701	5656	36.16	7135	5608	27.23
3	18.3	10617	14990	-29.17	9689	13301	-27.16
4	48.4	18835	17651	6.71	16895	16141	4.67
5	8.1	5938	4693	26.53	5566	4608	20.79
6	25.4	12964	12990	-0.20	11735	11837	-0.86
7	48.9	18953	18373	3.16	17000	16694	1.83
8	13.0	8432	7050	19.60	7773	6810	14.14
9	26.4	13283	13193	0.68	12013	11769	2.07
10	43.8	17792	17146	3.77	15974	15520	2.93
11	26.7	13393	11867	12.86	12111	11066	9.44
12	17.1	10099	9503	6.27	9231	8612	7.19
13	51.4	19438	19824	-1.95	17430	17874	-2.48
14	7.3	5467	7190	-23.96	5145	6591	-21.94
15	83.0	25408	24248	4.78	22771	21874	4.10
16	10.4	7216	5327	35.46	6706	5390	24.42
17	70.1	23079	21568	7.01	20677	19637	5.30
18	26.7	13293	11564	14.95	12024	10840	10.92
19	9.2	6587	5818	13.22	6149	5569	10.41
20	17.1	10115	10255	-1.37	9243	9534	-3.05
21	32.1	14905	13889	7.32	13437	12549	7.08
22	14.3	8951	7052	26.93	8226	6940	18.53
23	4.9	4130	4598	-10.18	3943	4376	-9.89
24	54.1	20016	18574	7.76	17944	16695	7.48
25	32.0	14902	16377	-9.01	13434	14310	-6.12
26	14.9	9260	15414	-39.92	8499	13469	-36.90
27	53.1	19799	24930	-20.58	17753	21310	-16.69
28	67.7	22722	24656	-7.84	20359	21944	-7.22
29	17.6	10160	9113	11.49	9285	8398	10.56
30	131.6	32241	31332	2.90	29011	29070	-0.20
average	33.3	13828	13730	4.44	12519	12507	2.63

Table C.15. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP3. Results for models ROTPANEL5c and RANTIME5c.

		ROTPANEL5c			RANTIME5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	5344	4854	10.09	4899	5135	-4.60
2	11.4	4611	4171	10.55	4235	4509	-6.08
3	18.3	6058	7030	-13.83	5558	7156	-22.33
4	48.4	10412	9942	4.73	9667	10683	-9.51
5	8.1	3762	3399	10.68	3469	3625	-4.30
6	25.4	7250	7263	-0.18	6661	7365	-9.56
7	48.9	10479	10214	2.59	9727	10658	-8.74
8	13.0	4985	4894	1.86	4575	5102	-10.33
9	26.4	7416	7390	0.35	6820	7898	-13.65
10	43.8	9835	9656	1.85	9110	10202	-10.70
11	26.7	7467	7223	3.38	6866	7467	-8.05
12	17.1	5800	5505	5.36	5319	5843	-8.97
13	51.4	10762	10915	-1.40	10004	11403	-12.27
14	7.3	3526	3962	-11.00	3255	4028	-19.19
15	83.0	14290	14261	0.20	13487	14725	-8.41
16	10.4	4377	4109	6.52	4024	4396	-8.46
17	70.1	12894	12254	5.22	12097	13139	-7.93
18	26.7	7442	7145	4.16	6835	7594	-9.99
19	9.2	4087	3837	6.52	3760	4104	-8.38
20	17.1	5842	5943	-1.70	5355	6062	-11.66
21	32.1	8263	7882	4.83	7615	8332	-8.61
22	14.3	5254	4933	6.51	4824	5236	-7.87
23	4.9	2836	2993	-5.25	2644	3097	-14.63
24	54.1	11078	10763	2.93	10318	11213	-7.98
25	32.0	8271	7811	5.89	7622	8363	-8.86
26	14.9	5406	6932	-22.01	4962	6761	-26.61
27	53.1	10952	11956	-8.40	10192	11788	-13.54
28	67.7	12658	13097	-3.35	11865	13691	-13.34
29	17.6	5847	5533	5.68	5361	5830	-8.04
30	131.6	18871	19281	-2.13	18158	20663	-12.12
average	33.3	7869	7838	1.02	7309	8202	-10.82

Table C.16. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under correct model in simulations from POP4.

region	$\bar{n}_i$	bias NESTEDc	bias PANEL5c	bias ROTPANEL5c	bias RANTIME5c
1	14.9	-0.18	0.01	0.05	-0.02
2	11.4	-0.07	0.21	0.25	0.17
3	18.3	-3.00	-1.28	-0.91	-0.92
4	48.4	-0.54	-0.17	-0.07	-0.14
5	8.1	1.59	0.70	0.55	0.45
6	25.4	1.62	0.70	0.58	0.48
7	48.9	0.91	0.32	0.29	0.21
8	13.0	1.11	0.62	0.51	0.42
9	26.4	-1.25	-0.42	-0.21	-0.26
10	43.8	0.55	0.21	0.13	0.05
11	26.7	-0.58	-0.13	-0.11	-0.17
12	17.1	-1.11	-0.34	-0.22	-0.27
13	51.4	0.68	0.23	0.11	0.02
14	7.3	-4.02	-1.98	-1.36	-1.35
15	83.0	0.33	0.13	0.05	-0.03
16	10.4	0.73	0.45	0.23	0.14
17	70.1	0.07	-0.06	-0.10	-0.17
18	26.7	-0.45	-0.26	-0.23	-0.30
19	9.2	-1.84	-0.96	-0.69	-0.75
20	17.1	2.09	0.98	0.68	0.58
21	32.1	0.54	0.13	0.03	-0.05
22	14.3	-0.75	-0.34	-0.24	-0.30
23	4.9	-3.84	-2.30	-1.70	-1.71
24	54.1	0.25	0.05	-0.01	-0.09
25	32.0	-1.41	-0.44	-0.29	-0.34
26	14.9	-4.05	-1.64	-1.19	-1.18
27	53.1	1.86	0.67	0.54	0.43
28	67.7	0.90	0.32	0.23	0.14
29	17.6	-1.40	-0.58	-0.47	-0.51
30	131.6	-0.30	-0.14	-0.10	-0.17
average	33.3	-0.39	-0.18	-0.12	-0.19

Table C.17. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under correct model in simulations from POP4.

region	$\bar{n}_i$	MARE NESTEDc	MARE PANEL5c	MARE ROTPANEL5c	MARE RANTIME5c
1	14.9	2.03	1.60	1.30	1.30
2	11.4	2.09	1.73	1.51	1.52
3	18.3	3.30	1.76	1.40	1.41
4	48.4	1.53	0.99	0.85	0.85
5	8.1	2.56	2.05	1.79	1.79
6	25.4	2.43	1.53	1.24	1.22
7	48.9	1.77	1.05	0.91	0.90
8	13.0	2.42	1.79	1.62	1.62
9	26.4	2.07	1.26	1.08	1.09
10	43.8	1.74	1.12	0.92	0.92
11	26.7	1.79	1.24	1.05	1.05
12	17.1	2.20	1.47	1.24	1.25
13	51.4	1.71	1.06	0.92	0.92
14	7.3	4.25	2.49	2.00	2.01
15	83.0	1.29	0.82	0.71	0.71
16	10.4	2.21	1.90	1.62	1.63
17	70.1	1.36	0.89	0.73	0.75
18	26.7	1.87	1.26	1.09	1.11
19	9.2	2.64	1.99	1.58	1.62
20	17.1	2.82	1.78	1.51	1.49
21	32.1	1.82	1.22	1.06	1.07
22	14.3	2.06	1.57	1.35	1.37
23	4.9	4.08	2.84	2.31	2.34
24	54.1	1.48	0.95	0.82	0.82
25	32.0	2.09	1.19	0.94	0.96
26	14.9	4.24	2.04	1.58	1.58
27	53.1	2.36	1.22	1.07	1.03
28	67.7	1.67	1.01	0.84	0.83
29	17.6	2.22	1.48	1.26	1.28
30	131.6	1.03	0.68	0.60	0.61
average	33.3	2.24	1.47	1.23	1.24

Table C.18. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under correct model in simulations from POP4.

region	$\bar{n}_i$	coverage NESTEDc	coverage PANEL5c	coverage ROTPANEL5c	coverage RANTIME5c
1	14.9	0.991	0.967	0.971	0.965
2	11.4	0.996	0.976	0.964	0.960
3	18.3	0.822	0.898	0.919	0.906
4	48.4	0.964	0.948	0.962	0.952
5	8.1	0.988	0.972	0.974	0.969
6	25.4	0.954	0.943	0.945	0.944
7	48.9	0.952	0.948	0.949	0.950
8	13.0	0.981	0.965	0.959	0.950
9	26.4	0.960	0.949	0.959	0.956
10	43.8	0.950	0.943	0.955	0.951
11	26.7	0.986	0.959	0.961	0.958
12	17.1	0.982	0.967	0.971	0.967
13	51.4	0.949	0.950	0.951	0.951
14	7.3	0.878	0.909	0.941	0.932
15	83.0	0.963	0.952	0.955	0.950
16	10.4	0.990	0.971	0.967	0.959
17	70.1	0.962	0.963	0.973	0.961
18	26.7	0.973	0.965	0.951	0.948
19	9.2	0.975	0.951	0.960	0.954
20	17.1	0.956	0.946	0.946	0.939
21	32.1	0.973	0.959	0.954	0.949
22	14.3	0.988	0.963	0.967	0.961
23	4.9	0.955	0.938	0.942	0.927
24	54.1	0.960	0.958	0.962	0.959
25	32.0	0.934	0.945	0.962	0.948
26	14.9	0.716	0.875	0.895	0.883
27	53.1	0.881	0.929	0.920	0.921
28	67.7	0.930	0.939	0.949	0.952
29	17.6	0.970	0.966	0.960	0.951
30	131.6	0.951	0.942	0.945	0.944
average	33.3	0.948	0.949	0.953	0.947

Table C.19. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP4. Results for models NESTEDc and PANEL5c.

		NESTEDc			PANEL5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	9164	6979	31.31	5990	5511	8.69
2	11.4	7711	5521	39.67	5196	4590	13.20
3	18.3	10628	14743	-27.91	6813	8234	-17.26
4	48.4	18847	17464	7.92	11615	11359	2.25
5	8.1	5944	4644	27.99	4196	3763	11.51
6	25.4	12974	13196	-1.68	8126	8386	-3.10
7	48.9	18964	18777	1.00	11688	11519	1.47
8	13.0	8447	7089	19.16	5597	5291	5.78
9	26.4	13297	13145	1.16	8306	8119	2.30
10	43.8	17805	17172	3.69	10975	10969	0.05
11	26.7	13403	11427	17.29	8371	7949	5.31
12	17.1	10112	9020	12.11	6515	6068	7.37
13	51.4	19449	19395	0.28	11992	11955	0.31
14	7.3	5475	7152	-23.45	3928	4547	-13.61
15	83.0	25419	24257	4.79	15896	15315	3.79
16	10.4	7225	5387	34.12	4928	4583	7.53
17	70.1	23089	21439	7.70	14336	13927	2.94
18	26.7	13305	11716	13.56	8315	7875	5.59
19	9.2	6603	5815	13.55	4585	4494	2.02
20	17.1	10130	10388	-2.48	6522	6646	-1.87
21	32.1	14917	13363	11.63	9251	8788	5.27
22	14.3	8966	7067	26.87	5877	5476	7.32
23	4.9	4139	4690	-11.75	3162	3491	-9.42
24	54.1	20026	18553	7.94	12355	11843	4.32
25	32.0	14914	16486	-9.54	9251	9512	-2.74
26	14.9	9276	15265	-39.23	6057	7833	-22.67
27	53.1	19809	25240	-21.52	12221	13249	-7.76
28	67.7	22732	24559	-7.44	14098	14735	-4.32
29	17.6	10171	9265	9.78	6543	6180	5.87
30	131.	32253	31639	1.94	20859	21169	-1.46
average	33.3	13840	13695	4.95	8785	8779	0.62

Table C.20. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under correct model in simulations from POP4. Results for models ROTPANEL5c and RANTIME5c.

		ROTPANEL5c			RANTIME5c		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	5023	4439	13.16	4899	4465	9.72
2	11.4	4337	4020	7.89	4233	4038	4.83
3	18.3	5699	6570	-13.26	5558	6637	-16.26
4	48.4	9896	9610	2.98	9671	9724	-0.55
5	8.1	3548	3227	9.95	3466	3233	7.21
6	25.4	6830	6891	-0.89	6662	6754	-1.36
7	48.9	9958	9840	1.20	9731	9737	-0.06
8	13.0	4689	4657	0.69	4574	4647	-1.57
9	26.4	6991	6895	1.39	6821	6958	-1.97
10	43.8	9329	8889	4.95	9113	8903	2.36
11	26.7	7040	6662	5.67	6867	6709	2.36
12	17.1	5454	5102	6.90	5319	5147	3.34
13	51.4	10239	10244	-0.05	10008	10244	-2.30
14	7.3	3328	3615	-7.94	3252	3638	-10.61
15	83.0	13764	13292	3.55	13493	13256	1.79
16	10.4	4120	3840	7.29	4022	3876	3.77
17	70.1	12359	11527	7.22	12102	11755	2.95
18	26.7	7010	6781	3.38	6836	6898	-0.90
19	9.2	3849	3593	7.12	3758	3659	2.71
20	17.1	5492	5645	-2.71	5355	5568	-3.83
21	32.1	7805	7628	2.32	7617	7681	-0.83
22	14.3	4945	4610	7.27	4824	4672	3.25
23	4.9	2694	2883	-6.56	2641	2920	-9.55
24	54.1	10556	10201	3.48	10321	10225	0.94
25	32.0	7812	7463	4.68	7623	7616	0.09
26	14.9	5086	6179	-17.69	4961	6192	-19.88
27	53.1	10429	11561	-9.79	10196	11225	-9.17
28	67.7	12125	12329	-1.65	11870	12185	-2.59
29	17.6	5498	5296	3.81	5361	5382	-0.39
30	131.6	18465	18392	0.40	18167	18763	-3.18
average	33.3	7479	7396	1.49	7311	7424	-1.32

D Appendix: Simulation study with incorrect model. Result tables for longitudinal data of 5 occasions

Table D.1. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP1.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL5i	bias ROTPANEL5i	bias RANTIME5i
1	14.9	0.04	0.11	-0.02	-0.15
2	11.4	0.06	0.19	0.19	0.07
3	18.3	-2.50	-2.34	-0.72	-0.66
4	48.4	-0.24	-0.20	-0.04	-0.16
5	8.1	1.63	1.53	0.54	0.29
6	25.4	1.53	1.44	0.47	0.27
7	48.9	0.55	0.46	0.15	-0.04
8	13.0	1.22	1.17	0.50	0.31
9	26.4	-1.02	-0.89	-0.19	-0.22
10	43.8	0.54	0.49	0.12	-0.02
11	26.7	-0.60	-0.52	-0.09	-0.17
12	17.1	-1.10	-0.98	-0.40	-0.44
13	51.4	0.60	0.57	0.13	0.02
14	7.3	-4.28	-4.02	-1.40	-1.21
15	83.0	0.40	0.35	0.07	-0.02
16	10.4	0.92	0.91	0.29	0.17
17	70.1	0.13	0.11	-0.01	-0.11
18	26.7	-0.26	-0.28	-0.11	-0.17
19	9.2	-2.03	-1.94	-0.73	-0.73
20	17.1	1.94	1.86	0.56	0.39
21	32.1	0.38	0.32	0.07	-0.05
22	14.3	-0.95	-0.94	-0.31	-0.34
23	4.9	-4.84	-4.61	-1.81	-1.58
24	54.1	0.27	0.21	0.05	-0.04
25	32.0	-1.03	-0.91	-0.24	-0.24
26	14.9	-3.54	-3.29	-1.08	-0.90
27	53.1	1.41	1.30	0.39	0.20
28	67.7	0.71	0.65	0.19	0.10
29	17.6	-0.93	-0.88	-0.31	-0.33
30	131.6	-0.10	-0.09	-0.05	-0.14
average	33.3	-0.37	-0.34	-0.13	-0.20

Table D.2. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP1.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL5i	MARE ROTPANEL5i	MARE RANTIME5i
1	14.9	3.07	2.99	1.78	1.87
2	11.4	3.37	3.33	2.01	2.14
3	18.3	3.35	3.22	1.60	1.67
4	48.4	1.91	1.84	1.00	1.10
5	8.1	4.00	3.93	2.46	2.59
6	25.4	2.93	2.80	1.59	1.66
7	48.9	2.21	2.16	1.20	1.31
8	13.0	3.64	3.53	2.11	2.23
9	26.4	2.49	2.38	1.39	1.50
10	43.8	2.21	2.12	1.18	1.25
11	26.7	2.42	2.32	1.40	1.47
12	17.1	3.10	2.93	1.68	1.79
13	51.4	2.23	2.14	1.15	1.24
14	7.3	5.06	4.86	2.46	2.53
15	83.0	1.60	1.53	0.91	0.98
16	10.4	3.49	3.51	2.18	2.36
17	70.1	1.74	1.65	0.97	1.05
18	26.7	2.63	2.56	1.46	1.58
19	9.2	3.71	3.64	2.11	2.25
20	17.1	3.57	3.47	1.96	2.05
21	32.1	2.53	2.42	1.34	1.43
22	14.3	3.05	2.98	1.81	1.93
23	4.9	5.59	5.40	2.95	3.09
24	54.1	1.90	1.79	1.05	1.12
25	32.0	2.35	2.19	1.12	1.22
26	14.9	4.06	3.88	1.75	1.77
27	53.1	2.63	2.52	1.34	1.37
28	67.7	2.04	1.96	1.05	1.14
29	17.6	2.80	2.70	1.54	1.65
30	131.6	1.12	1.08	0.67	0.73
average	33.3	2.89	2.79	1.57	1.67

Table D.3. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP1.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL5i	coverage ROTPANEL5i	coverage RANTIME5i
1	14.9	0.975	0.969	0.966	0.911
2	11.4	0.970	0.967	0.967	0.915
3	18.3	0.895	0.891	0.944	0.880
4	48.4	0.952	0.969	0.959	0.901
5	8.1	0.973	0.967	0.967	0.928
6	25.4	0.959	0.958	0.952	0.920
7	48.9	0.950	0.950	0.951	0.903
8	13.0	0.968	0.962	0.958	0.898
9	26.4	0.945	0.945	0.940	0.891
10	43.8	0.962	0.965	0.962	0.926
11	26.7	0.963	0.963	0.945	0.911
12	17.1	0.962	0.962	0.950	0.895
13	51.4	0.948	0.951	0.955	0.911
14	7.3	0.896	0.894	0.922	0.881
15	83.0	0.950	0.956	0.958	0.932
16	10.4	0.970	0.970	0.963	0.916
17	70.1	0.951	0.952	0.943	0.915
18	26.7	0.951	0.951	0.952	0.894
19	9.2	0.967	0.957	0.953	0.907
20	17.1	0.959	0.958	0.939	0.893
21	32.1	0.956	0.954	0.954	0.910
22	14.3	0.963	0.963	0.944	0.897
23	4.9	0.923	0.921	0.940	0.890
24	54.1	0.955	0.968	0.953	0.922
25	32.0	0.947	0.953	0.959	0.908
26	14.9	0.836	0.858	0.909	0.869
27	53.1	0.944	0.935	0.959	0.923
28	67.7	0.948	0.948	0.959	0.920
29	17.6	0.975	0.976	0.965	0.912
30	131.6	0.964	0.957	0.960	0.929
average	33.3	0.949	0.950	0.952	0.907

Table D.4. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP1. Results for models NESTEDi and PANEL5i.

		NESTEDi			PANEL5i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	11751	10444	12.51	11315	10232	10.58
2	11.4	10177	8919	14.10	9817	8847	10.96
3	18.3	13362	16432	-18.68	12849	15808	-18.72
4	48.4	22303	21718	2.69	21397	20926	2.25
5	8.1	8174	7066	15.68	7910	6926	14.21
6	25.4	15886	15415	3.06	15256	14812	3.00
7	48.9	22432	22457	-0.11	21520	21816	-1.36
8	13.0	10970	10355	5.94	10572	10086	4.82
9	26.4	16229	16573	-2.08	15584	15833	-1.57
10	43.8	21156	20485	3.28	20299	19500	4.10
11	26.7	16348	15645	4.49	15698	14990	4.72
12	17.1	12780	12746	0.27	12294	12200	0.77
13	51.4	22967	23956	-4.13	22034	22981	-4.12
14	7.3	7637	9432	-19.03	7400	9100	-18.68
15	83.0	29603	28925	2.34	28421	27751	2.41
16	10.4	9642	8219	17.31	9310	8214	13.34
17	70.1	27001	26467	2.02	25914	25171	2.95
18	26.7	16241	16459	-1.32	15595	15937	-2.15
19	9.2	8948	8401	6.51	8651	8211	5.36
20	17.1	12796	12553	1.94	12308	12177	1.08
21	32.1	18001	17863	0.77	17278	17087	1.12
22	14.3	11530	10592	8.86	11103	10343	7.35
23	4.9	6081	7002	-13.15	5922	6813	-13.08
24	54.1	23602	23222	1.64	22646	21960	3.12
25	32.0	18005	19080	-5.63	17282	17859	-3.23
26	14.9	11880	16510	-28.04	11438	15757	-27.41
27	53.1	23370	25182	-7.20	22422	24039	-6.73
28	67.7	26610	27741	-4.08	25538	26567	-3.87
29	17.6	12832	11654	10.11	12344	11242	9.80
30	131.6	37262	35140	6.04	35841	34261	4.61
average	33.3	16853	16888	0.54	16199	16248	0.19

Table D.5. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP1. Results for models ROTPANEL5i and RANTIME5i.

		ROTPANEL5i			RANTIME5i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	6373	6015	5.95	5727	6426	-10.88
2	11.4	5553	5299	4.79	4988	5621	-11.26
3	18.3	7174	7883	-8.99	6463	8315	-22.27
4	48.4	12060	11613	3.85	11095	12723	-12.80
5	8.1	4614	4326	6.66	4144	4521	-8.34
6	25.4	8512	8274	2.88	7700	8655	-11.03
7	48.9	12137	12081	0.46	11163	13101	-14.79
8	13.0	5973	5979	-0.10	5367	6292	-14.70
9	26.4	8697	9093	-4.35	7879	9797	-19.58
10	43.8	11415	10978	3.98	10464	11658	-10.24
11	26.7	8754	8987	-2.59	7931	9329	-14.99
12	17.1	6884	6995	-1.59	6195	7432	-16.64
13	51.4	12452	12342	0.89	11476	13252	-13.40
14	7.3	4351	4730	-8.01	3909	4908	-20.35
15	83.0	16388	16245	0.88	15428	17324	-10.94
16	10.4	5292	5032	5.17	4755	5406	-12.04
17	70.1	14835	14736	0.67	13850	15863	-12.69
18	26.7	8730	8985	-2.84	7896	9673	-18.37
19	9.2	4976	4855	2.49	4466	5154	-13.35
20	17.1	6935	7139	-2.86	6238	7475	-16.55
21	32.1	9649	9454	2.06	8775	10137	-13.44
22	14.3	6273	6303	-0.48	5646	6718	-15.96
23	4.9	3596	3834	-6.21	3251	4017	-19.07
24	54.1	12805	12679	0.99	11832	13502	-12.37
25	32.0	9659	9294	3.93	8783	10128	-13.28
26	14.9	6445	7358	-12.41	5799	7484	-22.51
27	53.1	12664	12736	-0.57	11690	13028	-10.27
28	67.7	14571	14365	1.43	13587	15397	-11.76
29	17.6	6936	6558	5.76	6241	7009	-10.96
30	131.6	21450	20864	2.81	20740	22817	-9.10
average	33.3	9205	9168	0.16	8449	9772	-14.13

Table D.6. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP2.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL5i	bias ROTPANEL5i	bias RANTIME5i
1	14.9	-0.10	0.02	-0.07	-0.18
2	11.4	0.04	0.30	0.25	0.15
3	18.3	-2.43	-1.73	-0.76	-0.73
4	48.4	-0.31	-0.19	-0.05	-0.16
5	8.1	1.54	1.08	0.45	0.26
6	25.4	1.57	1.12	0.52	0.33
7	48.9	0.61	0.31	0.16	0.01
8	13.0	1.25	0.96	0.48	0.30
9	26.4	-1.05	-0.64	-0.16	-0.21
10	43.8	0.56	0.38	0.11	-0.02
11	26.7	-0.60	-0.35	-0.11	-0.20
12	17.1	-1.06	-0.63	-0.30	-0.36
13	51.4	0.54	0.38	0.10	-0.02
14	7.3	-4.19	-3.03	-1.24	-1.12
15	83.0	0.39	0.26	0.08	-0.02
16	10.4	0.86	0.74	0.28	0.16
17	70.1	0.11	0.04	-0.05	-0.16
18	26.7	-0.25	-0.24	-0.14	-0.21
19	9.2	-1.95	-1.48	-0.66	-0.69
20	17.1	1.95	1.50	0.59	0.43
21	32.1	0.40	0.19	0.04	-0.08
22	14.3	-0.89	-0.71	-0.25	-0.31
23	4.9	-4.80	-3.73	-1.70	-1.57
24	54.1	0.27	0.13	0.04	-0.06
25	32.0	-1.08	-0.64	-0.20	-0.24
26	14.9	-3.48	-2.40	-0.96	-0.87
27	53.1	1.42	0.91	0.43	0.24
28	67.7	0.72	0.50	0.23	0.12
29	17.6	-0.94	-0.66	-0.32	-0.35
30	131.6	-0.13	-0.10	-0.09	-0.17
average	33.3	-0.37	-0.26	-0.11	-0.19

Table D.7. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP2.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL5i	MARE ROTPANEL5i	MARE RANTIME5i
1	14.9	3.05	2.70	1.71	1.76
2	11.4	3.33	3.08	1.94	2.00
3	18.3	3.30	2.67	1.54	1.58
4	48.4	1.89	1.60	1.01	1.07
5	8.1	4.12	3.61	2.44	2.48
6	25.4	3.00	2.45	1.59	1.59
7	48.9	2.22	1.86	1.18	1.25
8	13.0	3.62	3.12	2.05	2.10
9	26.4	2.48	2.07	1.34	1.39
10	43.8	2.21	1.82	1.14	1.17
11	26.7	2.41	2.00	1.37	1.40
12	17.1	3.03	2.53	1.64	1.69
13	51.4	2.19	1.81	1.16	1.19
14	7.3	5.00	4.07	2.39	2.42
15	83.0	1.59	1.31	0.89	0.91
16	10.4	3.46	3.19	2.08	2.18
17	70.1	1.74	1.41	0.95	0.99
18	26.7	2.58	2.22	1.42	1.48
19	9.2	3.70	3.27	2.07	2.15
20	17.1	3.61	3.01	1.96	1.98
21	32.1	2.46	2.09	1.34	1.38
22	14.3	3.08	2.67	1.80	1.87
23	4.9	5.60	4.71	2.87	2.94
24	54.1	1.93	1.58	1.04	1.07
25	32.0	2.35	1.86	1.11	1.17
26	14.9	4.02	3.11	1.67	1.68
27	53.1	2.61	2.09	1.33	1.32
28	67.7	2.04	1.67	1.05	1.07
29	17.6	2.79	2.38	1.55	1.60
30	131.6	1.14	0.94	0.67	0.70
average	33.3	2.88	2.43	1.54	1.59

Table D.8. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP2.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL5i	coverage ROTPANEL5i	coverage RANTIME5i
1	14.9	0.973	0.963	0.960	0.934
2	11.4	0.981	0.957	0.955	0.923
3	18.3	0.892	0.914	0.936	0.897
4	48.4	0.959	0.961	0.952	0.919
5	8.1	0.979	0.966	0.962	0.937
6	25.4	0.958	0.954	0.958	0.935
7	48.9	0.952	0.944	0.949	0.913
8	13.0	0.964	0.960	0.951	0.915
9	26.4	0.945	0.943	0.936	0.904
10	43.8	0.953	0.960	0.947	0.938
11	26.7	0.968	0.961	0.946	0.921
12	17.1	0.957	0.955	0.943	0.916
13	51.4	0.949	0.947	0.957	0.925
14	7.3	0.908	0.913	0.935	0.896
15	83.0	0.960	0.962	0.959	0.939
16	10.4	0.971	0.958	0.972	0.944
17	70.1	0.957	0.962	0.951	0.930
18	26.7	0.948	0.948	0.943	0.903
19	9.2	0.963	0.958	0.951	0.917
20	17.1	0.954	0.957	0.932	0.919
21	32.1	0.959	0.948	0.952	0.920
22	14.3	0.958	0.956	0.951	0.924
23	4.9	0.917	0.918	0.937	0.901
24	54.1	0.947	0.953	0.952	0.932
25	32.0	0.930	0.948	0.962	0.933
26	14.9	0.832	0.887	0.921	0.898
27	53.1	0.927	0.953	0.948	0.933
28	67.7	0.944	0.939	0.955	0.930
29	17.6	0.969	0.974	0.962	0.921
30	131.6	0.958	0.954	0.970	0.953
average	33.3	0.948	0.949	0.950	0.922

Table D.9. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP2. Results for models NESTEDi and PANEL5i.

		NESTEDi			PANEL5i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	11752	10360	13.44	9802	9162	6.99
2	11.4	10180	8729	16.62	8559	8075	5.99
3	18.3	13361	16215	-17.60	11090	13170	-15.79
4	48.4	22298	21636	3.06	18365	18201	0.90
5	8.1	8176	7097	15.20	6973	6332	10.12
6	25.4	15884	15678	1.31	13116	12838	2.17
7	48.9	22427	22467	-0.18	18471	18834	-1.93
8	13.0	10973	10366	5.86	9187	8923	2.96
9	26.4	16227	16595	-2.22	13392	13725	-2.43
10	43.8	21152	20740	1.99	17418	16798	3.69
11	26.7	16346	15538	5.20	13490	12955	4.13
12	17.1	12782	12460	2.58	10624	10582	0.40
13	51.4	22962	23574	-2.60	18916	19381	-2.40
14	7.3	7641	9313	-17.95	6552	7789	-15.88
15	83.0	29596	28700	3.12	24500	23566	3.96
16	10.4	9644	8223	17.28	8143	7483	8.82
17	70.1	26994	26389	2.29	22295	21386	4.25
18	26.7	16238	16413	-1.07	13402	13946	-3.90
19	9.2	8953	8461	5.81	7599	7451	1.99
20	17.1	12798	12763	0.27	10634	10568	0.62
21	32.1	18000	17468	3.05	14833	14726	0.73
22	14.3	11533	10712	7.66	9627	9276	3.78
23	4.9	6087	7037	-13.50	5339	6084	-12.25
24	54.1	23595	23627	-0.14	19445	19177	1.40
25	32.0	18002	19508	-7.72	14836	15312	-3.11
26	14.9	11884	16396	-27.52	9910	12800	-22.58
27	53.1	23365	25207	-7.31	19252	20098	-4.21
28	67.7	26604	27805	-4.32	21965	22722	-3.33
29	17.6	12832	11826	8.51	10662	9952	7.13
30	131.6	37251	35427	5.15	31163	29831	4.47
average	33.3	16851	16891	0.54	13985	14038	-0.44

Table D.10. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP2. Results for models ROTPANEL5i and RANTIME5i.

		ROTPANEL5i			RANTIME5i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	6208	5813	6.80	5726	6002	-4.60
2	11.4	5406	5185	4.26	4987	5291	-5.75
3	18.3	6996	7669	-8.78	6462	7863	-17.82
4	48.4	11869	11752	1.00	11094	12319	-9.94
5	8.1	4490	4237	5.97	4144	4316	-3.99
6	25.4	8317	8243	0.90	7699	8266	-6.86
7	48.9	11944	12017	-0.61	11161	12523	-10.88
8	13.0	5817	5787	0.52	5366	5941	-9.68
9	26.4	8502	8815	-3.55	7878	9135	-13.76
10	43.8	11217	10745	4.39	10463	11031	-5.15
11	26.7	8558	8753	-2.23	7930	8919	-11.09
12	17.1	6710	6853	-2.09	6194	7049	-12.13
13	51.4	12264	12229	0.29	11475	12599	-8.92
14	7.3	4234	4580	-7.55	3908	4656	-16.07
15	83.0	16288	16032	1.60	15426	16373	-5.78
16	10.4	5152	4785	7.67	4754	5011	-5.13
17	70.1	14691	14593	0.67	13848	15125	-8.44
18	26.7	8530	8836	-3.46	7895	9238	-14.54
19	9.2	4842	4767	1.57	4465	4929	-9.41
20	17.1	6758	7013	-3.64	6237	7107	-12.24
21	32.1	9449	9357	0.98	8774	9666	-9.23
22	14.3	6113	6146	-0.54	5645	6376	-11.46
23	4.9	3504	3760	-6.81	3251	3846	-15.47
24	54.1	12624	12612	0.10	11830	12884	-8.18
25	32.0	9458	9181	3.02	8782	9692	-9.39
26	14.9	6280	7035	-10.73	5798	7094	-18.27
27	53.1	12481	12675	-1.53	11688	12617	-7.36
28	67.7	14422	14324	0.68	13585	14526	-6.48
29	17.6	6760	6528	3.55	6240	6759	-7.68
30	131.6	21571	20502	5.21	20737	21485	-3.48
average	33.3	9049	9027	-0.08	8448	9288	-9.64

Table D.11. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP3.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL5i	bias ROTPANEL5i	bias RANTIME5i
1	14.9	-0.14	-0.01	0.00	-0.14
2	11.4	-0.28	-0.10	0.15	0.04
3	18.3	-4.15	-3.75	-1.41	-1.23
4	48.4	-0.36	-0.28	-0.05	-0.19
5	8.1	1.29	1.23	0.68	0.41
6	25.4	2.17	1.98	0.79	0.51
7	48.9	1.10	0.92	0.34	0.10
8	13.0	1.53	1.47	0.79	0.54
9	26.4	-1.80	-1.54	-0.43	-0.43
10	43.8	0.84	0.74	0.24	0.06
11	26.7	-1.15	-0.98	-0.26	-0.31
12	17.1	-1.92	-1.68	-0.70	-0.69
13	51.4	0.93	0.84	0.25	0.09
14	7.3	-5.44	-5.07	-2.42	-2.08
15	83.0	0.64	0.55	0.15	0.02
16	10.4	0.65	0.68	0.35	0.20
17	70.1	0.33	0.27	0.05	-0.09
18	26.7	-0.61	-0.57	-0.21	-0.27
19	9.2	-2.78	-2.60	-1.21	-1.14
20	17.1	2.27	2.14	0.87	0.62
21	32.1	0.55	0.46	0.15	-0.01
22	14.3	-1.78	-1.67	-0.63	-0.62
23	4.9	-5.46	-5.19	-2.83	-2.52
24	54.1	0.31	0.24	0.08	-0.04
25	32.0	-1.96	-1.67	-0.51	-0.47
26	14.9	-5.54	-5.04	-2.03	-1.68
27	53.1	2.40	2.09	0.73	0.44
28	67.7	1.12	0.98	0.33	0.18
29	17.6	-1.40	-1.28	-0.50	-0.50
30	131.6	-0.15	-0.12	-0.06	-0.17
average	33.3	-0.63	-0.57	-0.24	-0.31

Table D.12. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP3.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL5i	MARE ROTPANEL5i	MARE RANTIME5i
1	14.9	2.27	2.26	1.71	1.84
2	11.4	2.31	2.35	1.88	2.06
3	18.3	4.44	4.10	2.00	2.01
4	48.4	1.83	1.77	1.07	1.18
5	8.1	2.53	2.55	2.15	2.32
6	25.4	2.85	2.69	1.63	1.66
7	48.9	2.16	2.05	1.22	1.32
8	13.0	2.79	2.74	2.04	2.15
9	26.4	2.60	2.43	1.49	1.62
10	43.8	2.08	1.97	1.22	1.28
11	26.7	2.31	2.18	1.46	1.53
12	17.1	2.93	2.72	1.74	1.86
13	51.4	2.14	2.02	1.19	1.27
14	7.3	5.59	5.25	2.92	2.88
15	83.0	1.61	1.52	0.97	1.02
16	10.4	2.31	2.41	2.00	2.23
17	70.1	1.69	1.58	1.01	1.09
18	26.7	2.31	2.26	1.49	1.63
19	9.2	3.32	3.22	2.12	2.26
20	17.1	3.07	2.97	1.93	1.99
21	32.1	2.20	2.11	1.34	1.45
22	14.3	2.75	2.65	1.82	1.97
23	4.9	5.58	5.34	3.28	3.33
24	54.1	1.79	1.67	1.09	1.17
25	32.0	2.71	2.45	1.26	1.37
26	14.9	5.66	5.21	2.39	2.26
27	53.1	3.00	2.72	1.39	1.37
28	67.7	2.06	1.92	1.08	1.15
29	17.6	2.49	2.40	1.59	1.73
30	131.6	1.19	1.13	0.74	0.81
average	33.3	2.75	2.62	1.64	1.73

Table D.13. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP3.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL5i	coverage ROTPANEL5i	coverage RANTIME5i
1	14.9	0.993	0.991	0.972	0.935
2	11.4	0.996	0.988	0.976	0.938
3	18.3	0.762	0.787	0.908	0.855
4	48.4	0.970	0.973	0.964	0.904
5	8.1	0.996	0.990	0.982	0.958
6	25.4	0.962	0.953	0.954	0.921
7	48.9	0.960	0.954	0.951	0.910
8	13.0	0.988	0.987	0.960	0.917
9	26.4	0.946	0.944	0.944	0.905
10	43.8	0.971	0.975	0.961	0.932
11	26.7	0.975	0.973	0.950	0.923
12	17.1	0.968	0.966	0.950	0.907
13	51.4	0.958	0.960	0.959	0.921
14	7.3	0.799	0.819	0.895	0.863
15	83.0	0.962	0.959	0.962	0.930
16	10.4	0.995	0.992	0.977	0.947
17	70.1	0.963	0.961	0.954	0.928
18	26.7	0.974	0.971	0.959	0.901
19	9.2	0.977	0.972	0.957	0.915
20	17.1	0.973	0.967	0.941	0.898
21	32.1	0.969	0.966	0.957	0.921
22	14.3	0.979	0.973	0.953	0.918
23	4.9	0.881	0.890	0.919	0.881
24	54.1	0.962	0.973	0.959	0.933
25	32.0	0.914	0.933	0.954	0.913
26	14.9	0.633	0.669	0.845	0.826
27	53.1	0.868	0.897	0.941	0.918
28	67.7	0.940	0.941	0.950	0.923
29	17.6	0.986	0.990	0.969	0.926
30	131.6	0.968	0.960	0.956	0.932
average	33.3	0.940	0.942	0.949	0.913

Table D.14. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP3. Results for models NESTEDi and PANEL5i.

		NESTEDi			PANEL5i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	10837	7878	37.56	10382	7838	32.46
2	11.4	9010	6273	43.63	8680	6408	35.46
3	18.3	12681	19417	-34.69	12092	18084	-33.13
4	48.4	23240	20834	11.55	21836	19947	9.47
5	8.1	6829	4689	45.64	6628	4721	40.39
6	25.4	15693	15496	1.27	14868	14712	1.06
7	48.9	23390	23015	1.63	21976	21908	0.31
8	13.0	9921	8163	21.54	9527	8078	17.94
9	26.4	16105	16669	-3.38	15248	15608	-2.31
10	43.8	21902	20092	9.01	20600	18853	9.27
11	26.7	16243	14656	10.83	15378	13942	10.30
12	17.1	12028	11693	2.86	11483	11059	3.83
13	51.4	24018	24061	-0.18	22559	22808	-1.09
14	7.3	6264	9128	-31.38	6095	8699	-29.93
15	83.0	31649	30166	4.92	29672	28440	4.33
16	10.4	8401	5653	48.61	8111	5845	38.77
17	70.1	28675	26590	7.84	26892	24936	7.84
18	26.7	16126	14428	11.77	15269	14077	8.47
19	9.2	7625	7118	7.12	7383	6945	6.31
20	17.1	12056	11233	7.33	11505	10912	5.43
21	32.1	18191	16132	12.76	17175	15443	11.22
22	14.3	10584	9245	14.48	10140	9001	12.65
23	4.9	4648	6193	-24.95	4558	5988	-23.88
24	54.1	24754	22454	10.24	23243	21014	10.61
25	32.0	18188	20911	-13.02	17172	19006	-9.65
26	14.9	10963	19971	-45.11	10494	18560	-43.46
27	53.1	24479	31471	-22.22	22989	28716	-19.94
28	67.7	28220	29890	-5.59	26470	27963	-5.34
29	17.6	12116	10078	20.22	11564	9718	19.00
30	131.6	40347	36862	9.45	37884	35408	6.99
average	33.3	16839	16682	5.33	15929	15821	4.11

Table D.15. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($\widehat{MRMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $\widehat{MRMSE}$ (in %) under incorrect model in simulations from POP3. Results for models ROTPANEL5i and RANTIME5i.

		ROTPANEL5i			RANTIME5i		
region	$\bar{n}_i$	$\widehat{MRMSE}$	$ERMSE$	RE	$\widehat{MRMSE}$	$ERMSE$	RE
1	14.9	6653	5856	13.61	6087	6382	-4.62
2	11.4	5712	5013	13.94	5241	5467	-4.13
3	18.3	7566	9219	-17.93	6922	9372	-26.14
4	48.4	13092	12358	5.94	12125	13582	-10.73
5	8.1	4619	3920	17.83	4263	4226	0.88
6	25.4	9089	9021	0.75	8322	9210	-9.64
7	48.9	13178	13073	0.80	12201	13938	-12.46
8	13.0	6191	5964	3.81	5673	6304	-10.01
9	26.4	9299	9569	-2.82	8523	10334	-17.52
10	43.8	12366	11735	5.38	11421	12406	-7.94
11	26.7	9364	9323	0.44	8582	9765	-12.11
12	17.1	7239	7205	0.47	6620	7703	-14.06
13	51.4	13533	13389	1.08	12551	14230	-11.80
14	7.3	4315	5204	-17.08	3991	5234	-23.75
15	83.0	17956	17732	1.26	16946	18674	-9.25
16	10.4	5411	4737	14.23	4973	5229	-4.90
17	70.1	16211	15844	2.32	15192	17007	-10.67
18	26.7	9334	9249	0.92	8543	10065	-15.12
19	9.2	5035	4804	4.81	4633	5148	-10.00
20	17.1	7289	7390	-1.37	6665	7696	-13.40
21	32.1	10376	9832	5.53	9530	10564	-9.79
22	14.3	6536	6321	3.40	5990	6825	-12.23
23	4.9	3424	3991	-14.21	3207	4091	-21.61
24	54.1	13931	13584	2.55	12947	14424	-10.24
25	32.0	10385	10084	2.98	9538	10914	-12.61
26	14.9	6730	9142	-26.38	6164	8807	-30.01
27	53.1	13773	14830	-7.13	12789	14530	-11.98
28	67.7	15915	15914	0.01	14899	16749	-11.05
29	17.6	7299	6658	9.63	6674	7210	-7.43
30	131.6	23651	22898	3.29	22835	24920	-8.37
average	33.3	9849	9795	0.94	9135	10367	-12.09

Table D.16. Mean sample size $\bar{n}_i$ in the last month and observed relative bias (%) of the EBLUP total estimates under incorrect model in simulations from POP4.

region	$\bar{n}_i$	bias NESTEDi	bias PANEL5i	bias ROTPANEL5i	bias RANTIME5i
1	14.9	-0.28	-0.05	-0.06	-0.18
2	11.4	-0.26	0.12	0.23	0.12
3	18.3	-4.07	-2.76	-1.40	-1.31
4	48.4	-0.43	-0.23	-0.06	-0.19
5	8.1	1.21	0.95	0.59	0.39
6	25.4	2.24	1.54	0.82	0.59
7	48.9	1.14	0.62	0.33	0.15
8	13.0	1.58	1.26	0.76	0.54
9	26.4	-1.82	-1.07	-0.38	-0.42
10	43.8	0.87	0.56	0.22	0.06
11	26.7	-1.13	-0.64	-0.26	-0.34
12	17.1	-1.87	-1.12	-0.57	-0.60
13	51.4	0.86	0.55	0.21	0.06
14	7.3	-5.41	-4.04	-2.22	-2.02
15	83.0	0.63	0.39	0.15	0.02
16	10.4	0.61	0.63	0.34	0.20
17	70.1	0.31	0.15	0.00	-0.14
18	26.7	-0.62	-0.43	-0.23	-0.31
19	9.2	-2.75	-2.06	-1.11	-1.11
20	17.1	2.30	1.76	0.88	0.67
21	32.1	0.56	0.30	0.11	-0.04
22	14.3	-1.70	-1.22	-0.54	-0.58
23	4.9	-5.53	-4.45	-2.72	-2.55
24	54.1	0.29	0.16	0.06	-0.06
25	32.0	-2.00	-1.14	-0.46	-0.48
26	14.9	-5.47	-3.74	-1.84	-1.66
27	53.1	2.41	1.43	0.72	0.48
28	67.7	1.13	0.71	0.35	0.21
29	17.6	-1.38	-0.92	-0.48	-0.52
30	131.6	-0.17	-0.11	-0.10	-0.21
average	33.3	-0.62	-0.43	-0.22	-0.31

Table D.17. Mean sample size $\bar{n}_i$ in the last month and MARE (%) of the EBLUP total estimates under incorrect model in simulations from POP4.

region	$\bar{n}_i$	MARE NESTEDi	MARE PANEL5i	MARE ROTPANEL5i	MARE RANTIME5i
1	14.9	2.28	2.20	1.66	1.72
2	11.4	2.28	2.35	1.81	1.90
3	18.3	4.38	3.22	1.90	1.90
4	48.4	1.83	1.56	1.09	1.15
5	8.1	2.59	2.54	2.15	2.23
6	25.4	2.94	2.33	1.63	1.60
7	48.9	2.19	1.75	1.21	1.25
8	13.0	2.79	2.56	2.00	2.04
9	26.4	2.62	2.05	1.43	1.48
10	43.8	2.09	1.70	1.17	1.19
11	26.7	2.28	1.89	1.42	1.46
12	17.1	2.84	2.35	1.68	1.75
13	51.4	2.09	1.69	1.18	1.21
14	7.3	5.56	4.35	2.81	2.76
15	83.0	1.61	1.29	0.94	0.94
16	10.4	2.32	2.41	1.91	2.03
17	70.1	1.69	1.35	0.99	1.02
18	26.7	2.30	2.02	1.46	1.53
19	9.2	3.31	2.89	2.09	2.18
20	17.1	3.13	2.65	1.92	1.91
21	32.1	2.14	1.87	1.35	1.39
22	14.3	2.77	2.39	1.80	1.89
23	4.9	5.67	4.73	3.21	3.23
24	54.1	1.81	1.49	1.08	1.11
25	32.0	2.72	2.00	1.25	1.31
26	14.9	5.62	4.03	2.23	2.17
27	53.1	3.00	2.13	1.38	1.32
28	67.7	2.04	1.60	1.07	1.07
29	17.6	2.50	2.17	1.60	1.67
30	131.6	1.21	0.98	0.73	0.77
average	33.3	2.75	2.28	1.60	1.64

Table D.18. Mean sample size $\bar{n}_i$ in the last month and coverage of the 95 % confidence intervals under incorrect model in simulations from POP4.

region	$\bar{n}_i$	coverage NESTEDi	coverage PANEL5i	coverage ROTPANEL5i	coverage RANTIME5i
1	14.9	0.993	0.982	0.974	0.948
2	11.4	0.996	0.986	0.966	0.949
3	18.3	0.755	0.838	0.887	0.862
4	48.4	0.970	0.970	0.952	0.931
5	8.1	0.998	0.988	0.979	0.968
6	25.4	0.960	0.944	0.949	0.935
7	48.9	0.953	0.953	0.942	0.921
8	13.0	0.984	0.977	0.962	0.937
9	26.4	0.949	0.946	0.938	0.918
10	43.8	0.960	0.967	0.952	0.946
11	26.7	0.978	0.968	0.953	0.934
12	17.1	0.971	0.961	0.956	0.928
13	51.4	0.961	0.951	0.966	0.943
14	7.3	0.820	0.867	0.906	0.884
15	83.0	0.962	0.959	0.958	0.944
16	10.4	0.993	0.986	0.975	0.961
17	70.1	0.962	0.962	0.953	0.933
18	26.7	0.975	0.969	0.953	0.920
19	9.2	0.973	0.965	0.956	0.928
20	17.1	0.970	0.960	0.939	0.921
21	32.1	0.975	0.963	0.956	0.932
22	14.3	0.977	0.971	0.960	0.940
23	4.9	0.875	0.885	0.925	0.896
24	54.1	0.965	0.958	0.956	0.940
25	32.0	0.894	0.936	0.952	0.932
26	14.9	0.632	0.758	0.867	0.855
27	53.1	0.859	0.927	0.935	0.929
28	67.7	0.945	0.938	0.950	0.933
29	17.6	0.986	0.989	0.963	0.938
30	131.6	0.963	0.954	0.972	0.956
average	33.3	0.938	0.946	0.948	0.929

Table D.19. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP4. Results for models NESTEDi and PANEL5i.

		NESTEDi			PANEL5i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	10846	7909	37.13	9140	7526	21.45
2	11.4	9020	6110	47.63	7754	6271	23.65
3	18.3	12691	19167	-33.79	10537	14561	-27.64
4	48.4	23246	20791	11.81	18485	17542	5.38
5	8.1	6836	4663	46.60	6038	4626	30.52
6	25.4	15699	15848	-0.94	12777	12837	-0.47
7	48.9	23394	23206	0.81	18600	18703	-0.55
8	13.0	9936	8226	20.79	8449	7570	11.61
9	26.4	16115	16689	-3.44	13084	13372	-2.15
10	43.8	21910	20451	7.13	17462	16390	6.54
11	26.7	16246	14453	12.41	13188	12122	8.79
12	17.1	12039	11377	5.82	10035	9621	4.30
13	51.4	24022	23585	1.85	19084	19092	-0.04
14	7.3	6273	9081	-30.92	5592	7481	-25.25
15	83.0	31649	29899	5.85	25094	23904	4.98
16	10.4	8412	5726	46.91	7287	5793	25.79
17	70.1	28675	26459	8.38	22724	21136	7.51
18	26.7	16133	14480	11.42	13101	12703	3.13
19	9.2	7640	7148	6.88	6687	6378	4.84
20	17.1	12069	11459	5.32	10054	9796	2.63
21	32.1	18198	15682	16.04	14654	13619	7.60
22	14.3	10601	9325	13.68	8946	8205	9.03
23	4.9	4657	6299	-26.07	4280	5462	-21.64
24	54.1	24757	22767	8.74	19654	18556	5.92
25	32.0	18194	21313	-14.63	14653	15772	-7.09
26	14.9	10981	19873	-44.74	9240	14761	-37.40
27	53.1	24481	31522	-22.34	19444	22635	-14.10
28	67.7	28220	29842	-5.44	22366	23420	-4.50
29	17.6	12128	10208	18.81	10097	8816	14.53
30	131.6	40343	37250	8.30	32292	30564	5.65
average	33.3	16847	16694	5.33	13693	13641	2.10

Table D.20. Mean sample size $\bar{n}_i$ in the last month, average estimated RMSE ($M\widehat{RMSE}$), empirical RMSE ($ERMSE$) and approximate relative error (RE) of $M\widehat{RMSE}$ (in %) under incorrect model in simulations from POP4. Results for models ROTPANEL5i and RANTIME5i.

		ROTPANEL5i			RANTIME5i		
region	$\bar{n}_i$	$M\widehat{RMSE}$	$ERMSE$	RE	$M\widehat{RMSE}$	$ERMSE$	RE
1	14.9	6473	5674	14.08	6083	5916	2.82
2	11.4	5560	4930	12.78	5236	5103	2.61
3	18.3	7364	8936	-17.59	6918	8965	-22.83
4	48.4	12834	12518	2.52	12123	13080	-7.32
5	8.1	4503	3878	16.12	4258	4011	6.16
6	25.4	8856	8989	-1.48	8319	8825	-5.73
7	48.9	12917	13036	-0.91	12198	13319	-8.42
8	13.0	6025	5809	3.72	5669	5934	-4.47
9	26.4	9065	9223	-1.71	8520	9562	-10.90
10	43.8	12105	11454	5.68	11418	11649	-1.98
11	26.7	9128	9063	0.72	8579	9300	-7.75
12	17.1	7044	6988	0.80	6617	7226	-8.43
13	51.4	13275	13161	0.87	12548	13408	-6.41
14	7.3	4209	4994	-15.72	3987	4986	-20.04
15	83.0	17767	17381	2.22	16943	17551	-3.46
16	10.4	5269	4550	15.80	4968	4807	3.35
17	70.1	15984	15579	2.60	15189	16068	-5.47
18	26.7	9094	9104	-0.11	8540	9554	-10.61
19	9.2	4904	4725	3.79	4628	4918	-5.90
20	17.1	7092	7286	-2.66	6661	7309	-8.87
21	32.1	10127	9723	4.16	9527	10011	-4.83
22	14.3	6363	6148	3.50	5986	6429	-6.89
23	4.9	3352	3933	-14.77	3202	3972	-19.39
24	54.1	13679	13428	1.87	12944	13669	-5.30
25	32.0	10135	9963	1.73	9535	10451	-8.76
26	14.9	6551	8603	-23.85	6161	8421	-26.84
27	53.1	13518	14642	-7.68	12786	14092	-9.27
28	67.7	15685	15736	-0.32	14897	15671	-4.94
29	17.6	7102	6652	6.76	6670	6931	-3.77
30	131.6	23670	22361	5.85	22833	23352	-2.22
average	33.3	9655	9616	0.63	9131	9816	-7.20

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