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DEPARTMENT OF MATHEMATICS
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Foreword to the translation of "Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients"

A remarkable feature of quasiconformal mappings in the plane is the interplay between analytic and geometric arguments in the theory. The utility of the analytic approach is principally due to the explicit representation formula

$$f = z + C\mu(z) + C(\mu T\mu)(z) + C(\mu T(\mu T\mu))(z) + \ldots, \tag{1}$$

valid for a suitably normalized quasiconformal homeomorphism f of the plane, with compactly supported dilatation μ. Here T stands for the Beurling-Ahlfors singular integral operator and C is the Cauchy transform. It was previously known that T is an isometry on L^2, and the above formula easily yields that in this case f belongs to $W^{1,2}_{loc}(\mathbf{R}^2)$. However, this knowledge alone does not bring much new, since this fact may be already read from the integrability of the Jacobian of f.

The full power of the above representation was not realized until Bogdan Bojarski's fundamental paper [*Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients,* Mat. Sb. N.S. 43 (85), 1957, 451–503]. This publication was preceded by two short notes by the same author, but the new point of view is fully developed and worked out in this paper. A fundamental new approach due to Bojarski is to derive L^p-estimates for the derivatives of f by first showing that $\|T\|_{L^p \to L^p} \to 1$ as $p \to 2$. For this purpose he invokes, in a beautiful manner, new tools from harmonic analysis of 1950's, including interpolation and the fresh theory of singular integrals due to Calderon and Zygmund. This breakthrough idea opens up a direct pathway to many important issues in the theory, e.g. existence, regularity and Hölder continuity estimates of the solution. Bojarski's paper actually deals with solutions of general elliptic first order PDE-systems and contains a wealth of material on existence and properties of their solutions.

Bojarski's paper mentioned above was written in Russian around 50 years ago. Taking into account the importance of this work in the theory of quasiconformal mappings, it is a very welcome event that the paper now appears in English for the first time. The translation is due to V.Y.Gutlianskii, Denis Koftoniuk, Robin Krauze, V.A.Ryazanov, and Evgenii Sevostyanov, and follows the original paper without changes. Let us finally mention for the reader's benefit that a comprehensive account of modern planar quasiconformal theory, especially in relation to elliptic PDE's, is given in the recent monograph [K. Astala, T. Iwaniec and G. Martin: *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.* Princeton Mathematical Series, 2009].

Eero Saksman

Generalized solutions of a system of differential equations of the first order and elliptic type with discontinuous coefficients †

B. V. Bojarski, Warszawa

In the present work we investigate properties of the generalized solutions of the linear elliptic system of two partial differential equations of the following type:

$$(*)\qquad \begin{aligned} v_y &= \alpha u_x + \beta u_y + au + bv + e, \\ -v_x &= \gamma u_x + \delta u_y + cu + dv + f. \end{aligned}$$

It is assumed that the coefficients α, β, γ, δ are measurable functions satisfying the condition of uniform ellipticity, see § 2, and a, b, c, d, e, f are integrable with some exponent $p > 2$ in the domain where the system $(*)$, is considered.

In [1] I. N. Vekua proposed a new method for investigating differential equations of elliptic type based on the employment of properties of a two-dimensional singular integral. This method was applied, by the author, in papers [2] and [3] to the study of systems of differential equations with discontinuous coefficients. In the present paper, this method is applied to systems of equations of the general form $(*)$. An inequality of Zygmund and Calderon lies at the basis of all our arguments, see [4]. This inequality makes possible, in contrast to the usual way with $p = 2$, to work in the space L_p, $p > 2$, which drastically simplifies proofs and leads to more precise results.

The applied method allows one to investigate the properties of the solutions of system $(*)$ directly relying on the properties of analytic functions. The theory of systems of type $(*)$ developed below, in a formal sense does not employ classical studies in the theory of partial differential equations of the second order or systems of equations of the first order; neither it is based on

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the theory of quasiconformal mappings. Conversely, in a suitable presentation, it can be considered as the foundation for a theory admitting the most general assumptions on the coefficients of the equations or characteristics of quasiconformal mappings (linear as well as quasilinear).

First of all, we study the structure of solutions and get various representation formulas for them that lead to numerous properties and precise estimates. These representation formulas establish a close connection between solutions of $(*)$ and analytic functions of one complex variable.

On the basis of these results, in § 5 we prove a general existence theorem which allows us to construct a solution of the system $(*)$ associated, in a natural way, with any prescribed analytic function. This existence theorem, as shown in the sequel, allows us to construct the required solutions in many important cases (quasiconformal mappings, fundamental solutions, the construction of solutions with prescribed singularities etc.)

We also study the connection of the generalized solutions of system $(*)$ $(a = b = c = d = e = f \equiv 0)$ with the geometric theory of quasiconformal mappings. The equivalence of the class of univalent generalized solutions of the systems $(*)$ and the so-called general quasiconformal mappings, see [5], is established. We also prove that the generalized solutions of systems $(*)$ $(a = b = c = d = e = f \equiv 0)$ with continuous coefficients coincide with the class of quasiconformal mappings in the sense of M.A. Lavrent'ev.

In § 6 the uniqueness theorem for quasiconformal mappings of simply connected domains onto the unit disk is proved. In the general case, the proof of uniqueness is here published for the first time. For the so-called $p-$elliptic systems, a proof is given in a paper by Gergen and Dressel [6]. Our proof also uses an idea of the authors mentioned. Note that, even for equations with smooth coefficients, our proof of uniqueness is constructed only when the theory of systems $(*)$ with discontinuous coefficients is available.

Some applications and complementary results are given in § 8. The following problems are considered: the Dirichlet problem for a disk, correctness of the Cauchy problem, behavior of the mapping at a boundary point and quasilinear equations. In particular, some theorems on uniqueness, preparing to the quasilinear equations, are proved as well.

To the best of our knowledge, the first work devoted to the study of elliptic systems with discontinuous coefficients is the paper by Morrey [7]. As a matter of fact, the Beltrami system was investigated by him. Later, the methods of Morrey were applied in works [6], [8], [9], [10] and [11]. With the same general assumptions as ours, the system $(*)$ was studied by Bers and Nirenberg [10]. In this work, some results are merely formulated and proofs are sketched. The authors proceed along a different route systematically considering solutions of the system $(*)$ as limits of the corresponding solutions of equations with smooth coefficients. The main result of these authors is identical with one of the forms of the representation formulas which may be deduced from our Theorem 4; theorem 4 contains a series of another versions of the representation formulas and estimates which are absent in [10].

A part of § 3 of the present work is a detailed account of paper [2].

1 Preliminaries

1. The class $W_p(G)$. Denote by $C^1(G)$ the class of all complex–valued functions of the complex variable $z = x + iy$ defined in domain G of the complex plane z and having continuous derivatives of the first order. For the complex–valued function $f(z) \in C^1(G)$, we define the differential operators

$$f_{\overline{z}} \equiv \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(f_x + if_y\right), \quad f_z \equiv \frac{\partial f}{\partial z} = \frac{1}{2}\left(f_x - if_y\right).$$

Then we can write the usual Riemann formula in the following form:

$$(1.1) \qquad \int\limits_{G_1} \frac{\partial f}{\partial \overline{z}}\, dG = \frac{1}{2i}\int\limits_{L} f\, dz \quad \text{or} \quad \int\limits_{G_1} \frac{\partial f}{\partial z}\, dG = -\frac{1}{2i}\int\limits_{L} f\, d\overline{z}$$

for any domain G_1, $\overline{G_1} \subset G$, bounded by the rectifiable contour L and for any function $f \in C^1(G)$. We have $f_{\overline{z}} \equiv 0$, $f_z = f'(z)$ for holomorphic functions in G.

Denote by $L_p(G)$ or L_p the class of all functions which are Lebesgue integrable with the exponent p, $p \geq 1$ in G; $L_1(G) = L(G)$. As usual, we write $f \in C^1_0(G)$ if $f \in C^1(G)$ and $f \equiv 0$ outside some (generally speaking, depending on f) compact subset of G.

In accordance with the formula (1.1), we introduce the class $W_p(G)$ of all functions having generalized partial derivatives of the first order in the Sobolev sense [12]. Namely, we say that an integrable function $f(z)$, $z \in G$, belongs to the class $W_p(G)$ if there exist functions ω_1, ω_2 which are L_p-integrable on every compact subset of the domain G, such that

$$\int\limits_{G} f\frac{\partial \varphi}{\partial \overline{z}}\, dG = -\int\limits_{G} \omega_1\varphi\, dG \ \text{ and } \int\limits_{G} f\frac{\partial \varphi}{\partial z}\, dG = -\int\limits_{G} \omega_2\varphi\, dG$$

for every function $\varphi(z) \in C^1_0(G)$. By definition, for $f \in W_p(G)$, we set

$$f_{\overline{z}} \equiv \frac{\partial f}{\partial \overline{z}} \equiv \omega_1, \quad f_z \equiv \frac{\partial f}{\partial z} \equiv \omega_2.$$

It is clear that, if $f \in C^1(G)$, then $f \in W_p(G)$, $p > 0$, and the generalized derivatives of the function $f(z)$ coincide with its usual derivatives. The basic calculus formulas for generalized derivatives are identical with the corresponding formulas for usual derivatives. Properties of functions in the class $W_p(G)$ have been investigated in detail in [12].

Theorem 1.1. *If $f \in W_p(G)$, $p \geq 1$, and $f_{\overline{z}} \equiv 0$, then f is holomorphic in G, i.e. f coincides with a holomorphic function after modification on a subset of measure zero.*

For the proof of this theorem, see note [14].

It is easy to verify the following two formulas often used below:

$$\overline{f_{\overline{z}}} = \left(\overline{f}\right)_z, \quad \overline{f_z} = \left(\overline{f}\right)_{\overline{z}}. \tag{1.2}$$

2. Some integral operators. In what follows, the properties of two special operators studied in this section are of great importance.

First of all we consider the integral operator

$$\varphi(z) \equiv T(\omega) \equiv -\frac{1}{\pi} \int\limits_G \frac{\omega(t)}{t - z}\, dG. \tag{1.3}$$

The integral (1.3) is well defined for any function $\omega \in L_p(G)$ and a.e. $z \in G$ and all $z \notin \overline{G}$. In the particular case, $G \equiv K$ where K is the unit disk, we will use the integral

$$\psi(z) \equiv T_2(\omega) \equiv -\frac{1}{\pi} \int\limits_K \frac{z\overline{\omega}(t)}{1 - z\overline{t}}\, dK\ . \tag{1.4}$$

The integral (1.4) is a holomorphic function of the variable z for $|z| < 1$. If $\omega \in L_p(K)$, then the integral (1.4) exists for a.e. $z \notin K$, as well.

The integrals (1.3) and (1.4) define the functions $\varphi(z)$ and $\psi(z)$ over the full complex plane of variable z. If $\omega \in L_p$, $p > 2$, then these functions are continuous. More precisely, for $p > 2$, we have the following inequalities

$$|\varphi(z)| \leq C||\omega||_{L_p}, \quad ||\omega||_{L_p} = \left(\int\limits_G |\omega|^p\, dG\right)^{\frac{1}{p}}, \tag{1.5}$$

and

(1.6) $$|\varphi(z+h)-\varphi(z)| \le C_1|h|^{\beta}||\omega||_{L_p}, \quad \beta = \frac{p-2}{p},$$

which hold for any complex z and h. The constants C and C_1 depend only on domain G and the number p. Similar inequalities also hold for the function $\psi(z)$ and can be obtained directly from (1.5) and (1.6) observing the formula $\psi(z) = -\overline{\varphi\left(\frac{1}{\overline{z}}\right)}$, valid for all z (if $G \equiv K$).

For the proofs of (1.5) and (1.6), see [12].

In the paper of Vekua [14], the following theorem is proved.

Theorem 1.2. *If $f(z)$ admits the generalized derivative $f_{\overline{z}} = \omega$, then $f(z)$ has the representation*

(1.7) $$f(z) = h(z) - \frac{1}{\pi}\int\limits_{G} \frac{\omega(t)}{t-z}\, dG$$

where the function $h(z)$ is holomorphic in G. Conversely, if $h(z)$ is holomorphic in G and $\omega \in L$, then the function $f(z)$ defined by (1.7), *possesses the generalized derivative $f_{\overline{z}}$ and $f_{\overline{z}} = \omega$.*

In particular, the formula (1.7) holds for functions in the class $W_p(G)$, $p \ge 1$, whose generalized derivatives are integrable in G.

Now we study the properties of some two-dimensional singular integrals.

Let the function ω be defined and integrable with exponent p, $p > 1$, on the whole plane E of variable z. The integral $-\frac{1}{\pi}\int\limits_{E} \frac{\omega(t)}{(t-z)^2}\, dE$, generally speaking, does not exist. However, we can consider its principal value in the sense of Cauchy: if E_δ denotes the plane E with the disk of radius $\delta > 0$ centered at z deleted, then the integrals

(1.8) $$f_\delta(z) = -\frac{1}{\pi}\int\limits_{E_\delta} \frac{\omega(t)}{(t-z)^2}\, dE$$

are well defined for any function $\omega \in L_p$, $p > 1$. The integrals (1.8) are a special case of the integrals in detail studied in [4]. From numerous results of that paper, we use only the following: for any $\omega \in L_p$, $p > 1$, the uniform estimate holds

$$||f_\delta||_{L_p} \le A'_p ||\omega||_{L_p} \tag{1.9}$$

where A'_p is an absolute constant (independent of δ), see ([4], Theorem 1).

It follows from (1.9) that the functions $f_\delta(z)$ converge in the metric of every L_p, $p > 1$, as $\delta \to 0$ to some function $f(z) \in L_p$. By definition, we call $f(z)$ the principal value in the sense of Cauchy of the integral $\int\limits_E \frac{\omega(t)}{(t-z)^2}\, dE$ and set

$$f(z) \equiv S(\omega) \equiv \lim_{\delta\to 0} f_\delta(z) \equiv -\frac{1}{\pi}\int\limits_E \frac{\omega(t)}{(t-z)^2}\, dE. \tag{1.10}$$

By (1.9) we get the following estimate for the principal value:

$$||S(\omega)||_{L_p} \le A'_p ||\omega||_{L_p}. \tag{1.11}$$

We know that the function $\varphi(z) = T(\omega)$ defined by the integral (1.3) has the generalized derivative with respect to $\overline{z}$ and $\varphi_{\overline{z}} = \frac{\partial T(\omega)}{\partial \overline{z}} = \omega$. Now, starting from the estimate (1.9) we prove that, if $\omega \in L_p$, $p > 1$, then $T(\omega)$ also has the generalized derivative with respect to z, which is expressed by the singular integral (1.10):

$$\frac{\partial T(\omega)}{\partial z} = S(\omega). \tag{1.12}$$

Indeed, let $\varphi \in C^1_0(G)$; then

$$\int\limits_G T(\omega)\frac{\partial\varphi}{\partial z}\, dG = -\lim_{\delta\to 0}\int\limits_G \omega(t)\left[\frac{1}{\pi}\int\limits_{|t-z|\ge\delta,\, z\in G} \frac{1}{t-z}\,\frac{\partial\varphi}{\partial z}\, dG_z\right] dG_t.$$

Now

$$\frac{1}{\pi}\int\limits_{|t-z|\ge\delta} \frac{1}{t-z}\,\frac{\partial\varphi}{\partial z}\, dG_z = -\frac{1}{\pi}\int\limits_{|t-z|\ge\delta} \frac{\varphi(z)}{(t-z)^2}\, dG_z - \frac{1}{2\pi i}\int\limits_{L_\delta} \frac{\varphi(z)}{t-z}\,\overline{dz}$$

where L_δ is a small circle of radius δ centered at point t. Therefore

$$\int\limits_G T(\omega)\frac{\partial\varphi}{\partial z}\, dG = \lim_{\delta\to 0}\frac{1}{\pi}\int\limits_G \varphi(z)\int\limits_{|t-z|\ge\delta} \frac{\omega(t)}{(t-z)^2}\; dG_t\, dG_z$$

since we have the uniform estimate

$$\left| \int\limits_{L_\delta} \frac{\varphi(z)}{t-z} \overline{dz} \right| \leq \left| \int\limits_{L_\delta} \frac{\varphi(z)-\varphi(t)}{t-z} \overline{dz} \right| \leq 2\pi\delta M$$

for some constant M. In view of (1.9) and (1.10) from the last formula we get

$$\int\limits_G T(\omega) \frac{\partial \varphi}{\partial z} \, dG = - \int\limits_G \varphi(z) S(\omega) \, dG,$$

i.e. the equality (1.12).

Incidentally, in view of Theorem 1.2, we deduce that, if a complex–valued function $f(z)$ has the derivative $f_{\overline{z}}$ in the class L_p, $p > 1$, then it has also the derivative f_z in L_p.

For the derivative $\Psi'(z)$ of the holomorphic function $\Psi(z)$ defined by the formula (1.4) for $|z| < 1$, we have the expression

$$S_2(\omega) = \Psi'(z) = -\frac{1}{\pi} \int\limits_K \frac{\overline{\omega(t)}}{(1 - z\overline{t})^2} \, dK. \tag{1.13}$$

We show now how, from inequality (1.9), we can obtain the inequality

$$||S_2(\omega)||_{L_p(K)} \leq B'_p ||\omega||_{L_p(K)} \tag{1.13$'$}$$

where B'_p is an absolute constant.

First, observe that

$$\Psi'\left(\frac{1}{\overline{\zeta}}\right) = \overline{\zeta}^{2}\, \overline{\Phi(\zeta)}, \quad |\zeta| > 1$$

where

$$\Phi(\zeta) = -\frac{1}{\pi} \int\limits_K \frac{\omega(t)}{(t-\zeta)^2} \, dK$$

and, moreover, in view of (1.9),

$$||\Phi||_{L_p} \leq A'_p ||\omega||_{L_p}.$$

The inequality (1.13$'$) follows from the following estimates:

$$\int\limits_K |\Psi'(z)|^p \, dK = \int\limits_{|z| \leq \frac{1}{2}} |\Psi'(z)|^p \, dK + \int\limits_{\frac{1}{2} < |z| < 1} |\Psi'(z)|^p \, dK \leq$$

$$\leq 4^p \pi^{\frac{p}{q}-p} ||\omega||_{L_p}^p + \int\limits_{\frac{1}{2}<|z|<1} |\zeta|^{2p} \, |\Phi(\zeta)|^p \, \frac{dK}{|\zeta|^4} \leq$$

$$\leq 4^p \pi^{\frac{p}{q}-p} ||\omega||_{L_p}^p + 2^{2p-4} (A_p')^p ||\omega||_{L_p}^p = (B_p')^p \cdot ||\omega||_{L_p}^p$$

since

$$\pi \cdot |\Psi'(z)| \leq 4 \int\limits_K |\omega| \; dK \leq 4\pi^{\frac{1}{q}} ||\omega||_{L_p}^p \quad \text{for} \quad |z| < \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the integrals (1.10) and (1.13) will be considered as linear operators in Banach spaces $L_p(G)$ and $L_p(K)$ respectively. The inequalities (1.11) and (1.13′) show that these operators are bounded. Denoting the norm of the operator $S(\omega)$ by A_p, we have:

$$(1.14) \qquad ||S(\omega)||_{L_p} \leq A_p ||\omega||_{L_p}.$$

Later on, we also consider the operator $S_1(\omega)$ defined in the space $L_p(K)$ by the formula

$$(1.15) \qquad S_1(\omega) = \frac{\partial T_1(\omega)}{\partial z} = -\frac{1}{\pi} \int\limits_K \left\{ \frac{\omega(t)}{(t-z)^2} + \frac{\overline{\omega(t)}}{(1-z\bar{t})^2} \right\} \; dK$$

where, by definition,

$$(1.16) \qquad T_1(\omega) = -\frac{1}{\pi} \int\limits_K \left\{ \frac{\omega(t)}{t-z} + \frac{\overline{z\omega(t)}}{1-z\bar{t}} \right\} \; dK.$$

In view of (1.11) and (1.13′), $S_1(\omega)$ is a bounded operator in $L_p(K)$ and, denoting its norm by B_p, we see that

$$(1.17) \qquad ||S_1(\omega)||_{L_p} \leq B_p ||\omega||_{L_p}.$$

The norm of operator $S_1(\omega)$ can be estimated in terms of the value of B_p'. However, the above estimates for B_p' are not precise. It should be explicitly stated that, in general, we have no approximate estimates for the constants A_p and B_p. Obtaining such estimates is related to a deeper study of integrals (1.12) and (1.18), a task to the best of our knowledge, not yet accomplished. For $p = 2$, the study of integrals (1.10) and (1.15) is essentially easier. This is shown in paper [1] where, in particular, the equality $A_2 = 1$ is proved. Now we will prove also that $B_2 = 1$.

For the proof, we remark that if $\omega \in C_0^1(K)$, then $S(\omega) = -\frac{1}{\pi}\int\limits_K \frac{\omega(t)}{(t-z)^2}\, dK$ is a very good function. For example, it is continuous. Indeed, this follows from equalities:

$$S(\omega) = -\frac{1}{\pi}\lim_{\delta\to 0}\int\limits_{K_\delta} \frac{\omega(t)}{(t-z)^2}\, dK =$$

$$= -\frac{1}{\pi}\lim_{\delta\to 0}\left[\int\limits_{K_\delta} \frac{\partial\omega}{\partial t}\cdot\frac{1}{t-z}\, dK - \frac{1}{2\pi i}\int\limits_{L_\delta}\frac{\omega(t)}{t-z}\,\overline{dt}\right] = -\frac{1}{\pi}\int\limits_K \frac{\partial\omega}{\partial t}\cdot\frac{1}{t-z}\, dK$$

where K_δ is the unit disk with a small disk of radius δ centered at z, bounded by the circle L_δ, deleted. In particular, we get

$$\frac{\partial S(\omega)}{\partial\overline{z}} = \frac{\partial S_1(\omega)}{\partial\overline{z}} = \frac{\partial\omega}{\partial z}. \tag{1.18}$$

Let now $\omega \in C_0^1(K)$; then, in view of (1.1),

$$\|S_1(\omega)\|^2_{L_2} = \int\limits_K S_1(\omega)\overline{S_1(\omega)}\, dK = \int\limits_K \frac{\partial T_1(\omega)}{\partial z}\overline{S_1(\omega)}\, dK =$$

$$= -\frac{1}{2i}\int\limits_L T_1(\omega)\overline{S_1(\omega)}\, d\overline{z} - \int\limits_K T_1(\omega)\frac{\partial\overline{S_1(\omega)}}{\partial z}\, dK$$

where L is the boundary of the unit disk. Consider each of the integrals in the right hand side of the last formula separately. Taking into account (1.2) and (1.18), the second of these integrals can be transformed as:

$$\int\limits_K T_1(\omega)\frac{\partial\overline{S_1(\omega)}}{\partial z}\, dK = \int\limits_K T_1(\omega)\frac{\partial\overline{\omega}}{\partial\overline{z}}\, dK = -\int\limits_K \frac{\partial T_1(\omega)}{\partial\overline{z}}\overline{\omega}\, dK = -\int\limits_K |\omega|^2\, dK.$$

For the first we have:

$$\int\limits_T T_1(\omega)\overline{S_1(\omega)}d\overline{z} = -\overline{\int\limits_L T_1(\omega)S_1(\omega)dz} = -\frac{1}{2}\overline{\int\limits_L \frac{\partial}{\partial z}\left[T_1(\omega)\right]^2\, dz} = 0$$

since it is obvious that $\overline{T_1(\omega)} = -T_1(\omega)$ on L and $T_1(\omega)$ is a holomorphic function in the neighborhood of line L because $\omega \in C_0^1$. Thus, for $\omega \in C_0^1(K)$, we have

$$\|S_1(\omega)\|_{L_2} = \|\omega\|_{L_2}.$$

Hence it follows that $B_2 = 1$ in view of the density of the set $C_0^1(K)$ in space $L_2(K)$.

By similar transformations we can get the equality $A_2 = 1$. For that, it suffices to take as ω continuously differentiable functions vanishing outside of a sufficiently large disk.

Some information on the constants A_p and B_p as functions of p, for $p \neq 2$, can be deduced from M. Riesz's theorem on convexity, see [15]. According to this theorem, A_p and B_p are logarithmically convex functions in p; in particular, they are upper semicontinuous. Hence we conclude that

$$A_p \leq 1 + \varepsilon \quad \text{and} \quad B_p \leq 1 + \varepsilon \tag{1.19}$$

as soon as $2 \leq p < 2 + \delta(\varepsilon)$ for small enough δ. The inequalities (1.19) are of fundamental significance for our purposes.

3. Differentiability of functions in the class $W_p(G)$. It is said that the complex–valued function $f(z)$ has a differential in the Stolz sense, or,equivalentely, is totally differentiable at the point z if there exist constants A and B such that

$$f(z+h) - f(z) = Ah + B\overline{h} + o(h) = f_x(z)\triangle x + f_y(z)\triangle y + o(h)$$

$$h = \Delta x + i\Delta y$$

It is known that, if $f(z)$ has first generalized derivatives in the Sobolev sense, which are integrable, then $f(z)$ has the usual partial derivatives a.e. However, generally speaking, the inclusion $f(z) \in W_p(G)$ does not guarantee the existence of the total differential of $f(z)$ in the Stolz sense on a set of full measure.

The following criterion for the existence of a Stolz differential on a set of full measure was proved by V.V. Stepanov: the function $f(z)$ defined in domain G has the differential in the Stolz sense a.e. in G if and only if

$$\overline{\lim_{|h|\to 0}} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty, \quad \text{a.e. in } G.$$

Using Stepanov's result we now present the proof of the following theorem.

Theorem 1.3. *If $f(z) \in W_p(G)$, $p > 2$, then f has the differential in the Stolz sense for a.e. $z \in G$.*

Proof. In view of Theorem 1.2, it is sufficient to consider the case when $f(z)$ is represented by the integral (1.3). Then the difference quotient takes the form

$$\frac{f(z+h) - f(z)}{h} = -\frac{1}{\pi} \int\limits_G \frac{\omega(t)dG}{(t-z)(t-z-h)} = -\frac{1}{\pi} \int\limits_{|t-z| \leq 3|h|} \frac{\omega dG}{(t-z)(t-z-h)} -$$

$$-\frac{1}{\pi}\int\limits_{\substack{|t-z|\geq 3|h|\\ t\in G}}\frac{\omega dG}{(t-z)(t-z-h)}\equiv\varphi_h(z)+\psi_h(z).$$

We first estimate integral $\varphi_h(z)$ at each point z where the integral $\int|\omega(t)-\omega(z)|^p\,dG$ is differentiable and has derivative equal to zero. At these points

$$\varepsilon(h)=\left[\frac{1}{9\pi h^2}\int\limits_{|t-z|\leq 3|h|}|\omega(t)-\omega(z)|^p dG\right]^{\frac{1}{p}}\to 0$$

as $h\to 0$.

As is known, points with this property form a set of full measure. Thus we get:

$$\pi\cdot\varphi_h(z)=\int\limits_{|t-z|\leq 3|h|}\frac{\omega(t)-\omega(z)}{(t-z)(t-z-h)}dG\;+$$

$$+\;\omega(z)\int\limits_{|t-z|\leq 3|h|}\frac{dG}{(t-z)(t-z-h)}=I_1+I_2.$$

The first integral is estimated by the Hölder inequality:

$$|I_1|\leq\left\{\int\limits_{|t-z|\leq 3|h|}|\omega(t)-\omega(z)|^p dG\right\}^{\frac{1}{p}}\left\{\int\limits_{|t-z|\leq 3|h|}\frac{dG}{|t-z|^q|t-z-h|^q}\right\}^{\frac{1}{q}}\leq$$

$$\leq\varepsilon(h)\cdot 3^{\frac{2}{p}}\left\{\int\limits_0^3\frac{d\rho}{\rho^{q-1}}\int\limits_0^{2\pi}\frac{d\theta}{|1-\rho' e^{i\theta}|^q}\right\}^{\frac{1}{q}}=C\cdot\varepsilon(h)$$

since $q<2$ and the last integral is convergent. The integral I_2 can be directly calculated and it is equal to $\pi\omega$. Therefore

$$\limsup_{h\to 0}|\varphi_h(z)|\leq|\omega|<+\infty,\quad \text{a.e.}$$

To prove the analogues statement for $\psi_h(z)$, note the following inequality:

$$(1.20)\qquad ||f_{n,h}||_{L_2}\leq\frac{C_1}{n}\cdot\frac{1}{(3|h|)^n}\,||\omega||_{L_2},\quad n>0$$

where

$$f_{n,h}=\int\limits_{|t-z|\geq 3|h|}\frac{\omega(t)}{(t-z)^{n+2}}\,dG$$

and C_1 is a constant. The simplest way to prove (1.20) is to remark, see [4], that $f_{n,k}$ is a convolution of the function ω with the function

$$g_n^h(t) = \begin{cases} \frac{1}{t^{2+n}}, & \text{for } |t| \geq 3|h|, \\ 0, & \text{for } |t| < 3|h|, \end{cases}$$

$$f_{n,h} = \omega * g_n^h.$$

By the Fourier transform we get

$$\widetilde{f}_{n,h} = \widetilde{\omega} \cdot \widetilde{g_n^h},$$

and hence, using the boundedness of $\widetilde{g_n^h}$ and Parseval's theorem, we come to (1.20).

Now, expanding $\psi_h(z)$ as the series in powers of h, we get that

$$\psi_h = \int\limits_{|t-z|\geq 3|h|} \frac{\omega(t)}{(t-z)^2}\, dG + \sum_{n=1}^{\infty} h^n \int\limits_{|t-z|\geq 3|h|} \frac{\omega(t)}{(t-z)^{n+2}}\, dG.$$

Hence, in view of (1.19) and (1.20),

$$||\psi_h||_{L_2} \leq A_2||\omega||_{L_2} + \sum_{n=1}^{\infty} \frac{C_1}{n} \cdot \frac{|h|^n}{(3|h|)^n}||\omega||_{L_2} \leq C_2||\omega||_{L_2}$$

where C_2 is independent of h. In view of the Fatou lemma, $\overline{\lim\limits_{h\to 0}}\, |\psi_h|$ is square integrable and hence it is finite a.e. Thus, we have proved that

$$\overline{\lim_{h\to 0}} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty$$

a.e. in G. The proof is completed by the application of Stepanov's theorem recalled above.

Remark. It follows from the remarks at the beginning of this section and from (1.12) that, for any function of type (1.3), $\omega \in L_p$, $p > 2$, the following representation

$$f(z+h) - f(z) = \omega \cdot \overline{h} + S(\omega) \cdot h + o(h) \tag{1.21}$$

holds a.e. Using deeper results from [4] than the inequality (1.9), the estimate (1.21) may be proved directly without reference to the aforementioned theorem of Stepanov (see [4], p. 137.)

2 Definition of the generalized solutions

1. The complex form of the system. The linear system of two equations with two unknown functions $u(x, y)$ and $v(x, y)$ of the form

$$v_y = \alpha u_x + \beta u_y + au + bv + e,$$

$$-v_x = \gamma u_x + \delta u_y + cu + dv + f \tag{2.1}$$

is called a system of *elliptic type* if

$$\alpha > 0, \quad K^2 = 4\alpha\delta - (\beta + \gamma)^2 > 0. \tag{2.2}$$

We consider the system (2.1) in the planar domain G, where the coefficients α, β, γ, δ, a, b, c, d, e, f as functions of $z = x + iy$ are defined. The system (2.1) is said to be *uniformly elliptic* in the domain G if the coefficients α, β, γ, δ are uniformly bounded and satisfy the inequality:

$$4\alpha\delta - (\beta + \gamma)^2 \geq K_0 > 0 \tag{2.3}$$

where K_0 is a fixed constant. We will mostly study uniformly elliptic systems.

As will be clear from what follows, the study of the system (2.1) is essentially simplified if we represent it in the complex form. For this aim, multiply the second of the equations in (2.1) by i and add it to the first one. Then, introducing the complex–valued function $w(z) = u + iv$, and using the notations introduced above, we get the complex equation

$$\overline{w}_z(1 - \lambda) - w_{\overline{z}}(1 + \lambda) - w_z\mu - \overline{w_{\overline{z}}}\mu = A_1 w + B_1\overline{w} + C_1 \tag{2.4}$$

where

$$\lambda = \frac{\alpha + \delta + i(\gamma - \beta)}{2}, \quad \mu = \frac{\alpha - \delta + i(\gamma + \beta)}{2} \; .$$

This equation is equivalent to a simpler one. Indeed, taking the complex conjugate to (2.4), we get

$$w_z(1 - \overline{\lambda}) - \overline{w_{\overline{z}}}(1 + \overline{\lambda}) - \overline{w}_z\overline{\mu} - w_{\overline{z}}\overline{\mu} = \overline{A}_1\overline{w} + \overline{B_1}w + \overline{C_1}. \tag{2.5}$$

We can exclude $\overline{w_{\overline{z}}}$ from formulae (2.4) and (2.5); then we get

$$(2.6) \qquad w_{\overline{z}} - q_1(z)w_z - q_2(z)\overline{w_z} = Aw + B\overline{w} + C$$

where the coefficients q_1 and q_2 are equal, respectively, to

$$q_1(z) = \frac{2\mu}{|\mu|^2 - |1+\lambda|^2}\ , \quad q_2(z) = -\frac{|\mu|^2 + (1+\overline{\lambda})(1-\lambda)}{|\mu|^2 - |1+\lambda|^2}\ .$$

The coefficients A, B, C are obtained from a, b, c, d, e, f by addition and multiplication by constants and the function $\frac{1}{|\mu|^2 - |1+\lambda|^2}$. In view of the inequality (2.3) this function is bounded in domain G. Indeed, by a simple calculation, we obtain

$$||\mu|^2 - |1+\lambda|^2| = 1 + \frac{K^2}{4} + \frac{(\beta-\gamma)^2}{4} + \alpha + \delta \geq 1 + \frac{K_0}{4}.$$

We similarly check that

$$|q_1| + |q_2| = \frac{\sqrt{(\alpha+\delta) - K^2} + \sqrt{(1+\Delta)^2 - K^2}}{1 + \Delta + (\alpha+\delta)}\ , \quad \Delta = \alpha\delta - \beta\gamma\ .$$

¿From the last formula we see that the uniform ellipticity condition of the system (2.1) can be written in the form of the inequality (q_0 constant)

$$(2.7) \qquad |q_1| + |q_2| \leq q_0 < 1$$

valid for all $z \in G$.

Observe that, if a, b, c, d, e, f are bounded in G or in some $L_p(G)$, then A, B, C are bounded or in $L_p(G)$ as well.

In the sequel we only consider complex equations of the type (2.6) with coefficients satisfying (2.7).

2. Definition of solutions. In what follows we study equation (2.6), where the following assumptions mainly hold:

1) Domain G is bounded.

2) $q_1(z)$ and $q_2(z)$, defined in G, are measurable complex–valued functions in variable z satisfying the inequality:

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1$$

a.e. in G.

3) A, B, $C \in L_p(G)$ where p is some fixed number, $p > 2$.

The restriction 1) is imposed for simplicity only; if the boundary of domain G contains a continuum, we may get rid of 1) by applying elementary conformal transformations. The restrictions on q_1 and q_2 are essential. Without these restrictions the complexity of the theory drastically increases: in general, it can occur that the equation (2.6) has no bounded solutions etc. Without the above restrictions on A, B, C, equation (2.6) may have no continuous solutions. Therefore, the imposed restrictions seem well justified. Moreover, even under essentially stronger restrictions on the coefficients than in 2), equation (2.6) may have no continuously differentiable solutions (for example, if $A \equiv B \equiv C \equiv 0$, and q_1 and q_2 are continuous, see [16]).

Therefore, when studying systems of type (2.6) with continuous or, even more so, discontinuous coefficients, it is necessary to extend the concept of the solution. We accept the following definition.

A function $w = w(z)$ is said to be a *generalized regular solution (or just solution)* of equation (2.6) if:

1) $w = w(z)$ is defined a.e. in G;

2) $w = w(z)$ belongs to $W_2(G)$;

3) w, $w_{\bar{z}}$ and w_z satisfy the equation (2.6) for a.e. $z \in G$.

In what follows, we prove that, under our assumptions, any generalized solution of equation (2.6) is continuous (i.e. coincides a.e. with a continuous function) and belongs to the class $W_p(G)$ for some $p > 2$.

Besides regular generalized solutions, our methods allow us to study solutions with isolated singularities of the polar type or even essential singularities. Such solutions will not belong to the class $W_p(G)$ but to some $W_p(G')$ where $G' \subset G$ is a domain containing no singularities of the solution.

In what follows, it is shown that equation (2.6) always admits generalized solutions in G.

Moreover, many problems that may be asked in a reasonable way and naturally arise for system (2.6), admit a solution only in the sense given above. The generalized solutions, in the above sense, preserve many important properties of analytic and continuously differentiable functions.

Remark. It is easy to see, in view of Remark 1, p. 24, that we may assume, without extending the class of generalized solutions, that

$$w(z) \in W_{p'}, \quad 2 \geq p' > \frac{p}{p-1}, \quad \frac{p}{p-1} < 2$$

3 The Beltrami systems

In this section, we study equation (2.6) of the special form:

$$w_{\overline{z}} - q(z) w_z = 0. \tag{3.1}$$

We call it the *Beltrami equation.* In the real form it corresponds to the Beltrami system

$$\begin{aligned} v_y &= \alpha u_x + \beta u_y, \\ -v_x &= \beta u_x + \delta u_y, \\ \alpha\delta - \beta^2 &\equiv 1. \end{aligned}$$

It is convenient to consider equation (3.1) in the full complex plane E. Therefore, we extend the coefficient $q(z)$ in equation (3.1), defined in the bounded domain G, setting

$$q(z) \equiv 0 \quad \text{outside of } G \tag{3.2}$$

Then the inequality (2.7) holds for all z.

1. A special solution of the equation (3.1). Let K be a fixed disk containing the domain G strictly inside; $q \equiv 0$ outside of K. We denote by $L_p^0(K)$ the class of complex–valued functions which are integrable with exponent p on K and vanish outside of K. Let $\omega \in L_p^0(K)$. Then the function

$$\varphi(z) = T(\omega) \equiv -\frac{1}{\pi} \int\limits_K \frac{\omega(t)}{t-z}\, dK \;=\; -\frac{1}{\pi} \int\limits_E \frac{\omega(t)}{t-z}\, dE, \tag{3.3}$$

belongs to the class $W_p(E)$ in view of Theorem 1.2 and formula (1.12). Moreover,

$$\varphi_{\overline{z}} = \omega, \quad \varphi_z = S(\omega) = -\frac{1}{\pi} \int\limits_E \frac{\omega(t)}{(t-z)^2}\, dE \tag{3.4}$$

It follows from these equalities that the function

$$\chi(z) = z + T(\omega) = z - \frac{1}{\pi} \int\limits_K \frac{\omega(t)}{t-z}\, dK \tag{3.5}$$

is a generalized (regular) solution of equation (3.1) if and only if ω satisfies the singular integral equation

$$\omega - qS(\omega) = q. \tag{3.6}$$

We see from (1.11), (1.19) and (2.7) that the norm of operator $q(z)S(\omega)$ in $L_p(K)$ does not exceed the number $q_0 A_p$ and, for p which are close enough to 2, $p > 2$,

$$q_0 A_p < 1 \quad \text{for} \quad |2 - p| < \delta \tag{3.7}$$

for small enough δ. Therefore equation (3.6) has a unique solution in any space $L_p(K)$ for p satisfying the condition (3.7). This solution can be calculated by the method of successive approximations. Since ω belongs to $L_p(K)$, for p satisfying (3.7), the solution (3.5) belongs to $W_p(K)$ as well; in particular, it is Hölder continuous.

2. Equation (3.1) with the coefficient satisfying the Hölder condition. Such equations were repeatedly studied by various methods (see [17]). We use only some facts proven in a direct way by I. N. Vekua.

The idea of integrating the differential equation (3.1) using the singular equation (3.6) belongs to Vekua. Assuming that $q(z)$ satisfies the Hölder condition in the whole plane ($q(z) \equiv 0$ outside of K) I. N. Vekua [1] proves the following:

1) formula (3.5), if ω is the solution of equation (3.6), represents a continuously differentiable solution of the equation (3.1) mapping homeomorphically the plane z onto the plane w;

2) the Jacobian of mapping (3.5) is not zero for every z.

The last statement immediately extends to all univalent solutions of equation (3.1).

3. Some properties of the solution constructed in section 1. The following theorem generalizes the above result of I. N. Vekua to an arbitrary system of the form (3.1).

Theorem 3.1. *Let $q(z)$ be a measurable function, defined for all z, such that*

$$q(z) \equiv 0 \quad z \in CK = E \setminus K)$$

and

$$|q(z)| \leq q_0 < 1, \quad q_0 - \textit{a constant.}$$

Then the function $\chi(z)$ defined by formulas (3.6) *and* (3.5) *realizes a homeomorphic mapping of plane z onto plane χ; the function $\chi = \chi(z)$ and its inverse satisfy the Hölder condition in the full plane with the Hölder constant and Hölder exponent depending only on q_0 and the disk K.*

Proof. Let $q_n(z)$ $(n = 1, 2, \ldots)$ be a sequence of continuously differentiable functions defined in the full plane and satisfying the conditions

$$q_n(z) \to q(z), \quad |q_n| < q_0 \quad \text{a.e.},$$

$$q_n(z) \equiv 0 \quad z \in CK. \tag{3.8}$$

We may construct such a sequence, for instance, by convolutional averaging of the function $q(z)$. In view of (3.8), we have

$$||q_n - q||_{L_p} \to 0 \tag{3.9}$$

for each $p > 0$. Consider the sequence of functions

$$\chi_n(z) = z + T(\omega_n), \quad \frac{\partial \chi_n}{\partial \bar{z}} - q_n \frac{\partial \chi_n}{\partial z} = 0 \tag{3.10}$$

where ω_n is the solution of the integral equation

$$\omega_n - q_n S\omega_n = q_n. \tag{3.11}$$

It is obvious that

$$\omega_n \equiv 0 \quad \text{in} \quad CK.$$

¿From (3.11) we obtain the estimate:

$$||\omega_n||_{L_p} \leq q_0 A_p ||\omega||_{L_p} + ||q_n||_{L_p}$$

or, for p satisfying the condition (3.7),

$$||\omega_n||_{L_p} \leq \frac{||q_n||_{L_p}}{1 - q_0 A_p} < \frac{C}{1 - q_0 A_p} \tag{3.12}$$

where C is a constant which does not depend on n neither p. ¿From (3.11) we get:

$$\omega_n - \omega_m = q_n S(\omega_n - \omega_m) + (q_n - q_m)S\omega_m + q_n - q_m$$

and hence for p satisfying (3.7),

$$(1 - q_0 A_p)||\omega_n - \omega_m||_{L_p} \leq ||q_n - q_m||_{L_p} + ||(q_n - q_m)S\omega_m||_{L_p}.$$

But

$$||(q_n - q_m)S\omega_m||_{L_p} \leq ||q_n - q_m||_{L_{p \cdot q'}} \, ||S\omega_m||_{L_{p \cdot p'}}, \quad \frac{1}{p'} + \frac{1}{q'} = 1.$$

Therefore, choosing p' close enough to 1, such that $q_0 A_{pp'} < 1$ and taking into account (3.7), (3.9) and (3.12), we see that

$$(3.13) \qquad ||\omega_n - \omega_m||_{L_p} \leq \varepsilon_{n,m} \cdot C_1, \quad \varepsilon_{n,m} \to 0 \text{ as } n, m \to \infty$$

where $\varepsilon_{n,m} = ||q_n - q_m||_{L_{p \cdot q'}}$ and C_1 is a constant (depending on p but not on n nor m).

Thus ω_n converge in the mean in the space L_p for p satisfying condition (3.7). Let

$$(3.14) \qquad \omega = \lim_{n\to\infty} \omega_n, \quad \omega \in L_p^0(K).$$

It is obvious that $\omega - qS\omega = q$. Setting $\chi(z) = z + T(\omega)$, we conclude, using (1.5) and (3.10), that

$$(3.15) \qquad \chi_n(z) \to \chi(z)$$

uniformly in the whole plane. $\chi(z)$ is a generalized solution of the equation (3.1), which is Hölder continuous and belongs to the class $W_{p,\mathrm{loc}}(E)$ for some $p > 2$.

According to the above result of Vekua, $\chi_n(z)$ is a continuously differentiable homeomorphism of plane z onto plane χ. Thus, $\chi(z)$ is the limit of the uniformly convergent sequence $\chi_n(z)$ of homeomorphisms of plane z onto plane χ. We prove that $\chi = \chi(z)$ is also a homeomorphism. For this aim consider the sequence of the continuously differentiable functions $z = z_n(\chi)$ which are inverses to the functions of the sequence (3.10). We have

$$(3.16) \qquad z_n(\chi_n(z)) \equiv z \quad \text{and} \quad \chi_n(z_n(\chi)) \equiv \chi(z)$$

for all z and χ.

It is easy to verify the following formulas:

$$(3.17) \qquad \overline{\frac{\partial z_n}{\partial \chi}} = \frac{1}{J_n} \cdot \frac{\partial \chi_n}{\partial z}, \quad \frac{\partial z_n}{\partial \overline{\chi}} = -\frac{1}{J_n} \cdot \frac{\partial \chi_n}{\partial \overline{z}},$$

(where J_n is the Jacobian of the transformation $\chi = \chi_n(z)$) from which, in view of (3.10) it follows that $z_n(\chi)$ satisfies the quasilinear equation

$$(3.18) \qquad \frac{\partial z_n}{\partial \overline{\chi}} + q_n(z_n(\chi)) \overline{\frac{\partial z_n}{\partial \chi}} = 0.$$

¿From (1.5) and (3.11) we deduce that

$$|\chi_n(z) - z| < M \tag{3.19}$$

with a constant M independent of n and z. Thus, in view of (3.17), we see that $\frac{\partial z_n}{\partial \overline{\omega}} \equiv 0$ outside of some fixed disk K_1, which does not depend on n. Therefore, by Theorem 1.2, $z_n(\chi)$ admits the representation

$$z_n(\chi) = \chi + f_n(\chi) + T(\widetilde{\omega}_n) = \chi + f_n(\chi) - \frac{1}{\pi} \int\limits_E \frac{\widetilde{\omega}_n(t)}{t - \chi}\, dE \tag{3.20}$$

where $\widetilde{\omega}_n \equiv 0$ outside of K_1 and the function $f_n(\chi)$ is holomorphic in the whole plane. For a fixed n, $T(\widetilde{\omega}_n)$ is bounded and $\lim\limits_{\chi \to \infty} T(\widetilde{\omega}_n) = 0$. In view of (3.19), $f_n(\chi)$ is also bounded, and $f_n(\chi) \to 0$ as $\chi \to \infty$, i.e. $f_n(\chi) \equiv 0$, since $\lim\limits_{z \to \infty} |\chi_n(z) - z| = 0$, i.e. $\lim\limits_{\chi \to \infty} |\chi - z_n(\chi)| = 0$. Thus, we can write (3.20) in the following form:

$$z_n(\chi) = \chi + T(\widetilde{\omega}_n). \tag{3.21}$$

¿From this formula and (3.18) we obtain the following equation for $\widetilde{\omega}_n$:

$$\widetilde{\omega}_n + q_n(z_n(\chi))\overline{S\widetilde{\omega}_n} = -q_n(z_n(\chi)) \tag{3.22}$$

where

$$|q_n(z_n(\chi))| \le q_0 < 1.$$

Hence, similarly to (3.12), we deduce the estimate

$$||\widetilde{\omega}_n||_{L_p} \le C_2 \tag{3.23}$$

for p satisfying condition (3.7) (C_2 does not depend on n).

In accordance with the inequality (1.6), the operator $T(\omega)$, mapping the space L_p into the space of functions satisfying the Hölder condition, is completely continuous. Therefore, we can choose a subsequence $z_{n_k}(\chi)$ converging uniformly to a function $z = z(\chi)$ satisfying the Hölder condition as well. Passing to the limit in (3.16) along the subsequence n_k, $k \to \infty$, we get $\chi(z(\chi)) \equiv \chi$ and $z(\chi(z)) \equiv z$, i.e. $\chi = \chi(z)$ is a homeomorphic mapping of the plane z onto the plane χ which, together with its inverse mapping, has all the desired properties. Thus the theorem is proved.

Remark. It follows directly from the uniqueness of the limit $z(\chi)$ and from the compactness of the sequence $z_n(\chi)$ that $z_n(\chi) \to z(\chi)$ uniformly.

The above constructed mapping f of plane z onto plane χ, in general, will not be continuously differentiable. We know only that it belongs to the class $W_{p,\mathrm{loc}}(E)$ for some $p > 2$. Therefore, a direct application of this mapping to problems of analysis, for instance to problems considered in [1] would be difficult. However, as we prove below, the mapping $\chi = \chi(z)$ has a series of properties of continuously differentiable mappings with respect to the main operations of analysis (integration, differentiability etc.).

Theorem 3.2. 1) *The image $\chi(\Omega)$ of any measurable set Ω of plane z is measurable and*

$$mes\ \chi(\Omega) = \int\limits_{\Omega} J_\chi(z)\ d\Omega$$

where J_χ is the generalized Jacobian of the transformation $\chi = \chi(z)$:

$$J_\chi = \left|\frac{\partial \chi}{\partial z}\right|^2 - \left|\frac{\partial \chi}{\partial \overline{z}}\right|^2$$

2) *The function $\varphi(z) = f(\chi(z))$ is integrable on Ω for any function $f(\chi)$ which is integrable on $\chi(\Omega)$ and*

$$\int\limits_{\chi(\Omega)} f(\chi)\ d\Omega_\chi = \int\limits_{\Omega} \varphi(z) J_\chi(z)\ d\Omega.$$

3) *Let Ω be an open set, $f \in W_2(\chi(\Omega))$ and continuous. Then the function $\varphi(z) = f(\chi(z))$ belongs to $W_2(\Omega)$ and*

$$\varphi_{\overline{z}} = f_\chi \chi_{\overline{z}} + f_{\overline{\chi}} \overline{\chi}_{\overline{z}}, \quad \varphi_z = f_\chi \chi_z + f_{\overline{\chi}} \overline{\chi}_z \quad a.e.\ in\ \Omega.$$

4) *The inverse function $z = z(\chi)$ of the function $\chi = \chi(z)$ also belongs to $W_{p,\mathrm{loc}}(E)$ for p satisfying condition* (3.7) *and it also has the properties* 1), 2), 3).

5) *Let*

$$J_z = \left|\frac{\partial z}{\partial \chi}\right|^2 - \left|\frac{\partial z}{\partial \overline{\chi}}\right|^2$$

then $J_z \cdot J_\chi \equiv 1$ a.e.; in particular, $J_z > 0$ and $J_\chi > 0$ a.e.

Proof. Let $J_n(z) = \left|\frac{\partial \chi_n}{\partial z}\right|^2 - \left|\frac{\partial \chi_n}{\partial \overline{z}}\right|^2$ be the Jacobian of the continuously differentiable mapping $\chi_n(z)$. Then the image $\chi_n(\Omega)$ is measurable and

$$mes\ \chi_n(\Omega) = \int_\Omega J_n\ d\Omega \tag{3.24}$$

for any measurable set Ω. Let $mes\ \Omega \leq C_0$ where C_0 is an arbitrary positive fixed constant. Applying the Hölder inequality and taking into account (3.12), (3.10) and (1.14), we obtain from (3.24) the estimate

$$mes\ \chi_n(\Omega) \leq \int_\Omega |1 + S(\omega_n)|^2\ d\Omega \leq 2\left(\int_\Omega\ d\Omega + \int_\Omega |S\omega_n|^2\ d\Omega\right) \leq$$

$$2\left(mes\ \Omega + A_p||\omega_n||_{L_p}(mes\ \Omega)^{\frac{1}{q}}\right) \leq C_1(mes\ \Omega)^{\frac{1}{q}},$$

$$\frac{1}{q} + \frac{2}{p} = 1, \quad p > 2,$$

i.e.

$$mes\ \chi_n(\Omega) \leq C(mes\ \Omega)^{\frac{1}{q}}, \tag{3.25}$$

where C is a constant depending on q_0, p and C_0, only. From (3.25) follows the estimate

$$mes^*\ \chi(\Omega) \leq C(mes\ \Omega)^{\frac{1}{q}} \tag{3.26}$$

where $mes^*\ \chi(\Omega)$ denotes the outer measure of the set $\chi(\Omega)$.

We first prove the inequality (3.26) under the assumption that Ω is an open set. Let $\chi_0 \in \chi(\Omega)$; then $\chi_0 = \chi_0(z_0)$, $z_0 \in \Omega$ and Ω contains some disk $K(z_0)$ centered at z_0 of radius $\delta > 0$. If z_1 is a point on the boundary of this disk, then, in view of (1.6), (3.23) and (3.21)

$$\delta = |z_0 - z_1| \leq C_3|\chi_n(z_0) - \chi_n(z_1)|^\beta$$

where C_3 and β do not depend on n. Since $\chi = \chi_n(z)$ is a homeomorphism, we conclude from this inequality that the image of the disk $K(z_0)$ covers some disk centered at $\chi_n(z_0)$ with a radius not less than $\left(\frac{\delta}{C_3}\right)^{\frac{1}{\beta}}$. But $\chi_n(z_0) \to \chi(z_0)$. Therefore $\chi_0 = \chi(z_0)$ belongs to all images $\chi_n(\Omega)$ starting from some large enough n. Thus we have proved the inclusion

$$\chi(\Omega) \subset \sum_{k=1}^{\infty} E_k$$

where $E_k = \prod_{n=k}^{\infty} \chi_n(\Omega)$. From this formula and from (3.25) we immediately obtain the estimate (3.26) for open sets.

Now, let Ω be an arbitrary measurable set. For every $\varepsilon > 0$ there exists an open set Ω' containing Ω, $\Omega \subset \Omega'$, such that $mes\,\Omega' < mes\,\Omega + \varepsilon$. Then $mes^{*}\chi(\Omega) \le mes\,\chi(\Omega') \le C\,(mes\,\Omega')^{\frac{1}{q}}$ and (3.26) follows in view of the arbitrariness of ε. Thus the proof of estimate (3.26) is complete.

Let Ω_0 be a rectangle, $mes\ \Omega_0 < C_0$. Using (3.25) and (3.26), we verify by simple arguments that

$$\lim_{n\to\infty} mes\,\chi_n(\Omega_0) = mes\,\chi(\Omega_0). \tag{3.27}$$

In view of the formula $J_\chi = |1+S\omega|^2 - |\omega|^2$ and (3.14) we conclude that there exists a subsequence $n_k \to \infty$ (as $k \to \infty$) such that

$$J_{n_k} \to J_\chi \quad \text{a.e.} \tag{3.28}$$

The estimate (3.25) applied to an arbitrary measurable set $e \subset \Omega_0$ states that the set of the integrals $\int_e J_n\ d\Omega$, $e \subset \Omega_0$, is absolutely equicontinuous and, therefore, passing to the limit along a subsequence n_k as $k \to \infty$ is justified [18]. In view of (3.27), we get

$$mes\,\chi(\Omega_0) = \int_{\Omega_0} J_\chi\ d\Omega. \tag{3.29}$$

In view of the estimate (3.26), the last formula is directly extended to arbitrary measurable sets Ω. Thus, the statement 1) of the theorem is proved.

2) The statement 2) follows from 1) by standard arguments.

3) As is known, statement 3) is true whenever $\chi = \chi(z)$ is a continuously differentiable mapping. Statement 3) has a local nature [12]. Hence it is sufficient to prove it in the neighborhood of every point of the set Ω.

We consider only the case of z-derivatives, $\overline{z}$-derivatives being quite similar. Let $\chi_0 \in \chi(\Omega)$, $\chi_0 = \chi(z_0)$, $z_0 \in \Omega$. Then for a large enough n, in view of the equicontinuity of $\chi_n(z)$, the sequence of the functions $\varphi_n(z) = f(\chi_n(z))$ is well defined on a small enough disk S centered at point z_0. Obviously

$$\frac{\partial \varphi_n}{\partial z} = f_\chi(\chi_n(z))\,\frac{\partial \chi_n}{\partial z} + f_{\overline{\chi}}(\chi_n(z))\,\frac{\partial \overline{\chi}_n}{\partial z} \tag{3.30}$$

from which we obtain the following estimate:

$$(3.30') \qquad \int\limits_S \left|\frac{\partial \varphi_n}{\partial z}\right|^2 d\Omega_z \le \int\limits_{\chi_n(S)} \frac{|f_\chi(\chi)(\chi_n)_z + f_{\overline{\chi}}(\chi)(\overline{\chi}_n)_z|}{J_n}\, d\Omega_\chi$$

$$\le 2 \int\limits_{\chi_n(S)} \frac{|f_\chi|^2 + q_0^2|f_{\overline{\chi}}|^2}{1-q_0^2}\, d\Omega_\chi$$

which is, obviously, uniformly bounded, since for all large enough n the sets $\chi_n(S)$ will be contained in some fixed closed subdomain of Ω. Now $\varphi_n(z) \to \varphi(z) = f(\chi(z))$ uniformly on S, and we easily check the correctness of the statement 3), (see [12]).

4) To prove the same facts for the inverse transformation $z = z(\chi)$, we first prove that

$$(3.31) \qquad ||\widetilde{q}_n(\chi) - \widetilde{q}(\chi)||_{L_p} \to 0$$

as $n \to \infty$ for any $p > 1$ where $\widetilde{q}_n(\chi) = q(z_n(\chi))$, $\widetilde{q}(\chi) = q(z(\chi))$. For this purpose we show that $\widetilde{q}_n(\chi)$ tends to $\widetilde{q}(\chi)$ in measure, i.e. for every $\varepsilon > 0$

$$(3.32) \qquad mes\ \chi(E\{|\widetilde{q}(\chi) - \widetilde{q}_n(\chi)| > \varepsilon\}) \to 0 \ \text{ as } \ n \to \infty.$$

In view of (3.8), we may consider only points χ which belong to $\chi(K)$ since we have $\widetilde{q}(\chi) = \widetilde{q}_n(\chi) = 0$ outside of K for large a enough n. By Egorov's theorem, for any prescribed σ and μ (which will be fixed later) there exists a set $K_{\sigma,\mu}$ such that

$$(3.33) \quad mes\ (K - K_{\sigma,\mu}) < \sigma, \quad |q_n(z) - q(z)| < \mu \ \text{ for } \ n > N(\sigma,\mu),\ z \in K_{\sigma,\mu}.$$

Changing the variable $\chi = \chi_n(z)$, we have

$$|q_n(z_n(\chi)) - q(z_n(\chi))| < \mu \ \text{ for } \ \chi \notin \chi_n(K - K_{\sigma,\mu}), \quad n > N$$

and, in view of (3.25),

$$(3.34) \qquad mes\ \chi_n(K - K_{\sigma,\mu}) < C\sigma^{\frac{1}{q}}.$$

On the other hand, using Lusin's theorem, for arbitrary σ_1, μ_1 and δ_1 there is a set $K_{\sigma_1,\mu_1,\delta_1}$ such that

(3.35)

$$mes\,(K - K_{\sigma_1,\mu_1,\delta_1}) < \sigma_1, |q(z) - q(z')| < \mu_1 \text{ for } |z - z'| < \delta_1,\ z, z' \in K_{\sigma_1,\mu_1,\delta_1}.$$

As observed in the remark on page 19, $z_n(\chi) \to z(\chi)$ uniformly in the whole plane. Therefore, for a large enough n, in view of (3.35), we have

$$|q(z_n(\chi)) - q(z(\chi))| < \mu_1 \tag{3.36}$$

for all χ such that

$$\chi \notin \chi_n(K - K_{\sigma_1,\mu_1,\delta_1}) \ \text{ and } \ \chi \notin \chi(K - K_{\sigma_1,\mu_1,\delta_1}), \tag{3.37}$$

i.e., according to (3.25) and (3.26), outside of the set of measure $2C\sigma_1^{\frac{1}{q}}$. Taking $\varepsilon > 0$ and $\delta > 0$ arbitrary and setting $\mu_1 = \mu = \frac{\varepsilon}{2}$, $\sigma_1 = \sigma = \left(\frac{\delta}{3C}\right)^q$, from (3.34), (3.35), (3.36) and (3.37) we get

$$|\widetilde{q}(\chi) - \widetilde{q}_n(\chi)| < \varepsilon$$

outside of a set whose measure is less than or equal to $2C\sigma_1^{\frac{1}{q}} + C\sigma^{\frac{1}{q}} = \delta$ for all n such that $|z_n(\chi) - z(\chi)| < \delta$ and for all $\chi \in \chi(K)$. Thus, we verified the correctness of (3.32) and, consequently, also (3.31). The obvious inequalities $|\widetilde{q}_n(\chi)| \le q_0$, $|\widetilde{q}(\chi)| \le q_0$ and $q_0 < 1$ together with (3.31), (3.22) and (3.18), as shown in the proof of Theorem 3.1, allow us to conclude that $z(\chi)$ belongs to $W_{p\mathrm{loc}}(E)$, satisfies the equation $z_{\overline{\chi}} + \widetilde{q}(x)\overline{z}_{\chi} = 0$ and has all the properties proved above for the function $\chi(z)$. Statement 4) of the Theorem 3.2 is thus demonstrated.

5) Statement 5) follows from (3.28) and the similar fact for J_z by passing to the limit in the equality $J_\chi^n \cdot J_z^n \equiv 1$.

The proof of Theorem 3.2 is complete.

Now we give two important supplements to the theorem.

Remark 1. Statement 3) of Theorem 3.2 can be proved in the following form: *if* $f(\chi) \in W_{p_1}(\chi(\Omega))$, $p_1 > \frac{p}{p-1}$, *then* $\varphi(z) = f(\chi(z)) \in W_{q_1}(\Omega)$, *where* $q_1 = \frac{p_1(p-2)}{p-p_1} > 1$; *if* $p_1 = 2$, *then* $q_1 = 2$. For that goal, only minor changes are required in our proof: in the estimation of the integral (3.30′) it is sufficient to use the Hölder inequality; instead of the uniform convergence of the sequence $\varphi_n(z) = f(\chi_n(z))$, $\varphi_n(z) \to f(\chi(z))$ we may use the convergence to measure what is sufficient for our proof (see [12]).

Remark 2. The corresponding formulation of statement 3) and the differentiation formula remain valid for the superposition of the form $\chi(f(\eta)) = \varphi(\eta)$. The proof in this case is the same as above.

4. A general solution of the Beltrami system. The theorems proved in the previous section allow us to study the properties of any solution of equation (3.1). In this connection, the following theorem is fundamental.

Theorem 3.3. *Let $w(z)$ be a regular generalized solution of equation* (3.1). *Then*

$$w(z) = f(\chi(z)) \tag{3.38}$$

where $f(\chi)$ is a holomorphic function in domain $\chi(G)$. The function $\chi = \chi(z)$ extends over the whole complex plane of variable z as a continuous, univalent, holomorphic function in the exterior of G. If $w(z)$ is a generalized solution admitting isolated singularities in G, then $w(z)$ is also represented in the form (3.38) *and all singularities of $w(z)$ are transferred to $f(\chi)$ while preserving the type (pole to pole, essential singularity to essential singularity). Conversely, any function of the form* (3.38) *is a generalized solution of equation* (3.1).

The proof of this theorem follows by direct checking from Theorem 1.1, statement 3), Theorem 3.2 and from the properties of the function $\chi = \chi(z)$ which have been proved above.

It follows from Theorem 3.3 that every regular solution of equation (3.1), satisfying (2.7), belongs to $W_p(G)$, $p > 2$, and is Hölder continuous on any compact subset of G. In general, formula (3.38) allows us to transfer to solutions of system (3.1) many properties of complex analytic functions (the argument principle, behavior in the neighborhood of singularities, unique continuation property, and many others).

The function $w = w(z)$ defined by (3.38) will be univalent if, and only if, the function $f(\chi)$ is univalent. From the known properties of the univalent functions and the properties of the mapping $\chi = \chi(z)$ formulated in Theorem 3.2, we deduce the following corollary.

Corollary 3.1. *All statements of Theorem* 3.2 *hold true for any mapping $w = w(z)$ of plane z onto plane w realized by an arbitrary univalent regular solution of the equation* (3.1).

Immediately we obtain also the following strengthening of Theorem 3.3.

Theorem 3.4. *Let $w(z)$ be an arbitrary univalent solution of equation* (3.1). *Then any other solution $W = W(z)$ of this equation may be represented in the form*

$$W(z) = f(w(z)) \tag{3.39}$$

where $f(w)$ is a complex analytic function in $w(G)$ (which admits, possibly, singularities). The converse statement is also true.

Theorems 3.3 and 3.4 allow us to construct solutions of equation (3.1) with the prescribed properties or to reduce the construction of such solutions to the construction of the corresponding complex analytic functions.

In particular, we directly obtain Riemann's theorem on the existence of mappings of an arbitrary domain G onto canonical domains: the unit disk for G simply connected or the unit disk with deleted subdisks for G multiply connected. Instead of the above mentioned, we can consider canonical domains of other types. When properly normalized these mappings are unique. This follows directly from uniqueness theorems for conformal mappings.

Not dwelling longer on the obvious and known corollaries to Theorems 3.3 and 3.4, we stress that the mapping $\chi = \chi(z)$ in Theorem 3.3 maps the whole plane z onto the whole plane χ. This remark is every essential in the study of boundary correspondence for solutions of the system (3.1).

5. Solutions of equation (3.1) mapping the unit disk onto itself. In this section K will denote the open unit disk centered at the origin. Theorem 3.3 makes it possible to construct solutions of system (3.1) mapping the disk K onto itself. It follows from this theorem, together with the Caratheodory extension theorem for conformal mapping of a domain bounded by a Jordan curve to the closed domain, that every such mapping can be extended by continuity to the closed unit disk with the condition of preserving the homeomorphism.

However, in the case of solutions of equation (3.1) mapping the unit disk onto itself, much deeper results hold true than those which may be directly deduced from Theorem 3.3. For instance, Theorem 3.3 allows us only to state that every such mapping satisfies the Hölder condition on any disk strictly contained in the interior of K. As a matter of fact the following holds:

Theorem 3.5. *Let $w(z)$ be a regular solution of equation* (3.1) *mapping the disc K onto itself. Then $w(z)$ satisfies the Hölder condition in the closed disk $\overline{K}$ and w_z and $w_{\overline{z}}$ are integrable in the disk K with some exponent $p > 2$.*

Proof. We already know that $w(z)$ can be extended by continuity to the closed disk. Using a device of M.A. Lavrent'ev, see [19], we consider the extension $w^*(z)$ of the mapping $w = w(z)$ to the exterior of the unit disk by the formula

$$w^*(z) = \begin{cases} w(z), & |z| < 1, \\ \frac{1}{\overline{w(\frac{1}{\overline{z}})}}, & 1 < |z| < 1 + \eta. \end{cases} \tag{3.40}$$

Formulas (3.40) define a continuous function for small enough $|z| < 1 + \eta$ for $\eta > 0$. In view of the statement 3) of the Theorem 3.2 the generalized derivatives of the function $w^*(z)$ exist for $|z| < 1$ and $1 < |z| < 1 + \eta$ and can be calculated by the formulas

$$\frac{\partial w^*}{\partial z} = \frac{\partial w}{\partial z} \ , \qquad \frac{\partial w^*}{\partial \overline{z}} = \frac{\partial w}{\partial z} \qquad \text{for} \ \ |z| < 1$$

and

$$\frac{\partial w^*}{\partial z} = \frac{w^{*2}}{z^2} \cdot \overline{\frac{\partial w}{\partial \zeta}(\zeta)} \ , \qquad \frac{\partial w^*}{\partial \overline{z}} = \frac{w^{*2}}{\overline{z}^2} \cdot \overline{\frac{\partial w}{\partial \overline{\zeta}}(\zeta)} \qquad \text{for} \ \ |z| > 1 \ , \quad \zeta = \frac{1}{\overline{z}}$$

which imply that $w^*(z)$ satisfies the equation

$$\text{(3.41)} \qquad \frac{\partial w^*}{\partial \overline{z}} - q^*(z)\frac{\partial w^*}{\partial z} = 0 \ , \qquad q^*(z) = \begin{cases} q(z), & \text{for } |z| < 1, \\ \overline{q(\zeta)}\frac{z^2}{\overline{z}^2}, & \text{for } |z| > 1, \end{cases}$$

in the disk $|z| < 1 + \eta$; $|q^*(z)| \leq q_0 < 1$. As is easy to see, $w^*(z)$ has generalized derivatives which are square integrable in $|z| < 1 + \eta$. Thus, $w^*(z)$ is a generalized solution of the equation (3.1) in the disk $|z| < 1 + \eta$. Then Theorem 3.5 can be deduced immediately from Theorem 3.3 since the disk $|z| \leq 1$ is strictly contained in the interiour of the domain $|z| < 1 + \eta$.

The following theorem is a very important refinement of Theorem 3.5.

Theorem 3.6. *Under the hypothesis of Theorem* 3.5 *and the additional condition* $w(0) = 0$, $||w_z||_{L_p}$, $||w_{\overline{z}}||_{L_p}$ *and the Hölder constant are uniformly bounded by quantities depending on* q_0 *only and the Hölder exponent is bounded from below by a constant only depending on* q_0.

Proof. Let $\chi = \chi(z)$ be a solution of the equation (3.1) of the form (3.5). Let $z = z(\chi)$ be its inverse function. We have proved that $z(\chi)$ and $\chi(z)$ satisfy the Hölder condition with a constant and an exponent depending on q_0 only. Denote them by C and α, respectively. We prove first that $w(z)$ satisfies the inequality

$$\text{(3.42)} \qquad |w(z)| < C_1 \ |z|^\alpha \qquad \text{for} \ \ |z| < \beta$$

with C_1, α and β depending only on q_0. Indeed, let $\chi(0) = \chi_0$. In view of the inequality

$$\text{(3.43)} \qquad 1 = |z| \leq C|\chi(z) - \chi(0)|^\alpha \ ,$$

we conclude that $\chi(K)$ contains the disk $K(\chi_0, \delta)$ centered at χ_0 with the radius $\delta = C^{-\frac{1}{\alpha}}$. In view of Theorem 3.3, our solution $w(z)$ has the form

$w(z) = f(\chi(z))$ where $f(\chi)$ is a holomorphic univalent function mapping $\chi(K)$ onto K such that $f(\chi_0) = 0$. In particular, $|f(\chi)| < 1$ for $\chi \in \chi(K)$ and, even more so, for $\chi \in K(\chi_0, \delta)$. From this we at once obtain the estimate $|f'(\chi)| \le M$ for $\chi \in K(\chi_0, \frac{\delta}{2})$ where M depends only on δ. But the preimage of the disk $K(\chi_0, \frac{\delta}{2})$ under the mapping $\chi = \chi(z)$ contains some disk centered at the origin of radius $\beta = \left(\frac{\delta}{2C}\right)^{\frac{1}{\alpha}}$. For z in this preimage, we have

$$|w(z)| = \left|\frac{f(\chi(z)) - f(\chi_0)}{\chi(z) - \chi(0)}\right| \cdot |\chi(z) - \chi_0| \le M \cdot C \cdot |z|^\alpha.$$

Thus inequality (3.42) has been proved for any solution of equation (3.1) mapping the unit disk onto itself provided that $w(0) = 0$. In particular, this is true for the inverse transformation $z = z(\chi)$ satisfying, in view of the corollary to Theorem 3.3, the following equation:

$$z_{\overline{\chi}} + \widetilde{q}(\chi)\overline{z_\chi} \equiv z_{\bar{\chi}} + q_1(\chi) z_\chi = 0$$

where

$$\widetilde{q}(\chi) = q\left(z(\chi)\right) \quad \text{and} \quad q_1(\chi) = \frac{\widetilde{q}(\chi)\overline{z_\chi}}{z_\chi}, \qquad |q_1| \le q_0 < 1.$$

Thus,

$$|z(w)| \le C_1 |w|^\alpha \quad \text{for} \quad |w| < \beta. \tag{3.44}$$

For $|z| < \left(\frac{\beta}{C}\right)^{\frac{1}{\alpha}}$, we have $|w(z)| \le \beta$ and, in view of (3.44),

$$|w(z)| \ge \left(\frac{|z|}{C_1}\right)^{\frac{1}{\alpha}} \quad \text{for} \;\; |z| < \beta_1. \tag{3.45}$$

The last inequality allows us easily to derive our statement using arguments applied in the proof of Theorem 3.5. Indeed, it follows from (3.45) that $|w(z)| \ge \gamma_1$ on the circle $|z| = \beta_1$, therefore, in view of (3.40), we have on the circle $|z| = \frac{1}{\beta} = 1 + \eta$ the inequality

$$|w^*(z)| \le \frac{1}{\gamma_1}\,. \tag{3.46}$$

In this estimate γ_1 and η depend on q_0 only. However, $w^*(z)$ is a solution of the equation (3.41), $w^*(z) = f_1(\chi_1(z))$ where $\chi_1(z)$ is the solution of the form (3.5) for the equation (3.41) and $f_1(\chi)$ is a complex analytic function in $\chi_1(K_\eta)$. Since $|q^*(z)| \le q_0 < 1$, $\chi_1(z)$ with its inverse satisfy the Hölder condition with an exponent α and a constant C_2 which depend on q_0 only (and on β_1 which, in turn, depends on q_0 only). We conclude from this that the distance between the boundaries of $\chi_1(K_\eta)$ and $\chi_1(K)$ is not less than some δ_1 depending on q_0 only. The function $f_1(\chi)$ is holomorphic in $\chi_1(K_\eta)$ and $|f_1(\chi)| < \frac{1}{\gamma_1}$ for $\chi \in \chi_1(K_\eta)$. Therefore we have the following inequality in $\chi(K_1)$:

$$|f_1'(\chi)| \le M_1 \tag{3.47}$$

where M_1 depends on γ_1 and δ_1 only. Thus,

$$|w(z) - w(z_1)| = |w^*(z) - w^*(z_1)| = |f_1(\chi_1(z)) - f_1(\chi_1(z_1))| \le M_1 \cdot C_2 |z - z_1|^\alpha$$

for each $z, z_1 \in K$ as was to be proved.

The other statements of the theorem follow from the inequalities

$$|w_z(z)| \le M_1 \left|\frac{\partial \chi_1}{\partial z}\right|, \quad |w_{\bar z}| \le M_1 \left|\frac{\partial \chi_1}{\partial \bar z}\right| \qquad \text{for every } z \in K\ .$$

The proof of Theorem 3.6 is complete.

Theorem 3.6 implies the following.

Corollary 3.2. *Under the assumptions of Theorem* 3.6, *for the measure of the image $w(E)$ of a measurable set E in K, the following estimate holds:*

$$mes\, w(E) \le (mes\, E)^{\frac{1}{q}} \tag{3.48}$$

where C and q depend on q_0 only.

Note also one more lemma extending the classical Schwartz lemma to solutions of equation (3.1).

Lemma 3.1. *Let $w(z)$ be a solution of equation* (3.1) *mapping the unit disk into itself. If $w(0) = 0$, then*

$$f^{-1}(r) \le |w(r)| \le f(r) \qquad \text{for} \quad |z| = r$$

where $f(r)$ is some monotone function of variable r satisfying the conditions: $f(0) = 0$, $f(r) < 1$ for $r < 1$ and $\lim_{r\to 1} f(r) = 1$; $f(r)$ depends only on q_0. If $w(z)$ is an arbitrary solution of equation (3.1) *in the disk $|z| \le 1$ such that $w(0) = 0$ and $|w(z)| \le 1$, then $|w(z)| \le f(r)$, $r = |z|$.*

The proof of this lemma follows from Theorem 3.6.

4 General representations of the system's solutions (2.6)

1. Homogeneous equations with the principal Beltrami part. These are equations of the following type:

$$w_{\overline{z}} - q(z)w_z = Aw + B\overline{w}. \tag{4.1}$$

Parallel with the solutions of system (4.1), we will consider solutions of the corresponding Beltrami equation, which in this section will be denote by f :

$$f_{\overline{z}} - q(z)f_z = 0. \tag{4.2}$$

Let equations (4.1) and (4.2) be given in a domain G. We prove the following theorem.

Theorem 4.1. *Let $w = w(z)$ be a generalized solution (possibly, admitting isolated singularities) of equation* (4.1)*. Then $w(z)$ is represented in the form*

$$w(z) = f(z)e^{T(\omega)} = f(z)e^{\varphi(z)} \tag{4.3}$$

where $f(z)$ is a solution of equation (4.2),

$$\varphi(z) = T(\omega) = -\frac{1}{\pi}\int\limits_G \frac{\omega(t)}{t-z}\,dG, \quad \omega \in L_p(G),\ p > 2;$$

The function $\varphi(z)$ extends by continuity to the whole complex plane as a holomorphic function in the exterior of G, vanishing at infinity.

Proof. Let $w(z)$ be the considered solution. Set

$$h(z) = \begin{cases} A + B\frac{\overline{w}}{w}, & \text{for } w(z) \neq 0,\ w(z) \neq \infty, \\ A + B, & \text{for } w(z) = 0 \text{ and at singular points.} \end{cases}$$

Consider the integral equation

$$\omega - qS\omega = h. \tag{4.4}$$

In view of the assumptions in § 2, $h(z) \in L_p$, $p > 2$ (p is defined by the inequality (3.7)) and the equation (4.4) has the unique solution $\omega \in L_p(G)$ for each $h \in L_p$.

Set

$$\varphi(z) = T(\omega) = -\frac{1}{\pi} \int\limits_G \frac{\omega(t)}{t-z}\, dG$$

and consider the function

$$f(z) = w(z)e^{-\varphi(z)}.$$

We have

$$f_{\overline{z}} = w_{\overline{z}}e^{-\varphi} - we^{-\varphi}\omega, \quad f_z = w_z e^{-\varphi} - we^{-\varphi}S\omega$$

hence

$$[f_{\overline{z}} - q(z)f_z]\, e^{-\varphi} = w_{\overline{z}} - w\,[\omega - qS\omega] - qw_z = w_{\overline{z}} - wh - qw_z = 0,$$

i.e.

$$f_{\overline{z}} - q(z)f_z = 0.$$

Thus, $f(z)$ is a solution of equation (4.2). But $w = fe^{\varphi}$ and this proves the representation (4.3). Other statements of Theorem 4.1 follow from the given formulas for $\varphi(z)$.

The representation of form (4.3) is unique if its components are required to have the properties defined in Theorem 4.1. Indeed, assuming that $w(z) = f(z)e^{\varphi(z)} = f_1(z)e^{\varphi_1(z)}$, we observe that $\frac{f(z)}{f_1(z)} = e^{\varphi_1(z)-\varphi(z)}$ is a solution of equation (4.2) which admits analytic extension to the whole plane and is equal to 1 at infinity. In view of Theorem 3.3 and the Liouville theorem, such a solution is equal to 1 identically. The uniqueness follows.

Theorem (4.3) is a generalization of the representation formula for solutions to equations (4.1) proved, for the first time by Teodorescu, for systems of the form $w_{\overline{z}} = aw$, see [20]. This representation was, independently of Teodorescu, generalized by I. N. Vekua and L. Bers to systems of the form $w_{\overline{z}} = aw + b\overline{w}$, see [21] and [22].

Representation (4.3) is not the only possible one. It is characterized by the property that the function $\varphi(z)$ in (4.3) may be continuously and complex analytically extended to the whole plane and vanishes at infinity. One can give other representations of the form (4.3) dropping the above requirements on the exponent $\varphi(z)$. For instance, the following theorem holds (in the particular case when G is the unit disk).

Theorem 4.2. *Let K be the unit disk. Then every solution of the equation* (4.1) *in K can be represented in the form*

$$w(z) = f(z)e^{\psi(z)} \tag{4.5}$$

where

$$\psi(z) = T_1(\omega) = -\frac{1}{\pi}\int\limits_K \left[\frac{\omega(t)}{t-z} + \frac{z\overline{\omega}(t)}{1-z\overline{t}}\right] dK, \quad \omega \in L_p(G), p > 2$$

and $\Re\psi(z) = 0$ *for* $|z| = 1$ *and* $f(z)$ *is a solution of the equation* (4.2). *Such a representation is unique.*

The proof of this theorem repeats the proof of the previous theorem with the difference that, instead of the integral equation (4.4), we solve the equation of the form $\omega - qS_1(\omega) = h$ where operator $S_1(\omega)$ is defined by (1.15).

2. Representations for solutions of general homogeneous equations of form (4.6). In this section we generalize the theorems proved to solutions of equations of the form

$$w_{\overline{z}} - q_1(z)w_z - q_2(z)\overline{w_z} = Aw + B\overline{w}. \tag{4.6}$$

Parallel with (4.6) we will also consider the equation

$$w_{\overline{z}} - q_1(z)w_z - q_2(z)\overline{w_z} = 0. \tag{4.7}$$

We first consider equation (4.7). If $w = w(z)$ is its generalized solution, then $w(z)$ satisfies the equation

$$w_{\overline{z}} - q_0(z)w_z = 0 \tag{4.8}$$

where

$$q_0(z) = q_1(z) + q_2(z)\tfrac{\overline{w_z}}{w_z} \qquad \text{for} \quad w_z \neq 0,$$
$$q_0(z) = q_1(z) + q_2(z) \qquad \text{for} \quad w_z = 0.$$

Evidently $|q_0(z)| \leq q_0 < 1$ where q_0 is a constant. The following theorem follows directly from Theorems 3.3 and 3.4.

Theorem 4.3. *Any generalized solution of equation* (4.7) *can be represented in the form*

$$w(z) = f(\chi(z)) \tag{4.9}$$

where $f(\chi)$ *is a complex analytic function in* $\chi(G)$ *and* $\chi = \chi(z)$ *is an univalent solution of equation* (4.8). *It can be chosen in various ways, in particular, we may assume that* $\chi(z)$ *has the form* (3.5)*; we can also suppose that* $\chi(G)$ *is a canonical domain (the unit disk provided that* G *is simply connected and, in the general case, the plane with cuts etc.).*

In the same way, we obtain the following theorem from Theorems 3.4, 4.1 and 4.2:

Theorem 4.4. *Any generalized solution of equation* (4.6) *can be represented in the form*

$$w(z) = f(\chi(z))e^{\varphi(z)} \tag{4.10}$$

where $f(\chi)$ *is a complex analytic function in* $\chi(G)$ *and* $\chi = \chi(z)$ *is a univalent solution of equation* (4.8). *It can be chosen in various ways, in particular, we can require that* $\chi(G)$ *is either the unit disk (for a simply connected domain) or a canonical domain of another type.* $\chi(z)$ *can be also chosen in the form of* (3.5). $\varphi(z)$ *can also be chosen in various ways; in particular, as in Theorems* 4.1 *and* 4.2. *In both cases,* $\varphi(z)$ *is Hölder continuous in the closed domain* $\overline{G}$, $\varphi_{\overline{z}}$, $\varphi_z \in L_p(G)$, $p > 2$, *and the integrals* $\|\varphi_{\overline{z}}\|_{L_p}$ *and* $\|\varphi_z\|_{L_p}$ *have estimations not depending on the represented solution.*

Representation (4.9) is unique when suitably normalized. Representation (4.10) is also unique provided that $\chi(z)$ satisfies equation (4.8).

Representations (4.9) and (4.10) also hold for solutions admitting isolated singularities; in this case all the singularities are transferred to the complex analytic function $f(\chi)$.

The difference between Theorems 4.3 and 4.4 and Theorems 3.3 and 4.1 is that, in the former theorems, we cannot consider the homeomorphism $\chi(z)$ as fixed (i.e. as the same for all solutions). Indeed, $\chi(z)$ satisfies the equation (4.8) in which the coefficient $q_0(z)$ depends on the represented solution. However, $|q_0(z)| \leq q_0$ where q_0 is a constant not depending on the represented solution. In view of that, it is very important to study these properties of solutions for equations of form (4.8) that depend on q_0 only. This is the case in Theorem 3.6. Other properties of this type will be studied below.

With the help of representations of type (4.9) and (4.10), a number of properties of complex analytic functions is extended to solutions of equations (4.6) and (4.7). For instance, the following properties literally hold for systems of form (4.7): the maximum principle, the argument principle, the theorem on unique continuation and on isolated zeros, analogs of the theorems on removable singularities, on the behavior of solutions in the neighborhood of poles or essential singularities, criteria of the univalence of the mapping etc. For solutions of systems of type (4.6) the following hold true: the argument principle, the theorem on the unique continuation, on isolated zeros, etc.

In the following some other corollaries of Theorems 4.3 and 4.4 will be derived.

Remark. Theorems 4.3 and 4.4 also hold for solutions of the inequalities

$$|w_{\overline{z}}| \leq |q_1| \cdot |w_z|, \quad |q_1| + |q_2| < q_0 < 1,$$

$$|w_{\overline{z}} - q_1 w_z - q_2 \overline{w_z}| \leq A|w|, \quad A \in L_p,\ p > 2.$$

3. On some particular solutions of system (2.6). Let equation (2.6) be defined in a bounded domain G. We prove the following theorem:

Theorem 4.5. *Under the assumptions of* § 2, *equation* (2.6) *always admits a solution* $w = w(z)$ *which may be complex analytically extended to the whole plane* E *in the class* $W_{p,\mathrm{loc}}(E)$, $p > 2$, *such that*

$$w(z) \sim f(z) \quad \textit{for} \quad z \to \infty$$

where $f(z)$ *is a prescribed entire function. Such a solution is unique.*

Proof. We will look for the solution of equation (2.6) satisfying the conditions of the theorem in the following form

$$w(z) = f(z) - \frac{1}{\pi} \int_G \frac{\omega(t)}{t - z}\, dG \equiv f(z) + T(\omega). \tag{4.11}$$

Substituting (4.11) in (2.6), we get for ω the equation

$$\omega - q_1 S(\omega) - q_2 \overline{S(\omega)} = AT(\omega) + B\overline{T(\omega)} + C_* \tag{4.12}$$

where

$$C_* = Af + B\overline{f} + C + q_1 f' + q_1 \overline{f'}, \quad z \in G, \quad C_* \equiv 0 \ \text{ outside of } \ G.$$

We can apply the Fredholm theory to equation (4.12). Indeed, denoting by R the inverse operator to

$$\omega - q_1 S(\omega) - q_2 \overline{S(\omega)} = h,$$

we see that equation (4.12) is equivalent to the equation $\omega = R(A \cdot T(\omega)) + R(B \cdot \overline{T(\omega)}) + RC_*$ in which the operator $R(A \cdot T(\omega)) + R(B \cdot \overline{T(\omega)})$ is completely continuous since the operator $T(\omega)$ is completely continuous. Hence, it is sufficient to consider the homogeneous equation (4.12). Let $\omega \in L_p$ be a solution of this equation. Then $w_1(z) = T(\omega)$ will be a solution to the homogeneous equation (4.6) which is analytic outside of G and vanishes at infinity. In view of (4.10), $w_1(z) = f(\chi(z)) e^{\varphi(z)}$, moreover, without loss of generality we may assume that function $\varphi(z)$ may be extended to the whole complex plane and is holomorphic outside of G and equal to zero at infinity. The function $\chi(z)$ can be chosen in the form (3.5). Changing variables $z : \chi = (z)$ and taking into account that $\chi(z)$ holomorphically depends on z outside of G, we see that $f(\chi)$ may be complex analytically extended to the whole plane with $f(\chi) = 0$ for $\chi = \infty$. Therefore $f(\chi) \equiv 0$, i.e. $w_1(z) \equiv 0$, hence $\omega(z) = \frac{\partial w_1}{\partial \bar{z}} = 0$.

Thus we have proved that the homogeneous equation (4.12) admits only trivial solution. In view of what has been said above, we conclude that the non-homogeneous equation (4.12) admits the unique solution ω. Then formula (4.11) gives the desired solution of equation (2.6). The uniqueness follows, since, according to Theorem 1.2. any solution satisfying the conditions of Theorem 4.5 may be represented in the form (4.11). If $f(z)$ is not an entire function but satisfies only the condition $f'(z) \in L_p(G)$, $p > 2$, then we can also get solutions of equation (2.6) of the class $W_p(G)$ by the method described above, which, however, in general, may not be complex analytically extended to the whole plane. Additionally, other theorems on the existence of solutions for the equation (2.6) can be proved by similar methods. We shall prove the following

Theorem 4.6. *Let K be the unit disk. Then equation* (2.6) *has a unique solution satisfying the following conditions:*

1) $w(z) \in W_p(K)$, $p > 2$, $|w_z|$ *and* $|w_{\bar{z}}| \in L_p(K)$;
2) $\Re w(z) = 0$ *for* $|z| = 1$;
3) $w(1) = 0$.

Proof. The proof is similar to the proof of the previous theorem. We seek the solution $w(z)$ in the form

$$w(z) = -\frac{1}{\pi} \int_K \left[\frac{\omega}{t-z} + \frac{z\overline{\omega}}{1-z\bar{t}} - \frac{\omega}{t-1} - \frac{\overline{\omega}}{1-\bar{t}} \right] dK \equiv T_2(\omega) \tag{4.13}$$

where $w(z)$ automatically satisfies conditions 2) and 3) of Theorem 4.6. If $\omega \in L_p(K)$, $p > 2$, then $w(z)$ also satisfies condition 1). Substituting (4.13) in (2.6), we get the equation

$$\omega - q_1 S_1(\omega) - q_2 \overline{S_1(\omega)} = A T_2(\omega) + B \overline{T_2(\omega)} + C. \tag{4.14}$$

As in the previous proof, we see that the Fredholm theory in $L_p(K)$ can be applied to equation (4.14). Therefore, to prove solvability, it is sufficient to consider the homogeneous equation $C \equiv 0$. Let $\omega \in L_p(K)$ be a solution of the homogeneous equation (4.14); then the corresponding $w(z)$ will be a Hölder continuous solution of equation (4.6) satisfying conditions 1), 2), 3). In view of (4.10), $w(z) = f(\chi(z))e^{\varphi(z)}$ where $f(\chi)$ is a holomorphic function in $|\chi| < 1$, $\chi(z)$ and $\varphi(z)$ are Hölder continuous for $|z| \le 1$ and $\Re\varphi(z) = 0$ for $|z| = 1$. In view of 2), the equality $\Re\{f(\chi(z))e^{\varphi(z)}\} = 0$ holds for $|z| = 1$ and, passing to the variable χ, we obtain $\Re\{f(\chi)e^{\varphi(z(\chi))}\} = 0$ for $|\chi| = 1$, i.e. $f(\chi)$ is a solution of the homogeneous Riemann–Hilbert problem for the disk $|\chi| \le 1$. The index of this problem is 0. Hence, (see [23]), all of its solutions have the form $f(\chi) = iC_0e^{p(\chi)}$ where $p(\chi)$ is a function holomorphic for $|\chi| < 1$ and Hölder continuous for $|\chi| < 1$ and C_0 is some real constant. From condition 3) we obtain: $C_0 = 0$, i.e. $f(\chi) \equiv 0$ and $w(z) \equiv 0$. Hence $\omega \equiv 0$ also. Thus, we have proved that the homogeneous equation (4.14) admits only the trivial solution. Therefore equation (4.14) is always uniquely solvable, i.e. equation (2.6) admits a solution of form (4.13). The uniqueness follows since the desired solution may always be represented in the form (4.13), (see [24]).

Remark. It is easy to see that equations (4.12) and (4.14) can be solved by the iteration scheme

$$\omega_{n+1} - q_1S(\omega_{n+1}) - q_2\overline{S(\omega_{n+1})} = AT(\omega_n) + B\overline{T(\omega_n)} + C_*. \tag{4.14'}$$

4. Representation of solutions of non–homogeneous equations of form (2.6). Combining Theorem 4.4 with Theorems 4.5 and 4.6, or with their slight modifications, various representation formulas for solutions of equation (2.6) can be obtained. In general, these representations will have the form

$$w(z) = f(\chi(z))e^{\varphi(z)} + w_0(z). \tag{4.15}$$

We can require that functions $f(\chi)$, $\chi(z)$, $\varphi(z)$, participating in formula (4.15), have properties of the same type as mentioned in Theorems 3.3, 3.4, 4.1, 4.2, 4.3 and 4.4: $f(\chi)$ is always an analytic function of variable χ in the domain $\chi(G)$ (possibly, admitting singularities); $\chi(z)$ is a homeomorphism of G onto $\chi(G)$ satisfying the homogeneous equation of form (4.8). Many properties of the functions $\chi(z)$, $\varphi(z)$ and $w_0(z)$ do not depend on the represented solution but depend only on the constants bounding the coefficients of equation (2.6). As an example, we present one of the respective theorems in a precise formulation:

Theorem 4.7. *Let $w = w(z)$ be a solution of equation* (2.6) *satisfying the assumptions of* § 2. *Then $w(z)$ can be represented in form* (4.15). *Here $\chi(z)$ is a homeomorphism of form* (3.5) *of the z–plane onto the χ–plane satisfying equation* (4.8)*; $\varphi(z)$ is a continuous function in the whole plane, holomorphic outside of G and equal to zero at infinity; χ and $\varphi \in W_{p,\mathrm{loc}}(E)$, $p > 2$ (E is the full plane). Moreover the norms, $\|\varphi_z\|_{L_p}$, $\|\varphi_{\overline{z}}\|_{L_p}$, $\|\chi_z\|_{L_p}$, $\|\chi_{\overline{z}}\|_{L_p}$ admit* *bounds by quantities depending only on domain* G*, the constant* q_0 *and the integrals* $\int_G |A|^p \, dG$*,* $\int_G |B|^p \, dG$*. In particular,* χ *and* φ *satisfy a Hölder condition with exponent and Hölder constant depending only on the quantities mentioned above.* $w_0(z)$ *is a solution of the non–homogeneous equation* (2.6) *in the class* $W_{p,\mathrm{loc}}(E)$*, Hölder continuous in the whole plane, analytic outside of* G *and such that* $w_0(z) \sim 1$ *as* $z \to \infty$*.*

In general, the representation (4.15) is certainly not unique. However, if we require all components of this representation to satisfy the conditions mentioned in the theorem (in particular, $\chi(z)$ should be a solution of equation (4.8)), the representation (4.15) is unique. In this case, even more may be proven: all components of representation (4.15) continuously depend on the represented solution. In the weakest form this can be made more precise in the following way: if $w_n(z) = f_n(\chi_n(z))e^{\varphi_n(z)}$ tend uniformly on compact subsets of G to $w(z) = f(\chi(z))e^{\varphi(z)}$ and, moreover, $\frac{\partial w_n}{\partial z} \to \frac{\partial w}{\partial z}$, $\frac{\partial w_n}{\partial \bar{z}} \to \frac{\partial w}{\partial \bar{z}}$ a.e. in G, then $\chi_n(z) \to \chi(z)$, $\varphi_n(z) \to \varphi(z)$ uniformly on compact subsets of $\chi(G)$ and $f_n(\chi) \to f(\chi)$ uniformly in $\chi(G)$. (We have given the formulation of our proposition only for solutions of homogeneous equations.)

A proof of the above proposition may be obtained by a more detailed inspection of the above formulas for all components of representation (4.15).

We skip detail formulations of other theorems describing alternative representations of form (4.15). It is clear how this can be done.

Representation (4.15), revealing the structure of solutions of equation (2.6) makes possible to reduce investigation of the properties of solutions to the study of properties of the corresponding analytic functions. In most cases, this essentially simplifies the problem under investigation. We already met some examples of this type above, others will be given in the following.

Individual components of representation (4.15) are interconnected; they all depend on the represented solution. Therefore, given in advance and combined in formula (4.15), they do not generate a solution of the equation (2.6). Only in the case when $B \equiv 0$ and $q_2 \equiv 0$, the functions $\chi(z)$ and $\varphi(z)$ may be chosen independently of the particular represented solution. However, for the important case of a simply connected domain G, we show below that solutions of form (4.15) always exist.

A representation of form (4.15) for the general equation of the form (2.6) was first given in the paper of L. Bers and L. Nirenberg [10]. They considered the case when G is the unit disk, $\chi(G) = K$, $\Im w_0(z) = 0$ and $\Im \varphi(z) = 0$ for $|z| = 1$. There is no complete proof of this result in the cited paper; the authors promised to present such a proof in a following paper. The scheme of the proof presented in [10] shows that the method of these authors is different from ours.

A formulation of the representation theorem corresponding to a slight modification of Theorem 4.6 shows that the theorem of Bers and Nirenberg is a particular case of our representations.

5. Compactness of solutions of system (4.7). The representations (4.9) and (4.15) contain a number of criteria of compactness for solutions of systems (2.6) and (4.6). However, from these representation formulas we can also derive some criteria of compactness for families of derivatives of the solutions. In this direction, we prove the following lemma.

Lemma 4.1. *Let $\chi^n(z)$ be a sequence of homeomorphisms of the unit disk onto itself satisfying the equations*

$$\frac{\partial \chi^n(z)}{\partial \overline{z}} - q_n^1(z)\frac{\partial \chi^n(z)}{\partial z} - q_n^2(z)\overline{\frac{\partial \chi^n(z)}{\partial z}} = 0 \tag{4.16}$$

and the normalization condition $\chi^n(0) = 0$. Let

$$q_n^1 \to q_1, \quad q_n^2 \to q^2, \quad |q_n^1| + |q_n^2| \le q_0 < 1 \qquad a.e. \ .$$

Then the families of the derivatives (where $z = z^n(\chi)$ are the inverse homeomorphisms)

$$\frac{\partial \chi^n}{\partial z}, \quad \frac{\partial \chi^n}{\partial \overline{z}}, \quad \frac{\partial z^n}{\partial \chi}, \quad \frac{\partial z^n}{\partial \overline{\chi}}$$

are compact in $L_p(K)$, $p > 2$.

Proof. In view of Theorem 3.6, $\left|\left|\frac{\partial \chi^n}{\partial z}\right|\right|$, $\left|\left|\frac{\partial \chi^n}{\partial \overline{z}}\right|\right|$, $\left|\left|\frac{\partial z^n}{\partial \chi}\right|\right|$, $\left|\left|\frac{\partial z^n}{\partial \overline{\chi}}\right|\right|$ are uniformly bounded. We first prove that one can choose subsequences of the sequences $\frac{\partial \chi^n}{\partial \overline{z}}$ and $\frac{\partial \chi^n}{\partial z}$ which are convergent in the norm of L_p. Indeed, by Theorem 4.3, $\chi^n(z) = f_n(\chi_1^n(z))$ where $\chi_1^n(z)$ is a homeomorphism of the plane z onto the plane χ of the form

$$\chi_1^n(z) = z - \frac{1}{\pi}\int\limits_K \left[\frac{\omega_n(t)}{t-z} - \frac{\omega_n(t)}{t}\right] dK, \quad \omega_n \in L_p, \quad p > 2,$$

and $f_n(\chi)$ is a holomorphic univalent function mapping the domain $\chi_1^n(K)$ onto the unit disk; in view of the chosen formula, we have that $\chi_1^n(0) = 0$. Therefore, the origin $\chi = 0$ belongs to all domains $\chi_1^n(K)$ and $f_n(0) = 0$. Substituting $\chi^n(z) = f_n\left(\chi_1^n(z)\right)$ into the equation (4.16), we obtain the non-linear equation for $\chi_1^n(z)$

$$\frac{\partial \chi_1^n}{\partial \overline{z}} - q_n^1\frac{\partial \chi_1^n}{\partial z} - q_n^2\frac{\overline{f_n'(\chi_1^n)}}{f_n'(\chi_1^n)} \cdot \overline{\frac{\partial \chi_1^n}{\partial z}} = 0$$

in which

$$|q_n^1| + |\widetilde{q}_n^2| \le q_0 < 1, \quad \widetilde{q}_n^2 = q_n^2 \frac{\overline{f_n'}}{f_n'}.$$

Substituting the expression for χ_1^n, we have

$$\omega_n - q_n^1(z)S(\omega_n) - \widetilde{q}_n^2\overline{S(\omega_n)} = q_n^1 + \widetilde{q}_n^2 \tag{4.17}$$

and we conclude that the norms $\|\omega_n\|_{L_p}$ are uniformly bounded. Thus, taking into account the inequalities (1.5) and (1.6), we see that the sequence of homeomorphisms $\chi_1^n(z)$ satisfies a Hölder condition with exponent and constant depending on q_0 only. The same can be said about the inverse homeomorphisms $z = z_1^n(\chi)$. Therefore we can choose uniformly convergent subsequences, $\chi_1^{n_k} \to \chi_1$ and $z_1^{n_k} \to z_1$, of the sequences χ_1^n and z_1^n, respectively; $\chi_1(z)$ and its inverse $z = z_1(\chi)$ are homeomorphisms satisfying the Hölder condition and $\chi_1(0) = 0$. It is easy to see that the sequence of domains $\chi_1^{n_k}(K)$ converges to a domain $\chi_1(K)$ in the sense of Caratheodory. Therefore we can choose a subsequence $f_{n_k}(\chi)$ of the sequence $f_n(\chi)$ converging uniformly, together with the sequence of derivatives, to some function $f(\chi)$ on each compact subset of $\chi_1(K)$; $f(\chi)$ is an univalent function mapping $\chi_1(K)$ onto K, such that $f(0) = 0$. For simplicity, we may assume in the following, that the original sequences have all the properties proved for the subsequences.

It is easy to verify that

$$\widetilde{q}_n^2(z) \to q_n^2 \frac{\overline{f_n'(\chi_1(z))}}{f_n'(\chi_1(z))} \quad \text{a.e.}$$

Therefore we derive from (4.17) that the sequence ω_n converges to some $\omega \in L_p(K)$ in the norm of the space L_p and $\chi_1(z) = z + T(\omega)$. We show that $\frac{\partial \chi^n}{\partial z}$ converges to the function $f'(\chi_1)\frac{\partial \chi_1}{\partial z}$ in the norm of L_p. For this purpose consider the expression

$$\left\| \frac{\partial \chi^n}{\partial z} - f'(\chi_1^m)\frac{\partial \chi_1^m}{\partial z} \right\|_{L_p}^p = \int\limits_K \left| f_n'(\chi_1^n)\frac{\partial \chi_1^n}{\partial z} - f'(\chi_1^n)\frac{\partial \chi_1^m}{\partial z} \right|^p \, dK =$$

$$= \int\limits_{K_1} |\ldots|^p \, dK + \int\limits_{K_2} |\ldots|^p \, dK$$

where K_1 is the disk of radius $r < 1$ centered at 0 and K_2 is the annulus $r < |z| < 1$. In view of Lemma 3.1, we can state that $\chi_1^n(K_1)$ are contained in some closed fixed subdomain of $\chi_1(K)$ for large enough n. Since in this subdomain f_n' tends to f' uniformly and $\left\| \frac{\partial \chi_1^n}{\partial z} - \frac{\partial \chi_1^m}{\partial z} \right\|_{L_p} \to 0$, the first integral in the last formula tends to zero for every fixed $r < 1$. For the second integral, as is easy to see, we have the following estimate

$$\int_{K_2} \left| \frac{\partial \chi^n}{\partial z} - \frac{\partial \chi^m}{\partial z} \right|^p dK \leq C(mes\, K_2)^{\frac{1}{q_1}}$$

for large enough q_1. The constant C bounds from above the norms $\left\|\frac{\partial \chi^n}{\partial z}\right\|$ and $\left\|\frac{\partial \chi^m}{\partial z}\right\|$ in the metric of the space L_{pp_1} where p_1 is close enough to 1 and $\frac{1}{q_1} + \frac{1}{p_1} = 1$, $p > 2$. By Theorem 3.6, these norms are bounded by a quantity depending only on q_0 (but not on n and m). Thus, we have $\left\|\frac{\partial \chi^n}{\partial z} - \frac{\partial \chi}{\partial z}\right\|_{L_p} \to 0$. Similarly we verify that $\left\|\frac{\partial \chi^n}{\partial \overline{z}} - \frac{\partial \chi}{\partial \overline{z}}\right\|_{L_p} \to 0$.

The simplest way to verify the relations $\left\|\frac{\partial z^{n_k}}{\partial \chi} - \frac{\partial z}{\partial \chi}\right\|_{L_p} \to 0$ and $\left\|\frac{\partial z^{n_k}}{\partial \overline{\chi}} - \frac{\partial z}{\partial \overline{\chi}}\right\|_{L_p} \to 0$ for a subsequence $n_k \to \infty$ is the following: in view of the formulas $\frac{\partial z^n}{\partial \chi} = -\frac{1}{J_n} \overline{\frac{\partial \chi^n}{\partial z}}$, we can assert that $\frac{\partial z^{n_k}}{\partial \chi}$ tends to $\frac{\partial z}{\partial \chi}$ a.e. since J_{n_k} cannot tend to zero on a set of a positive measure. Theorem 3.6 obviously implies the absolute equicontinuity of the integrals $\int_e \left|\frac{\partial z^n}{\partial \chi}\right|^p dK$, $e \in K$. Therefore the chosen subsequence tends to $\frac{\partial z}{\partial \chi}$ in the norm of L_p, as well. Lemma 4.1 is proved.

Lemma 4.1 implies.

Corollary 4.1. *In the assumptions of Lemma* 4.1*, the limit function satisfies the equation*

$$\chi_{\overline{z}} - q_1 \chi_z - q_2 \overline{\chi}_z = 0.$$

Remark. One can conclude from the uniqueness theorems proved below that the limit function, properly normalized, does not depend on the choice of the subsequence $\chi^{n_k} \to \chi$. It follows then, that under suitable normalization (for instance, $\chi^n(0) = 0$, $\chi^n(1) = 1$) the full sequence χ^n converges to χ and $\left\|\frac{\partial \chi^n}{\partial z} - \frac{\partial \chi^m}{\partial z}\right\|_{L_p} \to 0$. Similar relations hold for the other derivatives.

In the same way we can prove the following statement.

Lemma 4.2. *Let $w_n(z)$ be a sequence of solutions of the systems*

$$\frac{\partial w_n}{\partial \overline{z}} - q_n^1 \frac{\partial w_n}{\partial z} - q_n^2 \overline{\frac{\partial w_n}{\partial z}} = 0$$

in a domain G. Let $w_n(z) \to w(z)$ uniformly on compact subsets of G, $|w_n(z)| < M$ in G and $q_n^1 \to q^1$, $q_n^2 \to q^2$, $|q_n^1| + |q_n^2| \leq q_0 < 1$.

Then:

1) *$w = w(z)$ is a solution of the system*

$$\frac{\partial w}{\partial \overline{z}} - q^1 \frac{\partial w}{\partial z} - q^2 \overline{\frac{\partial w}{\partial z}} = 0$$

2) *if $w_n(z)$ are univalent, then $w(z)$ is either a constant or an univalent function.*

5 The existence of solutions to the equation (4.6)

The existence problem for solutions of the Beltrami system is rather completely solved by Theorem 3.4. In the general case, the situation is more complex. This section is devoted to the proof of a theorem related to this question.

Theorem 5.1. *Let K be the (open) unit disk and $F(\chi)$ be an arbitrary analytic function in K, possibly admitting isolated singularities in K. Then there exist two functions $\chi(z)$ and $\varphi(z)$ of the class $W_p(K)$, $p > 2$, which are Hölder continuous in the closed disk $\overline{K}$ and such that the formula*

$$w(z) = F(\chi(z))e^{\varphi(z)}$$

represents a solution to equation (4.6). *$\chi = \chi(z)$ realizes a homeomorphic mapping of $\overline{K}$ onto itself; for any given points z_0, χ_0, z_1 and χ_1 in the interior and on the boundary of K, respectively, we can require that $\chi_0 = \chi(z_0)$, $\chi_1 = \chi(z_1)$. φ is required to admit an extension to the whole plane as a continuous function of variable z, holomorphic outside of K and vanishing at infinity. Moreover, $\chi(z)$ and $\varphi(z)$ admit nearly universal estimates: $|\chi(z)|$, $|\varphi(z)|$ and the norms of their derivatives $\|\chi_{\overline{z}}\|$, $\|\chi_z\|$, $\|\varphi_{\overline{z}}\|$, $\|\varphi_z\|$ in $L_p(K)$ are bounded by constants that do not depend on the properties of the function F (for fixed z, q_0).*

If $A = B \equiv 0$, then we can choose $\varphi(z) \equiv 0$.

All the statements of the theorem are valid if, instead of the above requirements on φ, we require that $Re\,\varphi(z) = 0$ for $|z| = 1$ and $\varphi(1) = 0$.

For simplification the proof is organized into a series of steps.

Without loss of generality, we may assume that $z_0 = \chi_0 = 0$, $z_1 = \chi_1 = 1$. We first consider the case $A = B \equiv 0$ everywhere and $q_2(z) \equiv 0$ in a neighborhood of zero.

By a simple calculation it can easily be seen that the function $w(z) = F(\chi(z))$ is a solution of the homogeneous equation (4.7) if and only if $\chi(z)$ is a solution of the quasi-linear equation of the form

$$\frac{\partial\chi}{\partial\overline{z}} - q_1(z)\frac{\partial\chi}{\partial z} - q_2(z)\frac{\overline{F'(\chi)}}{F'(\chi)}\frac{\overline{\partial\chi}}{\partial z} = 0 \ . \tag{5.1}$$

We prove that equation (5.1) always admits a solution $\chi = \chi(z)$ mapping the unit disk $|z| \leq 1$ onto the unit disk $|\chi| \leq 1$ and such that $\chi(0) = 0$, $\chi(1) = 1$. We construct the solution of (5.1) in the form

$$\chi(z) = f(z)e^{T_1(\omega)} \tag{5.2}$$

where $f(z)$ is the solution of the homogeneous equation $f_{\overline{z}} - q(z)f_z = 0$ mapping the unit disk $|z| \leq 1$ onto the unit disk $|f| \leq 1$, satisfying the conditions $f(0) = 0$, $f(1) = 1$, and

$$T_1(\omega) = -\frac{1}{\pi}\int_K \left\{ \frac{\omega(t)}{t-z} - \frac{\omega(t)}{t-1} + \frac{z\overline{\omega(t)}}{1-z\overline{t}} - \frac{\overline{\omega(t)}}{1-\overline{t}} \right\} dK \tag{5.3}$$

where $\omega(t)$ is a complex-valued function, $\omega \in L_p$. Obviously, we have $\chi(0) = 0$, $\chi(1) = 1$ for any $\omega \in L_p$. Substituting (5.2) in (5.1), we get the following nonlinear equation for ω

$$\omega - q_1 S_1(\omega) - q_2\frac{\overline{F'(\chi)}}{F'(\chi)}\frac{\overline{\chi}}{\chi}\overline{S_1(\omega)} = q_2\frac{\overline{F'(\chi)}}{F'(\chi)}\frac{\overline{f_z}}{f}e^{\overline{T_1(\omega)} - T_1(\omega)} \tag{5.4}$$

in which $\chi = \chi(z)$ should be considered as a nonlinear operator (5.2) acting on ω. We fix $p > 2$ according to the inequality $q_0 B_p < 1$, and work henceforward in a fixed Banach space $L_p(K)$. The operator $T_1(\omega)$ maps compactly the space $L_p(K)$ into the space of functions continuous in the closed disk $|z| \leq 1$. For all solutions of equation (5.4), we clearly have the estimate $||\omega|| \leq C$ with some C not depending on ω. Denote by Ω the set of all $\omega \in L_p(K)$, $||\omega|| \leq C$, such that $|\chi(z)| \leq 1$ for $|z| \leq 1$. Ω is a convex closed set in $L_p(K)$. Let $h > 0$ and $F_h(\chi)$ be the average of $\frac{\overline{F'(\chi)}}{F'(\chi)}$, $|F_h(\chi)| \leq 1$, see [12].

Consider the functional equation

$$\Phi - q_1 S_1(\Phi) - q_2 F_h(\chi)\frac{\overline{\chi}}{\chi}\overline{S_1(\Phi)} = q_2 F_h(\chi)\frac{\overline{f_z}}{f}e^{\overline{T_1(\omega)} - T_1(\omega)} \tag{5.5}$$

in which $\chi = \chi_\omega \equiv f(z)e^{T_1(\omega)}$ is considered as an operator acting on ω. Equation (5.5) defines an operator $\Phi = \Phi(\omega)$ on the set Ω. Indeed, this equation is always uniquely solvable since the principle of the contracting mappings may be applied to it. (The operator $\Phi(\omega)$ and other quantities considered below depend on h. However, for simplicity, we will not mark this dependence with the subscript h.) We prove first that the operator $\Phi(\omega)$ maps Ω into itself. The estimate $||\Phi||_{L_p} \leq C$ is obvious. Consider the function $\chi_\Phi = f(z)e^{T_1(\Phi)}$ where $\Phi = \Phi(\omega)$, $\omega \in \Omega$. We immediately verify that it satisfies equation $(\chi_\Phi)_{\bar{z}} - q_1(\chi_\Phi)_z - \widetilde{q_2}\overline{(\chi_\Phi)_z} = 0$ in which $\widetilde{q_2} = q_2(z)F_h(\chi)\frac{\overline{\chi_\omega}}{\chi_\omega}\frac{\chi_\Phi}{\overline{\chi_\Phi}}$, i.e. $|q_1| + |\widetilde{q_2}| \leq q_0 < 1$. By the strong maximum principle proved above for solutions of such equations, we get $|\chi_\Phi| < 1$ for $|z| < 1$ since $|\chi_\Phi| = 1$ for $|z| = 1$, i.e. $\Phi \in \Omega$. Let ω_n be a sequence of functions belonging to Ω, and let $\Phi_n = \Phi(\omega_n)$. Since the operator $T_1(\omega)$ is completely continuous, we can choose a subsequence n_k, $k = 1, 2, \ldots$, such that $\chi_{n_k} = f(z)e^{T_1(\omega_{n_k})}$ converge uniformly in the closed disk $|z| \leq 1$. Then the sequences $q_2F_h(\chi_{n_k})\frac{\overline{\chi_{n_k}}}{\chi_{n_k}}$ and $q_2F_h(\chi_{n_k})\,\frac{\overline{f_z}}{f}\,e^{\overline{T_1(\omega_{n_k})}-T_1(\omega_{n_k})}$ will converge at every point of the disk $|z| < 1$. As in the proof of Theorem 3.1, we show that Φ_{n_k} converge in the norm of $L_p(K)$. Thus, the equation (5.5) defines a compact operator $\Phi = \Phi(\omega)$ mapping the bounded closed convex set Ω of the Banach space $L_p(K)$ into itself. By the Schauder theorem, (see [25]), the operator $\Phi(\omega)$ has at least one fixed point. Thus, we have proved the solvability of the equation

$$\omega - q_1S_1(\omega) - q_2F_h(\chi)\,\frac{\overline{\chi}}{\chi}\,\overline{S_1(\omega)} = q_2F_h(\chi)\,\frac{\overline{f_z}}{f}\,e^{\overline{T_1(\omega)}-T_1(\omega)} \tag{5.6}$$

where $\chi = \chi_\omega \equiv f(z)e^{T_1(\omega)}$. Since χ_ω satisfies the equation $\chi_{\bar{z}} - q_1\chi_z - q_2F_h(\chi)\overline{\chi}_z = 0$ and the conditions $\chi(0) = 0$, $\chi(1) = 1$, it can be seen that $\chi(z)$ homeomorphically maps the disk K onto itself: indeed, by the representation theorems $\chi(z)$ has the form $\chi(z) = \widetilde{f}(\chi_1(z))$ where $\chi_1(z)$ is a homeomorphism of the disk K onto itself, $\chi_1(0) = 0$, $\chi_1(1) = 1$, and $\widetilde{f}(\chi_1)$ is an analytic function. In view of the conditions $|\widetilde{f}(\chi_1)| = 1$ for $|\chi_1| = 1$, $\widetilde{f}(0) = 0$, $\widetilde{f}(1) = 1$, the function $\widetilde{f}(\chi_1)$ can only have the form $\widetilde{f}(\chi_1) = \chi_1^n$, $n \geq 1$. By the argument principle $2\pi = \Delta \arg_{|\chi_1|=1}\chi(z) = \Delta \arg_{|z|=1}\widetilde{f}(\chi_1)$, hence $n = 1$ and we are done. By the same argument, Lemmas 3.1, 4.1 and Theorem 3.6 may be applied as well. Recall now that we agreed not to mark the dependence of the considered quantities on h; in fact, we have constructed the functions ω_h and $\chi_h = f(z)E^{T_1(\omega_h)}$ which are solutions of equation (5.6). Choosing an uniformly convergent subsequence $\chi_{h_k} \to \chi$ for $h_k \to 0$, it is easy to verify that $F_{h_k}(\chi_{h_k})$ tends to $\frac{\overline{F'(\chi)}}{F'(\chi)}$ a.e. and, consequently, also functions ω_{h_k} converge in the norm of L_p to some ω. ω is a solution of the non-linear equation (5.4) where $\chi = \chi(z) = f(z)e^{T_1(\omega)}$ and $\chi(z)$ itself is the desired solution of the equation (5.1). Thus, the case $A = B \equiv 0$ and $q_2 \equiv 0$ in a neighborhood of the origin is settled completely.

Now lets get rid of our first assumption still keeping for a while the condition that "$q_2 \equiv 0$ in a neighborhood of the origin". As lareviously done, we verify by a simple calculation that our theorem will be proved if we show the existence of solutions admitting all properties required in the theorem, for the following system of non-linear functional equations

$$\chi_{\bar z} - q_1(z)\chi_z - q_2(z)\frac{\overline{F'(\chi)}}{F'(\chi)}e^{\overline{\varphi}-\varphi}\overline{\chi_z} = 0,$$

$$\varphi_{\bar z} - q_1\varphi_z - q_2\frac{\overline{w}}{w}\overline{\varphi_z} = A + B\frac{\overline{w}}{w} \tag{5.7}$$

where $w(z) = F(\chi(z))e^{\varphi(z)}$, $\chi(z)$ and $\varphi(z)$ are the unknowns. We again search for solutions in the form

$$\chi(z) = f(z)e^{T_1(\omega)}, \quad \varphi(z) = T(\omega_1)$$

where

$$T(\omega) = -\frac{1}{\pi}\int\limits_K \left[\frac{\omega}{t-z} + \frac{z\overline{\omega}}{1-z\overline{t}} - \frac{\omega}{t-1} - \frac{\overline{\omega}}{1-\overline{t}}\right] dK, and$$

$$T(\omega_1) = -\frac{1}{\pi}\int\limits_K \frac{\omega_1(t)}{t-z}\, dK, \tag{5.8}$$

and $f(z)$ is the solution of the homogeneous equation $f_{\bar z} - q_1(z)f_z = 0$ mapping the disk $|z| \le 1$ onto itself, such that $f(0) = 0$, $f(1) = 1$. All the requirements imposed on the unknowns $\chi(z)$ and $\varphi(z)$ by our theorem will be automatically satisfied as long as we establish an estimate of the norms $||\omega||_{L_p}$ and $||\omega_1||_{L_p}$, $p > 2$, not depending on $F(\chi)$.

Substituting (5.8) in (5.7), we get the system of the following equations for the new unknowns ω and ω_1:

$$\omega - q_1S_1(\omega) - q_2\frac{\overline{F'(\chi)}}{F'(\chi)}\frac{\overline{\chi}}{\chi}e^{\overline{\varphi}-\varphi}\,\overline{S_1(\omega)} = q_2\frac{\overline{F'(\chi)}}{F'(\chi)}\frac{\overline{\chi}}{\chi}\frac{\overline{f_z}}{f}e^{\overline{\varphi}-\varphi}\cdot e^{\overline{T_1(\omega)}-T_1(\omega)},$$

$$\omega_1 - q_1S(\omega_1) - q_2\frac{\overline{w}}{w}\overline{S(\omega_1)} = A + B\frac{\overline{w}}{w} \tag{5.9}$$

where $w = F(\chi_\omega)e^{\varphi_{\omega_1}}$, and $\chi = \chi_\omega$, and $\varphi = \varphi_{\omega_1}$ should be considered as operators on ω and ω_1.

For the proof of solvability of system (5.9), introduce the space $L_p \times L_p$ of ordered pairs (ω, ω_1) with the norm $||(\omega,\omega_1)||_{L_p\times L_p} = ||\omega||_{L_p} + ||\omega_1||_{L_p}$. $L_p \times L_p$ is a Banach space. We immediately get an estimate $||(\omega,\omega_1)|| \le C$ for solutions (ω,ω_1) of the system (5.9) with C not depending on (ω,ω_1). Denote by Ω^* the set of all (ω, ω_1), $||(\omega, \omega_1)|| \le C$, such that $|\chi_\omega| \le 1$ for every $|z| \le 1$. Ω^* is a convex closed set in $L_p \times L_p$.

As before, we replace system (5.9) by a system with the averaged coefficients $\frac{\overline{F'(\chi)}}{F'(\chi)} \sim F_h(\chi)$ and $\frac{\overline{F(\chi)}}{F(\chi)} = F_h^*(\chi)$:

$$\omega - q_1 S_1(\omega) - q_2 F_h(\chi) \frac{\overline{\chi}}{\chi} e^{\overline{\varphi} - \varphi} \, \overline{S_1(\omega)} = q_2 F_h(\chi) \frac{\overline{\chi}}{\chi} \frac{\overline{f_z}}{f} e^{\overline{\varphi} - \varphi} \cdot e^{\overline{T_1(\omega)} - T_1(\omega)},$$

$$\omega_1 - q_1 S(\omega_1) - q_2 F_h^*(\chi) \, e^{\overline{\varphi} - \varphi} \, \overline{S(\omega_1)} = A + B F_h^*(\chi) e^{\overline{\varphi} - \varphi} \tag{5.10}$$

and consider the functional system

$$\Phi - q_1 S_1(\Phi) - q_2 F_h(\chi_\omega) \frac{\overline{\chi_\omega}}{\chi_\omega} e^{\overline{\varphi}_{\omega_1} - \varphi_{\omega_1}} \, \overline{S_1(\Phi)} = q_2 F_h(\chi_\omega) \frac{\overline{\chi_\omega}}{\chi_\omega} \frac{\overline{f_z}}{f} e^{\overline{\varphi}_{\omega_1} - \varphi_{\omega_1}} . e^{\overline{T_1(\omega)} - T_1(\omega)},$$

$$\Phi_1 - q_1 S(\Phi_1) - q_2 F_h^*(\chi_\omega) \, e^{\overline{\varphi}_{\omega_1} - \varphi_{\omega_1}} \, \overline{S(\Phi_1)} = A + B F_h^*(\chi_\omega) e^{\overline{\varphi}_{\omega_1} - \varphi_{\omega_1}} \tag{5.11}$$

which defines an operator $\Psi((\omega, \omega_1)) = (\Phi, \Phi_1)$ on the set Ω^*. Following the method described in the proof of analogous properties of operator $\Phi(\omega)$ above, we show that Ψ compactly maps Ω^* into itself.

Applying the Schauder fixed point theorem, we obtain the existence of a solution to system (5.10). Passing to the limit for a suitable subsequence $h_k \to 0$ we establish the existence of solutions for system (5.9) satisfying all the requirements of the theorem. The limiting process is justified in the same way as in the analogous case above.

We still must the last restriction: $q_2(z) \equiv 0$ in a neighborhood of the origin. This may be done by passing to the limit in equation (5.1) or in system (5.7). For a sequence $q_2^n(z) \to q_2(z)$ where q_2^n vanishes in a neighborhood of the origin, we solve equations (5.1) or (5.7) for $q_2 = q_2^n$, and obtain the sequences of solutions $\chi^n(z)$ and $\varphi^n(z)$. Relying on Lemma 4.1, we then perform the limiting process, generally speaking, along some subsequence $n_k \to \infty$. The necessary estimates are given by Theorem 3.6 and the inequalities (1.5) and (1.6).

The case of the boundary condition $\Re\varphi(z) = 0$ for $|z| = 1$ is considered analogously. In this case we look for $\varphi(z)$ in the form

$$\varphi(z) = T_1(\omega_1) = -\frac{1}{\pi} \int\limits_K \left[\frac{\omega}{t - z} + \frac{z\overline{\omega}}{1 - z\overline{t}} - \frac{\omega}{t - 1} - \frac{\overline{\omega}}{1 - \overline{t}} \right] dK_t.$$

The theorem is thus proved.

When $q_2 \equiv 0$ in equation (4.6), i.e. when (4.6) has the form

$$w_{\overline{z}} - q(z)w_z = Aw + B\overline{w}, \quad z \in G, \tag{5.12}$$

Theorem 5.1 may be strengthened: then the precise converse to Theorem 4.1 may be proved.

Theorem 5.2. *Let $f(z)$ be an arbitrary solution of the homogeneous equation*

$$f_{\overline{z}} - q(z)f_z = 0$$

in domain G. Then there exists a Hölder continuous function $\varphi(z)$ such that $w(z) = f(z)e^{\varphi(z)}$ is a solution of the equation (5.12). Moreover, we can require that $\varphi(z)$ may be extended to the entire plane as a continuous function of the variable z, holomorphic outside of G, equal to zero at infinity or $\Re\varphi(z) = 0$ for $|z| = 1$, $\varphi(1) = 0$ (if $G \equiv K$). In both cases $\varphi(z) \in W_p(K)$, $p > 2$, and the integrals $||\varphi_z||_{L_p}$, $||\varphi_{\overline{z}}||_{L_p}$ have an estimate from above not depending on $f(z)$.

The proof of this theorem can be obtained by the same method as the proof of the previous theorem.

In the case $A = B \equiv 0$, Theorem 5.1 also allows us to construct multi–valued solutions corresponding to the multi–valued functions $F(\chi)$ such that $F'(\chi)$ is single–valued. Using the above method the complex–valued Green function for equation (4.7) may be directly constructed.

Theorems 5.1 and 5.2 are generalizations of theorems proved in [13] and [22] for the case when $q_1 \equiv q_2 \equiv 0$.

6 The uniqueness theorem

We say that a complex–valued function $w = w(z)$ realizes a quasiconformal mapping of the domain Δ onto the domain D, corresponding to a given system of partial differential equations if $w = w(z)$ maps Δ onto D homeomorphically and satisfies the system.

In this section we prove the uniqueness theorem for quasiconformal mappings which correspond to the linear systems of the form

$$w_{\overline{z}} - q_1(z)w_z - q_2\overline{w_z} = 0 \tag{6.1}$$

with the most general assumptions on the coefficients $q_1(z)$ and $q_2(z)$. We allow $q_1(z)$ and $q_2(z)$ to be measurable functions satisfying the only condition (2.7) in the domain where the equation is considered.

As we remarked above, the case when $q_2 \equiv 0$, i.e., the case of the Beltrami equations, is trivial: the corresponding uniqueness theorem is obtained directly from the uniqueness theorems for conformal mappings.

For a special class of equations, beyond the Beltrami class, the uniqueness theorem was proven by Gergen and Dressel [16] (for the so-called $p-$elliptic systems). These authors consider the equations with smooth coefficients. In the framework of the method applied by Gergen and Dressel, their arguments cannot be extended to the general case of non–p–elliptic systems. We show below that the general uniqueness theorem can be obtained by the methods applied in the present paper. In the proof below, among other considerations, is also used an idea employed in the work of Gergen and Dressed.

The basis for our uniqueness theorem is the following.

Lemma 6.1. *Let $f = f(z)$ be a generalized solution of the equation*

$$f_{\bar z} - q(z) f_z + q(z)\overline{f_z} = 0, \qquad |q(z)| < q_0' < \frac{1}{2}, \tag{6.2}$$

mapping the unit disk $|z| \le 1$ onto the disk $|f| \le 1$. Then $f(z) \equiv z$ whenever one of the following conditions holds:

1) *$f(z) = z$ at three different points in the circle $|z| = 1$;*

2) *$f(z) = z$ at one point in the disk $|z| < 1$ and at one point on the circle $|z| = 1$;*

3) *$f(z) = z$ at two different points in the disk $|z| < 1$.*

Proof. For the proof we extend our function to some disk of radius $\rho_0 > 1$ by the formula

$$f^* = f(z) \quad \text{for} \quad |z| \le 1,$$

$$f^* = \frac{1}{f\left(\frac{1}{\bar z}\right)} \quad \text{for} \quad 1 < |z| \le \rho_0.$$

We choose the radius ρ_0 in such a way that $f^*(z)$ is not equal to infinity in the disk $|z| \le \rho_0$. Thus, we get a continuous function in the disk $|z| \le \rho_0$.

Now we write the differential equation for $f^*(z)$ on the exterior of the unit disk. For the partial derivatives we get:

$$f^*_{\bar z} = \frac{\zeta^2}{\overline{f(\zeta)^2}}\, \overline{f_{\bar\zeta}(\zeta)}, \qquad \zeta = \frac{1}{\bar z}, \qquad |z| > 1,$$

$$f^*_z = \frac{\overline{\zeta^2}}{\overline{f(\zeta)^2}}\, \overline{f_\zeta(\zeta)}.$$

Hence, in view of equation (6.2), we obtain

$$f^*_{\overline{z}} - \overline{q(\zeta)}\left[f^*_z\,\frac{\zeta^2}{\overline{\zeta}^2} - \overline{f^*_z}\,\frac{f^2(\zeta)}{\overline{f^2(\zeta)}}\right] = 0 \quad \text{for} \quad |z| > 1,\ \zeta = \frac{1}{\overline{z}}. \tag{6.3}$$

It is obvious that $f^*(z)$ has the first generalized derivatives in the Sobolev sense in the disk $|z| < \rho_0$. f satisfies equation (6.2) inside the unit disk and the equation (6.3) outside of it. Thus, setting

$$\widetilde{q}_1(z) = -\widetilde{q}_2(z) = q(z) \quad \text{for} \quad |z| < 1,$$

$$\widetilde{q}_1(z) = \overline{q(\zeta)}\,\frac{\zeta^2}{\overline{\zeta}^2} \quad \text{and} \quad q_2(z) = -\overline{q(\zeta)}\,\frac{f^2(\zeta)}{\overline{f^2(\zeta)}} \quad \text{for} \quad |z| > 1, \tag{6.4}$$

we see that f^* is a generalized solution of the equation

$$f^*_{\overline{z}} - \widetilde{q}_1 f^*_z - \widetilde{q}_2\overline{f^*_z} = 0$$

in the disk $|z| < \rho_0$. In view of (6.4), we have $|\widetilde{q}_1| + |\widetilde{q}_2| < 2q_0' < 1$.

Consider the function $w(z) = f^*(z) - z$. From the equations above we derive the equation for $w(z)$:

$$w_{\overline{z}} - \widetilde{q}_1(z)w_z - \widetilde{q}_2(z)\overline{w_z} = 0 \quad \text{for} \quad |z| < 1,$$

$$w_{\overline{z}} - \widetilde{q}_1 w_z - \widetilde{q}_2\overline{w_z} = \widetilde{q}_1(z) + \widetilde{q}_2(z) \quad \text{for} \quad |z| > 1;$$

however, for $|z| > 1$

$$\widetilde{q}_1 + \widetilde{q}_2 = \overline{q(\zeta)}\left[\frac{\zeta^2}{\overline{\zeta}^2} - \frac{f^2(\zeta)}{\overline{f^2(\zeta)}}\right] = \frac{q(\zeta)z^2}{f^{*2}}\,[\overline{\varphi} - \varphi]\,[\overline{\varphi} + \varphi]$$

where

$$\varphi = \frac{f^*(z)}{z}.$$

Set

$$\frac{\widetilde{q}_1(z) + \widetilde{q}_2(z)}{f^*(z) - z} = \frac{\overline{q(\zeta)}z^2}{f^{*2}\cdot z}\left[\frac{\overline{\varphi} - \varphi}{\varphi - 1}\right][\overline{\varphi} + \varphi] = A(z), \quad |z| > 1.$$

In view of the choice of ρ_0, we have $C_1 < |\varphi| < C$ for $1 < |z| < \rho_0$ with some constants C_1 and C.

Moreover, $\left|\frac{\overline{\varphi}-\varphi}{\varphi-1}\right| < 2$ for all φ; hence

$$|A| \leq \frac{4q_0'C}{C_1^2}.$$

Setting $A(z) \equiv 0$ for $|z| < 1$, we obtain the equation for $w(z)$

$$(6.5) \qquad w_{\overline{z}} - \widetilde{q_1} w_z - \widetilde{q_2} \overline{w_z} = A(z) w \quad \text{for} \quad |z| < \rho_0,$$

with $|\widetilde{q_1}| + |\widetilde{q_2}| < 2q_0' < 1$ and $A(z)$ bounded. In view of the properties of solutions of the equations of the form (6.5) proved above, we can assert that either $w(z)$ is identically zero or it vanishes on a discrete set of points in any disk $|z| \leq \rho_0' < \rho_0$. If $w(z)$ is not identically zero, the argument principle can be applied to the function $w(z)$. If $w(z) \equiv 0$, then there is nothing to prove. Suppose that $w(z)$ is not identically zero. We show that this assumption is not compatible with any of the conditions 1) – 3) of the lemma.

Indeed, let N_1 be the number of zeros of $w(z)$ in the disk $|z| < 1$, N_2 be the number of zeros of $w(z)$ on the circle $|z| = 1$. The function $w(z)$ has no zeros in the annulus $1 < |z| < 1 + \eta < \rho_0'$ for a small enough η. Therefore the increment of the argument of the function $w(z)$ along the circle $C : |z| = 1 + \eta$ equals to $\Delta_C \arg w(z) = 2\pi(N_1 + N_2)$. On the other hand, on the circle C we have

$$w(z) = \frac{1}{\overline{f\left(\frac{1}{\overline{z}}\right)}} - z = \frac{z}{\overline{f\left(\frac{1}{\overline{z}}\right)}} \left[\overline{\frac{1}{\overline{z}} - f\left(\frac{1}{\overline{z}}\right)} \right] = -\frac{z}{\overline{f\left(\frac{1}{\overline{z}}\right)}} \, \overline{w\left(\frac{1}{\overline{z}}\right)}.$$

Suppose that η is so small that all N_1 zeros of $w(z)$ in the disk $|z| < 1$ and the unique zero of $f(z)$ lie in the disk $|z| < \frac{1}{1+\eta}$. Then the last formula gives

$$\Delta_C \arg w(z) = 2\pi + 2\pi - 2\pi N_1 = 4\pi - 2\pi N_1.$$

Therefore

$$(6.6) \qquad 4\pi - 2\pi N_1 = 2\pi(N_1 + N_2) \quad \text{or} \quad 2N_1 + N_2 = 2.$$

It can easily be seen that the equality (6.6) is incompatible with any of the assumptions of our lemma. Thus, in those cases $w(z) \equiv 0$. The lemma is proved.

Now we prove the fundamental uniqueness theorem for the system of the form (6.2).

Theorem 6.1. *Let $w = w(z)$ be a solution of the system*

$$w_{\overline{z}} - q_1(z) w_z - q_2(z) \overline{w_z} = 0, \quad |q_1| + |q_2| \leq q_0 < 1,$$

mapping the unit disk onto itself. Then such a solution is unique provided that at least one of the following conditions holds:

1) $w(z_i) = w_i$ $(i = 1, 2, 3)$ *at three assigned points on the circle* $|z| = 1$*; here* w_1*,* w_2 *and* w_3 *are also prescribed;*

2) $w(z_0) = w_0$*,* $w(z_1) = w_1$*,* $|z_0| < 1$*,* $|w_0| < 1$*,* $|z_1| = |w_1| = 1$ *where* z_0, z_1, w_0 *and* w_1 *are prescribed.*

Proof. We argue by contradiction. Suppose there exist two solutions $w = w(z)$ and $v = v(z)$ satisfying the conditions of the lemma. Consider the function $z = z(v)$ which is inverse to the solution $v = v(z)$. It maps the disk $|v| \leq 1$ onto the disk $|z| \leq 1$. We have for its derivatives the following equalities

$$z_v = \frac{\overline{v_z}}{J}, \quad z_{\overline{v}} = -\frac{v_{\overline{z}}}{J}$$

where $J = |v_z|^2 - |v_{\overline{z}}|^2$ is the Jacobian. It follows that the function $z = z(v)$ satisfies the equation

$$z_{\overline{v}} + q_1(z(v))\overline{z_v} + q_2(z(v))z_v = 0.$$

Consider the function $f(v) = w(z(v))$. It obviously belongs to the class $W_2(K)$ and maps the unit disk $|v| \leq 1$ onto the disk $|f| \leq 1$. We write the differential equation for the function $f(v)$. For this purpose we calculate the derivatives:

$$f_{\overline{v}} = w_z z_{\overline{v}} + w_{\overline{z}} \overline{z_{\overline{v}}} = q_2(z)\left[\overline{w_z}\,\overline{z_v} - w_z z_v\right],$$

$$f_v = w_z z_v + w_{\overline{z}}\overline{z}_v = w_z z_v(1 - |q_1(z)|^2) - q_1\overline{q}_2 w_z \overline{z}_v - q_2\overline{q}_1\overline{w}_z z_v - |q_2|^2\overline{w}_z\,\overline{z}_v,$$

$$f_v - \overline{f_v} = -(\overline{w_z}\,\overline{z_v} - w_z z_v)(1 + |q_2|^2 - |q_1|^2)$$

and we get

$$f_{\overline{v}} - \widetilde{q}(f_v - \overline{f_v}) = 0, \quad \widetilde{q} = \frac{-q_2(z(v))}{1 + |q_2|^2 - |q_1|^2},$$

i.e. $f(v)$ satisfies the equation of the type considered in the lemma. It is simple to verify that $|\widetilde{q}(v)| \leq \widetilde{q}_0 < \frac{1}{2}$ for $|v| < 1$. In view of conditions 1) and 2) of our theorem, the function $f(v)$ satisfies all the conditions of the lemma. Therefore $f(v) \equiv v$, hence $v(z) \equiv w(z)$ as was to be proved.

Corollary 6.1. *Let* G *be a domain bounded by a Jordan curve* Γ*. If* $w(z)$ *is a solution of equation* (6.1) *mapping* G *onto* K *and satisfying one of the following normalization conditions:* 1) $w(z)$ *maps three given points on* Γ *to the three prescribed points of the circle* $|w| = 1$ *or 2)* $w(z)$ *maps the two given points* $z_0 \in G$ *and* $z_1 \in \Gamma$ *onto the two assigned points* w_0*,* w_1 *of the disk* $\overline{K}$*:* $w_0 \in K$*,* $|w_1| = 1$*, then such a solution is unique.*

7 Generalized solutions of system (4.7) and quasiconformal mappings

In this section, we briefly consider the connection between the theory of solutions of equations(4.7) and the geometric theory of quasiconformal mappings.

In the classical case, the class of quasiconformal mappings is defined as the so–called mappings with continuous characteristics (with one or two pairs of characteristics), which are denoted by p, θ or p, θ, p_1 and θ_1, respectively, (introduced by M. A. Lavrent'ev in 1935). Since the corresponding precise definitions are rather long, we don't give them here and refer the reader Lavrent'ev's paper [19]. A quasiconformal mapping is said to be Q–quasiconformal if it's characteristics are bounded by the number $Q : p \leq Q$, $p_1 < Q$ everywhere in the considered domain. Lavrent'ev also introduced the so-called class A_Q of mappings with bounded distortion, i.e. the class of homeomorphic mappings for which at each point z_0, the following inequality holds

$$\overline{\lim_{\rho\to 0}}\ \frac{\max\limits_{\varphi_1} |w(z_0+\rho e^{i\varphi_1}) - w(z_0)|}{\min\limits_{\varphi} |w(z_0+\rho e^{i\varphi}) - w(z_0)|} \leq Q. \tag{7.1}$$

The classical definition of quasiconformal mapping was generalized by Pesin. He introduced the concept of so-called general quasiconformal mappings. In particular, he requires that the inequality $p \leq Q$ holds almost everywhere only (for a precise definition see [5]).

We recall some properties of the general Q–quasiconformal mappings. Let $w(z) = u + iv$ be such a mapping. It is known by the general theorems of D. E. Men'shov that u and v are differentiable in the Stolz sense a.e., it's partial derivatives are square integrable and at a.e. point satisfy the uniformly elliptic system of the partial differential equations (4.7). It is also known that u and v are absolutely continuous on a.e. lines parallel to the coordinate axes. It follows from these properties that $w = u+iv$ has the generalized derivatives in z and $\overline{z}$, satisfying system (4.7). We combine all that was said above in the following proposition.

Theorem 7.1. *A general Q–quasiconformal mapping is a generalized solution of a uniformly elliptic system* (4.7). *In the general case, the coefficients of the corresponding system are measurable functions. They are continuous if the characteristics of the mapping are continuous.*

We note that, the case of mappings with one pair of characteristics ($p_1 \equiv 1$) corresponds to the Beltrami systems. In this case, it is not difficult to show the inequality

$$|q(z)| \leq \frac{Q-1}{Q+1} \tag{7.2}$$

which illustrates the connection between the uniform boundedness of characteristics and the uniform ellipticity of the equations.

Theorem 7.1 allows us to ascertain that all properties proved above of solutions of equations (4.9) extend to the class of general Q - quasiconformal mappings. In this way we immediately obtain a series of partly known, and proved earlier by other methods, and partly new properties of general Q - quasiconformal mappings.

In particular, our methods allow us to assert the integrability of the derivatives of quasiconformal mappings with some exponent $p > 2$. Hence also follow the estimates of the Jacobian of a Q–quasiconformal mapping or estimates of the measure of the image of an arbitrary measurable set under Q–quasiconformal mapping etc.

In this section, our main goal is to prove an inverse theorem to Theorem 7.1. In this way we show the full equivalence of the class of univalent generalized solutions of equation (4.7) and the corresponding class of Q–quasiconformal mappings.

A series of distortion theorems from the geometric theory of quasiconformal mappings can be easily transformed to mappings realized by generalized solutions of the system (4.7). First of all, we note a simple proof of a distortion estimate for mappings realized by solutions of system (4.7) and conformal mappings. This proof we obtain as a straightforward consequence of some estimates for conformal mappings. A more precise estimate may be obtained if the corresponding result from the theory of quasiconformal mappings is used (see [26]).

Lemma 7.1. *Let $w(z)$ be the solution of equation* (4.7) *mapping the unit disk onto itself such that $w(0) = 0$, $w(1) = 1$. Then*

$$|w(z) - z| \leq K\varepsilon \ln \frac{1}{\varepsilon} + K_1 \varepsilon^{\alpha} \tag{7.3}$$

where K, K_1 and α are some universal constants, depending on q_0 only and $\varepsilon = \| |q_1(z)| + |q_2(z)| \|_{L_p}$.

Proof. According to Theorem 3.3, $w(z)$ may be represented in the form $w(z) = f(\chi(z))$ where

$$\chi(z) = z - \frac{1}{\pi} \int\limits_K \left[\frac{\omega}{t-z} - \frac{\omega}{t} \right] \, dK,$$

K is the unit disk, ω is a solution of the equation $\omega - qS(\omega) = q$ with $q = q_1 + q_2\,\frac{\overline{w_z}}{w_z}$. From the above equation we have:

$$|\chi(z) - z| \le C\|\omega\|_{L_p} \le C_1\|q\|_{L_p} \le C_1\varepsilon,$$

i.e. $\chi(z)$ maps K onto a Jordan domain, close enough to the disk. The above estimates for $\chi = \chi(z)$ and $z = z(\chi)$, where $z = z(\chi)$ is the inverse function for $\chi = \chi(z)$, and a theorem of Warszawski on the distortion of conformal mappings lead to the inequality (7.3), (see [27]). The applicability of the Warszawski estimate in our case can be easily established by the estimates for the function $z = z(\chi)$ following from Theorem 3.2. In view of the inequality (7.2), we have

$$\varepsilon \le \left\|\frac{p-1}{p+1}\right\|_{L_p} + \left\|\frac{p_1-1}{p_1+1}\right\|_{L_p}$$

where p and p_1 are characteristics of the mapping.

Analogously to the theory of ε–quasiconformal mappings (see [19] and [28]), a class of equations may be introduced, whose solutions have properties analogous to the properties of ε–quasiconformal mappings. These will be equations for which

$$\text{vrai}\ \max\left(|q_1| + |q_2|\right) < \varepsilon \tag{7.4}$$

in the considered domain. We will call such equations the ε–equations. In view of the inequality (7.2),

$$|q_1| + |q_2| \le \left|\frac{p-1}{p+1}\right|_{L_p} + \left|\frac{p_1-1}{p_1+1}\right|_{L_p},$$

it is obvious that ε–quasiconformal mappings are solutions of the ε'–equations and the quantities ε and ε' are small quantities of the same order. A fundamental property of the ε–equations is that they are invariant under conformal transformations of the independent (z) and dependent (w) variable. (In general, the above property does not hold for equations for which $\|q_1\|_{L_p} + \|q_2\|_{L_p} < \varepsilon$ and this is crucial in some respects.) The known lemmas on the distortion of rings for ε-quasiconformal mappings can be literally transferred to univalent solutions of the ε–equations. Moreover, in view of the above mentioned property of the class of ε–equations, the proofs directly apply to our case. In the following, we will use one of such lemmas that are stated below in the formulation of Volkovyskii (see Lemma 8.4 in [28]).

Lemma 7.2. *Let $w = w(z)$ be a generalized solution of the ε–equation* (4.7) *mapping the unit disk $|z| \le 1$ onto the disk $|w| \le 1$ with $w(0) = 0$. Then*

$$\left|\frac{w(z^1) - w(z^0)}{w(z^2) - w(z^0)}\right| < 1 + \eta(\varepsilon,\, \varepsilon'),$$

$$\left|\arg\frac{w(z^1)-w(z^0)}{w(z^2)-w(z^0)}-\arg\frac{z^1-z^0}{z^2-z^0}\right| < \eta(\varepsilon,\,\varepsilon') \tag{7.5}$$

for an arbitrary choice of the point z^0 *in the disk* $|z|<1$, *and points* z^1 *and* z^2 *in an annulus* $(1-\varepsilon')\rho \le |z-z^0| \le \rho$, *where* $0<\rho\le 1-|z^0|$, $0\le\varepsilon'<1$, *and* $\eta(\varepsilon,\,\varepsilon')$ *is finite for all* ε *and* ε', *such that* $\lim\limits_{\varepsilon,\varepsilon'\to 0}\eta(\varepsilon,\varepsilon')=0$.

¿From Theorems 1.3 and 3.3 follows

Lemma 7.3. *Any univalent generalized solution of equation* (4.7) *is differentiable in the Stolz sense, with the non-zero Jacobian, on a set of full measure.*

Let $w(z)$ be an univalent generalized solution of equation (4.7). Consider a point z_0 such that the mapping $w=w(z)$ is differentiable with the non-zero Jacobian and satisfies (4.7). The points of this type form a set of full measure. It can easily be seen that, the limit

$$\lim_{\rho\to 0}\frac{\max\limits_{\varphi}|w(z_0+\rho\mathrm{e}^{i\varphi})-w(z_0)|}{\min\limits_{\varphi_1}|w(z_0+\rho\mathrm{e}^{i\varphi_1})-w(z_0)|} \tag{7.6}$$

exists at point z_0 and does not exceed $\frac{1+q_0}{1-q_0}$. By Theorem 3.3 and Lemma 7.2, for $\varepsilon=q_0$ and $\varepsilon'=0$ we conclude that the upper limit

$$\overline{\lim_{\rho\to 0}}\frac{\max\limits_{\varphi}|w(z_0+\rho\mathrm{e}^{i\varphi})-w(z_0)|}{\min\limits_{\varphi_1}|w(z_0+\rho\mathrm{e}^{i\varphi_1})-w(z_0)|} < +\infty. \tag{7.7}$$

exists everywhere in the mapped domain. If the mapped domain does not coincide with the unit disk, to establish inequality (7.7) we first map the domain conformally onto the unit disk. From (7.6) and (7.7) we get the following theorem.

Theorem 7.2. *Any univalent generalized solution of equation* (4.7) *belongs to the class of general Q–quasiconformal mappings (moreover,* $Q\le\frac{1+q_0}{1-q_0}$*).*

Incidentally, it follows from the above considerations that the class of general Q–quasiconformal mappings coincides with the generalized class A'_Q of mappings with bounded distortion: for a mapping to belong to A'_Q fulfilment of the inequality (7.1) a.e. and the finiteness of the upper limit (7.1) everywhere is required (or everywhere except in a finite or countable set).

On the basis of the properties of the generalized solutions proved above, it is not difficult to establish other geometric properties of the mappings realized by solutions of system (4.7). However, in distinction to the boundedness of the distortion, which holds everywhere in each point of the mapped domain, these properties hold a.e. only. Thus, it is clear that the solutions of general systems (4.7) with measurable coefficients map infinitesimal ellipses with characteristics $p(z)$ and $\theta(z)$, centered at a.e. point of the mapped domain, to infinitesimal ellipses with characteristics $p_1(z)$ and $\theta_1(z)$. The quantities p, θ, p_1 and θ_1 are calculated from the coefficients q_1 and q_2 of equation (4.7), by simple formulas.

If the coefficients q_1 and q_2 of the equation (4.7) satisfy the Hölder condition, then the generalized solution is continuously differentiable and is a solution in the classical sense. Moreover, it is not difficult to show, that univalent solutions of such equations have a non zero Jacobian at each point of the mapped domain. Hence the geometric properties of the mappings are immediately established at each point of the domain.

If the coefficients of the equation are continuous only, then, in general, the generalized solution may not be differentiable at every point. Nevertheless, the geometric properties of the mapping are preserved at each point, even at those points where the Jacobian equals zero, or the mapping is not differentiable.

Theorem 7.3. *If $w = w(z)$ is a generalized solution of equation* (4.7) *with continuous coefficients, mapping the unit disk onto itself, then the mapping maps an infinitesimal ellipse with characteristics $p(z)$ and $\theta(z)$ into an infinitesimal ellipse with characteristics $p_1(z)$ and $\theta_1(z)$ at every point of the disk. The characteristics p, θ, p_1 and θ_1 are continuous functions of the point z.*

We briefly sketch a proof of this theorem, which can be obtained on the basis of our theory. As is well known, any solution of the general equation (4.7) can be represented as a superposition of a solution of the Beltrami equation and of an equation of the form

$$w_{\overline{z}} - q(z)\overline{w_z} = 0. \tag{7.8}$$

It is sufficient to consider each of these cases separately. Without loss of generality, we can assume that $w(0) = 0$ and $w(1) = 1$. Let $w(z)$ be a solution of the equation $w_{\overline{z}} - q(z)w_z = 0$ satisfying the conditions of the theorem. Let $w^n(z)$ be a sequence of solutions of the equations $w^n_{\overline{z}} - q_n(z)w^n_z = 0$, $w^n(0) = 0$ and $w^n(1) = 1$ where $q_n(z)$ are continuously differentiable, $|q_n| \leq q_0$ and $q_n \to q$ uniformly in the unit disk. By the uniqueness theorem, we have: $w^n(z) \to w(z)$ uniformly in the unit disk.

To apply the proof, which for an analogous problem may be found in the work of Lavrent'ev or in the book of Volkovysski (see [28]), to our case, it is enough to establish that the superpositions $f_{n,m} = w_n\left(z_m(w)\right)$ and $\widetilde{f}_{n,m} =$

$z_m(w_n(z))$ are solutions of the $\varepsilon_{n,m}$–equations to which Lemma 7.2 is applicable. We have the following equations for $f_{n,m}$

$$\frac{\partial f_{n,m}}{\partial \overline{w}} = \frac{q_n(z_m) - q_m(z_m)}{1 - q_n(z_m)\overline{q_m(z_m)}} \cdot \frac{\overline{\frac{\partial z_m}{\partial w}}}{\frac{\partial z_m}{\partial w}} \frac{\partial f_{n,m}}{\partial w}, \tag{7.9}$$

which are the $\varepsilon_{n,m}$–equations with $\varepsilon_{n,m} \to 0$. In the same way we argue in the case of the equation (7.8). Then, for $\widetilde{f}_{n,m} = z_m(w_n(z))$, we have

$$\frac{\partial \widetilde{f}_{n,m}}{\partial \overline{z}} - \frac{q_n(z) - q_m\left(\widetilde{f}_{n,m}(z)\right)}{1 - \overline{q_n(z)}q_m(\widetilde{f}_{n,m}(z))} \frac{\overline{\frac{\partial w_n}{\partial z}}}{\frac{\partial w_n}{\partial z}} \frac{\partial \widetilde{f}_{n,m}}{\partial z} = 0$$

which are the $\varepsilon_{n,m}$–equations as well, in view of the general uniqueness theorem ($\varepsilon_{n,m} \to 0$ for $n, m \to \infty$).

Next in the proof we use Lemma 7.2 and repeat the arguments from the just quoted works.

The following statement follows directly from Theorem 7.3.

Corollary 7.1. *The class of mappings realized by the generalized solutions of the system* (4.7) *with continuous coefficients coincides with the class of classical quasiconformal mappings with continuous characteristics.*

By the considerations of this section, the existence Theorem 5.1 contains the solution of the following classical problem of the theory of quasiconformal mappings (or non classical, when the characteristics are discontinuous): to map quasiconformally, with assigned characteristics $p(z)$, $\theta(z)$, $p_1(z)$, $\theta_1(z)$ a given simply connected domain D onto another such domain, fixing the correspondence of one pair of inner points and one pair of boundary points. The characteristics are supposed to be arbitrary measurable functions of z and to be uniformly bounded in the mapped domain. In this formulation, the theorem is presented in the author's paper [3].

Theorem 5.1 allows us also to map arbitrary simply connected domains, which are not necessarily Jordan domains. By Theorem 5.1, we can also solve the problem of quasiconformal mapping of two Jordan domains with a given correspondence of three pairs of similarly oriented boundary points of these domains, in the most general form. For this purpose, it is sufficient to apply the continuity method of Brauer. The uniqueness theorem and Lemmas 4.1 and 4.2 give all that is needed for the application of this method, (see [29]).

In the general formulation as above, the solution of the quasiconformal mapping problem, though, without the full detailed proof, can be found in work [10].

In connection with the geometric theory of quasiconformal mappings, we note the following geometric interpretation of Lemma 6.1 from which we obtained the uniqueness of the quasiconformal mapping of an arbitrary Jordan domain onto the unit disk. From the formulas presented in the book [28] it is easy to see that, systems of type (6.2) correspond to the quasiconformal mappings with two pairs of characteristics p, θ; p_1, θ_1, such that $p = p_1$, $\theta = \theta_1$. This means that the infinitesimal ellipses with characteristics p and θ are transformed by a parallel translation under the corresponding quasiconformal mapping. Lemma 6.1 asserts in this case that, if three points on the boundary of a disk stay fixed, then all the points of the disk stay fixed. In this formulation Lemma 6.1 is a generalization of the known result by D. E. Menshov who considered the case where infinitesimal circles are transformed by a parallel translation.

8 Some applications

The representations and estimates obtained above can be successfully used in the solution of a series of problems which naturally arise when studying elliptic systems of the form (2.6). Various types of representations may be useful for different problems. This was seen above when some special solutions of equation (2.6) in Section 4 were constructed. Below we briefly present some additorial examples.

1. *The Dirichlet problems for a disk.* In addition to the results of Subsection 3, Section 4, we now construct the solution of the Dirichlet problem, namely, to find a solution of equation (2.6) satisfying the conditions $\Re w(z) = f(z)$ for $|z| = 1$ and $\Im w(1) = 0$, where $f(z)$ is a given real valued function. We consider the most simple case when the problem can be solved by the method of successive approximations. Let $\Phi(z)$ be a holomorphic function for $|z| < 1$ such that $\Re\Phi(z) = f(z)$ for $|z| = 1$ and $\Im\Phi(1) = 0$. Suppose that $\Phi'(z) \in L_p(K)$ for some $p > 2$. Then we search the solution in the form

$$(8.1) \qquad w(z) = \Phi(z) - \frac{1}{\pi}\int\limits_K \left[\frac{\omega(t)}{t-z} + \frac{z\overline{\omega}}{1-z\overline{t}} - \frac{\omega}{t-1} - \frac{\overline{\omega}}{1-\overline{t}}\right] dK.$$

Substituting (8.1) in equation (2.6), we get for ω an integral equation of the form (4.14) for which the Fredholm theory is applicable, as shown in Section 4. This equation can be solved by the method of successive approximations according to the scheme (4.14′).

In the general case the uniqueness of the solution of the Dirichlet problem can be obtained from the representation (8.1) as shown above in the proof of

Theorem 4.5. If $f(z)$ is continuous only, then the desired solution may be obtained as a limit of the uniformly convergent sequence $w_n(z) \to w(z)$ where $w_n(z)$ is the solution of the problem $\Re w_n(z) = f_n(z)$, $\Im w_n(1) = 0$, $f_n \to f$ uniformly and f_n are sufficiently smooth functions. The convergence $w_n \to w$ follows from the maximum principle for homogeneous equations (4.6), which may be obtained, from the representation (4.10), in the following formulation: if $|w(z)| \leq m$ on $|z| = 1$, then $|w(z)| \leq M \cdot m$ for $|z| \leq 1$, with M depending on the coefficients of the equation only.

2. *Correctness of the Cauchy problem.* Let $w(z)$ be a solution of equation (4.6) such that $|w(z)| \leq M$ for $|z| \leq 1$ and $|w(z)| \leq m$ on an arc s of the circle $|z| = 1$. Then, from the known estimates for holomorphic functions and from Theorems 3.6 and 4.4 and Lemma 3.1, we get the estimate of the form $|w(z)| \leq f_r(m, M)$ where $f_r(m, M) \to 0$ as $m \to 0$ uniformly in every inner disk $|z| \leq r < 1$. This estimate establishes the stability of the Cauchy problem in the class of bounded solutions.

3. *Behavior of the mapping at a boundary point.* Let $w = w(z)$ be a solution of the equation (4.7) mapping the unit disk $|z| \leq 1$ onto itself, $w_0 = w(z_0)$, $|z_0| = 1$. Then the following is true.

Theorem 8.1. *If*

$$\left|\frac{q_1(z) - q_1(z_0)}{z - z_0}\right| \in L_p(K) \quad \textit{and} \quad \left|\frac{q_2(z) - q_2(z_0)}{z - z_0}\right| \in L_p$$

for some $p > 2$, then the mapping $w = w(z)$ is differentiable at the point z_0 and the Jacobian $J = |w_z|^2 - |w_{\overline{z}}|^2$ is not zero at z_0.

Proof. We consider the case $q_1(z_0) = q_2(z_0) = 0$. The general case is reduced to this one by a linear affine and elementary conformal transformations of dependent and independent variables in equation (4.7). We extend $w(z)$ by symmetry outside of the disk $|z| \leq 1$ setting $w^*(z) = w(z)$ for $|z| < 1$ and $w^*(z) = \frac{1}{\overline{w(\frac{1}{\overline{z}})}}$ for $|z| > 1$. Consider the function

$$\Psi = \frac{w^*(z) - w_0}{z - z_0}.$$

¿From equation (4.7) and from the extension formulas we get the following equation for $\Psi(z)$

$$\Psi_{\overline{z}} - \widetilde{q}_1 \Psi_z - \widetilde{q}_2 \overline{\Psi_z} = A\Psi + B\overline{\Psi} \tag{8.2}$$

with discontinuous coefficients. For $|z| < 1$, we have

$$\widetilde{q}_1 = q_1, \quad \widetilde{q}_2 = q_2 \frac{\overline{z} - \overline{z_0}}{z - z_0}, \quad A = \frac{q_1}{z - z_0}, \quad B = \frac{q_2}{z - z_0}.$$

Analogous formulae hold for $|z| > 1$. Therefore $|\widetilde{q}_1| + |\widetilde{q}_2| < q_0 < 1$ and $A, B \in L_p$ in a neighborhood of the point z_0. By the representation theorem we have $\Psi(z) = f(\chi(z))\, e^{\varphi(z)}$ where $\varphi(z)$ is continuous at the point z_0, $\chi = \chi(z)$ is a homeomorphism of a neighborhood of z_0 onto itself such that $\chi(z_0) = z_0$, and $f(\chi)$ is an analytic function in the neighborhood of $\chi_0(z_0) = z_0$. It is clear that χ_0 cannot be an essential singular point of $f(\chi)$. But, in view of the univalentness of $w(z)$, the increment of the argument $\Delta \arg_{|z-z_0|=r} \Psi = \Delta \arg_{|\chi-\chi_0|=r} f(\chi) = 0$ for sufficiently small r, therefore χ_0 is a regular point of the function $f(\chi)$ and $f'(\chi_0) \neq 0$, what was to be proved.

Remark. With the help of conformal mappings, Theorem 8.1 can be extended to mappings of the unit disk onto an arbitrary domain with a sufficiently smooth boundary curve. The theorem holds also if z_0 is an inner point of the domain and $w = w(z)$ is a univalent solution of the equation (4.7) in some annulus around z_0. Then the proof can be simplified since we don't need to extend the function $w(z)$ to the exterior of the unit disk. In this formulation, for mappings with one pair of the characteristics, Theorem 8.1 is proved in another way in paper [30].

4. *Quasilinear equations.* Theorems 4.4 and 5.1 preserve their validity for wide classes of quasilinear equations of the type

$$w_{\overline{z}} - q_1(z,w) w_z - q_2(z,w)\overline{w_z} = F(z,w)\ , \tag{8.3}$$

and the proof, under some assumptions on q_1, q_2 and F as functions of w ($|q_1| + |q_2| \leq q_0 < 1$, $|F(z,w)| \leq A(z)|w|$, $A(z) \in L_p$, $p > 2$, and, in the simplest case, when q_1 and q_2 are continuous in w) does not need almost any change.

We consider the case $F(z,w) \equiv 0$ in some detail. Let the equation be given in the unit disk K, $q_1(z,w)$ and $q_2(z,w)$ defined for $z \in K$ and $w \in \Delta$, where Δ is a domain of the plane (it can be the full complex plane) and let$|q_1| + |q_2| \leq q_0 < 1$ for $z \in K$ and $w \in \Delta$. Then the following holds.

Theorem 8.2. *Let $f(\chi)$ be any analytic function for $|\chi| < 1$ such that $f(K) \subset \Delta$. Then there exists a homeomorphic mapping of K onto itself, $\chi = \chi(z)$, $\chi(0) = 0$, $\chi(1) = 1$, such that $w(z) = f(\chi(z))$ is a solution of the equation* (8.3) *(with $F \equiv 0$) and $\chi = \chi(z)$ is a solution of an equation of the form* (4.7). *It is assumed, that $q_1(z,w)$ and $q_2(z,w)$ are continuous in w for a.e. $z \in K$, or, if they are of the form $q_1 = q_1'(z) \cdot q_1''(w)$ and $q_2 = q_2'(z) \cdot q_2''(w)$, that they are measurable only.*

The proof of the above theorem is analogous to the proof of Theorem 5.1. In the case when $q_1 = q_1'(z) \cdot q_1''(w)$ and $q_2 = q_2'(z) \cdot q_2''(w)$, the estimate (3.48), for the measure of the image under mappings realized by solutions of equations (4.7), is to be used.

The important case of Theorem 8.2, when $f(\chi)$ is univalent and maps K on to a simply connected domain Δ, gives a new proof of Shapiro's theorem on the existence of a quasiconformal mapping of the unit disk K on Jordan domain with a given correspondence of a pair of boundary points and a pair of inner points, or three pairs of boundary points, see [31]. Our conditions on the coefficients of the equation are more general than in [31].

When considering univalent solutions of equation (8.3) ($F \equiv 0$), the roles of the variables z and w are symmetric; therefore, Theorem 8.2 holds if we interchange the variables z and w.

We now present some uniqueness theorems for quasiconformal mappings corresponding to quasilinear systems of the type (8.3) ($F \equiv 0$).

Theorem 8.3. *Let* $|q_1(z,w)-q_1(z',w)| \le M_1|z-z'|$ *and* $|q_2(z,w)-q_2(z',w)| \le M_2|z-z'|$ *where* M_1, M_2 *are constants not depending on* z, z' *and* w. *Then the solution of equation* (8.3) *mapping the unit disk on to the Jordan domain is unique provided that it is normalized as in Theorem* 6.1.

Proof. Let $w = w(z)$ and $v = v(z)$ be two such solutions. Denote by $z = z(v)$ the inverse function to $v = v(z)$, and consider the superposition $f(z) = z\,(w(z))$. Obviously $f(z)$ maps the unit disk onto itself. We construct two equations for $f(z)$. We have the following formulae for the derivatives $f_{\overline{z}}$ and f_z:

$$
\left.\begin{aligned}
f_{\overline{z}} = z_v w_{\overline{z}} + z_{\overline{v}}\overline{w_z} = [q_2(z,w) - q_2(f(z),w)]\, z_v \overline{w_z} + \\
+ [q_1(z,w) - q_1(f(z),w)]\, z_v w_z + q_1(f(z),w)\, [z_v w_z - \overline{z_v}\ \overline{w_z}]\,, \\
f_z = z_v w_z + z_{\overline{v}} \overline{w_z}, \\
f_z - \overline{f_z} = (z_v w_z - \overline{z_v}\ \overline{w_z})\,(1 + |q_1|^2 - |q_2|^2)\,.
\end{aligned}\right\} \tag{8.4}
$$

¿From these we see that $f(z)$ satisfies the equations

$$
\begin{aligned}
f_{\overline{z}} - \widetilde{q}(f_z - \overline{f_z}) = [q_2(z,w) - q_2(f(z),w)]\, z_v \overline{w_z} + \\
+ [q_1(z,w) - q_1(f(z),w)]\, z_v w_z,
\end{aligned} \tag{8.5$'$}
$$

$$
f_{\overline{z}} - Q f_z = 0 \tag{8.5$''$}
$$

where

$$
\widetilde{q} = \frac{q_1\,(f(z),w)}{1 + |q_1|^2 - |q_2|^2}, \quad |\widetilde{q}| \le \widetilde{q}_0 < \frac{1}{2}
$$

and Q is a coefficient such that $|Q| \leq q_0' < 1$ for $|z| < 1$. Extending the function $f(z)$ to a disk $|z| \leq 1 + \eta$ by symmetry and denoting the obtained function by $f^*(z)$, we get equation of the type (8.5′) for $f^*(z)$. Introducing the function $F(z) = f^*(z) - z$ we observe that it is a continuous generalized solution of the equation

$$F_{\bar{z}} - \widetilde{q}_1 F_z - \widetilde{q}_2 \overline{F_z} = AF \tag{8.5}$$

in the disk $|z| \leq 1 + \eta$ for sufficiently small η. Moreover, $|\widetilde{q}_1| + |\widetilde{q}_2| \leq q_0 < 1$. From (8.5), (8.5′) and (8.5″) and from the conditions of Theorem 8.3 we see that the only unbounded terms in the expression of $A(z)$, for sufficiently small η, may be the products $|z_v \overline{w_z}|$ by constants. But it is obvious that these products are integrable in the disk $|z| \leq 1 + \eta$ with some exponent $p > 2$. Indeed, we have $|z_v w_z| \leq \frac{f_z}{1 - {q_0'}^2}$ for $|z| \leq 1$ and in view of Theorem 3.5, $|f_z|$ is integrable with some exponent $p > 2$ since $f(z)$ maps the unit disk onto itself and satisfies equation (8.5″). This also implies the integrability of $A(z)$ in the annulus $1 < |z| < 1 + \eta$ with some exponent $p > 2$, and the p-integrability of $A(z)$ in the disk $|z| \leq 1 + \eta$.

Thus $F(z)$, either vanishes in the disk $|z| < 1 + \eta$ for all z, or vanishes at most on an isolated set of points; in this case to $F(z)$ the argument principle can be applied. As in the proof of Theorem 6.1, we see that the normalization conditions of the mappings exclude the last possibility. Therefore $F(z) \equiv 0$, i.e. $w(z) \equiv v(z)$ as was to be proved.

Interchanging the roles of z and w in Theorem 8.3, we obtain the following.

Corollary 8.1. *A solution of equation* (8.3)*, mapping the Jordan domain* Δ *onto the unit disk, and satisfying the same normalization conditions as in Theorem* 6.1*, is unique, provided that* $q_1(z,w)$ *and* $q_2(z,w)$ *satisfy a Lipschitz condition with respect to* w*, uniformly for* $z \in \Delta$*.*

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