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QUANTUM KINETIC THEORY WITH NONLOCAL COHERENCE

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Preface

The work reviewed in this thesis has been carried out during the years 2005-2009 at the Department of Physics in the University of Jyväskylä.

I am most grateful to my supervisor Doc. Kimmo Kainulainen for excellent guidance and support during these years. I have been privileged to share his profound ideas and wide knowledge of particle physics and cosmology. A major part of this work has been done in collaboration with Pyry Rahkila, to whom I want to extend my warmest thanks. I also want to express my gratitude to Prof. emer. Vesa Ruuskanen who introduced me to Kimmo many years ago, and to Prof. Mikko Laine and Prof. Kari Rummukainen for careful reading of the manuscript and valuable comments. I also wish to thank the staff and friends at the Department of Physics for creating an inspiring and pleasant working atmosphere.

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Finally, I wish to thank Jenni and my family for their love, support and patience.

List of publications

This thesis is based on the work contained within the following publications:

I **Towards a kinetic theory for fermions with quantum coherence**
M. Herranen, K. Kainulainen and P. M. Rahkila,
Nucl. Phys. B **810** (2009) 389 [arXiv:0807.1415 [hep-ph]].

II **Quantum kinetic theory for fermions in temporally varying backgrounds**
M. Herranen, K. Kainulainen and P. M. Rahkila,
JHEP **0809** (2008) 032 [arXiv:0807.1435 [hep-ph]].

III **Kinetic theory for scalar fields with nonlocal quantum coherence**
M. Herranen, K. Kainulainen and P. M. Rahkila,
JHEP **0905** (2009) 119 [arXiv:0812.4029v2 [hep-ph]].

The author has participated equally with K .Kainulainen and P. M. Rahkila in the development of papers I-III. A large part of the analytical calculations, in particular in papers II and III, was carried out by the author. The draft versions of papers II and III were largely written by the author.

Contents

Chapter 1

Introduction

A wide range of problems in modern high energy physics and cosmology involves the dynamics of quantum fields in highly out-of-equilibrium conditions, including relativistic heavy ion collisions [1–3], quantum fluctuations in inflationary cosmology [4–6], preheating after inflation [6], and baryogenesis [7]. It turns out that the traditional methods of (vacuum or thermal) quantum field theory (QFT) are not suited to describe these complicated non-thermal processes, and a new framework of nonequilibrium QFT [8,9] is needed. Even though a knowledge of the complete dynamics of interacting quantum fields is clearly beyond reach, there is a variety of approximative methods that are able catch the essentials of the problems under study. In a so called kinetic regime with weak interactions and slowly varying classical backgrounds the standard methods of quantum kinetic theory reduce the problem considerably to solving the famous (quantum) Boltzmann transport equations. For many problems of interest these transport equations provide a remarkably good approximation for the essentials of nonequilibrium quantum dynamics. However, certain problems are inherently very sensitive to quantum coherence (or interference), quantum reflection from a potential being a typical example. The Boltzmann equation approach inevitably loses the effects of nonlocal quantum coherence, and thus is not very well suited to study for example the quantum reflection problem in electroweak baryogenesis or the particle production in preheating.

In this thesis we present a novel approximation scheme related to the quantum kinetic theory, that enables us to treat nonlocal quantum coherence in the presence of decohering collisions with simple enough Boltzmannian-type transport equations. The key element in our scheme is the finding of new singular shell solutions in the phase space of 2-point correlation function, that are located at $k_0 = 0$ for spatially homogeneous problems and at $k_z = 0$ for a static planar symmetric case. When the complete phase space structure, including these new coherence solutions in addition to the standard mass-shell contribution, is inserted in the Kadanoff-Baym (KB) equations for the correlator, we obtain a closed set of transport equations for the corresponding on-shell distribution functions, thus giving an extension to the standard quantum Boltzmann equation to include nonlocal coherence.

The thesis consists of three original research papers [I]- [III] and an introductory and summary part presented below. In chapter 2 we introduce the basic mechanisms of (electroweak) baryogenesis and preheating, that are good examples of highly nonequilibrium processes in the early universe. In chapter 3 we present the basic formalism of nonequilibrium QFT needed to derive the KB-equations for two point correlation functions, using the two-particle irreducible (2PI) effective action method. Chapter 4 then presents a detailed survey of the main contents of our work, the novel approximation scheme. In chapter 5 we review a few applications that we have so far considered using our formalism, including the Klein problem, (collisionless) quantum reflection from a CP-violating mass wall, and examples of coherent production of decaying fermionic and scalar particles relevant for preheating. Finally, chapter 6 contains conclusions and outlook.

Chapter 2

Nonequilibrium processes in the early universe

According to modern theories of cosmology and particle physics the expanding universe has once been in an extremely dense and hot state (Hot Big Bang scenario), consisting of quantum plasma, which during the major part of the early evolution is very close to thermal equilibrium. Besides this overall picture of thermal plasma, however, many crucial processes in the early universe are inherently highly non-thermal, including inflation, preheating, and baryogenesis. The careful understanding of these processes is of primary importance in modern cosmology, providing an important field of applications to the methods of nonequilibrium quantum field theory. In this chapter we introduce the basic mechanisms of baryogenesis, focusing on a model called electroweak baryogenesis (EWBG), and the process of preheating after inflation.

2.1 Baryogenesis

The visible matter content of the universe, such as planets, stars and interstellar gas, consists of protons, neutrons and electrons. In astrophysics it is classified as baryonic matter, since the bulk of the mass is in protons and neutrons that are baryons. There is strong evidence that no large domains of antimatter exists in the universe [10, 11], implying that the universe has an excess of baryons compared to antibaryons which is called baryon asymmetry. The combination of data including several experiments of the fluctuations of cosmic microwave background (CMB) gives the following experimental measure of this asymmetry, the average baryon to photon number ratio in the universe [12]:

$$\frac{n_B}{n_\gamma} = \left(6.1^{+0.3}_{-0.2}\right) \times 10^{-10} \,. \tag{2.1}$$

Of course it might be possible that the baryon asymmetry is an initial condition in the evolution of the universe. Despite being very unnatural, this explanation for the baryon asymmetry is not consistent with the cosmological inflation [4, 5], which is one of the backbones of modern cosmology explaining the homogeneity and flatness of the universe as well as the primordial density fluctuations that will give rise to structure formation and the observed fluctuations in the CMB spectrum. The problem is that the exponential increase in the size of the universe by at least a factor e^{60} during the inflationary period dilutes any prior baryon number to totally negligible level. For this reason, we are very tempted to seek out different ways of creating the baryon asymmetry in the universe after the period of inflation.

A process that gives rise to a permanent baryon asymmetry at cosmological scales is called *baryogenesis*. The idea of such a process originates from Sakharov [13], who presented three conditions that any model for baryogenesis should necessarily fulfill:

1. Baryon number violation.
2. C and CP symmetry violations.
3. Departure from thermal equilibrium.

The first condition is obvious. If the second is not fulfilled, then for every reaction producing particles there is a counter-reaction that produces antiparticles at the same rate. The third condition is the most interesting for the scope of this work. It follows from the CPT-theorem that the masses of particles and antiparticles are equal, and consequently the thermal average of the baryon number will vanish in equilibrium. We conclude that every scenario for baryogenesis must be a nonequilibrium process.

Several models for baryogenesis have been proposed that fulfill the Sakharov conditions (for a recent review see *e.g.* . [7]), including GUT baryogenesis [4], electroweak baryogenesis [14], leptogenesis [15] and Afflect-Dine baryogenesis [16] as the most prominent candidates. The first of these, GUT baryogenesis, is based on decays of heavy gauge bosons with masses of order $M_{\rm GUT} \approx 10^{16}$ GeV. While it provides a scenario fulfilling all the Sakharov conditions, it has serious problems with inflationary models related to the high reheating temperature required, and the consequent overproduction of gravitinos [7]. The latter models, electroweak baryogenesis, leptogenesis and some variants of Affleck-Dine baryogenesis are based (directly or indirectly) on electroweak baryon number violation [17], which is a quantum anomaly in the electroweak sector of the standard model allowing the baryon number to be badly violated at high temperatures. These models differ however substantially in the mechanisms of how the required out-of-equilibrium conditions are reached. In what follows we will focus on EWBG in more detail, trying to elaborate the basic mechanism and the necessity to use the methods of nonequilibrium quantum field theory in its study.

2.1.1 Electroweak baryogenesis

Electroweak baryon number violation

Baryon and lepton numbers are classically conserved in the standard model. However, at the quantum level this is not the case. It can be shown that the axial current in the electroweak sector of the standard model and consequently the total baryon and lepton number currents j_B^μ and j_L^μ are anomalous *i.e.* not exactly conserved [18, 19]:

$$\partial_\mu j_B^\mu = \partial_\mu j_L^\mu = N_f \left(\frac{g^2}{32\pi^2} \tilde{W}^a_{\mu\nu} W^{a,\mu\nu} - \frac{g'^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) , \tag{2.2}$$

where N_f is the number of fermionic families, $W^a_{\mu\nu}$ and $F_{\mu\nu}$ are the field strength tensors of the $SU(2)_L$ and $U(1)_Y$ gauge symmetries with the duals $\tilde{W}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} W^a_{\rho\sigma}$ and an analogous expression for $\tilde{F}$, and g and g' are the associated coupling constants, respectively. Equation (2.2) implies that the total change in baryon (lepton) number from time $t = 0$ to some arbitrary final time t_f is given by:

$$\Delta B = \Delta L = N_f \left[N_{CS}(t_f) - N_{CS}(0) \right] - N_f \left[n_{CS}(t_f) - n_{CS}(0) \right] , \tag{2.3}$$

where

$$\begin{aligned} N_{CS} &= \frac{g^2}{32\pi^2} \int d^3x \epsilon^{ijk} \left(W^a_{ij} A^a_k - \frac{1}{3} g \epsilon_{abc} A^a_i A^b_j A^c_k \right) \\ n_{CS} &= \frac{g'^2}{32\pi^2} \int d^3x \epsilon^{ijk} F_{ij} B_k \end{aligned} \tag{2.4}$$

are called the Chern-Simons numbers of the $SU(2)_L$ and $U(1)_Y$ gauge symmetries. We proceed by considering the vacuum structure of the gauge fields A^a_μ and B_μ. It turns out that the abelian $U(1)_Y$ sector has a trivial nondegenerate vacuum with $\vec{B} = 0$, but the $SU(2)_L$ sector instead has a discrete set of degenerate vacua with

$$\vec{A} = \frac{1}{i} g_n^{-1} \vec{\nabla} g_n , \tag{2.5}$$

where $g_n(\vec{x}) = e^{inf(\vec{x})\hat{x}\cdot\tau/2}$, n is an integer and τ_i are the generators of the $SU(2)$ gauge group. Using the vacuum structure Eq. (2.5) in Eqs. (2.3)-(2.4) we see that the change in baryon number in transitions between the different vacua is given by

$$\begin{aligned} \Delta B &= N_f \Delta N_{CS} = N_f \frac{g^2}{32\pi^2} \int d^3x \, \epsilon^{ijk} \mathrm{Tr} \left[g_n^{-1} \partial_i g_n g_n^{-1} \partial_j g_n g_n^{-1} \partial_k g_n \right] \\ &= N_f n . \end{aligned} \tag{2.6}$$

We see that the baryon number changes in integer multiples of N_f in the transitions between different vacua. The structure of the effective potential

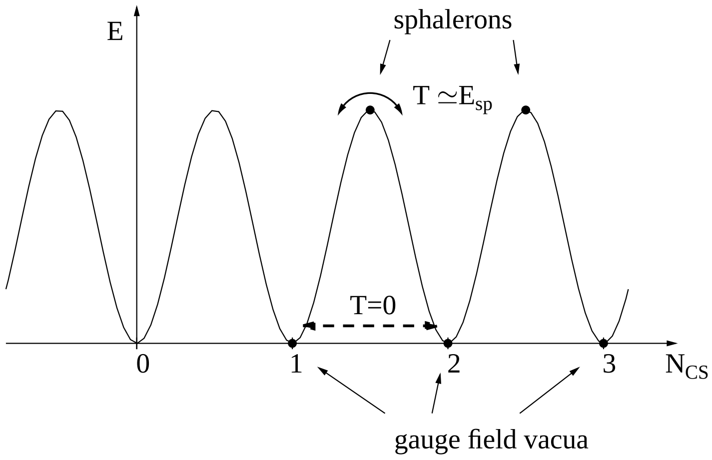

Figure 2.1: The vacuum structure of the $SU(2)_L$ gauge fields. The set of degenerate vacua can be labelled with an integer Chern-Simons number N_{CS}. At zero temperature the transitions between different vacua by quantum tunneling through the potential barrier are extremely suppressed. At high temperatures the transitions are possible by thermal activation via sphaleron configurations.

for the gauge fields is sketched in Fig. 2.1, where the minima correspond to different vacuum configurations labelled by the (integer) Chern-Simons number N_{CS}. But how could the transitions between different vacua actually take place? At zero temperature the only possibility is by quantum tunneling through the potential barrier. This corresponds to the so called *instanton* configuration [17], but it turns out that the tunneling rate is negligible: $\Gamma_{\text{in}} < e^{-4\pi/\alpha_W} \sim 10^{-170}$. With this rate not a single proton could have been produced in the lifetime of the universe! At high temperature the situation is better. The transitions between the vacua can take place through a thermal activation over the potential barriers, via the so called *sphaleron* field configuration [20, 21]. The thermal transition rate corresponding to this process is shown to be [14, 22]

$$\Gamma_{\text{sp}} \sim T^4 e^{-E_{\text{sp}}(T)/T}, \tag{2.7}$$

where E_{sp} is the energy of the sphaleron configuration, which is related to the vacuum expectation value of the Higgs field $\langle\phi\rangle$ by $E_{\text{sp}}/T \simeq 40\langle\phi\rangle/T$. This result is valid only in the broken phase *i.e.* when the gauge symmetry is spontaneously broken and the gauge bosons are massive. In the symmetric phase with massless gauge bosons the sphaleron rate is instead given by (in the minimal standard model) [23]

$$\Gamma_{\text{sp}} \simeq (25.4 \pm 2.0)\alpha_W^5 T^4 \,, \tag{2.8}$$

where $\alpha_W \approx 1/30$ is the weak coupling constant. To see if the sphaleron transitions are fast at the time scales of the expanding early universe, the sphaleron rate of a unit comoving volume: $\Gamma_{\rm sp}/T^3$, needs to be compared with the Hubble expansion rate of the (radiation dominated) universe [4]

$$H = 1.66\, g_*^{1/2} \frac{T^2}{m_{\rm pl}}\,, \tag{2.9}$$

where $g_*(T)$ counts the total number of effectively massless degrees of freedom and $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV is the Planck mass. In the minimal standard model at high temperatures of order $T \gtrsim 100$ GeV we have $g_* = 106.75$ so that the symmetric phase sphaleron rate in Eq. (2.8) is very large compared to the Hubble rate. Moreover, if $\langle\phi\rangle/T \lesssim \mathcal{O}(1)$ then also the broken phase sphaleron rate in Eq. (2.7) is large, and the first Sakharov condition is fulfilled in both of the phases. Actually, it is crucial in EWBG that the broken phase sphaleron rate is smaller than the Hubble rate so that the the generated baryon asymmetry is not washed out. Next we will briefly consider the electroweak phase transition between the symmetric and broken phases, where the gauge bosons and fermions become dynamically massive. This transition takes place at the (electroweak) temperature scale $T \sim 100$ GeV, and it provides a scheme for the creation of a permanent baryon asymmetry through EWBG, if the transition is of first order.

Electroweak phase transition

In the electroweak theory of the standard model [24–26] the masses of the gauge bosons $W^\pm$ and Z^0 and all fermions are generated through spontaneous symmetry breaking of the gauge symmetry $SU(2)_L \times U(1)_Y$. The "classical" potential of the ($SU(2)_L$ doublet) Higgs scalar field Φ:

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\,, \tag{2.10}$$

where $\phi \equiv \sqrt{2\,\Phi^\dagger\Phi}$, is minimized for $\phi = v_0 \equiv \mu/\sqrt{\lambda}$ corresponding to a degenerate set of vacuum configurations. Choosing a particular vacuum from this set breaks the gauge symmetry spontaneously giving rise to masses for the gauge bosons, which are proportional to the vacuum expectation value v_0 of the Higgs field. This is how the Higgs mechanism works at zero temperature.

At finite temperatures the "classical" Higgs potential Eq. (2.10) gets temperature dependent quantum corrections that will become more and more important when the temperature increases. These corrections are taken in account by computing the free energy (or the effective potential) of the Higgs field [7]:

$$V_{\rm eff}(\phi, T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda}{4}\phi^4 + ... \tag{2.11}$$

where the values of the parameters T_0, D and E depend on the considered model. We have only given the dominant terms from perturbative calculations. The

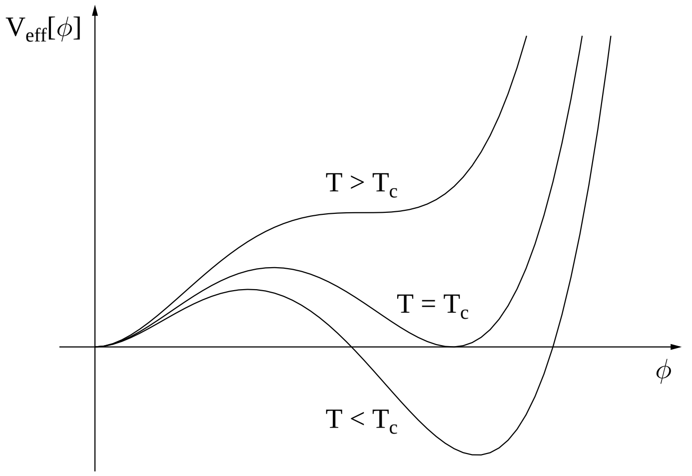

Figure 2.2: The Higgs field effective potential for a first order phase transition at different temperatures. Because of the barrier between the local minima, the phase transition takes place at a temperature $T < T_c$ releasing latent heat.

temperature dependence of the potential Eq. (2.11) gives rise to an interesting behaviour of the Higgs field. Let us first consider the case with vanishing third order term $E = 0$: when $T > T_0$ the second order term is positive implying that the minimum of the potential and the corresponding vacuum expectation value (VEV) of the Higgs field is zero: $\langle\phi\rangle = 0$, while for $T < T_0$ it is finite: $\langle\phi\rangle \neq 0$, since the negative second order term dominates for small ϕ. At the critical temperature $T = T_c = T_0$ there will be a phase transition from the former symmetric phase to the latter broken phase. This is called *electroweak phase transition*, and for the case $E = 0$ it is a second order transition with smoothly increasing VEV $\langle\phi\rangle$ as the temperature decreases. The case with $E > 0$ presented in Fig. 2.2 is more interesting for us. At the critical temperature $T = T_c$ there is now a potential barrier between the symmetric and broken minima. Because of this barrier the phase transition actually takes place at a lower temperature $T < T_c$, and latent heat is released due to the energy difference between the minima. Hence the transition is of first order and it happens by nucleation of broken phase bubbles inside the symmetric phase bulk. These bubbles start to grow rapidly reaching soon a stationary expansion speed. This scenario with first order phase transition is of crucial importance for the electroweak baryogenesis. It is just around the edge (or wall) of these broken phase bubbles where the last two Sakharov conditions of CP violation and departure from thermal equilibrium are fulfilled. Unfortunately, it has been shown by nonperturbative lattice calculations that for the minimal standard model the phase transition is not of first order [27]. For this and other reasons (*e.g.* not enough CP violation) we have to seek new possibilities for electroweak baryogenesis in extensions of the

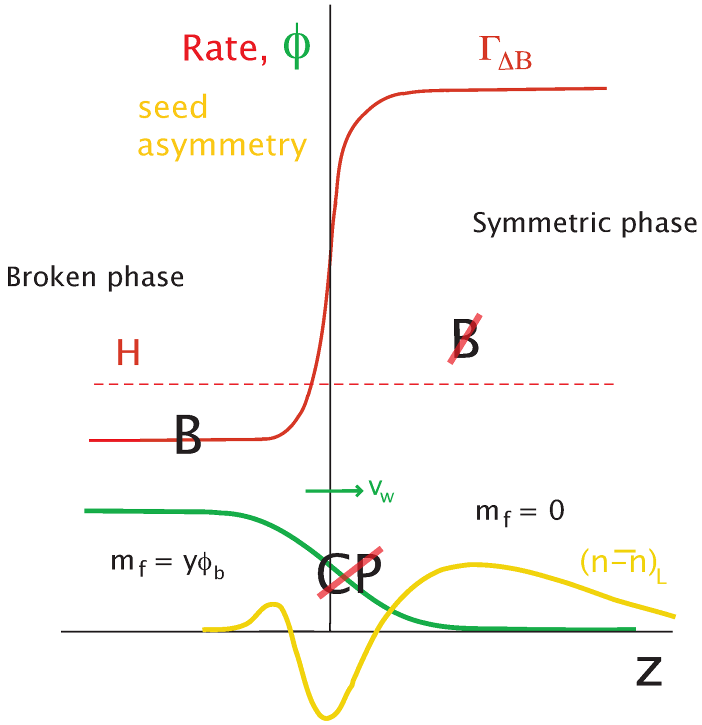

Figure 2.3: Sphaleron rate (red), vacuum expectation value of the Higgs field (green), and the source asymmetry (yellow) in the bubble wall region as a function of the coordinate normal to the wall front.

standard model, such as the minimal supersymmetric standard model (MSSM). Next we will give a brief conceptual description of the mechanism of (eletroweak) baryogenesis for a generic model with a strong first order phase transition.

Description of the EWBG mechanism

In Fig. 2.3 we present a schematic cross-section from the expanding bubble wall front where baryogenesis takes place. In the bubble wall region the VEV of the Higgs field and consequently the masses of the fermions (green line) are spatially dependent changing smoothly from zero in the symmetric phase to a finite value in the broken phase. The varying complex mass of a fermionic eigenstate gives rise to the required CP violating effects that will generate an asymmetry in left chiral number densities between particles and antiparticles (yellow line). This source asymmetry then creates a pseudo chemical potential that biases the sphaleron transitions in front of the wall in the symmetric phase to produce more baryons than antibaryons. This net baryon number is not washed away in the broken phase, because the sphaleron rate (red line) is too small there compared to the Hubble expansion rate H. At the end, when the bubble wall has passed and the plasma is back in thermal equilibrium, a net baryon number density has been created.

One of the most difficult problems in the actual calculations of the baryon asymmetry is to find out the source asymmetry due to the CP violating effects. An accurate calculation would necessarily involve the use of nonequilibrium quantum field theory, and this is the background motivation for our work. In earlier works the problem has been studied in the semiclassical WKB approach [28–33] and later with the methods of quantum kinetic theory in [34–38], both approaches using the Boltzmann transport equations for CP-violating phase space densities. These methods should provide a solid approximation in the semiclassical limit *i.e.* the case of a thick wall compared to the mean free path of the interacting fermions. However, in the thin wall limit the dominant source for the asymmetry comes from the quantum reflection processes, which are inherently nonlocal and absent in the standard WKB and kinetic approaches. Attempts have been made to treat the reflection phenomena by including collisions in the Dirac equation [39–43], but no consistent framework based on quantum field theory has been introduced. In this work we present an approximation scheme based on the kinetic approach that enables us to treat the quantum reflection in a simple but consistent way in the presence of decohering collisions.

2.2 Preheating

Cosmological inflation [4, 5, 44–52] (for a recent review see *e.g.* [6]) is a period of rapid exponential expansion in the very early universe during which the size of the universe increases by a huge factor (at least e^{60}). As mentioned in the beginning of last section, it provides a natural explanation for the homogeneity (horizon problem) and flatness of the universe. It also explains the (almost) scale invariant primordial density fluctuations that will give rise to large-scale structure formation and the observed anisotropies in the CMB spectrum [12]. Most of the inflationary models are based on the peculiar dynamics of one or several scalar fields, called inflaton(s), whose vacuum condensate dominates the evolution of the universe in a so called “slow roll” phase, causing the exponential growth. Because of the huge and very rapid expansion the universe is typically[1] in a highly non-thermal and very cold state at the end of inflation. However, the baryogenesis scenarios require energies greater than the electroweak scale and the primordial nucleosynthesis requires that the universe is close to thermal

[1]The warm inflation scenario with particle production during inflation is an exception [53, 54].

equilibrium at the temperature ~ 1 MeV at some stage after inflation. A mechanism for reheating the universe after inflation is thus needed to retain the Hot Big Bang scenario.

The modern scenarios of reheating consist of a *preheating* stage followed by thermalization. In the preheating stage [55–60] the inflaton condensate goes through rapid oscillations, so that the couplings to other matter fields give rise to particle production via parametric resonance. The basic mechanism is simple: the coupling to the inflaton(s) gives rise to rapidly oscillating effective masses for the matter fields that will bump up the particle numbers exponentially for certain momentum modes in close analogy with the Floquet theory of growing exponents [61,62]. The modes in these resonance bands will quickly obtain huge occupation numbers (for scalars with no Fermi blocking). This rapid growth of perturbations is followed by backreaction and rescatterings. *Backreaction* means the effects of the growing perturbations (particle production) back on the dynamics of the inflaton condensate. In most scenarios it will rather quickly shut off the inflaton oscillations (faster than the Hubble expansion rate of the universe) and consequently terminate the particle production. Before that the fluctuations of the inflaton field itself will grow and give rise to *rescatterings i.e.* couplings between different momentum modes leading to the growth of occupation numbers for non-resonance modes as well. When the oscillations of inflaton field shut off completely the preheating stage ends and the fields start to thermalize. This thermalization process can be very complex, including regimes of driven and free turbulence [63], which makes it difficult to estimate the final reheat temperature. Moreover, in certain cases the universe might enter a "quasi-thermal" phase with a kinetic equilibrium reached much before the full chemical equilibrium [64].

In addition to scalar particles, the parametric resonance during preheating may produce a significant amount of fermionic particles. This resonant production could possibly lead to dangerous relic abundances of problematic particles such as gravitinos [65,66]. Fortunately, extensive studies have shown that gravitino over-production can be avoided during the preheating in realistic supersymmetric theories [67–73]. Another interesting aspect in the fermionic preheating is the possibility to generate heavy fermions with masses of order 10^{17}-10^{18} GeV, that could be important in *e.g.* leptogenesis [74,75]. In chapter 5 we will apply our approximation scheme with decohering interactions to study a simple model of fermionic preheating, during which the fermion is subjected to decays.

Chapter 3

Basic formalism of quantum transport theory

The standard methods of (vacuum or thermal) quantum field theory (QFT) are not well suited for the study of nonequilibrium quantum fields for several reasons. First of all, the basic quantities of interest in nonequilibrium QFT are expectation values of operators in contrast to transition amplitudes in the standard vacuum QFT, requiring the formal extension of the time variable into a closed time path with two different branches. Other specific issues in nonequilibrium QFT are related to for example secularity[1] causing the complete failure of the standard perturbative expansion in many cases of interest [76–78]. In this chapter we introduce the basic concepts of nonequilibrium quantum field theory in order to derive the Kadanoff-Baym (KB) transport equations for fermions and scalar bosons. We start by introducing the closed time path formalism with four basic propagators. Then, we use the two-particle irreducible (2PI) effective action methods to derive the self-consistent (Schwinger-Dyson) equations of motion for the full 2-point correlation functions of the system, which we then write in the form of KB-equations. Finally, we list some physical observables that can be expressed in terms of these 2-point functions.

3.1 Closed time path formalism

The closed time path (CTP or Schwinger-Keldysh) formalism was developed by Schwinger [79] and Keldysh [80] and further refined by many influential works, including [81–89]. The basic idea of the formalism is simple: In order to study the expectation values instead of transition amplitudes by the methods of quantum field theory, the time coordinate must be extended to a *closed time path* from

[1]The standard perturbative approach suffers from secular terms giving rise to big late-time contributions in all orders of perturbation expansion, no matter how small the coupling constant is [76–78]. Heuristically, this can be seen in the sense that $\epsilon\, t \sim 1$ for a big enough t for every ϵ (no matter how small).

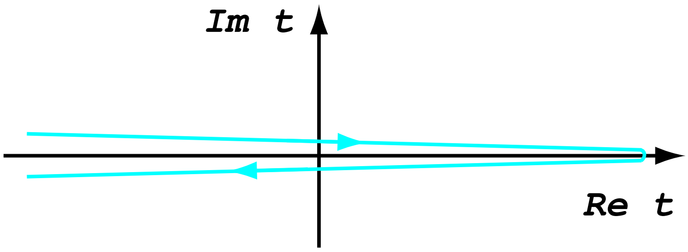

Figure 3.1: The closed time path of the CTP formalism.

initial time t_0 to final time t_f (often taken to be ∞) and then back to t_0[2] (see Fig. 3.1). The need for this closed time path can be demonstrated by writing the expectation value of a real scalar field $\phi(x)$ in a state defined by arbitrary density operator $\hat{\rho}$ in terms of a path integral (we choose here $t_0 = 0$):

$$\begin{aligned}
\langle\hat{\phi}(x)\rangle &\equiv \mathrm{Tr}\{\hat{\phi}(x)\hat{\rho}\} = \mathrm{Tr}\{U(t_f,0)U(0,t)\hat{\phi}_S(\vec{x})U(t,0)\hat{\rho}_S(0)U(0,t_f)\} \\
&= \int d\phi_f\, d\phi_t\, d\phi_0\, d\phi_0' \Big[\langle\phi_f|U(t_f,t)|\phi_t\rangle\phi(x)\langle\phi_t|U(t,0)|\phi_0\rangle\Big] \\
&\qquad\qquad\qquad \times\langle\phi_0|\hat{\rho}_S(0)|\phi_0'\rangle\langle\phi_0'|U(0,t_f)|\phi_f\rangle \\
&= \int d\phi_f\, d\phi_0\, d\phi_0'\; \rho\big[\phi_0(\vec{x}),\phi_0'(\vec{x})\big] \int_{\phi_0(\vec{x})}^{\phi_f(\vec{x})} D\phi^+\, \phi^+(x) \exp\{iS[\phi^+]\} \\
&\qquad\qquad\qquad\qquad \times \int_{\phi_0'(\vec{x})}^{\phi_f(\vec{x})} D\phi^- \exp\{iS[\phi^-]^*\} \\
&= \int D\phi^+ D\phi^-\; \rho\big[\phi^+(0,\vec{x}),\phi^-(0,\vec{x})\big]\, \phi^+(x) \exp\{i(S[\phi^+]-S[\phi^-]^*)\}\,,
\end{aligned} \tag{3.1}$$

where $\rho\big[\phi^+(0,\vec{x}),\phi^-(0,\vec{x})\big] \equiv \langle\phi^+|\hat{\rho}_S(0)|\phi^-\rangle$ is the initial density matrix and the subscript $\mathcal{O}_S$ denotes an operator in the Schrödinger picture in contrast to the Heisenberg picture without a subscript. We see that the path integral representation involves two "histories", for which the evolution is chronological from 0 to t_f and antichronological from t_f to 0, respectively. The field values in these $+/-$ branches are independent except the boundary condition $\phi^+(t_f,\vec{x}) = \phi^-(t_f,\vec{x})$ closing the path at $t = t_f$, which we did not write explicitly in the last row of Eq. (3.1).

[2]For this reason the CTP formalism is often called "in-in" formalism on the contrary to the traditional "in-out" formalism of quantum field theory with transition amplitudes between incoming and outgoing states.

3.1.1 Propagators

It turns out that the higher n-point Green's functions of the system will automatically become time ordered along the closed time path. Of special interest in quantum field theory are the 2-point functions or propagators, which are now defined as

$$i\Delta_{\mathcal{C}}(u,v) = \langle T_{\mathcal{C}}\left[\phi(u)\phi(v)\right]\rangle \equiv \mathrm{Tr}\left\{\hat\rho\; T_{\mathcal{C}}\left[\phi(u)\phi(v)\right]\right\} \tag{3.2}$$

$$iG_{\mathcal{C}}(u,v) = \left\langle T_{\mathcal{C}}\left[\psi(u)\bar\psi(v)\right]\right\rangle \equiv \mathrm{Tr}\left\{\hat\rho\; T_{\mathcal{C}}\left[\psi(u)\bar\psi(v)\right]\right\} \tag{3.3}$$

for a real scalar field ϕ and a fermionic field ψ, respectively. $\hat\rho$ is some unknown quantum density operator describing the statistical properties of the system, and $T_{\mathcal{C}}$ defines the time ordering along the closed time path $\mathcal{C}$, shown in Fig. 3.1, in the sense that the points on the lower (negative) branch are "later" than those on the upper (positive) branch. When written in terms of the ordinary real time variable running from $-\infty$ to $+\infty$, the closed time path propagators (3.2)-(3.3) contain four distinct contributions depending on the time branches of the "complex" CTP-coordinates u and v. That is, using indices $a,b=\pm$ to label the positive/negative branches, the scalar propagators are decomposed as:

$$\begin{aligned}
i\Delta^{+-}(u,v) \equiv i\Delta^{<}(u,v) &= \langle\phi(v)\phi(u)\rangle \\
i\Delta^{-+}(u,v) \equiv i\Delta^{>}(u,v) &= \langle\phi(u)\phi(v)\rangle \\
i\Delta^{++}(u,v) \equiv i\Delta_{F}(u,v) &= \theta(u_0-v_0)i\Delta^{>}(u,v)+\theta(v_0-u_0)i\Delta^{<}(u,v) \\
i\Delta^{--}(u,v) \equiv i\Delta_{\bar F}(u,v) &= \theta(v_0-u_0)i\Delta^{>}(u,v)+\theta(u_0-v_0)i\Delta^{<}(u,v)\,,
\end{aligned} \tag{3.4}$$

where now u^0 and v^0 are ordinary time coordinates. Similarly, for fermionic propagators we have:

$$\begin{aligned}
iG^{+-}(u,v) \equiv -iG^{<}(u,v) &= -\langle\bar\psi(v)\psi(u)\rangle \\
iG^{-+}(u,v) \equiv \quad iG^{>}(u,v) &= \langle\psi(u)\bar\psi(v)\rangle \\
iG^{++}(u,v) \equiv \quad iG_{F}(u,v) &= \theta(u_0-v_0)G^{>}(u,v)-\theta(v_0-u_0)G^{<}(u,v) \\
iG^{--}(u,v) \equiv \quad iG_{\bar F}(u,v) &= \theta(v_0-u_0)G^{>}(u,v)-\theta(u_0-v_0)G^{<}(u,v)\,.
\end{aligned} \tag{3.5}$$

Using the generic notation $\mathcal{G}=\{G,\Delta\}$ to denote fermionic/scalar popagators we see that $\mathcal{G}_F$ and $\mathcal{G}_{\bar F}$ are the chronological (Feynman) and anti-chronological (anti-Feynman) propagators, respectively, while $\mathcal{G}^{<}$ and $\mathcal{G}^{>}$ are the so called *Wightman functions*. In our further analysis we are especially interested in the dynamics of these Wightman functions, which contain the essential thermal or out-of-equilibrium statistical information of the quantum system under study, in order to compute for example the expectation values of the number current j^μ and the energy momentum tensor $T^{\mu\nu}$.

Before we start to build the calculational scheme of the CTP formalism, let us introduce a few more Green's functions, which are useful in the following analysis, and list some of their properties. First, we define the retarted and advanced propagators:

$$
\begin{aligned}
\mathcal{G}^r(u,v) &\equiv \mathcal{G}_F \mp \mathcal{G}^< = \theta(u^0 - v^0)(\mathcal{G}^> \mp \mathcal{G}^<) \\
\mathcal{G}^a(u,v) &\equiv \mathcal{G}_F - \mathcal{G}^> = -\theta(v^0 - u^0)(\mathcal{G}^> \mp \mathcal{G}^<),
\end{aligned} \tag{3.6}
$$

where now $\mp$ refers to bosons/fermions. The definitions (3.4), (3.5) and (3.6) then imply that the propagators have the following hermiticity properties:

$$
\begin{aligned}
\left[i\Delta^{<,>}(u,v)\right]^\dagger &= i\Delta^{<,>}(v,u) \\
\left[iG^{<,>}(u,v)\gamma^0\right]^\dagger &= iG^{<,>}(v,u)\gamma^0\,,
\end{aligned} \tag{3.7}
$$

and further $[i\Delta^r(u,v)]^\dagger = -i\Delta^a(v,u)$ and $[iG^r(u,v)\gamma^0]^\dagger = -iG^a(v,u)\gamma^0$. These latter identities for retarted and advanced propagators suggests to decompose them into hermitian and antihermitian parts:

$$
\begin{aligned}
\mathcal{G}_H &\equiv \frac{1}{2}\left(\mathcal{G}^a + \mathcal{G}^r\right) \\
\mathcal{A} &\equiv \frac{1}{2i}\left(\mathcal{G}^a - \mathcal{G}^r\right) = \frac{i}{2}\left(\mathcal{G}^> \mp \mathcal{G}^<\right).
\end{aligned} \tag{3.8}
$$

The antihermitian part $\mathcal{A}$ is called the *spectral function*. Based on Eqs. (3.6) it is easy to show that $\mathcal{G}_H$ and $\mathcal{A}$ obey the spectral relation:

$$
\mathcal{G}_H(u,v) = -i\,\mathrm{sgn}(u^0 - v^0)\mathcal{A}(u,v)\,. \tag{3.9}
$$

3.2 Two-particle irreducible effective action and Schwinger-Dyson equations

In a nonlinear quantum field theory, including the majority of the interacting theories, the 2-point correlation functions necessarily couple to higher order correlators and so on, to form an infinite Schwinger-Dyson hierarchy of equations analogous to Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in classical statistical mechanics [9]. For this reason it is an impossible task to solve the 2-point correlators exactly; that would correspond to a full solution of the nonlinear QFT. A major paradigm in practical applications thus is to truncate this hierarchy (slave the higher order correlators) in an appropriate way. The truncation can be done in many ways, for example by a brute use of perturbation theory or a loop expansion. However, these standard methods do not generally provide a good approximation for out-of-equilibrium dynamics, because of several problems *e.g.* with secularity [78].

A way to evade these problems is a method of obtaining the (truncated) equations of motion from variational principles of increasing complexity. On the first level we obtain an equation of motion for the field expectation value $\langle\phi\rangle$

only, in the next level to $\langle\phi\rangle$ and 2-point function $\mathcal{G} \sim \langle(\phi - \langle\phi\rangle)^2\rangle$, and so forth. The effective action corresponding to the n-th level of this hierarchy is called *n-particle irreducible* (nPI) effective action $\Gamma_{\rm nPI}$. The heuristic difference between the nPI-method and the standard perturbation theory is that in the latter the *solutions* are written as an expansion in a small parameter, say coupling constant, while in the former the *equations* themselves are expanded. This difference is of crucial importance in nonequilibrium quantum field theory; it is the very reason for the problems of *e.g.* secularity with the standard perturbation method. The truncation procedure using the nPI-method is practically feasible as it turns out that the higher order nPI effective actions become redundant, once the order of expansion is fixed. That is, for example in a loop expansion[3] at m-loop order all nPI effective actions with $n \geq m$ are equivalent. In addition to these reductions there may be further simplifications depending on special conditions, such as vanishing of the average field [78].

3.2.1 From a generating functional to 2PI effective action

In this work we will concentrate on the two-particle irreducible (2PI) effective action [83,88,90–94], which will lead to a self-consistent dynamics for the 2-point correlation function $\mathcal{G}$ as well as the 1-point function $\langle\phi\rangle$. We show, following ref. [9], how the 2PI effective action and the corresponding equations of motion are derived for a real scalar field. For fermions we will only give the appropriate results. We start by defining 2PI *generating functional* on the closed time path:

$$Z[J,K] = e^{iW[J,K]} = \int D\phi^a \, \rho\big[\phi^a(0,\vec{x})\big] \, \exp\Big\{i\big(S[\phi^a] + J_a\phi^a + \frac{1}{2}K_{ab}\phi^a\phi^b\big)\Big\}\,, \tag{3.10}$$

where we use notation with a branch doublet $\phi^a = (\phi^+, \phi^-)$, so that *e.g.* $D\phi^a = D\phi^+ D\phi^-$, and define a "metric" $c_{ab} = \mathrm{diag}(1,-1)$, so that $J_1(x) = J^1(x)$ and $J_2(x) = -J^2(x)$ and repeated indices are summed over. The CTP action is defined as $S[\phi^a] = S[\phi^+] - S[\phi^-]^*$, and $J_a(x)$ and $K_{ab}(x,x')$ are local and nonlocal Gaussian sources, respectively. We also use de Witt summation convention to leave out integrals in the notation for the source terms: $K_{ab}\phi^a\phi^b \equiv \int d^4x\, d^4x'\, K_{ab}(x,x')\phi^a(x)\phi^b(x')$ and similarly for $J_a\phi^a$. All n-point Green's functions are obtained from $Z[J,K]$ through functional differentiation with respect to sources J^a, while $W[J,K]$ generates the connected n-point functions. Especially, the *average field*[4] is defined as

$$\bar{\phi}^a(x) = \frac{\delta W[J,K]}{\delta J_a(x)}\,. \tag{3.11}$$

[3] A loop expansion in the nPI effective action corresponds to an expansion of the *equations*, not to the perturbative expansion of the *solutions*.

[4] This is usually called mean field, but in this work we have a different notion for mean field, explained later in chapter 4.

If we set $J_a = K_{ab} = 0$ after the variation, then $\bar{\phi}^+ = \bar{\phi}^- = \langle\phi\rangle$ is the physical expectation value without sources. The 2-point functions can be obtained form $W[J,K]$ either through a double derivative with respect to J_a or through a derivative with respect to the nonlocal source K_{ab}. For later purposes we use the latter way, and define the full propagators Δ^{ab} as

$$\frac{\delta W[J,K]}{\delta K_{ab}(x,x')} = \frac{1}{2}\big[\bar{\phi}^a(x)\bar{\phi}^b(x') + \Delta^{ab}(x,x')\big]\,. \tag{3.12}$$

From the definition we see that $\Delta^{ab} = \langle T_{\mathcal{C}}\left[(\phi^a - \bar{\phi}^a)(\phi^a - \bar{\phi}^a)\right]\rangle$ *i.e.* it corresponds to fluctuations with respect to the average field, and thus it actually reduces to Eq. (3.4) only for vanishing average field $\bar{\phi}$ (and vanishing sources). To proceed, we define the 2PI *effective action* as a double Legendre transformation of $W[J,K]$:

$$\Gamma_{\rm 2PI}[\bar{\phi},\Delta] = W[J,K] - J_a\bar{\phi}^a - \frac{1}{2}K_{ab}\big[\bar{\phi}^a\bar{\phi}^b + \Delta^{ab}\big]\,, \tag{3.13}$$

where it is understood that the sources $J_a(x)$ and K_{ab} are eliminated through the relations between them and the correlators $\bar{\phi}^a$ and Δ^{ab} arising from Eqs. (3.11)-(3.12). These relations are always invertible following from the general properties of Legendre transformation. The desired equations of motion for the correlators $\bar{\phi}^a$ and Δ^{ab} are now obtained by functional differentiation:

$$\frac{\delta\Gamma_{\rm 2PI}}{\delta\bar{\phi}^a} = -J_a - K_{ab}\bar{\phi}^b\,, \qquad \frac{\delta\Gamma_{\rm 2PI}}{\delta\Delta^{ab}} = -\frac{1}{2}K_{ab}\,, \tag{3.14}$$

so that in the case of physical (sourceless) dynamics we get the equations: $\delta\Gamma_{\rm 2PI}/\delta\bar{\phi}^a = 0$ and $\delta\Gamma_{\rm 2PI}/\delta\Delta^{ab} = 0$, which corresponds to finding the extremum for the effective action $\Gamma_{\rm 2PI}[\bar{\phi},\Delta]$.

3.2.2 Formula for the 2PI effective action in terms of the fluctuation field

In order to use the equations of motion (3.14), we want to find a practical method to compute the 2PI effective action $\Gamma_{\rm 2PI}$. To implement the so called background field method, we write the effective action in the form:

$$e^{i\Gamma_{\rm 2PI}} = \int D\phi^a\,\exp\Big\{i\big[S[\phi^a] + J_a(\phi^a - \bar{\phi}^a) + \frac{1}{2}K_{ab}(\phi^a\phi^b - \bar{\phi}^a\bar{\phi}^b - \Delta^{ab})\big]\Big\}\,, \tag{3.15}$$

which follows directly from Eqs. (3.10) and (3.13). Here we have left out the initial density matrix contribution $\rho\big[\phi^a(0,\vec{x})\big]$. The justification for this can be seen in two ways. First, if the initial state is Gaussian, the initial density matrix can be written as $\rho\big[\phi^a(0,\vec{x})\big] = \exp(-\frac{1}{2}R_{ab}\phi^a\phi^b)$, and the new initial-time Kernel R_{ab} can be absorbed in the source K_{ab}. But this inclusion seems to ruin the desired condition that the physical evolution is given by vanishing sources J_a and K_{ab}. However, since R_{ab} vanishes for all but initial time, we see that it affects only the initial conditions for the 1- and 2-point functions $\bar\phi^a$ and Δ^{ab}, and hence can be neglected in the dynamical equations if we do adjust these initial conditions correctly [78]. The other possibility when the neglection of $\rho\big[\phi^a(0,\vec{x})\big]$ is justified is to consider such initial conditions, where the initial state in the distant past $t_0 \to -\infty$ is in the *in* vacuum. This condition is implemented by just shifting the mass m^2 to $m^2 - i\epsilon$ in the first first branch and to $m^2 + i\epsilon$ in the second branch[5] *i.e.* "tilting" the time path in the complex plane in the same way as in the standard vacuum quantum field theory [9]. For an initial state that is neither of these cases the omitting of $\rho\big[\phi^a(0,\vec{x})\big]$ is not strictly justified and the following developments in this section provide only an approximation. If one wishes to consider those non-Gaussian initial states more accurately one needs to use higher nPI effective actions.

To come back to equation (3.15), we see that using Eqs. (3.14) and the symmetry of the source K_{ab} the exponent becomes

$$S[\phi^a] - \frac{\delta\Gamma_{2\rm PI}}{\delta\bar\phi^a}(\phi^a - \bar\phi^a) - \frac{\delta\Gamma_{2\rm PI}}{\delta\Delta^{ab}}\big[(\phi^a - \bar\phi^a)(\phi^b - \bar\phi^b) - \Delta^{ab}\big]\,. \tag{3.16}$$

Next we shift the integration variable in Eq. (3.15) by the average fields: $\phi^a = \bar\phi^a + \varphi^a$ and expand the classical action in powers of the new *fluctuation field* φ^a:

$$S[\bar\phi^a + \varphi^a] = S[\bar\phi^a] + S_{,a}\varphi^a + \frac{1}{2}S_{,a,b}\varphi^a\varphi^b + S_2\,, \tag{3.17}$$

where $S_{,a}$ means functional derivatives of S with respect to ϕ^a evaluated at $\phi^a = \bar\phi^a$ and similarly for the second derivative $S_{,a,b}$, and S_2 denotes the collection of the higher order terms (cubic and so forth) in the fluctuation field φ^a. Furthermore, we separate the trivial (lowest orders) and nontrivial parts of $\Gamma_{2\rm PI}$ by writing it in the form

$$\Gamma_{2\rm PI}[\bar\phi,\Delta] = S[\bar\phi^a] + \frac{1}{2}i\Delta^{-1}_{0,ab}(\bar\phi)\Delta^{ab} - \frac{1}{2}i{\rm Tr}[\ln\Delta] + \Gamma_2[\bar\phi,\Delta] + {\rm const}\,, \tag{3.18}$$

where the infinite constant (often discarded since it does not affect the equations of motion) is $-\frac{1}{2}\int d^4x\,\delta(0)$ and we denote $i\Delta^{-1}_{0,ab}(\bar\phi) \equiv S_{,a,b}$[6]. Plugging these expressions in Eq. (3.15) we find that the nontrivial part Γ_2 can be expressed as:

$$\begin{aligned} e^{i\Gamma_2} &= [\det\Delta]^{-1/2}\int D\varphi^a \\ &\quad\times\exp\Big\{-\frac{1}{2}\Delta^{-1}_{ab}\varphi^a\varphi^b + i\big[S_2[\varphi^a] - \tilde J_a\varphi^a - \tilde K_{ab}\big(\varphi^a\varphi^b - \Delta^{ab}\big)\big]\Big\}\,, \end{aligned} \tag{3.19}$$

[5]This is why the complex conjugate is explicitly written in the second branch action $S[\phi^-]^*$, even though the classical action is always real.

[6]The motivation for this notation is that $\Delta^{-1}_{0,ab}(\bar\phi)$ becomes the inverse free propagator in the limit of vanishing $\bar\phi^a$.

where
$$\tilde{J}_a = \frac{1}{2} S_{,a,b,c}\Delta^{bc} + \frac{\delta\Gamma_2}{\delta\bar{\phi}^a}\,, \qquad \tilde{K}_{ab} = \frac{\delta\Gamma_2}{\delta\Delta^{ab}}\,. \tag{3.20}$$

We see that despite the $\tilde{K}_{ab}\Delta^{ab}$-term Γ_2 has the form of a generating functional for a new theory with classical action $\frac{i}{2}\Delta^{-1}_{ab}\varphi^a\varphi^b + S_2[\varphi^a]$ and sources $\tilde{J}_a$ and $\tilde{K}_{ab}$. Next we show that these sources with the additional term $\tilde{K}_{ab}\Delta^{ab}$ will just fix the 1- and 2-point functions of this new theory. To show that, we start by considering the matrix

$$\begin{pmatrix} \frac{\delta J_a}{\delta\phi^c} & \frac{\delta J_a}{\delta\Delta^{cd}} \\ \frac{\delta K_{ab}}{\delta\phi^c} & \frac{\delta K_{ab}}{\delta\Delta^{cd}} \end{pmatrix}, \tag{3.21}$$

which we know is invertible, because of the general invertibility of Legendre transformations. In terms of the derivatives of Γ_2 this becomes (also subtracting a singular matrix which does not affect invertibility)

$$\begin{pmatrix} \frac{\delta^2\Gamma_2}{\delta\phi^a\delta\phi^c} - 2\frac{\delta\Gamma_2}{\delta\Delta^{ac}} & \frac{\delta^2\Gamma_2}{\delta\phi^a\delta\Delta^{cd}} \\ 2\frac{\delta^2\Gamma_2}{\delta\Delta^{ab}\delta\phi^c} & \frac{\delta^2\Gamma_2}{\delta\Delta^{ab}\delta\Delta^{cd}} \end{pmatrix}. \tag{3.22}$$

On the other hand, by taking variational derivatives of Eq. (3.19) we get the equations

$$\begin{pmatrix} \frac{\delta^2\Gamma_2}{\delta\phi^a\delta\phi^c} - 2\frac{\delta\Gamma_2}{\delta\Delta^{ac}} & \frac{\delta^2\Gamma_2}{\delta\phi^a\delta\Delta^{cd}} \\ 2\frac{\delta^2\Gamma_2}{\delta\Delta^{ab}\delta\phi^c} & \frac{\delta^2\Gamma_2}{\delta\Delta^{ab}\delta\Delta^{cd}} \end{pmatrix} \begin{pmatrix} \langle\varphi^c\rangle \\ \langle\varphi^c\varphi^d\rangle - \Delta^{cd} \end{pmatrix} = 0\,. \tag{3.23}$$

So, since the coefficient matrix is invertible, we conclude that

$$\langle\varphi^c\rangle = 0\,, \qquad \langle\varphi^c\varphi^d\rangle = \Delta^{cd}\,. \tag{3.24}$$

This result has tremendous implications: As the sources $\tilde{J}_a$ and $\tilde{K}_{ab}$ just kill the 1-point function and fix the 2-point function to Δ^{cd}, it follows that we can neglect these sources in the practical calculations, if we include only the *vacuum* contribution[7] to the effective action Γ_2 using Δ^{cd} as the full propagator of this φ-theory. Furthermore, because the *full* propagator is fixed to Δ^{cd}, it follows that in the diagrammatic calculations we need to consider only *two-particle irreducible* (2PI) graphs *i.e.* graphs that do not become disconnected while cutting two internal lines (see Fig. 3.2), hence the name for the 2PI action. So, we conclude that the 2PI effective action for the original theory is given, besides the terms explicit in equation (3.18) (classical and one-loop contribution), by the sum of all 2PI vacuum graphs in a theory with action $\frac{i}{2}\Delta^{-1}_{ab}\varphi^a\varphi^b + S_2[\varphi^a]$. For example, for the real scalar field with quartic interaction:

$$S[\phi^a] = S[\phi^+] - S[\phi^-] = \int d^4x \Big[\frac{1}{2}c_{ab}\big(\partial_\mu\phi^a\partial^\mu\phi^b - m^2\phi^a\phi^b\big) - \frac{\lambda}{4!}h_{abcd}\phi^a\phi^b\phi^c\phi^d\Big]\,, \tag{3.25}$$

[7] In non-vacuum graphs the external legs are connected to the average field of φ-theory that vanishes

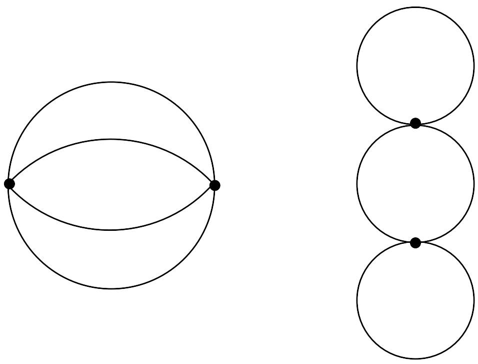

Figure 3.2: Three-loop vacuum graphs in a theory with quartic interaction. Only the left one is two-particle irreducible.

where $h_{1111} = -h_{2222} = 1$ and the other components are zero, we find by performing the shift $\phi^a = \bar{\phi}^a + \varphi^a$ that the interaction part for the φ-theory is given by

$$S_2[\varphi^a] = \int d^4x \Big[-\frac{\lambda}{6} h_{abcd} \bar{\phi}^a \varphi^b \varphi^c \varphi^d - \frac{\lambda}{4!} h_{abcd} \varphi^a \varphi^b \varphi^c \varphi^d \Big] . \qquad (3.26)$$

Note that a cubic interaction with an effective vertex depending on the average field $\bar{\phi}^a$ is generated.

3.2.3 Schwinger-Dyson equations for the propagators

Using the expression (3.18) for $\Gamma_{2\mathrm{PI}}$ in the equations of motion (3.14) we get the following equation for the propagator Δ^{ab} in the (physical) case of vanishing sources:

$$\frac{\delta \Gamma_{2\mathrm{PI}}[\Delta]}{\delta \Delta^{ab}(x,y)} = \frac{1}{2} i \Delta^{-1}_{0,ab}(x,y) - \frac{1}{2} i \Delta^{-1}_{ab}(x,y) + \frac{\delta \Gamma_2[\Delta]}{\delta \Delta^{ab}(x,y)} = 0 \,, \qquad (3.27)$$

where the second term follows from variation: $\delta \mathrm{Tr}[\ln \Delta]/\delta \Delta^{ab} = \Delta^{-1}_{ab}$. By defining the *self energy*:

$$\Pi_{ab}(x,y) \equiv 2i \frac{\delta \Gamma_2[\Delta]}{\delta \Delta^{ab}(x,y)} \,, \qquad (3.28)$$

we see that this equation (3.27) is of the form of famous *Schwinger-Dyson equation* for the full propagator:

$$\Delta^{-1}_{ab}(x,y) = \Delta^{-1}_{0,ab}(x,y) + \Pi_{ab}(x,y) \,, \qquad (3.29)$$

which (upon inverting) is presented graphically in Fig. 3.3. To get this equation in a form that is feasible for practical calculations we multiply it from the right by Δ^{ab} to obtain

$$\int d^4z \Delta^{-1}_{0,ac}(x,z) \Delta^{cb}(z,y) = \delta_{ab} \delta^4(x-y) + \int d^4z \Pi_{ac}(x,z) \Delta^{cb}(z,y) \,. \qquad (3.30)$$

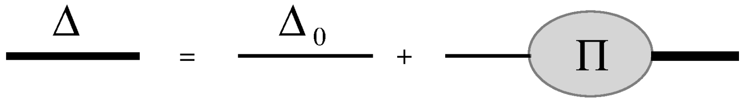

Figure 3.3: A generic form of a Schwinger-Dyson equation for the full 2-point function Δ^{ab}.

This is an integro-differential equation for the full propagator Δ^{ab}, because the inverse free propagator Δ_0^{-1} on the LHS contains explicit spacetime derivatives acting on Δ^{ab}, while on the other hand the full propagator appears inside the integral on the RHS, and also the self energy Π is (typically) a nontrivial functional of Δ^{ab}. One should stress that this Schwinger-Dyson equation is formally exact in the case of Gaussian (or distant past vacuum) initial conditions. However, the actual computation of the self energy via the 2PI action Γ_2 is a nontrivial task and in most cases of interest necessarily involves some truncation, such as the loop expansion or a large N expansion for $\mathcal{O}(N)$-invariant theories.

For fermions the construction of the 2PI effective action is very similar to scalar fields. However, due to the fact that fermionic fields are Grassmann numbers, there are some differences in the final formula, which for vanishing fermionic average field is given by [78]:

$$\Gamma_{2\mathrm{PI}}[G] = -iG^{-1}_{0,ab}G^{ba} + i\mathrm{Tr}[\ln G] + \Gamma_2[G] + \mathrm{const}\,, \tag{3.31}$$

where Γ_2 is again the sum of two-particle irreducible vacuum graphs in the theory with propagator G^{ab} and the vertices are the same as in the original action, because of the vanishing average field. By comparing this with the scalar expression (3.18) we see a difference of a factor $-1/2$ in the first two terms. This is due to different values of the functional determinant arising from the scalar and fermionic path integrals. In the same way as for scalar fields we obtain a completely analogous Schwinger-Dyson equation for the propagator G^{ab}:

$$\int d^4z G^{-1}_{0,ac}(x,z)G^{cb}(z,y) = \delta_{ab}\delta^4(x-y) + \int d^4z \Sigma_{ac}(x,z)G^{cb}(z,y)\,, \tag{3.32}$$

where the self energy Σ is now defined as:

$$\Sigma_{ab}(x,y) \equiv -i\frac{\delta\Gamma_2[G]}{\delta G^{ba}(y,x)}\,. \tag{3.33}$$

Note that for a *complex* scalar field the 2PI effective action would be similar to the fermionic expression (3.31), without 1/2 factors, but with different signs in the first two terms (see *e.g.* [37]). Consequently the definition of the self energy would be the same as in Eq. (3.33) except for the sign difference. The reason why the complex scalar field differs from the real one is basically just the doubling of the degrees of freedom for a complex field. Finally, we note that for an interacting theory that combines fermionic and scalar fields *e.g.* by a Yukawa interaction, we would just need to combine the nontrivial Γ_2-parts of the 2PI effective actions and include contribution from the Yukawa interaction. This would naturally lead to cross-couplings in the fermionic and scalar Schwinger-Dyson equations.

3.3 Kadanoff-Baym transport equations

Let us use again the generic notation $\mathcal{G} = \{G, \Delta\}$ for fermionic and scalar propagators and denote both self energies simply by Σ, and also adopt the same notation: $\Sigma^{++} \equiv \Sigma_F$ etc., as for scalar and fermionic propagators Eqs. (3.4)-(3.5). It follows that the different components of Schwinger-Dyson equations (3.30) and (3.32) are consistent provided that the self energies can be divided in local (singular) and nonlocal parts

$$\Sigma^{ab}(u,v) = c^{ab}\delta^4(u-v)\Sigma_{\rm sg}(u) + \bar{\Sigma}^{ab}(u,v)\,, \tag{3.34}$$

where the nonlocal part obeys similar relations as the propagators:

$$\begin{aligned}\bar{\Sigma}_F(u,v) &= \theta(u_0-v_0)\bar{\Sigma}^>(u,v) \pm \theta(v_0-u_0)\bar{\Sigma}^<(u,v)\\ \bar{\Sigma}_{\bar{F}}(u,v) &= \theta(v_0-u_0)\bar{\Sigma}^>(u,v) \pm \theta(u_0-v_0)\bar{\Sigma}^<(u,v)\end{aligned} \tag{3.35}$$

for scalars and fermions, respectively. These relations (3.34)-(3.35) should hold generally for any reasonable approximation of the self energy [37]. The singular term $\Sigma_{\rm sg}$ can be absorbed into the inverse propagator $\mathcal{G}^{-1}_{0,ab}$ on the LHS of the Schwinger-Dyson equations either to the mass renormalization or to a classical background field (for example for gauge interactions). From now on we assume that this absorption is made and denote simply: $\Sigma^{ab} = \bar{\Sigma}^{ab}$. We further define the retarted and advanced self energies and their (anti)hermitian parts analogously to the propagators in Eqs. (3.6) and (3.8)[8]. The antihermitian part is denoted as Γ:

$$\Gamma \equiv \frac{1}{2i}\left(\Sigma^a - \Sigma^r\right) = \frac{i}{2}\left(\Sigma^> \mp \Sigma^<\right)\,, \tag{3.36}$$

corresponding to the scattering width of the field excitations. For scalar fields Eq. (3.36) is conventionally defined as $\omega\tilde{\Gamma}$, when $\tilde{\Gamma}$ is directly the scattering width with the correct dimension. For fermions Γ is a 4×4 (spinor) matrix and the physical meaning of various elements is more obscure and will be discussed later in section 5.3 in the case of interaction with a thermal background.

Next we want to write the Schwinger-Dyson equations in a different form to make a separation between the dynamical and spectral properties of the system more evident. Using the above definitions and the corresponding ones for the propagators in Eqs. (3.6) and (3.8), and the fact that for vanishing average fields the inverse free propagator obeys $\mathcal{G}^{-1}_{0,ab} = c_{ab}\mathcal{G}^{-1}_{0,F}$, it is a matter of simple algebra

[8]The defined self energies clearly follow the hermiticity properties of the propagators, Eq. (3.7) and below

to show that the Schwinger-Dyson equations (3.30) and (3.32) can be written in the form:

$$
\begin{aligned}
&(\mathcal{G}_{0,F}^{-1} - \Sigma_H) \otimes \mathcal{A} - \Gamma \otimes \mathcal{G}_H = 0 \\
&(\mathcal{G}_{0,F}^{-1} - \Sigma_H) \otimes \mathcal{G}_H + \Gamma \otimes \mathcal{A} = \delta
\end{aligned}
\tag{3.37}
$$

and

$$
(\mathcal{G}_{0,F}^{-1} - \Sigma_H) \otimes \mathcal{G}^< - \Sigma^< \otimes \mathcal{G}_H = \frac{1}{2}\left(\Sigma^> \otimes \mathcal{G}^< - \Sigma^< \otimes \mathcal{G}^>\right) , \tag{3.38}
$$

where, as stated before, we have assumed that the singular self energy $\Sigma_{\rm sg}(u)$ is absorbed into $\mathcal{G}_{0,F}^{-1}$, and we use the notation $\otimes$ for the convolution integral:

$$
f \otimes g \equiv \int \mathrm{d}^4 z f(u,z) g(z,v) \, . \tag{3.39}
$$

The equations (3.37) are called *pole equations*, while Eq. (3.38) is one of the two *Kadanoff-Baym (KB) equations* [95]. The similar KB-equation for the other Wightman function $\mathcal{G}^>$ needs not to be considered, since from the definition (3.8) it immediately follows that $\mathcal{G}^> = \pm \mathcal{G}^< - 2i\mathcal{A}$. In general, the pole equations will fix the spectral properties of the theory, while the KB-equations will give the dynamical evolution, *i.e.* the quantum transport effects. Indeed, in the classical limit the KB-equations (3.38) for fermions and scalars will reduce to well known quantum Boltzmann transport equations for the phase space number densities (see *e.g.* [9, 34, 36–38]).

3.3.1 Mixed representation and gradient expansion

If there is a clear separation between internal (microscopic) and external (macroscopic) scales in the system, it is appropriate to analyze the pole- and KB-equations (3.37)-(3.38) in so called mixed or Wigner representation, where a partial Fourier transformation with respect to the internal coordinate $r = u - v$ is performed. This transformation leads to a gradient expansion in derivatives of the external (average) coordinate $x = (u+v)/2$, which contains, in general, infinitely many terms. However, if the separation of the scales is manifest, this expansion can be truncated (or resummed) to a good approximation.

To begin with, let us define the Wigner transformation for an arbitrary 2-point function $F(x,y)$:

$$
F(k,x) \equiv \int d^4 r \, e^{ik\cdot r} F(x + r/2, x - r/2) \, , \tag{3.40}
$$

where $x \equiv (u+v)/2$ is the average coordinate, and k is the internal momentum conjugate to the relative coordinate $r \equiv u - v$. Using this definition it is easy to transform the equations (3.37) and (3.38) into the mixed representation to get

$$
\tilde{\mathcal{G}}_{0,F}^{-1} \mathcal{A} - e^{-i\diamondsuit}\{\Sigma_H\}\{\mathcal{A}\} - e^{-i\diamondsuit}\{\Gamma\}\{\mathcal{G}_H\} \;=\; 0 \tag{3.41}
$$

$$
\tilde{\mathcal{G}}_{0,F}^{-1} \mathcal{G}_H - e^{-i\diamondsuit}\{\Sigma_H\}\{\mathcal{G}_H\} + e^{-i\diamondsuit}\{\Gamma\}\{\mathcal{A}\} \;=\; 1 \tag{3.42}
$$

and

$$\tilde{\mathcal{G}}^{-1}_{0,F}\mathcal{G}^< - e^{-i\Diamond}\{\Sigma_H\}\{\mathcal{G}^<\} - e^{-i\Diamond}\{\Sigma^<\}\{\mathcal{G}_H\} = \mathcal{C}_{\rm coll} \,, \tag{3.43}$$

where the collision term in Eq. (3.43) is given by

$$\mathcal{C}_{\rm coll} = -ie^{-i\Diamond}\left(\{\Gamma\}\{\mathcal{G}^<\} - \{\Sigma^<\}\{\mathcal{A}\}\right) \,, \tag{3.44}$$

and the $\Diamond$-operator is the following generalization of the Poisson brackets:

$$\Diamond\{f\}\{g\} = \frac{1}{2}\left[\partial_x f \cdot \partial_k g - \partial_k f \cdot \partial_x g\right] \,. \tag{3.45}$$

The differential operators $\tilde{\mathcal{G}}^{-1}_{0,F}$ are related to the inverse propagators:

$$\begin{aligned} \tilde{\Delta}^{-1}_{0,F} &\equiv k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - m^2 e^{-\frac{i}{2}\partial^m_x\cdot\partial_k} && (3.46) \\ \tilde{G}^{-1}_{0,F} &\equiv \not{k} + \frac{i}{2}\not{\partial}_x - m_R e^{-\frac{i}{2}\partial^m_x\cdot\partial_k} - i\gamma^5 m_I e^{-\frac{i}{2}\partial^m_x\cdot\partial_k} \,, && (3.47) \end{aligned}$$

for bosons/fermions respectively, where ∂^m_x means that the derivative is acting on the left to $m^2, m_{R,I}$. Note that these operators are not the transformed inverse propagators, but follow from the identity: $e^{-i\Diamond}\{\mathcal{G}^{-1}_{0,F}\}\{F\} = \tilde{\mathcal{G}}^{-1}_{0,F}F$.

Eqs. (3.41)-(3.43) are the desired quantum transport equations for the 2-point correlation functions $\mathcal{G}^<$, $\mathcal{G}_H$ and $\mathcal{A}$. We are primarily interested in solving the Kadanoff-Baym equation (3.43) for the Wightman function $\mathcal{G}^<$, but in general, also the pole equations (3.41)-(3.42) have to be considered because of the cross-couplings between the equations. One can see that these equations indeed contain the derivatives of the masses m and the self-energies Σ up to infinite order, restraining their use in practical applications unless this gradient expansion can be truncated (or resummed) in some reasonable way.

In the standard approach to *quantum kinetic theory* the conditions of so called quasiparticle or on-shell approximation are assumed, including slowly varying (*i.e.* "nearly" translation invariant) background fields *and* correlators, as well as weak interactions [9]. This approach provide a consistent approximation to truncate the gradient expansion in KB-equations (3.43) to leading order, culminating in the derivation of the famous quantum Boltzmann equations. However, because of the assumptions made, it follows that the correlators whose dynamics we are studying, are "close" to local thermal equilibrium throughout the evolution. Especially the information on quantum coherence will be irrevocably lost. In the next chapter we start to build an extended approximation scheme that incorporates the good features of the standard kinetic approach with easy-to-use Boltzmannian-type transport equations, yet including the effects of nonlocal quantum coherence.

3.4 Physical quantities from 2-point correlation functions

We conclude this chapter by writing down some familiar physical observables, including the number currents and the energy momentum tensors for fermionic and scalar fields, in terms of the 2-point correlation functions $G^<$ and $\Delta^<$. These expressions follow directly from the definitions of the correlators in Eqs.(3.4-3.5) written in the mixed representation, so we simply list the results here. The expectation value of the fermionic number current is given by

$$\langle j_F^\mu(x)\rangle = \langle\bar{\psi}\gamma^\mu\psi\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu iG^<(k,x)\right]\,, \tag{3.48}$$

and for *complex*[9] scalar bosons we have:

$$\langle j_B^\mu(x)\rangle = -i\big\langle\partial^\mu\phi^\dagger\phi - \phi^\dagger\partial^\mu\phi\big\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,2k^\mu i\Delta^<(k,x)\,. \tag{3.49}$$

The symmetric (Belinfante) energy momentum tensor (see *e.g.* [96]) for fermions is

$$\begin{aligned}\langle\theta^{\mu\nu}(x)\rangle &= \frac{i}{4}\Big\langle\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi\Big\rangle + \mu\leftrightarrow\nu \\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\Big[\frac{1}{2}(\gamma^\mu k^\nu + \gamma^\nu k^\mu)iG^<(k,x)\Big]\,,\end{aligned} \tag{3.50}$$

while the bosonic tensor for real scalar field is given by

$$\begin{aligned}&\langle T^{\mu\nu}(x)\rangle = \Big\langle\partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\big[(\partial\phi)^2 - m^2\phi^2\big]\Big\rangle \\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big[k^\mu k^\nu + \frac{1}{4}\partial_x^\mu\partial_x^\nu - \frac{1}{2}g^{\mu\nu}\big(k^2 - m^2 + \frac{1}{4}\partial_x^2\big)\Big]i\Delta^<(k,x)\,,\end{aligned} \tag{3.51}$$

where $g^{\mu\nu}$ is the standard Minkowskian metric with signature $(+,-,-,-)$. For a complex scalar field we get a result with the last row of Eq. (3.51) multiplied by two. These relations demonstrate the importance of the 2-point Wightman functions in nonequilibrium quantum field theory. Later, in section 4.6 we will use these results to express the observables in terms of the on-shell distribution functions.

[9] No conserved Noether current can be defined for a real scalar field

Chapter 4

Quasiparticle picture including nonlocal quantum coherence

4.1 Extended quasiparticle approximation

In the literature (see *e.g.* [97]) a *quasiparticle approximation* (QPA) usually refers to a set of approximations leading to a spectral phase space structure for the 2-point correlation functions $\mathcal{G}^{<,>}$, composed of sharp singular shells with definite energy momentum dispersion relations, such as the standard free particle mass shell. An alternative, stronger definition for QPA [9] requires that the functional forms of the free theory propagators are preserved in (QPA) interacting theory, with only the mass parameters replaced by effective masses. For both of these definitions the necessary conditions for the quasiparticle approximation to be justified include weak interactions and slowly varying background fields. Moreover, the standard treatment of QPA relies on the assumption that system be close to thermal equilibrium and consequently nearly translation invariant, the famous example being the derivation of the quantum Boltzmann transport equation from KB-equations [9].

Our *extended quasiparticle approximation* (eQPA) scheme relinquishes the assumption that the system needs to be close to thermal equilibrium. The key observation in our approach is that under the (otherwise) same conditions of QPA with weak interactions and slowly varying background fields, the phase space of the 2-point correlators $\mathcal{G}^{<,>}$ contains novel and completely different singular shell solutions, in addition to the standard (quasi)particle mass-shell solutions. These new $k_{0,z} = 0$-shell solutions are unavoidably absent if we demand that the system is near thermal equilibrium, hence their lacking in the standard quasiparticle treatments. We will examine these new solutions in detail in section 4.3, where we interpret them as describing *the nonlocal quantum coherence between "opposite" (quasi)particle excitations*. After the complete spectral structure of the correlators is discovered, we feed it as an ansatz to the dynamical equations to find out the equations of motion for the corresponding on-shell distribution functions f, including the novel coherence shells. In this way we get an extension to the quantum Boltzmann transport equation to include the effects of nonlocal quantum coherence.

First, we examine the necessary conditions for the extended quasiparticle approximation to be justified. These conditions include weak interactions, slowly varying background fields, and existence of particular spacetime symmetries. If some of these conditions are not fulfilled, the spectral approximation for the phase space breaks down. At the end of this section we outline the procedure of using the dynamical equations to find the desired equations of motion for the on-shell functions. The actual derivation of the appropriate equations for fermions and scalar bosons is presented in section 4.4.

4.1.1 Weak interactions

The limit of weak interactions in the context of quasiparticle approximation means $\Gamma \to 0$ in Eqs. (3.41)-(3.43), where Γ is the interaction width defined in Eq (3.36). This limit is taken strictly whenever the *phase space properties* of the correlators are studied. However, when studying the *dynamical properties*, one has to include Γ to at least leading order to get any thermalization effects. This is precisely the way how the Boltzmann transport equation is obtained in the classical limit.

Neglecting the terms proportional to Γ in Eqs. (3.41)-(3.43), including the collision term $\mathcal{C}_{\rm coll}$ (in general $\Sigma^{<,>} \sim \Gamma$), leads to the free field equations except for the self energies Σ_H. It is not completely obvious how these self energies should be handled, however. For a controlled expansion in the coupling constant, all the self energies need to be treated in an equal footing; if we neglect $\Gamma \sim g^n$, we have to neglect also Σ_H at the same order. So, it would be justified to retain Σ_H only if it is of lower order in coupling constant than Γ. This is indeed the case for gauge interactions for example, with $\Sigma_H \sim g^2$ and $\Gamma \sim g^4$ in the lowest order. Another approach is to treat Σ_H and Γ completely independently, so that even if there is no coupling hierarchy, Σ_H will be retained in the equations when the phase space properties of the correlators are examined. The motivation for this non-controlled approximation is that retaining Σ_H will not ruin the spectral structure of the correlators; it merely modifies the dispersion relations of the excitations, so that the standard free particles become *quasiparticles.*

4.1.2 Slowly varying background, mean field limit

In general, it is not enough to neglect the terms proportional to Γ to obtain a spectral phase space structure for the 2-point correlators. This can be demonstrated for a scalar field with nonvanishing constant $\partial_t m^2 \equiv \epsilon > 0$, while other derivatives of the mass m are vanishing. The spatially homogeneous solution for the free field (Σ_H is also neglected) KB-equation (3.43) is then:

$$i\Delta^<(k_0, |\vec{k}|; t) \propto \mathrm{Ai}\left(\frac{4^{1/3}(k^2 - m^2(t))}{\epsilon^{2/3}}\right), \tag{4.1}$$

where $\mathrm{Ai}(x)$ is the Airy-function. This is not a singular distribution in momentum for nonvanishing ϵ, but indeed in the limit $\epsilon \to 0$ it reduces to $i\Delta^< \propto \delta(k^2 - m^2(t))$. This example illustrates that in order to obtain spectral phase space structure one needs to consider slowly varying background fields. Moreover, to actually get singular shell solutions *all* derivatives of the background field have to be neglected in the equations (3.41)-(3.43). This approximation of including only the zeroth order gradients of the background is called *mean field* (or adiabatic) limit. The resulting KB-equations for the study of the phase space properties of the correlators in combined weakly interacting and mean field limits are then

$$\begin{aligned}\big(\not{k} - \frac{i}{2} \not{\partial}_x - m_R - i m_I \gamma^5 - \Sigma_H \big) G^< &= 0 &(4.2)\\ \big(k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - m^2 - \Pi_H\big)\Delta^< &= 0 &(4.3)\end{aligned}$$

for fermions and scalars, respectively. The corresponding pole equations are completely the same with 0 replaced by 1 on the RHS of the equations for G_H.

4.1.3 Special spacetime symmetries

It turns out that even the mean field equations (4.2)-(4.3) do not in general have spectral solutions for the correlators. We see this by considering a $1+1$ dimensional free scalar field, for which the solution for Eq. (4.3) reads (for $k_0 \neq 0$) [98]:

$$i\Delta^<(k,x) = \big[\theta(-k^2) + \theta(k^2 - m^2)\big]\,\big[A(k)\cos(q\cdot x) + B(k)\sin(q\cdot x)\big]\,, \quad (4.4)$$

where $A(k)$ and $B(k)$ are real functions of k determined by the initial conditions, and q is defined as:

$$q \equiv 2|k_0|\sqrt{(k^2 - m^2)/k^2}\left(\frac{k_1}{k_0}, 1\right)\,. \quad (4.5)$$

This solution is not restricted to spectral form with support only on singular shells; instead it potentially has support everywhere inside the mass shell $k^2 \geq m^2$, or outside the light cone $k^2 < 0$, depending on the unspecified functions $A(k)$ and $B(k)$. For an arbitrary x-dependence this is in conflict with the quasiparticle approximation, which requires that the phase space structure is spectral. However, if we demand for example the complete translational invariance: $\partial_{x_0,x_1}\Delta^<(k,x) = 0$, then q must be zero, implying that $k^2 - m^2 = 0$. We conclude that there can be support only on the mass shell, and the solution is indeed spectral. In the same way, if we demand only time translational invariance, $\partial_{x_0}\Delta^<(k,x) = 0$, we find two different spectral solutions: the former mass-shell solution, but also a nonconstant solution in x_1 with $k_1 = 0$. Later, in section 4.3, we examine the spectral phase space solutions in more detail with a different approach by subjecting the solutions of Eqs. (4.2)-(4.3) on certain spacetime symmetries in the first place. At least for fermions, this seems to be the only reasonable method, because of the complex spinor structure. In section 4.3 we will find that the phase space structure of fermionic and scalar correlators is indeed spectral for two particular spacetime symmetries of interest: *spatial homogeneity* and *static planar symmetry.*

4.1.4 Spectral ansatz for dynamical equations

The next step in our approximation scheme is to insert the spectral solutions as an *ansatz* in the full KB-equations (3.43). Now we are interested in the dynamics of the spectral solutions, so we will consider only those of the resulting component equations that include direct spacetime derivatives. We will resort to some approximations also in this step. As stated before, we are assuming the limit of weak interactions. However, now we do not want to neglect the terms proportional to Γ completely, since that would lead to collisionless plasma dynamics. Instead we will include only the leading order terms in Γ. To see which terms are actually leading order can be somewhat difficult in practice. It has been shown that for a scalar field close to thermal equilibrium, the term proportional to $\mathcal{G}_H$ on the LHS of Eq. (3.43) is of higher order in Γ than the dominant contribution from the collision term [37], so in this limit it is justified to neglect that term. For more general situations, however, it is not evident that neglecting this term would be justified by any simple arguments. For fermions this hierarchy has not been shown even for systems close to thermal equilibrium, as far as we know. Nevertheless, neglecting the term $\propto \mathcal{G}_H$ may anyway be a good first approximation; for example the well-known Boltzmann transport equation is derived in this limit. Further investigations on the role of this term (in dynamical equations) are definitely needed to make any conclusion on its importance.

The role of the term proportional to Σ_H on the LHS of Eq. (3.43) poses another question. Clearly, if the mean field part has been included in the study of spectral properties in equations (4.2)-(4.3), the same term should be included in dynamical equations as well. The gradient corrections[1] to this term are yet another question, and in the sense of a controlled expansion in coupling constant, they should also be included. However, the role of these higher gradient contributions would probably be the same as the mean field contribution to Σ_H *i.e.* to affect the spectral properties by "modifying" parameters like masses and momenta in the equations. By these arguments, neglecting higher gradient terms of Σ_H when the dynamics of the spectral solutions are studied, seems to be quite well justified.

On the other hand, it is of course possible to include all of these interaction

[1]The higher gradient corrections in $\Sigma\,\mathcal{G}$-terms do *not* necessarily correspond to gradients of the background field, and thus are not on the same footing as the overall mean field approximation

dependent terms in Eq. (3.43). Based on the above discussion, it is appropriate to do this by including everything else than the mean field contribution of Σ_H formally into the collision term. The KB-equations for fermions and scalars then become

$$\Big(\not{k} - \frac{i}{2} \not{\partial}_x - m_R e^{-\frac{i}{2}\partial_x^m \cdot \partial_k} - i\gamma^5 m_I e^{-\frac{i}{2}\partial_x^m \cdot \partial_k} - \Sigma_H \Big) G^< = \tilde{\mathcal{C}}^{\psi}_{\rm coll} \,. \quad (4.6)$$

$$\Big(k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - m^2 e^{-\frac{i}{2}\partial_x^m \cdot \partial_k^\Delta} - \Pi_H \Big) \Delta^< = \tilde{\mathcal{C}}^{\phi}_{\rm coll} \,, \quad (4.7)$$

where the "extended" collision terms are defined as

$$\tilde{\mathcal{C}}^{\psi}_{\rm coll} = e^{-i\Diamond} \left(\{\Sigma_H - i\Gamma\}\{G^<\} + \{\Sigma^<\}\{G_H + i\mathcal{A}\} \right) - \Sigma_H G^< \,. \quad (4.8)$$

$$\tilde{\mathcal{C}}^{\phi}_{\rm coll} = e^{-i\Diamond} \left(\{\Pi_H - i\Gamma\}\{\Delta^<\} + \{\Pi^<\}\{\Delta_H + i\mathcal{A}\} \right) - \Pi_H \Delta^< \,. \quad (4.9)$$

The higher order gradients in the collision term are a delicate issue in our approximation scheme. Usually, when slowly varying backgrounds are studied and the solutions are close to thermal equilibrium, it is justified to neglect consistently all higher (than zeroth) order derivatives in collision term, since those will necessarily correspond to higher order gradients in the background field. However, in our scheme the coherence shell solutions are rapidly oscillating even in constant backgrounds, and consequently the higher derivatives in the collision terms *do not* necessarily correspond to higher order gradients in the background field and thus it is not *a priori* justified to neglect them. In practical calculations this gradient expansion has to be truncated, however, unless the different order gradients of the self energy terms can be resummed in some useful way. Later on in chapter 5, we show that this resummation is indeed possible for a scalar field interacting with a thermal background.

To summarize, our approximation scheme works as follows: First, we find out the spectral properties of the 2-point functions by using the weakly interacting mean field equations (4.2)-(4.3). Then, we substitute the obtained spectral solutions as an ansatz in the full interacting equations (4.6)-(4.7) (with the extended collision terms $\tilde{\mathcal{C}}_{\rm coll}$ or the standard ones of Eq. (3.44)) to find out the equations of motion for the on-shell functions in the presence of collisions. The crucial difference compared with the standard treatment based on the quasiparticle approximation is that we do not assume that the system is close to thermal equilibrium at any moment, yet we are using the spectral solutions for the correlators. This apparent paradox will be settled in section 4.3, where we find the novel coherence shell solutions that have completely different properties than the standard (quasi)particle mass-shell solutions. In what follows, we will neglect the Σ_H and Π_H terms on the LHS of Eqs. (4.6)-(4.7), since we are primarily interested in the general structure of the phase space and the dynamics of the on-shell functions, and not so much in the specific modifications of dispersion relations caused by interactions. Thus, while studying the phase space properties, the corresponding equations are effectively reduced to the *noninteracting mean field limit*.

4.2 Reduction of the spin structure in fermionic equations

Before we enter the study of the phase space shell structure, let us first simplify the fermionic KB-equation (4.6) further in the cases of two spacetime symmetries of interest: the spatial homogeneity and the static planar symmetry. To begin with, we write the equation (4.6) for the hermitian Wightman function, defined as

$$\bar{G}^{<}(k,x) \equiv iG^{<}(k,x)\gamma^0 \,. \tag{4.10}$$

By multiplying both sides of Eq. (4.6) by γ^0 we get

$$\Big(k_0 + \frac{i}{2}\partial_t - \vec{\alpha}\cdot(\vec{k} - \frac{i}{2}\vec{\nabla}) - \gamma^0\hat{m}_0 - i\gamma^0\gamma^5\hat{m}_5\Big)\bar{G}^{<} = \gamma^0 i\mathcal{C}^{\psi}_{\rm coll}\gamma^0 \,, \tag{4.11}$$

where we use the notation:

$$\hat{m}_{0,5}(x) \equiv m_{R,I}(x)e^{-\frac{i}{2}\partial_x^m\cdot\partial_k} \,, \tag{4.12}$$

and we have dropped the tilde in the collision term to denote either the extended collision term in Eq. (4.9) or the standard one in Eq. (3.44).

4.2.1 Spatially homogeneous case, helicity diagonal correlator

In a spatially homogeneous case the spatial gradients of $G^<$ and m in Eq. (4.11) vanish, and consequently the helicity operator $\hat{h} = \hat{k}\cdot\vec{S} = \hat{k}\cdot\gamma^0\vec{\gamma}\gamma^5$, where $\hat{k} \equiv \vec{k}/|\vec{k}|$, commutes with the differential operator on the LHS of Eq. (4.11). This implies that different helicity projections

$$\bar{G}^{<}_{hh'}(k,x) \equiv P_h\bar{G}^{<}(k,x)P_{h'} \,, \tag{4.13}$$

where P_h denotes the helicity projector:

$$P_h = \frac{1}{2}(1 + h\hat{h}), \qquad P_hP_{h'} = \delta_{hh'}, \qquad h = \pm 1 \,, \tag{4.14}$$

do not mix in a noninteracting theory[2], so that helicity is a good quantum number *i.e.* a conserved quantity. It follows that the helicity off-diagonals couple to the dynamics of the diagonal part only through the collision term. In this work we will not consider these cross-couplings, but use the helicity diagonal part of the correlator: $G^< = \sum_{h=\pm1} G^<_{hh}$, as an ansatz for the interacting theory. In the Weyl basis, where the gamma-matrices are given by the following direct

[2] By noninteracting theory we mean here that the self-energies Σ^{ab} and consequently the collision term are vanishing. The system still interacts with the varying classical background (giving rise to varying mass).

product expressions (both ρ^i and σ^i are the usual Pauli matrices referring here to chiral / spin d.o.f. respectively):

$$\gamma^0 = \rho^1 \otimes 1\,, \qquad \vec{\alpha} = -\rho^3 \otimes \vec{\sigma} \quad \text{and} \quad \gamma^5 = -\rho^3 \otimes 1\,, \tag{4.15}$$

the helicity diagonal correlator can be written as:

$$\bar{G}^<_{hh} \equiv g^<_h \otimes \frac{1}{2}(1 + h\hat{k}\cdot\vec{\sigma}), \tag{4.16}$$

where $g^<_h$ are (unknown) hermitian 2×2 matrices (for $h = \pm 1$) in chiral indices. By taking a helicity diagonal projection of Eq. (4.11) we get then an equation for $g^<_h$:

$$\Big(k_0 + \frac{i}{2}\partial_t + h|\vec{k}|\rho^3 - \hat{m}_0\rho^1 + \hat{m}_5\rho^2\Big) g^<_h = \mathcal{C}_h\,, \tag{4.17}$$

where the 2×2 matrix $\mathcal{C}_h$ is the chiral part of the helicity diagonal projection of the collision term:

$$P_h\left(\gamma^0 i\mathcal{C}^{\psi}_{\rm coll}\gamma^0\right)P_h \equiv \mathcal{C}_h \otimes \frac{1}{2}(1 + h\hat{k}\cdot\vec{\sigma})\,. \tag{4.18}$$

Given that $g^<_h$ is a hermitian matrix, it is useful to decompose the equation (4.17) into hermitian (H) and antihermitian (AH) parts:

$$\begin{aligned} (\mathrm{H}):\quad 2k_0 g^<_h &= \hat{H}g^<_h + g^<_h\hat{H}^\dagger + \mathcal{C}^+_h && (4.19)\\ (\mathrm{AH}):\quad i\partial_t g^<_h &= \hat{H}g^<_h - g^<_h\hat{H}^\dagger + \mathcal{C}^-_h\,, && (4.20)\end{aligned}$$

where

$$\hat{H} \equiv -h|\vec{k}|\rho^3 + \hat{m}_0\rho^1 - \hat{m}_5\rho^2 \tag{4.21}$$

can be interpreted as a local free field Hamiltonian operator, and $\mathcal{C}^\pm_h \equiv \mathcal{C}_h \pm \mathcal{C}^\dagger_h$ are the hermitian and antihermitian parts of $\mathcal{C}_h$. We see that in the noninteracting mean field limit with $\mathcal{C}_h = 0$ and $\hat{m}_{0,5} = m_{R,I}$ only the AH-equation contains an explicit time derivative of $g^<_h$, while the H-equation is a purely algebraic matrix equation. For this reason the AH-equation is called a *"kinetic equation"* describing the dynamical evolution of the helicity diagonal Wightman function in a varying background. The H-equation on the other hand is called a *"constraint equation"*, which will constrain the solutions of the kinetic equation in the 4-dimensional phase space and thus is the one to be used in determining the phase space properties of the (eQPA) interacting theory.

4.2.2 Static planar symmetric case, spin-z diagonal correlator

In a static planar symmetric case all but one spatial derivative (chosen to be z here) of $G^<$ and m vanish. Then, apart from the $\vec{\alpha}_{||}\cdot\vec{k}_{||}$-term, the differential operator on the LHS of Eq. (4.11) commutes with the spin in z-direction:

$\hat{S}^3 = \gamma^0\gamma^3\gamma^5$. We can proceed analogously to the previous section 4.2.1, by first performing a Lorentz boost to a frame where this term vanishes [34,36,37]. The Dirac representation of the desired boost is found to be

$$S_{||} = \mathrm{sgn(k_0)}\frac{k_0 + \tilde{k}_0 - \vec{\alpha}\cdot\vec{k}_{||}}{\sqrt{2\tilde{k}_0(k_0+\tilde{k}_0)}}\,. \tag{4.22}$$

The boosted correlator[3]

$$\bar{G}^<_{||}(\tilde{k}_0, k_z; z) \equiv S_{||}\bar{G}^<(k; z)S_{||} \tag{4.23}$$

then obeys an equation

$$\Big(\,\tilde{k}_0 - \alpha^3\,(k_z - \frac{i}{2}\partial_{z_w}) - \gamma^0\hat{m}_0 - i\gamma^0\gamma^5\hat{m}_5\,\Big)\,\bar{G}^<_{||}(\tilde{k}_0, k_z; z) = S^{-1}_{||}\gamma^0 i\mathcal{C}^\psi_{\rm coll}\gamma^0 S_{||}\,, \tag{4.24}$$

where $\tilde{k}_0 = \mathrm{sgn}(k_0)(k_0^2 - k_{||}^2)^{1/2}$. After this boost the differential operator on the LHS of Eq. (4.24) indeed commutes with the spin $\hat{S}^3$, as expected. Analogously to the spatially homogeneous case this implies that different spin projections

$$\bar{G}^<_{||ss'} \equiv P_s\bar{G}^<_{||}P_{s'}\,, \tag{4.25}$$

where P_s denotes the spin-z projector:

$$P_s = \frac{1}{2}(1 + s\hat{S}^3), \qquad P_sP_{s'} = \delta_{ss'}, \qquad s = \pm 1\,, \tag{4.26}$$

do not mix in a noninteracting theory, so that spin-z is a good quantum number. We again neglect the effects of spin off-diagonals and consider only the spin diagonal correlator, which is written in the Weyl basis as:

$$\bar{G}^<_{||s} \equiv g^<_s \otimes \frac{1}{2}(1 + s\sigma^3)\,, \tag{4.27}$$

where $g^<_s$ are hermitian 2×2 matrices (for $s = \pm 1$) in chiral indices. By taking the spin-z diagonal projection of Eq. (4.24) we get now the following equation for $g^<_s$:

$$\Big(\tilde{k}_0 + s(k_z - \frac{i}{2}\partial_z)\rho^3 - \hat{m}_0\rho^1 + \hat{m}_5\rho^2\Big)g^<_s = \mathcal{C}_s\,, \tag{4.28}$$

where the 2×2 matrix $\mathcal{C}_s$ is the chiral part of the spin-z diagonal projection of the boosted collision term:

$$P_s\left(S^{-1}_{||}\gamma^0 i\mathcal{C}^\psi_{\rm coll}\gamma^0 S_{||}\right)P_s \equiv \mathcal{C}_s \otimes \frac{1}{2}(1 + s\sigma^3)\,. \tag{4.29}$$

Now, because the ρ^3 matrix is multiplying the derivative ∂_z in Eq. (4.28), the straightforward division into hermitian and antihermitian parts does not lead

[3]Whereas $G^<$ transforms conventionally: $G^< \to S\,G^<S^{-1}$, the hermitian correlator $\bar{G}^< = iG^<\gamma^0$ obeys a peculiar transformation law: $\bar{G}^< \to S\,\bar{G}^<S$.

to a convenient separation of the derivatives. However, by first multiplying Eq. (4.28) from the left by ρ^3 and only then taking the hermitian and antihermitian parts gives the desired division:

$$\begin{aligned}
(\mathrm{H}): \quad -2sk_z g_s^< &= \hat{P} g_s^< + g_s^< \hat{P}^\dagger - (\rho^3 \mathcal{C}_s)^+ && (4.30)\\
(\mathrm{AH}): \quad is\partial_z g_s^< &= \hat{P} g_s^< - g_s^< \hat{P}^\dagger - (\rho^3 \mathcal{C}_s)^- \,, && (4.31)
\end{aligned}$$

where

$$\hat{P} \equiv k_0 \rho^3 + i(\hat{m}_0 \rho^2 + \hat{m}_5 \rho^1)\,, \tag{4.32}$$

and $(\rho^3 \mathcal{C}_s)^\pm \equiv \rho^3 \mathcal{C}_s \pm \mathcal{C}_s^\dagger \rho^3$ are the hermitian and antihermitian parts of $\rho^3 \mathcal{C}_s$. These equations are analogous to Eqs. (4.19)-(4.20) of the spatially homogeneous case. Again, the AH-equation is called a "kinetic equation" describing the dynamical z-evolution of the spin diagonal Wightman function, while the H-equation is called a "constraint equation" determining the phase space properties of the correlator.

4.3 Phase space shell structure

We now begin to examine the phase space structure of the fermionic and scalar Wightman functions $iG^<$ and $i\Delta^<$ in the (extended) quasiparticle limit discussed in section 4.1 for the case of $\Sigma_H = \Pi_H = 0$. We find out that in addition to the standard mass-shell excitations, with the dispersion relation $k^2 - |m|^2$, the phase space consists of novel singular shell solutions that are located at $k_0 = 0$ for a spatially homogeneous case and at $k_z = 0$ for a static planar symmetric case.

4.3.1 Fermions

As discussed above, the relevant equations that describe the phase space properties of the fermionic Wightman function are the constraint (H) equations Eqs. (4.19) and (4.30), that in the noninteracting mean field limit reduce to

$$\begin{aligned}
2k_0 g_h^< &= \{H, g_h^<\}\,, \qquad & H &\equiv -h|\vec{k}|\rho^3 + m_R \rho^1 - m_I \rho^2 && (4.33)\\
-2sk_z g_s^< &= P g_s^< + g_s^< P^\dagger\,, \qquad & P &\equiv \ k_0 \rho^3 + i(m_R \rho^2 + m_I \rho^1) && (4.34)
\end{aligned}$$

for the spatially homogeneous and the static planar symmetric cases, respectively.

Spatially homogeneous case

To further analyze the constraint equation (4.33) for the spatially homogeneous case, it is convenient to introduce the so called Bloch-representation for the chiral matrix $g_h^<$:

$$g_h^< \equiv \frac{1}{2}\left(g_0^h + g_i^h \rho^i\right)\,, \tag{4.35}$$

where ρ^i are the (chiral) Pauli matrices and $g^h_\alpha(k,t)$ are real functions, because of the hermiticity of $g^<_h$. It is easy to see that in the Bloch-representation the constraint equation (4.33) decomposes into a simple homogeneous matrix equation

$$B_\alpha{}^\beta g^h_\beta = 0\,, \tag{4.36}$$

where the 4×4 coefficient matrix is (index ordering is here defined as $\alpha = 0,3,1,2$):

$$B = \begin{pmatrix} k_0 & h|\vec{k}| & -m_R & m_I \\ h|\vec{k}| & k_0 & 0 & 0 \\ -m_R & 0 & k_0 & 0 \\ m_I & 0 & 0 & k_0 \end{pmatrix}. \tag{4.37}$$

A homogeneous matrix equation, such as Eq. (4.36), may have nonzero solutions only when the determinant of the matrix vanishes. Here the determinant is simply:

$$\det(B) = \left(k^2 - |m|^2\right) k_0^2\,, \tag{4.38}$$

which implies that the nonzero solutions are possible only when

$$k^2 - |m|^2 = 0 \qquad or \qquad k_0 = 0\,. \tag{4.39}$$

These constraints give rise to a *singular shell structure* for the solutions, since they need to be proportional to $\delta(k_0^2 - \vec{k}^2 - |m|^2)$ or $\delta(k_0)$. The former class is identified as the standard one particle mass-shell solutions, with the dispersion relation $k_0 = \pm\omega_k \equiv \pm(\vec{k}^2 + |m|^2)^{1/2}$, while the latter class of solutions with $k_0 = 0$ are completely novel in the context of quantum field theory (see Fig. 4.1). Based on the observation that the quantum interference of the plane waves $\sim e^{\pm i\omega t}$ contains a contribution with $k_0 = 0$, we make an interpretation that *these additional ($k_0 = 0$)-shell solutions describe the quantum coherence between the particles and antiparticles (positive and negative energy states) with opposite momenta and spin.*

The explicit matrix structure of these solutions is found easily by setting $k_0 \neq 0$ and $k_0 = 0$ in the matrix equation (4.36) for the mass-shell and the coherence shell solutions, respectively [I]. The full chiral matrix corresponding to the mass-shell solution is given by:

$$g^<_{h,\mathrm{m-s}}(k_0,|\vec{k}|;t) = 2\pi f^h_{s_{k_0}}(|\vec{k}|,t)|k_0| \begin{pmatrix} 1 - h|\vec{k}|/k_0 & m/k_0 \\ m^*/k_0 & 1 + h|\vec{k}|/k_0 \end{pmatrix} \delta(k^2 - |m|^2)\,, \tag{4.40}$$

where $s_{k_0} \equiv \mathrm{sgn}(k_0)$ and $f^h_{s_{k_0}}(|\vec{k}|,t)$ are real functions parametrizing this solution. The on-shell distribution functions $f^h_\pm$ are called the phase space densities for positive and negative energy modes, respectively. Indeed, in the thermal limit they are related to the number densities of physical particles, as we will see later. For $k_0 = 0$-shell solution we get on the other hand

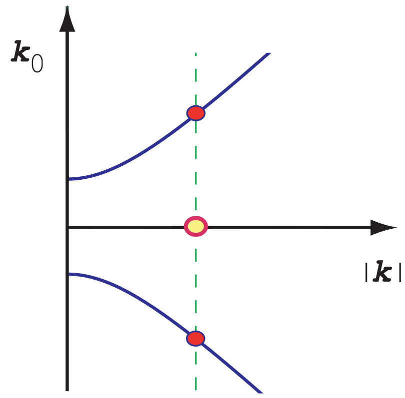

Figure 4.1: Shell structure of the correlators $G^<$ and $\Delta^<$ in the spatially homogeneous case. The dark filled (red) blobs show the mass-shell contributions for a given $|\vec{k}|$, and the light (yellow) blob shows the corresponding coherence contribution from the new $k_0 = 0$-shell.

$$\begin{aligned} g^<_{h,0-\mathrm{s}}(k_0, |\vec{k}|; t) \;=\; \pi \Bigg[& f^h_1(|\vec{k}|, t) \begin{pmatrix} h\, m_R/|\vec{k}| & 1 \\ 1 & -h\, m_R/|\vec{k}| \end{pmatrix} \\ & + f^h_2(|\vec{k}|, t) \begin{pmatrix} -h\, m_I/|\vec{k}| & -i \\ i & h\, m_I/|\vec{k}| \end{pmatrix} \Bigg] \delta(k_0)\,, \end{aligned} \tag{4.41}$$

where $f^h_1(|\vec{k}|, t)$ and $f^h_2(|\vec{k}|, t)$ are new undetermined real functions corresponding to the degrees of freedom of this coherence solution. The most general solution satisfying the constraint equation (4.33) (or equivalently the matrix equation (4.36)) for a spatially homogeneous case is the linear combination of Eqs. (4.40) and (4.41):

$$g^<_h = g^<_{h,\mathrm{m-s}} + g^<_{h,0-\mathrm{s}}\,. \tag{4.42}$$

This general solution contains four independent on-shell distribution functions $f^h_{\pm,1,2}$ (for both $h = \pm 1$), which is just the number of independent components in a hermitian 2×2 matrix, such as the chiral matrix $g^<_h$. Indeed, in section 4.4 we find that there is a *one-to-one mapping* between these on-shell functions and the components of the k_0-integrated chiral matrix $\langle g^<_h \rangle$.

Static planar symmetric case

The analysis of the static planar symmetric case proceeds in complete analogy. By introducing a Bloch representation for $g^<_s$:

$$g^<_s \equiv \frac{1}{2}\left(g^s_0 + g^s_i \rho^i\right) \tag{4.43}$$

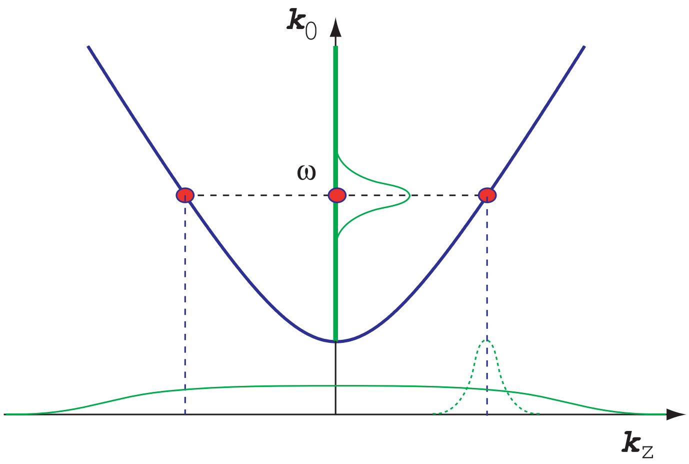

Figure 4.2: Shell structure of the correlators $G^<$ and $\Delta^<$ in the static planar symmetric case. The filled blobs show the contributions from the mass-shell and the new $k_z = 0$-shell for a given $k_0 = \omega$. Peaked functions on each axis represent the accuracy of possible exterior knowledge or "measurement" on k_0 and k_z allowing coherent mixing (wide solid peak in k_z-axis) or reducing the system to a noncoherent evolution (dashed peak).

with real g^h_α, the constraint equation (4.34) decomposes again into a homogeneous matrix equation

$$\tilde{B}_\alpha{}^\beta g^s_\beta = 0\,, \tag{4.44}$$

where the 4×4 coefficient matrix is now (index ordering is again $\alpha = 0, 3, 1, 2$):

$$\tilde{B} = \begin{pmatrix} k_0 & sk_z & -m_R & m_I \\ sk_z & k_0 & 0 & 0 \\ 0 & m_R & sk_z & 0 \\ 0 & -m_I & 0 & sk_z \end{pmatrix}. \tag{4.45}$$

The nonzero solutions of this matrix equation are found at the zeros of the determinant

$$\det(\tilde{B}) = k_z^2(k^2 - |m|^2)\,, \tag{4.46}$$

which are now:

$$k^2 - |m|^2 = 0 \qquad \mathit{or} \qquad k_z = 0\,. \tag{4.47}$$

The former condition gives again the standard one particle mass-shell solutions, with $k_z = \pm k_m \equiv \pm(k_0^2 - |m|^2)^{1/2}$, while the latter condition $k_z = 0$ gives now a different novel class of solutions (see Fig. 4.2). By the same argument as in the spatially homogeneous case, we interpret that *these additional ($k_z = 0$)-shell solutions describe the quantum coherence between the states of same spin and energy travelling in opposite z-directions.*

The matrix structure of these solutions is found by setting $k_z \neq 0$ (mass-shell) and $k_z = 0$ (coherence shell) in the matrix equation (4.44). For the mass-shell solution we get

$$g^{<}_{s,\mathrm{m-s}}(k_0,k_z;z) = 2\pi|k_0|\, f^s_{s_{k_z}}(k_0,z) \begin{pmatrix} 1-sk_z/k_0 & m/k_0 \\ m^*/k_0 & 1+sk_z/k_0 \end{pmatrix} \delta(k^2-|m|^2)\,, \tag{4.48}$$

where $s_{k_z} \equiv \mathrm{sgn}(k_z)$ and $f^s_{s_{k_z}}(k_0,z)$ are real on-shell distribution functions parametrizing this solution. The coherence shell solution is given by

$$\begin{aligned} g^{<}_{s,0-\mathrm{s}}(k_0,k_z;z) \;&=\; \pi\left[f^s_1(k_0,z) \begin{pmatrix} m_R/k_0 & 1 \\ 1 & m_R/k_0 \end{pmatrix} \right. \\ &\quad \left. + \, f^s_2(k_0,z) \begin{pmatrix} -m_I/k_0 & -i \\ i & -m_I/k_0 \end{pmatrix} \right] \delta(k_z)\,, \end{aligned} \tag{4.49}$$

where $f^s_1(k_0,z)$ and $f^s_2(k_0,z)$ are new real distribution functions. The most general solution satisfying the constraint equation (4.34) for a static planar symmetric case is now the linear combination of Eqs. (4.48) and (4.49):

$$g^{<}_s = g^{<}_{s,\mathrm{m-s}} + g^{<}_{s,0-\mathrm{s}}\,. \tag{4.50}$$

In this case also, we later find that the four independent on-shell functions $f^s_{\pm,1,2}$ will uniquely correspond to the components of the integrated chiral matrix $\langle g^{<}_s \rangle$, where the integration is now over k_z instead of k_0.

We later see that even in the simplest possible example of constant mass and no interactions, the new $k_{0,z}$-shell solutions for the spatially homogeneous and static planar symmetric cases are not constant, but oscillate rapidly with the frequencies $2\omega_{\vec{k}}$ and $2k_m$, respectively. Thus they break the translational invariance of the correlator $G^{<}$ badly even in this trivial limit. This is the very reason why these coherence solutions have not been found (used) in the standard treatments of quasiparticle approximation, where it is assumed that the correlator is close to thermal equilibrium.

4.3.2 Scalar bosons

Next, we discuss the singular shell structure for scalar fields. It appears that the method of finding these solutions is not so straightforward, since there is no purely algebraic equation in this case. To analyze the phase space properties of the scalar Wightman function $i\Delta^{<}$, we consider the KB-equation (3.43) in the noninteracting mean field limit:

$$\Big(k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - m^2\Big) i\Delta^{<} = 0\,. \tag{4.51}$$

Like for fermions, we first consider the spatially homogeneous case and then the static planar symmetric case.

Spatially homogeneous case

In the spatially homogeneous case the spatial gradients of the correlator $\Delta^<$ and the mass m vanish. Splitting the equation (4.51) into real and imaginary parts gives then

$$\begin{aligned}\Big(k^2 - m^2 - \frac{1}{4}\partial_t^2\Big) i\Delta^<(k,t) &= 0 && (4.52)\\ k_0\partial_t i\Delta^<(k,t) &= 0\,. && (4.53)\end{aligned}$$

In contrast to the fermionic case we cannot divide these equations into "kinetic" and "constraint" equations, since both of them contain time derivatives. However, we can now determine the phase space structure indirectly by using both of the equations (4.52) and (4.53) in an appropriate way.

We begin by setting $k_0 \neq 0$. Then Eq. (4.53) requires that $\partial_t i\Delta^< = 0$ at all times implying that also $\partial_t^2 i\Delta^< = 0$. Substituting this to Eq. (4.52) now leads to an algebraic equation

$$\big(k^2 - m^2\big)\, i\Delta^<_{\rm m-s} = 0\,, \tag{4.54}$$

which has the spectral solution:

$$i\Delta^<_{\rm m-s}(k_0,|\vec{k}|,t) = 2\pi\,\mathrm{sgn}(k_0) f_{s_{k_0}}(|\vec{k}|,t)\delta\big(k^2-m^2\big)\,, \tag{4.55}$$

corresponding to the standard one particle mass-shell solution, with the dispersion relation $k_0 = \pm\omega_k \equiv \pm(\vec{k}^2+m^2)^{1/2}$. Note that this solution satisfies the equation (4.53) *only* in the mean field limit *i.e.* neglecting all the time derivatives of the mass m. This is expected, since based on the discussion in section 4.1, the solutions will spread in phase space when the gradients are taken in account (see Eq. 4.1).

On the other hand, by setting $k_0 = 0$ in the first place, Eq. (4.53) is identically satisfied, and no extra constraint for the time derivatives of $i\Delta^<$ follows. Then Eq. (4.52) simply becomes

$$(\partial_t^2 + 4\omega_{\vec{k}}^2)\Delta^<_{0-\rm s} = 0\,, \tag{4.56}$$

which has the (mean field) solution,

$$i\Delta^<_{0-\rm s}(k_0,|\vec{k}|,t) = 2\pi\big[A(|\vec{k}|,t)\cos(2\omega_{\vec{k}}t) + B(|\vec{k}|,t)\sin(2\omega_{\vec{k}}t)\big]\,\delta(k_0) \tag{4.57}$$

where $A(|\vec{k}|,t)$ and $B(|\vec{k}|,t)$ are real functions that become constants when the mass m (and $\omega_{\vec{k}}$) is a constant. The $\delta(k_0)$-factor explicitly fixes the restriction to the shell $k_0 = 0$. From now on, we will call (parametrize) the factor in the square brackets in Eq. (4.57) as $f_c(|\vec{k}|,t)$[4], so that the solution (4.57) is written simply as

$$i\Delta^<_{0-\rm s}(k_0,|\vec{k}|,t) = 2\pi\, f_c(|\vec{k}|,t)\delta(k_0)\,. \tag{4.58}$$

[4]Note that f_c is not dimensionless here, but has the dimension of $1/M$

The full spectral solution satisfying the (mean field) equations (4.52) and (4.53) is then the combination of Eqs. (4.55) and (4.58) (see Fig. 4.1):

$$i\Delta^{<} = i\Delta^{<}_{\rm m-s} + i\Delta^{<}_{\rm 0-s}\,. \tag{4.59}$$

This complete solution has three independent on-shell distribution functions $f_{\pm,c}$, which are now one-to-one related to the three lowest k_0-moments of $i\Delta^{<}$, as we will see in section 4.4.

Static planar symmetric case

For the static planar symmetric case with $\partial_{t,x,y} i\Delta^{<} = \partial_{t,x,y} m = 0$ the real and imaginary parts of Eq. (4.51) are

$$\begin{aligned}\Big(k^2 - m^2 + \frac{1}{4}\partial_z^2\Big) i\Delta^{<}(k,z) &= 0 & (4.60)\\ k_z\partial_z i\Delta^{<}(k,z) &= 0\,. & (4.61)\end{aligned}$$

The analysis proceeds now in complete analogy with the spatially homogeneous case. For $k_z \neq 0$ we have $\partial_z i\Delta^{<} = \partial_z^2 i\Delta^{<} \equiv 0$, so that we find the same mass-shell solution as before:

$$i\Delta^{<}_{\rm m-s}(k_0, |\vec{k}_{||}|, k_z, z) = 2\pi\,{\rm sgn}(k_0) f_{s_{k_z}}(k_0, |\vec{k}_{||}|, z)\delta\big(k^2 - m^2\big)\,. \tag{4.62}$$

A new solution is found by setting $k_z = 0$ in the first place, leading to (the other equation is identically satisfied)

$$\big(\partial_z^2 + 4k_m^2(z)\big)\, i\Delta^{<}_{\rm 0-s} = 0\,, \tag{4.63}$$

where $k_m \equiv (k_0^2 - \vec{k}_{||}^2 - m^2)^{1/2}$. Analogously to the spatially homogeneous case, this equation has a (mean field) solution that can be parametrized as

$$i\Delta^{<}_{\rm 0-s}(k_0, |\vec{k}_{||}|, k_z, z) = 2\pi\, f_c(k_0, |\vec{k}_{||}|, z)\delta(k_z)\,, \tag{4.64}$$

where $f_c(k_0, |\vec{k}_{||}|, z)$ is a real on-shell distribution function corresponding to this coherence solution. The most general solution satisfying the (mean field) equations (4.60) and (4.61) is once again the combination of Eqs. (4.62) and (4.64) (see Fig. 4.2):

$$i\Delta^{<} = i\Delta^{<}_{\rm m-s} + i\Delta^{<}_{\rm 0-s}\,. \tag{4.65}$$

In this case the three independent on-shell distribution functions $f_{\pm,c}$ are one-to-one related to the three lowest k_z-moments of $i\Delta^{<}$.

Like for fermions, we interpret the new $k_{0,z} = 0$-shell solutions for scalar fields as describing the nonlocal quantum coherence between the states with opposite momenta and z-momenta in the spatially homogeneous and the static planar symmetric cases, respectively. For a complex scalar field the $k_0 = 0$-coherence is between particles and antiparticles, while for a real scalar field it is between the different modes of the same particle[5]. The oscillatory behaviour of the coherence shell solution with the frequency $2\omega_{\vec{k}}$ in the constant mass limit is now directly seen from Eq. (4.57).

[5] For a real scalar field particles are their own antiparticles

4.4 Equations of motion with collisions

Having determined the singular phase space shell structure in the (extended) quasiparticle limit, we now want to use the corresponding spectral correlators as an ansatz for the dynamical equations including the gradients and collisions. Our goal is to find a closed set of equations of motion for the on-shell functions f (or some generic quantities related to these). We will only consider the spatially homogeneous case here, but the static planar symmetric case would be highly analogous to this[6]. To proceed, the singular shell structure of the correlator naturally suggests to integrate the dynamical equations over the momentum. This procedure is supported also in the sense that (local) physical quantities, like currents, energy density and pressure, are obtained from the correlator $\mathcal{G}^<$ by a full 4-momentum integration (see section 3.4). There is one profound difference between the procedures for fermions and scalar bosons, however. For fermions, it is enough to consider the bare integral (zeroth moment) of the correlator to find out a closed set of equations for the on-shell functions f_α. For scalars, on the other hand, the different moment integrals couple to each other, and due to our decomposition with three on-shell functions $f_{\pm,c}$ we need to consider the three lowest moment integrals to get a closure.

4.4.1 Integrated matrix equations for fermions

To derive the equations of motion for the on-shell functions $f_{\pm,1,2}$ for fermions in the spatially homogeneous case, we start with the kinetic (AH) equation Eq. (4.20) for the chiral 2×2 matrix $g_h^<$:

$$i\partial_t g_h^< = \hat{H} g_h^< - g_h^< \hat{H}^\dagger + \mathcal{C}_h^- \,, \tag{4.66}$$

where the operator $\hat{H}$ is given by Eq. (4.21) and $\mathcal{C}_h^-$ is the antihermitian chiral part of the collision term: $P_h\left(\gamma^0 i\mathcal{C}^\psi_{\rm coll}\gamma^0\right)P_h$, defined in Eq. (4.18). Integration over k_0 gives now the following equation:

$$i\partial_t \rho_h = [H,\rho_h] + \langle \mathcal{C}_h^- \rangle \,, \tag{4.67}$$

where we use notation $\langle\ldots\rangle \equiv \int \frac{{\rm d}k_0}{2\pi}(\ldots)$ for the k_0-integral, $\rho_h \equiv \langle g_h^<\rangle$ denotes the (zeroth moment) integral of the correlator $g_h^<$, and the local Hamiltonian operator is given by

$$H \equiv -h|\vec{k}|\rho^3 + m_R\rho^1 - m_I\rho^2 \,. \tag{4.68}$$

The k_0-derivatives appearing in Eq. (4.21) have disappeared as total derivatives due to the integration. When we substitute the spectral correlator Eq. (4.42)

[6] For the free theory the static planar symmetric case will be considered in connection with the Klein problem in section 5.1

as an ansatz for $g_h^<$, the (chiral) components of the integrated correlator ρ_h are simply related to the on-shell functions f_α, $\alpha = \pm, 1, 2$:

$$\begin{aligned}
\rho_{h,LL} &= \frac{1}{2}\Big(1 - h\frac{|\vec{k}|}{\omega_{\vec{k}}}\Big) f_+^h + \frac{1}{2}\Big(1 + h\frac{|\vec{k}|}{\omega_{\vec{k}}}\Big) f_-^h + h\Big(\frac{m_R}{2|\vec{k}|} f_1^h - \frac{m_I}{2|\vec{k}|} f_2^h\Big) \\
\rho_{h,RR} &= \frac{1}{2}\Big(1 + h\frac{|\vec{k}|}{\omega_{\vec{k}}}\Big) f_+^h + \frac{1}{2}\Big(1 - h\frac{|\vec{k}|}{\omega_{\vec{k}}}\Big) f_-^h - h\Big(\frac{m_R}{2|\vec{k}|} f_1^h - \frac{m_I}{2|\vec{k}|} f_2^h\Big) \\
\rho_{h,LR} &= \frac{m}{2\omega_{\vec{k}}}(f_+^h - f_-^h) + \frac{1}{2}(f_1^h - i f_2^h) \\
\rho_{h,RL} &= \frac{m^*}{2\omega_{\vec{k}}}(f_+^h - f_-^h) + \frac{1}{2}(f_1^h + i f_2^h) \,.
\end{aligned} \tag{4.69}$$

It is easy to show that these linear relations can be inverted whenever $|\vec{k}| \neq 0$. Moreover, because of the singular shell structure of the spectral correlator Eq. (4.42), the collision integral $\langle \mathcal{C}_h^- \rangle$ is composed of projections of the self energies $\Sigma^{<,>}$ on these singular shells, and is thus depending on the correlator $g_h^<$ only through the on-shell functions f_α. Based on these observations, we conclude that the matrix equation (4.67) can be used to find the desired closed set of equations of motion for the on-shell functions f_α. We will not give the resulting equations explicitly here, since they would be more messy than Eq. (4.67), and in any case the specifications of the self energies in the collision term are needed to get the equations in the final form. Instead, we refer to the matrix equation (4.67) together with the relations (4.69) as our *master equations* to study the dynamics of either the on-shell functions f_α or the components of the chiral matrix ρ_h. When the self energy functionals $\Sigma^{<,>}$ are specified, these equations can be used in practical calculations. In chapter 5 we use them to study coherent particle production in an oscillating background field in the presence of decohering collisions.

4.4.2 Moment equations for scalar bosons

For scalar bosons the relevant dynamical equation is Eq. (3.43), which in the spatially homogeneous case reduces to the following coupled equations (real and imaginary parts):

$$\Big(k^2 - \frac{1}{4}\partial_t^2 - m^2 \cos(\tfrac{1}{2}\partial_t^m \partial_{k_0}^\Delta)\Big) i\Delta^< = \mathcal{C}^+ \tag{4.70}$$

$$\Big(k_0 \partial_t + m^2 \sin(\tfrac{1}{2}\partial_t^m \partial_{k_0}^\Delta)\Big) i\Delta^< = \mathcal{C}^- \,, \tag{4.71}$$

where $\mathcal{C}^\pm$ are the real and imaginary parts of $i\mathcal{C}_{\rm coll}^\phi$, respectively. We now see that because of the explicit k_0-factors in Eqs. (4.70)-(4.71) the k_0-integration of these equations will couple different moments

$$\rho_n(|\vec{k}|, t) \equiv \langle k_0^n \, i\Delta^< \rangle = \int \frac{{\rm d}k_0}{2\pi} \, k_0^n \, i\Delta^<(k_0, |\vec{k}|, t) \tag{4.72}$$

in the same equation. That is, by taking the zeroth moment of Eq. (4.70) and the zeroth and first moments of Eq. (4.71) we find the following coupled moment equations:

$$
\begin{aligned}
\frac{1}{4}\partial_t^2\rho_0 + \omega_{\vec{k}}^2\rho_0 - \rho_2 &= \langle \mathcal{C}^+ \rangle \\
\partial_t \rho_1 &= \langle \mathcal{C}^- \rangle \\
\partial_t \rho_2 - \frac{1}{2}\partial_t(m^2)\rho_0 &= \langle k_0 \mathcal{C}^- \rangle \,.
\end{aligned}
\tag{4.73}
$$

Moreover, when the spectral correlator Eq. (4.65) is used as an ansatz for $\Delta^<$, the moments $\rho_{0,1,2}$ are related to the on-shell functions $f_{\pm,c}$ in a simple way:

$$
\begin{aligned}
\rho_0 &= \frac{1}{2\omega_{\vec{k}}}(f_+ - f_-) + f_c \\
\rho_1 &= \frac{1}{2}(f_+ + f_-) \\
\rho_2 &= \frac{\omega_{\vec{k}}}{2}(f_+ - f_-) \,.
\end{aligned}
\tag{4.74}
$$

We see that this linear set of equations is invertible whenever $|\vec{k}| \neq 0$, and also the collision integrals $\langle \mathcal{C}^\pm \rangle$ and $\langle k_0 \mathcal{C}^- \rangle$ are depending on the correlator $\Delta^<$ only through the on-shell functions $f_{\pm,c}$, in complete analogy with fermions. So, we conclude now that the moment equations (4.73) can be used to obtain the desired closed set of equations of motion for the on-shell functions $f_{\pm,c}$. Again, we will not give these equations here explicitly, but refer to Eqs. (4.73)-(4.74) as our *master equations* for the study of the dynamics of either the on-shell functions $f_{\pm,c}$ or the three lowest moments $\rho_{0,1,2}$. In chapter 5 we use these equations to study the coherent production of unstable particles in an oscillating background.

4.5 Spectral function and thermal limit

Let us next consider the fermionic and scalar spectral functions $\mathcal{A}$ in terms of our eQPA scheme. In our approach these are needed in the evaluation of the collision terms in Eqs. (4.67) and (4.73). We consider here only the spatially homogeneous case, since all the problems we are going to study with collisions in chapter 5 have this particular symmetry. The relevant equations for the study of the phase space properties of spectral functions (in the noninteracting mean field limit) are the same as the corresponding ones for the Wightman functions $G^<$ and $\Delta^<$, Eqs. (4.2)-(4.3). Consequently, the spectral functions have spectral solutions given by Eqs. (4.42) and (4.59), with yet undefined on-shell functions $f^{h\mathcal{A}}_{\pm,1,2}$ for fermions and $f^{\mathcal{A}}_{\pm,c}$ for scalars. *In addition*, however, the spectral functions must obey the sum rules that can be derived from the pole equations (3.41)-(3.42) using the spectral relation (3.9) (or alternatively from the canonical equal time (anti)commutation relations of the fields):

$$\int \frac{\mathrm{d}k_0}{\pi} \mathcal{A}(k,x)\gamma^0 = 1 \tag{4.75}$$

for fermions, and

$$\int \frac{\mathrm{d}k_0}{\pi} \big(k_0 + \frac{i}{2}\partial_t\big)\mathcal{A}(k,x) = 1 \tag{4.76}$$

for scalars. It is easy to show that for the fermionic spectral function the conditions from the sum rule (4.75) completely fix the values of all the on-shell functions

$$f^{h\,\mathcal{A}}_{\pm} = \frac{1}{2} \qquad \text{and} \qquad f^{h\,\mathcal{A}}_{1,2} = 0 \tag{4.77}$$

for both helicities, so that the spectral function reduces to the familiar local thermal equilibrium form:

$$\mathcal{A} = \pi \mathrm{sgn}(k_0)(\not{k} + m_R - i\gamma^5 m_I)\delta(k^2 - |m|^2)\,. \tag{4.78}$$

For the scalar fields, on the other hand, the sum rule (4.75) does not immediately fix the values of the on-shell functions $f^{\mathcal{A}}_{\pm,c}$. However, by use of the dynamical equations for the moment functions $\rho^{\mathcal{A}}_n \equiv \int \frac{\mathrm{d}k_0}{2\pi}\, k_0^n\, \mathcal{A}$ that are identical to (4.73) with vanishing collision terms, the sum rule gives that $\rho^{\mathcal{A}}_0 = \rho^{\mathcal{A}}_2 = 0$ and $\rho^{\mathcal{A}}_1 = 1/2$ in order to get a continuous constant mass limit. The connection relations identical to Eq. (4.74) between $\rho^{\mathcal{A}}_n$ and $f^{\mathcal{A}}_{\pm,c}$ then give:

$$f^{\mathcal{A}}_{\pm} = \frac{1}{2}\,, \qquad f^{\mathcal{A}}_{c} = 0\,, \tag{4.79}$$

so that the spectral function $\mathcal{A}$ again reduces to its standard thermal form

$$\mathcal{A} = \pi \mathrm{sgn}(k_0)\,\delta\left(k^2 - m^2\right)\,. \tag{4.80}$$

Note that the contributions from the coherence shell $k_0 = 0$ are completely absent in the (eQPA) spectral functions (4.78) and (4.80), because of the strong constraints imposed by the spectral sum rule.

Let us conclude this section by giving the Wightman functions $G^<$ and $\Delta^<$ in the thermal limit. It is easy to show that the full translation invariance in thermal equilibrium kills the coherence shell contributions completely, and furthermore implies the Kubo-Martin-Schwinger (KMS) conditions for the Wightman functions [99–101]:

$$\mathcal{G}^>_{\mathrm{eq}}(k_0) = e^{\beta k_0}\mathcal{G}^<_{\mathrm{eq}}(k_0)\,. \tag{4.81}$$

The KMS-conditions and the relation $\mathcal{G}^> = \pm\mathcal{G}^< - 2i\mathcal{A}$ as well as the thermal spectral functions Eqs. (4.78) and (4.80) will then fix the mass-shell distribution functions to Fermi-Dirac and Bose-Einstein distributions for fermions and bosons, respectively:

$$\begin{aligned} f^h_{s_{k_0}}(|\vec{k}|) &\to n^F_{\mathrm{eq}}(k_0) \equiv \frac{1}{e^{\beta k_0}+1} \\ f_{s_{k_0}}(|\vec{k}|) &\to n^B_{\mathrm{eq}}(k_0) \equiv \frac{1}{e^{\beta k_0}-1}\,, \end{aligned} \tag{4.82}$$

leading to the following thermal equilibrium correlators [97]:

$$\begin{aligned} iG^<_{\rm eq} &= 2\pi\,{\rm sgn}(k_0)(\not{k} + m_R - i\gamma^5 m_I)\, n^F_{\rm eq}(k_0)\,\delta(k^2 - |m|^2) \\ i\Delta^<_{\rm eq} &= 2\pi\,{\rm sgn}(k_0) n^B_{\rm eq}(k_0)\delta\left(k^2 - m^2\right)\,. \end{aligned} \tag{4.83}$$

These expressions will be used in chapter 5 when we compute the collision terms in the case of interaction with a thermal background.

4.6 Physical quantities in terms of the on-shell functions

Having found the spectral phase space structure of the correlators $G^<$ and $\Delta^<$, we want to write some of the physical observables considered in section 3.4 in terms of the on-shell functions f. In addition, we introduce the concept of phase space particle number.

4.6.1 Particle number and fluxes

In contrast to standard vacuum QFT, the concept of particles is not very well defined in nonequilibrium quantum field theory [8, 9]. The problem is that different observers will see different vacuum states, and on the other hand vacuum states are not remaining "empty" even in the free field evolution, because of the nontrivial background. That said, in the case of clear asymptotic regions in the background field, the problem of particle production from the classical background is well understood. Most definitions for particle number are relying on some sort of diagonalization of the Hamiltonian [8, 9, 102, 103], the idea being that the energy of the $\vec{k}$-mode would entirely consists of a vacuum part plus (anti)particle excitations: $\mathcal{E}_{\vec{k}} \propto \omega_{\vec{k}}(n_{\vec{k}} + \bar{n}_{\vec{k}} \mp 1)$ for fermions and bosons, respectively. This idea is of course supported by thermal equilibrium, where this relation is satisfied.

Here we adopt a somewhat different concept for the particle number. At the end of the previous section we saw that in thermal equilibrium the mass-shell distribution functions of $G^<$ and $\Delta^<$ are just the Fermi-Dirac and Bose-Einstein distributions, which correspond to particle number densities for positive energies: $n^{F,B}_{\vec{k}} = n^{F,B}_{\rm eq}(k_0 = \omega_{\vec{k}})$, and to antiparticle number densities for negative energies according to the Feynman-Stuckelberg interpretation: $\bar{n}^{F,B}_{\vec{k}} = \pm 1 - n^{F,B}_{\rm eq}(k_0 = -\omega_{\vec{k}})$. Within our eQPA scheme the Wightman functions $G^<$ and $\Delta^<$ always have a spectral phase space structure including the singular mass-shell, even in out-of-equilibrium conditions. This suggests us to generalize these relations and use them as definitions for the *phase space particle and antiparticle numbers*, *i.e.* for a spatially homogeneous system we define:

$$\begin{aligned} n_{\vec{k}h}(t) &\equiv f^h_+(|\vec{k}|, t) \qquad \text{and} \qquad \bar{n}_{\vec{k}h}(t) \equiv 1 - f^h_-(|\vec{k}|, t) \\ n_{\vec{k}}(t) &\equiv f_+(|\vec{k}|, t) \qquad \text{and} \qquad \bar{n}_{\vec{k}}(t) \equiv -1 - f_-(|\vec{k}|, t) \end{aligned} \tag{4.84}$$

for fermions and scalars, respectively. The soundness of these definitions is consolidated by expressing the number current densities in Eqs. (3.48)-(3.49) in terms of the on-shell functions. We get

$$\begin{aligned}\langle j_F^0(t)\rangle &= \sum_h \int \frac{\mathrm{d}^3k}{(2\pi)^3}(f_+^h + f_-^h) = \sum_h \int \frac{\mathrm{d}^3k}{(2\pi)^3}(n_{\vec{k}h} - \bar{n}_{\vec{k}h} + 1)\\ \langle j_B^0(t)\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}(f_+ + f_-) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}(n_{\vec{k}} - \bar{n}_{\vec{k}} - 1)\,, \end{aligned} \tag{4.85}$$

which are just the expected results with the correct vacuum energy contributions. Now, using the inverse relations of Eq. (4.69) we can express the fermionic (anti)particle numbers in terms of the Bloch components of the integrated chiral matrix $\rho_h = \frac{1}{2}(\langle g_0^h\rangle + \langle \vec{g}^h\rangle \cdot \vec{\sigma})$:

$$\begin{aligned} n_{\vec{k}h} &= \frac{1}{2\omega}\Big(-h|\vec{k}|\langle g_3^h\rangle + m_R\langle g_1^h\rangle - m_I\langle g_2^h\rangle\Big) + \frac{1}{2}\langle g_0^h\rangle\\ \bar{n}_{\vec{k}h} &= \frac{1}{2\omega}\Big(-h|\vec{k}|\langle g_3^h\rangle + m_R\langle g_1^h\rangle - m_I\langle g_2^h\rangle\Big) - \frac{1}{2}\langle g_0^h\rangle + 1\,. \end{aligned} \tag{4.86}$$

For scalar fields, on the other hand, the (anti)particle numbers can be related to the three lowest moments $\rho_{0,1,2}$ by the inverse relations of Eq. (4.74):

$$\begin{aligned} n_{\vec{k}} &= \frac{1}{\omega_{\vec{k}}}\rho_2 + \rho_1\\ \bar{n}_{\vec{k}} &= \frac{1}{\omega_{\vec{k}}}\rho_2 - \rho_1 - 1\,. \end{aligned} \tag{4.87}$$

We see that the particle and antiparticle numbers coincide if $\langle g_0^h\rangle = 1$ for fermions and $\rho_1 = -1/2$ for scalars[7]. These values thus correspond to zero "chemical potential", for which the number current densities (4.85) have only vacuum contributions.

Let us now compare our phase space particle number to other definitions in literature. Apart from some sign conventions our fermionic particle number density in Eq. (4.86) with zero chemical potential reduces to the one obtained in ref. [102] by using the Bogoliubov transformation to diagonalize the Hamiltonian. Moreover, using the free field equations of motion (4.73), we see that our scalar particle number in Eq. (4.87) agrees with the one in ref. [103], namely

$$\Big(n_{\vec{k}} + \frac{1}{2}\Big)^2 = \rho_0\Big(\omega_{\vec{k}}^2\rho_0 + \frac{1}{2}\partial_t^2\rho_0\Big)\,, \tag{4.88}$$

if we take the adiabatic limit: $\partial_t^2\rho_0 \ll \omega_{\vec{k}}^2\rho_0$. These comparisons further strengthen the soundness of our definitions.

[7]For a real scalar field particle is its own antiparticle, thus $\rho_1 = -1/2$ is the only sensible value

For spatially dependent problems one is often more interested in fluxes, *i.e.* the spatial components of the number currents, than densities. Using Eqs. (3.48)-(3.49) we find that the z-fluxes in a *static planar symmetric case* are given by:

$$\begin{aligned}
\langle j^3_F(z)\rangle &= \sum_s \int \frac{\mathrm{d}k_0\,\mathrm{d}^2k_{||}}{(2\pi)^3}\,\mathrm{sgn}(k_0)\,s\,(f^s_+ - f^s_-) \\
\langle j^3_B(z)\rangle &= \int \frac{\mathrm{d}k_0\,\mathrm{d}^2k_{||}}{(2\pi)^3}\,\mathrm{sgn}(k_0)(f_+ - f_-)\,. \qquad (4.89)
\end{aligned}$$

In this case it is natural to interpret or define $f_\pm$ as *right/left moving particle fluxes* per unit volume in $(k_0, \vec{k}_{||})$-phase space. We use these definitions in sections 5.1 and 5.2 for solving quantum reflection problems with our methods.

4.6.2 Energy density and pressure for spatially homogeneous systems

Let us proceed by considering the energy density and pressure in a spatially homogeneous case. These are by definition the 00- and ii-components of the energy momentum tensor, respectively. Using Eqs. (3.50)-(3.51) we get for the energy density (in terms of n and $\bar{n}$ instead of $f_\pm$):

$$\begin{aligned}
\langle \mathcal{E}_F(t)\rangle = \langle \theta^{00}(t)\rangle &= \sum_h \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_{\vec{k}}\big(n_{\vec{k}h} + \bar{n}_{\vec{k}h} - 1\big) \\
\langle \mathcal{E}_B(t)\rangle = \langle T^{00}(t)\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2}\omega_{\vec{k}}\big(n_{\vec{k}} + \bar{n}_{\vec{k}} + 1\big)\,, \qquad (4.90)
\end{aligned}$$

for fermions and scalars, respectively. For the pressure we get instead:

$$\begin{aligned}
\langle P_F(t)\rangle = \langle \theta^{ii}(t)\rangle &= \sum_h \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{3}\left[\frac{\vec{k}^2}{\omega_{\vec{k}}}\big(n_{\vec{k}h} + \bar{n}_{\vec{k}h} - 1\big) - m_R f^h_1 + m_I f^h_2\right] \\
\langle P_B(t)\rangle = \langle T^{ii}(t)\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{3}\left[\frac{\vec{k}^2}{\omega_{\vec{k}}}\frac{1}{2}\big(n_{\vec{k}} + \bar{n}_{\vec{k}} + 1\big) - \big(3\omega^2_{\vec{k}} - \vec{k}^2\big)f_c\right]. \qquad (4.91)
\end{aligned}$$

The expressions for energy densities in Eq. (4.90) are, as expected, consisting of particle and antiparticle contributions and the vacuum energies, which are opposite for fermions and scalars. The expressions for pressure in Eq. (4.91), however, involve explicit contributions from the coherence shell as well. Thus, in the presence of coherence the "quantum pressure" may be drastically different from the standard classical one. However, because of the typical oscillatory behavior of the coherence solutions at microscopic time-scales $\Delta t_{\mathrm{osc}} \sim 1/\omega$, this deviation would average out in most cases of interest at time-scales much larger than Δt_{osc}.

4.7 On the validity of eQPA approximation

We conclude this chapter by discussing the main concerns regarding the validity of our eQPA approximation scheme, the breakdown (spreading) of the singular shell picture and the handling of the gradient expansion in the collision term. To begin with, we recall that the basic requirements of the quasiparticle approximation, weak (quantum) interactions and a slowly varying background field, should be fulfilled for the singular shell picture to be a good approximation. However, in our eQPA scheme we relax the assumption that the system is close to a local thermal equilibrium, and hence nearly translation invariant. This allows us to include the solutions describing nonlocal quantum coherence, which are oscillatory (and thus translationally non-invariant) on quantum scales with frequencies of order $2\omega_{\vec{k}}$ (or $2k_m$ in a static planar symmetric case). To that end, it is worth noticing that in this work we have considered the coherence only in the cases of two particular spacetime symmetries: ($k_0 = 0$-shell) in a spatially homogeneous case and ($k_z = 0$-shell) in a static (or stationary) planar symmetric case. For other spacetime symmetries, it is expected that similar coherence solutions would exist, living in different singular shells in the phase space. One interesting symmetry, worth studying in the future, is the rotationally symmetric geometry.

Of course, the singular shell picture is not exact for any (nontrivial) practical application, as the shells will spread out in the phase space when the interaction width Γ and the gradients of the background field are nonvanishing. Thus, an important question concerning the eQPA scheme is whether the new $k_{0,z} = 0$-shells are spreading similarly (in magnitude) to the well-known mass-shell solutions, when the strict quasiparticle limit is lifted. To elaborate on this question, we review a simple nonrelativistic example from ref. [III], where we computed the exact free-field Wightman function from the one-particle wave functions in the case of an infinite square well potential. In this case, the standing waves of bound states are highly coherent superpositions of the opposite travelling plane waves. The essential phase space structure is plotted in Fig. 4.3. We clearly see that the spreads of the mass-shell contributions (at $k_z = \pm k_n$) and the $k_z = 0$-shell are of the same order. This observation is confirmed by an analytical calculation, which shows that the functional forms of individual peaks are actually the same in this specific case. That is, the plotted phase space density in the middle of the square well is given by

$$\sum_{\pm} f(k_z \mp k_n) + 2f(k_z)\,, \tag{4.92}$$

where the density corresponding to each "shell" is $f(q) \equiv \sin(qL)/q$. Although this is just one simple example, it supports the expectation that the different phase space shells are spreading in an equal manner also more generally. Indeed, the spreads for all shells should be controlled by the same parameters, the interaction width Γ and the gradients of the background.

Another important issue in our eQPA scheme is the treatment of the gradient

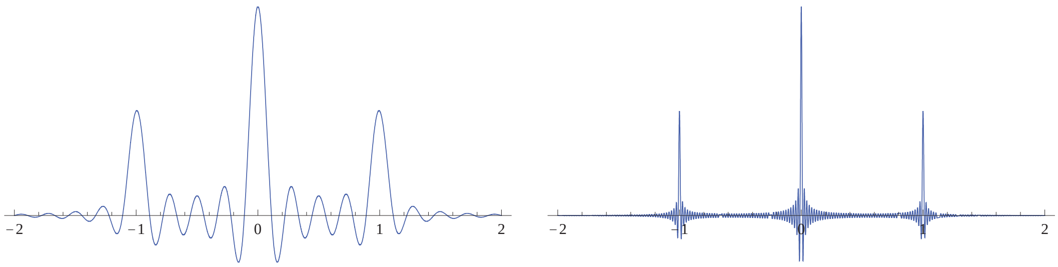

Figure 4.3: The phase space structure of the 2-point Wightman $i\Delta^<$ in the case of a 1-dimensional infinite square well potential. Shown is the phase space density Eq. (4.92) as a function of k_z/k_n at the middle of the square well at $z = 0$, for odd bound states with $n = 9$ (left) and $n = 99$ (right). For more details, see ref. [III].

expansion in the collision term. For the flow term this expansion is trivial as the k-derivatives can be handled easily in the k-integration (partial integrations are straightforward). For the collision term, on the other hand, nontrivial contributions arise that are proportional to spacetime derivatives of the correlator $\partial_x^n \mathcal{G}^<$. In the standard kinetic approach these terms would necessarily be proportional to Γ or the gradients of the background field $\partial^n m$[8], as the correlator is adiabatic. In our eQPA approach, in contrast, this is not the case: because of the oscillatory coherence solutions, one typically has (in a spatially homogeneous case): $\partial_t^2 \mathcal{G}^< \sim -4\omega_{\underline{k}}^2 \mathcal{G}^<$, such that all orders of the expansion involve terms that are not controlled by any small parameter. This severe-looking problem can be cured, however, by a recursive use of the equations of motion and a resummation of these coherence contributions, to get a controlled expansion in powers of Γ and $\partial^n m$. In an example of coherent production of decaying scalar particles in section 5.4, this expansion is worked out for a thermal interaction to the leading order, neglecting terms of order $\mathcal{O}(\Gamma^2, \Gamma\,\partial m)$ and higher.

[8]This notation represents the n-th gradient order, where it is assumed that the n-th derivative and the first derivative to the power n are of the same gradient order

Chapter 5

Applications

5.1 Klein problem

As our first application for the eQPA scheme introduced in chapter 4 we consider the famous Klein problem *i.e.* quantum reflection from a step potential. The problem with noninteracting quantum fields can be solved conveniently using the Dirac and Klein-Gordon equations for fermions and scalars, respectively. However, here we want to illustrate the necessity to include the coherence shell solutions when the problem is solved with the methods of quantum transport theory. Moreover, when the interacting fields are considered, the wave equation approach is not possible anymore, and one needs to use some more advanced methods, like our approximation scheme, to solve the problem.

The setup for the problem is illustrated in figure 5.1. We consider a step potential

$$V(z) = V\theta(-z)\,, \tag{5.1}$$

such that the mass-shell momenta are $k \equiv k_m^{\rm I} = (k_0^2 - |m|^2)^{1/2}$ and $q \equiv k_m^{\rm II} = ((k_0 - V)^2 - |m|^2)^{1/2}$ in regions I and II, respectively. The momentum k is always real, while q can be either real or imaginary, depending on the values of energy k_0 and potential V. The latter case describes the quantum tunneling inside the potential barrier. Without collisions energy is a conserved quantity, and we fix now: $k_0 \equiv \omega > 0$. In our approach the particle fluxes are just the mass-shell distribution functions $f_\pm$. The *asymptotic* boundary conditions for this setup are the following: We normalize the incoming flux from the right to unity $f_-^{\rm I} = 1$, and we have no incoming flux form the left $f_+^{\rm II} = 0$. Also, since there is no incoming flux from the left, we set the coherence asymptotically to zero in region II. In addition, in the case of imaginary mass-shell momentum q in region II, we cannot have any asymptotic mass-shell solution there, since no propagating wave would penetrate a potential barrier to infinite distance. The reflected and transmitted fluxes we want to compute are $f_+^{\rm I}$ and $f_-^{\rm II}$, respectively. We will first consider fermions and then scalar bosons. For both, we analyze the cases with real and imaginary q separately.

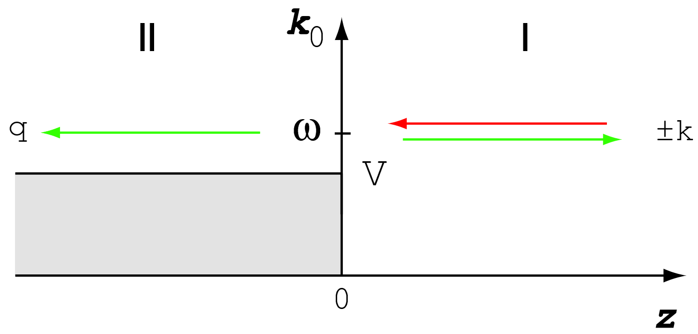

Figure 5.1: Reflection from a step potential $V(z) = V\theta(-z)$ with the momenta of incoming (red) and outgoing (green) particles described by arrows.

5.1.1 Fermions

For a varying potential instead of the mass, the fermionic free field KB-equation becomes in the mixed representation

$$\left(\not{k} + \frac{i}{2}\not{\partial}_x - m_R - i\gamma^5 m_I - \gamma_0 V(x) e^{-\frac{i}{2}\partial_x^V \cdot \partial_k}\right) G^<(k,x) = 0\,. \tag{5.2}$$

In this case the potential is planar symmetric, and the reduction of the spin structure is completely analogous to section 4.2.2 leading to the collisionless equations

$$\begin{aligned} (\mathrm{H}): \quad -2sk_z g_s^< &= \hat{P}_V g_s^< + g_s^< \hat{P}_V^\dagger \qquad (5.3)\\ (\mathrm{AH}): \quad is\partial_z g_s^< &= \hat{P}_V g_s^< - g_s^< \hat{P}_V^\dagger \qquad (5.4)\end{aligned}$$

with

$$\hat{P}_V \equiv (k_0 - \hat{V})\rho^3 + i(m_R\rho^2 + m_I\rho^1)\,, \tag{5.5}$$

where we denote $\hat{V}(z) \equiv V(z)e^{\frac{i}{2}\partial_z^V \partial_{k_z}}$. The spectral phase space structure of the correlator $g_s^<$ that is obtained by analyzing the H-equation (5.3) in the mean field limit, is now identical to the case of a varying mass given by Eq. (4.50), except for the replacement $k_0 \to k_0 - V$. By integrating the kinetic (AH) equation (5.4) over k_z we get

$$is\partial_z \rho_s = P_V \rho_s - \rho_s P_V^\dagger\,, \tag{5.6}$$

where we denote $\rho_s \equiv \int \frac{\mathrm{d}k_z}{2\pi} g_s^<$. The k_z-derivatives in $\hat{V}$ have disappeared due to the integration reducing $\hat{V}$ and $\hat{P}_V$ to their corresponding mean field forms V and P_V. When the spectral correlator is used as an ansatz for $g_s^<$, we get the following linear relations between the components of the chiral matrix ρ_s and the on-shell functions f_α, $\alpha = \pm, 1, 2$:

$$\begin{aligned}
\rho^s_{LL} &= \frac{1}{2}(\frac{\tilde{k}_0}{k_m} + s)f^s_- + \frac{1}{2}(\frac{\tilde{k}_0}{k_m} - s)f^s_+ + \frac{m_R}{2\tilde{k}_0}f^s_1 - \frac{m_I}{2\tilde{k}_0}f^s_2 \\
\rho^s_{RR} &= \frac{1}{2}(\frac{\tilde{k}_0}{k_m} - s)f^s_- + \frac{1}{2}(\frac{\tilde{k}_0}{k_m} + s)f^s_+ + \frac{m_R}{2\tilde{k}_0}f^s_1 - \frac{m_I}{2\tilde{k}_0}f^s_2 \\
\rho^s_{LR} &= \frac{m}{2k_m}(f^s_- + f^s_+) + \frac{1}{2}(f^s_1 - if^s_2) \\
\rho^s_{RL} &= \frac{m^*}{2k_m}(f^s_- + f^s_+) + \frac{1}{2}(f^s_1 + if^s_2)\,,
\end{aligned} \tag{5.7}$$

where $\tilde{k}_0 \equiv k_0 - V(z)$ and $k_m \equiv (\tilde{k}_0^2 - m^2)^{1/2}$. Formally these linear relations are invertible whenever $k_m \neq 0$. However, the on-shell momentum k_m may become imaginary, depending on the value of k_0 with respect to V. In this case, one needs to set $f_\pm = 0$, since it actually corresponds to a situation where the mass-shell functions do not contribute in k_z-integration at all due to imaginary roots in the $\delta(k^2 - m^2)$-factor. In general, the spectral approximation for the correlator $g^<_s$, based on the mean field limit with vanishing gradients, is expected to become better when the distance from the potential wall increases. *Asymptotically* far away from the wall the spectral decomposition and thus the relations (5.7) should be *exact*. Now, if we formally replace $k_m \to |k_m|$ in Eq. (5.7), we see that the resulting relations are invertible for all $k_m \neq 0$ and there are no problems with imaginary k_m. Thus, we can always parametrize ρ_s in terms of effective f-functions via these connection relations, which will reduce to the actual on-shell functions only asymptotically when k_m is real (or $f_\pm = 0$). The use of such a parametrization is that the equations of motion for the effective f-functions, resulting from the matrix equation (5.6), become very simple inside the regions I and II (we set here $m_I = 0$ for simplicity):

$$\begin{aligned}
s\partial_z f^s_\pm &= 0 \\
s\partial_z f^s_1 &= -2\tilde{k}_0 f^s_2 \\
s\partial_z f^s_2 &= \frac{2k_m^2}{\tilde{k}_0} f^s_1\,.
\end{aligned} \tag{5.8}$$

Moreover, the structure of the dynamical matrix equation (5.6) readily implies that the only consistent condition for matching ρ_s at $z = 0$ is that all of its components are continuous, while the off-diagonals have kinks *i.e.* their derivatives have finite discontinuities.

Real q, partial reflection

Let us first consider the case with real q. The equations (5.8) now imply that inside the regions I and II the functions $f^s_\pm$ are constants, so the asymptotic boundary conditions for the incoming fluxes fix $f^{s(\mathrm{I})}_- = 1$ and $f^{s(\mathrm{II})}_+ = 0$ throughout the regions. Moreover, the functions $f^{s(\mathrm{I,II})}_{1,2}$ are oscillatory in both regions. Hence, the asymptotic boundary condition for the coherence to vanish as $z \to -\infty$, kills it completely in region II: $f^{s(\mathrm{II})}_{1,2} = 0$. Using these conditions and the four matching conditions for the components of ρ^{I}_s and ρ^{II}_s at $z = 0$, we can fix all the remaining constants: $f^{s(\mathrm{I})}_{+}$, $f^{s(\mathrm{II})}_{-}$, as well as two constants related to the oscillatory coherence solutions in region I. We find that the reflected and transmitted fluxes are related by flux conservation:

$$f^{s(\mathrm{I})}_{+} = \frac{1-x}{1+x} \qquad \text{and} \qquad f^{s(\mathrm{II})}_{-} = 1 - f^{s(\mathrm{I})}_{+} = \frac{2x}{1+x}\,, \tag{5.9}$$

while the oscillatory coherence solution in region I is given by

$$f^{s(\mathrm{I})}_{1}(z) = \frac{m\omega V}{k^2 q}\frac{2x}{1+x}\cos(2kz) \qquad \text{and} \qquad f^{s(\mathrm{I})}_{2}(z) = -\frac{1}{2\omega}s\partial_z f^{s(\mathrm{I})}_{1}(z)\,, \tag{5.10}$$

where we denote

$$x \equiv \frac{qk}{\omega(\omega - V) - m^2}\,. \tag{5.11}$$

Imaginary q, total reflection

In the case of imaginary q we have an additional boundary condition. That is, we cannot have any asymptotic mass-shell solution in region II: $f^{s(\mathrm{II})}_{\pm} = 0$ throughout (since they are constants). However, we see from Eq. (5.8) that the coherence solutions are exponentials in region II for imaginary q, so that the asymptotic boundary condition at $z \to -\infty$ does not kill the exponentially decaying mode. An otherwise similar analysis gives now the result with a complete reflection:

$$f^{s(\mathrm{I})}_{+} = 1 \qquad \text{and} \qquad f^{s(\mathrm{II})}_{-} = 0\,, \tag{5.12}$$

while the coherence solutions are

$$f^{s(\mathrm{I})}_{1}(z) = \left(\frac{k(\omega - V)}{mV} - \frac{2m}{k}\right)\cos(2kz) + \frac{|q|\omega}{mV}\sin(2kz) \tag{5.13}$$

with $f^{s(\mathrm{I})}_{2}(z) = -s\partial_z f^{s(\mathrm{I})}_{1}(z)/2\omega$ in region I (there was a misprint in Eq. (118) in ref. [I]), and

$$f^{s(\mathrm{II})}_{1}(z) = \frac{k(\omega - V)}{mV}e^{2|q|z} \tag{5.14}$$

with $f^{s(\mathrm{II})}_{2}(z) = -s\partial_z f^{s(\mathrm{II})}_{1}(z)/2(\omega - V)$ in region II. Note that the results in Eqs. (5.9-5.14), which agree with the standard Dirac equation approach [104], would not have been obtained, should we have dropped the coherence shell solutions $f^{s}_{1,2}$ from our analysis.

5.1.2 Scalar bosons

For scalar fields the free field KB-equation with a z-dependent potential becomes

$$\Big([k_0 - V(z)]^2 e^{-\frac{i}{2}\partial_z^V \partial_{k_z}} - k_z^2 - \frac{1}{4}\partial_z^2 + ik_z\partial_z - m^2\Big) i\Delta^< = 0\,, \tag{5.15}$$

where we have taken $\vec{k}_{||} = 0$. Taking the real and imaginary parts of Eq. (5.15) we find:

$$\begin{aligned}\Big(k_z^2 - \frac{1}{4}\partial_z^2 - k_m^2 \cos(\tfrac{1}{2}\partial_z^V\partial_{k_z})\Big) i\Delta^< &= 0 &(5.16)\\ \Big(k_z\partial_z + k_m^2 \sin(\tfrac{1}{2}\partial_z^V\partial_{k_z})\Big) i\Delta^< &= 0\,, &(5.17)\end{aligned}$$

where $k_m(z) \equiv ((k_0 - V(z))^2 - m^2)^{1/2}$. We find again that the spectral phase space structure of the correlator $\Delta^<$, obtained by analyzing these equations in the mean field limit, is identical to the case of a spatially varying mass given by Eq. (4.65), except for the replacement $k_0 \to k_0 - V$. We proceed now analogously to the spatially homogeneous case in section 4.4.2. The n-th moments of the correlator $\Delta^<$ are defined as integrals over k_z:

$$\rho_n(k_0, z) \equiv \int \frac{\mathrm{d}k_z}{2\pi}\, k_z^n\, i\Delta^<(k_0, \vec{k}_{||} = 0, k_z, z)\,. \tag{5.18}$$

By taking the 0th moment of Eq. (5.16) and the 0th and 1st moments of Eq. (5.17) we get the following closed set of equations for the three lowest moments $\rho_{0,1,2}$:

$$\begin{aligned}\frac{1}{4}\partial_z^2\rho_0 + k_m^2\rho_0 - \rho_2 &= 0\\ \partial_z\rho_1 &= 0\\ \partial_z\rho_2 - \frac{1}{2}(\partial_z k_m^2)\rho_0 &= 0\,.\end{aligned} \tag{5.19}$$

When the spectral correlator Eq. (4.65) is used as an ansatz for $\Delta^<$, we get the following relations between the lowest moments $\rho_{0,1,2}$ and the on-shell functions $f_{\pm,c}$:

$$\begin{aligned}\rho_0 &= \mathrm{sgn}(k_0 - V(z))\frac{1}{2k_m}(f_+ + f_-) + f_c\\ \rho_1 &= \mathrm{sgn}(k_0 - V(z))\frac{1}{2}(f_+ - f_-)\\ \rho_2 &= \mathrm{sgn}(k_0 - V(z))\frac{k_m}{2}(f_+ + f_-)\,.\end{aligned} \tag{5.20}$$

These linear relations are invertible whenever $k_m \neq 0$, but for the case of imaginary k_m we have to set $f_\pm = 0$, as for fermions. Again, the spectral approximation for the correlator, and consequently the relations (5.20), become exact asymptotically far away from the wall. The matching conditions at $z = 0$ induced by the moment equations (5.19) are now more complicated than with fermions. First, we note that $\partial_z \rho_1$ vanishes everywhere, so ρ_1 is a constant throughout. The last Eq. (5.19) implies that ρ_2 must have a finite discontinuity over the barrier, which can be computed by integrating it over a step from $z = -\epsilon$ to $z = \epsilon$ to give $\rho_2^{\mathrm{I}} - \rho_2^{\mathrm{II}} = \frac{1}{2}[k^2 - q^2]\rho_0(z = 0)$. Using this in the first Eq. (5.19), we see that also $\partial_z^2 \rho_0$ has at most a finite discontinuity over the step, implying finally that ρ_0 and its derivative $\partial_z \rho_0$ are continuous at $z = 0$.

Real q, partial reflection

We consider first the case with real q. The two latter Eqs. (5.19) imply that $\partial_z \rho_{1,2} = 0$, *i.e.* $\rho_{1,2}$ are constants inside the regions I and II. These constants are partially fixed by the asymptotic boundary conditions $f_-^{\mathrm{I}} = 1$ and $f_+^{\mathrm{II}} = 0$ through the relations (5.20). Moreover, from the first Eq. (5.19) we now find that ρ_0 is oscillatory with a constant shift of amount ρ_2/k_m^2 in both regions I and II, respectively. Combining Eqs. (5.20) and (5.19) we then find that asymptotically the coherence solutions are also oscillatory: $f_c = -\partial_z^2 \rho_0/(4k_m^2)$, so the asymptotic vanishing of the coherence as $z \to -\infty$ kills the oscillatory part of ρ_0 completely in region II. Using these conditions and the four matching conditions for $\rho_{0,1,2}$ and $\partial_z \rho_0$ at $z = 0$ described above, we can fix all the remaining constants to obtain the solution

$$f_+^{\mathrm{I}} = \frac{(k-q)^2}{(k+q)^2} \qquad \text{and} \qquad f_-^{\mathrm{II}} = 1 - f_+^{\mathrm{I}} = \frac{4kq}{(k+q)^2}\,, \tag{5.21}$$

while the coherence solution in the region I is given by

$$f_c^{\mathrm{I}} = \frac{1}{k}\sqrt{f_+^{\mathrm{I}}}\cos(2kz)\,. \tag{5.22}$$

Note that these solutions describe the shell structure of the correlator $\Delta^<$ exactly only asymptotically as $z \to \pm\infty$. However, as in the case of fermions, replacing $k_m \to |k_m|$ the relations are invertible for all $k_m \neq 0$, so that the moments $\rho_{0,1,2}$ can be completely parametrized with these f-functions. In this (effective) sense the solutions (5.21) and (5.22) are exact throughout the regions I and II.

Imaginary q, total reflection

As for fermions, in the case of imaginary q we cannot have any mass-shell solutions in region II, so that asymptotically both $f_\pm^{\mathrm{II}} = 0$. Now the coherence solution $f_c = -\partial_z^2 \rho_0/(4k_m^2)$ is a superposition of decaying and increasing exponentials in region II, so that the asymptotic boundary condition at $z \to -\infty$ does not kill the decaying mode. Because of these differences we get the solution with total reflection:

$$f_+^{\mathrm{I}} = 1 \qquad \text{and} \qquad f_\pm^{\mathrm{II}} = 0\,, \tag{5.23}$$

while the coherence solution is given by

$$f_c^{\mathrm{I}}(z) = \frac{1}{k}\frac{k^2 - |q|^2}{k^2 + |q|^2}\cos(2kz) - \frac{2|q|}{k^2 + |q|^2}\sin(2kz) \tag{5.24}$$

in region I, and

$$f_c^{\mathrm{II}}(z) = \frac{2k}{k^2 + |q|^2}e^{2|q|z} \tag{5.25}$$

in region II. Again, we would not have obtained the results in Eqs. (5.21-5.25) that agree with the standard Klein-Gordon approach, should we have dropped the coherence shell solutions f_c from our analysis.

5.2 Quantum reflection from a CP-violating mass wall

Let us next consider the quantum reflection of free fermionic fields from a smooth CP-varying mass wall. This simple example is of relevance for electroweak baryogenesis (see refs. [105–107] and section 2.1.1), where the complex spatially varying mass function arises from Yukawa couplings to the vacuum expectation value of the Higgs field(s):

$$m(z) = y\phi(z)\,, \tag{5.26}$$

where y is a Yukawa coupling and $\phi(z)$ is a complex scalar field corresponding to the total effect of the VEVs of (possibly) multiple Higgs fields [107]. For the complex mass wall we have used the following parametrization (see Fig. 5.2):

$$|\phi(z)| = \frac{1}{2}\big(1 - \tanh(z/\ell_w)\big)\,, \qquad \arg[\phi(z)] = \frac{1}{2}\Delta\theta\big(1 + \tanh(z/\ell_w)\big)\,, \tag{5.27}$$

where ℓ_w is the width of the wall, and $\Delta\theta$ is the total magnitude of the change of the phase of ϕ. The problem is described by the same integrated evolution equation as for the fermionic Klein problem, Eq. (5.6), and the connection relations Eq. (5.7) between the components of the chiral matrix ρ_s and the on-shell functions f_α with the replacements $V(z) \to 0$ and $m_{R,I} \to m_{R,I}(z)$. For the asymptotic boundary conditions we set the incoming flux from deep in the symmetric phase at $z \to \infty$ to unity and take no incoming flux from deep in the broken phase at $z \to -\infty$, and correspondingly set the asymptotic coherence to zero as well:

$$\begin{aligned} f_-^s &= 1, &\qquad z &\to \infty \\ f_+^s = f_1^s = f_2^s &= 0, &\qquad z &\to -\infty\,. \end{aligned} \tag{5.28}$$

Let us summarize the results of the numerical calculations presented in ref. [I]. The calculations were performed using Eq. (5.6) in Bloch representation: $\rho_s \equiv \frac{1}{2}(\langle g_0^s\rangle + \langle \vec{g}^s\rangle \cdot \vec{\sigma})$, together with the relations Eq. (5.7) and the boundary conditions Eq. (5.28). In figure 5.3 we plot the results for the case of $s = 1$ and

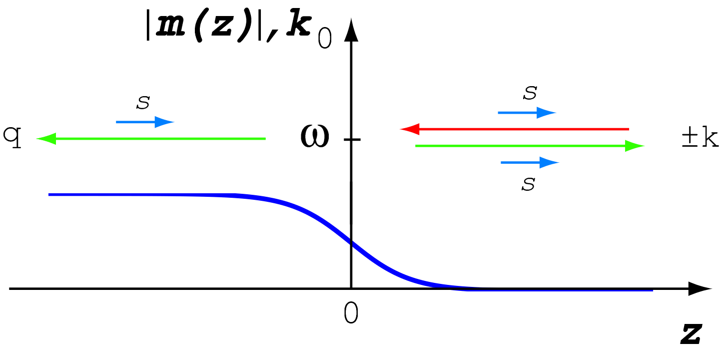

Figure 5.2: Reflection from a spatially varying mass wall, with the momenta of incoming (red) and outgoing (green) particles described by arrows. Spin s is conserved in the collisionless case.

$q/|m_{-\infty}| = 0.088$, where $q \equiv (\omega^2 - |m_{-\infty}|^2)^{1/2}$ is the asymptotic momentum in the broken phase. In the left panel we show the values of the on-shell functions f_α while in the right panel we plot the components of the chiral matrix ρ. We can see that in the symmetric phase (to the right from the wall) the system is a coherent superposition of left and right moving states with opposite k_z-momenta, and the k_z=0-shell functions are oscillating coherently. In the broken phase however, all but the f_--function die off and the state soon becomes a pure transmitted left moving state. Note that the physical flux is conserved: $f_+(\infty) + f_-(-\infty) = f_-(\infty) \equiv 1$. For the chiral components we see that the imaginary part of $\rho_{\rm LR}$ goes to zero when $z \to -\infty$, as a result of our choice that m becomes asymptotically real in the broken phase. The diagonal components of ρ become large in the broken phase, because they represent chiral densities that are enhanced in the regime of small local velocity $v_z = k_z/\omega$ due to flux conservation.

In figure 5.4 we show the particle-antiparticle flux-asymmetry $\Delta j_+ \equiv (f_+ - \bar{f}_+)_{z=\infty}$ as a function of $q/|m_{-\infty}|$. For antiparticles we simply need to make a replacement $m \to m^*$ in all equations. The characteristic peaked shape of the flux-asymmetry can be understood as follows: The reflection amplitudes for both particles and antiparticles tend to unity in the limit of total reflection $q \to 0$, so the asymmetry must tend to zero. For large q on the other hand, the (anti)particles start to behave classically compared to the width (and height) of the wall, and hence the reflection amplitudes and the asymmetry start to decrease exponentially.

These results have been derived earlier using the Dirac equation approach (see *e.g.* ref. [107]). Here we just wanted to demonstrate how the results are obtained using our formalism. We emphasize that for noninteracting fields our method is *exact*, as in the case of Klein problem. That is, all gradients of the mass are vanishing asymptotically at $z \to \pm\infty$, and consequently the mean field approximation becomes exact in that limit. So the $f_\pm$-functions indeed correspond to the physical fluxes asymptotically, and this is the only place where

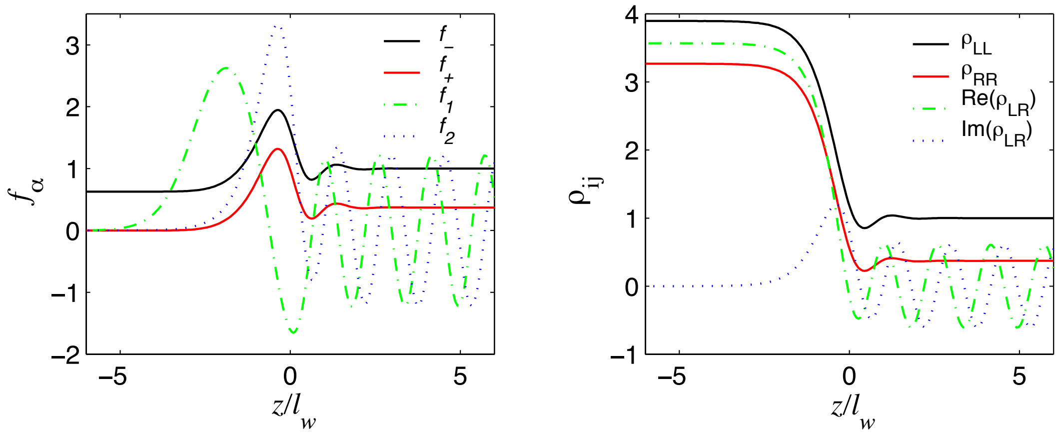

Figure 5.3: a) Shown are the mass-shell functions $f_\pm$ corresponding to the left and right moving fluxes and functions $f_{1,2}$, which encode the quantum coherence. We have taken $s = 1$ and $q/|m_{-\infty}| = 0.088$. b) The chiral density matrix components for the same solution. For the wall width and the total change of the phase we used $\ell_w = 2$ and $\Delta\theta = -1$.

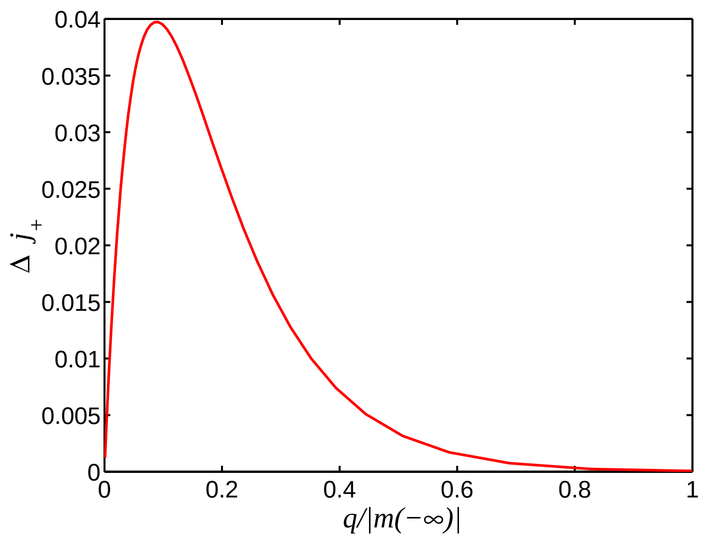

Figure 5.4: Shown is the current asymmetry of reflected states as a function of the asymptotic momentum to mass ratio in the broken phase. The wall parametrization is the same as in figure 5.3.

this identification is needed in our analysis. Between the boundary regions the f-functions are just a parametrization of the chiral matrix ρ. However, the whole point of our approach is the ability to include the effects of collisions together with nonlocal coherence. With collisions the picture changes as we need the on-shell identifications of the f-functions for the reliable computation of the collision term[1]. For this reason our method becomes an approximation in the case with collisions, with the better *quantitative* results expected the thicker the wall is. But even in the thin wall limit we expect to get at least the correct *qualitative* behaviour, yet it would not be surprising if the results would prove to be quite accurate also quantitatively, since the region of (rapidly) varying mass is very narrow in this case, and outside the wall region the mean field approximation should work well. Moreover, the f-functions are presumably smooth also in the wall region, as suggested by the plots in Fig. 5.3, thus the overall error in the dynamics arising from the (approximate) computation of the collision term in that region should be rather small.

5.3 Preheating of decaying fermions

As a first example of our formalism including collisions, we consider the production of fermionic particles at preheating, during which the fermion is subjected to decays. A similar fermion production scenario in out-of-equilibrium conditions has been recently studied using the 2PI approach in ref. [108], where a complete numerical next-to-leading order calculation in a $1/N$-expansion of the 2PI effective action was performed for a $SU(2)_L \times SU(2)_R \sim O(4)$ symmetric theory with Yukawa interactions between fermions and bosons. As discussed in section 2.2, in preheating the (coupled) fermions obtain a rapidly oscillating effective mass due to the couplings to the inflaton condensate, for which we use here a simple cosine function (here t corresponds to a time coordinate in the conformal frame, and the expansion of the universe is neglected during the preheating) [75, 102]:

$$m(t) = m_0 + A\cos(2\omega_\varphi t) + iB\sin(2\omega_\varphi t), \tag{5.29}$$

where m_0, A, B and ω_φ (the inflaton oscillation frequency) are real constants. The decay of the fermion into daughter particles is modelled here by a left-chiral non-diagonal Yukawa interaction defined by the Lagrangian:

$$\mathcal{L}_{\rm int} = -y\,\bar{\psi}_L \phi\, q_R + h.c. \tag{5.30}$$

where ψ is the fermion field considered, q is some other fermion (quark) field and ϕ is a complex scalar field. For simplicity, we assume that the fields q and ϕ form a thermal background *i.e.* they are in thermal equilibrium throughout

[1] Also the corrections from mass gradients that arise from the gradient expansion of the collision term become important

the preheating. A more realistic but complex scenario could be modelled by taking into account the full dynamics of the daughter particles with appropriate equations of motion. By the assumption of thermal equilibrium for the daughter fields, the self energies $\Sigma^{<,>}$ for the fermion ψ are related by the KMS relation $\Sigma^{>}(k) = e^{\beta k_0}\Sigma^{<}(k)$, and the interaction width Γ can be written in the form [II]

$$\Gamma(k) = \frac{1}{2}(1+e^{\beta k_0})i\Sigma^{<}(k) = \Big[\Gamma_0\,\gamma^0 - \Gamma_3\left(\hat{k}\cdot\vec{\gamma}\right)\Big]\,P_L\,, \tag{5.31}$$

where $P_L = \frac{1}{2}(1-\gamma^5)$ is the left chiral projector and $\Gamma_{0,3}(k)$ are real functions. Using these expressions and the mean field spectral function Eq. (4.78) we find that the zeroth order gradient contribution to the collision term $\mathcal{C}_h^-$ in the dynamical equation (4.66) is given by

$$\mathcal{C}_h^- = -i\left\{D,\, g_h^< - (g_h^<)_{\rm eq}\right\}\,, \tag{5.32}$$

where

$$D \equiv \frac{1}{2}(1+\rho_3)\,\Gamma_h \qquad \text{with} \qquad \Gamma_h \equiv \Gamma_0 - h\Gamma_3 \tag{5.33}$$

and $(g_h^<)_{\rm eq}$ is the thermal equilibrium limit of $g_h^<$ with $f_\pm^h = f_\pm^{\rm eq} \equiv 1/(e^{\pm\beta\omega_{\vec{k}}}+1)$. Upon integration over k_0 we get then the matrix equation (4.67):

$$\partial_t\rho_h = -i[H,\rho_h] - i\langle\mathcal{C}_h^-\rangle\,, \tag{5.34}$$

where the collision integral is now given by

$$\begin{aligned}
i\langle\mathcal{C}_h^-\rangle \;&\equiv\; \int\frac{{\rm d}k_0}{2\pi}\left\{D\,,\,g_h^< - (g_h^<)^{\rm eq}\right\}\\
&=\; \Gamma_{m0}\begin{pmatrix} (f_0^h - f_0^{\rm eq}) - h\frac{k}{\omega}(f_3^h - f_3^{\rm eq}) & \frac{m}{2\omega}(f_3^h - f_3^{\rm eq}) \\ \frac{m^*}{2\omega}(f_3^h - f_3^{\rm eq}) & 0 \end{pmatrix}\\
&-\; h\Gamma_{m3}\begin{pmatrix} (f_3^h - f_3^{\rm eq}) - h\frac{k}{\omega}(f_0^h - f_0^{\rm eq}) & \frac{m}{2\omega}(f_0^h - f_0^{\rm eq}) \\ \frac{m^*}{2\omega}(f_0^h - f_0^{\rm eq}) & 0 \end{pmatrix}\\
&+\; \Gamma_{00}\begin{pmatrix} h\frac{m_R}{k}f_1^h - h\frac{m_I}{k}f_2^h & \frac{1}{2}(f_1^h - if_2^h) \\ \frac{1}{2}(f_1^h + if_2^h) & 0 \end{pmatrix},
\end{aligned} \tag{5.35}$$

where we use the shorthand notations $f_{0,3}^h \equiv f_+^h \pm f_-^h$ and $k \equiv |\vec{k}|$. The Γ_i-functions appearing as coefficients are now projections on the different singular shells:

$$\begin{aligned}
\Gamma_{m(0,3)}(|\vec{k}|,t) \;&\equiv\; \Gamma_{0,3}(k_0=\omega(t),|\vec{k}|) \qquad &&\text{(positive mass-shell)}\\
\Gamma_{00}(|\vec{k}|) \;&\equiv\; \Gamma_0(k_0=0,|\vec{k}|) \qquad &&(k_0 = 0\text{ -shell})\,.
\end{aligned} \tag{5.36}$$

The explicit expressions for these Γ-functions are not presented here, but they can be read from the relations $\Gamma_{0,3}(k) = \frac{1}{2}(1 + e^{\beta k_0}) i\Sigma^<_{0,3}(k)$ and the equations (6.23-6.28) in ref. [II]. Note that Γ on the negative mass-shell is related to that on the positive with $\Gamma_{0,3}(-k_0, |\vec{k}|) = \pm\Gamma_{0,3}(k_0, |\vec{k}|)$, and furthermore that Γ_3 vanishes on the $k_0 = 0$ -shell.

Let us now summarize the results of the numerical calculations for particle production through a parametric resonance. A more detailed discussion is found in ref. [II]. In the calculations we have used the equations (5.34)-(5.35) with the connection relations (4.69), and the oscillatory mass function Eq. (5.29). We have studied the time-evolution of the fermionic particle number and coherence with varying interaction strengths. In all cases, the initial condition for the evolution has been an uncorrelated vacuum state with $n_{\vec{k}h} \equiv \bar{n}_{\vec{k}h} = f_{1,2} = 0$. The results of our calculations for helicity $h = -1$ are presented in Fig. 5.5. In the upper panel we have considered the noninteracting case with $\Gamma = 0$. We see that the produced particle number (thick black line) as well as the "total amount" of coherence $f_c \equiv \sqrt{f_1^2 + f_2^2}$ (dotted blue line) increases steadily as a function of time. This increase takes place around the resonance peaks while between the peaks the particle number is essentially constant and the coherence oscillates with a constant amplitude, saturating to a maximum after a few resonance crossings. This picture of parametric resonance depends sensitively on the parameters of the mass oscillation and the size of momentum $|\vec{k}|$ and helicity h. For example for the opposite helicity $h = +1$ with otherwise the same parameters we would get a completely different figure without a clear resonance behavior.

In the lower panel of Fig. 5.5 we have considered the case with interactions, $\Gamma \neq 0$, with otherwise the same parametrization as in the noninteracting case. The difference compared with the noninteracting cases is quite dramatic. Now we see that the particle number drops between the resonance peaks, which results from the decays to the daughter particles, only to be regenerated again in the next resonance crossing. Also the growth of the coherence is now damped in comparison with the noninteracting case, and both the particle number and the coherence evolution settle into a stationary pattern after a few oscillation periods. In Fig. 5.6 we plot the particle number evolution for varying interaction strengths, again with otherwise the same parametrization as before. We see that the same pattern remains with increase in particle number during the resonance peaks, while the magnitude of damping (decays) between the peaks depends on the strength of the interactions, as expected.

These results show that if the fermion has strong enough interactions with other fields during the preheating, the effects (especially decoherence) on the amount of produced particles can be significant. Caution must be exercised in that our approximation scheme with singular shell decompositions is not guaranteed to produce correct quantitative results in this example, because the rapidly oscillating mass function gives rise to big gradient corrections. However, at least the qualitative picture is expected to be correct.

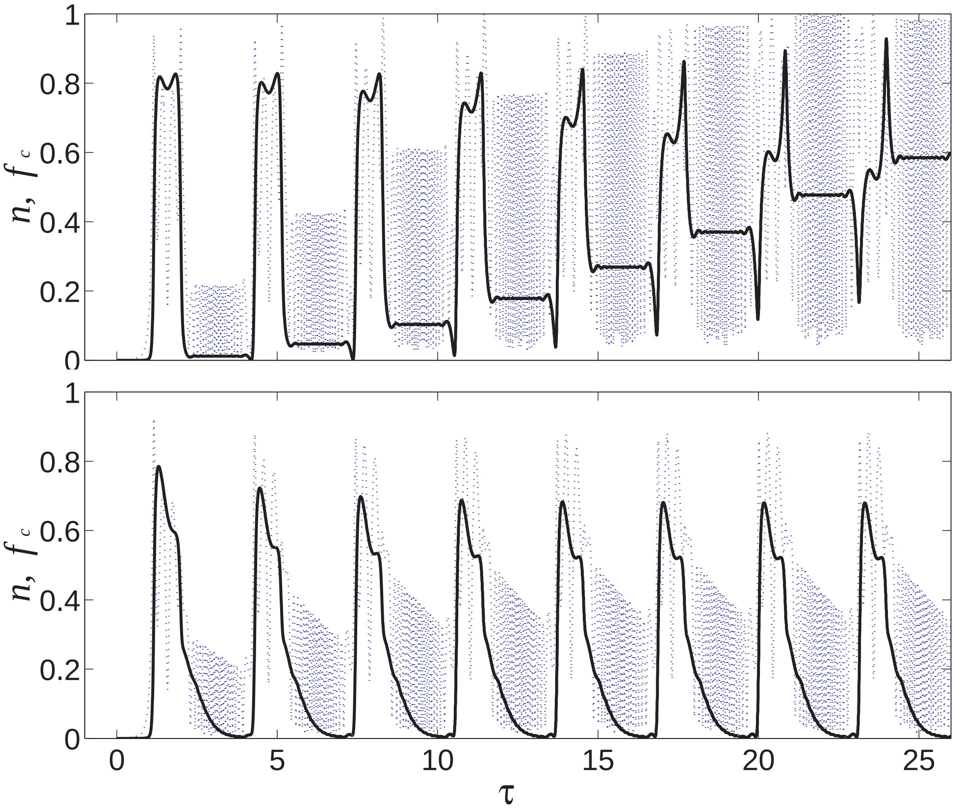

Figure 5.5: Shown is the mean field number density $n_{\vec{k}h}$ of produced fermions, and the total amount of coherence $f_c^h \equiv ((f_1^h)^2 + (f_2^h)^2)^{1/2}$ (thin dotted blue line) for the negative helicity state $h = -1$. Effects of inflaton oscillations are modelled by a varying mass $m(t) = (10+15\cos(2\omega_\varphi t)+i\sin(2\omega_\varphi t))T$. The upper panel corresponds to the collisionless case, while in the lower panel the collision terms of Eq. (5.35) were used, with the parameters $|\vec{k}| = T$, $y = 5, m_q = 0.02T$, $m_\phi = 0.1T$, and $\omega_\varphi = T$, where the temperature T sets the scale. Initially, at $\tau \equiv \omega_\varphi t = 0$ the fermion system is taken to be an uncorrelated vacuum state.

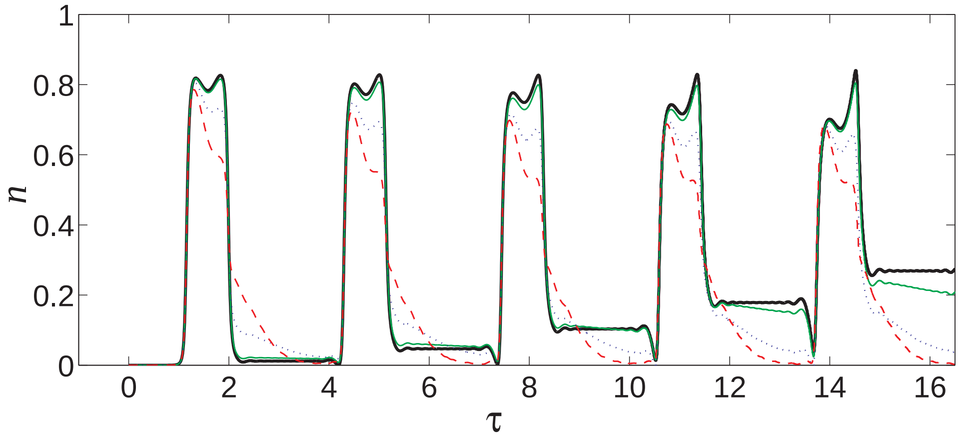

Figure 5.6: Shown is the number density $n_{\vec{k}h}$ in the same setting as before with changing interaction strengths. The thick black line is the free field case with a coupling constant $y = 0$. The other lines are interacting cases, with $y = 1$ (green line), $y = 3$ (blue dotted line) and $y = 5$ (red dashed line).

5.4 Coherent production of decaying scalar particles

In this section we consider a similar coherent particle production with decays as in the previous section 5.3, but now for scalar fields. A comprehensive study of resonant particle production using the 2PI approach has been carried out in ref. [109], where a complete numerical next-to-leading order calculation in a $1/N$-expansion of the 2PI effective action was performed for an $O(N)$-symmetric scalar theory. We are not trying to model the scalar preheating scenario here, we simply want to compare the behaviour and the results of decaying scalar fields with that of fermions. For this purpose we take the mass term driving the scalar particle production to the absolute value of the fermionic mass in Eq. (5.29):

$$m^2(t) \equiv \left| m_0 + A\cos(2\omega_\varphi t) + iB\sin(2\omega_\varphi t)\right|^2 , \tag{5.37}$$

where m_0, A, B and ω_φ (oscillation frequency of the driving field φ) are real constants. For the decays we use the following Yukawa interaction:

$$\mathcal{L}_{\text{int}} = -y\,\bar{\psi}\psi\phi\,, \tag{5.38}$$

where ϕ is the real scalar field considered and ψ is some fermion field, which is assumed to be in thermal equilibrium throughout the evolution. Hence the scalar field self energies $\Pi^{<,>}$ are related by the KMS relation $\Pi^{>}(k) = e^{\beta k_0}\Pi^{<}(k)$ and the interaction width Γ is given by

$$\Gamma(k) = \frac{1}{2}(1 + e^{\beta k_0})i\Pi^{<}(k)\,. \tag{5.39}$$

Using these expressions and the mean field spectral function in Eq. (4.80) we find that up to first order gradients in the collision term $i\mathcal{C}^{\phi}_{\rm coll}$, the collision integrals in the dynamical equations (4.73) are given by

$$
\begin{aligned}
\langle \mathcal{C}^+ \rangle &= \frac{1}{2}\partial_{k_0}\Gamma_0\partial_t f_c \\
\langle \mathcal{C}^- \rangle &= -\frac{1}{2\omega_{\vec{k}}}\Gamma_m\left[(f_+ + f_-) - (f_+^{\rm eq} + f_-^{\rm eq})\right] \\
\langle k_0\mathcal{C}^- \rangle &= -\frac{1}{2}\Gamma_m\left[(f_+ - f_-) - (f_+^{\rm eq} - f_-^{\rm eq})\right] \,,
\end{aligned}
\tag{5.40}
$$

where we have now defined $f_\pm^{\rm eq} \equiv 1/(e^{\pm\beta\omega_{\vec{k}}} - 1)$, and we have neglected terms of order[2] $\mathcal{O}(\Gamma^2, \Gamma\partial_t m^2)$. The Γ_i-functions appearing in Eq. (5.40) are again projections onto the mass- and the coherence shells:

$$
\begin{aligned}
\Gamma_m(|\vec{k}|, t) &\equiv \Gamma(k_0 = \omega_{\vec{k}}(t), |\vec{k}|) && \text{(positive mass-shell)} \\
\partial_{k_0}\Gamma_0(|\vec{k}|) &\equiv \partial_{k_0}\Gamma(k_0 = 0, |\vec{k}|) && (k_0 = 0 \text{ -shell})\,.
\end{aligned}
\tag{5.41}
$$

As in the previous section, we present no explicit expressions for these Γ_i-functions here, but they can be read from the equations (5.39) and (A.5-A.11) in ref. [III]. Note that Γ on the negative mass-shell is again simply related to that on the positive mass-shell: $\Gamma(-k_0, |\vec{k}|) = -\Gamma(k_0, |\vec{k}|)$ and it vanishes on $k_0 = 0$-shell. This is the reason that we need to include the first order gradients of the collision term in the scalar case, as we see that $\partial_{k_0}\Gamma_0$ is the lowest order nonvanishing $k_0 = 0$-shell contribution, giving rise to decoherence effects. Using the relations (4.74) we can now express the on-shell functions $f_{\pm,c}$ in terms of the moments $\rho_{0,1,2}$ to find a closed set of equations for these moment functions:

$$
\begin{aligned}
\frac{1}{4}\partial_t^2\rho_0 + \omega_{\vec{k}}^2\rho_0 - \rho_2 &= -\frac{1}{2}\partial_{k_0}\Gamma_0\,\partial_t\rho_0 \\
\partial_t\rho_1 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_1 - \rho_{1,{\rm eq}}) \\
\partial_t\rho_2 - \frac{1}{2}\partial_t(m^2)\rho_0 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_2 - \rho_{2,{\rm eq}})\,,
\end{aligned}
\tag{5.42}
$$

where $\rho_{i,{\rm eq}}$ are the the thermal equilibrium values for the moments with $f_\pm = f_\pm^{\rm eq}$ and $f_c = 0$.

Let us now summarize the results of the numerical calculations for particle production through a parametric resonance. A more detailed discussion can be found in ref. [II]. It turns out that the equations of motion (5.42) become numerically unstable for rapidly oscillating driving mass terms in Eq. (5.37). For that reason we have written Eq. (5.42) in a different form with nonlinear terms and an integration constant, given by Eqs. (8.3)-(8.5) in ref. [III], for which the stability problems do not occur. Using these equations we have performed

[2]These terms are neglected also indirectly, by recursive use of equations of motion (4.73)

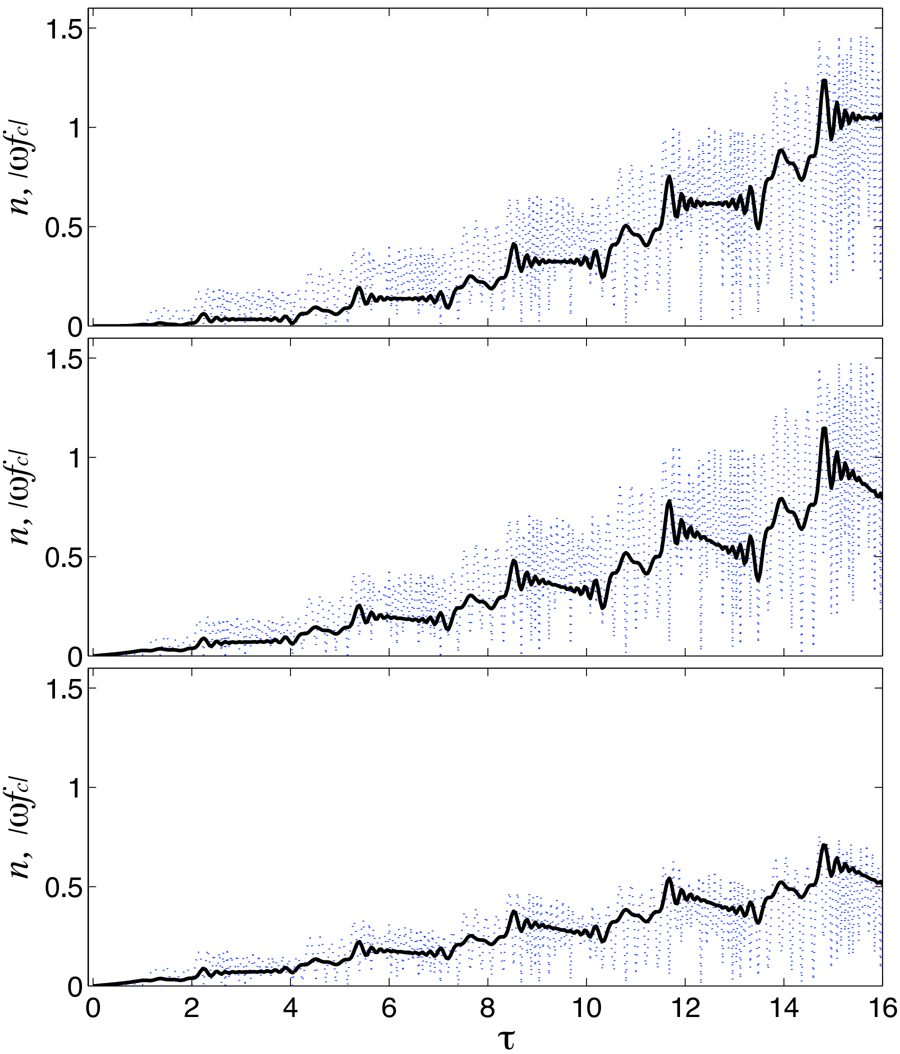

Figure 5.7: Shown is the number density $n_{\vec{k}}$ (thick solid line) and the coherence function $f_c(|\vec{k}|)$ (dotted(blue) line) with changing interactions. The driving mass function is taken to be $m^2(t) = |(1 + 1.5\cos(2\omega_\varphi t) + i\,0.1\sin(2\omega_\varphi t))T|^2$. The upper panel corresponds to the case without collisions. In the central panel we have included the collision terms on the mass-shells, but kept $\partial_{k_0}\Gamma_0 = 0$. In the lowest panel the full interaction terms were kept for all shells. For parameters we have used $|\vec{k}| = 0.6\,T$, $y = 1$, $m_\psi = 0.1\,T$ and $\omega_\varphi = 0.1T$, where the temperature T sets the scale. Initially, at $\tau \equiv \omega_\varphi t = 0$ the system is in the adiabatic vacuum.

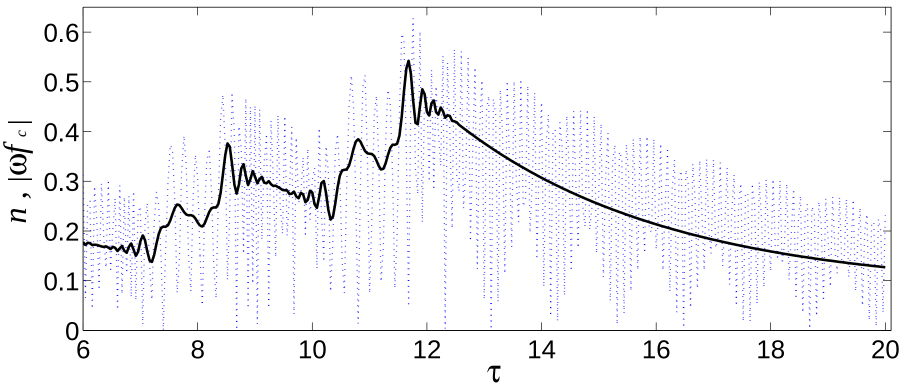

Figure 5.8: The same configuration as in the lowest panel of Fig. 5.7 but with the driving mass term smoothly set to a constant $m = 2.5\,T$ after $\tau > 4\pi$.

similar calculations as for fermions in section 5.3 *i.e.* we have studied the time-evolution of the particle number and coherence from an uncorrelated vacuum state $n_{\vec{k}} \equiv \bar{n}_{\vec{k}} = f_c = 0$, with varying interaction strengths.

The results of the calculations are presented in Fig. 5.7. In the upper panel we have considered the noninteracting case with $\Gamma = 0$. The increase of the number density (thick solid line) is seen to be accompanied by a steady growth of the amplitude of the coherence (thin dotted blue line). Similar to fermions there are regions (resonance "peaks") where the increase takes place, while between the peaks the particle number and the amplitude of the coherence are relatively constant. In the middle panel we considered the case where only the mass-shell collision terms were included but we set artificially $\partial_{k_0}\Gamma_0 \equiv 0$, while in the lowest panel all collision terms were included properly. In these cases with interactions we see again that the decrease of the particle number (decays to daughter particles) and of the amplitude of the coherence (decoherence) takes place between the resonance peaks. By comparing the two lowest panels we can conclude that the dominant effect of interactions comes from the coherence shell contribution, $\partial_{k_0}\Gamma_0$, which would be absent in the standard quantum Boltzmann approach.

To see the effects of thermalization and decoherence more clearly, we consider finally a case where the driving term was switched off smoothly at a specific moment. This scenario is depicted in Fig. 5.8. We can see a smooth thermalization of the particle number towards the equilibrium value, accompanied by decoherence towards zero in the form of damped oscillations of the coherence solution f_c. From the master equations (5.42) we can identify that the corresponding thermalization and decoherence rates are given by Γ_m and $\partial_{k_0}\Gamma_0$, respectively. This example of thermalization and decoherence of a highly coherent initial state provides a more suited application for our eQPA scheme, as the big gradients of the background field are no longer a problem. It would be interesting to study a more realistic example of scalar field thermalization using our methods, *e.g.* with a model of quartic self interaction.

5.4.1 Resummation of the coherence contributions in the collision term

In the examples of sections 5.3 and 5.4 the gradient expansion of the collision term was truncated to the lowest order that includes decoherence. For fermions this was the zeroth order and for scalars the first order. However, as discussed in section 4.7, also the higher order coherence contributions in the collision term should be taken into account, since their expansion is not (completely) controlled by small parameters. In this section we present an analysis that shows how the coherence contributions in the collision term can be resummed for a scalar field with a thermal interaction. The results of this section were not presented in refs. [II, III], and a new publication [110] on these and the corresponding results for fermions is being prepared.

To be specific, we want to expand the collision term to leading order in Γ (and $\partial_t m^2$): $\mathcal{O}_1 \equiv \mathcal{O}(\Gamma, \partial_t m^2)$, neglecting the terms of order $\mathcal{O}_2 \equiv \mathcal{O}(\Gamma^2, \Gamma\,\partial_t m^2)$ and higher. We begin by an observation that the on-shell functions obey the zeroth order equations:

$$\partial_t f_\pm = \mathcal{O}_1 \tag{5.43}$$

$$\partial_t^2 f_c + 4\omega_{\vec{k}}^2 f_c = \mathcal{O}_1\,, \tag{5.44}$$

which follow directly from Eqs. (4.73) and the relations (4.74). Furthermore, the first part of the (standard) collision term (3.44) gives

$$e^{-i\diamond}\{\Gamma\}\{i\Delta^<\} = \exp\left(\frac{i}{2}\partial_{k_0}^{\Gamma}\partial_t^{\Delta}\right)\Gamma\, i\Delta^< + \mathcal{O}_2\,, \tag{5.45}$$

since $\partial_t\Gamma$ is proportional to $\partial_t m^2$ (as well as to Γ) for the interaction with thermal background. When the k_0-integration is performed for a term of order $n \geq 1$ of this expansion, we see that the only leading order contribution is the projection to the k_0-shell:

$$\int \frac{\mathrm{d}k_0}{2\pi}\,\partial_{k_0}^n\Gamma\,\partial_t^n i\Delta^< = \partial_{k_0}^n\Gamma|_{k_0=0}\,\partial_t^n f_c + \mathcal{O}_2\,, \tag{5.46}$$

where we have used Eqs. (5.43). Now, using Eq. (5.44) we find that to zeroth order all the derivatives $\partial_t^n f_c$ are proportional to either f_c or $\partial_t f_c$: $\partial_t^{2n} f_c = (-4\omega_{\vec{k}}^2)^n f_c + \mathcal{O}_1$ and $\partial_t^{2n+1} f_c = (-4\omega_{\vec{k}}^2)^n \partial_t f_c + \mathcal{O}_1$. Using these relations we get then

$$\int \frac{\mathrm{d}k_0}{2\pi}\,e^{-i\diamond}\{\Gamma\}\{i\Delta^<\} = \cosh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0}\,f_c + i\frac{1}{2\omega_{\vec{k}}}\sinh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0}\,\partial_t f_c + \text{regular} + \mathcal{O}_2\,, \tag{5.47}$$

where regular refers to standard first order mass-shell contribution. The second part of the collision term (3.44), proportional to the spectral function $\mathcal{A}$, gives only the regular contribution, since the spectral function does not involve the coherence contributions in our approach. Putting things together, we find the following generalizations of Eq. (5.42):

$$
\begin{aligned}
\frac{1}{4}\partial_t^2\rho_0 + \omega_{\vec{k}}^2\rho_0 - \rho_2 &= -\frac{1}{2\omega_{\vec{k}}}\sinh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0}\,\partial_t\rho_0 \\
\partial_t\rho_1 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_1-\rho_{1,\mathrm{eq}}) - \cosh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0}\Big(\rho_0 - \frac{1}{\omega_{\vec{k}}^2}\rho_2\Big) \\
\partial_t\rho_2 - \frac{1}{2}\partial_t(m^2)\rho_0 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_2-\rho_{2,\mathrm{eq}})\,,
\end{aligned}
\tag{5.48}
$$

where the terms of order $\mathcal{O}_2$ are neglected. We see that the infinite k_0-derivative expansions of Γ in Eq. (5.48) are not controlled by the small parameters Γ and $\partial_t m^2$; instead they provide expansions in powers of $\beta\omega_{\vec{k}}$ or $\omega_{\vec{k}}/|\vec{k}|$ etc. that are typically of order unity. Moreover, because of the generally nontrivial k_0-dependence of the (thermal) Γ, a brute force calculation of the (infinitely many) derivatives seems impossible. However, it turns out that these expansions can be resummed by a neat trick. That is, by denoting $\tilde{\Gamma}(r_0)$ the Fourier transform of Γ with respect to k_0 and using the identity $\sinh(ix) = i\sin(x)$, we find

$$
\begin{aligned}
\sinh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0} &= \bigg[\int \mathrm{d}r_0\,\sinh(i\omega_{\vec{k}}r_0)\tilde{\Gamma}(r_0)e^{ik_0r_0}\bigg]_{k_0=0} \\
&= \int \mathrm{d}r_0\,\frac{1}{2}\big(e^{i\omega_{\vec{k}}r_0} - e^{-i\omega_{\vec{k}}r_0}\big)\tilde{\Gamma}(r_0) \\
&= \frac{1}{2}\big[\Gamma(\omega_{\vec{k}}) - \Gamma(-\omega_{\vec{k}})\big]\,,
\end{aligned}
\tag{5.49}
$$

and similarly

$$
\cosh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0} = \frac{1}{2}\big[\Gamma(\omega_{\vec{k}}) + \Gamma(-\omega_{\vec{k}})\big]\,. \tag{5.50}
$$

For the thermal interaction considered we furthermore have: $\Gamma(-\omega_{\vec{k}}) = -\Gamma(\omega_{\vec{k}})$, so that we get simply: $\sinh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0} = \Gamma(\omega_{\vec{k}}) \equiv \Gamma_m$ and $\cosh(\omega_{\vec{k}}\partial_{k_0})\Gamma|_{k_0=0} = 0$. When these amazingly simple results are substituted in Eq. (5.48) we get the final equations to the leading order as

$$
\begin{aligned}
\frac{1}{4}\partial_t^2\rho_0 + \omega_{\vec{k}}^2\rho_0 - \rho_2 &= -\frac{1}{2\omega_{\vec{k}}}\Gamma_m\,\partial_t\rho_0 \\
\partial_t\rho_1 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_1-\rho_{1,\mathrm{eq}}) \\
\partial_t\rho_2 - \frac{1}{2}\partial_t(m^2)\rho_0 &= -\frac{1}{\omega_{\vec{k}}}\Gamma_m\,(\rho_2-\rho_{2,\mathrm{eq}})\,,
\end{aligned}
\tag{5.51}
$$

where the terms of order $\mathcal{O}(\Gamma^2, \Gamma\,\partial_t m^2)$ and higher have been consistently neglected. We see that due to the resummation of the coherence contributions, the resulting equations have actually simplified, as Γ_m is now the only interaction

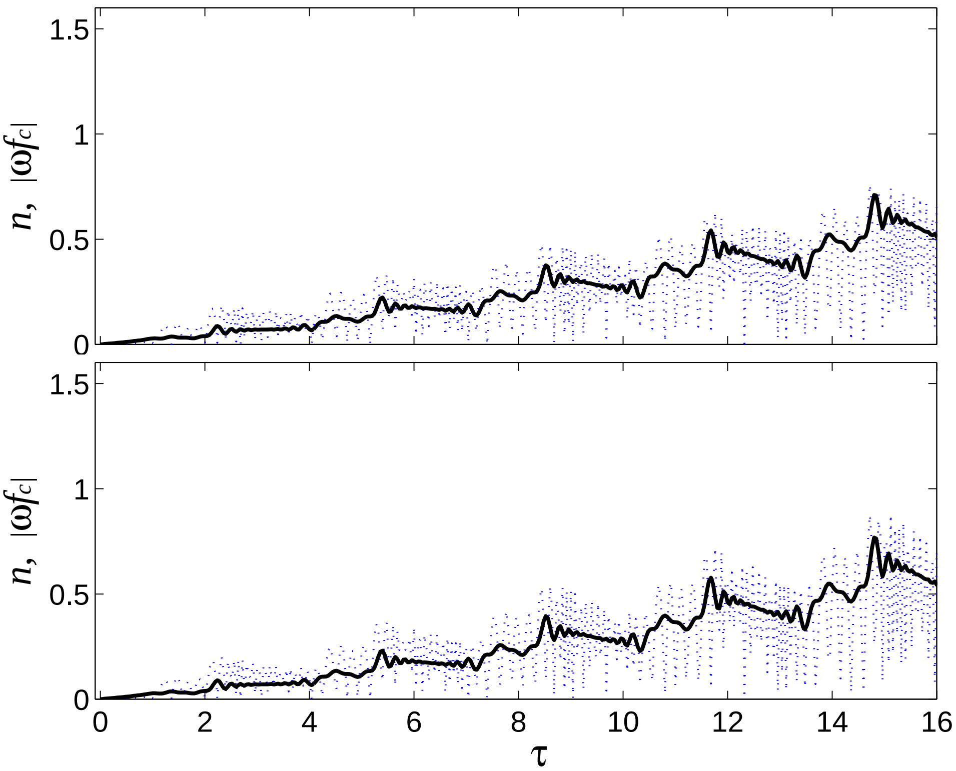

Figure 5.9: The number density $n_{\vec{k}}$ (thick solid line) and the coherence function $f_c(|\vec{k}|)$ (dotted(blue) line) in the cases of non-resummed (upper panel) and resummed (lower panel) coherence shell collision terms. The configuration is the same as in the lowest panel of Fig. 5.7.

term to be evaluated. However, what is more important, these equations provide a controlled expansion of the collision term to the leading order in Γ and $\partial_t m^2$.

Let us now compare the numerical results obtained by using the resummed equations (5.51) to the original ones, presented in Figs. 5.7-5.8. In figure 5.9 we show the results with the original (non-resummed) collision term in the upper panel and the resummed one in the lower panel. We have used the same configuration as in Fig. 5.7, so the upper panel here is just a copy of the lowest panel in Fig. 5.7. We see only a slight difference in the results, with the net amount of produced particles slightly increased in the resummed case. This is explained in that the average value of the oscillating resummed decoherence rate Γ_m/ω is approximately equal to the constant non-resummed rate $\partial_{k_0}\Gamma_0$ in this particular case. In the example of thermalization and decoherence, however, a bigger difference can be seen. The case with the resummed collision term and otherwise the same configuration as in Fig. 5.8 is presented in Fig. 5.10. We see that decoherence effects are clearly stronger in the resummed case, because now Γ_m/ω freezes to a value that is significantly larger than $\partial_{k_0}\Gamma_0$. The thermalization rate of the particle number, on the other hand, is not changed at all,

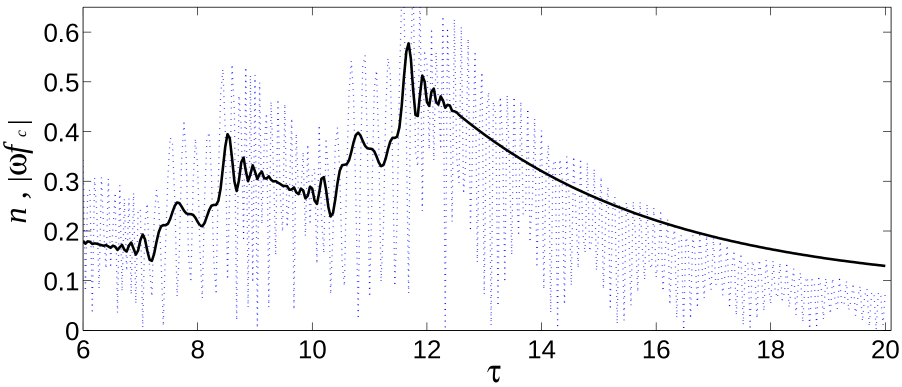

Figure 5.10: The same configuration as in Fig. 5.8 but with a resummed coherence shell collision term.

because the particle number and the coherence solution decouple in the constant mass limit in the case of a simple thermal interaction, and thus the (increased) decoherence rate does not affect the thermalization of the particle number.

Chapter 6

Conclusions and outlook

We have developed a novel approximation scheme (eQPA) that enables us to include the effects of nonlocal quantum coherence in the standard kinetic approach to nonequilibrium quantum dynamics. The key element in our scheme is the finding of new singular shell solutions in the phase space of 2-point Wightman function, located at $k_0 = 0$ for spatially homogeneous problems and at $k_z = 0$ for a static planar symmetric case. We have interpreted these new solutions as describing the nonlocal coherence between the "opposite" mass-shell excitations with either $k_0 = \pm\omega_{\vec{k}}$ or $k_z = \pm k_m$, respectively. This nontrivial phase space structure is then inserted in the dynamical Kadanoff-Baym equation for the 2-point correlator, leading to a closed set of transport equations for the corresponding on-shell distribution functions f, that provides an extension of the standard quantum Boltzmann equation to include nonlocal coherence.

We have shown how the eQPA scheme emerges from first principles in chapters 3-4. We used the 2PI effective action method to derive the self-consistent Schwinger-Dyson equations for the 2-point correlation functions of the system. We then performed a partial Fourier transformation to obtain the Kadanoff-Baym transport equations for the Wightman function in the mixed representation. When these equations were analyzed in detail in the kinetic regime with weak interactions and a slowly varying background field, the nontrivial singular shell picture with the new coherence solutions was found.

The analysis for fermions was originally presented in refs. [I, II], and for scalar fields in ref. [III]. In these papers we have considered several applications to demonstrate the use of our formalism. The (collisionless) Klein problem was considered in refs. [I,III] for fermions and scalars, respectively. In ref. [I] we also considered the (fermionic) quantum reflection from a CP-violating mass wall, and a topic omitted in the introductory part of this thesis, a generalization of our formalism to flavour mixing. The coherent production of decaying fermionic and scalar particles was considered in refs. [II, III], respectively. Finally, the nonrelativistic limit of our formalism was considered in ref. [III].

In the collisionless examples of the Klein problem and the fermionic quantum reflection from a smooth mass wall, presented in sections 5.1 and 5.2, our formalism was seen to reproduce the same (exact) results as the standard approach using the Dirac and Klein-Gordon equations. Of course, the whole point of our approach is the ability to include the effects of collisions together with the nonlocal coherence. We studied the role of collisions in the examples of coherent production of decaying particles in sections 5.3 and 5.4. We saw a clear decrease in the amount of produced particles due to smooth thermalization and decoherence effects caused by the collisions. A caution must be given, in that the parameters of the models were chosen such that the gradients of the background field were not necessarily small, and thus the numerical results of these particular examples are perhaps only qualitatively correct. In section 5.4.1 we have shown how a resummation can be carried out for the coherence contribution of the (infinite) gradient expansion of the collision term. This procedure leads to a controlled leading order result in powers of the interaction width Γ and gradients of the background field. These new results are not presented in refs. [I, II, III] but will be published elsewhere [110], along with the corresponding results for fermions.

One of the most interesting future applications for our eQPA scheme is a realistic calculation of the baryon asymmetry in electroweak baryogenesis, where the nonequilibrium dynamics of fermions in the neighbourhood of an expanding CP-violating phase transition wall needs to be considered. As discussed above, the relevant quantum reflection effects from the wall have not yet been considered in a consistent framework based on quantum field theory, including the effects of decohering collisions. Our eQPA scheme provides a tool for the study of such a problem with a smoothly varying background field and well-defined asymptotic regions. Indeed, the $k_z = 0$-shell coherence solutions are just the missing piece for describing the nonlocal quantum reflection effects. In the case of a thick wall with small gradients, the mean field approximation works well throughout, and in the opposite limit of a thin wall at least the correct qualitative behaviour is expected. However, the results in this case should be quite accurate also quantitatively, since the region of (rapidly) varying mass is very narrow, and outside the wall region the mean field approximation works well again.

Other interesting applications for our formalism include neutrino flavour oscillations in inhomogeneous background, where the relevant coherence effects occur between different flavour states travelling in the same direction. More generally, we expect that a number of other problems that can be studied in the standard kinetic approach with the Boltzmann equations, could be studied using our eQPA scheme including the coherence effects.

References

[I] M. Herranen, K. Kainulainen and P. M. Rahkila, "Towards a kinetic theory for fermions with quantum coherence," Nucl. Phys. B **810** (2009) 389 [arXiv:0807.1415 [hep-ph]].

[II] M. Herranen, K. Kainulainen and P. M. Rahkila, "Quantum kinetic theory for fermions in temporally varying backgrounds," JHEP **0809** (2008) 032 [arXiv:0807.1435 [hep-ph]].

[III] M. Herranen, K. Kainulainen and P. M. Rahkila, "Kinetic theory for scalar fields with nonlocal quantum coherence," JHEP **0905** (2009) 119 [arXiv:0812.4029v2 [hep-ph]].

[1] E. V. Shuryak, "The QCD vacuum, hadrons and the superdense matter," World Sci. Lect. Notes Phys. **71** (2004) 1 [World Sci. Lect. Notes Phys. **8** (1988) 1].

[2] L. P. Csernai, "Introduction to relativistic heavy ion collisions," *Chichester, UK: Wiley (1994) 310 p*

[3] C. Y. Wong, "Introduction to high-energy heavy ion collisions," *Singapore, Singapore: World Scientific (1994) 516 p*

[4] E. W. Kolb and M. S. Turner, "The Early universe," Front. Phys. **69** (1990) 1.

[5] A. D. Linde, "Particle Physics and Inflationary Cosmology," *Chur, Switzerland: Harwood (1990) 362 p.*

[6] B. A. Bassett, S. Tsujikawa and D. Wands, "Inflation dynamics and reheating," Rev. Mod. Phys. **78** (2006) 537 [arXiv:astro-ph/0507632].

[7] M. Dine and A. Kusenko, "The origin of the matter-antimatter asymmetry," Rev. Mod. Phys. **76** (2004) 1 [arXiv:hep-ph/0303065].

[8] N. D. Birrell and P. C. W. Davies, "Quantum Fields In Curved Space," *Cambridge, Uk: Univ. Pr. (1982) 340 p*

[9] E. Calzetta and B. L. Hu, "Nonequilibrium Quantum Field Theory," *Cambridge, UK: Univ. Pr. (2008) 535 p.*

[10] G. Steigman, "Observational Tests Of Antimatter Cosmologies," Ann. Rev. Astron. Astrophys. **14** (1976) 339.

[11] A. G. Cohen, A. De Rujula and S. L. Glashow, "A matter-antimatter universe?," Astrophys. J. **495** (1998) 539 [arXiv:astro-ph/9707087].

[12] C. L. Bennett *et al.* [WMAP Collaboration], "First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results," Astrophys. J. Suppl. **148** (2003) 1 [arXiv:astro-ph/0302207].

[13] A. D. Sakharov, "Violation of CP Invariance, c Asymmetry, and Baryon Asymmetry of the Universe," Pisma Zh. Eksp. Teor. Fiz. **5** (1967) 32 [JETP Lett. **5** (1967 SOPUA,34,392-393.1991 UFNAA,161,61-64.1991) 24].

[14] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, "On The Anomalous Electroweak Baryon Number Nonconservation In The Early Universe," Phys. Lett. B **155** (1985) 36.

[15] M. Fukugita and T. Yanagida, "Baryogenesis Without Grand Unification," Phys. Lett. B **174** (1986) 45.

[16] I. Affleck and M. Dine, "A New Mechanism For Baryogenesis," Nucl. Phys. B **249** (1985) 361.

[17] G. 't Hooft, "Symmetry breaking through Bell-Jackiw anomalies," Phys. Rev. Lett. **37** (1976) 8.

[18] S. L. Adler, "Axial vector vertex in spinor electrodynamics," Phys. Rev. **177** (1969) 2426.

[19] J. S. Bell and R. Jackiw, "A PCAC puzzle: $\pi_0 \to \gamma\gamma$ in the sigma model," Nuovo Cim. A **60** (1969) 47.

[20] N. S. Manton, "Topology In The Weinberg-Salam Theory," Phys. Rev. D **28** (1983) 2019.

[21] F. R. Klinkhamer and N. S. Manton, "A Saddle Point Solution In The Weinberg-Salam Theory," Phys. Rev. D **30** (1984) 2212.

[22] P. Arnold and L. D. McLerran, "The Sphaleron Strikes Back," Phys. Rev. D **37** (1988) 1020.

[23] D. Bodeker, G. D. Moore and K. Rummukainen, "Chern-Simons number diffusion and hard thermal loops on the lattice," Phys. Rev. D **61** (2000) 056003 [arXiv:hep-ph/9907545].

[24] S. L. Glashow, "Partial Symmetries Of Weak Interactions," Nucl. Phys. **22** (1961) 579.

[25] S. Weinberg, "A Model Of Leptons," Phys. Rev. Lett. **19** (1967) 1264.

[26] A. Salam, "Weak And Electromagnetic Interactions," *Originally printed in *Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden*, Stockholm 1968, 367-377*

[27] K. Kajantie, M. Laine, K. Rummukainen and M. E. Shaposhnikov, "Is there a hot electroweak phase transition at $m(H) \gtrsim m(W)$?," Phys. Rev. Lett. **77** (1996) 2887 [arXiv:hep-ph/9605288].

[28] M. Joyce, T. Prokopec and N. Turok, "Electroweak baryogenesis from a classical force," Phys. Rev. Lett. **75** (1995) 1695 [Erratum-ibid. **75** (1995) 3375] [arXiv:hep-ph/9408339].

[29] M. Joyce, T. Prokopec and N. Turok, "Nonlocal electroweak baryogenesis. Part 1: Thin wall regime," Phys. Rev. D **53** (1996) 2930 [arXiv:hep-ph/9410281].

[30] M. Joyce, T. Prokopec and N. Turok, "Nonlocal electroweak baryogenesis. Part 2: The Classical regime," Phys. Rev. D **53** (1996) 2958 [arXiv:hep-ph/9410282].

[31] J. M. Cline, M. Joyce and K. Kainulainen, "Supersymmetric electroweak baryogenesis in the WKB approximation," Phys. Lett. B **417** (1998) 79 [Erratum-ibid. B **448** (1999) 321] [arXiv:hep-ph/9708393].

[32] J. M. Cline and K. Kainulainen, "A new source for electroweak baryogenesis in the MSSM," Phys. Rev. Lett. **85** (2000) 5519 [arXiv:hep-ph/0002272].

[33] J. M. Cline, M. Joyce and K. Kainulainen, "Supersymmetric electroweak baryogenesis," JHEP **0007** (2000) 018 [arXiv:hep-ph/0006119].

[34] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, "First principle derivation of semiclassical force for electroweak baryogenesis," JHEP **0106** (2001) 031 [arXiv:hep-ph/0105295].

[35] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, "Quantum Boltzmann equations for electroweak baryogenesis including gauge fields," arXiv:hep-ph/0201293.

[36] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, "Semiclassical force for electroweak baryogenesis: Three-dimensional derivation," Phys. Rev. D **66** (2002) 043502 [arXiv:hep-ph/0202177].

[37] T. Prokopec, M. G. Schmidt and S. Weinstock, "Transport equations for chiral fermions to order $\hbar$ and electroweak baryogenesis. I," Annals Phys. **314** (2004) 208 [arXiv:hep-ph/0312110].

[38] T. Prokopec, M. G. Schmidt and S. Weinstock, "Transport equations for chiral fermions to order $\hbar$ and electroweak baryogenesis. II," Annals Phys. **314** (2004) 267 [arXiv:hep-ph/0406140].

[39] M. B. Gavela, P. Hernandez, J. Orloff and O. Pene, "Standard Model CP-violation and Baryon asymmetry," Mod. Phys. Lett. A **9** (1994) 795 [arXiv:hep-ph/9312215].

[40] M. B. Gavela, P. Hernandez, J. Orloff, O. Pene and C. Quimbay, "Standard model CP violation and baryon asymmetry. Part 2: Finite temperature," Nucl. Phys. B **430** (1994) 382 [arXiv:hep-ph/9406289].

[41] P. Huet and E. Sather, "Electroweak baryogenesis and standard model CP violation," Phys. Rev. D **51** (1995) 379 [arXiv:hep-ph/9404302].

[42] P. Huet and A. E. Nelson, "Electroweak baryogenesis in supersymmetric models," Phys. Rev. D **53** (1996) 4578 [arXiv:hep-ph/9506477].

[43] N. Rius and V. Sanz, "Supersymmetric electroweak baryogenesis," Nucl. Phys. B **570** (2000) 155 [arXiv:hep-ph/9907460].

[44] D. Kazanas, "Dynamics Of The Universe And Spontaneous Symmetry Breaking," Astrophys. J. **241** (1980) L59.

[45] A. A. Starobinsky, "A new type of isotropic cosmological models without singularity," Phys. Lett. B **91** (1980) 99.

[46] A. H. Guth, "The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems," Phys. Rev. D **23** (1981) 347.

[47] K. Sato, "First Order Phase Transition Of A Vacuum And Expansion Of The Universe," Mon. Not. Roy. Astron. Soc. **195** (1981) 467.

[48] K. Sato, "Cosmological Baryon Number Domain Structure And The First Order Phase Transition Of A Vacuum," Phys. Lett. B **99** (1981) 66.

[49] V. F. Mukhanov and G. V. Chibisov, "Quantum Fluctuation And Nonsingular Universe. (In Russian)," JETP Lett. **33** (1981) 532 [Pisma Zh. Eksp. Teor. Fiz. **33** (1981) 549].

[50] A. H. Guth and S. Y. Pi, "Fluctuations In The New Inflationary Universe," Phys. Rev. Lett. **49** (1982) 1110.

[51] S. W. Hawking, "The Development Of Irregularities In A Single Bubble Inflationary Universe," Phys. Lett. B **115** (1982) 295.

[52] A. A. Starobinsky, "Dynamics Of Phase Transition In The New Inflationary Universe Scenario And Generation Of Perturbations," Phys. Lett. B **117** (1982) 175.

[53] A. Berera, "Interpolating the stage of exponential expansion in the early universe: A possible alternative with no reheating," Phys. Rev. D **55** (1997) 3346 [arXiv:hep-ph/9612239].

[54] A. Berera and T. W. Kephart, "Ubiquitous inflaton in string-inspired models," Phys. Rev. Lett. **83** (1999) 1084 [arXiv:hep-ph/9904410].

[55] A. D. Dolgov and D. P. Kirilova, "Production of particles by a variable scalar field," Sov. J. Nucl. Phys. **51** (1990) 172 [Yad. Fiz. **51** (1990) 273].

[56] J. H. Traschen and R. H. Brandenberger, "Particle production during out-of-equilibrium phase transition," Phys. Rev. D **42** (1990) 2491.

[57] L. Kofman, A. D. Linde and A. A. Starobinsky, "Reheating after inflation," Phys. Rev. Lett. **73** (1994) 3195 [arXiv:hep-th/9405187].

[58] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, "Universe reheating after inflation," Phys. Rev. D **51** (1995) 5438 [arXiv:hep-ph/9407247].

[59] D. Boyanovsky, H. J. de Vega, R. Holman and J. F. J. Salgado, "Analytic and numerical study of preheating dynamics," Phys. Rev. D **54** (1996) 7570 [arXiv:hep-ph/9608205].

[60] L. Kofman, A. D. Linde and A. A. Starobinsky, "Towards the theory of reheating after inflation," Phys. Rev. D **56** (1997) 3258 [arXiv:hep-ph/9704452].

[61] E. Whittaker and G. Watson, "A Course of Modern Analysis," *Cambridge, UK: Univ. Pr. (1940).*

[62] E. Ince, "Ordinary Differential Equations," *Dover, New York (1956).*

[63] R. Micha and I. I. Tkachev, "Turbulent thermalization," Phys. Rev. D **70** (2004) 043538 [arXiv:hep-ph/0403101].

[64] R. Allahverdi and A. Mazumdar, "Quasi-thermal universe and its implication for gravitino production, baryogenesis and dark matter," arXiv:hep-ph/0505050.

[65] J. Baacke, K. Heitmann and C. Patzold, "Nonequilibrium dynamics of fermions in a spatially homogeneous scalar background field," Phys. Rev. D **58** (1998) 125013 [arXiv:hep-ph/9806205].

[66] P. B. Greene and L. Kofman, "Preheating of fermions," Phys. Lett. B **448** (1999) 6 [arXiv:hep-ph/9807339].

[67] G. F. Giudice, A. Riotto and I. Tkachev, "Thermal and non-thermal production of gravitinos in the early universe," JHEP **9911** (1999) 036 [arXiv:hep-ph/9911302].

[68] R. Kallosh, L. Kofman, A. D. Linde and A. Van Proeyen, "Gravitino production after inflation," Phys. Rev. D **61** (2000) 103503 [arXiv:hep-th/9907124].

[69] R. Kallosh, L. Kofman, A. D. Linde and A. Van Proeyen, "Superconformal symmetry, supergravity and cosmology," Class. Quant. Grav. **17** (2000) 4269 [Erratum-ibid. **21** (2004) 5017] [arXiv:hep-th/0006179].

[70] A. L. Maroto and A. Mazumdar, "Production of spin 3/2 particles from vacuum fluctuations," Phys. Rev. Lett. **84** (2000) 1655 [arXiv:hep-ph/9904206].

[71] H. P. Nilles, M. Peloso and L. Sorbo, "Nonthermal production of gravitinos and inflatinos," Phys. Rev. Lett. **87** (2001) 051302 [arXiv:hep-ph/0102264].

[72] H. P. Nilles, M. Peloso and L. Sorbo, "Coupled fields in external background with application to nonthermal production of gravitinos," JHEP **0104** (2001) 004 [arXiv:hep-th/0103202].

[73] P. B. Greene, K. Kadota and H. Murayama, "Supergravity inflation free from harmful relics," Phys. Rev. D **68** (2003) 043502 [arXiv:hep-ph/0208276].

[74] G. F. Giudice, M. Peloso, A. Riotto and I. Tkachev, “Production of massive fermions at preheating and leptogenesis,” JHEP **9908** (1999) 014 [arXiv:hep-ph/9905242].

[75] M. Peloso and L. Sorbo, “Preheating of massive fermions after inflation: Analytical results,” JHEP **0005** (2000) 016 [arXiv:hep-ph/0003045].

[76] J. Berges and J. Serreau, “Progress in nonequilibrium quantum field theory,” arXiv:hep-ph/0302210.

[77] J. Berges and J. Serreau, “Progress in nonequilibrium quantum field theory. II,” arXiv:hep-ph/0410330.

[78] J. Berges, “Introduction to nonequilibrium quantum field theory,” AIP Conf. Proc. **739** (2005) 3 [arXiv:hep-ph/0409233].

[79] J. S. Schwinger, “Brownian motion of a quantum oscillator,” J. Math. Phys. **2** (1961) 407.

[80] L. V. Keldysh, “Diagram technique for nonequilibrium processes,” Zh. Eksp. Teor. Fiz. **47** (1964) 1515 [Sov. Phys. JETP **20** (1965) 1018].

[81] P. M. Bakshi and K. T. Mahanthappa, “Expectation value formalism in quantum field theory. 1,” J. Math. Phys. **4** (1963) 1.

[82] G. z. Zhou, Z. b. Su, B. l. Hao and L. Yu, “Closed time path Green’s functions and critical dynamics,” Phys. Rev. B **22** (1980) 3385.

[83] K. c. Chou, Z. b. Su, B. l. Hao and L. Yu, “Equilibrium And Nonequilibrium Formalisms Made Unified,” Phys. Rept. **118** (1985) 1.

[84] Z. b. Su, L. y. Chen, X. t. Yu and K. c. Chou, “Influence Functional And Closed-Time-Path Green’s Function,” Phys. Rev. B **37** (1988) 9810.

[85] B. S. DeWitt, “Effective action for expectation values,” *In *Oxford 1984, Proceedings, Quantum Concepts In Space and Time*, 325-336*

[86] R. D. Jordan, “Effective Field Equations for Expectation Values,” Phys. Rev. D **33** (1986) 444.

[87] E. Calzetta and B. L. Hu, “Closed Time Path Functional Formalism in Curved Space-Time: Application to Cosmological Back Reaction Problems,” Phys. Rev. D **35** (1987) 495.

[88] E. Calzetta and B. L. Hu, “Nonequilibrium Quantum Fields: Closed Time Path Effective Action, Wigner Function and Boltzmann Equation,” Phys. Rev. D **37** (1988) 2878.

[89] E. Calzetta and B. L. Hu, “Dissipation Of Quantum Fields From Particle Creation,” Phys. Rev. D **40** (1989) 656.

[90] J. M. Luttinger and J. C. Ward, “Ground state energy of a many fermion system. 2,” Phys. Rev. **118** (1960) 1417.

[91] C. de Dominicis and P. C. Martin, "Stationary Entropy Principle And Renormalization In Normal And Superfluid Systems. I," J. Math. Phys. **5** (1964) 14.

[92] C. de Dominicis and P. C. Martin, "Stationary Entropy Principle And Renormalization In Normal And Superfluid Systems. II," J. Math. Phys. **5** (1964) 31.

[93] H. D. Dahmen and G. Jona Lasino, "Variational formulation of quantum field theory," II Nuovo Cim. A **LIIA** (1967) 807.

[94] J. M. Cornwall, R. Jackiw and E. Tomboulis, "Effective Action For Composite Operators," Phys. Rev. D **10** (1974) 2428.

[95] L. Kadanoff and G. Baym, "Quantum Statistical Mechanics" *Benjamin, New York (1962).*

[96] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," *Cambridge, UK: Univ. Pr. (1995) 609 p.*

[97] M. Le Bellac, "Thermal Field Theory," *Cambridge, UK: Univ. Pr. (2000) 256 p.*

[98] S. Mrowczynski, "Transport theory of massless fields," Phys. Rev. D **56** (1997) 2265 [arXiv:hep-th/9702022].

[99] R. Kubo, "Statistical Mechanical Theory Of Irreversible Processes. I," J. Phys. Soc. Jap. **12** (1957) 570.

[100] R. Kubo, M. Yokota and S. Nakajima, "Statistical Mechanical Theory Of Irreversible Processes. II," J. Phys. Soc. Jap. **12** (1957) 1203.

[101] P. C. Martin and J. S. Schwinger, "Theory of many particle systems. I," Phys. Rev. **115** (1959) 1342.

[102] B. Garbrecht, T. Prokopec and M. G. Schmidt, "Particle number in kinetic theory," Eur. Phys. J. C **38** (2004) 135 [arXiv:hep-th/0211219].

[103] G. Aarts and J. Berges, "Nonequilibrium time evolution of the spectral function in quantum field theory," Phys. Rev. D **64** (2001) 105010 [arXiv:hep-ph/0103049].

[104] C. Itzykson and J-B. Zuber, "Quantum Field Theory," *McGraw-Hill, 1980.*

[105] G. R. Farrar and M. E. Shaposhnikov, "Baryon Asymmetry Of The Universe In The Minimal Standard Model," Phys. Rev. Lett. **70** (1993) 2833 [Erratum-ibid. **71** (1993) 210] [arXiv:hep-ph/9305274].

[106] G. R. Farrar and M. E. Shaposhnikov, "Baryon Asymmetry Of The Universe In The Standard Electroweak Theory," Phys. Rev. D **50** (1994) 774 [arXiv:hep-ph/9305275].

[107] J. M. Cline, K. Kainulainen and A. P. Vischer, "Dynamics of two Higgs doublet CP violation and baryogenesis at the electroweak phase transition," Phys. Rev. D **54** (1996) 2451 [arXiv:hep-ph/9506284].

[108] J. Berges, J. Pruschke and A. Rothkopf, "Instability-induced fermion production in quantum field theory," arXiv:0904.3073 [hep-ph].

[109] J. Berges and J. Serreau, "Parametric resonance in quantum field theory," Phys. Rev. Lett. **91** (2003) 111601 [arXiv:hep-ph/0208070].

[110] M. Herranen, K. Kainulainen and P. M. Rahkila, in preparation.