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ZERO-RANGE PROCESSES AND THEIR APPLICATIONS

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Preface

The work reviewed in this Thesis has been carried out at the Department of Physics in the University of Jyväskylä during the years 2000-2007.

I would like to thank Docent Juha Merikoski for supervision, collaboration and patience. The years at the department of physics have allowed me to gain knowledge on multiple facets of statistical physics, theory of probability and stochastic processes. I feel that this makes a strong combination and the results of the effort are beginning to show.

It would have been impossible to make this research without proper guidance in mathematics. I want to express my gratitude to Dr. Pekka Kekäläinen and Prof. Stefan Geiss for listening to my problems and for explaining the intricacies of probability in a way understandable to a physicist.

Prof. Jussi Timonen first gave me the opportunity to work on statistical physics, and his help is gratefully acknowledged.

Finally, I wish to thank my family, the Kilpelä family, and most of all Tuuli for love and support.

Jyväskylä, August 2007

Otto Pulkkinen

Abstract

A class of models for interacting particles, known as zero-range processes, is considered in this Thesis. The Thesis consists of an introductory part and three publications, in which applications of zero-range processes are studied. These include statistics of extremes in canonical ensembles, uncorrelated diluted networks and transport in random media. The theoretical framework is presented in a separate chapter of the introductory part.

Sizes of largest clusters in closed zero-range processes are analyzed for two types of interactions in this Thesis. The main findings, namely the various deviations from classical extreme value distributions, are not limited to zero-range processes alone, but apply to more general systems of random partitions as well.

A graph-valued Markov process, whose evolution is governed by zero-range interaction, is introduced. The existence and size of the giant component is studied. Some general features of the phase diagrams are announced, and a detailed analysis of a specific model is presented. Some mathematical issues often neglected in physics studies are also clarified.

The last application of this Thesis is a model for transport of fluids in random media. A quantitative analysis of stationary fugacity profiles is presented and estimates for certain functionals of tagged particle paths are given. The study opens some new questions concerning other interacting particle systems and Bayesian analysis.

Keywords: interacting particle systems, zero-range process, extreme order statistics, random graph, random media, tagged particle

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List of Publications

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III O. Pulkkinen, Boundary driven zero-range processes in random media, in print, *Journal of Statistical Physics* (2007).

The author of this thesis has done the theoretical analysis presented in articles I–III and has also written these articles.

Contents

1 Introduction

Spitzer's article [1] from 1970 can be seen as the true start of mathematical modeling of interacting particle systems. He had realized that certain Markov processes mimicking the motion of gas particles have tractable stationary states that often resemble Gibbs distributions. These systems included exclusion and zero-range processes, which are nowadays considered central both in the interacting particle systems branch of mathematics and as a test ground for nonequilibrium theories of theoretical physics.

The symmetric simple exclusion process is perhaps the simplest interacting system on a lattice. Particles jump from one lattice site to another in a random fashion that excludes the possibility of double occupancy of any site. Each of the particles attempts a jump at exponentially distributed intervals of time, which makes the process Markovian, and the jumps that would lead to multiply occupied sites are simply suppressed. In one dimension, there is a mapping from the exclusion process to the nearest-neighbor zero-range process [2]: The number of empty spaces to the right of the exclusion process particle labeled i equals the number of particles at the site i of the zero-range process. The dynamics of this new process then allows an unlimited amount of particles at any site. The movement of particles occurs by nearest-neighbor jumps in such a way that, for any occupied site, one particle is moved out of the site to its randomly selected neighbor at exponential intervals of time. The class of zero-range processes considered in this Thesis is obtained by altering the rate according to the occupation numbers. The exponential rate, at which a particle leaves the site i with total of η_i particles, equals a positive real number $g_i(\eta_i)$. The transitions need not be symmetric or restricted to nearest neighbors either. In the special case $g_i(\eta_i) = \eta_i$, the particles do not interact.

The startling feature of zero-range processes is that the stationary distributions on finite sets are of the product form even for asymmetric transition probabilities, in that the stationary particle numbers are either independent (open systems) or at least exchangeable (closed systems). More generally, it was shown by Andjel [3] that under the assumption of boundedness of g, all invariant measures of nearest-neighbor zero-range processes on $\mathbb{Z}$ are mixtures of product measures.

Zero-range processes are often used as minimal models for complex phenomena. Their stationary states are analytically tractable, but their formulation is suitable for various applications. One feature of these processes is generic: Zero-range

processes with particle conservation and homogeneous interaction ($g_i \equiv g$) exhibit condensation transitions for decreasing interaction functions g [4]. More precisely, a non-zero fraction of the particles condense on a single lattice site for interactions of the type $g(k) \sim 1 + b/k^\sigma$, $\sigma \geq 1$ [5], and practically every particle is swallowed by the condensate for asymptotically vanishing functions [6]. Using these ideas, Kafri *et al.* [7] formulated a general criterion for phase separation in driven one-dimensional systems. The criterion concerns the effective rate, at which domains of given size exchange mass with their neighbors. If the rates are of the forms given above, a coarsening process, in which small domains coalesce with larger ones, takes place. Thus the processes studied in this Thesis are significant not only as tractable models of transport, but also as a framework for non-equilibrium phase transitions.

Zero-range processes were directly applied to verify the existence of the jamming transition in the asymmetric exclusion process with random rates by Krug and Ferrari [8], and independently by Evans [9]. The model of Lipowski and Droz [10] for spatial separation of shaken sand is also a zero-range process. There are also numerous applications in other fields than physics. However, most of them are studied in a queueing theoretic framework because zero-range processes and the basic models of queueing networks are very much alike [11].

In addition to applications, a major reason for the interest towards zero-range processes lies in the theory of scaling limits leading to hydrodynamic transport equations. Partial differential equation descriptions can be rigorously justified in many cases, as explained in the monograph of Kipnis and Landim [12], but there are still open questions even for zero-range processes. The hydrodynamic descriptions, considered alongside with the microscopic dynamics, have been used to predict and test the general theories for fluctuations on macroscopic scales in stationary non-equilibrium states [13, 14]. Also in this respect, the efforts of mathematicians and physicists on these processes have helped in building a concise theory of non-equilibrium physics.

In this Thesis, properties and applications of zero-range processes are considered. The theoretical framework is first presented in chapter 2. In chapter 3, the classical extreme value distributions are reviewed and the statistics of extreme cluster sizes in zero-range processes are discussed. A graph-valued Markov process, whose evolution is governed by the zero-range interaction, is analyzed in chapter 4, and the transport of interacting particles in random media is considered in chapter 5. Conclusions are drawn and open problems are presented in the last chapter.

2 Zero-range processes

An informal description of zero-range processes was given in the introduction, yet suitable notation must be introduced and the precise assumptions must be written down to cover the cases that will be met in this Thesis. Therefore the present chapter contains a formal definition of these processes. After this, the stationary states are derived for both open and closed systems and the equivalence of canonical and grand-canonical ensembles is given a proof. The approximation of the canonical measures by grand-canonical ones also provides us the means to understand the condensation transitions in homogeneous systems. Finally, scaling limits are briefly discussed.

2.1 Zero-range processes on finite sets: definition and basic properties

The dynamics defining a zero-range process on a finite set S is stated as follows (see also figure 1): The particles reside in the set S, the elements of which will be called sites, and move within the set by jumping from site to site in a Markovian manner. Time is taken to be continuous. The total number of particles in S need not be fixed, and may change because of jumps of particles between S and an extra site Γ called the particle reservoir. The number of particles in the reservoir is not a measurable quantity of the model. The jumps of particles are characterized by parameters p and $\{g_i : i \in S\}$, where $p : (S \cup \{\Gamma\})^2 \to [0,1]$ is the single particle transition probability and $g_i : \mathbb{Z}_+ \to [0,\infty)$ describes the strength of interaction between the particles at a site i. More precisely, given that there are k particles at a site $i \in S$, the exponential rate of an occurrence of a jump of a particle from i to $j \in S \cup \{\Gamma\}$ equals $g_i(k)p(i,j)$. The transition probability p is assumed stochastic and irreducible. We also set $p(i,i) = 0$ for every $i \in S \cup \{\Gamma\}$. The interaction is such that $0 = g_i(0) < g_i(k) < \infty$ for all $k > 0$. To avoid unnecessary complications, we also assume that $\lim_{k\to\infty} g_i(k)$ exists in $[0,\infty]$ for every i, and that the increments of g_i are uniformly bounded, $\max_{i\in S} \sup_{k\in\mathbb{Z}_+} |g_i(k+1) - g_i(k)| \leq K < \infty$. Particles are allowed to enter the system at finite rates $p(\Gamma,i)\psi$, where i is the site, where the extra particle is placed. The number ψ is referred to as the *fugacity* of the particle reservoir Γ.

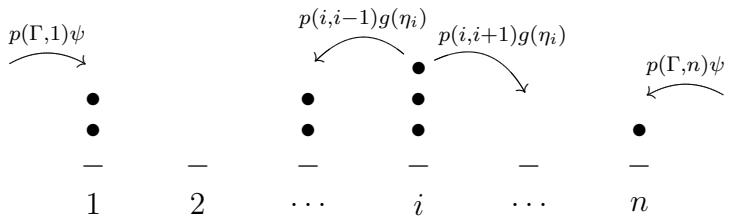

Figure 2.1: A zero-range process on the set $\{1, 2, \ldots, n\}$. Some nearest-neighbor transitions and the input of particles to the system through boundary sites are shown.

Example 2.1.1. There are two choices of the functions g_i that we will encounter frequently in this thesis: $g_i(k) = k$ corresponds to particles moving independently according to the single-particle transition matrix p, whereas $g_i(k) = c_i I\{k \geq 1\}$, where $c_i > 0$ and I stands for the indicator function, yields a network of queues (known as the Jackson network [16]) with a server at $i \in S$ processing the units at a constant speed c_i.

Let $\eta = (\eta_i)_{i\in S}$ denote a particle configuration (*i.e.* an element of the state space), and let $\eta^{i,j}$ be the state obtained from η by moving one particle from i to j. The state $\eta^{i,\Gamma}$ is defined to be equal to η with one particle removed from the site i, whereas $\eta^{\Gamma,i}$ is η with a particle added to i. The zero-range process is formally defined by the generator

$$(Lf)(\eta) = \sum_{i\in S} \sum_{j\in S\cup\{\Gamma\}} p(i,j) g_i(\eta_i)[f(\eta^{i,j}) - f(\eta)] + \sum_{j\in S} p(\Gamma,j)\psi[f(\eta^{\Gamma,j}) - f(\eta)] \quad (2.1)$$

acting on bounded measurable functions. The first double sum describes the movement of particles within S and the jumps out to Γ. The second term is the input of particles to the system from the reservoir. The time evolution of the vector of particle numbers $X(t) = (X_i(t))_{i\in S}$ is given by the Markov semigroup $\{P_t : t \geq 0\}$ such that

$$E^\eta f(X(t)) = P_t f(\eta) = \lim_{k\to\infty} \left(I - \frac{t}{k} L\right)^{-k} f(\eta). \quad (2.2)$$

Here E^η is the expectation with respect to the path measure P^η of the Markov process, under which the starting configuration is almost surely η. The semigroup property means that $P_{t+s} = P_t P_s$. For the general construction for finite S, see Liggett's monograph [15], chapter IX.

We will now investigate the existence and forms of stationary distributions. Originally, Spitzer [1] gave a rigorous derivation of the stationary state for a zero-range process with a fixed number of particles but closely related results can be found in the queueing networks literature from the 1950's and 60's [16, 17] (see also [11] for

modern queueing theoretical aspects). From the perspective of physical modeling, there are two cases to be considered: open systems that involve nonzero boundary sources $p(\Gamma, i)\psi$ and closed systems with the total number of particles conserved. The distinction is crucial for the theory of condensation.

We begin with **open systems**.

Theorem 2.1.1. Suppose that

$$\max_{i\in S}\Big(\phi_i - \lim_{k\to\infty} g_i(k)\Big) < 0, \tag{2.3}$$

where the *fugacities* ϕ_i, $i \in S$, are given by the unique positive solution to the *traffic equations*

$$\phi_i = \sum_{j\in S\cup\{\Gamma\}} p(j,i)\phi_j, \tag{2.4}$$

with the convention that $\phi_\Gamma = \psi$. Then the stationary distribution of the open zero-range process with parameters (p, g_i, ψ) is

$$\nu(\eta) = \prod_{i\in S}\nu_{i,\phi_i}(\eta_i) = \prod_{i\in S}\frac{1}{\mathcal{Z}_i(\phi_i)}\frac{\phi_i^{\eta_i}}{g_i!(\eta_i)}, \qquad \mathcal{Z}_i(\phi_i) = \sum_{k=0}^{\infty}\frac{\phi_i^k}{g_i!(k)}, \tag{2.5}$$

where $\eta \in \mathbb{Z}_+^S$ and the shorthand notation $g_i!(k) = \prod_{j=1}^k g_i(j)$, with $g_i!(0) = 1$, is used.

Proof. By the irreducibility of the transition matrix p, the traffic equations have a unique positive solution [18], and a necessary and sufficient condition for a probability distribution ν to be the stationary distribution is that it satisfies the global balance equation $\nu[Lf] = 0$, where f is bounded and measurable [12, 18]. To help the reader, we check this condition here in full detail.

$$\begin{aligned}
\nu[Lf] &= \sum_\eta \nu(\eta)\Big\{\sum_{i\in S}\sum_{j\in S} p(i,j)g_i(\eta_i)[f(\eta^{i,j}) - f(\eta)] \\
&\quad + \sum_{i\in S} p(i,\Gamma)g_i(\eta_i)[f(\eta^{i,\Gamma}) - f(\eta)] + \sum_{j\in S} p(\Gamma,j)\psi[f(\eta^{\Gamma,j}) - f(\eta)]\Big\} \\
&= \sum_\eta f(\eta)\Big\{\sum_{i\in S}\sum_{j\in S} p(i,j)[g_i(\eta_i+1)\nu(\eta^{j,i}) - g_i(\eta_i)\nu(\eta)] \\
&\quad + \sum_{i\in S} p(i,\Gamma)[g_i(\eta_i+1)\nu(\eta^{\Gamma,i}) - g_i(\eta_i)\nu(\eta)] + \sum_{j\in S} p(\Gamma,j)\psi[\nu(\eta^{j,\Gamma}) - \nu(\eta)]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_\eta f(\eta)\Big\{ \sum_{i\in S}\sum_{j\in S}[p(j,i)g_j(\eta_j+1)\nu(\eta^{i,j}) - p(i,j)g_i(\eta_i)\nu(\eta)] \\
&\qquad + \sum_{i\in S}[p(\Gamma,i)\psi\nu(\eta^{i,\Gamma}) - p(i,\Gamma)g_i(\eta_i)\nu(\eta)] \qquad (2.6)\\
&\qquad + \sum_{j\in S}[p(j,\Gamma)g_j(\eta_j+1)\nu(\eta^{\Gamma,j}) - p(i,\Gamma)g_i(\eta_i)\nu(\eta)]\Big\} \\
&= \sum_\eta f(\eta)\nu(\eta)\sum_{i\in S} g_i(\eta_i)\Big\{ \sum_{j\in S\cup\{\Gamma\}}[p(j,i)\frac{\phi_j}{\phi_i} - p(i,j)] \\
&\qquad + \sum_{j\in S}[p(j,\Gamma)\phi_j - p(\Gamma,j)\psi]\Big\} = 0,
\end{aligned}
$$

where changes of summation variables, *e.g.* $\eta^{i,j} \mapsto \eta$, were performed in the second equality. The indices i and j were swapped in the first double sum inside the curly brackets, and the other terms were rearranged to obtain the third. The fourth equality is a consequence of the property $g_j(\eta_j+1)\nu(\eta^{i,j}) = g_i(\eta_i)\phi_j\nu(\eta)/\phi_i$ of the distribution ν. The last equality then follows because the fugacities ϕ_i satisfy the traffic equations (2.4). Condition (2.3) now guarantees that the distribution can be normalized, in that it is a probability, and hence the process is ergodic. □

The third equality, numbered (2.6), in the previous calculation shows that it would have been sufficient to check that ν fulfills the *partial balance equations*

$$\sum_{j\in S}[p(j,i)g_j(\eta_j+1)\nu(\eta^{i,j}) - p(i,j)g_i(\eta_i)\nu(\eta)] = 0, \quad i\in S, \qquad (2.7)$$

$$\sum_{j\in S}[p(j,\Gamma)g_j(\eta_j+1)\nu(\eta^{\Gamma,j}) - p(i,\Gamma)g_i(\eta_i)\nu(\eta)] = 0. \qquad (2.8)$$

These equations represent the balance of probability flows through the sites locally, in contrast to the global balance of flows imposed by the *full balance equation* $\nu[Lf] = 0$. In the partial balance equations, there is an equation for each site and one for the reservoir, and each of these equations require that the mean rate, at which particles leave the location in question, is balanced by the mean rate at which particles arrive at this location from the other sites or the particle reservoir. While being more stringent condition for stationarity than the full balance equation, the partial balance equations are on the other hand weaker than the *detailed balance equations* that require that the probability flows between any two sites cancel: the mean rate, at which of particles move from i to j is equal to the mean rate of the movement from j to i. The detailed balance equations are fulfilled by time reversible (equilibrium) processes only.

Notice that for the zero-range process with open boundaries to be ergodic, the rate functions g_i must be bounded away from zero. The breaking of ergodicity is sometimes referred to as condensation even if one is speaking of an open system [19].

The most noticeable property of the stationary distribution (2.5) is that it is a *product measure*, in that no correlations between the numbers of particles at disjoint sets of sites exist. This considerably facilitates any calculation because the structure of the stationary state is essentially determined by the one-site marginal distributions ν_{i,ϕ_i}. The normalizing series $\mathcal{Z}_i(\phi) = \sum_{k\geq 0} \phi^k / g_i!(k)$ are known as *partition functions*, and their radii of convergence will be denoted by Φ_i. The expectation of the number of particles at site i in the stationary state is

$$R_i(\phi_i) = E^\nu X_i(t) = \frac{1}{\mathcal{Z}_i(\phi_i)} \sum_{k=1}^{\infty} \frac{k\phi_i^k}{g_i!(k)} = \phi_i \frac{\mathcal{Z}_i'(\phi_i)}{\mathcal{Z}_i(\phi_i)}, \tag{2.9}$$

and

$$\sigma_i^2(\phi_i) = E^\nu \left(X_i(t) - E^\nu X_i(t)\right)^2 = \phi_i^2 \frac{\mathcal{Z}_i''(\phi_i)}{\mathcal{Z}_i(\phi_i)} + \phi_i \frac{\mathcal{Z}_i'(\phi_i)}{\mathcal{Z}_i(\phi_i)} - \phi_i^2 \left(\frac{\mathcal{Z}_i'(\phi_i)}{\mathcal{Z}_i(\phi_i)}\right)^2 \tag{2.10}$$

is its variance.

To ease the notation, the extra site index in $\nu_{i,\phi_i}, \mathcal{Z}_i(\phi_i), R_i(\phi_i)$ and $\sigma_i^2(\phi_i)$ will be dropped whenever discussing homogeneous interactions $g_i \equiv g$. Thus we will write $\nu_{\phi_i}, \mathcal{Z}(\phi_i), R(\phi_i)$ and $\sigma^2(\phi_i)$ for the one site marginal distribution, its normalization, expectation and variance. The radius of convergence of $\mathcal{Z}$ will be denoted by Φ.

Example 2.1.2. (a) For noninteracting particles in homogeneous environment, *i.e.* $g_i(k) = k$, the particle number at $i \in S$ has a Poisson distribution with parameter $\phi_i = R_i(\phi_i) = \sigma_i^2(\phi_i)$. By the stability of the Poisson distribution, the total number of particles in the system is also Poisson distributed with the parameter equal to the sum of the fugacities. For the Jackson queueing network, obtained by setting $g_i(k) = c_i > 0$, the particle numbers obey a geometric distribution with mean $R_i(\phi_i) = c_i\phi_i / (c_i - \phi_i)$ and variance $\sigma_i^2(\phi_i) = c_i\phi_i \left(c_i^2 - c_i\phi_i - \phi_i^2\right) / \left(c_i - \phi_i\right)^4$. The process is ergodic provided that $\max\left(\phi_i - c_i\right) < 0$.

(b) Consider a process of homogeneous interaction, in that $g_i \equiv g$, on the cube $S = [n]^d$. The particles move by symmetric nearest-neighbor jumps within and out of S: $p(i,j) = I\{|i-j| = 1\}/(2d)$, $i \in S$, where we set $|i - \Gamma| = 1$ for i at the boundary of the cube and zero elsewhere. The rate of input of particles from the reservoir to the boundary sites is given by the set of positive parameters $p(\Gamma, i)\psi$. Then the fugacities ϕ_i satisfy the discrete Laplace equation with boundary conditions given by the sources $p(\Gamma, i)\psi$ for each i at the

boundary of the cube. Under a slightly more stringent condition than (2.3), namely $\psi < \lim g(k)$, the tails of the one-site marginals are at least exponentially thin with the exponential rate of decrease bounded uniformly in i and n. In particular, the moments of the particle numbers exist for all sites and system sizes, and they are bounded by the moments of the distribution ν_ψ. Assuming further that $p(\Gamma, i)\psi \geq \varepsilon > 0$ for all boundary sites, the variables $X_i(t)$ satisfy Lyapunov's condition (see *e.g.* [20]) under P^ν, and the scaling of the total number of particles is given by

$$\frac{\sum_{i=1}^n \left(X_i(t) - R(\phi_i)\right)}{\sqrt{\sum_{i=1}^n \sigma^2(\phi_i)}} \stackrel{n\to\infty}{\longrightarrow} N(0,1), \tag{2.11}$$

where the convergence is in distribution.

The zero-range process is one of the canonical models of interacting non-equilibrium systems. Thus the time reversed process frequently appears in applications. The content of the next proposition is that the stationary process viewed backwards in time differs from the original process by its one-particle transition matrix only.

Proposition 2.1.1. The time reversal of $X(t)$ under P^ν is generated by

$$\left(L^* f\right)(\eta) = \sum_{i\in S} \sum_{j\in S\cup\{\Gamma\}} p^*(i,j) g_i(\eta_i)[f(\eta^{i,j}) - f(\eta)] + \sum_{j\in S} p^*(\Gamma, j)\psi[f(\eta^{\Gamma,j}) - f(\eta)], \tag{2.12}$$

where the adjoint transition probabilities are given by $p^*(i,j) = p(j,i)\phi_j/\phi_i$.

Proof. The time reversed dynamics is generated by the adjoint of the extension of L to the space $L^2(\nu)$ [12]. That is to say $\nu[(L^*h)\, f] = \nu[hLf]$ for square ν-integrable h and f. Changing variables in the summations over the state space and using theorem 2.1.1 yields the result. □

How does the reversal of time affect the currents in the system? Remember that we chose the matrix p of transition probabilities to be stochastic, in that the row sums add up to 1. The same holds for p^* because the traffic equations (2.4) are satisfied. This just reflects the conservation of particles in the combined system $S \cup \{\Gamma\}$ – a fact that cannot depend on the direction of time. We next show that reversing the direction of time alters the transition probabilities in such a way that the particle currents are reversed. Indeed, by Lévy's formula (theorem 4.6 of reference [11])

$$E^\nu \int_{(a,b]} f\left(X(t-), X(t)\right) N(\mathrm{d}t) = (b-a) \sum_\eta \nu(\eta) \sum_{\xi\neq\eta} L(\eta,\xi) f(\eta,\xi), \tag{2.13}$$

where the measure N counts the transitions of the Markov process, so choosing $f(\eta, \xi) = I\{\xi = \eta^{i,j}\}$, where $i, j \in S \cup \{\Gamma\}$, yields

$$E^\nu N_{i,j}(a, b] = (b - a)p(i, j)\phi_i \tag{2.14}$$

for the expected number of particles moving from i to j on the time interval $(a, b]$. Similarly, if $N^*_{i,j}$ counts the transitions from i to j of the time reversed process, proposition 2.1.1 implies that $E^\nu N^*_{i,j}(a, b] = (b - a)p(j, i)\phi_j = E^\nu N_{j,i}(a, b]$. Observe that formula (2.14) gives a reason to call the parameters ϕ_i fugacities: They directly measure the activity at the sites.

For a zero-range process with open boundaries, condition (2.3) guaranteed the ergodicity of the process. The condition is sufficient but not necessary because the series $\mathcal{Z}_i$ may be convergent exactly at the radius convergence as well. Ergodicity is harder to break in **closed systems** with $p(i, \Gamma) = p(\Gamma, i) = 0$ and $\sum_{j\in S} p(i, j) = 1$, for $i \in S$. In fact, zero-range processes that conserve the number of particles are ergodic under the mild condition of positivity of interaction functions g_i for positive arguments, which we took as an integral part of the definition of these processes. The stationary state of a closed zero-range process is characterized in the next theorem. The state space for m particles moving within the set S is $\Omega(S, m)$, the set of all possible allocations of m identical objects into the $|S|$ bins.

Theorem 2.1.2. Let the fugacities $(\phi_i)_{i\in S}$ denote a positive solution to the traffic equations

$$\phi_i = \sum_{j\in S} p(j, i)\phi_j, \tag{2.15}$$

Then the stationary distribution of the closed zero-range process with parameters (p, g_i) is

$$\mu(\eta) = \frac{1}{Z(S, m)} \prod_{i\in S} \frac{\phi_i^{\eta_i}}{g_i!(\eta_i)}, \qquad Z(S, m) = \sum_{\eta\in\Omega(S,m)} \prod_{i\in S} \frac{\phi_i^{\eta_i}}{g_i!(\eta_i)}, \tag{2.16}$$

where $\eta \in \Omega(S, m)$.

Proof. For a closed system, there is no boundary condition for the traffic equations. Hence the solution to these equations is not unique: the fugacities are determined up to a multiplicative factor only. Any positive solution will nevertheless do because the partition function Z is proportional to the factor chosen for the fugacities. Thus the stationary distribution is unique. Otherwise the statement is proven exactly in the same way as in the case of an open system. □

The main difference to the case of an open system is that the stationary distribution is not a product measure anymore, but the particle numbers are correlated

because of the global constraint $\sum_{i\in S} X_i(t) = m$. This is evident if one compares the expressions for the one-dimensional marginal distributions,

$$P^\mu\left(X_i(t) = k\right) = \frac{Z(S\setminus\{i\}, m-k)}{Z(S,m)}\frac{\phi_i^k}{g_i!(k)}, \tag{2.17}$$

and the expected rate at which the particles move from i to j,

$$E^\mu N_{i,j}(a,b] = (b-a)p(i,j)\phi_i\frac{Z(S,m-1)}{Z(S,m)}, \tag{2.18}$$

to the corresponding formulae for open systems. However, the distribution is of *product form* (another term borrowed from the queueing theoretic literature [21], but beware, since it is sometimes used to denote independence [22]). This is equivalent to saying that, under P^μ, the random variables $X_i(t)$ are exchangeable: $P^\mu\{X_i(t) = \eta_i : i \in S\} = P^\mu\{X_{\sigma(i)}(t) = \eta_i : i \in S\}$ for any permutation $\sigma : S \to S$. Exchangeability can be seen as the first step away from independence, and a considerable amount of research has accumulated around this topic [23].

Example 2.1.3. (a) The stationary distribution of a system of independent particles is $\mu(\eta) = m!\prod_{i\in S}\phi_i^{\eta_i}/\eta_i!$, where a solution of the traffic equations is chosen such that $\sum_{i\in S}\phi_i = 1$. Here ϕ_i is the stationary probability for a tagged particle (single random walker on S) to be at i.

(b) The queueing system on the torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ with translation invariant transition probabilities, $p(i+k, j+k) = p(i,j)$, and uniform processing rates $g_i(k) \equiv c$ has the uniform stationary distribution on the set of all possible allocations. $\mu(\eta) = |\Omega(\mathbb{T}_n^d, m)|^{-1}$, where $|\Omega(\mathbb{T}_n^d, m)| = \binom{n^d+m-1}{m}$.

2.2 Equivalence of ensembles and condensation

The notion of an ensemble refers to the probabilistic setting, in which the model for a physical system is formulated. Practically, there is a set of conserved quantities, such as the number of particles or energy, associated with each ensemble. As we have already seen, in zero-range processes the number of particles may be either fixed or a fluctuating quantity depending on the contact with the particle reservoir. The closed system with particle conservation will be called the *canonical ensemble* and the corresponding stationary distributions *canonical measures*. This is opposed to the *grand-canonical ensemble* and the *grand-canonical measures* associated with zero-range processes that do not conserve the particle number. These two situations are simply related in the case of zero-range processes: $P^\mu\{X_i(t) = \eta_i : i \in S\} = P^\nu\{X_i(t) = \eta_i : i \in S | \sum_{i\in S} X_i(t) = m\}$.

The particular system, we will focus on in this section, is the zero-range process on the one-dimensional torus $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})$ with translation invariant transition probabilities and interaction, *i.e.* $p(i,j) = p(j-i)$ and $g_i \equiv g$. The higher dimensional generalizations can be treated similarly. We will use the density $\rho \in (0,\infty)$ of particles as a fixed parameter instead of the exact number of particles m. Thus $m = [\rho n]$. This choice approximately preserves the local characteristics of the stationary state as the large system size limit is approached. From the results of the previous section we gather that the canonical measure of this process is $\mu(\eta) = Z^{-1}\prod_i 1/g!(\eta_i)$, $\eta \in \Omega(\mathbb{T}_n, [\rho n])$.

As an introduction to the concept of equivalence of ensembles, let us first consider the simple case of $[\rho n]$ independent particles. Thus $\mu(\eta) = [\rho n]! n^{-[\rho n]} \prod_i 1/\eta_i!$. Now fix a k-element set $A \in S$. The marginal distribution on A reads

$$\begin{aligned}\mu_A\{\eta_i : i \in A\} &= \frac{Z(S\setminus A, m - \sum_{i\in A}\eta_i)}{Z(S,m)} \prod_{i\in A}\frac{1}{\eta_i!} \\ &= \frac{[\rho n]!(n-k)^{[\rho n]-\Sigma_{i\in A}\eta_i}}{\big([\rho n] - \sum_{i\in A}\eta_i\big)! n^{[\rho n]}} \prod_{i\in A}\frac{1}{\eta_i!}.\end{aligned}$$

Applying the Stirling approximation to the factorials in front of the product yields

$$\mu_A\{\eta_i : i \in A\} \longrightarrow \prod_{i\in A}\frac{\rho^{\eta_i}}{\eta_i!}e^{-\rho} \tag{2.19}$$

as $n \to \infty$, which is exactly the k-dimensional marginal of the grand-canonical measure ν for independent particles. Especially note that the fugacities appearing in (2.19) are such that the expectation of the grand-canonical marginal distribution coincides with the canonical particle density. Thus, up to a vanishing error as the system size increases, the k-dimensional canonical marginal measure of a system of noninteracting particles is approximated by the grand-canonical marginal measure of the same product form and a suitably adjusted fugacity. We then say that, in this particular case, the canonical and grand-canonical ensembles are equivalent. Based on this example, it is natural to ask whether such approximation by grand-canonical measures is valid for a general interaction. By the product form of both the canonical and grand-canonical measures and the global constraint $\sum_i X_i = m$ suggesting only weak correlations, this seems plausible. The next example shows that strong enough attractive interactions may complicate the approximation significantly.

Example 2.2.1. For $k > 0$, set $g(k) = 1 + b/k + o(k^{-1-\epsilon})$, where $b, \epsilon > 0$. This interaction has been considered in references [4–6]. What makes this choice of g so interesting is that the tails of the grand-canonical marginal measures are of the form $\nu_\phi(k) \sim \phi^k k^{-b}$, in that the decay is either exponential (for ϕ strictly less than

1, which is the radius of convergence of $\mathcal{Z}$) or a power-law with index b (for $\phi = 1$). In particular, the grand-canonical expected number of particles at a site, which we denoted by $R(\phi)$ in formula (2.9), is finite for all $\phi \in [0, 1]$ given $b > 2$, whereas the image of $[0, 1)$ is the non-negative real line under the map R for $b \leq 2$. There is a problem in the recipe for grand-canonical approximation in the first case because the density ρ of the canonical system may exceed $R(1)$, the supremum of the expectation of the grand-canonical measure. This problem is even more obvious for interaction functions converging to zero because then the grand-canonical partition functions have zero radius of convergence, and hence the distributions are concentrated on configurations with no mass. Nevertheless the canonical densities can be chosen arbitrarily.

If there is any hope of getting an equivalence of ensembles result for strongly attractive interactions, at least a new way of adjusting the grand-canonical fugacities must be invented for large canonical densities. To this end, we introduce the following: Let Φ denote the radius of convergence of the grand-canonical partition function $\mathcal{Z}$. Then $\rho_c = \lim_{\phi \nearrow \Phi} R(\phi) \in [0, \infty]$ will be called the critical density, and the function

$$\phi(\rho) = \begin{cases} R^{-1}(\rho) & \text{for } \rho < \rho_c, \\ \Phi & \text{for } \rho \geq \rho_c. \end{cases} \tag{2.20}$$

serves as the candidate for the parameter of the grand-canonical measure that is adjusted to mimic the canonical distribution. The following theorem shows that such truncated fugacity is indeed the right choice for interactions that do not produce arbitrarily small transition rates. The result is essential to the understanding of condensation phenomena. Thus, for the sake of completeness and readability, we give here the original proof by Großkinsky, Schütz and Spohn [5] (see also [24]).

Theorem 2.2.1 (Equivalence of ensembles; [5], theorem 1; [24], corollary 5.3)**.** Let the function g be bounded away from zero, and let μ denote the canonical measure for such zero-range process on $\Omega(\mathbb{T}_n, \lfloor \rho n \rfloor)$, and ν the corresponding grand-canonical measure with fugacities equal to $\phi(\rho)$. Then $\mu[f] \overset{n\to\infty}{\longrightarrow} \nu[f]$ for all bounded, continuous real functions f depending only on finite number of coordinates.

Proof. To give a proof for the theorem, we need to show that the $|A|$-dimensional marginal measures converge weakly, which on a countable state space is equivalent to the convergence of the marginals in total variation [25]. A sufficient condition for this is that the relative entropy $S(\mu_A|\nu_A) = \sum_{\eta \in \Omega(\mathbb{T}_n, \lfloor \rho n \rfloor)} \mu_A(\eta) \log \left(\mu_A(\eta)/\nu_A(\eta) \right)$ vanishes as $n \to \infty$ [12]. Here μ_A is absolutely continuous with respect ν_A because

the strength of the interaction is restricted. Noticing that

$$\begin{aligned}\frac{\mu(\eta)}{\nu(\eta)} &= \frac{P^\nu\{X_i(t)=\eta_i : i\in\mathbb{T}_n | \sum_{i=0}^{n-1} X_i(t)=[\rho n]\}}{P^\nu\{X_i(t)=\eta_i : i\in\mathbb{T}_n\}} \\ &= \frac{1}{P^\nu\{\sum_{i=0}^{n-1} X_i(t)=[\rho n]\}}\end{aligned}$$

for $\eta \in \Omega(\mathbb{T}_n, [\rho n])$, the relative entropy of the full canonical and grand-canonical measures equals $-\log P^\nu\{\sum_{i=0}^{n-1} X_i(t) = [\rho n]\}$. The joint superadditivity of the relative entropy in the decomposition into marginal measures on disjoint sets [26], $S(\mu|\nu) \geq S(\mu_B|\nu_B) + S(\mu_C|\nu_C)$, now implies that $S(\mu_A|\nu_A) \leq -\frac{1}{[n/k]} \log P^\nu\{\sum_{i=0}^{n-1} X_i(t) = [\rho n]\}$, so it is enough to show that the probability on the right hand side of the inequality decays more slowly than exponentially in n. For $\rho < \rho_c$ the summands in the probability have a distribution with an exponential tail (because the fugacity is strictly less than the radius of convergence), and have hence a finite second moment. Thus the the local limit theorem for the convergence to the normal law [27] shows that, in this case, the probability in question decays as $n^{-1/2}$. On the other hand, the tail of the of the summand distribution vanishes subexponentially for $\rho \geq \rho_c$, and the second moment of the distribution might not exist. Thus the convergence towards normal law is not possible. These cases are covered by the local limit theorems for attraction to non-normal distributions [27, 28]: exactly at $\rho = \rho_c$ we have $P^\nu\{\sum_{i=0}^{n-1} X_i(t) = [\rho_c n]\} \gtrsim n^{-2}$, and the proof in the last case with $\rho > \rho_c$ is recovered by noticing that

$$\begin{aligned}P^\nu\Big\{\sum_{i=0}^{n-1} X_i(t) = [\rho n]\Big\} &\geq P^\nu\Big\{X_0 = [\rho n] - [\rho_c(n-1)], \sum_{i=1}^{n-1} X_i(t) = [\rho_c(n-1)]\Big\} \\ &= P^\nu\Big\{X_0 = [\rho n] - [\rho_c(n-1)]\Big\} P^\nu\Big\{\sum_{i=1}^{n-1} X_i(t) = [\rho_c(n-1)]\Big\}\end{aligned}$$

because the first probability on the second line vanishes subexponentially by definition for $\rho \geq \rho_c$, and for the second probability the same lower bound holds as in the case $\rho = \rho_c$. □

The choice of the fugacity parameter for the approximating grand-canonical measure, equation (2.20), may have seemed arbitrary at first but the theorem just proved shows that it works. Some sort of interpretation of what is going on in systems with high particle densities and strong enough attractive interactions is definitely needed. The case of weak interactions, with $\rho_c = \infty$, can be readily understood on the basis of our discussion on the system of independent particles. The straightforward choice of the fugacity parameter as an inverse of the true particle density under the map R was always attainable and was successful because the total number of particles in the grand-canonical approximation is sharply peaked

around the fixed canonical number of particles, the fluctuations being negligible in the large n limit. This is also reflected by the local limit theorem applied in the proof of equivalence of ensembles theorem. One should notice that we are talking about approximations of *finite dimensional distributions* only. Had we chosen the set A, to which the approximation concerns, to grow without limit as $n \to \infty$, and in particular at least as fast as $n/\log n$, a more careful study avoiding the superadditivity argument would have been needed.

The cases with finite radius of convergence Φ of the grand-canonical partition function $\mathcal{Z}$ and $\rho_c = R(\Phi) < \infty$ are interesting. Obviously the inverse of R can be defined up to the radius of convergence only, and thus cannot be used as a fugacity parameter for higher densities than ρ_c. The solution to this problem given in the theorem above was the truncation of the fugacity to $\Phi = R^{-1}(\rho_c)$. To understand this remember again that we are dealing with a finite collection of sites, the marginal distributions of which are modeled by the measures with fluctuating number of particles. Since the system is homogeneous, in that the canonical marginals are identical, the chosen finite number of sites can be considered as typical representatives in the system. In fact, we could have just picked them blindfolded. The equivalence of ensembles theorem shows that, in the high density phase, these typical representatives of sites are well described by the grand-canonical measures with truncated fugacity. This is what the system looks like on small sets in the large system size limit. In particular, the expected number of particles on these typical sites is approximately $\rho_c = R(\Phi)$, which is less than the true expectation given by the canonical measure, $E^\mu X_i = \rho$. Thus the equivalence result tells us that, since the number of particles observed by picking any finite number of sites at random does not reach the level it should according to the canonical distribution, measuring the particle numbers at finite number of sites is not enough. In other words, the extra mass avoided by the finite dimensional approximation must be clumped on a set of sites that the probability of finding it blindfolded approaches zero as the system size increases. Hence the cardinality of this set is of strictly smaller order than n. This phenomenon is called condensation, and ρ_c is the critical density of this phase transition. We leave the question of the actual size of the set with the extra clump of particles to the next chapter.

The equivalence of ensembles theorem of Großkinsky, Schütz and Spohn given above works for all densities of particles. It guarantees that any bounded continuous canonical observable can be approximated by its grand-canonical counterpart with increasing precision as the system size tends to infinity. We have already met an unbounded variable that does not converge in this sense in the high density phase $\rho > \rho_c$: the particle density. Fortunately, this gave insight into the structure of the stationary state, and allowed us to deduce that a condensation transition takes place. In the low density phase (also called the fluid phase) a stronger equivalence of ensembles result exists. In the appendix of the book by Kipnis and Landim [12],

the equivalence is given proof for *square integrable* functions depending on finite number of coordinates, provided that $\rho < \rho_c = \infty$. This class of functions is of course much larger than the class of bounded continuous functions, and includes also the particle density. We refer to [12] for details.

The study of condensation transitions and equivalence of ensembles have been extended recently to systems with several species of particles [29,30].

2.3 Hydrodynamic limits

Even though the hydrodynamic descriptions are not needed in the applications presented in this Thesis, it would be unacceptable to write an introduction to zero-range processes without even mentioning the subject. The study of scaling limits is a major branch of the particle systems theory, and zero-range processes are, alongside with exclusion processes, the most prominent examples, for which the description on macroscopic time and scales can be rigorously justified. This description is also the starting point for the extension of the Onsager-Machlup theory of fluctuations in time reversible systems to stationary non-equilibrium states [13]. This section is just to give a taste of the results presented and proved in reference [12], which is a detailed account on the mathematics of the scaling limits of interacting particle systems, especially exclusion and zero-range processes. Also references [31,32] discuss the hydrodynamics of particle systems, but pay less attention to zero-range processes.

There are three basic settings of zero-range processes on the d-dimensional torus $\mathbb{T}_n^d$, the hydrodynamic limits of which are of interest: the time-reversible process with symmetric nearest-neighbor jumps, and irreversible processes which are either mean-zero asymmetric, *i.e.* without a mean drift, or simply asymmetric, in that a non-zero drift of particles takes place. Informally, the hydrodynamic equation for the density field $\rho(x,t)$ in the first case is the non-linear heat equation,

$$\partial_t \rho = \frac{1}{2}\nabla^2 \phi(\rho), \tag{2.21}$$

where the appearance of $\phi(\rho) = R^{-1}(\rho)$ shows that the fugacity differences generate the movement of particles also on the macroscopic scale. The second case is the asymmetric process with zero-mean transition probabilities, *i.e.* $\sum_x x_j p(x) = 0$ for $j \in \{1, \dots d\}$. The equation for the motion on the macroscopic scale now takes the correlations in the single-particle transitions into account:

$$\partial_t \rho = \frac{1}{2}\sum_{i,j} \sigma_{i,j} \partial_{u_i} \partial_{u_j} \phi(\rho), \tag{2.22}$$

where $\sigma_{i,j} = \sum_x x_i x_j p(x)$. The particles feel a drift in the asymmetric system, the i-th component of which is $\gamma_i = \sum_x x_i p(x)$, and the equation on the macroscopic scale is of the first order,

$$\partial_t \rho = -\sum_i \gamma_i \partial_{u_i} \phi(\rho). \tag{2.23}$$

It is quite easy to convince oneself that these are the right equations on some physically relevant spatio-temporal length scales and heuristic derivations of similar equations can be found in physics textbooks. The real issue is to find rigorous proofs of the validity of such descriptions. Naturally, some restrictions must be imposed on the rate function g, guaranteeing for example the existence of exponential moments, but even more stringent conditions may be needed. Also the sense, in which the hydrodynamic description is valid, must be made precise. It is customary to investigate the time evolution of the empirical measures $\pi(\eta, \mathrm{d}u) = N^{-d} \sum_{i \in \mathbb{T}^d} \eta_i \delta_{\{i/N\}}(\mathrm{d}u)$ and prove that these measures converge weakly in probability to a deterministic measure $\rho(t,u)\mathrm{d}u$ in the space of right continuous measure trajectories with left-hand limits. This can be achieved by showing the relative compactness of sequences of probability measures for the empirical measures by applying the Prohorov theorem [33], and then proving the uniqueness of the limits of the convergent sequences. Notice that also the limiting measure needs to be appropriately interpreted. For example, its density is the unique weak solution of the nonlinear heat equation in the case of symmetric system. Interested readers are encouraged to take a look at reference [12].

It is clear that to carry out a program as complex as sketched above, deep knowledge of several branches of mathematics is needed, which makes the theory of scaling limits a very challenging area of research. The limits of zero-range processes are still actively studied: The authors of reference [34] give a proof of the hydrodynamic description on the Euler scale of zero-range processes with multiple species of particles.

3 Statistics of extremes

The condensation transition exhibited by certain zero-range processes were discussed in the previous chapter. Equivalence of ensembles provided us estimates for the typical cluster sizes in the system, but we noticed that the measures associated with these typical representatives cannot be used to predict the size of the largest cluster in supercritical systems. Hence an analysis of extreme cluster sizes is required.

The literature on the condensate sizes is reviewed in the last section of this chapter. Before this, the problem of extreme cluster sizes in subcritical systems is studied. The rate of growth of the largest cluster is shown to be at most logarithmic as a function of the system size. A deeper question concerns the distributional convergence of suitably normalized maximal cluster sizes, which is discussed for two particular interactions. Connection to the theory of extreme value distributions is made. The results can be applied to general one-dimensional driven systems in the spirit of reference [7]: The dynamics of clusters is associated with the movement of domain walls.

3.1 Classical theory of extreme order statistics

The aim of extreme order statistics is to characterize the outcomes of a few of the highest and lowest values in a set of data modeled by a set of random variables. The data could, for example, consist of wind speeds measured on one-hour intervals in a given period of time. The classical theory for extremes starts from the assumption of independent random variables, which would of course be a very crude model for the wind speed data - a storm is likely to have an effect on the wind speeds on much longer time scales than one hour. The data is also likely to contain some peaks and lows occurring annually due to seasonal variations. But for measurements performed only once a year, the approximation by independent variables might not be that bad.

There are a few books written on the statistics of extremes. Galambos [35] presents the basic theorems for independent and identically distributed (i.i.d.) variables clearly, and also discusses the extensions where the assumptions are loosened. The books by Leadbetter, Lindgren and Rootzén [36], and Resnick [37] are also stan-

dard references. Kotz and Nadarajah [38], and especially Coles [39], put emphasis on statistical modeling.

Parallelling the theory of sums, also the extremes of i.i.d random variables have universal scaling limits. In the case of extremes, only three nondegenerate limit laws are possible both for linearly scaled maxima and minima of $X_1, X_2, \ldots, X_n$ as $n \to \infty$. In the following, we will concentrate on the properties of maxima only. Distributions for the minima can be obtained by investigating the negated variables. Let $F(x) = P(X_i \leq x)$ denote the distribution function of the i.i.d. random variables $(X_i)_{i \geq 1}$. The trinity of extreme value distributions is the following (theorems 2.1.1–2.1.3 and 2.4.3 of [35]):

1. **Fréchet distribution**
   There exists a termwise positive sequence (b_n) such that $b_n^{-1} \max_{1 \leq i \leq n} X_i$ converges in distribution to a random variable with the distribution function

$$H_{1,\gamma}(z) = \begin{cases} \exp(-z^{-\gamma}) & \text{for } z \geq 0, \\ 0 & \text{for } z < 0, \end{cases} \tag{3.1}$$

   if and only if $\omega := \sup\{x : F(x) < 1\} = \infty$ and there is a parameter $\gamma > 0$ such that

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\gamma} \tag{3.2}$$

   for $x > 0$. Moreover, the constant b_n can be chosen to be equal to $\inf\{x : 1 - F(x) \leq 1/n\}$.

2. **Weibull distribution**
   A necessary and sufficient condition for the existence of sequences of constants a_n and $b_n > 0$ such that $b_n^{-1} (\max_{1 \leq i \leq n} X_i - a_n)$ converges in distribution to a random variable with the distribution function

$$H_{2,\gamma}(z) = \begin{cases} \exp(-(-z)^{\gamma}) & \text{for } z \leq 0, \\ 1 & \text{for } z > 0, \end{cases} \tag{3.3}$$

   is that $\omega < \infty$ and the auxiliary function $F(\omega - 1/x)$ satisfies condition (3.2). The normalizing sequences can be chosen as $a_n \equiv \omega$ and $b_n = \omega - \inf\{x : 1 - F(x) \leq 1/n\}$.

3. **Gumbel (Fisher-Tippett) distribution**
   There are sequences of constants a_n and $b_n > 0$ such that the scaled maximum $b_n^{-1} (\max_{1 \leq i \leq n} X_i - a_n)$ converges in distribution to a random variable with the distribution function

$$H_{3,0}(z) = \exp(-\exp(-z)), \qquad z \in \mathbb{R}, \tag{3.4}$$

if and only if

$$\int_a^\omega (1 - F(y))\, \mathrm{d}y < \infty \tag{3.5}$$

for some $a < \infty$ and the function $r(t) = \int_t^\omega \frac{1-F(y)}{1-F(t)}\, \mathrm{d}y$, with $\inf\{x : F(x) > 0\} < t < \omega$, satisfies

$$\lim_{t\to\omega} \frac{1 - F(t + xr(t))}{1 - F(t)} = \exp(-x) \tag{3.6}$$

for $x \in \mathbb{R}$. The constants can be chosen as $a_n = \inf\{x : 1 - F(x) \leq 1/n\}$ and $b_n = r(a_n)$.

These are the only nondegenerate limiting distributions that can be obtained as the limit of $F^n(a_n + b_n x)$ (see *e.g.* [35]). We briefly comment on the meaning of the conditions for convergence. Clearly, the assumptions on $\omega = \sup\{x : F(x) < 1\}$ divide the random variables X_i into two categories: They are bounded from above if the scaled maximum is to converge to the Weibull distribution and necessarily unbounded in the Fréchet case. Both bounded and unbounded variables may converge to the Gumbel limit. The other conditions characterize the tail of the distribution function F. In the convergence to the Fréchet distribution, condition (3.2) requires that the variables X_i have a heavy-tailed distribution, the decay of which is parameterized by γ. Similarly, the condition for the convergence to the Weibull distribution states that there must be a unique algebraic rate, at which the distribution function F tends to 1 as the upper bound for the variables X_i is approached. There are two assumptions on F for the convergence to the Gumbel limit. The first of these, inequality (3.5), just sets a lower bound on the rate of decay of the distribution of X_i and guarantees that the auxiliary function $r(t)$ is finite for t sufficiently close to ω. The second condition (3.6) is a technical requirement for the convergence to the doubly exponential limit.

Common examples of random variables whose maximum converges to the Fréchet limit are those with stable distributions, *i.e.* $Ee^{-\lambda X_i} = e^{-c\lambda^\gamma}$. Notice that solving x from $nP(X_i > x) = 1$ gives a rough estimate for the size of the maximum because the left-hand side of the equation equals the expected number of variables X_i that exceed x. The underlying stable distributions are sometimes referred to as scale free because they lack any exponential or even sharper cut-off. Thus, due to lack of any intrinsic scale, $b_n = \inf\{x : 1 - F(x) \leq 1/n\}$ is apparently the right candidate for normalization in order to get convergence to a nontrivial distribution.

The convergence to the Weibull distribution occurs for example if the random variables are beta distributed, *i.e.* with density $f(x) = B(a, b)^{-1} x^{a-1}(1 - x)^{b-1}$ where $x \in [0, 1]$ and $a, b \geq 1$. Clearly, setting $a = b = 1$ yields a uniform distribution on the unit interval. In beta distributions, the parameter b characterizes the rate of decay of the distribution on the right boundary, and hence determines the rate of growth

of the maximum. This parameter then shows up in the Weibull distribution. As an example of random variables with a finite upper bound whose maximum does not converge to the Weibull distribution, and therefore to none of the three, consider the distribution function given by $F(x) = 1 - \exp(1/x)$ with $x \in (-\infty, 0]$ because then $(1 - F(-1/xt)) / (1 - F(-1/x)) = e^{-x(t-1)} \nrightarrow x^{-\gamma}$ for any $\gamma > 0$. The reason for lack of convergence is that the density $f(x) = x^{-2} \exp(1/x)$ simply vanishes too fast to be compensated by any linear scaling with parameters a_n and b_n as $x \to 0$ from below. In fact, exponential scaling of the random variables X_i would yield the Weibull distribution.

The Gumbel-Fisher-Tippett distribution appears most frequently of the three in applications and is therefore sometimes called simply *the* extreme value distribution. Some of the most common distributions belong to its domain of attraction. For example, the maximum of normally distributed random variables converge to it, given a scaling with parameters $a_n = \sqrt{2 \log n} - (\log \log n + \log 4\pi)/\sqrt{8 \log n}$ and $b_n = 1/\sqrt{2 \log n}$. Notice in this case that, contrary to the stable variables, there is an intrinsic scale for X_i and two scaling parameters are needed: One must subtract the value given by the rough estimate $a_n = \inf\{x : 1 - F(x) \leq 1/n\}$ from the maximum and scale the fluctuations of the resulting random variable by the value of the auxiliary function r at the point given by the rough estimate, $b_n = R(a_n)$.

As in the case bounded random variables, there are examples that avoid both the Fréchet and Gumbel limits. One of them was given by Galambos [35], $F(x) = 1 - 1/\log x$ with $x \in (e, \infty)$. This time the decay of the density is very slow as $x \to \infty$, and F fails to satisfy the integrability condition for Gumbel convergence. The tail of the distribution is also heavier than any power-law, and hence the Fréchet limit fails. However, now a logarithmic scaling would do the trick and give the Fréchet distribution for the limiting maximum.

There is also a large class of distributions that avoid the three limits due to a special reason, namely discreteness. This topic will be pursued further in the next section– with applications to zero-range processes.

3.2 Statistics of the largest cluster in subcritical zero-range processes

The maxima of cluster sizes in one-dimensional closed interacting particle systems were investigated in Article [I] of this Thesis. Two cases of zero-range processes were considered: the queueing model with $g(k) = I\{k \geq 1\}$ and noninteracting particles. The third system considered in the article was motivated by a model for

growing one-dimensional interfaces and has a non-local interaction. In that case, the dynamics was totally asymmetric (particles jump to the right only) and the jump rates depended not only on the mass at present site but also on the empty space ahead. This case will not be discussed further.

As we saw in chapter 1, the stationary numbers of particles X_i in zero-range processes on tori are, in contrast to to the setting of the previous section, not independent but only exchangeable due to conservation of particles. This obviously complicates the analysis of extreme value statistics. In Article [I] some exact results for the distributions of maxima were first derived. For example, in the queueing model the distribution function for the maximum of X_i, $i = 1, \dots n$, was written down,

$$P(\max_{1\le i\le n} X_i \le k) = \sum_{i=0}^{n\wedge[[\rho n]/(k+1)]} (-1)^i \frac{\binom{n}{i}\binom{n+[\rho n]-1-(k+1)i}{n-1}}{\binom{n+[\rho n]-1}{[\rho n]}}. \tag{3.7}$$

The formula can be obtained by noticing that the stationary distribution of the model is the uniform distribution on the set of solutions to the equation $m_1 + \dots + m_n = [\rho n]$ in non-negative integers (the current state space differs a little from that of Article [I] of this Thesis, which was written in the exclusion process framework). Thus the distribution function for the maximum equals the ratio of the number of solutions to $m_1 + \dots + m_n = [\rho n]$ with $m_i \le k$ for $i = 1, \dots n$, and the number of solutions to the same equation without restrictions, which is the number $\binom{n+[\rho n]-1}{[\rho n]}$ appearing in the denominator of formula (3.7). The first number simply equals the coefficient of the $[\rho n]$-th term of the generating function $(1 + x + \dots + x^k)^n = \big((1 - x^{k+1})/(1 - x)\big)^n$. Also an expression for the mean of the maximum was derived from the formula (3.7).

The derivation of limiting distributions for scaled maxima from the exact expression (3.7) was deemed too complicated and the method of approximations by independent random variables was used instead. The heuristic was that the approximation would be valid since the ensemble equivalence holds and the system is homogeneous for processes without condensates. This argument is faulty. Even the strong L_2 version of the equivalence of ensembles theorem does not guarantee an approximation for global quantities such as the maximum because it discusses the convergence of the finite dimensional distributions only. However, the approximation by independent components can be shown to be valid as I learned later after discovering the article of Jeon, March and Pittel [6]. The following idea can be found in [40]: Let A_n be a statement considering the random variables X_i and let B_n be the same statement for independent (grand-canonical) variables M_i, which are related to X_i by $(X_1, \dots, X_n) = (M_1, \dots, M_n | \sum_{i=1}^n M_i = [\rho n])$ where the equality holds in law. Then

$$P(A_n) = \frac{P(B_n, \sum M_i = [\rho n])}{P(\sum M_i = [\rho n])} = P(B_n)\frac{P(\sum M_i = [\rho n]|B_n)}{P(\sum M_i = [\rho n])}. \tag{3.8}$$

Hence the the grand-canonical approximation is valid if the ratio on the rightmost formula can be shown to be of the order $1 + o(1)$. The denominator is in the subcritical zero-range processes covered by the local limit theorem, but to estimate the numerator, one has to consider the effect of conditioning by B_n. For $B_n = \{\max_i M_i \leq m(n)\}$ with $m(n) = O(\log n)$, this has been carried out in [6].

Let us now go back to the statistics of the largest cluster taken that the independent approximation can be used. Since in the queueing model, the one-site grand-canonical marginals are geometrically distributed with parameter $\rho/(1+\rho)$, one may approximate

$$P(\max_{1\leq i\leq n} X_i \leq k) \approx [1 - e^{-(k+1\log(1+1/\rho))}]^n \approx e^{-e^{-k\log(1+1/\rho)+\log n}}, \tag{3.9}$$

which suggests that under the scaling $\log(1+1/\rho)[\max_{1\leq i\leq n} X_i - \log n/(\log(1+1/\rho)]$ the maximum would converge to the Gumbel distribution. To this end, we can check whether the conditions (3.5) and (3.6) for the Gumbel convergence are met. We have $F(x) = 1 - (\rho/(1+\rho))^{\lfloor x+1\rfloor}$, so

$$\int_a^\infty (1 - F(x))\, \mathrm{d}x = \sum_{i=a+1}^{\infty} (\rho/(1+\rho))^i < \infty$$

and the first condition is satisfied. However, with $r(t) = 1 + \rho$, we get

$$\frac{1 - F(t + xr(t))}{1 - F(t)} = \exp[-\log(1/\rho)\,(\lfloor t + x(1+\rho) + 1\rfloor - \lfloor t + 1\rfloor)],$$

which does not converge as $t \to \infty$ unless $x(1+\rho)$ is an integer. Thus the second condition fails! Remarkably, this condition is satisfied by the exponential distribution, which is the continuous cousin of the geometric distribution. What we did not take into account in deriving the naive approximation for the limit law was that the following inequalities hold:

$$\left(1 - \frac{1}{n} e^{-(x+1)\log(1+1/\rho)+\log n}\right)^n \leq F^n(x) \leq \left(1 - \frac{1}{n} e^{-x\log(1+1/\rho)+\log n}\right)^n \tag{3.10}$$

and that the bounds are sharp in the sense that both of them are attained for infinitely many n for any $\rho \in (0, \infty)$ and $x > 0$. Thus we get by changing variables to $y = x\log(1+1/\rho) - \log n$ that

$$e^{-e^{-y+\log(1+1/\rho)}} \leq \liminf_{n\to\infty} P\,(Z_n \leq y) \leq \limsup_{n\to\infty} P\,(Z_n \leq y) \leq e^{-e^{-y}}, \tag{3.11}$$

for $Z_n = \log(1+1/\rho)[\max_{1\leq i\leq n} X_i - \log n/(\log(1+1/\rho)]$, as first shown rigorously by Anderson [41]. The scaled maximum thus has a distribution function with Gumbel

envelopes. Again both bounds are sharp: They are both observed as the density ρ is varied.

The phenomenon just described came as a surprise to us when we first looked at the data for the mean and variance of the largest cluster as a function of the system size. In particular, we observed that the mean was significantly larger than predicted by the naive analysis and the variance showed anomalous fluctuations. We soon learned that the same phenomena was observed for the statistics of the largest cluster in subcritical percolation [42]. These effects came even more evident when we studied the data for independent particles, *i.e.* $g(k) = k$. In that case the stationary grand-canonical marginals are Poissonian with parameter ρ, which is a very sharply peaked distribution, and the intrinsic scale of the mass variables is more pronounced than in the queueing model. The idea of envelope functions is here taken to the extreme: Anderson [41] proves for general discrete parent distributions that there exists a sequence of integers (I_n) such that $\lim_{n\to\infty} P\left(\max_{1\le i\le n} X_i \in \{I_n, I_{n+1}\}\right) = 1$ if and only if the tail of the parent distribution satisfies $\lim_{n\to\infty} \left(1 - F(n)\right) / \left(1 - F(n+1)\right) = \infty$, which can be checked for Poisson variables: a Chernoff estimate yields

$$P(X_i > k) \sim e^{-k\log(k/\rho)+k-\rho}$$

for k large. So

$$\left(1 - F(n)\right) / \left(1 - F(n+1)\right) \sim e^{\log n + O(1)}.$$

Thus the maximum concentrates on two consecutive integers with high probability. In his article, Anderson also shows a way to choose the sequence I_n, from which Kimber [43] derived $I_n \sim \log n / \log\log n$ in his short note. A heuristic for this result was given by the author of this Thesis: Although there is no convergence towards the Gumbel law, one may nevertheless use the double exponential function as an approximation for the distribution function for moderately large system sizes (see also [44] for Gumbel approximations for discrete random variables). This gives

$$P\left(\max_{1\le i\le n} X_i \le k\right) \sim e^{-\left(\frac{k}{\rho}\right)^{-k} n e^{-\rho+k}}$$

in the leading order. The distribution being heavily peaked, the expected size of the largest cluster may be approximated by the point k_* of the most rapid increase of the distribution function. This takes place roughly at

$$k_*^{-k_*} \sim \frac{1}{n},$$

from which the result of Kimber follows.

In both of the these examples, the growth of the largest cluster is at most logarithmic in the system size n. This can be shown to be valid for all subcritical zero-range

processes. The argument of Jeon *et al.* [6] goes roughly as follows. Choosing a number $m(n)$ according to the criterion $\lim_{n\to\infty} n \sum_{k\geq m(n)} \nu_{\phi(\rho)}(k) = \lambda \in (0,\infty)$ implies that $m(n) = O(\log n)$ because the tail of the subcritical grand-canonical marginal distribution $\nu_{\phi(\rho)}$ vanishes exponentially (remember that $\phi(\rho) < \Phi$ for subcritical processes). Yet $\lim_{n\to\infty} n \sum_{k\geq \lfloor (1+\epsilon)m(n)\rfloor} \nu_{\phi(\rho)}(k) = 0$ meaning that the expected number of variables exceeding $\lfloor (1+\epsilon)m(n)\rfloor$ vanishes. This implies

$$P^{\nu_{\phi(\rho)}}\left(\max_i X_i < \lfloor (1+\epsilon)m(n)\rfloor\right) \to 1.$$

Use of the truncation argument of equation (3.8) then shows that the same holds for canonical measures. Jeon *et al.* further show that $m(n)^{-1}\max_i X_i$ converges to 1 under the canonical probability.

3.3 Size of the condensate

Below the condensation point the grand-canonical measures have exponential tails, and it was not too difficult to show using the idea (3.8) that in this case $\max_{1\leq i\leq n} X_i$ is at most of the order $\log n$ generally. Analyzing the size of the largest cluster is more complicated if the particle density exceeds the critical density ρ_c for condensation. The approximation by independent variables naturally fails because the system is not homogeneous anymore. By the discussion of section 2.2, there is a clump of size $(\rho-\rho_c)n$ hiding in a $o(n)$-set while the masses on the majority of sites are distributed according to the grand-canonical measure with the parameter set to produce the critical density for its mean.

The issue is pretty much settled for for interaction functions such as $g(k) = 1 + b/k + O(k^{-1-\epsilon})$ resulting in power-law distributed grand-canonical measures. This model has a condensation transition for $b > 2$, see example 2.2.1. Jeon, March, and Pittel [6] proved that, for $b > 3$, the extra mass avoided by the grand-canonical measure is essentially concentrated on one randomly located site. More precisely, $n^{-1}\max_{1\leq i\leq n} X_i$ converges to $\rho - \rho_c$ in probability, and the second largest cluster is of the order less than $\sqrt{n}(\log n)^2$. The first statement was extended to $b > 2$ by Großkinsky in his thesis [24]. In the same vein as in the subcritical case, Jeon *et al.* show that exactly at the critical density the size of the largest cluster scales as $n^{1/(b-1)}$, which is the same result as one obtains for independent variables with critical grand-canonical distributions.

Jeon *et al.* [6] consider vanishing interaction functions as well, in which case the grand-canonical partition functions have a zero radius of convergence. Their result states that under quite general assumptions on the manner that g vanishes, the

largest cluster contains all but $ng(n)$ particles with probability tending to 1 as $n \to \infty$. This result is further strengthened in reference [6].

4 Graph-valued process

In this chapter, a short introduction to random graphs and network modeling is given and, after this, a Markovian model for evolving networks, based on zero-range interaction, is introduced. The model combines a number of elements of modern network theory such as edge dilution, preferential dynamics and condensation of edges, and the tractability of its stationary degree sequence makes the grand-canonical analysis of the component structure possible. Mathematical problems with the equivalence of ensembles are discussed.

4.1 Statistical mechanics of random networks

Traditional graph theory has diverse applications in physics, ranging from crystal structures to Mayer graphs and Feynman diagrams [45]. In many of those applications, graph theory is merely used for bookkeeping, although crystallography is an exception. Physicists attention was more keenly drawn to modeling of networks at the end of the 1990's after the discovery of Barabási and Albert [46] that the usual Erdös–Renyi model (see *e.g.* [47,48]) of random graphs is not satisfactory for many real-world networks, but the growth of the network and the so-called preferential attachment of links are in many cases essential (see below). They devised a model that could produce network characteristics rather close to the ones observed for example in actor collaboration and power grid networks, and the world-wide web. Results on the structure of the world-wide web were also reported in [49]. The study of complex networks has now become a major field of its own.

In this chapter we discuss briefly some recent models for complex networks, and the general statistical mechanics for them. The chapter is largely based on the state of affairs in 2004, mostly because the author's interests have drifted (should I say diffused) to other areas since then, and the amount of literature in this area has grown rapidly. Furthermore, we concentrate only on the issues concerning the structure of the network, most notably the existence and size of a macroscopic connected component, while more general processes defined on complex networks, such as diffusions, models for spreading of diseases and the extensions of the usual models of statistical physics (Ising, Potts *etc.*) to general network geometries are left without comment. For a recent review discussing also these aspects, see for exam-

ple preprint [50].

In physics, one usually speaks of networks and network modeling – one may use a graph as a model for a physical network – but mathematicians study random graphs and there are several models for them. Thus we usually interpret networks as real world objects, while graphs exist on paper and in the heads of mathematicians only. This division is not strict, and sometimes products of imagination without a physical realization are called networks. The choices of words of the original references are preferred in this text.

Classical random graphs

An undirected multigraph $G = (V, E)$ consists of a finite set V of vertices (also known as nodes) and a finite multiset $E = \{ij \in V^2\}$ of edges (links) connecting the vertices. In the models we are going to study, the cardinality of the vertex set is a fixed constant, so we can take $V = \{1, 2, \ldots, n\}$. The set E is allowed to be random. Thus there is a probability space $(\Omega, \mathcal{F}, P)$ with the state space Ω consisting of all graphs, a sigma-algebra $\mathcal{F}$, and a probability P. We shall specify a random graph by giving its distribution. The *degree* of a vertex is defined as the number of edges incident at the vertex, *i.e.* $deg(v) = |\{ij \in E : i \neq j, i = v \text{ or } j = v\}| + 2|\{vv \in E\}|$. Here the self-loops, which are edges with both ends meeting the same vertex, give a double count to the degree.

In the standard models of random graphs [47, 48], it is customary to restrict the state space to simple graphs. In that case self-loops are forbidden, and two distinct vertices may be connected by a single edge only. The *uniform random graph* $G(n, m)$ is defined as a uniformly random sample from all graphs with n vertices and m edges. Thus $P(G) = \binom{\binom{n}{2}}{m}^{-1}$ On the other hand, the *binomial random graph* is obtained by fixing the number of vertices n and tossing a biased coin for every pair of vertices in the graph. Each pair of vertices is connected by an edge with probability p independent of whether the other pairs are connected or not. We get $P(G) = (1-p)^{\binom{n}{2}-m} p^m$ for any graph with exactly m edges. Hence the binomial model is a grand-canonical version of the uniform case. Equivalence of ensembles can be proved on a quite general level, see section 1.4 of Janson *et al.* [48]. Other models considered in mathematics literature are *regular graphs*, where each vertex has the same given degree, and graphs with a given degree sequence.

Sometimes it is also useful to consider a time dependent version of the binomial model. In that case there are n vertices and no edges at $t = 0$, and at each integer t a new edge is inserted uniformly randomly into one of the $\binom{n}{2} - (t-1)$ vacancies. This Markov process is simply known as the *graph process*.

Before discussing the models of complex networks we have to delve into the structure of the uniform and binomial random graphs. Knowing that for $m = 0$, or equivalently $p = 0$, the system consists of isolated vertices, and that for $m = \binom{n}{2}$ or $p = 1$ the graph is fully connected, one may infer that a percolation transition, where a macroscopic connected component (also called a giant component) appears, takes place for some intermediate parameter values. As is typical to the models of statistical physics, the phase transition is expected to be sharp in the limit $n \to \infty$ provided that the parameter m or p is correctly tuned. We will focus here on the model $G(n, m)$ only, the other case is largely covered by the ensemble equivalence by taking $p = 2m/n(n-1)$ [48]. The critical value for m is easily guessed. There must be a macroscopic number of vertices of degree at least equal to two in order to a path of macroscopic length to exist. By the uncorrelated positioning of the edges, this is attained if there is on average at least one edge incident at each vertex because then a macroscopic amount of vertices are of degree zero, *i.e.* isolated, and in compensation a macroscopic amount of them have a degree greater than one. For a moderate number of edges, the occurrences of very large degrees are also highly improbable, since growing the degree of a given vertex involves a random pick whose probability is of the order $1/n$. Thus the threshold for the transition is $m = n/2$.

The following result goes back to Erdös and Renyi [51], but see refs. [47, 48]: If $m < n/2$, the size Z_n of the largest component of $G(n, m)$ is bounded by $C \log n$ with high probability (meaning that the probability approaches certainty as $n \to \infty$. Some say 'asymptotically almost surely'). Exactly at $m = n/2$, the size of the largest component is exactly of the order $n^{2/3}$ in probability in the following sense: $\forall \delta > 0\ \exists n_0, c_\delta, C_\delta > 0$ such that $P(c_\delta n^{2/3} \leq Z_n \leq C_\delta n^{2/3}) > 1 - \delta$ for all $n > n_0$. Finally, above the critical threshold, there are constants $c, C > 0$ such that $c < Z_n/n < C$ with high probability. This phenomenon with three different types of scalings is known as the double jump. These results were further refined by Bollobás [52] by allowing $m = n/2 + o(n)$, in which case more continuous changes are observed. Note the similarity of the double jump phenomenon with the statistics of the largest cluster in zero-range processes, *cf.* section 3.3.

Models of complex networks

What does it take for a network to be complex? One definition would be that the network must have characteristics that cannot be reproduced by the two basic models of random graphs, but it is too general because practically none of the large real-world networks are decently described by one of the two models in every respect. Examples of such real-world networks are [53,54] the World Wide Web and the Internet, power grids, phone call graphs, networks of collaborations, acquain-

tances and sexual relations, food webs and metabolic networks to name a few.

Some typical features of real-world networks are listed in Newman's review [54] and can be used as a guideline in defining complexity. These include the small-world effect, transitivity, mixing patterns, community structure, and scale-free degree distributions. Of these the small world effect is perhaps the most well known to the public because of the 'the six degrees of separation' hypothesis, stating that on average a message between every two people on Earth can be transmitted by using a chain of no more than six links of personal acquaintances. What is usually meant by the small-world effect in network modeling is that the mean vertex-to-vertex distance grows very slowly as a function of the number of vertices. Typically the scaling is logarithmic, as in the classical models of random graphs, but some network models show even slower growth. Although the small-world effect is common to classical random graphs and real-world networks, they often have different transitivity, which essentially measures the number of triangles in the network. In some real-world networks, such as social networks, a vertex being linked to vertices a and b implies an increased probability that a and b are joined together. In some other cases, the effect may be opposite with anticorrelation in linking. These correlations are absent in the binomial and uniform models.

Transitivity is also closely related to so-called mixing patterns: In assortative mixing, a vertex more likely linked to vertices of the same characteristic (for example degree, or an additional parameter associated with each vertex in an augmented model, such as race in social networks). The opposite case is called dissortative mixing.

One of the differences between the classical models and many real-world networks is the distribution of degrees. The degrees are asymptotically Poisson distributed in the binomial and uniform models, whereas some real networks such as the WWW and the networks of citations between scientific articles have heavy-tailed degree distributions whose decay is a power-law. Networks of the latter type are called *scale-free* for obvious reason. As compared to the classical random graphs, these networks contain very highly connected nodes known as *hubs*.

Barabási and Albert [46] went to consider the mechanisms that would cause the absence of scale and introduced a model that captured this property (see reference [55] for rigorous analysis). They argued that there are two generic features in the evolution of real networks that are not present in the graph process, namely growth and *preferential attachment* of links (the latter is also known as *cumulative advantage* by the much earlier work of Price [56]). The first of these means that also new nodes are brought to the system rather than just new links as in the graph process. It is also assumed that the new nodes are immediately connected to the already existing nodes by some nonzero (mean) number of links upon their introduction to

the system. The second feature concerns the way the points of contact of these new links are chosen. This preferential attachment is familiar to every scientist. Picking a bunch of articles on a subject one is not familiar with – from a preprint server say – so that their scientific relevance is not clear a priori, one is quickly lead to the most cited articles of that field on the basis of lists of references in that bunch. There is a high probability that one ends up citing those basic references again, hence increasing the probability of future citations to those articles on ones part. The idea of preferential attachment is now clear: A new link is more likely joined to an already well connected node than to a node with a small degree.

While the absence of scale in networks of scientific citations is simply explained by the preferential attachment of the links from the new nodes, and the same mechanism is probably behind the existence of hubs in the WWW, the growth of the network is not a necessary condition for the emergence of heavy-tailed degree distributions. In particular, the same effect can be obtained for instance by rewiring dynamics, in which the existing links reorganize in a manner that favors the formation of highly connected nodes [57]. More generally, one may define equilibrium statistical ensembles of *uncorrelated random networks* for fixed numbers of nodes that are capable of providing various degree distributions [58]. The processes that consist of transitions where only one edge end is relocated in each transition and whose transition rates depend on the degree of the node at either end of the move then result in canonical stationary distributions. The grand-canonical measures are the stationary distributions of processes that contain deletion and insertion of edges too. The grand-canonical ensemble has been used to model polymerization [59–61]. We shall study a process that leads to canonical measures in the next section.

4.2 Model with zero-range interaction

In Article [II] of this thesis, a model with edge rewiring dynamics with zero-range interaction was introduced. The setting of the model is that of figure 4.1 (a), in that the state space is bipartite multigraphs with vertex sets W and V such that $|W| = L$ and $|V| = M_1 + M_2$. The degrees, with a number M_i of them equal to $i = 1, 2$, are static on the set V and movement of edge ends takes place on the set W. The choice that degrees greater than two are not allowed is made to facilitate the analysis only (see also below). The interaction of the total of $N = M_1 + 2M_2$ edge ends is zero-range: the rate at which a vertex $w \in W$ loses an (uniformly randomly chosen) edge end is $g(\eta_w)$ given that the degree of the vertex is η_w. The new location of the moving edge end is then determined according to an irreducible transition probability that, although not need be for the analytical tractability of the stationary state, is assumed symmetric. The fact that the edge end movement on the set W can be seen also as a two-species zero-range process, with numbers of particles M_1 and

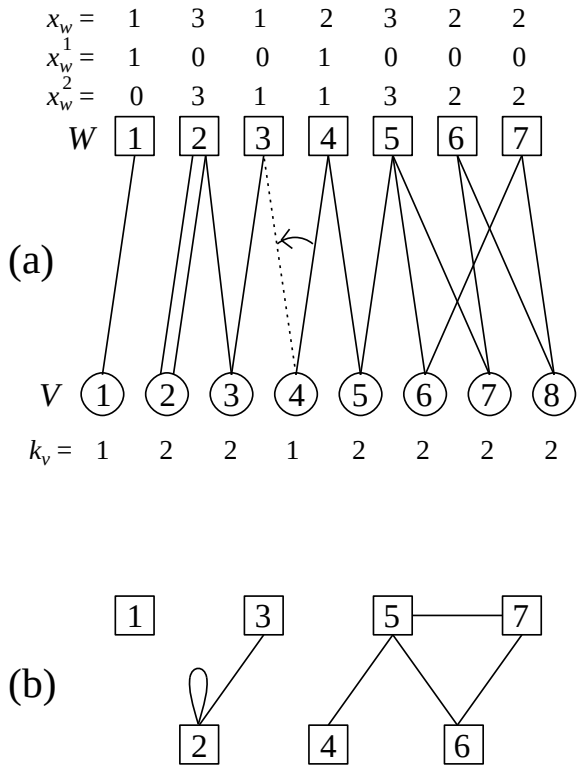

Figure 4.1: (a) Example of a bipartite multigraph with vertex sets V and W. The static degrees of $v \in V$ are denoted by K_v and the evolving degrees of $w \in W$ by $X_w = X_w^1 + X_w^2$ with X_w^1 and X_w^2 the numbers of edges leading to vertices of degree one and two, respectively. The arrow and the dotted line show a possible transition. (b) The projection of the bipartite graph in (a) to the vertex set W.

$2M_2$ of the first and second species, will be used in the sequel.

One is interested in the component structure on the projection of the graph to the set W, also called the one-mode network [62] (see figure 4.1 (b)). The choice of the state space can now be motivated. If one observes the motion of the edges on the one-mode network only, a zero-range interacting unipartite graph-valued Markov process with node set V is obtained in the special case $M_1 = 0$ because then the nodes of the set W are redundant. The process on V only is not Markovian for densities $r = 2M_2/N$ strictly less than one because the dangling ends (edges leading to one of the M_1 vertices) do not show up on the projection but still affect the transition rates. Thus the increasing the density of these hidden variables can be used to dilute the projection graph.

4.2.1 Stationary state

Due to the zero-range interaction, the stationary state of the model can be easily written down. However, instead of examining the numbers of species 1 and 2 edges on the nodes, it proves to be useful to study the numbers $D_{m,n}$ of vertices with exactly m and n particles of species 1 and 2, respectively. In the stationary state, the probability of the event $\{D_{m,n} = d_{m,n} : m, n \geq 0\}$ is given by

$$\sigma(d) = \frac{1}{Z_\sigma} \prod_{m=0}^{M_1} \prod_{n=0}^{M_2} \frac{1}{d_{m,n}!} \left(\binom{m+n}{n} \frac{1}{g!(m+n)} \right)^{d_{m,n}} \tag{4.1}$$

provided that the three conditions

$$\begin{aligned} \sum_{m=0}^{M_1} \sum_{n=0}^{M_2} d_{m,n} &= L, \\ \sum_{m=0}^{M_1} \sum_{n=0}^{M_2} (m+n) d_{m,n} &= M_1 + 2M_2, \\ \sum_{m=0}^{M_1} \sum_{n=0}^{M_2} m d_{m,n} &= M_1 \end{aligned} \tag{4.2}$$

are satisfied and is zero otherwise.

The random variables $D_n = \sum_{m=0}^{M_1} D_{m,n}$ are the numbers of vertices degree n on the projection graph. To study the structure of this graph, and especially whether a macroscopic component exists on it, one needs to show that the conditions of Molloy and Reed [63, 64] for a well-behaved degree sequence (they are reproduced in Article [II] of this thesis) are satisfied by (D_n). Given the conditions (4.2), this would be difficult to do just using the distribution (4.1) because the random variables D_n are not independent. One of the key points of Article [II] is the construction of a sequence of independent variables C_n that approximates the sequence (D_n), *i.e.* an equivalent grand-canonical ensemble. We briefly illustrate the idea here (see also reference [65] for approximations by independent variables in general).

Taking $C_{m,n}$ independent Poisson distributed random variables with mean

$$\lambda_{m,n} = \frac{1}{Z(\phi)} \binom{m+n}{n} \frac{\alpha(1-\gamma)^m \gamma^n \phi^{m+n}}{g!(m+n)}, \tag{4.3}$$

where the normalization is $Z(\phi) = \sum_{k=0}^{\infty} \frac{\phi^k}{g!(k)}$, and the parameters α, γ and ϕ are to be determined appropriately. Then the joint distribution of $C_{m,n}$, given that they satisfy the same conditions (4.2) as the dependent sequence $(D_{m,n})$, is the distribution σ given in eq. (4.1). The sequence $(C_{m,n})$ is thus a good candidate for a precise

approximation for the dependent variables. It is further argued in Article [II] of the thesis that one must choose $\alpha = L$, $\gamma = r = 2M_2/N$ and, with $\rho = (M_1 + 2M_2)/L$ denoting the total density of edges,

$$\phi = \begin{cases} R^{-1}(\rho) & \text{for } \rho < \rho_c \\ \Phi & \text{for } \rho \geq \rho_c, \end{cases} \tag{4.4}$$

where $\rho_c = \lim_{\phi \nearrow \Phi} R(\phi) = \lim_{\phi \nearrow \Phi} \phi Z'(\phi)/Z(\phi) \in [0, \infty]$ and Φ is the radius of convergence of the grand-canonical partition function Z, just as in chapter 2.2, where we discussed the equivalence of ensembles for zero-range processes. The effect of condensation of edges is large on the structure of the projection graph. As we know from the discussions in chapters 1 and 2, there are two types of condensation transitions: One for interaction functions bounded away from zero and one for vanishing rates. The vanishing case is not that interesting for the graph application because, by the result of Jeon *et al.* [6], the condensate contains all but an amount of roughly $ng(n)$ edges. Thus the volume fraction of the largest component vanishes. On the other hand, in the case of positively lower bounded interaction a condensate implies that a macroscopic component exists. This is a consequence of nonzero radius of convergence of the grand-canonical partition function. Thus it is sufficient to consider the subcritical and critical zero-range processes.

Let now $\rho = (M_1 + 2M_2)/L < \rho_c$, and let us introduce

$$\nu_\phi^r(n) \frac{1}{Z(\phi)} \sum_{k=n}^{\infty} \binom{k}{n} r^n (1-r)^{k-n} \frac{\phi^k}{g!(k)}, \tag{4.5}$$

which equals $\lambda_{m,n}$ summed over m, the number of dangling ends. By the stability of the Poisson distribution, a Poisson random variable with density $\nu_\phi^r(n)L$ is then a good candidate for the degree of a randomly chosen vertex on the diluted projection graph.

In Article [II] of this thesis it is shown that $D_n/L \to \nu_\phi^r(n)$ in probability, and that the deviations from this limit are at most of the order $L^{-1/2}$. Strictly speaking, the argument is not strong enough to show that the degree sequence is well-behaved in the sense of Molloy and Reed [63,64]. This shortcoming can be fixed to some extent, in particular for subcritical zero-range processes, but even then, not completely. For instance, Molloy and Reed assume that $n(n-2)D_n/L$ converges *uniformly* to $n(n-2)\nu_\phi^r(n)$ as $L \to \infty$. This can be shown to hold for subcritical zero-range processes with high probability as follows. Let A_1, A_2 and A_3 denote the three conditions (4.2) for Poisson distributed independent random variables C_n with intensities $\nu_\phi^r(n)$. We

get

$$P\left(\sup_{n\geq 0}|n(n-2)\frac{D_n}{L}-n(n-2)\nu_\phi^r(n)|\geq\epsilon\right)$$
$$\leq\frac{P\left(\sup_{n\geq 0}|n(n-2)\frac{C_n}{L}-n(n-2)\nu_\phi^r(n)|\geq\epsilon\right)}{P\left(A_1\cap A_2\cap A_3\right)}, \tag{4.6}$$

so it is enough to get an upper bound for the probability of the same event for the independent variables C_n such that the ratio of the probabilities will vanish in the limit $L\to\infty$. For subcritical zero-range processes, it is shown in Article [II] of this thesis that $P\left(A_1\cap A_2\cap A_3\right)\geq cL^{3/2}$. In particular, any bound for the probability in the numerator with faster than power-law decay will do the job. To this end, let $(K(L))$ be a sequence of integers such that $K(L)=o(L^{1/4})$ and $\log L=o(K(L))$, and let $Z=\max\{k:C_k\neq 0\}$ denote the largest degree in the independent approximation. Then

$$\begin{aligned}
&P\left(\sup_{n\geq 0}\left|n(n-2)\frac{C_n}{L}-n(n-2)\nu_\phi^r(n)\right|\geq\epsilon\right)\\
&\leq\sum_{n=1}^{K(L)}P\left(|\frac{C_n}{L}-\nu_\phi^r(n)|\geq\frac{\epsilon}{K(L)^2}\right)+\sum_{n=K(L)+1}^{\infty}P\left(n^2|\frac{C_n}{L}-\nu_\phi^r(n)|\geq\epsilon\right)\\
&\leq K(L)e^{-\frac{LK(L)^{-4}\epsilon^2}{2+\epsilon}}+\sum_{n=K(L)+1}^{\infty}\Bigg[P\left(n^2|\frac{C_n}{L}-\nu_\phi^r(n)|\geq\epsilon,Z<n\right)\\
&\qquad\qquad+P\left(n^2|\frac{C_n}{L}-\nu_\phi^r(n)|\geq\epsilon,Z\geq n\right)\Bigg]\\
&\leq K(L)e^{-\frac{LK(L)^{-4}\epsilon^2}{2+\epsilon}}+|\{n:n\geq K(L)+1,n^2\nu_\phi^r(n)>\epsilon\}|+\sum_{n=K(L)+1}^{\infty}P\left(Z\geq n\right).
\end{aligned}\tag{4.7}$$

The second inequality is obtained by the Chernoff bound given in Article [II] of this thesis. Hence the first term on the last inequality vanishes because $M=o(L^{1/4})$. The set on the last line, obtained by the simple estimate $P(Z<n)\leq 1$, is empty for any ϵ for large enough L provided that $n^2\nu_\phi^r(n)$ vanishes as $n\to\infty$. This is the case for subcritical grand-canonical zero-range measures because then the tail is exponentially small. Using the same fact we get for the last term

$$P\left(Z\geq n\right)=1-\prod_{k\geq n}e^{-\nu_\phi^r(k)}\leq 1-e^{-e^{-cn}}\leq e^{-cn}$$

for some $c>0$ and n large, and therefore there is a $\tilde{c}>0$ such that $\sum_{n=K(L)+1}^{\infty}P\left(Z\geq n\right)\leq e^{-\tilde{c}K(L)}$, which vanishes faster than any power because $\log L=o(K(L))$. Observe that substantially stronger bounds would be needed for

grand-canonical measures with power-law tails, such as in the critical zero-range processes with interaction of the form $g(k) = 1 + b/k$, because then the largest degree has power-law deviations, and so does the last term in inequality (4.7).

The previous calculation shows that $\sup_n |n(n-2)D_n/L - n(n-2)\nu_\phi^r(n)| = o(\log^{2+\delta} L/\sqrt{L})$ with high probability but this is not enough to show the convergence of the sum $\sum_n n(n-2)D_n/L$, which is essential in determining whether a macroscopic component exists. Moreover, Molloy and Reed demand that the sum converges even uniformly, which further complicates the rigorous analysis. It seems that some exponential bounds for the deviations of these sums can be derived for fast decaying Poisson densities $\nu_\phi^r(n)$, but the general problem for subcritical zero-range processes seems too hard to tackle by the techniques known to the author of this thesis.

The previous story has a moral: One has to carefully specify the level on which the equivalence of ensembles hold. It is now quite obvious that the convergence of finite dimensional distributions of the particle system is not enough to ensure that other features of the graph, which typically depend on the largest degrees, coincide.

4.2.2 Size of the macroscopic component

In the following, we discuss the properties of stationary graphs believing that the approximation by independent degrees is precise. Article [II] was also written in this vague sense, relying completely on pointwise estimates. That kind of heuristics are given by Molloy and Reed themselves for classical random graphs, see the examples in references [63,64]. However, the theoretical results presented here and in Article [II] of this thesis coincided with the ones from numerical simulations, so the problems are expected to be technical only.

Let us discuss the existence and size of a macroscopic component. The heuristic is that such giant component exists if the mean number of second nearest neighbors of a randomly chosen vertex is larger than the mean number of nearest neighbors, and does not exist if on average there are fewer second than first neighbors. Thus for a graph with an asymptotic degree distribution $\nu_\phi^r(n)$ a macroscopic component exists if $\sum_{n\geq 0} n(n-1)\nu_\phi^r(n) > \sum_{n\geq 0} n\nu_\phi^r(n)$, in that $\sum_{n\geq 0} n(n-2)\nu_\phi^r(n) > 0$, which is just the condition given by Molloy and Reed. Substituting expression (4.5) for the distribution and changing the order of summations over n and k yields that the giant component exists if

$$0 < \sum_{k\geq 0} \big(k(k-1)r^2 - kr\big)\, \frac{1}{Z(\phi)} \frac{\phi^k}{g!(k)} = r^2\phi^2 \frac{Z''(\phi)}{Z(\phi)} - r\phi \frac{Z'(\phi)}{Z(\phi)}. \tag{4.8}$$

The critical value for the dilution parameter r is

$$r_c = \frac{Z'(\phi)}{\phi Z''(\phi)}. \tag{4.9}$$

Remember that r is the fraction of edges that will show up on the projection graph. So for r greater than r_c, a macroscopic component exists.

Below r_c the size of the largest component is expected to be at most of the order $\log^3 L$ [63]. For supercritical r, the giant component contains a nonzero fraction Δ of the vertices. This fraction was calculated in reference [64]: In the case of zero-range measures, it equals

$$\Delta = 1 - Z((1-\beta r)\phi)/Z(\phi), \tag{4.10}$$

where β is the positive solution to

$$\beta = 1 - Z'((1-\beta r)\phi)/Z'(\phi). \tag{4.11}$$

A heuristic for general degree distributions, based on generating functions, was given by Newman, Strogatz and Watts [62]. It goes as follows for zero-range grand-canonical measures: The generating function for the degree distribution on the one-mode network is

$$F_V(z) = \sum_{n\geq 0} z^n \nu_\phi^r(n) = \frac{Z\left((1-r(1-z))\phi\right)}{Z\left(\phi\right)} \tag{4.12}$$

and the generating function for the degree of a vertex that is reached by following a uniformly randomly chosen edge is calculated from a Palm probability,

$$F_E(z) = \frac{\sum_{n\geq 1} n z^n \nu_\phi^r(n)}{\sum_{n\geq 1} n \nu_\phi^r} = z\frac{Z'\left((1-r(1-z))\phi\right)}{Z'\left(\phi\right)}. \tag{4.13}$$

The sizes of *small* components, *i.e.* excluding the giant component, can then be estimated because they are supposed to have few cycles (closed paths). The generating function G_E for the size of such tree whose root is reached by following a randomly chosen edge then satisfies $G_E(z) = zF_E(G_E(z))/G_E(z)$ because the process of exposing the neighbors of the root just reveals new trees that have the same size distribution as the whole tree before we had any information on the number of neighbors of the root vertex. The factor z appears because there is at least one vertex, the root, and the division by $G_E(z)$ excludes the root vertex from being counted twice. The generating function for the size of the component obtained by picking a random vertex then reads $G_V(z) = zF_V(G_E(z))$, the factor z in front of which again indicates that there is at least one vertex in the component. Observe that $G_V(1)$ is just the probability of hitting any small cluster in the system by picking a

vertex at random. Thus the fraction of vertices belonging to the giant component is $\Delta = 1-G_V(1) = 1-F_V(G_E(1))$ where $G_E(1) = F_E(G_E(1))/G_E(1)$. Using the expressions (4.12) and (4.13) for F_V and F_E and introducing the notation $\beta = 1 - G_E(1)$ then yields equations (4.10) and (4.11).

For subcritical zero-range processes, in that there is no condensation of edges, it can be shown shown by Taylor expansions of $Z((1-\beta r)\phi)$ and $Z'((1-\beta r)\phi)$ in equations (4.10) and (4.11) that the growth of the fraction of vertices occupied by the giant component starts linearly as r increases beyond $r_c(\phi)$. More precisely,

$$\Delta \approx \frac{2\phi Z''(\phi)^2}{Z(\phi)Z'''(\phi)} \left(r - r_c(\phi)\right). \tag{4.14}$$

Thus the critical exponent for the growth of the macroscopic component equals 1.

Some general facts about the phase diagrams can be inferred from formula (4.9) for the critical dilution. Rather than using the partition function $Z(\phi)$, it can be alternatively written in terms of the function $R(\phi) = \phi Z'(\phi)/Z(\phi)$, which was used to determine the fugacity parameter ϕ from the density of edges $\rho = (M_1 + 2M_2)/L$ moving on the vertex set W, *cf.* equation (4.4). In this case it reads

$$r_c(\phi) = \frac{1}{R(\phi) - 1 + \phi\frac{R'(\phi)}{R(\phi)}}, \tag{4.15}$$

which implies that, in terms of the edge density, $r_c(\rho) := r_c(\phi(\rho)) \leq 1/(\rho - 1)$. In addition to this bound, the following observations about the phase diagrams on the (ρ, r)-plane were presented in Article [II] of this thesis:

1. When $R(\phi) \to \infty$ as $\phi \to \Phi$, *i.e* ρ_c is infinite, the critical curve decreases to zero as a function ρ in such a way that $r_c(\rho) > 0$ for ρ arbitrarily large. The high density asymptotic behavior of the curve is either $r_c(\rho) \sim 1/\rho$ or given by the nontrivial part in the denominator of formula (4.15).

2. When the first derivative of the partition function converges as $\phi \to \Phi$, in that ρ_c is finite, but the second derivative diverges, we have a condensation transition, and the critical curve approaches zero as ρ tends to ρ_c from below. Therefore, a giant component exists at and above the zero-range critical point for any non-zero r. Near the condensation transition, where ϕ is just below Φ, the critical curve goes like

$$r_c(\rho) \approx \frac{\rho}{\phi(\rho)R'(\phi(\rho))}. \tag{4.16}$$

3. If also the second derivative of the partition function converges, $\lim_{\rho\to\rho_c} r_c(\rho)$ is non-zero. In fact, the values of the dilution parameter r being restricted

to the unit interval, this case is further divided into two categories according to whether this non-zero limiting value is greater or less than 1. For $\lim_{\rho\to\rho_c} r_c(\rho) > 1$, no phase transition can be achieved by varying the dilution parameter r. In both cases, if the condensation occurs on a single vertex, we know that above ρ_c there is a giant component, but whether it exists exactly at the zero-range critical point is not known.

The three types of phase diagrams were then demonstrated by a detailed study of the model with zero-range interaction of the form $g(k) = 1 + b/k$, which has a condensation transition for $b > 2$, *cf.* example 2.2.1 and the discussion in 3.3. Figure 4.2 shows the critical curves for various values of b. Curves A and B are examples the first case in the list above. Curve A decays in inverse proportion to the density, whereas the high density end of curve B is dominated by the term $\phi(\rho)R'(\phi(\rho))/R(\phi(\rho))$ in formula (4.15). Curves C_1 and C_2 are of the type discussed in the second point of the list, and the remaining (D–F) represent the third category. The very last of these shows that for high enough values of b the emergence of a macroscopic component may occur only by condensation of edges.

Conjectures for the critical exponent describing the growth of the giant exactly at the point of condensation of edges were also given in Article [II]. Remember that for subcritical zero-range processes, the growth started linearly. On the other hand, expansions of the grand-canonical partition functions near the radius of convergence provided a set of exponents: $\Delta \sim r^{1/(3-b)}$ for $2 < b < 3$ (curves C_1 and C_2 of figure 4.2), $\Delta \sim \exp(-1/(4r))$ at $b = 3$, *i.e.* a transition of infinite order, and $\Delta \sim (r - (b-3)/4)^{1/(b-3)}$ for $3 < b < 4$ (curve D). Finally, the linear growth was recovered for $4 < b < 7$, in which case $\Delta \sim (r - (b-3)/4)$ (curve E). The same set of exponents appears also in refs. [66] and [67], which discuss the effect of edge dilution to scale-free networks and the general statistical mechanics of networks, respectively.

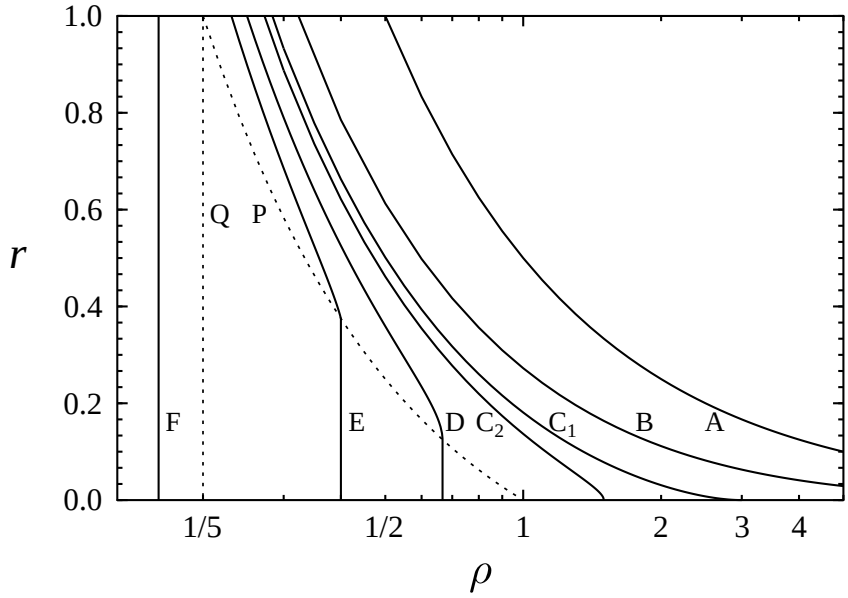

Figure 4.2: The curves A-F show phase diagrams for the interaction $g(k) = 1 + b/k$ with various values of b: From A to F, $b = 0, 3/2, 7/3, 8/3, 7/2, 9/2$ and 8. The dotted line P is the curve of limiting points for $3 < b \leq 7$, in which case the second derivative of the partition function converges, and the dotted line Q marks the value of density ρ, below which the birth of the giant component is possible only through zero-range condensation.

5 Zero-range process in random media

The study of processes in random environments has a long history going back at least to Dyson's article [68] on chains of randomly coupled oscillators. Nonetheless, there is still plenty to discover in the area of interacting non-equilibrium systems with disordered transition rates. Some recent advances in this direction were already mentioned in the introduction: Krug and Ferrari [8], and Evans [9] have demonstrated the occurrence of a phase transition in asymmetric disordered exclusion processes on one-dimensional tori. In the current chapter, boundary driven zero-range processes in random media are analyzed.

This chapter is divided into two parts: The analysis of the stationary state of a non-equilibrium zero-range process with random rates is first presented, and the motion of a tagged particle is then studied in this steady state. Before the introduction of the model in section 5.2, we briefly review some motivating and related studies.

5.1 Random walks and diffusions in random media

Random walks and diffusions in random environments are commonly used as models for transport in non-crystalline, porous or fractured materials, and in topologically complex flows. The literature on this topic is vast (see [69–72] and references therein) and we have no intention of going through the numerous applications and models here, but merely want to discuss some aspects that are most closely related to the model presented below.

We start with the models for a single particle moving on a lattice with random rates or transition probabilities. The most studied of these models is perhaps the one-dimensional random walk with random position dependent bias. This is sometimes called Temkin's model [71], and in mathematical literature often just a random walk in random environment (RWRE) [73]. In this model, every lattice site $x \in \mathbb{Z}$ is assigned a probability ω_x such that, given that the random walker is located at x, it takes its next step to the right with this probability and to the left with probability $1 - \omega_x$. Each ω_x is itself random in the sense that ω_x takes a value

$a \in (0,1)$ with probability $p(a)$, independent of the values of the other $\omega_y, y \neq x$. This is done for all $x \in \mathbb{Z}$ prior to the release of the walker from the origin, and the setup is fixed for the whole duration of the experiment. This type of spatial randomness is known as *quenched disorder* in physics literature. Observe that Temkin's model can be considered as a model for diffusion in an effective potential with increments $\log[(1-\omega_x)/\omega_x]$: According to the simple transition state theory [74], $\omega_x \propto \exp(-\Delta_x)$, where Δ_x stands for the potential difference between sites x and $x+1$ and the force exerted on the particle at x by the effective potential equals in linear approximation $\Delta_{x-1} - \Delta_x = \log\exp(-\Delta_x + \Delta_{x-1}) = -\log[(1-\omega_x)/\omega_x]$.

The increments of the effective potential show up in the phase diagram of the model: There are regions where the walk is either left or right recurrent, or both, depending on the expected values of $[(1-\omega_x)/\omega_x]$ and $\log[(1-\omega_x)/\omega_x]$. The case with $\mathbb{E}\log[(1-\omega_x)/\omega_x] = 0$, known as Sinai's problem, is particularly interesting because in that case the walk takes place in a random landscape, which itself is generated by a mean zero random walk, and the typical displacements of the particle are very small because of trapping into deep valleys.

We will formulate a model for transport of zero-range interacting particles between two very large particle reservoirs. However, with attractive interactions and potentials as in Temkin's model, one quickly runs into trouble with the accumulation of particles on bottoms of the valleys. The stationary state does not exist in many cases. There is a very natural model that is in one respect simpler than Temkin's model but is still rich enough to capture the effects of random media. It is sometimes called a random barrier model because the particles can be imagined surmounting potential barriers of random heights placed between the lattice sites, whereas the sites themselves all lie on the same potential level. Thus the one-particle transition rates are symmetric between two neighboring sites, but some borders between pairs of sites are harder to cross than the others. Physical motivation and some analysis of this model is presented for example in the review article of Alexander *et al.* [69]. Kawazu and Kesten [75] provide a rigorous analysis for the movement of the particle on macroscopic scales. A closely related, but suffering from the same problems as Temkin's model for zero-range use, is the random wells model (see *e.g.* [70]), in which all the barriers are of the same height but the potential variations are associated with sites, so that sites can be regarded as a wells with random depths. In this model, the probability of a particle jumping to either direction is the same. Only the occupation times at sites vary.

5.2 Boundary driven zero-range process in random media

As in Article [III] of this Thesis, we now study the motion of zero-range interacting particles in a one-dimensional conductor with random symmetric rates, *i.e.* a random barrier model. The situation is as depicted in figure 2.1 with the one-particle transition probabilities in the bulk given by the formulas $p(i, i+1) = v_i/(v_{i-1} + v_i)$ and $p(i, i-1) = 1-p(i, i+1) = v_{i-1}/(v_{i-1}+v_i)$ for $i = 2, \dots n-1$, where the sequence $(v_i)_{i=1}^{n-1}$ consists of independent and identically distributed, positive and bounded random variables. Thus the transition rate in the bulk equals $v_i g(\eta_i)$ for the jumps from i to $i+1$ and $v_i g(\eta_{i+1})$ from $i+1$ to i, where η_i is the number of particles at site i. The probability and expectations for the rate variables will be denoted by the blackboard symbols $\mathbb{P}$ and $\mathbb{E}$, whereas the usual P and E are reserved for the particle system in a fixed environment.

The disordered conductor is next placed in between two reservoirs and the contacts at the ends are carefully prepared, so we set $p(1, \Gamma) = 1/(1 + v_1)$ and $p(n, \Gamma) = 1/(v_{n-1}+1)$ at the boundaries. The rates of input of particles from the reservoirs are controlled by the reservoir fugacities, which are fixed with values v and u for the left and right reservoirs respectively. In the notation of chapter 2, this is achieved by taking $p(\Gamma, 1)\Psi = v$ and $p(\Gamma, n)\Psi = u$. We assume that $\max\{u, v\} < \liminf_k g(k)$ to make sure that a stationary distribution exists.

5.2.1 Stationary density profiles

By theorem 2.1.1, the stationary distribution for a fixed (deterministic) environment $(v_i)_{i=1}^n$ reads

$$\nu(\eta) = \prod_{i=1}^{n} \frac{1}{\mathcal{Z}_i(\phi_i)} \frac{\phi_i^{\eta_i}}{g_i!(\eta_i)}, \tag{5.1}$$

where the fugacities are given by

$$\phi_i = u\frac{s_i}{s_{n+1}} + v\left(1 - \frac{s_i}{s_{n+1}}\right), \qquad s_i = 1 + \sum_{j=1}^{i-1} v_j^{-1}. \tag{5.2}$$

Thus the fugacity profile is trivial in equilibrium, that is when $u = v$, and a monotone function bounded by the reservoir fugacities otherwise. Its increments are determined by the the inverse rates v_i^{-1}, which can be interpreted as resistances: formula (2.14) tells that the mean current of particles from the left reservoir to the

right is $c = N[0,1] = (v-u)/s_{n+1}$, where the denominator is just the total resistance for $n+1$ resistors in series.

The object of our study is to quantify the large scale structure of the stationary state for random transition rates. To this end, we introduce a scaled continuum analogue for s_i by

$$S_{n,\alpha}(x) = n^{-1/\alpha}\Big(1 + \sum_{i=1}^{\lfloor (n+1)x\rfloor - 1} v_i^{-1}\Big), \qquad x \in [0,1], \tag{5.3}$$

so that a suitably rescaled analogue of the fugacity is

$$\phi_{n,\alpha}(x) := u\frac{S_{n,\alpha}(x)}{S_{n,\alpha}(1)} + v\Big(1 - \frac{S_{n,\alpha}(x)}{S_{n,\alpha}(1)}\Big). \tag{5.4}$$

The parameter α provides some freedom in balancing the rate of growth of the sums involved. In fact, we can take $\alpha = 1$ for all models such that the expected resistances are finite, $\mathbb{E}v_i^{-1} < \infty$, because in this case $S_{n,1}(x)$ converges to $x\mathbb{E}v_i^{-1}$ almost surely as $n \to \infty$ by the law of large numbers. Hence $\phi_{n,1}(x) \to ux + v(1-x)$, and the convergence is uniform in x because the functions $\phi_{n,1}(x)$ are monotone and the limiting function is continuous on $[0,1]$. The convergence of the fugacity profile and the continuity of the map $R(\phi) = \phi\mathcal{Z}'(\phi)/\mathcal{Z}(\phi)$ then imply that the limiting particle density at $x \in [0,1]$ almost surely equals $R(ux + v(1-x))$ and the convergence is again uniform–a fact that will be needed when we study the motion of a tagged particle.

For $\mathbb{E}v_i^{-1} = \infty$, we are forced to take $\alpha < 1$ because the sum in (5.3) necessarily grows faster than the number of summands. We focus on a model with the resistances v_i^{-1} belonging to the domain of attraction of an α-stable law [28] with $\alpha \in (0,1)$, which means that there are sequences of constants such that the normalized sum $(\sum_1^n v_i^{-1} - a_n)/b_n$ converges weakly to an α-stable random variable, whose Laplace transform equals $e^{-c\lambda^\alpha}$. A necessary and sufficient condition for the convergence is that $\mathbb{P}(v_i^{-1} > r) = (c + o(1))\, h(r) r^{-\alpha}$ with a slowly varying h: $\lim_{r\to\infty} h(rt)/h(r) = 1$ for all positive t [28]. Then also the normalizing constant b_n is necessarily of the form $b_n = n^{1/\alpha}h(n)$.

A convergence result on the process level, similar to the uniform convergence in the case of finite mean resistances, will be needed in the next section for the fugacities for $\mathbb{E}v_i^{-1} = \infty$. However, the topology of uniform convergence is not suited for this purpose. The limiting paths are necessarily discontinuous because the largest of the summands v_i^{-1} is of the same order as the whole sum (5.3). In fact, Darling [76] has given proof that, as $n \to \infty$, $\sum_1^n v_i^{-1} / \max_{1\le i\le n} v_i^{-1}$ converges in distribution to a non-degenerate random variable with the characteristic function $\varphi(t) = e^{it}\left(1 - \alpha\int_0^1 (e^{ity} - 1) y^{-\alpha-1}\,\mathrm{d}y\right)^{-1}$, see [77] for a more detailed analysis. To

obtain convergence in the uniform topology, the positions of the discontinuities of the limiting paths should be known in advance, which is impossible since, due to large fluctuations of the bond conductivities, some terms are necessarily negligible in the above sums and the summands do not appear in any ranked order. The correct topology for the study of such discontinuous functions was introduced by Skorohod [78, 79]. By theorem 2.7 of reference [79], the process $S_{n,\alpha}(x)$, $x \in [0,1]$, converges to what is nowadays known as a strictly α-stable subordinator, which is a self-similar increasing Lévy process [80–82]. It has a representation

$$S(x) = \iint_{\mathbb{R}_+ \times [0,x]} u \, J(\mathrm{d}u \, \mathrm{d}y), \tag{5.5}$$

where the Poisson random measure J with the intensity measure $\nu(\mathrm{d}u \, \mathrm{d}y) = cu^{-\alpha-1} \, \mathrm{d}u \, \mathrm{d}y$ counts the jumps within $\mathrm{d}u \, \mathrm{d}y$.

By the results of Skorohod and a standard theorem of weak convergence (theorem 5.1 of reference [33]), the fugacity process $\phi_{n,\alpha}$ converges as well, and the limit is

$$\phi(x) = u\frac{S(x)}{S(1)} + v\Big(1 - \frac{S(x)}{S(1)}\Big). \tag{5.6}$$

Thus the limiting fugacity is determined by an α-stable subordinator normalized by its own value at 1. Such processes appear quite commonly in statistics literature, especially as priors for non-parametric inference (*e.g.* [83] and references therein), but also in partitioning problems [84–86] with applications to the excursion lengths of Bessel processes [87].

In our model, the jumps of the normalized subordinator describe the sizes and locations of bottlenecks in the conductor: The closer the stability parameter α is to zero, the more pronounced is the effect of the most weakly conducting bonds, and the density profile is more likely to be segregated into regions of nearly constant density between a few bonds, see figure 5.1 for typical realizations. The extreme limit would be $\alpha = 0$, in which case the sums are completely dominated by a single term [76], and the system is split into two parts. The order statistics for the jump sizes of a normalized subordinator with $0 < \alpha < 1$ has been studied by Perman [88].

The macroscopic scale fugacity process has *exchangeable* rather than independent increments, as is the case for the subordinator S. The starting and end points of the process are fixed by the reservoir fugacities, so a large increment in the fugacity at some point inevitably reduces the jump sizes elsewhere. This heuristic can be brought to more quantitative grounds by the following calculation. Let $(x_i)_{i=1}^k$, $(y_i)_{i=1}^k$, be two sequences of points such that $0 \leq x_1$, $x_k + y_k \leq 1$ and $x_i < x_i + y_i \leq x_{i+1}$ for $i = 1, \ldots, k-1$, and let $\mathcal{F}_x$ denote the natural filtration of the subordinator S. Let also $\Psi(\lambda) = c\lambda^\alpha$ denote its Laplace exponent at $x = 1$.

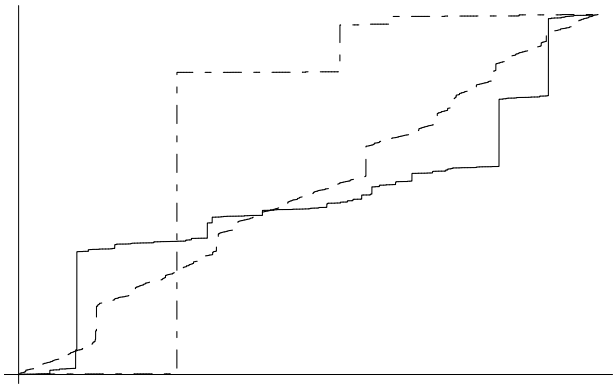

Figure 5.1: Typical fugacity profiles for $\alpha = 0.9$ (dashed line), 0.6 (solid line) and 0.3 (dash-dotted line)

Then we get by using the integral representation of the Γ-function and by successively conditioning on $\mathcal{F}_{x_k+y_k}, \mathcal{F}_{x_k}, \ldots, \mathcal{F}_{x_1}$ that

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{k}\frac{S(x_i+y_i)-S(x_i)}{S(1)} &= \mathbb{E}\left(\prod_{i=1}^{k}\left(S(x_i+y_i)-S(x_i)\right)\right)\int_0^\infty \frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda S(1)}\,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{\lambda^{k-1}}{(k-1)!}e^{-x_1\Psi(\lambda)}\left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}e^{-y_1\Psi(\lambda)}\right)e^{-(x_2-x_1-y_1)\Psi(\lambda)}\left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}e^{-y_2\Psi(\lambda)}\right) \\
&\qquad\qquad \cdots\left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}e^{-y_k\Psi(\lambda)}\right)e^{-(1-x_k-y_k)\Psi(\lambda)}\,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{\lambda^{k-1}}{(k-1)!}c^k\alpha^k\lambda^{(\alpha-1)k}e^{-c\lambda^\alpha}\,\mathrm{d}\lambda = \alpha^{-1}\prod_{i=1}^{k}\alpha y_i. \qquad (5.7)
\end{aligned}$$

Thus

$$\mathbb{E}\prod_{i=1}^{k}\frac{S(x_i+y_i)-S(x_i)}{S(1)} - \prod_{i=1}^{k}\mathbb{E}\frac{S(x_i+y_i)-S(x_i)}{S(1)} = \left(\alpha^{k-1}-1\right)\prod_{i=1}^{k}y_i \qquad (5.8)$$

and the increments are negatively correlated as already anticipated. Notice also the lack of dependence on the sequence (x_i), which means exchangeability.

In Article [III] of this thesis also the moments of the fugacity were calculated using similar techniques as above and the relation to the theory of exchangeable random partitions was briefly discussed. Moreover, the distribution function of the fugacity at a given point was provided,

$$\mathbb{P}(\phi(x)\le t) = \frac{1}{2} - \frac{\operatorname{sgn}(u-v)}{\pi\alpha}\arctan\left(\frac{x|u-t|^\alpha-(1-x)|t-v|^\alpha}{x|u-t|^\alpha+(1-x)|t-v|^\alpha}\tan\frac{\pi\alpha}{2}\right), \qquad (5.9)$$

to which there is a density

$$f(t) = \frac{\sin \pi\alpha}{\pi} \frac{(u-v)x(1-x)\lambda^{\alpha-1}\xi^{\alpha-1}}{(x\lambda^\alpha + (1-x)\xi^\alpha)^2 \cos^2 \frac{\pi\alpha}{2} + (x\lambda^\alpha - (1-x)\xi^\alpha)^2 \sin^2 \frac{\pi\alpha}{2}} \tag{5.10}$$

with $\lambda = t - v$ and $\xi = u - t$. There is a close resemblance to the generalized arcsine distribution. The derivation of these formulas was based on a result of Regazzini, Lijoi and Prünster [83] for the distributions of functions integrated with respect to normalized random measures, which themselves are derived from processes with independent increments. Their analysis was based on a specific Fourier inversion due to Gurland [89]. The machinery of reference [83] was rather heavy for our purposes, since the function integrated with respect to the random measure was just an indicator function, and I think prior formulas for this simple case must exist. However, those are not easily found in the literature, and Regazzini *et al.* also consider the α-stable case as an example, giving a nice compact formula for the distribution function. Examples of density (5.10) for $x = 1/2$ are given in figure 5.2 below. Notice the divergence of the density at the boundaries $t \to u, v$ and the formation of a peak at the expected value $\mathbb{E}\phi(x) = v + (u-v)x$ for high values of α.

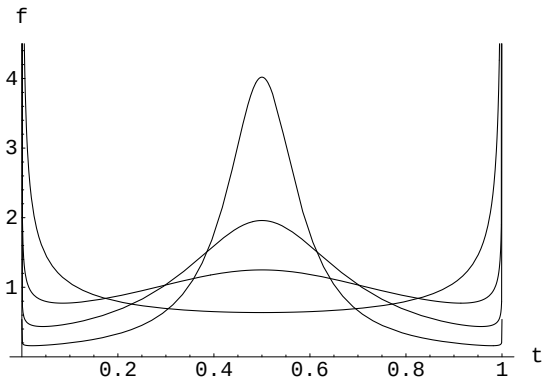

Figure 5.2: The probability density of eq. (5.10) for the fugacity $\phi(1/2)$ in the middle of the system. The reservoir fugacities are $u = 1$ and $v = 0$, and the values of α are $0.9, 0.8, 0.7$ and 0.5 for the curves in decreasing order at $t = 1/2$.

5.2.2 Dynamics of a tagged particle

There are a few studies on the movement of a tagged (marked, tracer) particle in zero-range processes with fixed rates. Saada [90] has proved a law of large numbers and a central limit theorem for the tagged particle position in a model with symmetric transition probabilities and interaction $g(k) = 1$. Later Siri [91] provided a full invariance principle (convergence of the path measures) for a class of increasing interaction functions, which were also allowed to have Lipschitz-type spatial variations. Recently Sethuraman [92] has considered asymmetric dynamics.

The motion of a tagged particle in our model of boundary driven zero-range process in random media is affected on one hand by the interaction, slowing down or speeding up the movement, and by the random symmetric environment, hindering the transport, on the other. There is also interplay of these effects: The structure of the media determines the fugacity profile, which shows up as spatial variations in the speed of the particle due to the interaction.

Given rates

In Article [III] of this thesis, the tagged particle dynamics in stationary boundary driven zero-range processes was investigated first in finite systems with deterministic rates, and those results were then applied to the motion in random media. The particular problem considered was the evaluation of integrals of the type

$$E \int_0^T f(Y_t)\,\mathrm{d}t,$$

where $f(Y_t)$ is a real-valued function of the tagged particle position at time $t \geq 0$ and T is some travel time (*e.g.* the time spent in a set $J \in [n] = \{1, \dots n\}$ before the time of exit of the particle from the system through the right boundary) measurable with respect to the filtration of an auxiliary process that carries the information on the route traversing status of the particles for a set of chosen particle routes. For the construction of this process in a queueing theoretic context, see reference [93]. The expectation E is with respect to a Palm probability of this auxiliary process, and it is conditioned on the starting point of the particle at $t = 0$ and on event that the trajectory of the particle belongs to one of the chosen routes. The particle system is also conditioned on being in its stationary state at $t = 0$.

Three types of route sets were considered in Article [III] of this thesis: Conditioning on the event that the particle starts a journey from some site a (including the reservoirs) at $t = 0$ and eventually arrives at some b without returning to its starting point was denoted by $E^{a\to b}$. The opposite case of returning back to a without visiting b was associated with $E^{a\leftarrow b}$. Finally, E^a was used for conditioning on the starting point a only.

Let now T_J be the exit time of the tagged particle from a set $J = \{a+1, \dots b-1\}$, which is a stopping time of the auxiliary process. The following results concerning the motion in deterministic environments were proved in Article [III] of the thesis. Let $s(x) = 1 + \sum_{i=1}^{\lfloor x \rfloor - 1} v_i^{-1}$ denote the continuum version of the sum s_i in equation (5.2), and let *Green's function* be given by

$$G_J(x,y) = \frac{[s(x \wedge y) - s(a)][s(b) - s(x \vee y)]}{s(b) - s(a)}\, I\{x, y \in (a,b)\}, \tag{5.11}$$

and the *speed measure* by

$$m_n(\mathrm{d}x) = \sum_{i=1}^{n} \frac{\mathcal{Z}'(\phi_i)}{\mathcal{Z}(\phi_i)}\, \delta_i(\mathrm{d}x), \tag{5.12}$$

where ϕ_i is the fugacity of equation (5.2) and δ_i is the Dirac measure at i. Then the following equalities hold:

$$E^{a\to b}\int_0^{T_J} f(Y_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(y)G_J(y,y)\,m_n(\mathrm{d}y) \tag{5.13}$$

$$E^{a\leftarrow b}\int_0^{T_J} f(Y_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(y)\frac{v_a(s(b)-s(y))^2}{(s(b)-s(a))(s(b)-s(a+1))}\,m_n(\mathrm{d}y) \tag{5.14}$$

$$E^{i}\int_0^{T_J} f(Y_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(y)G_J(i,y)\,m_n(\mathrm{d}y). \tag{5.15}$$

The proof relied on the generalized Little laws of Kook and Serfozo [93]. For example, with $T_J(i)$ denoting the time spent at $i \in J$ before the exit time T_J, we get for the first case that

$$\begin{aligned} E^{a\to b}\int_0^{T_J} f(Y_t)\,\mathrm{d}t &= E^{a\to b}\sum_{i=a+1}^{b-1} f(i)T_J(i) \\ &= \sum_{i=a+1}^{b-1} f(i)E^{a\to b}T_J(i) = \sum_{i=a+1}^{b-1} f(i)p_i(a\to b)\frac{EX_i}{EN_{a\to b}(0,1]}. \end{aligned} \tag{5.16}$$

The last equality is a consequence of the Little law [94, 95], relating the expected occupation time to the probability $p_i(a \to b)$ that the particle at i is making the journey from a to b, to the expected numbers of particles EX_i along the route, and to the mean number of particles $EN_{a\to b}(0,1]$ that start the particular journey in a time window of unit length. The probability $p_i(a \to b)$ was further split into a product of two factors: the probability of a particle at i reaching b according to the transition probabilities of the zero-range process and the probability of reaching a when it moves as determined by the adjoint generator L^* given in Proposition 2.1.1. Solving simple recursions for these probabilities then allowed one to carry out the proof of the first part. The second assertion is proved in the same manner.

The two results involving the paths within a and b were used as lemmas for proving the third statement: The time spent in $j \in J$ given that the travel started from $i \in J$ can be partitioned into occupation times in j during the 'bridges' that are travels towards j and return to i without leaving J, and the final journey from i to the boundary of J.

Random environment

The convergence results for the stationary fugacity profiles can now be combined with the formulas for the expected travel functionals of the preceding section. As usual, the movement of the tagged particle must be observed on a suitable temporal scale as the spatial resolution is taken to infinity. To this end, we set $J_n := \{\lfloor an \rfloor + 1, \dots, \lfloor bn \rfloor - 1\}$, where $0 \le a < b \le 1$, and observe the motion in the macroscopic coordinates $y \in (a, b)$ in the limit $n \to \infty$. The motion in this macroscopic limit is in the case of weak disorder, *i.e.* $\mathbb{E}v_i^{-1} < \infty$, expected to be nontrivial under diffusive scaling $n^{-1}Y_{n^2t}$. Indeed, denoting the exit time of the speeded up process Y_{n^2t} from J_n by $T^{(2)}_{J_n}$, we get that for almost all environments

$$\begin{aligned} &E^{\lfloor xn \rfloor} \int_0^{T^{(2)}_{J_n}} f\left(\frac{1}{n} Y_{n^2t}\right) \,\mathrm{d}t = n^{-2} \int_{-\infty}^{\infty} f(y/n) G_{J_n}(\lfloor xn \rfloor, y)\, m_n(\mathrm{d}y) \\ &= \frac{1}{n} \sum_{i=\lfloor an \rfloor + 1}^{\lfloor bn \rfloor - 1} f\left(\frac{i}{n}\right) \frac{[s(\lfloor xn \rfloor \wedge i) - s(\lfloor an \rfloor)][s(\lfloor bn \rfloor) - s(\lfloor xn \rfloor \vee i)]}{n[s(\lfloor bn \rfloor) - s(\lfloor an \rfloor)]} \cdot \frac{\mathcal{Z}'(\phi_i)}{\mathcal{Z}(\phi_i)} \\ &\longrightarrow \frac{1}{u - v} \int_a^b f(y) \frac{(y \wedge x - a)(b - y \vee x)}{b - a}\, m(\mathrm{d}y) \end{aligned} \tag{5.17}$$

as $n \to \infty$. Here the limiting speed measure is $m(\mathrm{d}y) = [\mathcal{Z}'(uy + v(1-y))/\mathcal{Z}(uy + v(1-y))]\,\mathrm{d}y$. The calculation suggests that, on the level of these travel functionals, the motion on the hydrodynamic scale coincides with a diffusion on natural scale with the speed measure m and absorbing boundaries at 0 and 1. See for example reference [96] for the theory of one-dimensional diffusions. The more complete statement for continuous, bounded real functions on the path space, with proper compactness proofs, would be called an *invariance principle*.

For our α-stable model with $0 < \alpha < 1$, percolation effects start to show and the transport is significantly slowed down. Therefore a subdiffusive scaling is needed for the travel functionals, *cf.* (5.3), which suggests that the process should be speeded up by $n^{1+1/\alpha}$. It was argued in Article [III] of this thesis that the mappings involved in constructing the travel functionals from $S_{n,\alpha}$ are almost surely continuous in the Skorohod topology, which implies that

$$E^{\lfloor xn \rfloor} \int_0^{T^{(1+1/\alpha)}_{J_n}} f\left(\frac{1}{n} Y_{n^{1+1/\alpha}t}\right) \,\mathrm{d}t \longrightarrow \int_a^b f(y) \tilde{G}_J(x, y)\, \tilde{m}(\mathrm{d}y) \tag{5.18}$$

in distributional sense. This time the scale function, defined by the relation $P(b \text{ hit before } a) = [\sigma(x) - \sigma(a)]/[\sigma(b) - \sigma(a)]$, of the limiting diffusion equals the subordinator S, and the Green's function reads

$$\tilde{G}_J(x, y) = \frac{[S(x \wedge y) - S(a)][S(b) - S(y \vee x)]}{S(b) - S(a)}. \tag{5.19}$$

The speed measure of the process is

$$\tilde{m}(dy) = \frac{\mathcal{Z}'(\phi(y))}{\mathcal{Z}(\phi(y))}\mathrm{d}y, \tag{5.20}$$

where the function ϕ is as in equation (5.6).

The same type of arguments were given for the measures associated with asymmetric paths $a \to b$ in Article [III] of the thesis, and it was shown that the above formulas (5.17) and (5.18) hold with a slight modifications, namely that x should be substituted with y in Green's functions. An example of a mean travel time from reservoir to reservoir, averaged over environments, was provided.

6 Conclusions and outlook

Zero-range processes on finite sets with three applications were considered in this thesis. The first one concerned the extremal statistics of domain sizes in driven one-dimensional systems. The most notable findings were that, even though the correlations in the subcritical systems were not strong enough to alter the predictions from systems of independent clusters, the discreteness of the state space caused deviations from the classical limiting distributions of extreme order statistics. Although the persistent fluctuations of the variance, as well as the other symptoms of discreteness, must be well known in statistics, I am not aware of any reports of these phenomena in addition to Article [I] of this thesis and M. Bazant's account on percolation clusters [42] in physics literature. A less studied area is the statistics of extremes, especially the limit laws of suitably normalized variables, in more strongly interacting systems.

As the second application, an ensemble of diluted uncorrelated networks induced by zero-range invariant measures was studied. The mathematical goals originally set in Article [II] of this thesis were not all met but this didn't prevent drawing conclusions on the structure of the graphs: The calculations were performed using the grand-canonical ensemble and checked against numerical simulations in the canonical ensemble. Some general features of the phase diagrams were discovered, and a specific example with power-law distributed degrees at the zero-range point of condensation was studied in detail. The study of complex networks has expanded in recent years, and the physics of uncorrelated graphs is thoroughly covered in the literature. However, one of the motivations in introducing a zero-range process based model was the inclusion of dynamics, and there is still plenty to do in the dynamics of graph-valued processes.

Boundary-driven zero-range processes in random media were studied in Article [III] of this Thesis. The stationary fugacity profiles were characterized and the motion of a tagged particle was analysed using queueing theoretic arguments. Although the results for the tagged particle were not mathematically conclusive, they strongly suggest a diffusion description. A complete proof would be desirable.

A related, theoretically more challenging system would be the boundary driven symmetric simple exclusion process with particlewise disorder. For α-stable rates, the stationary density profiles on macroscopic scales are again expected to be jump processes but the correlations are much stronger than in the case of a zero-range

process. This can be seen already from the matrix product solution for homogeneous rates [97] and from the fact that the large-deviation functional for stationary density profiles is non-local, as opposed to zero-range processes [14]. The time correlations of the macroscopic density would also be of interest, and the tagged particle movement is expected to be anomalously slow.

Another problem that I have been recently keen on is the Bayesian inference of the random media on basis of marker particle data. It is most easily explained for a gas of noninteracting particles under boundary drive. The idea is to dilute the gas at the other reservoir with a marker, and to approximate the structure of the media by its continuum limit in a way that the number of marker particles is finite. This is the limit of a very dilute mixture of the marker and the gas, and is in quantitative terms obtained by a $1/n$-scaling of the marker density, where n denotes the system size. Parametric and non-parametric questions can be asked: Given a vector $x^{(k)} = (x_1, \ldots, x_k)$ of marker positions in a stationary flow, what is the most likely value of the stability parameter α (*cf.* section 5.2.1), and what is the best estimate for the random Poisson measure that determines the barrier structure, if one suspects that the α-stable model correctly describes the problem. The first of these requires the estimation of the Janossy densities [98], essentially given by

$$\mathbb{E} \exp\left(- \int_0^1 \frac{S(x)}{S(1)} \, \mathrm{d}x \right) \prod_{i=1}^{k} \frac{S(x_i)}{S(1)}$$

with S denoting a stable subordinator, while the other can be attacked by the methods described in reference [99]. Notice that the difference in the marker density between the reservoirs is essential in the problem. With equal densities, the stationary profiles are uniform and no inference is possible.

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