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Maria Dementieva

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ABSTRACT

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This thesis deals with solutions of cooperative games. We describe the properties of the subcore and grand subcore. The main part of the work is dedicated to the time-consistency problem of a multistage cooperative games solution. This problem is closely connected with the imputation distribution procedure, which allocates the common benefit among the players at every step of the game. The reduced game property (or consistency) is also considered with respect to the modified Davis-Maschler-reduced game, and the property of dynamic consistency is introduced and investigated. We use a new approach to the time-consistency problem. The problem of minimal reduction is stated and we apply the results to regularization of the cooperative dynamic game in the case of no time-consistent imputations from the core in the balanced multistage cooperative game with transferable utility. At the end of the work a multistage cooperative model of the Kyoto Protocol realization is constructed and corresponding imputation distribution procedures are suggested for the game.

Keywords: dynamic games, cooperative games, core, subcore, time-consistency, reduced games, minimal reduction problem, Kyoto protocol.

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BASIC TERMS AND ACRONYMS

Balanced cooperative game	Page 17
Characteristic function $v(\cdot)$	Definition 1.11
Coalition acceptable for reduction	Definition 4.7
Conditionally minimal coalition	Definition 4.1
Consistent solution	Definition 3.1
Convex cooperative game	Definition 3.6
Cooperative differential game	Page 22
Cooperative game with transferable utility	Definition 1.11
Core $C(\cdot)$	Definitions 1.16 and 2.2
Davis $-$ Maschler reduced game	Page 40
Dynamically consistent solution	Definition 3.11
Feasible payoff vector	Definition 1.15
Grand subcore $GSC(\cdot)$	Definitions 2.4 and 2.20
Imputation	Definitions 1.15 and 2.1
Imputation distribution procedure (IDP)	Definitions 2.9 and 4.16
Minimal coalition	Definition 4.11
Mixed strategy Nash equilibrium	Page 15
Modified Davis $-$ Maschler reduced game	Page 40
Nash equilibrium	Definition 1.3
Reduced game	Page 39
$S-$feasible payoff vector	Definition 1.15
Shapley value $Sh(\cdot)$	Page 18
Strategic game	Definition 1.2
Strictly competitive (zero $-$ sum) game	Definition 1.7
Subcore $SC(\cdot)$	Definitions 2.3 and 2.19
Superadditive game	Definition 1.12
Time $-$ consistent grand subcore $TCGSC(\cdot)$	Definitions 2.5 and 2.21
Time $-$ consistent solution	Definition 2.7

CONTENTS

LIST OF FIGURES

LIST OF TABLES

PREFACE

The main topic of the thesis is time-consistency of the subcore in multistage cooperative games and the problem of regularization of time-inconsistent solutions. To provide time-consistency and construct the corresponding imputation distribution procedures we use two different algorithms based on delays of total payoffs and a new approach connected with a reduced game. Regularization algorithms are implied to realization of the Kyoto Protocol flexibility mechanisms.

The work consists of the following five chapters. The first one basically consists of the compiled materials and includes preliminary information on the game theory and its applications [73, 75, 74, 53]. The second chapter is devoted to the problem of time-consistency and contains the necessary and sufficient conditions of the subcore imputations' time-consistency and algorithms corresponding to the classical approach to the regularization problem [128, 129]. The third chapter is called "Consistency of the subcore" and encloses auxiliary results about modified Davis and Maschler reduced game [130], which are used in the next chapter. In the fourth chapter the problem of minimal reduction is formulated and applied to the regularization of multistage cooperative games [27]. The last chapter envelops of game-theoretical modelling of the Kyoto Protocol and applications of the previous results to this model [28].

1 BASIC IN GAME THEORY

1.1 Introduction

Game theory is a set of mathematical tools for understand the nature of conflict and its management. Since the birth of the game theory in 1928 [72] a lot of monographs have been written in different directions of the theory (see *e.g.* [13, 20, 22, 29, 39, 50, 54, 61, 66, 73, 79, 81, 85, 86, 95, 119]), and the results of investigations are used in a number of fields (see [16, 25, 26, 28, 29, 34, 35, 42, 5, 6, 52, 53, 55, 59, 65, 68, 88, 112]) (*e.g.* macro- and microeconomics, policy, environment, industry, energy, demography, psychology, *etc.*).

In fact, any dispute or a set at variance could be modelled as a system $< Players, players'\ feasible\ Behaviours, players'\ Utility\ function\ depending\ on\ the\ behaviours >$. Almost every model assumes that the players (or Decision Makers (DMs)) are rational and they take into account information about other DMs' actions. That is, a player should make a decision. A decision problem is the problem of choosing among a set of alternative behaviours to increase (an individual or common) outcome. The outcome (or utility) is a function of all players' choices and define the preferences of the player.

According to the possibility of players' co-operation, to the available information for every player, to the time or the number of repetitions, to a game representation there are several classifications of games.

Let us assume that there is a set of at least two players. The game begins when one or more players make a choice among a number of specified alternatives (strategies). In the case of the first player is a leader and he starts earlier than others, the game is called hierarchical. The choice made by the players may or may not become known to the other players. Game in which all the choices of all players are known to everyone as soon as they are made is called a game with perfect information. Otherwise it is a game with incomplete information or under uncertainty. If the players can act together (and form coalitions) in order to get larger outcome the game is called cooperative, and the problem is not to choose the best strategies but to allocate the common payoff fairly. Most of real-life conflicts are dynamic, *i.e.* the strategies and the utility functions are the functions of time. If time is continuous then the game is called differential; if time is discrete then the game is called multistage.

1.2 Strategic games

A strategic game, or a game in normal form, is a model of interactive decision making in which every DM chooses his plan of action once, and these choices are made simultaneously [73, 75]. The model consists of a finite set of players N and, for each player i, a set A_i of actions (strategies), and a set of utility functions $u_i(\cdot)$ over $A = \prod_{i \in N} A_i$. The set A is called the set of outcomes and an element $a \in A$ is a situation.

Example 1.1 (Cross-road game). *Let us consider two-person game with $N = \{1, 2\}$, where 1 and 2 are drivers. The ways of the drivers intersect on a tantamount cross-road. The sets of strategies are the same for every player: driver can prolong his motion (Go) or let pass to another one (Wait). The values of utility functions are the following*

$$\begin{array}{ccc} & Go & Wait \\ Go & -1,-1 & 1, 1-\varepsilon \\ Wait & 1-\varepsilon, 1 & 0, 0 \end{array} \tag{1.1}$$

In this table the first column contains the strategies of the first player, and the first row contains the strategies of the second player. Let us assume that driver 1 chooses "Go" and driver 2 chooses "Wait", then (Go, Wait) is a situation and $(1, 1-\varepsilon)$ is the corresponding utility vector. Variable ε is a measure of dissatisfaction of waiting.

The requirement that the utility function of every player i be defined over A, rather than A_i, is the feature that distinguishes a strategic game from a decision problem: each player may care not only about his own action but also about the actions taken by the other players. To summarize, the definition is the following.

Definition 1.2. *A strategic game is a triple $< N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} >$, where N is a finite set of players, A_i is a nonempty set of actions, and u_i is an utility function of the player $i \in N$ over $A = \prod_{i \in N} A_i$. If the strategy set A_i is finite for every player $i \in N$ then the game is called finite.*

The high level of abstraction of this model allows it to be applied to a wide variety of situations. A player may be an individual human being or any other DM entity like a government, a board of directors, or even a flower or an animal. The model places no restrictions on the set of actions available to a player, which may contain just a few elements or be a huge set containing complicated plans that cover a variety of contingencies. However, the range of applications of the model is limited by a preference relation (or utility function) associated to each player.

A common interpretation of a strategic game is that it is a model of an event that occurs only once; each player knows the details of the game and the fact that all the players are "rational", and the players choose their strategy simultaneously and independently. Another interpretation is that a player can form his expectation of the other players' behaviour on the basis of information about the way that the game or a similar game was played in the past [74].

The most used solution concept in game theory is Nash equilibrium [71].

Definition 1.3. *A Nash equilibrium of a game* $< N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} >$ *is a situation* $a^* \in A$ *such that for every player* $i \in N$ *we have*

$$u_i(a^*_{N \setminus \{i\}}, a^*_i) \geq u_i(a^*_{N \setminus \{i\}}, a_i) \quad for\ all\ a_i \in A_i. \tag{1.2}$$

Thus for a^* to be a Nash equilibrium it must be that no player $i \in N$ can profitably deviate off a^*_i, given the actions of the other players $a^*_{N \setminus \{i\}}$.

The following classical games represent a variety of strategic cases. As before In this table the first column contains the strategies of the first player, and the first row contains the strategies of the second player.

Example 1.4 (The battle of the sexes). *Woman and her husband wish to go out together either to a concert of classical music or to a football match [75]. Their main concern is to go out together but he prefers football and she prefers music. Representing the individuals' preferences by utility function as following*

	Music	*Football*
Music	2, 1	0, 0
Football	0, 0	1, 2

(1.3)

It models a case when players wish to coordinate their behaviour but have conflicting interests. The game has two Nash equilibria $(Music, Music)$ *and* $(Football, Football)$.

Example 1.5 (The prisoner's dilemma). *Two suspects in a crime are put into separate cells [53]. If they both confess, each will be sentenced to three years in prison. If only one of them confesses, he will be freed and used as a witness against the other, who will receive a sentence of four years. If neither of them confesses, they will both be convicted of a minor offense and spend one year in prison. Choosing a convenient payoff representation we have the following game*

	Don't confess	*Confess*
Don't confess	1, 1	4, 0
Confess	0, 4	3, 3

(1.4)

This is a game in which there are gains from cooperation (the best outcome for the players is that neither confesses) but each player has an temptation to be a "free rider". Whatever one player does, the other prefers $Confess$ *to* $Don't\ confess$ *so that the game has a unique Nash equilibrium* $(Confess,\ Confess)$.

Example 1.6 (Matching euros). *Each of two players chooses either Head or Tail [74]. If the choices differ, person 1 pays person 2 one euro; if they are the same person 2 pays person 1 one euro. Each person cares only about the money that he receives. A game that models this situation is the following*

	Head	*Tail*
Head	$1,-1$	$-1,1$
Tail	$-1,1$	$1,-1$

(1.5)

This game does not have a Nash equilibrium. Such a game in which the interests of the players are diametrically opposed is called "strictly competitive".

We can see that not every strategic game has a Nash equilibrium point. Only in limited classes of games we can say something about the qualitative character of the equilibria. One such class of games is strictly competitive games.

Definition 1.7. *A strategic game* $< \{1,2\}, A_1, A_2, u_1, u_2 >$ *is strictly competitive (or zero-sum) game if for any* $a \in A$ *we have* $u_1(a) = -u_2(a)$.

We denote such games by $< \{1,2\}, A_1, A_2, u_1 >$.

We say that player i maxminimizes if he chooses a strategy that is best for him on the assumption that whatever he does, player j will choose the action to hurt him as much as possible.

Definition 1.8. *Let* $< \{1,2\}, A_1, A_2, u_1 >$ *be a strictly competitive strategic game. The action* $x^* \in A_1$ *is a maxminimizer for player 1 if*

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \quad \text{for all } x \in A_1.$$

Similarly, the action $y^* \in A_2$ *is a maxminimizer for player 2 if*

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \quad \text{for all } y \in A_2.$$

The following result [74] gives the connection between the Nash equilibria of a strictly competitive game and the set of pairs of mixminimizers.

Theorem 1.9. *Let* $G =< \{1,2\}, A_1, A_2, u_1 >$ *be a strictly competitive strategic game.*

(1) *If* (x^*, y^*) *is a Nash equilibrium of* G *then* x^* *is a maxminimizer for player 1 and* y^* *is a maxminimizer for player 2.*

(2) *If* (x^*, y^*) *is a Nash equilibrium of* G *then*

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y),$$

and thus all Nash equilibria of G *yields the same payoffs.*

(3) *If*

$$\max_{x\in A_1}\min_{y\in A_2} u_1(x,y) = \min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$$

(and thus, in particular, if G has a Nash equilibrium), x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2, then (x^*, y^*) is a Nash equilibrium of G.

Proof. See [74, p. 22] □

The hypothesis that the game has a Nash equilibrium is essential in establishing the opposite inequality. For example, consider the game Matching euros, in which $\max_{x\in A_1}\min_{y\in A_2} u_1(x,y) = -1 < \min_{y\in A_2}\max_{x\in A_1} u_1(x,y) = 1$.

If $\max_{x\in A_1}\min_{y\in A_2} u_1(x,y) = \min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$ then this equilibrium payoff of player 1 is called value of the game.

Not every strategic game has an equilibrium point but if we study some kind of a Nash equilibrium in the mixed extension of the strategic game we can guaranty the existence of equilibrium in finite game [71, 74]. The notion of mixed strategy Nash equilibrium is designed to model a "stable state" of a game in which the players' choices are not deterministic but are regulated by probability rules.

Let $G =< N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N} >$ be a strategic game. Then the set ΔA_i of probability distributions [106] over A_i is the set of mixed strategies of player i. Thus the mixed extension of the game G is the strategic game $< N, \{\Delta A_i\}_{i\in N}, \{U_i\}_{i\in N} >$, where U_i assigns to each $\alpha \in \prod_{j\in N} \Delta A_j$ the expected value under u_i of the lottery over A. A mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.

Theorem 1.10. *Every finite strategic game has a mixed strategy Nash equilibrium.*

Proof. The original proof was given by John Nash [71]. □

Detailed comments and interpretations of the mixed strategy Nash equilibrium can be found, for example, in [4, 19, 22, 44, 74, 96, 97]. For the study of different extensions of the notion of the equilibrium, see *e.g.* [3, 22, 64, 65, 66, 67, 71, 74, 109, 110]. Strategic games and their modifications have a number of applications as a tool to analyse and control of the conflict situations, as well as evolution and learning in games, strategic planning, *etc.* (see [16, 42, 49, 53, 59, 66, 73, 88, 103]).

1.3 Cooperative games

Let us imagine that the players in a strategy game made arrangement to play together for the purpose of increasing of the total benefit [53, 73, 79]. In this case the conflict is not in how to get the maximal individual payoff by choosing strategies but to divide the whole gain fairly. And what is the fairness? This leads us to consideration of cooperative games. Here we describe a simple version of a coalitional game (coalition is a subset of player set) in which every group of players is associated with a single number, interpreted as the payoff that is available to the group. There are no restriction on how to allocate this payoff among the members of the coalition. The formal definition of such games is the following.

Definition 1.11. *A cooperative game with transferable utility (or a game in the form of characteristic function, TU-game) is a pair (N, v) where N is a finite set of players and v is a characteristic function that associates with every nonempty subset $S \subseteq N$ (a coalition) a real number $v(S)$ (the worth of S), $v(\emptyset) = 0$.*

In many situations the payoff that a coalition can achieve depends on the actions taken by the other players or coalitions. However, the interpretation of a coalitional game is that it models a situation in which the actions of the players who are not part of S do not influence $v(S)$. In the literature (see [73, 75]) other interpretations are given to a coalitional game; for example, $v(S)$ is sometimes interpreted to be the maximin value to S of the two-person game $< S, N \setminus S, A_S, A_{N\setminus S}, u_S, u_{N\setminus S} >$ played between S and $N \setminus S$, *i.e.* $\max_{x_S \in A_S} \min_{y_{N\setminus S} \in A_{N\setminus S}} u_S(x_S, y_{N\setminus S}) = v(S)$.

Usually, some natural assumptions are made.

Definition 1.12. *A cooperative game (N, v) is called superadditive if*

$$v(S \cup T) \geq v(S) + v(T), \quad \textit{for all } S, T \subset N, \ S \cap T = \emptyset.$$

A cooperative game (N, v) is called cohesive if

$$v(N) \geq \sum_{j=1}^{K} v(S_j) \textit{ for any partition } \{S_1, \dots, S_K\} \textit{ of } N : \bigcup_{j=1}^{K} S_j = N, \ \bigcap_{j=1}^{K} S_j = \emptyset.$$

In other words cooperative behaviour is profitable and the players have a possibility to receive better benefit than without coalition N.

Example 1.13 (A three-player majority game). *Suppose that three players can obtain one unit of payoff together, any two of them can obtain a payoff $\alpha \in [0, 1]$ independently of the actions of the third, and each player alone can obtain nothing, independently of the actions of the remaining two players. We can model this situation as the cooperative game (N, v) where $N = \{1, 2, 3\}$, $v(N) = 1$, $v(S) = \alpha$ whenever $|S| = 2$, and $v(\{i\}) = 0$ for all $i \in N$.*

Example 1.14. [1] *An expedition of n people has discovered treasure in the mountains; each pair of them can carry out one piece. A cooperative game that models such a situation is (N, v) with the following characteristic function*

$$v(S) = \begin{cases} |S|/2 & \textit{if } |S| \textit{ is even} \\ (|S| - 1)/2 & \textit{if } |S| \textit{ is odd.} \end{cases}$$

Definition 1.15. *A vector $x = (x_1, \dots, x_{|S|})$ is called S-feasible payoff vector if $\sum_{i \in S} x_i = v(S)$. A vector $x = (x_1, \dots, x_{|N|})$ is called a feasible payoff vector (or an imputation) if it is N-feasible.*

[1] Example 1.14 is due to Shapley (inspired by the 1948 movie "The Treasure of the Sierra Madre") [74].

A core is a classical solution concept in cooperative game theory. The idea is analogous to a Nash equilibrium of strategic games: an outcome is stable if no deviation is profitable. Originally the definition of the core was given in [40], and published in [41]. Later versions and axiomatization definitions appeared in [14, 70, 76, 77, 78, 105].

Definition 1.16. *The core of the cooperative game (N, v) is a set $C(N, v)$ of feasible payoff vectors x for which there is no coalition S and S-feasible payoff vector y that $y_i > x_i$ for all $i \in S$.*

The following theorem gives us a method of finding the core.

Theorem 1.17. *The core of a game (N, v) is a set of feasible payoff vectors x satisfying*

$$\sum_{i \in S} x_i \geq v(S) \quad \textit{for all } S \subset N.$$

Proof. See [75]. □

The core of the game in Example 1.13 is the set of all nonnegative payoff vectors (ξ_1, ξ_2, ξ_3) for which $\sum_{i \in N} \xi_i = 1$ and $\sum_{i \in S} \xi_i \geq \alpha$ for every two-player coalition S. In the game in Example 1.14 the core consists of the unique payoff vector $(0,5; \ldots ; 0,5)$ if $|N| \geq 4$ is even; the core is empty set if $|N| \geq 3$ is odd.

Another presentation of the core connected with balanced collection of weights [14, 74, 105]. Let 2^N be a set of all coalitions. For any $S \subseteq N$ let $\mathbb{R}^S$ be $|S|$-dimensional Euclidian space in which the dimensions are indexed by the members of S, and $1_S \in \mathbb{R}^N$ be the characteristic vector of S given by

$$(1_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

A collection $(\lambda_S)_{S \in 2^N}$ of numbers in $[0, 1]$ is called a balanced collection of weights if for every player the sum of λ_S over all coalitions containing i is 1. A game (N, v) is called balanced if

$$\sum_{S \in 2^N} \lambda_S v(S) \leq v(N)$$

for every balanced collection of weights.

The following result is referred to as the Bondareva–Shapley [14, 105] theorem.

Theorem 1.18. *A cooperative TU-game has a nonempty core if and only if it is balanced.*

Proof. See [74]. □

All imputations in the core are "good" because of their stability [75]. However, in general, the core is a multivalued solution and it can be an empty set. There are a number of one-point solutions like Shapley value [104], which is a feasible payoff vector based on the "power" of each of the players as reflected in the additional payoff resulting from the joining of this player to the coalitions not including him [102, 104]. Formally the Shapley value of a cooperative TU-game is the following

$$Sh_i(N,v) = \sum_{S \subseteq N} \gamma_n(S) \left[v(S \cup \{i\}) - v(S) \right],$$

where

$$\gamma_n(S) = \frac{|S|!(|N| - |S| - 1)!}{|N|!}$$

is the probability that a player i joins coalition S.

Example 1.19. *Let us consider a three-person game [53] with the following characteristic function:* $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 0.1$, $v(\{1,3\}) = 0.2$, $v(\{2,3\}) = 0.2$, $v(\{1,2,3\}) = v(N) = 1$. *The core of this game is a convex span of six points (see Theorem 1.17)*

$$C(N,v) = Co\{(1,0,0.9),\ (0,1,0.9),\ (0,0.8,0.2),\\ (0.2,0.8,0),\ (0.8,0.2,0),\ (0.8,0,0.2)\}$$

and the Shapley value is

$$Sh(N,v) = \left(\frac{19}{60}, \frac{19}{60}, \frac{22}{60} \right).$$

The application area of classical (static) and dynamic cooperative games is quite wide, see [1, 15, 17, 21, 38, 51, 52, 61, 85, 112].

1.4 Dynamic games

Dynamic (differential or multistage) games model the conflict or cooperative situations in which players choose their strategies over time (continuous or discrete). As before, the payoffs to each player depend on the dynamic strategies by all the players. The basic references for dynamic games are [12, 36, 46, 47, 54, 75, 108].

As well as in strategic, in dynamic games cooperation can be defined. It is also possible to consider zero-sum, or non zero-sum games, non-coalitional dynamic games, dynamic games with perfect information or under uncertainty, *etc.* Solution concepts from strategic and cooperative games are adapted for dynamic games (see [53, 75, 81, 82, 115]).

In the theory of dynamic games there is a problem of time-consistency (dynamic stability) of the solution corresponding to the optimality principles. Time-consistency and regularization of the dynamic cooperative games[2] to provide non-negative payoffs to every player is the main topic of this work.

As an example of the differential game we can consider a pursuit game as the most important from the point of view of both theory and applications (see

[2]For details of cooperation under dynamics see Chapter 2

[33, 48, 49, 53, 54, 81, 92, 99, 108]). Player 1 is the pursuer and player 2 is the evader. The game ends when the pursuer is sufficiently close to the evader, at which point the pursuer is said to “capture” the evader, the “time to capture” is the duration of the game. The goal of the pursuer is to minimize the time to capture, and the evader has the opposite purpose to maximize the time to capture. The evader “escapes” if the pursuer can not come sufficiently close to him, and the time to capture is infinite in this case. This description of the pursuit game is general enough to cover many instances of pursuit and evasion, including such diverse situations as the pursuit of the runner in a football game, or the pursuit of a missile by an anti-missile.

One can find other applications, for example, in [2, 7, 10, 29, 32, 56, 58, 101].

1.5 The main results of the work

The classical TU-cooperative game theory considers and treats many optimality concepts (the core, Shapley value, *etc.*). An important problem in a dynamic cooperative theory is the time-consistency of a solution [80]. As in the theory of non zero-sum differential games [10, 12, 29], the use of optimality principles from the static theory in dynamic TU-games leads to contradictions arising from loss of time-consistency. Time-consistency of the optimality principle means that any segment of an optimal trajectory determines the optimal motion with respect to relevant initial states of the process. This property holds for the overwhelming majority of classical optimal control problems and follows from the Bellman optimality principle [11].

The absence of time-consistency in the optimality principle involves the possibility that the previous “optimal” decision are abandoned at some current moment of time, thereby making meaningless the problem of seeking an optimal control as such. This is why particular emphasis is placed on the construction of time-consistent optimality principles. This problem has attracted much attention [18, 23, 60, 84, 85, 86, 88, 107, 115].

The problem of time-consistency of a solution in a differential TU-cooperative game was investigated for the first time in [80]. It is directly relevant to regularization methods of cooperative games [80, 123, 124]. We suggest constructing time-consistent optimality principles for multistage cooperative games on the basis of “regularization” of optimality principles from the differential cooperative game theory. The idea of regularization is based on constructing delays of the payoffs to the players along optimal trajectory of the game.

In Chapter 2 we study time-consistency of the subcore selectors and propose two imputation distribution procedures that provide non-negative payoffs at every moment of the game. Both algorithms are based on delays of total payoff at a current moment of the game to avoid debtors at the following steps. Theorem 2.10 and its reformulation Theorem 2.22 set necessary and sufficient conditions for the time-consistency of an imputation from the subcore in a multistage cooperative game. The results of this chapter were partially presented in [128, 129].

Chapter 3 has an auxiliary character and the main results, connected with the reduced game property (Theorem 3.10 and Theorem 3.12 [130]), are used in the Chapter 4.

In Chapter 4 we formulate a new problem of minimal reduction, and apply it to the regularization of dynamic TU-games. We apply a reduced game due to Davis and Maschler [24] and a modified Davis-Maschler reduced game (see Chapter 3) to get the appropriate IDPs. This approach we can use even in the case of no time-consistent imputation in the core of a balanced game. The results of this chapter were presented in [27].

In Chapter 5 we describe a cooperative model of relations of countries (or groups of countries) under Kyoto Protocol. The realization of the Kyoto Protocol flexibility mechanisms is a basis of the multistage cooperative game with transferable payoff. And we employ both classical approach (the algorithms from Chapter 2) and minimal reduction from Chapter 4. The results of this chapter were presented in [28].

2 THE PROBLEM OF TIME-CONSISTENCY

2.1 Introduction

In this chapter we treat time-consistency property of the subcore selectors. Several imputation distribution procedures (IDP) can be proposed with regard to the continuous dynamics of the process [118, 128]. Two of them are considered in this work. They provide time-consistency of some imputations from the subcore and non-negativity of the payoffs to the players at each moment of time along the optimal trajectory of a balanced game. It is not always appropriate to use the same IDP when we deal with a multistage TU-cooperative game. Using the proof of the theorem, that formulates necessary and sufficient conditions for the time-consistency of an imputation from the subcore in a multistage cooperative game, we suggest two algorithms for IDP and introduce the notion of a time-consistent grand subcore (TCGSC). The algorithms construct the procedures of nonnegative payoffs to the players in balanced multistage TU-cooperative games.

2.2 Dynamic cooperative games

Consider the differential game $\Gamma = \langle N, \{U_i\}_{i \in N}, \{H_i\}_{i \in N} \rangle$ with the initial state x_0 and duration $T - t_0$ [36, 118]. Here N is a finite set of players, a subset U_i of compact set from $\mathbb{R}^l$ is the strategy set of player $i \in N$, H_i is an utility function of player i. The motion equations are $\dot{x}(t) = f(x, u_1, \dots, u_n)$, where $u_i \in U_i$ denotes the player i's control, $x(t) = (x_1(t), \dots, x_{|N|}(t))$ is state variable, $x(t_0) = x_0$. The payoff function of player i from current time t till the moment T is defined as

$$K_i(x(t), T - t, u_1, \dots, u_n) = \int_t^T H_i(x(\tau))d\tau, \qquad (2.1)$$
$$t \in [t_0, T],\ H_i > 0,\ i \in N,$$

where $x(\tau)$ is the trajectory realized when the $|N|$-tuple u is used and the initial state is $x(t)$.

Players may cooperate in order to achieve a higher payoff. We suppose that before starting the game the players agree to use a combination of controls $\tilde{u}_1, \ldots, \tilde{u}_{|N|}$ such that for every $t \in [t_0, T]$ the corresponding trajectory $\tilde{x}(t)$ maximizes the sum of payoffs, that is,

$$\max \sum_{i=1}^{n} K_i(x(t), T-t, u_1, \ldots, u_n)$$

$$= \sum_{i=1}^{n} K_i(\tilde{x}(t), T-t, \tilde{u}_1, \ldots, \tilde{u}_n). \tag{2.2}$$

Let us denote this sum by $v(N,t)$. The characteristic function $v(S,t)$ ($v : 2^N \times \mathbb{T} \to \mathbb{R}$), $S \subset N$, of the game Γ could be introduced in different ways. In this work the method of construction is less important than the properties of v.

The pair (N, v), where N is a finite set of players, $v : 2^N \times [t_0, T] \to \mathbb{R}$ is a characteristic function, is called cooperative differential game over $t \in [t_0, T]$. A game $(N, v(t))$ is a subgame of (N, v), *i.e.* the differential cooperative game at the period $[t, T]$. In the case when instead of continuous time we study discrete time $\{t_0, \ldots, t_m = T\}$ then the game (N, v) with characteristic function $v : 2^N \times \{t_0, \ldots, t_m = T\} \to \mathbb{R}$ is called multistage. Let us use the notation $\mathbb{T}$ to denote the appropriate time set.

Now let us introduce some solution concepts from the static cooperative game theory and consider them for a dynamic game (N, v), $t \in \mathbb{T}$.

Definition 2.1. *A vector $\xi = (\xi_1, \ldots, \xi_n)$, whose components satisfy the conditions*

$$1)\ \xi_i \geq v(\{i\}, t),\ \ i \in N, \tag{2.3}$$

$$2)\ \sum_{i \in N} \xi_i = v(N, t), \tag{2.4}$$

is called an imputation in a game $(N, v(t))$, $t \in \mathbb{T}$.

Definition 2.2. *A subset $C(N, v(t))$, $t \in \mathbb{T}$, of imputation set is called core of the game $(N, v(t))$ if all its elements satisfy the inequalities*

$$\sum_{i \in S} \xi_i \geq v(S, t), S \subset N. \tag{2.5}$$

Following [125] and [131], we introduce multiple selectors of the core.

Let us denote by $X^0(t)$ the set of solutions of the following linear programming problem

$$\min \sum_{i \in N} \xi_i \tag{2.6}$$

$$\textit{subject to } \sum_{i \in S} \xi_i \geq v(S, t),\ \ \textit{for all } S \subset N,\ S \neq N. \tag{2.7}$$

Definition 2.3. *A set*

$$SC(v(t),\xi^0(t)) =$$
$$\big\{\xi(t) = (\xi_1(t),\dots,\xi_{|N|}(t)) \mid \xi_i(t) \geq \xi_i^0(t), i \in N, \sum_{i\in N}\xi_i(t) = v(N,t)\,\big\} =$$
$$\big\{\xi(t) = (\xi_1(t),\dots,\xi_{|N|}(t)) \mid \xi(t) = \xi^0(t) + \alpha\,\big(v(N,t) - \sum_{i\in N}\xi_i^0(t)\big),$$
$$\text{where } \alpha = (\alpha_1,\dots,\alpha_{|N|}):\ \alpha_i \geq 0, i\in N, \sum_{i\in N}\alpha_i = 1;\ \sum_{i\in N}\xi_i^0(t) \leq v(N,t)\,\big\}$$

is called a subcore of the dynamic cooperative game $(N, v(t))$*,* $t \in \mathbb{T}$*, with respect to* $\xi^0(t)$*,* $\xi^0(t) \in X^0(t)$*.*

Definition 2.4. *We call a set*

$$GSC(N, v(t)) = \bigcup_{\xi^0(t)\in X^0(t)} SC(v(t),\,\xi^0(t))$$

a grand subcore of the dynamic cooperative game $(N, v(t))$*,* $t \in \mathbb{T}$*.*

Subcore and grand subcore are subsets of the core in the balanced TU-games [125].

Let us now introduce a subset of a grand subcore, which we denote by $TCGSC$ (time-consistent grand subcore).

Definition 2.5. *The solution set* $TCGSC(N, v(t))$ *of a game* $(N, v(t))$*,* $t \in \mathbb{T}$*, is the set of all imputations* $\xi(t)$ *from* $GSC(N, v(t))$*, such that for any time* $t \leq \tau \leq T$ *there exists a vector* $\xi^0(\tau) \in X^0(\tau)$ *which satisfies the inequality* $\xi_i(t) \geq \xi_i^0(\tau)$ *for all* $i \in N$*.*

R. Villiger and A. Petrosjan proposed the concept of an undercore to choose a time-consistent imputation from the core (see [118]).

Definition 2.6. *The undercore of the game* $(N, v(t))$ *is defined by*

$$UC(N, v(t), \xi^0(\cdot)) = \Big\{x \mid x_i \geq \max_{\tau\geq t}\xi_i^0(\tau), i \in N, \sum_{i\in N} x_i = v(N,t)\Big\}.$$

Here $\xi^0(\cdot) = \{\xi^0(\tau), t \leq \tau \leq T\}$*.*

Definition 2.7. *A solution concept* $\phi(t)$ *of the dynamic cooperative game* $(N, v(t))$*,* $t \in \mathbb{T}$*, is called time-consistent if for every* $\xi \in \phi(t)$ *and for all* $t \leq t^* \leq T$ *there exists a vector* $\alpha(t^*) \geq 0$ *such that* $\xi - \alpha(t^*) \in \phi(t^*)$*.*

We assume that $\alpha(t)$ is a payoff vector to the players by the moment t. In this work we consider nonnegative non-decreasing vector-valued functions $\alpha(t)$, which satisfy a natural acceptance condition from the following definition.

Definition 2.8. *A vector $\alpha(t)$ with non-decreasing coordinates is called acceptable in the game $(N, v(t))$ if its elements satisfy $\alpha_i(t) \geq 0$ for all $i \in N$ and $\sum_{i\in N} \alpha_i(t) \leq (v(N, t_0) - v(N, t))$.*

Definition 2.9. *Imputation distribution procedure (IDP) in dynamic cooperative game (N, v), $t \in \mathbb{T}$, is a sequence of payoffs to the players during the game.*

2.3 Time-consistent imputations in $SC(N, v(t_0))$

In this section we consider the cooperative differential game $(N, v(t))$ with $t \in [t_0, T]$. We suppose $v(N, t)$ is a decreasing function.

It is not difficult to show that the core of the differential cooperative game $(N,\ v(t))$ is empty if and only if $GSC(N, v(t))$ is empty. Let us also note that $GSC(N, v(t)) \subset C(N, v(t))$ [131].

Theorem 2.10. *Assume that $C(N, v(t)) \neq \emptyset$ with any $t \in [t_0, T]$. An imputation $\xi(t_0) \in GSC(N, v(t_0))$ is time-consistent at $[t_0, T]$ if and only if for every $t \in [t_0, T]$ there exists a vector $\xi^0(t) \in X^0(t)$ such that $\xi_i(t_0) \geq \xi_i^0(t)$ for all $i \in N$.*

Proof. Let us remark again that the condition $C(N, v(t)) \neq \emptyset$ is equivalent to $GSC(N, v(t)) \neq \emptyset$.

Necessity. Suppose that the vector $\xi(t_0)$ is a time-consistent imputation from $GSC(N, v(t_0))$. We show that there exists such a vector $\xi^0(t) \in X^0(t)$ that $\xi_i(t_0) \geq \xi_i^0(t)$ for all $i \in N$.

By the time-consistency of $\xi(t_0)$ we have $\xi(t_0) - \alpha(t) \in GSC(N, v(t))$. It implies that there exists $\xi^0(t) \in X^0(t)$ which satisfies

$$\xi(t_0) - \alpha(t) \geq \xi^0(t). \tag{2.8}$$

Taking into account the non-negativity of $\alpha_i(t)$, we obtain the required inequality $\xi_i(t_0) \geq \xi_i^0(t)$ for all $i \in N$.

Sufficiency. Suppose that for every $t \in [t_0, T]$ there exists such a vector $\xi^0(t) \in X^0(t)$ that $\xi_i(t_0) \geq \xi_i^0(t)$ for all $i \in N$. To prove the time-consistency we have to construct an acceptable payoff vector $\alpha(t)$. Let us find $\alpha(t)$ in the form $\alpha(t) = \beta(t)[v(N, t_0) - v(N, t)]$. Here $\beta(t)$ is a vector which satisfies the following conditions:

(i) $\beta_i(t) \geq 0$ for all $i \in N$,

(ii) $\sum_{i\in N} \beta_i(t) \leq 1$.

It is obvious that the vector $\beta(t) = \frac{\xi(t_0) - \xi^0(t)}{v(N,t_0) - \sum_{i\in N} \xi_i^0(t)}$ satisfies (i) and (ii). Indeed, non-negativity of $\beta_i(t)$ is evident for all $i \in N$ and

$$\sum_{i\in N} \beta_i(t) = \frac{\sum_{i\in N} \xi_i(t_0) - \sum_{i\in N} \xi_i^0(t)}{v(N, t_0) - \sum_{i\in N} \xi_i^0(t)} = \frac{v(N, t_0) - \sum_{i\in N} \xi_i^0(t)}{v(N, t_0) - \sum_{i\in N} \xi_i^0(t)} = 1.$$

Thus, we have found the payoff vector $\alpha(t)$ providing (2.10)

$$\xi(t_0) - \alpha(t) = \xi(t_0) - \frac{\xi(t_0) - \xi^0(t)}{v(N,t_0) - \sum_{i \in N} \xi_i^0(t)}(v(N,t_0) - v(N,t)) \geq$$
$$\xi(t_0) - (\xi(t_0) - \xi^0(t)) \cdot 1 = \xi^0(t).$$

It completes the proof. □

Corollary 2.11. *An imputation $\xi(t_0)$ is time-consistent in $(N, v(t))$ if and only if it belongs to $TCGSC(N, v(t))$.*

Corollary 2.12. *In the balanced game $(N, v(t))$ a vector $\xi(t_0) \in GSC(N, v(t_0))$ is time-consistent if and only if for all $t \in [t_0, T]$ there exists $\xi^0(t) \in X^0(t)$ such that $\xi(t_0) \in UC(N, v(t_0), \xi^0(t))$.*

Remark 2.13. *The proof of Theorem 2.10 is constructive. Actually, for a balanced game $(N, v(t))$ we can configure the acceptable payoff vector*

$$\alpha(t^*) = \frac{\xi(t_0) - \xi^0(t^*)}{v(N,t_0) - \sum_{i \in N} \xi_i^0(t^*)} \cdot (v(N,t_0) - v(N,t^*)) \tag{2.9}$$

providing the time-consistency of $\xi(t_0)$ from $GSC(N, v(t_0))$ for every intermediate moment $t^* \in (t_0, T)$.

Remark 2.14. *If $\xi_i^0(t^*) = \max_{\tau \in [t^*, T]} \xi_i^0(\tau)$ for every moment $t^* \in [t_0, T]$ and for all $i \in N$ the payoff vector $\alpha(t^*)$ is analogous with another one, which was offered by Villiger and Petrosjan in [118].*

2.4 Construction of imputation distribution procedure in a multistage game

In this section we consider multistage cooperative game $(N, v(t))$ with $t \in \{t_0, t_1, \ldots, t_m = T\}$. Assume that $v(N, t)$ is a decreasing function and the conditions of Theorem 2.10 are fulfilled. We make use of the IDP proposed in the previous section.

Example 2.15. *We describe the cooperative 3-person game (N, v), $t \in \{t_0, \ldots, t_4 = T\}$. Player set $N = \{1, 2, 3\}$. The characteristic function $v(S, t)$ is presented in Table 2.15. We assume that $v(\{i\}, t) = 0$ for all t and $i \in N$. We choose $\alpha(T) = \xi(t_0) = (20, 40, 40)$ as a payoff vector, because it satisfies the conditions of Theorem 2.10 and it is time-consistent. Then total payoff of the player 1 is 20, total payoff of the player 2 is 40, and the player 3 receives also 40 by the end of the game.*

In this example the notion $\alpha(t^*)$ means acceptable vector from formula (2.9) by the moment t^*. Note that for this example IDP from the previous section coincides with the IDP from the paper [118] (see Remark 2.14).

t	$v(N,t)$	$v(\{1,2\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$	$X^0(t)$
t_0	100	40	50	30	(10, 30, 20)
t_1	80	40	40	30	(15, 25, 15)
t_2	60	30	40	20	(5, 25, 15)
t_3	40	20	20	10	(5, 15, 5)
t_4	0	0	0	0	(0, 0, 0)

Table 1: Characteristic function for Example 2.15.

The 1st step. Here we allocate the benefit at the period $[t_0, t_1]$*. By Theorem 2.10, that players can get* $\alpha(t_0, t_1) = \alpha(t_1) - \alpha(t_0) = \alpha(t_1) - (0, 0, 0)$*. Thus*

$$\alpha(t_0, t_1) = \frac{(20, 40, 40) - (15, 25, 15)}{100 - 55} \cdot (100 - 80) = \left(\frac{20}{9}, \frac{60}{9}, \frac{100}{9}\right),$$

i.e. *at the moment* t_1 *player 1 receives* $\frac{20}{9}$*, player 2 has* $\frac{60}{9}$*, and player 3 has* $\frac{100}{9}$*.*

The 2nd step. Here we allocate the benefit at the period $[t_1, t_2]$*. Consider the vector* $\alpha(t_2)$*. It provides the time-consistency of* $\xi(t_0)$*, so we can fix*

$$\alpha(t_1, t_2) = \alpha(t_2) - \alpha(t_0, t_1) = \alpha(t_2) - \alpha(t_1) =$$

$$\left(\frac{120}{11}, \frac{120}{11}, \frac{200}{11}\right) - \left(\frac{20}{9}, \frac{60}{9}, \frac{100}{9}\right) = \left(\frac{860}{99}, \frac{420}{99}, \frac{700}{99}\right).$$

Then at the moment t_2 *player 1 gets* $\frac{860}{99}$*, player 2 has* $\frac{420}{99}$*, and player 3 has* $\frac{700}{99}$*.*

The 3rd step. Here we allocate the gains at $([t_2, t_3]$*. In the same way as before we choose*

$$\alpha(t_2, t_3) = \alpha(t_3) - \alpha(t_2) = (12, 20, 28) - \left(\frac{120}{11}, \frac{120}{11}, \frac{200}{11}\right) = \left(\frac{12}{11}, \frac{100}{11}, \frac{108}{11}\right).$$

Player 1 gets $\frac{12}{11}$ *and player 2 gets* $\frac{100}{11}$*, but player 3 have to give back* $\frac{108}{11}$*.*

The 4th step. Here we allocate the gains at $(t_3, t_4]$*. We can choose*

$$\alpha(t_3, t_4) = \alpha(t_4) - \alpha(t_3) = (20, 40, 40) - (12, 20, 28) = (8, 20, 12).$$

As a result we have the sequence of nonnegative payoffs for every step of the game, total payoff vector is time-consistent, and in this multistage cooperative game IDP proposed for differential games is applicable.

Example 2.16. *Let us consider a three-person game (the characteristic function can be seen in Table 2.16).*

The 1st step. Here we allocate the benefit at $[t_0, t_1]$*. Using formula (2.9), we get*

$$\alpha(t_0, t_1) = \left(\frac{100}{7}, \frac{40}{7}, 0\right).$$

The 2nd step. Here we allocate the benefit at $[t_1, t_2]$*. We can fix*

$$\alpha(t_1, t_2) = \alpha(t_2) - \alpha(t_1) =$$

t	$v(N,t)$	$v(\{1,2\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$	$X^0(t)$
t_0	120	40	80	90	(25, 15, 65)
t_1	100	20	70	80	(15, 5, 65)
t_2	80	50	20	50	(40, 10, 10)
t_3	40	20	10	20	(15, 5, 5)
t_4	0	0	0	0	(0, 0, 0)

TABLE 2: Characteristic function For Example 2.16.

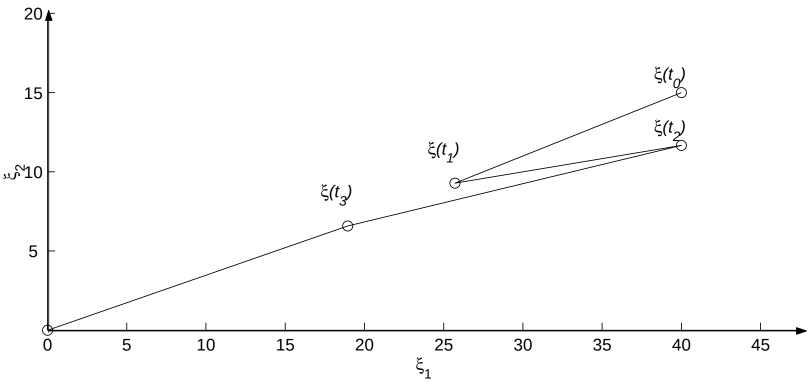


FIGURE 1: Projection to the plane (x_1, x_2) (Example 2.16).

$$= \left(0, \frac{10}{3}, \frac{110}{3}\right) - \left(\frac{100}{7}, \frac{40}{7}, 0\right) = \left(-\frac{100}{7}, -\frac{50}{21}, \frac{110}{3}\right).$$

We can see that $\alpha_i(t_1, t_2) < 0$, $i = 1, 2$, i.e. the IDP from the continuous dynamic (see [118]) can not work perfectly in multistage game. For the projections of the result calculation see Figures 1 and 2. When we try to provide the time-consistency we are loosing non-negativity of the payoffs to the players. Remark that one of the ways to reach it is to state $\alpha_i(t_1, t_2) = 0$, $i = 1, 2$, and these players remain the debtors for the next periods. The problem is avoidable if we detain payoffs $\frac{100}{7}$ and $\frac{50}{21}$ from the previous periods for the players 1 and 2 respectively. Formally it means the changes of the characteristic function of the grand coalition $v(N, t)$ at some moments. To know the value of the delay for any $t_k \in \mathbb{T}$ let us use the backward induction principle.

Algorithm 1

Let us consider now an algorithm for construction of IDP based on the backward induction principle to provide non-negative payoffs at every step.

Denote again the acceptable vector by a moment t in the form (2.9) by $\alpha(t)$, the payoffs at t_{k+1}, $k = 0, \ldots, m-1$, by $\alpha(t_k, t_{k+1})$, and the agreement imputation

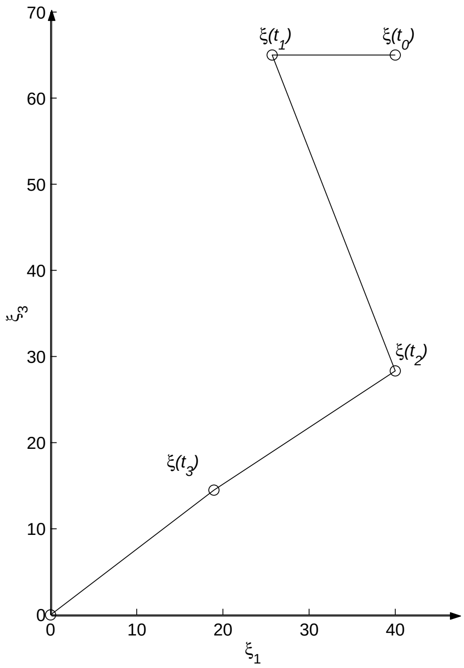


FIGURE 2: Projection to the plane (x_1, x_3) (Example 2.16).

by $\xi(t_0) \in X^0(t_0)$.

The 1st step. Find an imputation at the final moment $t_m = T$. Players receive $\alpha(T) = \xi(t_0)$ by the time T, and $\alpha(t_{m-1})$ is the acceptable payoff-vector by the time t_{m-1}. Note that $\xi_i(t_0) \geq \alpha_i(t_{m-1})$ for all $i \in N$. Hence we can state $\alpha(t_{m-1}, T) := \xi(t_0) - \alpha(t_{m-1})$. There is no debtor on this period.

The 2nd step. Find the allocation at the moment t_{m-1} with respect to the 1st step. We stated at the first step that players receive $\alpha(t_{m-1})$ by the time t_{m-1}. The vector $\alpha(t_{m-2})$ is the acceptable payoff-vector by the time t_{m-2}. Denote the set of potential debtors of this period by

$$M(t_{m-1}) = \{i \in N | \alpha_i(t_{m-1}) < \alpha_i(t_{m-2})\}.$$

Set

$$\alpha_i(t_{m-2}, t_{m-1}) := 0, \ \textit{for all } i \in M(t_{m-1}),$$
$$\alpha_i(t_{m-2}, t_{m-1}) := \alpha_i(t_{m-1}) - \alpha_i(t_{m-2}), \ \textit{for all } i \in N \setminus M(t_{m-1}).$$

The 3rd step. Find the allocation at the moment t_{m-2} with respect to the 2nd step. Here let us retain the differences $\alpha_i(t_{m-1}) - \alpha_i(t_{m-2})$, $i \in M(t_{m-1})$. Let us introduce new notion $\tilde{\alpha}(t_{m-2})$ in the following way

$$\tilde{\alpha}_i(t_{m-2}) := \alpha_i(t_{m-2}), \ \textit{for all } i \in N \setminus M(t_{m-1}),$$
$$\tilde{\alpha}_i(t_{m-2}) := \alpha_i(t_{m-2}) - (\alpha_i(t_{m-2}) - \alpha_i(t_{m-1})), \ \textit{for all } i \in M(t_{m-1}).$$

Now the set of potential debtors of this period is

$$M(t_{m-2}) = \{i \in N | \alpha_i(t_{m-3}) > \tilde{\alpha}_i(t_{m-2})\}.$$

Set

$$\alpha_i(t_{m-3}, t_{m-2}) := 0, \ \textit{for all}\ i \in M(t_{m-2}),$$
$$\alpha_i(t_{m-3}, t_{m-2}) := \tilde{\alpha}_i(t_{m-2}) - \alpha_i(t_{m-3}), \ \textit{for all}\ i \in N \setminus M(t_{m-2}).$$

The step number k. Find the allocation at the moment t_{m-k+1} with respect to the step number $k-1$. As above, we define

$$\tilde{\alpha}_i(t_{m-k+1}) := \alpha_i(t_{m-k+1}), \ \textit{for all}\ i \in N \setminus M(t_{m-k+2}),$$
$$\tilde{\alpha}_i(t_{m-k+1}) := \tilde{\alpha}_i(t_{m-k+2}), \ \textit{for all}\ i \in M(t_{m-k+2}),$$
$$M(t_{m-k+1}) = \{i \in N | \alpha_i(t_{m-k}) > \tilde{\alpha}_i(t_{m-k+1})\},$$

and set

$$\alpha_i(t_{m-k}, t_{m-k+1}) := 0, \ \textit{for all}\ i \in M(t_{m-k+1}),$$
$$\alpha_i(t_{m-k}, t_{m-k+1}) := \tilde{\alpha}_i(t_{m-k+1}) - \alpha_i(t_{m-k}) \ \textit{for all}\ i \in N \setminus M(t_{m-k+1}).$$

The last step. Find the allocation at the moment t_1 with respect to the step number $m-1$. Here we have payoffs $\alpha(t_{m-1}, T)$, $\ldots$, $\alpha(t_1, t_2)$ and the potential debtor set $M(t_2)$. As above,

$$\tilde{\alpha}_i(t_1) := \alpha_i(t_1), \ \textit{for all}\ i \in N \setminus M(t_2),$$
$$\tilde{\alpha}_i(t_1) := \tilde{\alpha}_i(t_2), \ \textit{for all}\ i \in M(t_2).$$

Since $\alpha_i(t_0) = 0$ we have $\tilde{\alpha}_i(t_1) \geq \alpha_i(t_0)$ for all $i \in N$. At last we set $\alpha(t_0, t_1) := \tilde{\alpha}(t_1)$.

As the result of the algorithm we have the payoff sequence $\{\alpha(t_k, t_{k+1})\}_{k=0}^{m-1}$ which guarantees the time-consistent $\xi(t_0)$.

Let us apply the algorithm to our example.

Example 2.17 (backward allocation construction). *Now consider the game (N, v) from Example 2.16. Let us calculate the payoff sequence by Algorithm 1.*

The 1st step. As in the 1st step in the previous example

$$\alpha(T) = (20, 40, 40), \ \ \alpha(t_3) = \Big(\frac{36}{5}, \frac{60}{5}, \frac{84}{5}\Big)$$

we have

$$\alpha(t_3, T) = \Big(\frac{64}{5}, \frac{140}{5}, \frac{116}{5}\Big).$$

The 2nd step. Since from formula (2.9)

$$\alpha(t_3) = \Big(\frac{36}{5}, \frac{60}{5}, \frac{84}{5}\Big), \ \ \alpha(t_2) = \Big(\frac{120}{11}, \frac{120}{11}, \frac{200}{11}\Big)$$

we have

$$\alpha(t_2, t_3) = \Big(0, \frac{12}{11}, 0\Big), \;\; M(t_3) = \{1, 3\}.$$

The 3rd step. Since

$$\tilde{\alpha}_1(t_2) = \alpha_1(t_3) = \frac{36}{5}, \;\; \tilde{\alpha}_2(t_2) = \alpha_2(t_2) = \frac{120}{11},$$
$$\tilde{\alpha}_3(t_2) = \alpha_3(t_3) = \frac{84}{5}, \;\; \alpha(t_1) = \Big(\frac{20}{9}, \frac{60}{9}, \frac{100}{9}\Big)$$

we have

$$\alpha(t_1, t_2) = \Big(\frac{224}{45}, \frac{420}{99}, \frac{256}{45}\Big), \;\; M(t_2) = \emptyset.$$

The 4th step. Since

$$\tilde{\alpha}(t_1) = \alpha(t_1) = \Big(\frac{20}{9}, \frac{60}{9}, \frac{100}{9}\Big), \;\; \alpha(t_0) = (0, 0, 0)$$

we have

$$\alpha(t_0, t_1) = \Big(\frac{20}{9}, \frac{60}{9}, \frac{100}{9}\Big).$$

We can see that the proposed algorithm constructs a time-consistent IDP with non-negative payoffs to the players at each period (see Figures 3, 4(a), and 4(b)).

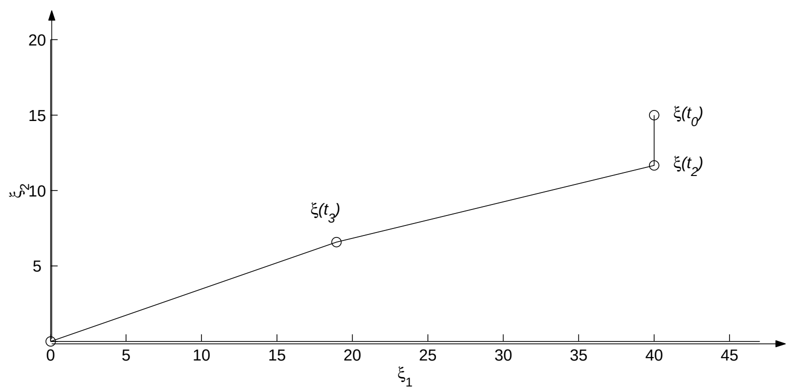
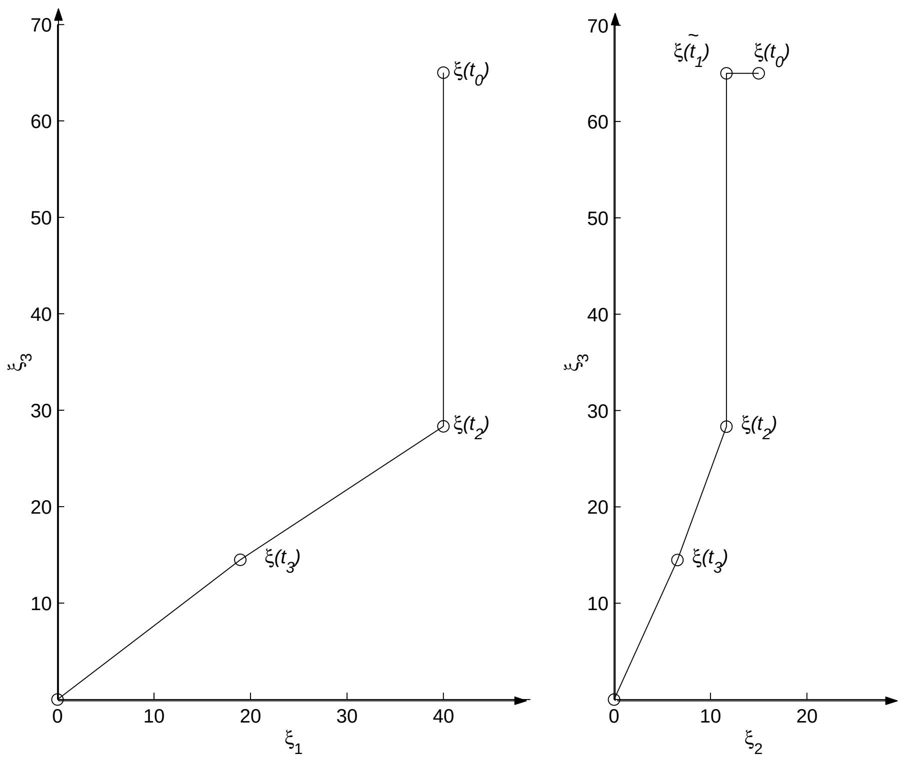


FIGURE 3: Projection to the plate (x_1, x_2) (Example 2.17).

However, let us note that this algorithm constructs only one solution, which depends on $\xi(t_0)$ and $\xi^0(t_k)$, $k = 1, \ldots, m-1$. Meanwhile, there is a set of imputation distribution procedures in general case. The following example illustrates even more complicated benefit allocation in our game.



(a) plate (x_1, x_3) (b) plate (x_2, x_3)

FIGURE 4: Projections to the plates (x_1, x_3) and (x_2, x_3) (Example 2.17).

Example 2.18. *Let us consider again the game from the previous example. As an acceptable vector $\alpha(t_k)$ we will take the difference $[\xi(t_0) - \xi(t_k)]$, $k = 1, \ldots, m-1$, and we will fix vectors $\xi(t_k)$ such that the inequality $\xi(t_{k-1}) \geq \xi(t_k)$ holds. In Table 2.18 the corresponding points from the subcore and allocations $\alpha(t_{k-1}, t_k)$ are presented.*

t_k	$X^0(t_k)$	$\xi(t_k)$	$\alpha(t_{k-1}, t_k)$
t_0	(10,30,20)	(20,40,40)	—
t_1	(15,25,15)	(15,35,30)	(5,5,10)
t_2	(5,25,15)	(12,29,19)	(3,6,11)
t_3	(5,15,5)	(10,20,10)	(2,9,9)
t_4	(0,0,0)	(0,0,0)	(10,20,10)

TABLE 3: Imputations from subcore and allocations $\alpha(t_{k-1}, t_k)$ (Example 2.18).

2.5 The second imputation distribution procedure for multistage game

As before (N, v) is a multistage cooperative game with $t \in \{t_0, \ldots, t_m = T\} =: \mathbb{T}$. We assume that $v(N, t)$ is a decreasing function with respect to t and that the game is balanced.

We denote by $Con(\xi^0)$ a cone defined by the inequality $\xi \geq \xi^0$.

Let $Y^0(t)$ be the union of the cones $Con(\xi^0)$

$$Y^0(t) = \bigcup_{\xi^0 \in X^0(t)} Con(\xi^0).$$

Now let us redefine some solution concepts for the multistage cooperative game (N, v) using $Y^0(t)$ (see Definitions 2.3, 2.4, 2.5).

Definition 2.19. *We call the set*

$$SC(v(t), \xi^0(t)) = \{\xi \in Con(\xi^0(t)) | \sum_{i \in N} \xi_i = v(N, t)\}$$

subcore of the game $(N, v(t))$ with respect to $\xi^0(t)$ from $X^0(t)$.

Definition 2.20. *We call the set*

$$GSC(N, v(t)) = \bigcup_{\xi^0(t) \in X^0(t)} SC(v(t), \xi^0(t)) = \{\xi \in Y^0(t) | \sum_{i \in N} \xi_i = v(N, t)\}$$

grand subcore of the game $(N, v(t))$.

Definition 2.21. *We call a set*

$$TCGSC(N, v(t_k)) = \{\xi \in \bigcap_{r=k}^{m} Y^0(t_r) | \sum_{i \in N} \xi_i = v(N, t_k)\}$$

time-consistent grand subcore of the game $(N, v(t_k))$, $t_k \in \mathbb{T}$.

Let us reformulate the condition for time-consistency of an imputation from the grand subcore (Theorem 2.10) and its proof. We assume again that $\alpha(t)$ is a payoff vector by the moment t.

Theorem 2.22. *In a balanced multistage game (N, v), $t \in \mathbb{T}$, a vector $\xi(t_0) \in GSC(N, v(t_0))$ is time-consistent if and only if $\xi(t_0) \in Y^0(t)$ for all $t \in \mathbb{T}$.*

Proof. Note that the condition of balancedness of (N, v) is equivalent to the nonemptiness of $GSC(N, v(t))$ for every $t \in \mathbb{T}$.

Necessity. Suppose that a vector $\xi(t_0)$ is a time-consistent imputation from $GSC(N, v(t_0))$. We show that there exists such a vector $\xi^0(t) \in X^0(t)$ that $\xi(t_0) \geq \xi^0(t)$.

By the time-consistency of $\xi(t_0)$ we have $\xi(t_0) - \alpha(t) \in GSC(N, v(t))$. It implies that there exists $\xi^0(t) \in X^0(t)$ which satisfies

$$\xi(t_0) - \alpha(t) \geq \xi^0(t). \tag{2.10}$$

Taking into account the non-negativity of $\alpha_i(t)$, we obtain the required inequality $\xi(t_0) \geq \xi^0(t)$ for all $t \in \mathbb{T}$. That is $\xi(t_0) \in Y^0(t)$.

Sufficiency. Suppose that for every $t \in [t_0, T]$ there exists such a vector $\xi^0(t) \in X^0(t)$ that $\xi(t_0) \geq \xi^0(t)$. To prove the time-consistency we have to construct an acceptable payoff vector $\alpha(t)$. Let us find $\alpha(t)$ in the form $\alpha(t) = \beta(t)[v(N, t_0) - v(N, t)]$. Here $\beta(t)$ is a vector which satisfies the following conditions:

(i) $\beta(t) \geq \mathbf{0}$,

(ii) $\sum_{i \in N} \beta_i(t) = 1$.

The vector $\beta(t) = \frac{\xi(t_0) - \xi^0(t)}{v(N,t_0) - \sum_{i \in N} \xi_i^0(t)}$ satisfies the conditions (i) and (ii). Indeed, non-negativity of $\beta_i(t)$ is evident (numerator is non-negative and denominator is positive) for all $i \in N$ and

$$\sum_{i \in N} \beta_i(t) = \frac{\sum_{i \in N} \xi_i(t_0) - \sum_{i \in N} \xi_i^0(t)}{v(N, t_0) - \sum_{i \in N} \xi_i^0(t)} = \frac{v(N, t_0) - \sum_{i \in N} \xi_i^0(t)}{v(N, t_0) - \sum_{i \in N} \xi_i^0(t)} = 1.$$

Thus, we have found the payoff vector $\alpha(t)$ providing (2.10)

$$\begin{aligned} \xi(t_0) - \alpha(t) &= \xi(t_0) - \frac{\xi(t_0) - \xi^0(t)}{v(N, t_0) - \sum_{i \in N} \xi_i^0(t)} (v(N, t_0) - v(N, t)) \geq \\ &\quad \xi(t_0) - (\xi(t_0) - \xi^0(t)) \cdot 1 = \xi^0(t). \end{aligned}$$

This completes the proof. □

Corollary 2.23. *An imputation $\xi(t_0) \in GSC(N, v(t_0))$ is time-consistent in a balanced multistage game (N, v) if and only if it belongs to $TCGSC(N, v(t_0))$ (or, equivalently, $\xi(t_0) \in \bigcap_{k=0,\dots,m} Y^0(t_k)$).*

It can be shown that IDPs suggested for differential cooperative games can lead to negative payoffs during the game (see Example 2.16). Informally, this means that players have to give back a part of received benefit to provide time-consistency of the optimality principle. However, the total payoffs that the players receive by the end of the game correspond to the imputation that was chosen at the beginning.

Corollary 2.24. *In the conditions of Theorem 2.22 the nonemptiness of $TCGSC(N, v(t_k))$, $k = 0, \dots, m$, and the decreasing of $v(N, t)$ with respect to t imply that there exists a sequence $\{\xi(t_k)\}_{k=0,\dots,m}$, such that $\xi(t_0) \geq \xi(t_1) \geq \dots \geq \xi(t_{m-1}) \geq 0$. Here $\xi(t_k) \in TCGSC(N, v(t_k))$.*

Proof. We have to show that for every vector $\xi(t_k) \in TCGSC(N, v(t_k))$ there exists a vector $\zeta \in TCGSC(N, v(t_{k+1}))$, such that $\zeta \leq \xi(t_k)$.

Indeed, by $\xi(t_k) \in TCGSC(N, v(t_k))$ we have

$$\xi(t_k) \in \bigcap_{r=k}^{m} Y^0(t_r).$$

Thus

$$\xi(t_k) \in \bigcap_{r=k+1}^{m} Y^0(t_r),$$

and there exists a non empty subset

$$M(t_{k+1}) \subset \bigcap_{r=k+1}^{m} Y^0(t_r),$$

such that a vector $\zeta^* \in M(t_{k+1})$ satisfies $\zeta^* \leq \xi(t_k)$.

Since $TCGSC(N, v(t_k + 1)) \neq \emptyset$ and $v(N, t_k) > v(N, t_{k+1})$ we can choose a vector $\zeta \in M(t_{k+1})$ with $\sum_{i \in N} \zeta_i = v(N, t_{k+1})$. Our reasoning holds for all $k = 0, \ldots, m-1$. This completes the proof. □

Now by Corollary 2.24 we have the following IDP insuring the non-negativity of payoffs to every player $i \in N$. In the condition of Corollary 2.24 we choose a vector $\xi(t_k) \in TCGSC(N, v(t_k))$ for every $k = 1, \ldots, m$, such that $\xi(t_k) \leq \xi(t_{k-1})$. After that we set the payoff vectors at every moment $t_k \in \mathbb{T}$ as

$$\alpha(t_{k-1}, t_k) = \xi(t_{k-1}) - \xi(t_k), \; k = 1, \ldots, m.$$

However, sometimes $TCGSC(N, v(t_k)) = \emptyset$ at a moment $t_k \in \mathbb{T}$. In such a case we can delay a part of difference $v(N, t_{k-1}) - v(N, t_k)$ to insure time-consistency.

Let us consider now an algorithm based on the maximization of total payoffs at every step. Assume that there is a time-consistent vector $\xi(t_0) \in GSC(N, v(t_0))$ in a multistage cooperative game (N, v), $t \in \mathbb{T}$.

Algorithm 2

Let us introduce a new notation. Let $\alpha(t_k)$ be a total payoff vector at a period $(t_k, t_m]$, $\alpha(t_{k-1}, t_k)$ a payoff vector at a moment t_k, and $\tilde{v}(N, t_k)$ a new guaranteed payoff at a period $[t_k, t_m]$.

We define an auxiliary set $Z^0(t_k)$ as the solution set of the following minimization problem

$$\text{minimize} \quad \sum_{i \in N} \omega_i, \tag{2.11}$$

$$\text{subject to} \quad \omega \in \bigcap_{r=k}^{m} Y^0(t_r). \tag{2.12}$$

Remark that for all $t_k \in \mathbb{T}$ there exists a solution of the problem (2.11), (2.12). It is evident from construction of $Y^0(t_k)$.

Initial step. We choose a vector $\xi(t_0) \in TCGSC(N, v(t_0))$. The players will receive this imputation by the end of the game (the moment t_m). We set

$$\alpha(t_0) := \xi(t_0),\ \tilde{v}(N, t_0) := v(N, t_0).$$

Step number k. At this step we find a non negative payoff vector to the players at the moment t_k with respect to the vector $\alpha(t_{k-1})$ and $\tilde{v}(N, t_{k-1})$ from the previous step.

Consider a set $Z^0(t_k)$. If for a vector $\omega \in Z^0(t_k)$ we have $\sum_{i\in N} \omega_i \leq v(N, t_k)$, then we set

$$\alpha(t_k) := \xi(t_k),\ \text{and}\ \tilde{v}(N, t_k) := v(N, t_k).$$

Here $\xi(t_k) \in TCGSC(N, v(t_k))$ and $\xi(t_k) \leq \xi(t_{k-1})$.

Otherwise we set

$$\alpha(t_k) := \omega,\ \text{and}\ \tilde{v}(N, t_k) := \sum_{i\in N} \omega_i.$$

Here ω is a vector from $Z^0(t_k)$. Finally, we set

$$\alpha(t_{k-1}, t_k) := \alpha(t_{k-1}) - \alpha(t_k).$$

Last step. By the definition of a multistage game we have

$$\alpha(t_m) := (0, 0),\ \tilde{v}(N, t_m) := 0,\ \text{and}\ \alpha(t_{m-1}, t_m) = \alpha(t_{m-1}).$$

Note that the existence of the applicable $\alpha(t_{k+1})$ with respect to $\alpha(t_k)$ is proved analogously to Corollary 2.24. Indeed, it is sufficient to replace $\xi(t_k)$ by $\alpha(t_k)$, ζ by $\alpha(t_{k+1})$ and $v(N, t)$ by $\tilde{v}(N, t)$ in the proof.

As the result of the algorithm we have the payoff sequence $\{\alpha(t_{k-1}, t_k)\}_{k=1}^m$, which guarantees the time-consistency of $\xi(t_0)$.

Let us apply the algorithm to an example.

Example 2.25. *Consider a multistage cooperative game (N, v) with $N = \{1, 2\}$ and $m = 5$. The values of the characteristic function are presented in Table 2.25. In Figure 5 one can find the sets $TCGSC(t)$ at $t = t_0,\ t_1,\ t_3,\ t_4$ (bold lines), stared points are the sets X^0.*

Initial step. We choose a vector $\xi(t_0) = (12, 13)$ from the set $TCGSC(N, v(t_0))$ as a final payoff vector (see Figure 5, circumscribed point on bold line $TCGSC(t_0)$). We set

$$\alpha(t_0) := \xi(t_0) = (12, 13),\ \tilde{v}(N, t_0) := v(N, t_0) = 25.$$

1st step. Here we find the allocation vector $\alpha(t_0, t_1)$. The solution vector of the problem (2.11), (2.12) at the moment t_1 is $(10, 12)$. Since $10 + 12 < 23$ we set

$$\alpha(t_1) := (11.5; 11.5),\ \textit{and}\ \tilde{v}(N, t_1) := v(N, t_1) = 23.$$

t	$v(\{1\},t)$	$v(\{2\},t)$	$v(N,t)$	$X^0(t)$
t_0	9	12	25	(9,12)
t_1	10	10	23	(10,10)
t_2	8	2	15	(8,2)
t_3	5	8	14	(5,8)
t_4	3	3	8	(3,3)
t_5	0	0	0	(0, 0)

TABLE 4: The characteristic function $v(S,t)$ for Example 2.25.

In Figure 5 $\alpha(t_1)$ is circumscribed point on bold line $TCGSC(t_1)$.

Thus, we have

$$\alpha(t_0,t_1) := \alpha(t_0) - \alpha(t_1) = (12,13) - (11.5;11.5) = (0.5;1.5).$$

2nd step. Here we find the payoff vector $\alpha(t_1,t_2)$. The solution vector of the problem (2.11), (2.12) at the moment t_2 is $(8,8)$ (the set $TCGSC(N,v(t_2))$ is empty and we use the set $Z^0(t_2)$ instead of this). Since $8+8>15$ we set

$$\alpha(t_2) := (8,8), \text{ and } \tilde{v}(N,t_2) := 16.$$

Thus, we have

$$\alpha(t_1,t_2) := \alpha(t_1) - \alpha(t_2) = (11.5;11.5) - (8,8) = (3.5;3.5).$$

3rd step. Here we find the payoff vector $\alpha(t_2,t_3)$. The solution vector of the problem (2.11), (2.12) at the moment t_3 is $(5,8)$. Since $5+8<14$ we set

$$\alpha(t_3) := (6,8), \text{ and } \tilde{v}(N,t_3) := v(N,t_3) = 14.$$

Thus, we have

$$\alpha(t_2,t_3) := \alpha(t_2) - \alpha(t_3) = (8,8) - (6,8) = (2,0).$$

4th step. Here we find the payoff vector $\alpha(t_3,t_4)$. The solution vector of the problem (2.11), (2.12) at the moment t_4 is $(3,3)$. Since $3+3<8$ we set

$$\alpha(t_4) := (5,3), \text{ and } \tilde{v}(N,t_4) := v(N,t_4) = 8.$$

Thus, we have

$$\alpha(t_3,t_4) := \alpha(t_3) - \alpha(t_4) = (6,8) - (5,3) = (1,5).$$

Last step. Here we have

$$\alpha(t_4,t_5) := \alpha(t_4) = (5,3).$$

The proposed algorithm constructs the following IDP with the nonnegative payoffs to the players at each period: at the moment t_1 player 1 gets payoff 0.5, player 2 gets 1.5; at the moment t_2 both player 1 and player 2 get 3.5; at the moment t_3 player 1 gets 2, player 2 gets zero; at the moment t_4 player 1 gets 1, player 2 gets 5; and at the moment t_5 player 1 gets 5, player 2 gets 3. The result of this IDP is time-consistent imputation (12,13) (see Figure 5).

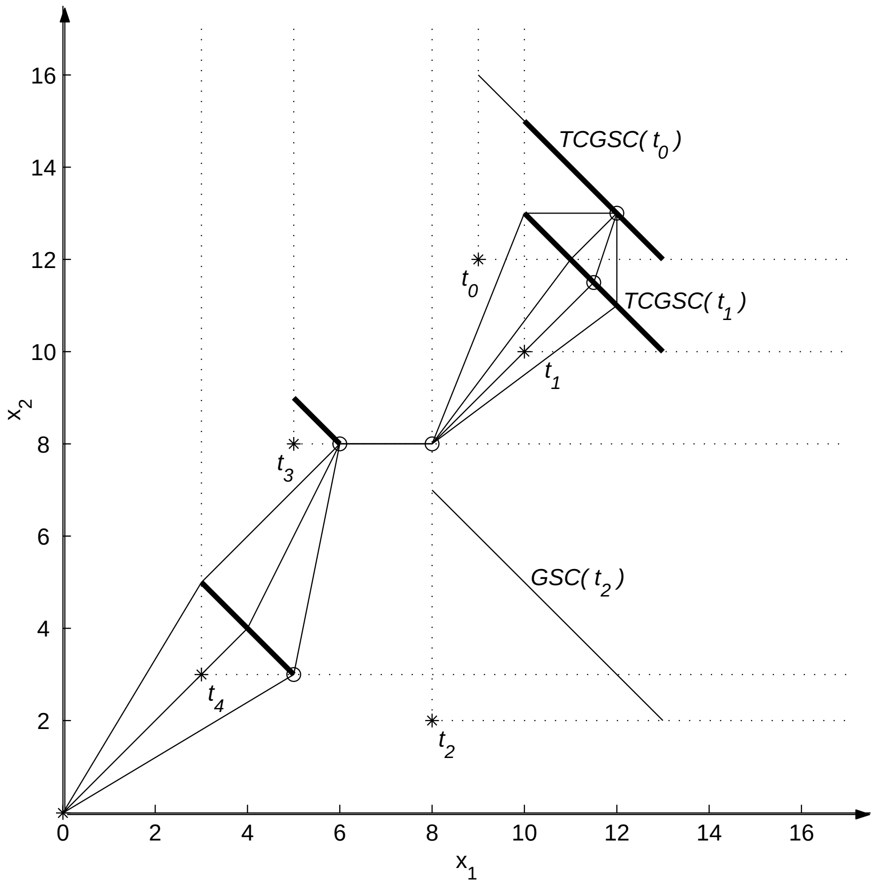


FIGURE 5: Imputation distribution procedures for Example 2.25.

2.6 Conclusion

In this chapter we considered the subcore and the grand subcore as optimality principles in multistage cooperative games. We used one of the classical approaches to the problem of time-consistency of solutions and constructed two algorithms based on delays of total payoffs to provide nonnegative payoffs at every step of the game. We use the backward induction to calculate the payoffs in the first algorithm, and in the second we created auxiliary sets that, if necessary, specifies the minimal delays in payoffs to make the debtors set empty during all the game.

3 CONSISTENCY OF THE SUBCORE

3.1 Introduction

One of the main properties of the solutions, corresponding to the optimality principles in cooperative games, is consistency, or reduced game property. The first definition of a reduced game was introduced by Davis and Maschler [24]. It states that if some coalition of players leaves the original game, the remaining players form the player set of a reduced game. The characteristic function of the reduced game is constructed by a rule, which depends on the payoffs of the removed players. The solution is consistent if every player has the same benefit in the reduced game as in the original game. In [30, 31, 45, 63, 70, 77, 98, 111] various reduced games and characterizations of optimality principles in TU-games are suggested. Here we consider a modification of the well known reduced game by Davis and Maschler [24] and formulate the condition the subcore to be consistent with respect to the MDM-reduction [127] in classical cooperative theory. We also introduce a notion of dynamic consistency for multistage TU-games and prove that time-consistent imputation from the subcore is dynamically consistent with respect to the MDM-reduction. We use the results of this chapter to provide time-consistency in a new way (Chapter 4).

3.2 Modified of the Reduced Game due to Davis and Maschler and Consistency Property

In this section we consider the reduced game property regarding the subcore. Let (N, v) be a classical cooperative TU-game; here N is the finite set of players, $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. Let us denote by n the number of players in the set N.

Given a cooperative game (N, v), a player $j \in N$, any optimality principle ϕ, and any payoff vector $\xi \in \phi(N, v)$, there are various ways to define a reduced game $(N \setminus \{j\}, v_\xi^j)$ with respect to ξ. The characteristic function $v_\xi^j(S)$ of coalition

$S \subset N \setminus \{j\}$ represents the total benefit that the players of S may achieve by mutual cooperation. The removed player j is paid according to the vector ξ.

Let us denote by ξ_S the restriction of the vector $\xi \in \mathbb{R}^N$ to a set $S \subset N$.

Definition 3.1. *We call a solution ϕ to be consistent with respect to the reduced game $(N \setminus \{j\}, v^j_\xi)$, if for all $\xi \in \phi(N, v)$ the condition $\xi_{N\setminus\{j\}} \in \phi(N \setminus \{j\}, v^j_\xi)$ is fulfilled.*

The original reduced game due to Davis and Maschler (DM-reduced game) [24] for a given player $j \in N$, and a payoff vector ξ is the following

$$v^j_\xi(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ v(N) - \xi_j, & \text{if } S = N \setminus \{j\}, \\ \max\{v(S \cup \{j\}) - \xi_j, v(S)\}, & \text{otherwise.} \end{cases}$$

It can be shown that the subcore appears to be inconsistent with respect to the DM-reduced game [127]. In this work we deal with the modified Davis-Maschler reduced game (MDM-reduced game). For a given player $j \in N$, $\xi^0 \in X^0(N, v)$ and a payoff vector ξ the characteristic function of the MDM-reduced game is the following

$$v^j_{\xi^0}(\cdot) = v^j_{\xi^0}(S, \xi_j) = \begin{cases} 0, & \text{if } S = \emptyset, \\ v(N) - \xi_j, & \text{if } S = N\setminus\{j\}, \\ \max\{v(S \cup \{j\}) - \xi^0_j, v(S)\}, & \text{otherwise.} \end{cases}$$

Now we consider the case where one player drops off from the grand coalition N of the original game. Let us formulate the linear programming problem (2.6), (2.7) for the MDM-reduced game

$$\min \sum_{i \in N\setminus\{j\}} x_i, \tag{3.1}$$

$$\sum_{i \in S} x_i \geq v^j_{\xi^0}(S, \xi_j),\ S \subset N\setminus\{j\},\ S \neq N\setminus\{j\}, \tag{3.2}$$

and find a condition for a subcore to be a consistent solution with respect to the MDM-reduced game. To prove this we formulate two lemmas.

Lemma 3.2. *If (N, v) is a balanced TU-game and*

$$\sum_{i \in N\setminus\{j\}} \eta^0_i \geq v(N \setminus \{j\}), \quad \eta^0 \in X^0(N \setminus \{j\}, v^j_{\xi^0}(\cdot)), \tag{3.3}$$

then $\xi^0_{N\setminus\{j\}} \in X^0(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$.

Proof. We assume (N, v) to be a cooperative TU-game and $(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$ to be the MDM-reduction of (N, v), $\xi \in SC(v, \xi^0)$.

Let us suppose that the vector $\eta^0 = \xi^0_{N\setminus\{j\}}$ is not a solution of the problem (3.1), (3.2). It means that either η^0 does not satisfy the inequality system (3.2), or does not minimize the sum (3.1). Since for all proper coalitions $S \subset N \setminus \{j\}$ it is true that $\sum_{i\in S}\xi^0_i \geq v(S)$ and $\sum_{i\in S}\xi^0_i \geq v(S\cup j) - \xi^0_j$ then $\sum_{i\in S}\xi^0_i \geq \max\{v(S), v(S\cup j) - \xi^0_j\}$. Hence,

$$\sum_{i\in S}\eta^0_i \geq \max\{v(S), v(S\cup j) - \xi^0_j\}.$$

Thus it holds that the vector η^0 is the acceptable in the linear programming problem (3.1), (3.2). Consequently, it is sufficient to show that it minimizes the sum (3.1). If it is not the case, there exists a vector $\overline{\eta}^0$, which satisfies the system (3.2) and minimizes the sum (3.1), and

$$\sum_{i\in N\setminus\{j\}}\overline{\eta}^0_i < \sum_{i\in N\setminus\{j\}}\eta^0_i.$$

Let us consider a vector, which is created from $\overline{\eta}^0$ and ξ^0_j by $\overline{\xi}^0 = (\overline{\eta}^0_1, \dots, \overline{\eta}^0_{j-1}, \xi^0_j, \overline{\eta}^0_{j+1}, \overline{\eta}^0_n)$. It is acceptable with respect to the problem (2.6), (2.7), because for every proper coalition $S \subset N\setminus\{j\}$

$$\sum_{i\in S}\overline{\eta}^0_i \geq \max\{v(S), v(S\cup j)\}$$

and

$$\left[\begin{array}{l} \left\{\begin{array}{l} v(S) \geq v(S\cup j) - \xi^0_j \\ \sum_{i\in S}\overline{\xi}^0_i \geq v(S) \end{array}\right. \\ \left\{\begin{array}{l} v(S) < v(S\cup j) - \xi^0_j \\ \sum_{i\in S}\overline{\xi}^0_i \geq v(S\cup j) - \xi^0_j \end{array}\right. \end{array}\right.$$

Taking into account the condition (3.3), we can write

$$\sum_{i\in T}\overline{\xi}^0_i \geq v(T), \quad \textit{for all } T \subset N.$$

However, it is clear that $\sum_{i\in N}\overline{\xi}^0_i < \sum_{i\in N}\xi^0_i$. This contradicts the original assumption $\xi^0 \in X^0(N,v)$. Hence, for every vector $\xi^0 \in X^0(N,v)$ there exists such a vector $\eta^0 \in X^0(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$ that $\eta^0 = \xi^0_{N\setminus\{j\}}$. This completes the proof. □

Lemma 3.3. *If the original game (N,v) is balanced and (3.3) is fulfilled, then MDM-reduced game $(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$ is also balanced.*

Proof. Since the game (N,v) is balanced, every vector $\xi^0 \in X^0(N,v)$ satisfies $\sum_{i\in N}\xi^0_i \leq v(N)$. To prove the lemma it is sufficient to show that $\sum_{i\in N\setminus\{j\}}\xi^0_i \leq v(N) - \xi_j$ (see Lemma 3.2). Let us present ξ_j in the following form

$$\xi_j = \xi^0_j + \alpha_j(v(N) - \sum_{i\in N}\xi^0_i), \ \alpha_j \in [0,1].$$

If $\alpha_j = 1$ then the proposition of the lemma is true. If $\alpha_j \neq 1$ we have

$$\sum_{i\in N\setminus\{j\}} \xi_i^0 \leq v(N) - \xi_j^0 - \alpha_j(v(N) - \sum_{i\in N}\xi_i^0)$$
$$\Leftrightarrow \sum_{i\in N\setminus\{j\}} \xi_i^0 \leq v(N)(1-\alpha_j) - \xi_j^0 + \alpha_j \cdot \sum_{i\in N}\xi_i^0$$
$$\Leftrightarrow \sum_{i\in N}\xi_i^0 - \alpha_j \cdot \sum_{i\in N}\xi_i^0 \leq v(N)(1-\alpha_j) \Leftrightarrow \sum_{i\in N}\xi_i^0 \leq v(N).$$

This completes the proof. □

Finally we can formulate the following theorem.

Theorem 3.4. *The set $SC(v, \xi^0)$ is consistent with respect to the MDM-reduced game $(N \setminus \{j\}, v^j_{\xi^0}(\cdot))$ if (3.3) is fulfilled.*

Proof. Let (N, v) be the original game, $\xi^0 \in X^0(N, v)$, $\xi \in SC(v, \xi^0)$, $(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$ be the MDM-reduction. To prove the theorem we have to show that for all fixed ξ^0 and ξ it is true that $\xi_{N\setminus\{j\}} \in SC(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$. By Lemma 3.3 the MDM-reduced game is balanced. Consequently the subcore of $(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$ is not an empty set. By Lemma 3.2 we have $\eta^0 = \xi^0_{N\setminus\{j\}} \in X^0(N\setminus\{j\}, v^j_{\xi^0}(\cdot))$, therefore there exists such a vector $\eta \in SC(v^j_{\xi^0}, \eta^0)$, that $\eta_i = \xi_i$ for all $i \in N\setminus\{j\}$. This completes the proof. □

Let us note also the superadditivity of the DM- and MDM-reduced games under the condition of the convex initial game (N, v).

Definition 3.5. *The game (N, v) is called superadditive if for all $S, T \subset N$ and $S \cap T = \emptyset$ the following condition is true*

$$v(S) + v(T) \leq v(S \cup T).$$

Definition 3.6. *The game (N, v) is called convex if for all $S, T \subset N$ the following condition is true*

$$v(S) + v(T) \leq v(S \cup T) - v(S \cap T).$$

Lemma 3.7. *The DM- and MDM-reduced games are superadditive if the initial game is convex.*

Proof. It is sufficient to prove the lemma at least for one of the reduced games. Let us show it for convex initial game (N, v) and DM-reduced game $(N \setminus \{i\}, v^i_\xi)$, where ξ is an imputation of the initial game.

Let us consider the sum

$$v^i_\xi(S) + v^i_\xi(T) = \max(v(S), v(S \cup \{i\}) - \xi_i) + \max(v(T), v(T \cup \{i\}) - \xi_i)$$

with $S, T \subset N \setminus \{i\}$, $S \cap T = \emptyset$. There are three variants:

1. if $v(S) > v(S \cup \{i\}) - \xi_i$ and $v(T) > v(T \cup \{i\}) - \xi_i$ then we can write

$$\begin{aligned} & v^i_\xi(S) + v^i_\xi(T) = v(S) + v(T) \le v(S \cup T) \\ & \le \max(v(S \cup T), v(S \cup T \cup \{i\}) - \xi_i) = v^i_\xi(S \cup T); \end{aligned}$$

2. if $v(S) > v(S \cup \{i\}) - \xi_i$ and $v(T) \le v(T \cup \{i\}) - \xi_i$ then

$$\begin{aligned} & v^i_\xi(S) + v^i_\xi(T) = v(S) + v(T \cup \{i\}) - \xi_i v(S \cup T \cup \{i\}) - \xi_i \\ & \le \max(v(S \cup T), v(S \cup T \cup \{i\}) - \xi_i) = v^i_\xi(S \cup T); \end{aligned}$$

3. if $v(S) \le v(S \cup \{i\}) - \xi_i$ and $v(T) \le v(T \cup \{i\}) - \xi_i$ we have

$$\begin{aligned} & v^i_\xi(S) + v^i_\xi(T) = v(S \cup \{i\}) - \xi_i + v(T \cup \{i\}) - \xi_i \\ & \le v(S \cup T \cup \{i\}) - 2\xi_i - v(\{i\}) \le v(S \cup T \cup \{i\}) - \xi_i \\ & \le \max(v(S \cup T), v(S \cup T \cup \{i\}) - \xi_i) = v^i_\xi(S \cup T). \end{aligned}$$

At the first and the second cases we use only superadditivity of the initial game to estimate the sum $v^i_\xi(S) + v^i_\xi(T)$, but at the third case we use the convexity of (N, v) and non-negativity of $v(\{i\})$ and ξ_i. This completes the proof. □

MDM-reduction $(N \setminus R, v^R_{\xi^0})$

Now let us introduce the MDM-reduction for a more general variant of the reduction, when a coalition R is removed from the initial game (N, v). In this section we show that the characteristic function of the final game $(N \setminus R, v^R_{\xi^0})$ does not depend on the order of removing $i \in R$ and it is equal to

$$v^R_{\xi^0}(S) = \begin{cases} 0, & S = \emptyset, \\ v(N) - \sum_{j \in R} x_j, & S = N \setminus R, \\ \max_{K \subset R} \left\{ v(S \cup K) - \sum_{j \in K} \xi^0_j \right\}, & S \subset N \setminus R. \end{cases} \tag{3.4}$$

Here x is an imputation of the initial game, and $\xi^0 \in X^0(N, v)$.

To prove this fact let us use the mathematical induction.

The induction base. Let us consider the MDM-reduced game $(N \setminus \{i\}, v^i_{\xi^0})$ and find the MDM-reduced game $\left(N \setminus \{i,j\}, \left(v^i_{\xi^0}\right)^j_{\xi^0}\right)$:

$$\begin{aligned} & \left(v^i_{\xi^0}\right)^j_{\xi^0}(\emptyset) = 0 = v^{i,j}_{\xi^0}(\emptyset), \\ & \left(v^i_{\xi^0}\right)^j_{\xi^0}(N \setminus \{i,j\}) = v^i_{\xi^0}(N \setminus \{i\}) - x_j = v(N) - x_i - x_j \\ & = v^{i,j}_{\xi^0}(N \setminus \{i,j\}), \\ & \left(v^i_{\xi^0}\right)^j_{\xi^0}(S) = \max\left\{ v^i_{\xi^0}(S \cup j) - \xi^0_j; v^i_{\xi^0}(S) \right\} \\ & = \max\left\{ v(S \cup \{i,j\}) - \xi^0_i - \xi^0_j; v(S \cup j) - \xi^0_j; v(S \cup i) - \xi^0_i; v(S) \right\} \\ & = v^{i,j}_{\xi^0}(S), \ S \subset N \setminus \{i,j\}. \end{aligned}$$

That means that in this case the order of the player removing i and j is not significant.

The inductive assumption. Let us assume that the players $i \in R \setminus \{j\}$ are removed from the initial game (N, v) one after another and the finishing MDM-reduced game is equal to $v_{\xi^0}^{R\setminus\{j\}}$. It is left to show that if the player j removes from the game we get the form (3.4). Indeed, the characteristic function of the new game is the following

$$\begin{aligned}
&\left(v_{\xi^0}^{R\setminus\{j\}}\right)_{\xi^0}^{j}(\emptyset) = 0 = v_{\xi^0}^{R}(\emptyset),\\
&\left(v_{\xi^0}^{R\setminus\{j\}}\right)_{\xi^0}^{j}(N \setminus R) = v(N) - \sum_{i\in R} x_i = v_{\xi^0}^{R}(N \setminus R),\\
&\left(v_{\xi^0}^{R\setminus\{j\}}\right)_{\xi^0}^{j}(S) = \max\left\{v_{\xi^0}^{R\setminus\{j\}}(S \cup j) - \xi_j^0; v_{\xi^0}^{R\setminus\{j\}}(S)\right\}\\
&= \max_{K\subset R}\left\{v(S \cup K) - \sum_{i\in K} \xi_i^0\right\}, \ S \subset N \setminus R.
\end{aligned}$$

Consequently, the identity (3.4) holds and the characteristic function of the resulting reduced game does not depend on the order of player removal.

The same method can be used to prove that the MDM-reduction $\left(N \setminus \{R \cup T\}, u_{\xi^0}^{T}\right)$ of the MDM-reduction $(N \setminus R, u)$, where $u = v_{\xi^0}^{R}$, is the MDM-reduction $\left(N \setminus \{R \cup T\}, v_{\xi^0}^{R\cup T}\right)$.

We can rewrite Lemmas 3.2 and 3.3, and Theorem 3.4 in the following way.

Lemma 3.8. *If (N, v) is a balanced TU-game and*

$$\sum_{i\in N\setminus R} \eta_i^0 \geq v(N \setminus R), \quad \eta^0 \in X^0(N \setminus R, v_{\xi^0}^{R}(\cdot)), \tag{3.5}$$

then $\xi_{N\setminus R}^0 \subset X^0(N\setminus R, v_{\xi^0}^{R}(\cdot))$.

Lemma 3.9. *If the original game (N, v) is balanced and (3.5) is fulfilled, then MDM-reduced game $(N\setminus R, v_{\xi^0}^{R}(\cdot))$ is also balanced.*

Theorem 3.10. *The set $SC(v, \xi^0)$ is consistent with respect to the MDM-reduced game $(N \setminus R, v_{\xi^0}^{R}(\cdot))$ if (3.5) is fulfilled.*

3.3 Dynamic Consistency

In this section we introduce a dynamic consistency property and show that the time-consistent imputation from the grand subcore is dynamically consistent with respect to the MDM-reduced game. Assume that $\phi(N, v)$ is an optimality principle for a multistage cooperative game and $\xi(t) \in \phi(N, v(t))$, $t \in \mathbb{T}$. Now we denote a reduced subgame for the period $\mathbb{T}^* = \{t^*, \dots, T\}$ with player $j \in N$ removed, $t \in \mathbb{T}^*$, by $(N \setminus \{j\}, v^j_{\xi^0}(S, \xi_j, t))$.

Definition 3.11. *We call a solution $\phi(N, v)$ dynamically consistent with respect to reduced game $(N\backslash\{j\}, v^j_\xi)$, if for all $x(t) \in \phi(N, v(t))$ at every moment $t \in \mathbb{T}^*$ we have $(x(t))_{N\backslash\{j\}} \in \phi(N\backslash\{j\}, v^j_\xi)$.*

Now let us redefine the MDM-reduction for the original multistage cooperative game (N, v), $t \in [t_0, T]$. For a given player $j \in N$ removed at time t^*, the payoff vector $\xi(t)$, and $\xi^0(t) \in X^0(N, v(t))$, the characteristic function of the MDM-reduced game for every $t \in \mathbb{T}^*$ is the following

$$\begin{aligned} v^j_{\xi^0}(\cdot) &= v^j_{\xi^0}(S, \xi_j, t) \\ &= \begin{cases} 0, & \text{if } S = \emptyset, \\ v(N, t) - \xi_j(t), & \text{if } S = N\backslash\{j\}, \\ \max\{v(S \cup \{j\}, t) - \xi^0_j(t), v(S, t)\}, & \text{otherwise.} \end{cases} \end{aligned}$$

The following theorem is useful for finding dynamically consistent imputations from the subcore.

Theorem 3.12. *In a balanced multistage cooperative game (N, v), $t \in \mathbb{T}$, an imputation $\xi(t) \in TCGSC(N, v(t_0)) \cap SC(v, \xi^0(t_0))$ is dynamically consistent with respect to the MDM-reduced game $(N \setminus \{j\}, v^j_{\xi^0}(\cdot))$, $t \in \mathbb{T}^*$, if the condition*

$$\sum_{i \in N\backslash\{j\}} \eta^0_i(t) \geq v(N \setminus \{j\}, t), \quad \eta^0(t) \in X^0(N \setminus \{j\}, v^j_{\xi^0}(\cdot)), \tag{3.6}$$

is fulfilled.

The proof of this theorem is analogous to the proof of Theorem 3.4.

Example 3.13. *Let us construct the MDM-reduced game of the multistage cooperative game (N, v) with $v(\{i\}) = 0$ for all $i \in N$ (the characteristic function of the initial game one can find in Table 3.13). In Table 3.13 the points from the set $X^0(N, v(t))$ are presented in the column $\xi^0(t)$, the column $\xi(t)$ contains the time-consistent solution from the subcore $SC(v, \xi_0(t))$.*

We assume that at the moment $t^* = t_2$ the player number 1 leaves the game (N, v), $t \in \mathbb{T}$. Table 3.13 includes the MDM-reduced game $(\{2, 3\}, v^1_{\xi^0})$, $t \in \mathbb{T}^*$, and the subcore of this game.

One can see that $\xi^0_{\{2,3\}}(t)$ belongs to the set $X^0(\{2, 3\}, v^1_{\xi^0}(t))$ as $t \in \mathbb{T}^*$, and $\xi_{\{2,3\}}(t)$ belongs to the subcore $SC(v^1_{\xi^0}, \xi^0_{\{2,3\}}(t))$, $t \in \mathbb{T}^*$. In other words, in the initial multistage cooperative game (N, v), $t \in \mathbb{T}$, the imputation $\xi(t_0)$ in Table 3.13 is dynamically consistent with respect to the MDM-reduced game $(\{2, 3\}, v^1_{\xi^0}(t))$, $t \in \mathbb{T}^*$.

t	$v(N,t)$	$v(\{1,2\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$
t_0	100	60	60	80
t_1	80	40	50	60
t_2	70	35	40	30
t_3	50	25	12	12
t_4	25	10	7	11
t_5	0	0	0	0

TABLE 5: The characteristic function for Example 3.13.

t	$\xi^0(t)$	$\xi(t)$
t_0	(40, 20, 40)	(40,20,40)
t_1	(25, 15, 35)	(26,5;17;36,5)
t_2	(12,5;**22,5**;**17,5**)	(18,26,26)
t_3	(12, **13**, **0**)	(15,15,20)
t_4	(7, **3**, **4**)	(8,8,9)
t_5	(0, 0, 0)	(0,0,0)

TABLE 6: Vectors $\xi^0(t) \in X^0(N, v(t))$ and $\xi(t) \in SC(v, \xi^0(t))$ of the initial game (N, v), $t \in \mathbb{T}$, in Example 3.13.

t	$v^1_{\xi^0}(\{2,3\},t)$	$v^1_{\xi^0}(\{2\},t)$	$v^1_{\xi^0}(\{3\},t)$	$X^0(\{2,3\}, v^1_{\xi^0}(t))$
t_2	52	22,5	17,5	(**22,5**;**17,5**)
t_3	35	13	0	(**13**,**0**)
t_4	17	3	4	(**3**,**4**)

TABLE 7: MDM-reduced game and the sets $X^0(\{2,3\}, v^1_{\xi^0}(t))$ in Example 3.13.

4 MINIMAL REDUCTION

4.1 Introduction

In this chapter we formulate the problem of minimal reduction, and apply it to the regularization of dynamic TU-games.

Up to now we used the classical approach to the regularization of games to provide time-consistency of a solution based on delays of total payoffs to the players (see Chapter 2). Here we employ a reduced game due to Davis and Maschler [24] and a modified Davis-Maschler reduced game (see Chapter 3) to get time-consistent imputations from the core and construct the corresponding IDPs.

The chapter is organized as follows. In Section 4.2 we formulate the minimal reduction problem. In Section 4.3 this problem is considered under the restriction of a feasible removing coalition. Conditionally minimal coalitions are defined with respect to the core and DM-reduction [24] and with respect to the subcore and MDM-reduction [127]. Section 4.4 is devoted to minimal reduction in a general case (when the removing coalition is chosen from the whole player set). Example 4.18 in Section 4.5 illustrates the application of minimal reduction to a multistage cooperative game.

4.2 Formulation of minimal reduction problem

Let us consider the following problem of minimal reduction of a balanced TU-game. Let (N, v) be a cooperative game with non-empty core, vector $\xi = (\xi_1, \ldots, \xi_n)$ be an imputation, which does not belong to the core. What is the minimal coalition $K \subset N$ such that $\xi_{N \setminus K} \in C(N \setminus K, v_\xi^K)$?

This problem has a solution. Indeed, a set $N \setminus \{i\}$, for all $i \in N$, could be taken as removing coalition K. Then the characteristic function of DM-reduced game is

$$v_\xi^{N\setminus\{i\}}(\{i\}) = v(N) - \sum_{j\in N\setminus\{i\}} \xi_j = \xi_i.$$

That is ξ_i belongs to the core $C(\{i\}, v_\xi^{N\setminus\{i\}})$. Since ξ is an imputation and the set of coalitions is finite, there exists at least one minimal size coalition.

We call this problem a problem of minimal reduction with respect to the imputation $\xi \notin C(N, v)$ and DM-reduced game.

Let us denote by $\{T_r\}_{r=1}^m$ the set of coalitions T_r, $r = 1, \dots, m$, for which the following inequalities hold

$$\sum_{i\in T_r} \xi_i < v(T_r),\ r = 1, \dots, m \tag{4.1}$$

$$\sum_{i\in S} \xi_i \geq v(S),\ \forall S \subset N,\ S \neq T_r,\ r = 1, \dots, m. \tag{4.2}$$

In this chapter we discuss two approaches to the minimal reduction. On one hand, it is natural to find a feasible (for reduction) coalition K as a subset of $U = \bigcup_{r=1}^m T_r$. The notion of a conditional minimal coalition is introduced (see Chapter 4). On the other hand, in a general case ($K \subset N$) the number of players in a minimal coalition could be less than in a conditionally minimal coalition (Chapter 5).

Analogously, we can formulate and consider the problem of minimal reduction with respect to an imputation $\xi \notin GSC(N, v)$ and MDM-reduced game.

4.3 Conditionally minimal coalition

Definition 4.1. *Coalition K is called conditionally minimal coalition with respect to an imputation $\xi \notin \phi(N, v)$ (ϕ is an optimality principle), coalition U, and reduction game $(N \setminus K, v_\xi^K)$ if*

1. $K \subseteq U \subset N$;
2. $\xi_{N\setminus K} \in \phi(N \setminus K, v_\xi^K)$;
3. *there is no such* $K' \subset K$, *that* $\xi_{N\setminus K'} \in \phi(N \setminus K', v_\xi^{K'})$.

Here U is a collection of players, which could depend on the optimality principle. In this paper we consider the following coalitions $U(C, \xi) = \bigcup_{r=1}^m T_r$ for the core and $U(GSC, \xi, \xi^0) = \{i \in N : \xi_i < \xi_i^0\}$ for the grand subcore[3].

The following example illustrates how to choose a conditionally minimal coalition with respect to $\xi \notin C(N, v)$, $U(C, \xi)$ and DM-reduced game in a cooperative game (N, v).

[3] In the case of a study of conditionally minimal coalition with respect to the grand subcore a set $U(GSC, \xi, \xi^0)$ depends on the vector ξ^0, then by "good enough" choice of $\xi^0 \in X^0(N, v)$ we can decrease a conditionally minimal coalition.

Example 4.2. *Let us consider a three person TU-game with* $v(\{i\}) = 0$ *for all* $i \in N$*, and* $0 \le v(\{i,j\}) \le v(N)$, $i,j \in N$*. Assume that for* $\xi = (\xi_1, \xi_2, \xi_3)$ *the following conditions hold*

$$\xi_1 + \xi_2 \ge v(\{1,2\});$$
$$\xi_1 + \xi_3 \ge v(\{1,3\});$$
$$\xi_2 + \xi_3 < v(\{2,3\}).$$

That is the vector ξ *does not belong to the core* $C(N,v)$ *because of the coalition* $\{2,3\}$ *(*$U(C,\xi) = \{2,3\}$*).*

Let us construct DM-reduced game $v_\xi^K(\cdot)$, $K = \{2,3\}$*. It is clear that* $v_\xi^K(\{1\}) = \xi_1$*. Hence, the removal of* $\{2,3\}$ *leads to imputation from the set* $C(\{1\}, v_\xi^{\{2,3\}}(\cdot))$*.*

If we set $K' := \{2\}$*, then the characteristic function of the reduced game is as follows*

$$v_\xi^{\{2\}}(\{1\}) = \max\{0, v\{1,2\} - \xi_2\};$$
$$v_\xi^{\{2\}}(\{3\}) = \max\{0, v\{2,3\} - \xi_2\} = v\{2,3\} - \xi_2;$$
$$v_\xi^{\{2\}}(\{1,3\}) = v(N) - \xi_2.$$

Vector ξ *is an imputation, therefore* $\xi_1 \ge 0$*. Moreover, from the inequality* $\xi_1 + \xi_2 \ge v(\{1,2\})$ *we have* $\xi_1 \ge v(\{1,2\}) - \xi_2$*. Then* $\xi_1 \ge v_\xi^{\{2\}}(\{1\})$*. However, the condition* $\xi_3 \ge v_\xi^{\{2\}}(\{3\})$ *does not apply in view of* $\xi_2 + \xi_3 < v(\{2,3\})$*. The situation is similar, if* $K' := \{3\}$*.*

Consequently, we obtained that the coalition $\{2,3\}$ *is conditionally minimal with respect to* $\xi \notin C(N,v)$, $U(C,\xi)$ *and DM-reduced game.*

Assume that there is such a coalition T in the set $\{T_r\}_{r=1}^m$, that $T_r \subseteq T$ for all $r = 1, \dots, m$. In that case T is the conditionally minimal with respect to $\xi \notin C(N,v)$, $U(C,\xi)$ and DM-reduced game.

Theorem 4.3. *Coalition* $K(\xi) = T$ *is conditionally minimal with respect to* $\xi \notin C(N,v)$, $U(C,\xi)$ *and DM-reduced game.*

Proof. We should show that

1.$K \subseteq U(C,\xi)$,

2.$\xi_{N \setminus K(\xi)} \in C(N \setminus K(\xi), v_\xi^{K(\xi)})$,

3. there is no such $K' \subset K(\xi)$ that $\xi_{N \setminus K'} \in C(N \setminus K', v_\xi^{K'})$.

Since $U(C,\xi) = \bigcup_{r=1}^m T_r$ and $T_r \subseteq T$ for $r = 1, \dots, m$ we have $T = U(C,\xi)$ (and $K(\xi) = U(C,\xi)$). By the definition of the reduced game due to Davis and Maschler, we have

$$v_\xi^{K(\xi)}(N \setminus K(\xi)) = v(N) - \sum_{i \in K(\xi)} \xi_i = \sum_{i \in N \setminus K(\xi)} \xi_i.$$

Moreover, for any $S \subsetneq N \setminus K(\xi)$, $S \neq \emptyset$, and $R \subseteq K(\xi)$ the coalition $S \cup R$ is not subset of T, hence

$$\sum_{i \in S \cup R} \xi_i \geq v(S \cup R),$$

consequently

$$\sum_{i \in S} \xi_i \geq v_\xi^{K(\xi)}.$$

That is $\xi_{N \setminus K(\xi)} \in C(N \setminus K(\xi), v_\xi^{K(\xi)})$.

Let us suppose that there exists a coalition $K' \subset K(\xi)$ and $\xi_{N \setminus K'} \in C(N \setminus K', v_\xi^{K'})$. That is for any coalition $S \subset N \setminus K'$, $S \neq N \setminus K'$, $S \neq \emptyset$ the inequality

$$\sum_{i \in S} \xi_i \geq v_\xi^{K'}(S) = \max_{R \subseteq K'} \left\{ v(S \cup R) - \sum_{i \in R} \xi_i \right\}$$

applies. Consequently, for $S = K(\xi) \setminus K'$ and $R = K'$ we can write

$$\sum_{i \in K(\xi) \setminus K'} \xi_i \geq v\left((K(\xi) \setminus K') \cup K'\right) - \sum_{i \in K'} \xi_i.$$

It is equivalent to

$$\sum_{i \in K(\xi)} \xi_i \geq v(K(\xi)).$$

This condition contradicts the original assumption. □

The condition $\xi \notin C(N, v)$ implies that $\xi \notin GSC(N, v)$. The following example shows that a conditional minimal coalition with respect to $\xi \notin C(N, v)$, $U(GSC, \xi, \xi^0)$ and MDM-reduced game can be less than a conditional minimal coalition with respect to $\xi \notin C(N, v)$, $U(C, \xi)$ and DM-reduced game.

Example 4.4. *Let us fix a vector $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0)$ from the solution set $X^0(N, v)$ of linear programming problem (2.6), (2.7) for the game in Example 4.2. We assume that $\xi_1 \geq \xi_1^0$, $\xi_2 < \xi_2^0$, $\xi_3 \geq \xi_3^0$, then $U(GSC, \xi, \xi^0) = \{2\}$. MDM-reduced game $(\{1\}, v_{\xi^0}^{\{2,3\}})$ with respect to ξ and ξ^0 coincides with DM-reduced game $(\{1\}, v_\xi^{\{2,3\}})$. If we take $K := \{2\}$ then*

$$v_{\xi^0}^{\{2\}}(\{1\}) = \max\{0, v\{1,2\} - \xi_2^0\};$$
$$v_{\xi^0}^{\{2\}}(\{3\}) = \max\{0, v\{2,3\} - \xi_2^0\}.$$

Since $\xi_1^0 + \xi_2^0 \geq v\{1,2\}$ and $\xi_1 \geq \xi_1^0 \geq 0$, then $\xi_1 \geq v_{\xi^0}^{\{2\}}(\{1\})$. Analogically, from $\xi_2^0 + \xi_3^0 \geq v\{2,3\}$ and $\xi_2 \geq \xi_2^0 \geq 0$ we have $\xi_3 \geq v_{\xi^0}^{\{2\}}(\{3\})$. Thus the vector (ξ_1, ξ_3) belongs to the core of MDM-reduced game $(\{1,3\}, v_{\xi^0}^{\{2\}})$.

For a vector $\xi^0 \in X^0(N,v)$ and an imputation $\xi \notin GSC(N,v)$ let us denote by $K(\xi^0,\xi)$ such a proper coalition of N, that

$$\xi_i^0 > \xi_i, \;\; i \in K(\xi^0,\xi), \tag{4.3}$$

$$\xi_i^0 \le \xi_i, \;\; i \in N \setminus K(\xi^0,\xi). \tag{4.4}$$

Theorem 4.5. *Assume that (N,v) is balanced, $\xi^0 \in X^0(N,v)$, $\xi \notin GSC(N,v)$, and conditions (3.3), (4.3) and (4.4) hold. Vector $\xi_{N\setminus K}$ belongs to the set $SC(v^K_{\xi^0}(\cdot),\xi^0_{N\setminus K})$ if $K = K(\xi^0,\xi)$.*

Proof. Directly from Lemma 3.2 we have $\xi^0_{N\setminus K} \in X^0(N \setminus K, v^K_{\xi^0}(\cdot))$. Hence to show that the reduced game $(N \setminus K, v^K_{\xi^0}(\cdot))$ is balanced we should prove that the inequality

$$\sum_{i\in N\setminus K} \xi_i^0 \le v^K_{\xi^0}(N \setminus K, \xi_K)$$

takes place. As the original game is balanced we obtain that

$$\sum_{i\in N} \xi_i^0 \le v(N).$$

It is equivalent to

$$\sum_{i\in N\setminus K} \xi_i^0 \le v(N) - \sum_{i\in K} \xi_i^0.$$

Taking into account the limitation $\xi_i^0 > \xi_i$ for all $i \in K$, we can write

$$\sum_{i\in N\setminus K} \xi_i^0 \le v(N) - \sum_{i\in K} \xi_i,$$

that is

$$\sum_{i\in N\setminus K} \xi_i^0 \le v^K_{\xi^0}(N \setminus K, \xi_K).$$

Consequently, the set $C(N \setminus K, v^K_{\xi^0}(\cdot))$ is not empty.

Since $\xi^0_{N\setminus K} \in X^0(N \setminus K, v^K_{\xi^0}(\cdot))$ and for all $i \in N \setminus K$ the condition $\xi_i^0 \le \xi_i$ is fulfilled, the inclusion $\xi_{N\setminus K} \in SC(v^K_{\xi^0}(\cdot), \xi^0_{N\setminus K})$ holds by Definition **??**. □

In this case coalition K is conditionally minimal with respect to the grand subcore and MDM-reduction of the original game.

4.4 Acceptable coalitions

The goal of this section is to determine the collection of all such coalitions K, that $\xi_{N\setminus K} \in C(N \setminus K, v^K_\xi)$ for an imputation $\xi \notin C(N,v)$. The following example demonstrates that in a general case when we choose the removed coalition from the set of all players it can be less than a conditionally minimal coalition.

Example 4.6. *Let us return again to the TU-game which was described in Example 4.2 and consider player* 1 *as a removing coalition* ($K := \{1\}$)*. DM-reduced game is as follows*

$$v_\xi^{\{1\}}(\{2,3\}) = v(N) - \xi_1,$$
$$v_\xi^{\{1\}}(\{2\}) = \max\{0, v(\{1,2\}) - \xi_1\},$$
$$v_\xi^{\{1\}}(\{3\}) = \max\{0, v(\{1,3\}) - \xi_1\}.$$

Vector (ξ_2, ξ_3) *belongs to the core of the reduced game* $(\{2,3\}, v_\xi^{\{1\}})$ *if and only if the inequalities*

$$\xi_2 \geq v_\xi^{\{1\}}(\{2\}),$$
$$\xi_3 \geq v_\xi^{\{1\}}(\{3\})$$

hold. Since ξ *is an imputation, then* $\xi_2 \geq 0$ *and* $\xi_3 \geq 0$*. That is, it remains to check the conditions*

$$\xi_2 \geq v(\{1,2\}) - \xi_1,$$
$$\xi_3 \geq v(\{1,3\}) - \xi_1.$$

It is equivalent to

$$\xi_1 + \xi_2 \geq v(\{1,2\}),$$
$$\xi_1 + \xi_3 \geq v(\{1,3\}).$$

Hence, $(\xi_2, \xi_3) \in C(\{2,3\}, v_\xi^{\{1\}})$.

Let us again fix a balanced TU-game (N, v) and an imputation $\xi \notin C(N, v)$.

Definition 4.7. *Coalition* K *is called acceptable for reduction with respect to* $\xi \notin C(N, v)$ *and DM-reduced game* $(N \setminus K, v_\xi^K)$*, if* $\xi_{N\setminus K} \in C(N \setminus K, v_\xi^K)$.

Theorem 4.8. *Coalition* $K \subset N$ ($|K| < |N| - 1$) *is acceptable for reduction with respect to* $\xi \notin C(N, v)$ *and DM-reduced game if and only if for all* T_r, $r = 1, \ldots, m$ *there does not exist a couple of subcoalitions* $R \subseteq K$ *and* $S \subset N \setminus K$ *such that* $S \cup R = T_r$, $S \neq \emptyset$, $R \neq \emptyset$.

Proof. Assume that K is a coalition of N, $|K| < |N| - 1$. Davis-Maschler reduced game $(N \setminus K, v_\xi^K)$ is the following

$$v_\xi^K(N \setminus K) = v(N) - \sum_{i \in K} \xi_i, \tag{4.5}$$

$$v_\xi^K(S) = \max_{R \subseteq K} \left\{ v(S \cup R) - \sum_{i \in R} \xi_i \right\}, \ \forall S \subset N \setminus K,\, S \neq N \setminus K. \tag{4.6}$$

By Definition 4.7 coalition K is acceptable for reduction if and only if $\xi_{N\setminus K} \in C(N\setminus K, v_\xi^K)$, i.e. $\xi_{N\setminus K}$ is an imputation of corresponding DM-reduced game and $\sum_{i\in S}\xi_i \geq v_\xi^K(S)$ for all $S\subset N\setminus K$, $S\neq N\setminus K$. It is equal to the requirement

$$\sum_{i\in S}\xi_i \geq \max_{R\subseteq K}\left\{v(S\cup R)-\sum_{i\in R}\xi_i\right\},\ \forall S\subset N\setminus K,\, S\neq N\setminus K$$

$$\Leftarrow: \sum_{i\in S}\xi_i \geq v(S\cup R)-\sum_{i\in R}\xi_i,\ \forall S\subset N\setminus K,\, S\neq N\setminus K,\ \forall R\subseteq K$$

$$\Leftarrow: \sum_{i\in S\cup R}\xi_i \geq v(S\cup R),\ \forall S\subset N\setminus K,\, S\neq N\setminus K,\ \forall R\subseteq K. \tag{4.7}$$

To satisfy the condition (4.7) it is necessary and sufficient that $S\cup R\neq T_r$ for $r=1,\dots,m$, and for any proper subcoalition $S\subset N\setminus K$, and for any $R\subseteq K$ ($S\neq\emptyset$, $R\neq\emptyset$). In other words, the inequality $\sum_{i\in S\cup R}\xi_i\geq v(S\cup R)$ is tantamount to $\sum_{i\in T_r}\xi_i\geq v(T_r)$, which contradicts (4.1). □

Corollary 4.9. *If $m=1$ then there are only two coalitions $K_1=T_1$ and $K_2=N\setminus T_1$ which are acceptable for reduction.*

Proof. Let us assume that coalition $K\subset N$ is acceptable for reduction. We denote the intersection $K\cap T_1$ by R and $K\cap(N\setminus T_1)$ by S. Then $S\cup R\neq T_1$ if and only if either $S=T_1$ or $R=T_1$. Otherwise K is not acceptable for reduction by Theorem 4.8. □

Corollary 4.10. *If $m=2$, $T_1\cap T_2=\emptyset$ and $T_1\cup T_2=N$ then there are only two coalitions $K_1=T_1$ and $K_2=T_2$ which are acceptable for reduction.*

The **proof** is similar to the previous one.

Definition 4.11. *Coalition $K^*\in\{K\}$ is called minimal with respect to $\xi\notin C(N,v)$ and DM-reduced game $(N\setminus K, v_\xi^K)$, if $|K^*|=\min_{\{K\}}|K|$. Here $\{K\}$ is the set of all acceptable coalitions.*

Let us denote by $\min_{|\cdot|}\{A\}$ the minimal size coalition of a collection $A=\{S_1,\dots,S_k\}$, where $S_j\subset N$, $j=1,\dots,k$. Corollaries 4.9 and 4.10 imply that under their assumptions the minimal coalition with respect to the core and DM-reduced game is $K_{min}=\min_{|\cdot|}\{K_1,K_2\}$.

Corollary 4.12. *If $\bigcap_{r=1}^m T_r\neq\emptyset$ then $K=N\setminus\bigcap_{r=1}^m T_r$ is acceptable for reduction.*

Proof. Suppose that $K=N\setminus\bigcap_{r=1}^m T_r$. There is no proper coalition $S\subset N\setminus K$ such that $S\cup R=T_r$ (for any $R\subseteq K$). Hence K is an acceptable coalition by Theorem 4.8. □

Corollary 4.13. *If $\bigcup_{r=1}^m T_r\neq N$ then $K=\bigcup_{r=1}^m T_r$ is acceptable for reduction.*

Proof. Suppose that $K = \bigcup_{r=1}^{m} T_r$. Since $N \setminus K$ does not contain any player $i \in T_r$ for $r = 1, \dots, m$ there is no such $S \subset N \setminus K$ that $S \cup R = T_r$ for any $R \subseteq K$ and $r = 1, \dots, m$. Consequently, K is acceptable for reduction. □

Corollary 4.14. *If* $T_1 \subset T_2 \subset \cdots \subset T_m$ *then the minimal coalition is* $K_{min} = \min_{|\cdot|}\{T_m, N \setminus T_1\}$.

Proof. It immediately follows from Corollary 4.12 and 4.13 that $K_1 = N \setminus T_1$ and $K_2 = T_m$ are acceptable coalitions. Another acceptable coalitions could be presented in the forms $N \setminus T$, $T \subset T_1$, or $\overline{T}$, $T_m \subset \overline{T}$. However this coalition consists of more players than K_1 and K_2 correspondingly. This implies that the statement of the corollary is true. □

The following corollary is essentially proved in Chapter 3.

Corollary 4.15. *The coalition* $K_i = N \setminus \{i\}$ *is acceptable for reduction for all* $i \in N$.

4.5 Dynamic example

Here we again deal with a multistage TU-game. Let us remind that we call a pair $(N, v(\cdot))$ a multistage cooperative game. Here $\mathbb{T} = \{t_r\}_{r=0}^{l}$ is a division of time period $t_0 < t_1 < \cdots < t_l$, N is a finite set of players and $v : 2^N \times \mathbb{T} \mapsto \mathbb{R}$ is a characteristic function of the game, $v(\emptyset, t) = 0$ for all $t \in \mathbb{T}$, $v(S, t_l) = 0$ for all $S \subset N$. By $(N, v(t^*))$ we means the subgame at a moment $t^* \in \mathbb{T}$. We assume that $v(N, t)$ is the decreasing function with respect to t.

In [89] the time-consistency was introduced for an optimality principle ϕ, which is given at every moment $t \in \mathbb{T}$. Let us reformulate Definition 2.7 in the following way.

Definition 4.16. *Suppose that* $\xi = (\xi_1, \dots, \xi_n) \in \phi(t_0)$. *Any matrix* $\alpha = \{\alpha_{ik}\}$, $i = 1, \dots, n$, $k = 0, \dots, l$, *such that*

$$\xi_i = \sum_{k=0}^{l} \alpha_{ik}, \quad \alpha_{ik} \geq 0,$$

is called the imputation distribution procedure (IDP).

In other words, an element α_{ik} of IDP α is a payoff to the player i at the moment t_k, and a column number k is equal to the payoff vector at the moment t_k.

We denote $\alpha_k = (\alpha_{1k}, \dots, \alpha_{nk})$ and $\alpha(t_k) = \sum_{r=0}^{k} \alpha_r$. That is, a vector α_k consists of summary payoffs to every player by the moment t_k, and the value $\alpha(t_k)$ is total payoff to all players.

Definition 4.17. *An optimality principle $\phi(t_0)$ is called time-consistent if for every $\xi \in \phi(t_0)$ there exists IDP α, such that*

$$\xi - \alpha(t_k) \in \phi(t_k), \quad k = 0, \dots, l.$$

The following example demonstrates the use of a conditionally minimal coalition and a minimal coalition to provide time-consistency of given imputation.

Example 4.18. *Let us consider the balanced multistage TU-game $(N, v(\cdot))$, $N = \{1, 2, 3\}$ (the characteristic function of the game see in Table 4.18). We assume that $v(\{i\}, t) = 0$ for $t \in \mathbb{T}$.*

t	$v(\{1,2,3\},t)$	$v(\{1,2\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$
t_0	80	34	68	58
t_1	60	30	50	40
t_2	40	35	18	19
t_3	0	0	0	0

TABLE 8: Multistage TU-game $(N, v(\cdot))$ for Example 4.18.

At the moment $t = t_0$ the core, the set $X^0(\cdot)$ and the grand subcore coincide and consist of one vector $\xi(t_0)$ (see Figure 6a)

$$C(N, v(t_0)) = GSC(N, v(t_0)) = X^0(v(\cdot), t_0) = \{(12, 22, 46)\}.$$

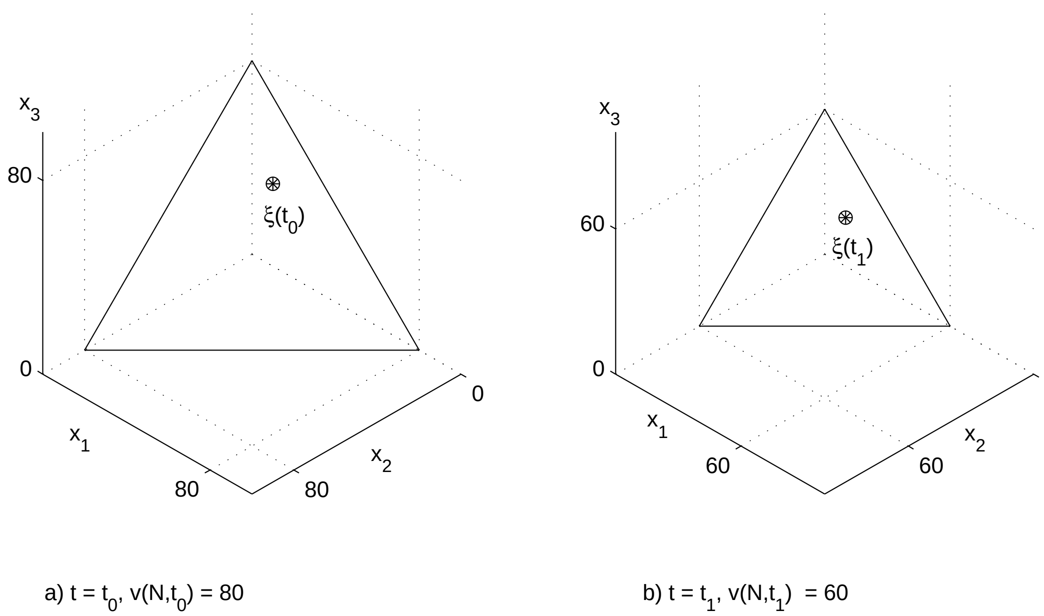


FIGURE 6: Imputation simplex for subgames at $t = t_0$ and at $t = t_1$.

Let us also compute the core, $X^0(\cdot)$ and the grand subcore at $t = t_1$ and $t = t_2$ (see Figure 6b and 7). At the moment t_1 the core and grand subcore coincide with *a vector* $\xi(t_1)$ *(Figure 6b)*

$$C(N, v(t_1)) = GSC(N, v(t_1)) = X^0(v(\cdot), t_1) = \{(10, 20, 30)\}$$

The core at the moment t_2 *is a convex hull of vectors* $(14, 21, 5)$, $(22, 13, 5)$, $(22, 18, 0)$ *and* $(19, 21, 0)$ *(see Figure 7, bold line)*

$$C(N, v(t_2)) = Co\{(14, 21, 5); (22, 13, 5); (22, 18, 0); (19, 21, 0)\}.$$

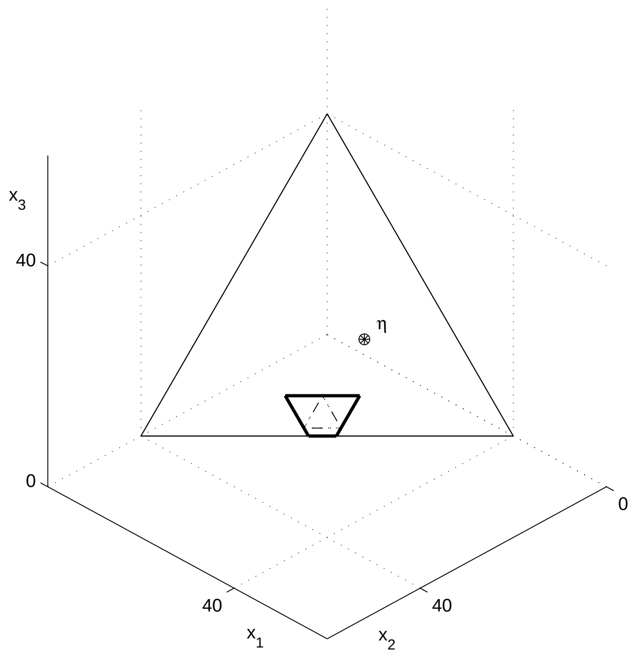


FIGURE 7: Imputation simplex for subgame at $t = t_2$, $v(N, t_2) = 40$.

The set $X^0(v(\cdot), t_2)$ *is the unique solution* ξ^0

$$X^0(v(\cdot), t_2) = \{\xi^0\} = \{(18, 17, 1)\}.$$

Grand subcore coincides with subcore $SC(v(t_2), \xi^0)$ *(see Figure 7, "dash-dot" line)*

$$GSC(N, v(t_2)) = Co\{(22, 17, 1); (18, 21, 1); (18, 17, 5)\}.$$

The imputation in $C(N, v(t_0))$ *is not time-consistent (i.e. there is no such IDP that can move players from the point* $(12, 22, 46)$ *into the set* $C(N, v(t_2))$*) because time-consistency is violated at* t_2*. However, we can fix the following IDP (note, that it only provides non-negative payoffs during every step of the game):*

$$\alpha = \begin{pmatrix} 0 & 2 & 0 & 10 \\ 0 & 2 & 2 & 18 \\ 0 & 16 & 18 & 12 \end{pmatrix}.$$

The corresponding trajectory of the payoffs is

$$(12, 22, 46) \to (10, 20, 30) \to (10, 18, 12) \to (0, 0, 0).$$

We denote an intermediate vector $(10, 18, 12)^4$ *by* η. *Then* $U(C(N, v(t_2)), \eta) = \{1, 2\}$ *and* $U(GSC(t_2), \eta) = \{1\}$.

A conditional minimal coalition $K(\xi^0, \eta)$ *with respect to the subcore and MDM-reduction is* $\{1\}$. *MDM-reduced subgame* $(\{2, 3\}, v_{\xi^0}^{\{1\}}(\cdot))$ *at* $t = t_2$ *is the following*

$$v_{\xi^0}^{\{1\}}(\{2, 3\}, t_2) = 30,$$
$$v_{\xi^0}^{\{1\}}(\{2\}, t_2) = 17,$$
$$v_{\xi^0}^{\{1\}}(\{3\}, t_2) = 1.$$

The solution set of linear programming problem (2.6), (2.7) for this subgame is

$$X^0(v_{\xi^0}^{\{1\}}(\cdot), t_2) = \{(17, 1)\}.$$

A conditional minimal coalition with respect to the core and DM-reduced game is $K(\eta) = \{1, 2\}$. *The characteristic function of DM-reduced game* $(\{3\}, v_\eta^{\{1,2\}}(\cdot))$ *is*

$$v_\eta^{\{1,2\}}(\{3\}, t_2) = 12.$$

Minimal coalition K *with respect to the core and DM-reduction is* $\{3\}$. *Corresponding DM-reduced subgame* $(\{1, 2\}, v_\eta^{\{3\}}(\cdot))$ *is*

$$v_\eta^{\{3\}}(\{1, 2\}, t_2) = 28,$$
$$v_\eta^{\{3\}}(\{1\}, t_2) = 7,$$
$$v_\eta^{\{3\}}(\{2\}, t_2) = 6.$$

Then the solution set of linear programming problem (2.6), (2.7) for the game $(\{1, 2\}, v_\eta^{\{3\}}(\cdot))$ *is*

$$X^0(v_\eta^{\{3\}}(\cdot), t_2) = \{(7, 6)\},$$

and

$$\eta_{\{1,2\}} = (10, 18) \in GSC(\{1, 2\}, v_\eta^{\{3\}}(t_2)) \subset C(\{1, 2\}, v_\eta^{\{3\}}(t_2)).$$

This example shows the possibility of the minimal reduction use to guarantee time-consistency of a chosen solution. The number of the removed players depends on the fixed optimality principle and reduced game rule as well as on the set $U(\cdot)$.

[4] Let us underscore that here we could take any vector η satisfying the assumptions $\eta \leq (10, 20, 30)$ and $\eta_1 + \eta_2 + \eta_3 = 40$ and consider an appropriate IDP.

4.6 Conclusion

The minimal reduction problem was considered in two ways. On the one hand, to regularize the original game it is natural to remove a coalition from the collection of the disturber players, and seeing this the notion of conditionally minimal coalition was introduced and investigated in Chapter 4. On the other hand, the minimal coalition in a general case could be less (in terms of number of players) than the conditionally minimal coalition, that is why the acceptable coalitions were discussed. Example 4.18 illustrated the method of the practical use of a minimal reduction to provide time-consistency of the imputation from the core of a multistage cooperative TU-game.

5 GAME-THEORETICAL MODELLING OF THE KYOTO PROTOCOL

5.1 Introduction

In this part we construct the time-consistent solutions in cooperative games of three country groups realizing the flexibility mechanisms of Kyoto Protocol[5].

Without a doubt, climate change is the first among the global environmental threats to civilization at the beginning of the XXI Century. The importance of this problem is demonstrated by the adaptation costs the global community pays to protect itself from a growing number of natural disasters. The United Nations Framework Convention on Climate Change was signed at the World Summit on the Environment and Development in Rio de Janeiro in 1992, and the Kyoto Protocol to the Convention was adopted in 1997. The Kyoto Protocol proposes six innovative "mechanisms:" joint implementation, clean development, emission trading, joint fulfilment, banking and sinks. The mechanisms aim to reduce the costs of curbing emissions by allowing Parties[6] to pursue opportunities to cut emissions more cheaply abroad than at home. The cost of curbing emissions varies considerably from region as a result of differences in, for example, energy sources, energy efficiency and waste management. It makes economic sense to cut emissions where it is cheapest to do so, given that the impact on the atmosphere is the same.

The Kyoto protocol defines six flexibility mechanisms and three of them have the following sense: "joint implementation" provides for Annex B Parties (mostly highly developed industry countries) to implement projects that reduce emission, or remove carbon from the air, in other Annex B Parties, in return for emission reduction units (ERUs); the "clean development" mechanism provides for Annex B Parties to implement projects that reduce emissions in non-Annex B Parties, in return for certified emission reductions (CERs), and assist the host Parties in

[5] For more details see e.g. http://unfccc.int/resource/docs/convkp/kpeng.pdf

[6] Party is a term of Kyoto Protocol and means a country, or group of countries, that has ratified the Kyoto Protocol.

achieving sustainable development and contributing to the ultimate objective of the Convention; "emission trading" provides for Annex B Parties to acquire units from other Annex B Parties. The emission reduction units and certified emission reductions generated by the flexibility mechanisms can be used by Annex B Parties to help meet their emission targets.

That flexibility mechanisms are the base of the cooperation because joint implementation, clean development, and emission trading comprehend that Parties work together and receive common "benefit" (emission reduction), which should be allocated fairly. It is natural to use the dynamic cooperative theory to model the Kyoto Protocol realization [28]. For other models connected with the flexibility mechanisms of the Kyoto Protocol see [9, 37, 87, 90, 91].

5.2 Kyoto Protocol model

In this section we describe a cooperative model of relations of countries (or groups of countries) under Kyoto Protocol. The players pursue two mail goals: to achieve the required amount of emission reduction units and to decrease the reduction costs. The participants of the corresponding projects can get significant income from realization of the flexibility mechanisms. To define the cooperative model we should set a method of calculation the characteristic function v of the game. We assume that $v(S)$, where S is a coalition of players, is the difference between the sum of the personal costs of players, when they act individually, and total cost of coalition S under co-operation. Here player is Party in Kyoto Protocol. In the model we use the following notations

K_i — emission quota of player i;

c_i^e — price of emission unit for player i;

c_i^q — price of emission unit on account of a pollution quota of player i;

ΔE_i — required emission reduction of player i;

ΔL_i — ecological sinks[7] of player i;

ΔK_i — a fraction of pollution quota that player i wants to use.

Let us consider a game with two players. The individual cost of player i is

$$H_i^0 = c_i^e(\Delta E_i - \Delta L_i - \Delta K_i). \tag{5.1}$$

Under co-operation a more developed country (player **1**) can invest money into the emission reduction in the territory of another country (player **2**). That is, $c_1^e > c_2^e$

[7] "Sinks" (Land use, land use change and forestry carbon units) The Protocol allows industrialized countries to meet part of their emissions targets through activities that absorb CO_2 so-called carbon 'sinks.' As with so many other details in the Protocol, the rules and modalities have yet to be worked out. http://europa.eu.int/comm/environment/press/bio00172.htm

and $c_1^q > c_2^q$. Let us assume that the reduction costs are

$$\begin{aligned} H_1 = c_2^e \cdot \delta_1(\Delta E_1 - \Delta L_1 - \Delta K_1) + c_2^q \cdot \delta_2(\Delta E_1 - \Delta L_1 - \Delta K_1) \\ + c_1^e(1 - \delta_1 - \delta_2)(\Delta E_1 - \Delta L_1 - \Delta K_1); \end{aligned} \tag{5.2}$$

$$\begin{aligned} H_2 = c_2^e(\Delta E_2 - \Delta L_2 - \Delta K_2 + \delta_2(\Delta E_1 - \Delta L_1 - \Delta K_1)) \\ - c_2^e \cdot \delta_1(\Delta E_1 - \Delta L_1 - \Delta K_1) - c_2^q \cdot \delta_2(\Delta E_1 - \Delta L_1 - \Delta K_1). \end{aligned} \tag{5.3}$$

Here δ_1 and δ_2 are parameters. It is possible to specify that parameters in different ways by some appropriate limits, for example by Q_1 (the limit of emission unites that the player **1** wants to buy from the player **2** at the price c_2^q on account of quota K_2), Q_2 (the limit of emission unites that the player **2** wants to sell to the player **1** on account of quota K_2), and M_1 (financial limit of the player **1**) in the following way

$$\min\{Q_1, Q_2\} = \delta_2(\Delta E_1 - \Delta L_1 - \Delta K_1) := Q,$$
$$M_1 \geq c_2^e \delta_1(\Delta E_1 - \Delta L_1 - \Delta K_1).$$

From (5.1)–(5.3) we have the value of characteristic function for the coalition of two players

$$\begin{aligned} v(\{1,2\}) &= H_1^0 + H_2^0 - H_1 - H_2 \\ &= (\Delta E_1 - \Delta L_1 - \Delta K_1)(\delta_1 c_1^e + \delta_2 c_1^e - \delta_2 c_2^q) \\ &= (\Delta E_1 - \Delta L_1 - \Delta K_1)(\delta_1 c_1^e + \delta_2(c_1^e - c_2^q)). \end{aligned} \tag{5.4}$$

In the case of three players' joint action we calculate

$$v(\{1,2,3\}) = H_1^0 + H_2^0 + H_3^0 - H_1 - H_2 - H_3. \tag{5.5}$$

We assume that $c_1^e > c_2^e > c_3^e$ and $c_1^q > c_2^q > c_3^q$. Then the player **1**'s cost under cooperation is

$$\begin{aligned} H_1 = c_3^q \cdot \delta_2(13)(\Delta E_1 - \Delta L_1 - \Delta K_1) + c_3^e \delta_1(13)(\Delta E_1 - \Delta L_1 - \Delta K_1) \\ + c_2^q \cdot \delta_2(12)(\Delta E_1 - \Delta L_1 - \Delta K_1) + c_3^e \cdot \delta_1(12)(\Delta E_1 - \Delta L_1 - \Delta K_1) \\ + c_1^e(1 - \delta_2(13) - \delta_1(13) - \delta_2(12) - \delta_1(12))(\Delta E_1 - \Delta L_1 - \Delta K_1). \end{aligned} \tag{5.6}$$

The first line of (5.6) is the cost due to the realization of the flexibility mechanisms between the players **1** and **3**, the second line is the cost due to the realization of the flexibility mechanisms between the players **1** and **2**, and the third is the cost of emission reduction in the territory of player **1**. Using the limits $Q_1(13)$, $Q_3(13)$, $Q_1(12)$, $Q_2(12)$, $Q_2(23)$, $Q_3(23)$, M_1 and M_2 defined as before (see p. 61) and the following notations

$$\min\{Q_1(13), Q_3(13)\} = \delta_2(13)(\Delta E_1 - \Delta L_1 - \Delta K_1) := Q(13),$$

$$\min\{Q_1(12), Q_2(12)\} = \delta_2(12)(\Delta E_1 - \Delta L_1 - \Delta K_1) := Q(12),$$

$$\min\{Q_2(23), Q_3(23)\} = \delta_2(23)(\Delta E_2 - \Delta L_2 - \Delta K_2 + Q(12)) := Q(23),$$

$$M_1 \geq c_3^e \delta_1(13)(\Delta E_1 - \Delta L_1 - \Delta K_1) + c_3^e \cdot \delta_1(12)(\Delta E_1 - \Delta L_1 - \Delta K_1),$$
$$M_2 \geq c_3^e \cdot \delta_1(23)(\Delta E_2 - \Delta L_2 - \Delta K_2 + Q(12)),$$

let us write down the costs of the players **2** and **3** under cooperation

$$\begin{aligned} H_2 = {} & c_3^q \cdot \delta_2(23)(\Delta E_2 - \Delta L_2 - \Delta K_2 + Q(12)) \\ & + c_3^e \cdot \delta_1(23)(\Delta E_2 - \Delta L_2 - \Delta K_2 + Q(12)) \\ & + c_2^e(1 - \delta_2(23) - \delta_1(23))(\Delta E_1 - \Delta L_1 - \Delta K_1 + Q(12)) \\ & - c_2^q \cdot Q(12) - c_2^e \cdot \delta_1(12)(\Delta E_2 - \Delta L_2 - \Delta K_2), \end{aligned} \tag{5.7}$$

$$\begin{aligned} H_3 = {} & c_3^e(\Delta E_3 - \Delta L_3 - \Delta K_3 + Q(13) + Q(23)) \\ & - c_3^q \cdot Q(13) - c_3^e \cdot \delta_1(13)(\Delta E_1 - \Delta L_1 - \Delta K_1) \\ & - c_3^q \cdot Q(23) - c_3^e \cdot \delta_1(23)(\Delta E_2 - \Delta L_2 - \Delta K_2 + Q(12)). \end{aligned} \tag{5.8}$$

Consequently from (5.5)–(5.8) and (5.1) we calculate the value $v(\{1, 2, 3\})$.

By analogy to the previous formulas we can define the characteristic function for any number of players. The values $v(\{i\}) = 0$ for every player i conclude the construction of the characteristic function.

5.3 Imputation Distribution Procedures for Kyoto Protocol model

In this section we consider two multistage cooperative games corresponding to the model of realization of flexibility mechanisms. The characteristic function $v(S, t)$ is the guaranteed economy in million dollars due to the co-operation (joint implementation, clean development and emission trading) during 5 years. The data for calculations were taken from [43] and [120]. Here S is a coalition of Parties (groups of Parties) in co-operation on a period $[t, T]$. Characteristic function values depend on a set of parameters: limitations of the emission reduction investment, emission quota, *etc.* Depending on the parameters we have the different variants of the game. In the following examples we have three players: **1** is European Union (EU), **2** is the new members of EU (EU-A), and **3** is Russian Federation.

5.3.1 Example with a time-consistent solution

Let us consider 3-person multistage cooperative game (the characteristic function is in Table 5.3.1).

The sets $X^0(t)$, $t \in \mathbb{T}$, are the following

$$X^0(t_0) = \{\xi^0(t_0) = (54050, 14100, 25850)\},$$

$$X^0(t_1) = \{\xi^0(t_1) = (35250, 11750, 23500)\},$$

$$X^0(t_2) = \{\xi^0(t_2) = (20000, 10000, 17000)\},$$

t	$v(\{1,2,3\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$	$v(\{1,2\},t)$
t_0	94000	39950	79900	68150
t_1	70500	35250	58750	47000
t_2	47000	27000	37000	30000
t_3	38000	17000	31000	28000
t_4	22500	10480	17250	14730
t_5	0	0	0	0

TABLE 9: Characteristic function of the multistage cooperative game.

$$X^0(t_3) = \{\xi^0(t_3) = (21000, 7000, 10000)\},$$
$$X^0(t_4) = \{\xi^0(t_4) = (10750, 3980, 6500)\},$$
$$X^0(t_5) = \{\xi^0(t_5) = (0, 0, 0)\}.$$

At the moments t_k, $k = 0, 1, 2, 3, 5$, the grand subcore is equal to the set $X^0(t_k)$

$$GSC(N, v(t)) = X^0(t)$$

and the unique imputation $\xi(t_0) = \xi^0(t_0)$ is time-consistent (see Theorems 2.10 and 2.22).

Let us now apply the algorithm to construct IDP for this game.

Initial step. We set

$$\alpha(t_0) = \xi(t_0) = (54050, 14100, 25850).$$

Step 1. We set

$$\alpha(t_1) = \xi(t_1) = (35250, 11750, 23500)$$

and

$$\alpha(t_0, t_1) = (18800, 2350, 2350).$$

Step 2. The solution of the problem (2.11), (2.12) is the vector $\omega = (21000, 1000, 17000)$. The sum of ω's components are greater than $v(N, t_2)$, hence we set

$$\alpha(t_2) = \omega, \quad \tilde{v}(N, t_2) = 48000,$$
$$\alpha(t_1, t_2) = (14250, 1750, 6500).$$

Step 3. At this step $\omega = \xi^0(t_3)$, hence we can set

$$\alpha(t_3) = \xi(t_3) = (21000, 7000, 10000)$$

and the payoff vector is

$$\alpha(t_2, t_3) = (0, 3000, 7000).$$

Step 4. We should find $\alpha(t_4) \in GSC(N, v(t_4))$ such that $\alpha(t_4) \leq \alpha(t_3)$. Let us choose

$$\alpha(t_4) = \xi^0(t_4) + (400, 470, 400) = (11150, 4450, 6900),$$

t	$v(\{1,2,3\},t)$	$v(\{2,3\},t)$	$v(\{1,3\},t)$	$v(\{1,2\},t)$
t_0	94000	39950	79900	68150
t_1	70500	35250	58750	47000
t_2	47000	41125	21150	22325
t_3	33950	15680	30520	20300
t_4	18800	6800	15680	10620
t_5	0	0	0	0

Table 10: Characteristic function of the multistage cooperative game.

then

$$\alpha(t_3, t_4) = (9850, 2550, 3100).$$

The last step. Here $\alpha(t_5) = (0, 0, 0)$ and

$$\alpha(t_4, t_5) = \alpha(t_4) = (11150, 4450, 6900).$$

Consequently, the corresponding payoff trajectory is

$$\begin{pmatrix} 54050 \\ 14100 \\ 25850 \end{pmatrix} \to \begin{pmatrix} 35250 \\ 11750 \\ 23500 \end{pmatrix} \to \begin{pmatrix} 21000 \\ 10000 \\ 17000 \end{pmatrix}$$
$$\to \begin{pmatrix} 21000 \\ 7000 \\ 10000 \end{pmatrix} \to \begin{pmatrix} 11150 \\ 4450 \\ 6900 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and the imputation distribution procedure α is

$$\alpha = \begin{pmatrix} 0 & 18800 & 14250 & 0 & 9850 & 11150 \\ 0 & 2350 & 1750 & 3000 & 2550 & 4450 \\ 0 & 2350 & 6500 & 7000 & 3100 & 6900 \end{pmatrix}.$$

This IDP provides the time-consistent imputation $\xi(t_0)$ by non-negative payoffs to every player at every step of the game.

5.3.2 Example without time-consistent solution

In this part we use a reduced game to construct the imputation distribution procedure in the multistage cooperative game presented in Table 5.3.2.

The sets $X^0(t)$, $t \in \mathbb{T}$, are the following

$$X^0(t_0) = \{\xi^0(t_0) = (54050, 14100, 25850)\},$$

$$X^0(t_1) = \{\xi^0(t_1) = (35250, 11750, 23500)\},$$
$$X^0(t_2) = \{\xi^0(t_2) = (1175, 21150, 19975)\},$$
$$X^0(t_3) = \{\xi^0(t_3) = (17570, 2730, 12950)\},$$

$$X^0(t_4) = \{\xi^0(t_4) = (9710, 910, 5890)\},$$
$$X^0(t_5) = \{\xi^0(t_5) = (0, 0, 0)\}.$$

The unique solution $\xi(t_0) \in GSC(N, v(t_0))$ is not time-consistent because there is no vector $\xi^0(t) \in X^0(t)$ such that $\xi_i(t_0) \geq \xi_i^0(t)$ for all $i \in N$ and $t \in \mathbb{T}$ (see Theorem 2.10). The property of time-consistency is violated at the moment $t = t_2$. Let us choose the vector $\theta = (18000, 9000, 20000)$ instead of $\xi(t_2) \in GSC(N, v(t_2))$. The vector θ does not belong to the grand subcore of the subgame $(N, v(t_2))$ due to the player 2; we call him a "disturbing" player. Let us create the MDM-reduced game $(\{1,3\}, v_{\xi^0}^{\{2\}})$, $t \in \{t_2, t_3, t_4, t_5\}$. To do this we fix the vectors $\xi(t_3) = (17770, 3030, 13150)$ and $\xi(t_4) = (10510, 1600, 6690)$. Then the characteristic function of the reduced game is the following

$$v_{\xi^0}^{\{2\}}(\{1,3\}, \theta, t_2) = v(N, t_2) - \theta_2 = 38000,$$
$$v_{\xi^0}^{\{2\}}(\{1\}, \theta, t_2) = v(\{1,2\}, t_2) - \xi_2^0(t_2) = 1175,$$
$$v_{\xi^0}^{\{2\}}(\{3\}, \theta, t_2) = v(\{2,3\}, t_2) - \xi_2^0(t_2) = 19875;$$

$$v_{\xi^0}^{\{2\}}(\{1,3\}, \xi(t_3)) = v(N, t_3) - \xi_2(t_3) = 30920,$$
$$v_{\xi^0}^{\{2\}}(\{1\}, \xi(t_3)) = v(\{1,2\}, t_3) - \xi_2^0(t_3) = 17570,$$
$$v_{\xi^0}^{\{2\}}(\{3\}, \xi(t_3)) = v(\{2,3\}, t_3) - \xi_2^0(t_3) = 12950;$$

$$v_{\xi^0}^{\{2\}}(\{1,3\}, \xi(t_4)) = v(N, t_4) - \xi_2(t_4) = 17200,$$
$$v_{\xi^0}^{\{2\}}(\{1\}, \xi(t_4)) = v(\{1,2\}, t_4) - \xi_2^0(t_4) = 9710,$$
$$v_{\xi^0}^{\{2\}}(\{3\}, \xi(t_4)) = v(\{2,3\}, t_4) - \xi_2^0(t_4) = 5890.$$

In the MDM-reduced multistage game the players 1 and 3 realize the solution from the grand subcore of this game. The corresponding payoff trajectory can be, for example, the following

$$\begin{pmatrix} 54050 \\ 14100 \\ 25850 \end{pmatrix} \to \begin{pmatrix} 35250 \\ 11750 \\ 23500 \end{pmatrix} \to \begin{pmatrix} 18000 \\ 9000 \\ 20000 \end{pmatrix}$$
$$\to \begin{pmatrix} 17770 \\ 9000 \\ 13150 \end{pmatrix} \to \begin{pmatrix} 10510 \\ 9000 \\ 6900 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and the imputation distribution procedure α is

$$\alpha = \begin{pmatrix} 0 & 18800 & 17250 & 230 & 7260 & 10510 \\ 0 & 2350 & 2750 & 0 & 0 & 9000 \\ 0 & 2350 & 3500 & 6850 & 6250 & 6200 \end{pmatrix}.$$

Decision maker should fix the moment when the player 2 can get the payoff of 9000. For example, it can be the moment $t = t_5$. This method combines both classical methods of regularization — regularization of the optimality principle and delays of total payoffs. It constructs an IDP in the time-inconsistent case.

5.4 Conclusion

In this chapter we considered two different approaches to the problem of time-consistency in the real-life multistage cooperative game. The first method let us to construct the imputation distribution procedure providing a time-consistent solution from the grand subcore of the game. The second method works if there is no time-consistent imputations in a balanced game and it helps to regularize the game and the optimality principle. We applied both methods to the problem connected with Kyoto Protocol realization.

6 CONCLUSION

In this work multistage cooperative games with transferable utility are investigated. We consider the core [40, 41] and the grand subcore [131] as optimality principles. One of the main problems in the theory of dynamic games is the problem of time-consistency of solutions corresponding to the optimality principle.

Chapters 1 and 3 have an auxiliary character. The first one consists of the basic knowledge in game theory to make the work all-sufficient. This mostly contains materials from [53, 73, 74, 75] and references connected with application fields of game theory.

Chapter 3 is based on the paper [130]. There we study the consistency property of the grand subcore with respect to the modification [127] of the reduced game due to Davis and Maschler [24] and then we introduce a dynamical analogue of the consistency property for multistage cooperative games.

Chapters 2 and 4 are devoted to the problem of time-consistency of the grand subcore selectors. In Chapter 2, which is based on [128, 129, 130], we describe two algorithms for allocation of a common benefit from cooperation of players in such a way that the resulting payoff vector to be time-consistent and payoffs at every step of the game to be nonnegative. The indicated properties are provided through the delays of total payoffs in intermediate moments in order to the players do not become debtors during the game.

In Chapter 4, based on [27], we suggest a new approach to construction of imputation distribution procedure in multistage cooperative games related to the results of Chapter 3. In particular, we formulate a problem of minimal reduction (Section 4.2), propose two variants of solving of this problem (conditionally minimal coalition and acceptable coalitions for reduction). Here the results of Chapter 3 are in use.

In Chapter 5 we discuss a real-life cooperative game, which models Kyoto Protocol realization [28]. In this part of the work we apply the results of [27, 128, 129, 130]. Flexibility mechanisms of Kyoto protocol are the basis of multistage cooperative model. In our example there are three players: European Union, Russian Federation and the new members of European Union. As a benefit we consider the values of pollution reduction obtained due to collaboration of players under “joint implementation”, “emission trading”, and “clean development” [120].

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