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# ON THE BBM-PHENOMENON IN FRACTIONAL POINCARÉ–SOBOLEV INEQUALITIES WITH WEIGHTS

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ABSTRACT. In this paper we unify and improve some of the results of Bourgain, Brezis and Mironescu and the weighted Poincaré–Sobolev estimate by Fabes, Kenig and Serapioni. More precisely, we get weighted counterparts of the Poincaré–Sobolev type inequality and also of the Hardy type inequality in the fractional case under some mild natural restrictions.

A main feature of the results we obtain is the fact that we keep track of the behaviour of the constants involved when the fractional parameter approaches to 1. Our main method is based on techniques coming from harmonic analysis related to the self-improving property of generalized Poincaré inequalities.

## 1. INTRODUCTION: BACKGROUND AND MOTIVATION

Let  $p \geq 1$ . The classical  $(1, p)$ -Poincaré inequality states the existence of a dimensional constant  $c_n > 0$  such that, for any Sobolev function  $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ ,

$$\int_Q |u(x) - u_Q| dx \leq c_n \ell(Q) \left( \int_Q |\nabla u(x)|^p dx \right)^{1/p}, \quad Q \in \mathcal{Q}, \quad (1.1)$$

where  $\mathcal{Q}$  is the family of cubes (i.e. a cartesian product of  $n$  intervals of the same side length  $\ell(Q)$  in  $\mathbb{R}^n$  and  $u_Q := \int_Q u(x) dx := \frac{1}{|Q|} \int_Q u(x) dx$ ). The mean oscillations of Sobolev functions  $u$  over cubes are then controlled by the local Sobolev seminorm

$$[u]_{W^{1,p}(Q)} := \ell(Q) \left( \int_Q |\nabla u(x)|^p dx \right)^{1/p}.$$

More recently (see for instance [DD20, DM21b, ACPS20, ACPS21, HSLG19]), a fractional counterpart of this Poincaré inequality has attracted the attention of many authors. A naive version of it states the existence of a dimensional positive constant  $c_n$  such that, given  $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ ,

$$\int_Q |u(x) - u_Q| dx \leq c_n \ell(Q)^\delta \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}, \quad Q \in \mathcal{Q} \quad (1.2)$$

for any  $p \geq 1$  and  $\delta \in (0, 1)$ . This inequality allows to control the mean oscillation of a function  $u$  over a cube  $Q$  by the fractional Sobolev seminorm

$$[u]_{W^{\delta,p}(Q)} := \ell(Q)^\delta \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}. \quad (1.3)$$

Inequality (1.2) is an easy-to-get estimate but it turns out that it encodes a lot of information. Indeed, it can be shown, by using the methods first proved in [FPW98] and then improved in [CP21],

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that the left hand side can be replaced by the normalized weak or Marcinkiewicz norm (see (3.1)). We then get

$$\|u - u_Q\|_{L^{p_\delta^*, \infty}(Q, \frac{dx}{|Q|})} \leq c_n p_\delta^* [u]_{W^{\delta, p}(Q)}, \quad Q \in \mathcal{Q}, \quad (1.4)$$

whenever  $\delta \in (0, 1)$ ,  $p \in [1, \frac{n}{\delta})$ , and where  $p_\delta^*$  is the fractional Sobolev exponent defined by

$$\frac{1}{p} - \frac{1}{p_\delta^*} = \frac{\delta}{n}. \quad (1.5)$$

Observe that  $p < p_\delta^*$ . Inequality (1.4) readily follows from the following property of the functional  $a : \mathcal{Q} \rightarrow [0, \infty)$ ,  $a(Q) = [u]_{W^{\delta, p}(Q)}$ ,

$$\left( \sum_i a(Q_i)^{p_\delta^*} \frac{|Q_i|}{|Q|} \right)^{\frac{1}{p_\delta^*}} \leq a(Q) \quad (1.6)$$

which holds for any  $Q \in \mathcal{Q}$  and any family of disjoint dyadic subcubes  $\{Q_i\} \in \mathcal{D}(Q)$ . We will denote the collection of all dyadic cubes by  $\mathcal{D}$  and by  $\mathcal{D}(Q)$  the collection of all dyadic cubes relative to the cube  $Q$ .

Moreover, the well-known truncation argument (see Section 3.4) for getting a strong inequality from the corresponding weak one in the classical case of the gradient works for the functional  $[u]_{W^{\delta, p}(Q)}$  too, as shown in [DIV16]. Then we can replace the weak norm in (1.4) by the strong norm to get

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L^{p_\delta^*}(Q, \frac{dx}{|Q|})} \leq c_n p_\delta^* [u]_{W^{\delta, p}(Q)}. \quad (1.7)$$

Despite of the presence of the  $L^q$  norm at the left-hand side of this inequality it turns out that (1.7) is far from being optimal in the parameter  $\delta$  as  $\delta \rightarrow 1^-$ . Indeed, it follows from [BBM02],

$$\int_Q |u - u_Q| dx \leq c_n (1 - \delta)^{\frac{1}{p}} [u]_{W^{\delta, p}(Q)}, \quad (1.8)$$

where the highly interesting extra gain  $(1 - \delta)^{\frac{1}{p}}$  appears in front of  $[u]_{W^{\delta, p}(Q)}$ . We are going to get improvements of inequality (1.7) by making use of this inequality. A different and interesting approach was considered later in [Mil05] combining ideas from interpolation theory and extrapolation theory [JM91], [DM]. We remit to [KMX05] for interesting extensions to the context of higher-order Besov norms. See also [DM21a] for more about the central role played by interpolation theory and extrapolation theory. See also [CMPR21] for some related results within the context of product spaces.

The estimate (1.8) will be the “key initial” starting point in most of our proofs.

Note that, according to [Bre02, Proposition 2], a measurable non-constant function  $u$  on a cube  $Q$  would satisfy,

$$\left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+p}} dy dx \right)^{\frac{1}{p}} = \infty,$$

and so inequality (1.2) does not provide any information about the function  $u$  when  $\delta \rightarrow 1$ . This is corrected in estimate (1.8) or (1.9) below, where the factor  $(1 - \delta)^{1/p}$  balances this behaviour when  $\delta \rightarrow 1$  and so its presence in the inequality is in fact essential.

Now, exactly as outlined above, to prove (1.7), the results in [CP21] combined with the truncation method in this context obtained in [DIV16], yield the following result.

**Theorem A.** *Let  $0 < \delta < 1$ ,  $1 \leq p < \frac{n}{\delta}$  and let  $p_\delta^*$  be the fractional Sobolev exponent (1.5). Then, for any locally integrable function  $u$ ,*

$$\left( \int_Q |u(x) - u_Q|^{p_\delta^*} dx \right)^{\frac{1}{p_\delta^*}} \leq c_n p_\delta^* (1 - \delta)^{\frac{1}{p}} [u]_{W^{\delta, p}(Q)} \quad Q \in \mathcal{Q}. \quad (1.9)$$

One of the main results of this paper is to extend (1.9) into the context of  $A_p$  weights. Unfortunately we can only derive sharp results, preserving the  $\delta \rightarrow 1$ , in the case of  $A_1$  weights.

**Remark 1.1.** *Standard arguments can be used to obtain from (1.9) the global estimate in  $\mathbb{R}^n$  with the correction factor  $(1 - \delta)^{\frac{1}{p}}$  in front, namely*

$$\left( \int_{\mathbb{R}^n} |u(x)|^{p_\delta^*} dx \right)^{\frac{1}{p_\delta^*}} \leq c_n p_\delta^* (1 - \delta)^{1/p} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p},$$

which holds for, say, any function  $u \in L_{\text{loc}}^1(\mathbb{R}^n)$  such that for an increasing family of cubes  $\{Q_j\}$  with  $\mathbb{R}^n = \bigcup_{j=1}^{\infty} Q_j$  we have  $u_{Q_j} \rightarrow 0$  as  $j \rightarrow \infty$ , see Definition 2.5 in Section 2.1 below. Corresponding inequalities are considered for functions defined on unbounded John domains in [HSV15] without keeping track of the constants' exact dependence on the parameter  $\delta$ . Different and interesting approaches to prove this global result were provided in [MS02] (see [MS03] too) and [KL05].

A global version of Theorem A was obtained in [MS02] using appropriate global Hardy type estimates which are also interesting on its own and that we state next.

**Theorem B** ([MS02, Theorem 2]). *Let  $1 \leq p < \infty$  and  $0 < \delta < 1$  such that  $\delta p < n$ . There exists  $c_{n,p} > 0$  such that, for any function in the completion  $W_0^{\delta,p}(\mathbb{R}^n)$  of the space of compactly supported smooth functions under the seminorm  $[\cdot]_{W^{\delta,p}(\mathbb{R}^n)}$ , there holds*

$$\left( \int_{\mathbb{R}^n} |u(x)|^p \frac{dx}{|x|^{\delta p}} \right)^{1/p} \leq c_{n,p} \delta^{1/p} (1 - \delta)^{1/p} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}. \quad (1.10)$$

We improve this inequality in this work. First we will replace the class of power weights  $\frac{1}{|x|^{\delta p}}$  at the left hand side of (1.10) by the class of  $A_\infty$  weights. Then, we will present a similar result with no assumption on the weight, at the price of a worse constant in the right-hand side. Our method is different and more general than the one in [MS02].

In the following section we state and discuss the main results of this paper. We distinguish between the weighted variants of (1.9) and (1.10), which we will call fractional Poincaré–Sobolev and Hardy type inequalities, respectively.

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## 2. STATEMENT OF THE MAIN RESULTS

**2.1. Results involving Muckenhoupt weights.** Our first main result is an extension of Theorem A to include weights from the  $A_1$  class. The method of proof in [BBM02] cannot be used at all due to the presence of the weight. We use ideas from [PR19] instead, where a general “self-improving” argument is introduced, thus avoiding completely the use of any representation formula.

Actually, we will state two type of Poincaré–Sobolev inequalities. The first one involves a *weighted fractional Sobolev exponent*,  $p_{\delta,w}^*$  and the second one involves the usual fractional Sobolev exponent  $p_\delta^*$  (1.5).

**Theorem 2.1.** *Let  $0 < \delta < 1$  and  $1 \leq p < \frac{n}{\delta}$ . Let  $w \in A_1$ , and let  $p_{\delta,w}^*$  be the weighted fractional Sobolev exponent defined by*

$$\frac{1}{p} - \frac{1}{p_{\delta,w}^*} = \frac{\delta}{n} \frac{1}{1 + \log[w]_{A_1}}. \quad (2.1)$$

Then there is a dimensional constant  $c_n$  such that, for every cube  $Q$  in  $\mathbb{R}^n$  and for any  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left( \frac{1}{w(Q)} \int_Q |u - c|^{p_{\delta,w}^*} w dx \right)^{\frac{1}{p_{\delta,w}^*}} \\ & \leq c_n p_{\delta}^* (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{1+\frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}} dy w dx \right)^{\frac{1}{p}}. \end{aligned} \quad (2.2)$$

**Remark 2.2.** We emphasize that in spite of the singularity introduced by the weight  $w$ , the smallness factor  $(1 - \delta)^{\frac{1}{p}}$  is kept as if there were no such singularity.

This theorem and next one can be seen as improvement of the celebrated Poincaré–Sobolev inequalities obtained by Fabes, Kenig and Serapioni [FKS82] (see also [HKM06]) by combining this result and the “reverse type” results obtained in [HSMPPV], at least in the case  $w \in A_1$ .

Observe that  $p < p_{\delta,w}^* \leq p_\delta^*$  where  $p_\delta^*$  is the fractional Sobolev exponent defined above and note that  $p_{\delta,w}^*$  is of the form

$$p_{\delta,w}^* := \frac{pn(1 + \log[w]_{A_1})}{n(1 + \log[w]_{A_1}) - \delta p},$$

and so, by comparing with the expression of  $p_\delta^* = \frac{np}{n-p\delta}$ , the term  $1 + \log[w]_{A_1}$  may be regarded as a distortion of the dimension  $n$  introduced by the presence of the weight  $w$ . Nevertheless, the largest possible borderline exponent  $p_\delta^*$  can also be attained with the presence of an  $A_1$  weight. The cost of this better improvement in the scale of the  $L^p$  spaces is the loss of some extra  $A_1$  constant in front.

**Theorem 2.3.** Let  $0 < \delta < 1$  and  $1 \leq p < \frac{n}{\delta}$ . Let  $w \in A_1$ , and let  $p_\delta^*$  be the fractional Sobolev exponent (1.5). Then, there exists a constant  $c_n > 0$  such that for every  $Q \in \mathcal{Q}$  and for any  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left( \frac{1}{w(Q)} \int_Q |u - c|^{p_\delta^*} w dx \right)^{\frac{1}{p_\delta^*}} \\ & \leq c_n p_\delta^* (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{\frac{\delta}{n} + 1 + \frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}} dy w dx \right)^{\frac{1}{p}}. \end{aligned}$$

These two theorems hold for any  $A_p$  weight without the gain  $(1 - \delta)^{\frac{1}{p}}$ . It may well be the case that both results with  $A_p$  weights are true with the gain.

Another satisfactory result is the following fractional Hardy type inequality which follows, as mentioned before, from a general self-improving argument which avoids completely the use of any representation formula.

We will use next the local fractional maximal function  $M_{\alpha,Q}$ ,  $\alpha \geq 0$ ,  $Q \in \mathcal{Q}$ , defined for any measurable function  $u$  by

$$M_{\alpha,Q} u(x) = \sup_{x \in P} \ell(P)^\alpha \int_P |u(y)| dy, \quad x \in Q,$$

where the supremum is taken over all cubes  $P \subset Q$  such that  $x \in Q$ .

**Theorem 2.4.** Let  $0 < \delta < 1$  and let  $1 \leq p < \infty$ . There exists a dimensional constant  $c_n > 0$  such that, for any  $w \in A_\infty$  and for any  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,

$$\inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^p w(x) dx \right)^{\frac{1}{p}} \leq c_n p (1 - \delta)^{\frac{1}{p}} [w]_{A_\infty} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy M_{\delta p,Q}(w)(x) dx \right)^{\frac{1}{p}} \quad (2.3)$$

for every cube  $Q$  in  $\mathbb{R}^n$ .

Observe that this result is of different nature since the fractional part has been absorbed by the fractional maximal function  $M_{\delta p,Q}$ . This is the reason why we cannot get an “ $L^{p_\delta^*}$ ”, namely a Poincaré–Sobolev, version as in the other theorems. However, we can derive a global type result for which the following family of class of functions is relevant.

**Definition 2.5.** Let  $w$  be a weight in  $\mathbb{R}^n$ . We define  $\mathcal{F}_w$  as the class of functions  $u$  such that  $u \in L^1_{\text{loc}}(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n; w dx)$  for which there exists an increasing sequence of cubes  $\{Q_j\}_{j \in \mathbb{N}}$  such that

$$\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} Q_j \quad \text{and} \quad \lim_{j \rightarrow \infty} u_{Q_j, w} = 0,$$

where  $u_{Q, w} = \frac{1}{w(Q)} \int_Q u w dx$ .

Notice that  $L^\infty_c(\mathbb{R}^n) \subset \mathcal{F}_w$  whenever  $w$  is a weight outside of  $L^1(\mathbb{R}^n)$ . Doubling weights, that is, weights satisfying  $0 < w(B(x, 2r)) \leq C(w)w(B(x, r))$  for all  $x \in \mathbb{R}^n$  and  $r > 0$ , are typical examples of weights that do not belong to  $L^1(\mathbb{R}^n)$ . We refer to [BB11, Corollary 3.8].

**Corollary 2.6.** Let  $0 < \delta < 1$  and let  $1 \leq p < \infty$ . There exists a dimensional constant  $c_n > 0$  such that, for any  $w \in A_\infty$ ,

$$\left( \int_{\mathbb{R}^n} |u(x)|^p w dx \right)^{1/p} \leq c_n p (1 - \delta)^{1/p} [w]_{A_\infty} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy M_{\delta p}(w)(x) dx \right)^{1/p} \quad (2.4)$$

whenever  $u \in \mathcal{F}_w$ .

The  $A_\infty$  restriction is sufficiently mild to allow many interesting examples. In particular,  $A_1$  weights can be chosen for our result including power weights. The fact that the result in Theorem 2.4 does not involve any geometric constant depending on  $\ell(Q)$  is what allows to get a global variant of it, thus getting Theorem B as a corollary of ours at least in the case  $\delta \rightarrow 1$ .

More precisely, as a consequence of Theorem 2.4 we get the following general local Hardy type inequality.

**Corollary 2.7.** Let  $0 < \delta < 1$  and  $\beta > 0$ , and let  $1 \leq p < \frac{n}{\delta} \min\{1, \beta\}$ . Let  $\mu$  be any non-negative Borel measure such that  $M_{n - \frac{p\delta}{\beta}}(\mu)$  is finite almost everywhere. Then, there exists a positive dimensional constant  $c_n$  such that, for any  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^p M_{n - \frac{p\delta}{\beta}}(\mu)(x)^\beta dx \right)^{\frac{1}{p}} \\ & \leq c_n^{\beta+1} (1 - \delta)^{\frac{1}{p}} \frac{p}{(n - \delta p)^{1 + \frac{1}{p}}} \mu(\mathbb{R}^n)^{\frac{\beta}{p}} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy dx \right)^{\frac{1}{p}}. \end{aligned} \quad (2.5)$$

In particular if we choose  $\beta = 1$  and  $\mu$  as the Dirac mass at the origin, we have that

$$\inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^p \frac{1}{|x|^{\delta p}} dx \right)^{\frac{1}{p}} \leq c_n (1 - \delta)^{\frac{1}{p}} \frac{p}{(n - \delta p)^{1 + \frac{1}{p}}} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy dx \right)^{\frac{1}{p}}. \quad (2.6)$$

We notice that  $M_{n - \frac{p\delta}{\beta}}(\mu)^\beta$  is a weight that belongs to the  $A_1$  class since  $p < \frac{n}{\delta} \min\{1, \beta\}$ . Indeed, whenever  $0 \leq \alpha < n$ ,  $\rho > 0$ , it is known that  $M_\alpha(\mu)^\rho \in A_1$  if  $0 < \rho < \frac{n}{n - \alpha}$  (see Lemma 3.6 (2) for more precise details).

The global versions follow easily.

**Corollary 2.8.** Let  $0 < \delta < 1$  and  $\beta > 0$ , and let  $1 \leq p < \frac{n}{\delta} \min\{1, \beta\}$ . Let  $\mu$  be any non-negative Borel measure such that  $M_{n - \frac{p\delta}{\beta}}(\mu)$  is finite almost everywhere. Then there exists a dimensional positive constant  $c_n$  such that

$$\begin{aligned} & \left( \int_{\mathbb{R}^n} |u(x)|^p (M_{n - \frac{p\delta}{\beta}}(\mu)(x))^\beta dx \right)^{\frac{1}{p}} \\ & \leq c_n^{\beta+1} (1 - \delta)^{\frac{1}{p}} \frac{p}{(n - \delta p)^{1 + \frac{1}{p}}} \mu(\mathbb{R}^n)^{\frac{\beta}{p}} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy dx \right)^{\frac{1}{p}} \end{aligned} \quad (2.7)$$

whenever  $u \in \mathcal{F}_w$  where  $w = (M_{n-\frac{p\delta}{\beta}}\mu)^\beta$ . Hence, if  $\beta = 1$  and  $\mu$  is the Dirac mass at the origin, we have that

$$\left( \int_{\mathbb{R}^n} |u(x)|^p \frac{1}{|x|^{\delta p}} dx \right)^{\frac{1}{p}} \leq c_n (1-\delta)^{\frac{1}{p}} \frac{p}{(n-\delta p)^{1+\frac{1}{p}}} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}} dy dx \right)^{\frac{1}{p}}, \quad (2.8)$$

whenever  $u \in \mathcal{F}_w$  where  $w = \frac{1}{|x|^{\delta p}}$ .

We conjecture that the correct bound in terms of  $\delta \in (0, 1)$  is actually  $\delta^{\frac{1}{p}}(1-\delta)^{\frac{1}{p}}$ . We remit to Section 4 for the proofs of the results of this section.

**2.2. Weighted fractional Poincaré type inequalities without the  $A_\infty$  condition.** In this section, we state several extensions of Theorem B where power weights are replaced by a general weight  $w$ . In particular we do not assume that the weight satisfies the  $A_\infty$  condition. However, we cannot obtain the factor  $\delta^{1/p}$  in (1.10) from Theorem B.

The results of this section are motivated by the classical Fefferman-Stein inequality [FS71],

$$\|Mf\|_{L^{1,\infty}(w)} \leq c_n \int_{\mathbb{R}^n} |f(x)| Mw(x) dx$$

from which we deduce, for  $p \in (1, \infty)$ , that

$$\|Mf\|_{L^p(w)} \leq c_n p' \|f\|_{L^p(Mw)},$$

where no assumption is made on the weight  $w$ . As already known, this suggests the definition of the class  $A_1$  of pairs of weights  $(w, v)$  which satisfy

$$[w, v]_{A_1} := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{v(x)} < \infty.$$

**Theorem 2.9.** *Let  $1 \leq p < \infty$  and let  $w$  be any weight.*

a) *Let  $0 < \delta < 1$  and let  $1 \leq p < \infty$ . Let also  $0 < \varepsilon \leq \delta$ , then there is a positive dimensional constant  $c_n$  such that for any weight  $w$ ,*

$$\begin{aligned} & \left( \int_Q |u(x) - u_Q|^p w dx \right)^{1/p} \\ & \leq c_n \frac{(1-\delta)^{\frac{1}{p}}}{\varepsilon} \ell(Q)^\varepsilon \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy M_{(\delta-\varepsilon)p, Q}(w\chi_Q)(x) dx \right)^{1/p}. \end{aligned} \quad (2.9)$$

b) *Hence, if  $\frac{1}{2} < \delta < 1$ ,*

$$\begin{aligned} & \left( \int_Q |u(x) - u_Q|^p w dx \right)^{1/p} \\ & \leq c_n (1-\delta)^{\frac{1}{p}} \ell(Q)^\delta \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy M(w\chi_Q)(x) dx \right)^{1/p} \end{aligned} \quad (2.10)$$

and in particular, for any  $(w, v) \in A_1$ , we have

$$\begin{aligned} & \left( \int_Q |u(x) - u_Q|^p w dx \right)^{1/p} \\ & \leq c_n (1-\delta)^{\frac{1}{p}} \ell(Q)^\delta [w, v]_{A_1}^{\frac{1}{p}} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy v dx \right)^{1/p}. \end{aligned} \quad (2.11)$$

**2.3. Weighted fractional isoperimetric inequalities with one sharp gain.** We also consider local and global *fractional* versions of the following weighted Gagliardo-Nirenberg inequality

$$\left( \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} w(x) dx \right)^{\frac{n-1}{n}} \leq c_n \int_{\mathbb{R}^n} |\nabla u(x)| (Mw(x))^{\frac{n-1}{n}} dx, \quad (2.12)$$

which is a result that already appeared in [PR19] and was also implicit in [FPW00]. This generalizes the well known Gagliardo-Nirenberg-Sobolev inequality in the limiting case  $p = 1$ , which in turn is equivalent to the well known isoperimetric inequality, see [Oss78, FF60, Maz60].

We obtain a local fractional version of (2.12), with the presence of the sharp gain phenomenon when  $\delta \rightarrow 1$ . Observe that the admissible weights here are more singular since there is no assumption on the weight  $w$ .

**Theorem 2.10.** *Let  $w$  be a weight in  $\mathbb{R}^n$  and let  $\frac{1}{2} \leq \delta < 1$ . Then there exists a dimensional constant  $c_n$  such that*

$$\inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \leq c_n (1 - \delta) \int_Q \int_Q \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy Mw(x)^{\frac{n-\delta}{n}} dx, \quad (2.13)$$

for any cube  $Q$  and any  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ .

The absence of any quantity depending on the size of the cube yields the following global results.

**Corollary 2.11.** *Let  $w$  be a weight in  $\mathbb{R}^n$  and let  $\frac{1}{2} \leq \delta < 1$ . There is a positive dimensional constant  $c_n$  such that,*

$$\left( \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \leq c_n (1 - \delta) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx,$$

whenever  $u \in \mathcal{F}_w$ .

**Corollary 2.12.** *Let  $w$  be a weight in  $\mathbb{R}^n$  and let  $\frac{1}{2} \leq \delta < 1$ . Assume further that  $w \notin L^1(\mathbb{R}^n)$ . Then there is a positive dimensional constant  $c_n$  such that*

$$w(E)^{\frac{n-\delta}{n}} \leq c_n (1 - \delta) \left( \int_E \int_{\mathbb{R}^n \setminus E} \frac{1}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx + \int_{\mathbb{R}^n \setminus E} \int_E \frac{1}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx \right)$$

for every bounded measurable set  $E \subset \mathbb{R}^n$ .

To finish the section we provide another consequence of Theorem 2.10.

**Corollary 2.13.** *Let  $Q$  be a cube and let  $E$  be any measurable set  $E \subset Q$ . Then, for any weight  $w$  and any  $0 < \varepsilon \leq \frac{1}{2}$ ,*

$$w(Q \setminus E) w(E) \leq w(Q) \left( c_n \varepsilon \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-1+\varepsilon}{n}}}{|x - y|^{n+1-\varepsilon}} dy dx + c_n \varepsilon \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-1+\varepsilon}{n}}}{|x - y|^{n+1-\varepsilon}} dy dx \right)^{\frac{n}{n-1+\varepsilon}}.$$

This result is an extension of [BBM02, Corollary 1] with  $w = 1$  and  $Q$  the unit cube. This, in turn, is related to an improvement of an estimate from [BBM01]. The improved version was used to prove a conjecture formulated in [BBM00]. Our result should play a similar role.

### 3. PRELIMINARIES AND KNOWN RESULTS

**3.1. Some basic facts about the  $A_p$  theory of weights.** In this section we recall first some well known definitions.

We recall briefly some concepts about the classes of Muckenhoupt weights. A weight is a function  $w \in L^1_{\text{loc}}(\mathbb{R}^n)$  satisfying  $w(x) > 0$  for almost every point  $x \in \mathbb{R}^n$ .

**Definition 3.1.** *Let  $w \in L^1_{\text{loc}}(\mathbb{R}^n)$  be a weight.*



(1) For  $1 < p < \infty$  we say that  $w \in A_p$  if

$$[w]_{A_p} := \sup_{Q \in \mathcal{Q}} \int_Q w(x) dx \left( \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty.$$

(2) We say that  $w \in A_1$  if there is a constant  $C > 0$  such that, for any cube  $Q$  in  $\mathbb{R}^n$ ,

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \operatorname{ess\,inf}_{x \in Q} w(x).$$

and the  $A_1$  constant  $[w]_{A_1}$  is defined as the smallest of these  $C$ .

(3) The  $A_\infty$  class is defined as the union of all the  $A_p$  classes, that is,

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p,$$

and the  $A_\infty$  constant is defined as

$$[w]_{A_\infty} = \sup_{Q \in \mathcal{Q}} \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x) dx.$$

We will use the following notation for the weighted local  $L^q$  average over a set  $E$  defined as follows:

$$\|f\|_{L^q\left(E, \frac{w dx}{w(E)}\right)} := \left( \int_E |f|^q w dx \right)^{\frac{1}{q}} := \left( \frac{1}{w(E)} \int_E |f|^q w dx \right)^{\frac{1}{q}}.$$

Similarly, we will use the standard notation for the normalized weak  $(q, \infty)$  (quasi)norm: for any  $0 < q < \infty$ , measurable  $E$  and a positive weight  $w$ , we define the normalized weak or Marcinkiewicz norm

$$\|f\|_{L^{q,\infty}\left(E, \frac{w dx}{w(E)}\right)} := \sup_{t>0} t \left( \frac{1}{w(E)} w(\{x \in E : |f(x)| > t\}) \right)^{\frac{1}{q}}, \quad (3.1)$$

where, for a measurable set  $E$ , we denote its weighted measure by  $w(E) := \int_E w(x) dx$ . Similarly we define the global weak or Marcinkiewicz norm

$$\|f\|_{L^{q,\infty}(w)} := \sup_{t>0} t \left( w(\{x \in \mathbb{R}^n : |f(x)| > t\}) \right)^{\frac{1}{q}}. \quad (3.2)$$

We will also use few times the class of pair of  $A_p$  weights.

**Definition 3.2.** A pair of weights  $(w, v)$  belongs to the  $A_p$  class,  $1 < p < \infty$ , if

$$[w, v]_{A_p} := \sup_{Q \in \mathcal{Q}} \left( \int_Q w dx \right) \left( \int_Q v^{1-p'} dx \right)^{p-1} < \infty. \quad (3.3)$$

We denote this by  $(w, v) \in A_p$ . Similarly, we say that  $(w, v) \in A_1$  if

$$[w, v]_{A_1} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{v(x)} < \infty,$$

where  $M$  is the usual Hardy-Littlewood maximal function (see Definition 3.3 with  $\alpha = 0$ ).

It is also well known that  $(w, v) \in A_p$  if and only if

$$\frac{1}{|Q|} \int_Q f dx \leq [w, v]_{A_p}^{\frac{1}{p}} \left( \frac{1}{w(Q)} \int_Q f^p v dx \right)^{\frac{1}{p}} \quad f \geq 0. \quad (3.4)$$

**3.2. Weighted oscillation.** We are often estimating the following weighted  $L^q$  oscillation

$$\left( \frac{1}{w(Q)} \int_Q |u - u_{Q,w}|^q w dx \right)^{\frac{1}{q}}, \quad (3.5)$$

where  $1 \leq q < \infty$  and  $w$  is a weight. Recall that for a weight  $w$ , we write  $u_{Q,w} = \frac{1}{w(Q)} \int_Q u w dx$ . When estimating (3.5) from above, it is possible and often more convenient to replace  $u_{Q,w}$  by any other constant  $c$  instead. This is a consequence of inequalities

$$\inf_{c \in \mathbb{R}} \left( \frac{1}{w(Q)} \int_Q |u - c|^q w dx \right)^{\frac{1}{q}} \leq \left( \frac{1}{w(Q)} \int_Q |u - u_{Q,w}|^q w dx \right)^{\frac{1}{q}} \leq 2 \inf_{c \in \mathbb{R}} \left( \frac{1}{w(Q)} \int_Q |u - c|^q w dx \right)^{\frac{1}{q}}, \quad (3.6)$$

the latter of which follows by combining triangle and Jensen's inequalities.

**3.3. Review of fractional integrals and related maximal functions.** We introduce now both the fractional and the classical variants of the maximal function.

**Definition 3.3.** Let  $\alpha \geq 0$  and consider a locally integrable function  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ . The fractional maximal function of  $u$  is defined by

$$M_\alpha u(x) := \sup_{Q \ni x} \ell(Q)^\alpha \int_Q |u(y)| dy,$$

where the supremum is taken over all cubes  $Q$  in  $\mathbb{R}^n$  satisfying  $x \in Q$ . The case  $\alpha = 0$  corresponds to the non-centered Hardy-Littlewood maximal function and we will denote  $M = M_0$ . Similarly, for any non-negative Borel measure  $\mu$ , the fractional maximal function  $M_\alpha \mu$  is defined by

$$M_\alpha \mu(x) := \sup_{x \in Q: Q \in \mathcal{Q}} \ell(Q)^\alpha \frac{\mu(Q)}{|Q|}.$$

**Definition 3.4.** Let  $\alpha \geq 0$ . If  $Q_0 \subset \mathbb{R}^n$  is a cube, then the local fractional maximal function  $M_{\alpha, Q_0}$  is the operator defined by

$$M_{\alpha, Q_0} u(x) = \sup_{x \in Q} \ell(Q)^\alpha \int_Q |u(y)| dy, \quad x \in Q_0,$$

where the supremum is taken over all cubes  $Q \subset Q_0$  such that  $x \in Q$  and  $u$  is a measurable function. When the supremum is taken over all cubes  $Q \in \mathcal{D}(Q_0)$  such that  $x \in Q$  we get the dyadic local fractional maximal function  $M_{\alpha, Q_0}^d$ .

Another important operator in the theory, which is related to the fractional maximal function is the fractional integral operator, or the Riesz potential.

**Definition 3.5.** For  $0 < \alpha < n$ , we define the Riesz potential of a non-negative measurable function  $u$  by

$$I_\alpha u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x - y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

We finish this part of the section by proving some elementary properties about weights in relation to the fractional maximal function and the fractional integral operator.

**Lemma 3.6.** Let  $0 \leq \alpha < n$ .

(1) Let  $\mu$  be a non-negative Borel measure. Then,

$$\|M_\alpha \mu\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} \leq 5^{n-\alpha} \mu(\mathbb{R}^n). \quad (3.7)$$

(2) If  $0 < \varepsilon \leq \frac{n}{n-\alpha}$ , then  $(M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon} \in A_1$  for any non-negative Borel measure  $\mu$  such that  $M_\alpha \mu(x) < \infty$  for almost every  $x \in \mathbb{R}^n$ . The  $A_1$  constant does not depend on  $\mu$ . More precisely,

$$[(M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon}]_{A_1} \leq 2^{\frac{n}{n-\alpha}-\varepsilon} \frac{2n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon}. \quad (3.8)$$

(3) If  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ , then if further  $\alpha > 0$ ,

$$M_\alpha(f)(x) \leq \left(\frac{n}{\alpha}\right)' \|f\|_{L^{\frac{n}{\alpha}, \infty}(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n.$$

*Proof.* The proof of part (1) follows from the classical 5-covering lemma: If  $\{Q_i\}_{i=1, \dots, N}$  is a finite family of cubes in  $\mathbb{R}^n$ . Then we can extract a subsequence of pairwise disjoint cubes  $\{Q_j\}_{j=1, \dots, M}$  such that

$$\bigcup_{i=1}^N Q_i \subset \bigcup_{j=1}^M 5Q_j.$$

Now if we let  $\Omega_t = \{x \in \mathbb{R}^n : M_\alpha \mu(x) > t\}$  for a given  $t$  and let  $K$  to be any compact subset contained in  $\Omega_t$ . Let  $x \in K$  then by definition of the maximal function there is a cube  $Q = Q_x$  containing  $x$  such that

$$\frac{|Q|^{\frac{\alpha}{n}}}{|Q|} \mu(Q) > t. \quad (3.9)$$

Then  $K \subset \bigcup_{x \in K} Q_x$  and by compactness we can extract a finite family of cubes  $\{Q_i\}_{i=1, \dots, N}$  such that  $K \subset \bigcup_{i=1, \dots, N} Q_i$  and where each cube satisfies (3.9). Then by the 5-covering lemma we can extract a subsequence of pairwise disjoint family of cubes  $\{Q_j\}_{j=1}^M$  such that  $\bigcup_{i=1}^N Q_i \subset \bigcup_{j=1}^M 5Q_j$ . Then, since  $|\lambda Q| = \lambda^n |Q|$  and since  $0 < \frac{n-\alpha}{n} \leq 1$

$$|K|^{\frac{n-\alpha}{n}} \leq \sum_{j=1}^M |5Q_j|^{\frac{n-\alpha}{n}} \leq 5^{n-\alpha} \sum_{j=1}^M |Q_j|^{\frac{n-\alpha}{n}} \leq \frac{5^{n-\alpha}}{t} \sum_{j=1}^M \mu(Q_j) \leq \frac{5^{n-\alpha}}{t} \mu(\mathbb{R}^n).$$

This yields (3.7) immediately, since  $K$  is arbitrary.

We now prove Part (2). The fact that  $(M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon}$  is an  $A_1$  weight under our assumptions on the parameters is known but we need a proof with precise bounds as given by (3.8). Actually we will assume  $0 < \varepsilon < \frac{n}{n-\alpha}$  since the case  $\varepsilon = \frac{n}{n-\alpha}$  is trivial.

We will use the following estimate known sometimes as Kolmogorov's inequality, if  $(X, \mu)$  is a probability space (or even  $\mu(X) \leq 1$ ), and if  $0 < q < r$

$$\|f\|_{L^q(X, \mu)} \leq \left(\frac{r}{r-q}\right)^{\frac{1}{q}} \|f\|_{L^r, \infty(X, \mu)}. \quad (3.10)$$

Let  $x \in \mathbb{R}^n$  be such that  $M_\alpha \mu(x) < \infty$  and let  $Q$  be a cube containing  $x$ . Then

$$M_\alpha \mu(x) \leq M_\alpha(\mu \chi_{3Q})(x) + M_\alpha(\mu \chi_{(3Q)^c})(x).$$

Consider first the case  $\frac{n}{n-\alpha} - \varepsilon \leq 1$ . Then,

$$\int_Q (M_\alpha \mu)(y)^{\frac{n}{n-\alpha}-\varepsilon} dy \leq \int_Q (M_\alpha(\mu \chi_{3Q})(y))^{\frac{n}{n-\alpha}-\varepsilon} dy + \int_Q (M_\alpha(\mu \chi_{(3Q)^c}))(y)^{\frac{n}{n-\alpha}-\varepsilon} dy =: I + II.$$

For  $I$  we use (3.10) combined with (3.7),

$$\begin{aligned} I &\leq \frac{n}{n-\alpha} \frac{1}{\varepsilon} \|M_\alpha(\mu \chi_{3Q})\|_{L^{\frac{n}{n-\alpha}, \infty}(Q, \frac{dx}{|Q|})}^{\frac{n}{n-\alpha}-\varepsilon} \leq \frac{n}{n-\alpha} \frac{1}{\varepsilon} \left(5^{n-\alpha} \frac{|Q|^{\frac{\alpha}{n}}}{|Q|} \mu(3Q)\right)^{\frac{n}{n-\alpha}-\varepsilon} \\ &= \frac{n}{n-\alpha} \frac{1}{\varepsilon} \left(15^{n-\alpha} \frac{|3Q|^{\frac{\alpha}{n}}}{|3Q|} \mu(3Q)\right)^{\frac{n}{n-\alpha}-\varepsilon} \leq \frac{n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon} (M_\alpha \mu(x))^{\frac{n}{n-\alpha}-\varepsilon}. \end{aligned}$$

For  $II$  we use the following geometrical argument. Let  $y \in Q$  and consider  $M_\alpha(\mu \chi_{(3Q)^c})(y)$ . If  $R$  is a cube such that  $y \in R \subset 3Q$  then  $\frac{|R|^{\frac{\alpha}{n}}}{|R|} \mu((3Q)^c \cap R) = 0$ , hence we only must take cubes  $R$  containing  $y$  such that  $(3Q)^c \cap R \neq \emptyset$ . Thus these cubes  $3R \supseteq Q$ , and therefore

$$\frac{|R|^{\frac{\alpha}{n}}}{|R|} \mu((3Q)^c \cap R) \leq 3^{n-\alpha} \frac{|3R|^{\frac{\alpha}{n}}}{|3R|} \mu((3Q)^c \cap 3R) \leq 3^{n-\alpha} M_\alpha(\mu)(x).$$

Hence

$$II \leq \left(3^{n-\alpha} M_\alpha \mu(x)\right)^{\frac{n}{n-\alpha}-\varepsilon} = 3^{n-\varepsilon(n-\alpha)} (M_\alpha \mu(x))^{\frac{n}{n-\alpha}-\varepsilon}.$$

Combining, since  $0 < \varepsilon < \frac{n}{n-\alpha}$

$$\int_Q (M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon} dx \leq \frac{2n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon} \left( M_\alpha \mu(x) \right)^{\frac{n}{n-\alpha}-\varepsilon}.$$

In the case  $\frac{n}{n-\alpha} - \varepsilon > 1$ , using that  $(a+b)^p \leq 2^{p-1}(a^p + b^p)$  for any  $a, b > 0, p \geq 1$ ,

$$\int_Q (M_\alpha \mu)(y)^{\frac{n}{n-\alpha}-\varepsilon} dy \leq 2^{\frac{n}{n-\alpha}-\varepsilon-1} (I + II) \leq 2^{\frac{n}{n-\alpha}-\varepsilon} \frac{n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon} \left( M_\alpha \mu(x) \right)^{\frac{n}{n-\alpha}-\varepsilon}.$$

Finally, combining both estimates, for  $0 < \varepsilon < \frac{n}{n-\alpha}$ , we have

$$\int_Q (M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon} dy \leq 2^{\frac{n}{n-\alpha}-\varepsilon} \frac{2n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon} \inf_{x \in Q} \left( M_\alpha \mu(x) \right)^{\frac{n}{n-\alpha}-\varepsilon},$$

which yields

$$[(M_\alpha \mu)^{\frac{n}{n-\alpha}-\varepsilon}]_{A_1} \leq 2^{\frac{n}{n-\alpha}-\varepsilon} \frac{2n}{n-\alpha} \frac{15^{n-\varepsilon(n-\alpha)}}{\varepsilon}.$$

For the proof of (3), we may assume  $f$  to satisfy  $\|f\|_{L^{\frac{n}{\alpha}, \infty}(\mathbb{R}^n)} = 1$ . Consider  $x \in \mathbb{R}^n$  and let  $Q$  be a cube containing  $x$ . Then, for any  $L > 0$ ,

$$\begin{aligned} \int_Q |f(y)| dy &= \int_0^\infty |\{y \in Q : |f(y)| > \lambda\}| d\lambda \\ &= \int_0^L |\{y \in Q : |f(y)| > \lambda\}| d\lambda + \int_L^\infty |\{y \in Q : |f(y)| > \lambda\}| d\lambda \\ &\leq L|Q| + \frac{1}{\frac{n}{\alpha}-1} L^{1-\frac{n}{\alpha}}, \end{aligned}$$

and then

$$\begin{aligned} |Q|^{\frac{\alpha}{n}-1} \int_Q |f(y)| dy &\leq |Q|^{\frac{\alpha}{n}-1} \left( L|Q| + \frac{1}{\frac{n}{\alpha}-1} L^{1-\frac{n}{\alpha}} \right) \\ &= |Q|^{\frac{\alpha}{n}} \left( L + \frac{1}{|Q|(\frac{n}{\alpha}-1)} L^{1-\frac{n}{\alpha}} \right), \end{aligned}$$

which equals  $(\frac{n}{\alpha})'$  when choosing  $L = |Q|^{-\frac{\alpha}{n}}$ . This concludes the proof of (3).  $\square$

**3.4. The truncation method.** Another important tool that we will be using is the so called ‘‘fractional truncation method’’, see [DIV16, Theorem 4.1] and [Chu19, Proposition 2.14]. Similar truncation methods are originally used by Maz’ya [Maz85]. The proof of [DLV21, Theorem 3.2] can be easily adapted to show the following variant of this method.

**Theorem 3.7.** *Let  $1 \leq p \leq q < \infty$  and let  $w, v$  be a weights in  $\mathbb{R}^n$ . Let  $Q \subset \mathbb{R}^n$  be a cube. Then the following conditions are equivalent:*

(A) *There is a constant  $C_1 > 0$  such that inequality*

$$\inf_{c \in \mathbb{R}} \sup_{t > 0} t \left( w(\{x \in Q : |u(x) - c| > t\}) \right)^{\frac{1}{q}} \leq C_1 \left( \int_Q \int_Q \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz v(y) dy \right)^{\frac{1}{p}}$$

*holds for every  $u \in L^\infty(Q; w dx)$ .*

(B) *There is a constant  $C_2 > 0$  such that inequality*

$$\inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^q w(x) dx \right)^{\frac{1}{q}} \leq C_2 \left( \int_Q \int_Q \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz v(y) dy \right)^{\frac{1}{p}}$$

*holds for every measurable function  $u$  in  $Q$  such that  $|u| < \infty$  almost everywhere in  $Q$ .*

*Moreover, in the implication from (A) to (B) the constant  $C_2$  is of the form  $CC_1$ , where  $C$  does not depend on any of the parameters, and in the implication from (B) to (A) we have  $C_1 = C_2$ .*

## 4. HARMONIC ANALYSIS AND SELF-IMPROVING THEORY

In this section we will provide the proofs for Theorems 2.1, 2.3, and 2.4. We recall first some definitions and results.

**4.1. Some recent results.** The plan is to derive Poincaré–Sobolev type inequalities from a Poincaré type inequality within a general context. This will be accomplished by using some improvement results from the literature. Being more precise, we will be considering, as a starting point estimates, of the form

$$\int_Q |f - f_Q| dx \leq a(Q), \quad Q \in \mathcal{Q}, \quad (4.1)$$

where  $a : \mathcal{Q} \rightarrow [0, \infty)$  is some general functional with no restriction. The functional may or may not depend on  $f$ . In the case it does, we will write  $a_f$ , but the required condition will be uniform on  $f$ .

The key idea is that inequality (4.1) enjoys a self-improving property. It is remarkable that this property is not so much related to the presence of a gradient on the right hand side of (4.1), but to a discrete geometrical summation condition associated to the functional  $a$ . The theory is very flexible and can be developed within the contexts of other geometries, see [CMPR21].

We will prove Theorems 2.1 and 2.3 using ideas from [PR19] applying a recent improvement of [FPW98, Theorem 2.3], namely [CP21, Theorem 1.5]. The following definition is the key in the theory.

**Definition 4.1.** Let  $0 < p < \infty$ , let  $w$  be a weight and let  $a : \mathcal{Q} \rightarrow [0, \infty)$ .

1) For  $s > 0$ , we denote  $a \in SD_p^s(w)$ , if there is a positive constant  $c$  such that, for any family of disjoint dyadic subcubes  $\{Q_i\}_i$  of any given cube  $Q$ , the following inequality holds

$$\left( \sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \leq c \left( \frac{|\bigcup_i Q_i|}{|Q|} \right)^{\frac{1}{s}} a(Q). \quad (4.2)$$

The best possible constant  $c$ , the infimum of the constants in the last inequality, is denoted by  $\|a\|_{SD_p^s(w)}$ .

2) We say that the functional  $a$  satisfies the weighted  $D_p(w)$  condition if there is a constant  $c$  such that, for any family of disjoint dyadic cubes  $\{Q_i\}_i$  of any given cube  $Q$ , the following inequality holds

$$\left( \sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \leq c a(Q). \quad (4.3)$$

The best possible constant  $c$  above is denoted by  $\|a\|_{D_p(w)}$ . We will write in this case that  $a \in D_p(w)$ .

Next result from [CP21] will be relevant. It is an improved version of the main result in [FPW98] but with linear bounds in both  $[w]_{A_\infty}$  and  $p$ , instead of exponential.

**Theorem 4.2.** Let  $w$  be any  $A_\infty$  weight in  $\mathbb{R}^n$  and  $a \in D_p(w)$  from (4.3). Let  $f$  be a locally integrable function such that,

$$\int_Q |f - f_Q| \leq a(Q) \quad Q \in \mathcal{Q}.$$

Then there exists a dimensional constant  $c_n > 0$  such that for every  $Q \in \mathcal{Q}$ ,

$$\|f - f_Q\|_{L^{p,\infty}(Q, \frac{w dx}{w(Q)})} \leq c_n p [w]_{A_\infty} \|a\|_{D_p(w)} a(Q).$$

Using the stronger condition (4.2) we can get a better result. We need to state the following very interesting improvement of one of the main results in [PR19], which can be found as a consequence of the main theorem in [LLO21].

**Theorem 4.3.** Let  $w$  be any weight and let  $a$  such that for some  $p \geq 1$  it satisfies the weighted condition  $SD_p^s(w)$  from (4.2) with  $s > 1$  and constant  $\|a\|_{SD_p^s(w)}$ . Let  $f$  be a locally integrable function such that

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq a(Q),$$

for every cube  $Q$ . Then, there exists a dimensional constant  $c_n > 0$  such that for any cube  $Q$  the following inequality holds

$$\|f - f_Q\|_{L^p(Q, \frac{w dx}{w(Q)})} \leq c_n s \|a\|_{SD_p^s(w)} a(Q). \quad (4.4)$$

**4.2. Proofs of Theorems 2.1 and 2.3.** The starting point for both results is the same. We look for the appropriate functional. Indeed, from the key initial estimate (1.8)

$$\int_Q |u(x) - u_Q| dx \leq c_n (1 - \delta)^{\frac{1}{p}} [u]_{W^{\delta,p}(Q)},$$

which holds for every  $Q \in \mathcal{Q}$  and any  $0 < \delta < 1$ . Now, by definition of the  $A_1$  class of weights, we have

$$[u]_{W^{\delta,p}(Q)} \leq [w]_{A_1}^{\frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}} dy w dx \right)^{\frac{1}{p}},$$

and hence,

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \leq a_u(Q), \quad (4.5)$$

where

$$a_u(Q) := \lambda \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q g(x, y) dy w dx \right)^{\frac{1}{p}} \quad (4.6)$$

and where  $\lambda$  is the constant  $\lambda = c_n (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{\frac{1}{p}}$  and  $g(x, y) = \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}}$ .

We will need the following properties of the functional  $a$  adapting [PR19, Lemma 5.2] or more precisely Lemma 6.2 in [CMPR21]. To simplify the presentation we recall that for  $M \geq 1$  we define  $p_M^*$  by

$$\frac{1}{p} - \frac{1}{p_M^*} = \frac{\delta}{nM}. \quad (4.7)$$

where  $M$  will be chosen. When  $M = 1$ ,  $p_1^*$  equals the fractional Sobolev exponent  $p_1^* = p_\delta^*$  which corresponds to the unweighted known case  $w = 1$  in Theorem A.

**Lemma 4.4.** *Let  $w \in A_1$  and let  $a$  be defined as in (4.6).*

1) *Let  $M > 1$  and let  $p_M^*$  defined as above. Then,  $a_u \in SD_{p_M^*}^{\frac{nM'}{\delta}}(w)$ , namely,*

$$\left( \sum_i a_u(Q_i)^{p_M^*} \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p_M^*}} \leq [w]_{A_1}^{\frac{\delta}{nM}} \left( \frac{|\bigcup_i Q_i|}{|Q|} \right)^{\frac{\delta}{nM'}} a_u(Q). \quad (4.8)$$

and hence  $\|a_u\|_{SD_{p_M^*}^{\frac{nM'}{\delta}}(w)} \leq [w]_{A_1}^{\frac{\delta}{nM}}$ .

2) *Let  $M = 1$ . Then  $a_u \in D_{p_\delta^*}(w)$ , namely*

$$\left( \sum_i a_u(Q_i)^{p_\delta^*} \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p_\delta^*}} \leq [w]_{A_1}^{\frac{\delta}{n}} a_u(Q), \quad (4.9)$$

and hence  $\|a_u\|_{D_{p_\delta^*}(w)} \leq [w]_{A_1}^{\frac{\delta}{n}}$ .

The small difference is that the functional (4.6) is slightly different from the usual one defined as

$$a(Q) = \ell(Q)^\alpha \left( \frac{1}{w(Q)} \mu(Q) \right)^{1/p},$$

with  $\alpha, p > 0$ , that was considered in [PR19]. In our case  $\mu(Q)$  is replaced by  $\int_Q \int_Q g(x, y) dy w dx$  with  $g \geq 0$ . The inner integral is increasing in  $Q$ , which is enough and the same proof can be applied here.

*Proof.* Recall that

$$a_u(Q) := \lambda \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q g(x, y) dy w dx \right)^{\frac{1}{p}}$$

and where  $\lambda$  is a parameter,  $g$  is a function defined in (4.6) and  $w \in A_1$ . We will use the following key geometric property,

$$\frac{|E|}{|Q|} \leq [w]_{A_1} \frac{w(E)}{w(Q)}, \quad (4.10)$$

which is valid for any measurable subset  $E \subset Q$ . It is a particular case of (3.4) with  $w = v$ ,  $p = 1$ ,  $f = \chi_E$ , although we will always restrict ourselves to subcubes rather than subsets of  $Q$ .

1) Case  $M > 1$ . For simplicity in the exposition, we will omit the subindex  $M$  and just use  $p^*$  instead of  $p_M^*$ . Then using the definition of  $p^*$  and the fact that  $p^* > p$ , we have

$$\begin{aligned} \sum_i a_u(Q_i)^{p^*} w(Q_i) &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{\ell(Q_i)^\delta}{w(Q_i)^{\frac{1}{p} - \frac{1}{p^*}}} \right)^{p^*} \\ &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{|Q_i|^{\frac{\delta}{n}}}{w(Q_i)^{\frac{\delta}{nM}}} \right)^{p^*} \\ &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{|Q_i|}{w(Q_i)^{\frac{1}{M}}} \right)^{\frac{\delta p^*}{n}} \\ &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( |Q_i|^{\frac{1}{M'}} \left( \frac{|Q_i|}{w(Q_i)} \right)^{\frac{1}{M}} \right)^{\frac{\delta p^*}{n}}. \end{aligned}$$

By applying (4.10), we can continue to estimate

$$\begin{aligned} \sum_i a_u(Q_i)^{p^*} w(Q_i) &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{nM}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{nM}} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} |Q_i|^{\frac{\delta p^*}{nM'}} \\ &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{nM}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{nM}} \sup_i |Q_i|^{\frac{\delta p^*}{nM'}} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\ &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{nM}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{nM}} |\bigcup_i Q_i|^{\frac{\delta p^*}{nM'}} \left( \sum_i \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\ &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{nM}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{nM}} |\bigcup_i Q_i|^{\frac{\delta p^*}{nM'}} \left( \int_Q \int_Q g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\ &= [w]_{A_1}^{\frac{\delta p^*}{nM}} \left( \frac{|\bigcup_i Q_i|}{|Q|} \right)^{\frac{\delta p^*}{nM'}} a_u(Q)^{p^*} w(Q). \end{aligned}$$

This proves part 1) of the lemma.

The second part corresponds to the case  $M = 1$ , which corresponds formally to the case  $M' = \infty$  before and hence without "smallness", namely without the factor  $\frac{|\bigcup_i Q_i|}{|Q|}$ .

$$\begin{aligned}
 \sum_i a_u(Q_i)^{p^*} w(Q_i) &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{\ell(Q_i)^\delta}{w(Q_i)^{\frac{1}{p} - \frac{1}{p^*}}} \right)^{p^*} \\
 &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{|Q_i|^{\frac{\delta}{n}}}{w(Q_i)^{\frac{\delta}{n}}} \right)^{p^*} \\
 &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{|Q_i|}{w(Q_i)} \right)^{\frac{\delta p^*}{n}} \\
 &= \lambda^{p^*} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \left( \frac{|Q_i|}{w(Q_i)} \right)^{\frac{\delta p^*}{n}} \\
 &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{n}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{n}} \sum_i \left( \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\
 &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{n}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{n}} \left( \sum_i \int_{Q_i} \int_{Q_i} g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\
 &\leq \lambda^{p^*} [w]_{A_1}^{\frac{\delta p^*}{n}} \left( \frac{|Q|}{w(Q)} \right)^{\frac{\delta p^*}{n}} \left( \int_Q \int_Q g(x, y) dy w dx \right)^{\frac{p^*}{p}} \\
 &= [w]_{A_1}^{\frac{\delta p^*}{n}} a_u(Q)^{p^*} w(Q).
 \end{aligned}$$

□

*Proof of Theorem 2.1.* Let  $1 \leq p < \frac{n}{\delta}$  and recall that  $p_{\delta, w}^*$  is defined by the relationship

$$\frac{1}{p} - \frac{1}{p_{\delta, w}^*} = \frac{\delta}{n} \frac{1}{1 + \log[w]_{A_1}}. \quad (4.11)$$

Fix a cube  $Q$  in  $\mathbb{R}^n$ . The goal is to prove the strong type  $(p_{\delta, w}^*, p)$  Poincaré–Sobolev estimate with  $A_1$  weights:

$$\begin{aligned}
 &\|u - u_Q\|_{L^{p_{\delta, w}^*, w}(Q, \frac{w dx}{w(Q)})} \\
 &\leq c_n (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{1 + \frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy w dx \right)^{\frac{1}{p}}.
 \end{aligned} \quad (4.12)$$

We will distinguish two cases:  $[w]_{A_1} > e^{\frac{1}{\delta}}$  or the contrary.

Recall that we start with estimate (4.5) using the functional (4.6) which by Lemma 4.4 part 1) satisfies an  $SD_{p_M^*}^{\frac{nM'}{\delta}}$ -condition from Definition 4.1 for appropriate  $M > 1$  with  $s = \frac{nM'}{\delta} > 1$ , namely

$$\|a_u\| := \|a_u\|_{SD_{p_M^*}^{\frac{nM'}{\delta}}(w)} \leq [w]_{A_1}^{\frac{\delta}{nM}}. \quad (4.13)$$



If we choose  $M = 1 + \log[w]_{A_1}$ , the exponent  $p_M^*$  is exactly the value  $p_{\delta,w}^*$  from (4.11). Hence applying Theorem 4.3 with  $p$  replaced by  $p_{\delta,w}^*$  and using estimate (4.13) we obtain

$$\begin{aligned} \left( \frac{1}{w(Q)} \int_Q |u - u_Q|^{p_{\delta,w}^*} w dx \right)^{\frac{1}{p_{\delta,w}^*}} &\leq c_n s \|a_u\| a_u(Q) \\ &\leq c_n \frac{M'}{\delta} [w]_{A_1}^{\frac{\delta}{nM}} a_u(Q) \\ &\leq c_n \frac{M'}{\delta} [w]_{A_1}^{\frac{1}{1+\log[w]_{A_1}}} a_u(Q) \\ &\leq c_n \frac{1 + \log[w]_{A_1}}{\log[w]_{A_1}} \frac{1}{\delta} a_u(Q). \end{aligned}$$

Hence, if we assume first that  $[w]_{A_1} > e^{\frac{1}{\delta}}$ , we have

$$\begin{aligned} \left( \frac{1}{w(Q)} \int_Q |u - u_Q|^{p_{\delta,w}^*} w dx \right)^{\frac{1}{p_{\delta,w}^*}} &\leq c_n (1 + \log[w]_{A_1}) a_u(Q) \\ &\leq c_n [w]_{A_1} a_u(Q). \end{aligned}$$

This gives (4.12) when  $[w]_{A_1} > e^{\frac{1}{\delta}}$ . Assume now that  $[w]_{A_1} \leq e^{\frac{1}{\delta}}$ . Note that in this case we do not know how to prove the result we are looking for using the argument we just used directly from the strong norm. To overcome this difficulty we will use the truncation method. That is, by Theorem 3.7, it is enough to prove the weak norm version of (4.12), namely

$$\|u - u_Q\|_{L^{p_{\delta,w}^*, \infty}(Q, \frac{w}{w(Q)})} \leq c a_u(Q). \quad (4.14)$$

We observe first that for any family  $\{Q_i\}_i$  of pairwise disjoint dyadic subcubes of  $Q$ , the following inequality follows from Lemma 4.4 part 2):

$$\left( \sum_i a(Q_i)^{p_1^*} \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p_1^*}} \leq e^{\frac{1}{n}} a_u(Q),$$

since  $[w]_{A_1} \leq e^{\frac{1}{\delta}}$ . Recall from (4.7) that the exponent  $p_1^*$  is defined by  $\frac{1}{p} - \frac{1}{p_1^*} = \frac{\delta}{n}$ .

Now, applying Theorem 4.2, we get

$$\begin{aligned} \|u - u_Q\|_{L^{p_1^*, \infty}(Q, \frac{w dx}{w(Q)})} &\leq c p_1^* [w]_{A_\infty} e^{\frac{1}{n}} a_u(Q). \\ &\leq c_n p_1^* [w]_{A_1} a_u(Q). \end{aligned}$$

Consider here the same choice as before of  $M = 1 + \log[w]_{A_1}$ . Since  $p_1^* > p_M^* = p_{\delta,w}^*$  (just note that  $p_M^*$  is a decreasing function on  $M$ ), by Jensen's inequality which holds at weak level (simply use that the inner part of what is inside the power  $\frac{1}{p}$  is less than or equal to one) we have,

$$\begin{aligned} \|u - u_Q\|_{L^{p_{\delta,w}^*, \infty}(Q, \frac{w dx}{w(Q)})} &\leq \|u - u_Q\|_{L^{p_1^*, \infty}(Q, \frac{w dx}{w(Q)})} \\ &\leq c_n p_1^* [w]_{A_1} a_u(Q). \end{aligned}$$

This gives the claim (4.14) finishing the proof of the theorem.  $\square$

We now proceed with the proof of Theorem 2.3, in which we allow ourselves to lose some sharpness in the constant regarding the  $A_1$  constant of the weight involved to be able to get the usual fractional Sobolev exponent  $p_\delta^*$ .

*Proof of Theorem 2.3.* We have to prove

$$\begin{aligned} & \left( \frac{1}{w(Q)} \int_Q |u - u_Q|^{p_\delta^*} w dx \right)^{\frac{1}{p_\delta^*}} \\ & \leq c_n p_\delta^* (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{\frac{\delta}{n} + 1 + \frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy w(x) dx \right)^{\frac{1}{p}}, \end{aligned}$$

where  $p_\delta^*$  is the (unweighted) fractional Sobolev exponent  $\frac{1}{p} - \frac{1}{p_\delta^*} = \frac{\delta}{n}$ .

To do this, we start as in the proof of Theorem 2.1, that is, we begin with (4.5), namely

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \leq a_u(Q),$$

where  $a_u$  is the functional

$$a_u(Q) := \lambda \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q g(x, y) dy w dx \right)^{\frac{1}{p}},$$

with  $\lambda = c_n (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{\frac{1}{p}}$  and  $g(x, y) = \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}}$ .

Also, as in the proof of Theorem 2.1 we can use part 2) of Lemma 4.4, namely if  $\{Q_i\}$  is a family of pairwise disjoint subcubes of  $Q$ , the following inequality holds

$$\left( \sum_i a_u(Q_i)^{p_\delta^*} \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p_\delta^*}} \leq [w]_{A_1}^{\frac{\delta}{n}} a_u(Q) \quad (4.15)$$

uniformly in  $u$ . That is, the functional  $a$  satisfies the  $D_{p_\delta^*}(w)$  condition and further we have that  $\|a_u\|_{D_{p_\delta^*}(w)} \leq [w]_{A_1}^{\frac{\delta}{n}}$ . Hence, by applying Theorem 4.2 we have

$$\begin{aligned} \|u - u_Q\|_{L^{p_\delta^*, \infty}(Q, \frac{w}{w(Q)})} & \leq c_n p_\delta^* [w]_{A_1}^{\frac{\delta}{n} + 1} a_u(Q) \\ & = c_n p_\delta^* (1 - \delta)^{\frac{1}{p}} [w]_{A_1}^{\frac{\delta}{n} + 1 + \frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy w dx \right)^{\frac{1}{p}}. \end{aligned}$$

An application of the truncation method finishes the proof, see Theorem 3.7.  $\square$

**4.3. Proof of Theorem 2.4.** We begin this section with the last of our applications of the self-improving methods.

*Proof of Theorem 2.4.* Our starting point is similar as in Theorems 2.1 and 2.3,

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \leq c_n (1 - \delta)^{\frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy dx \right)^{\frac{1}{p}}.$$

Then, we bound the integral term in the right-hand side as follows,

$$\begin{aligned} & \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy dx \right)^{\frac{1}{p}} \\ & = \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy \frac{M_{p\delta, Q}(w)(x)}{M_{p\delta, Q}(w)(x)} dx \right)^{\frac{1}{p}} \\ & \leq \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy M_{p\delta, Q}(w)(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

by definition of  $M_{p\delta, Q}$ . Define now the new functional

$$a_{u, w}(Q) := c_n (1 - \delta)^{\frac{1}{p}} \left( \frac{1}{w(Q)} \int_Q \int_Q g(x, y) dy M_{p\delta, Q}(w)(x) dx \right)^{\frac{1}{p}}, \quad (4.16)$$

where  $g(x, y) = \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}}$ .

Observe that the functional  $a$  satisfies readily the  $D_p(w)$  condition from Definition 4.1 part 2) uniformly in  $u$ . Furthermore,  $\|a_{u,w}\|_{D_p(w)} \leq 1$ . Then as in the proofs of Theorems 2.1 and 2.3, we use Theorem 4.2 since we are assuming  $w \in A_\infty$ :

$$\|u - u_Q\|_{L^{p,\infty}(Q, \frac{w}{w(Q)})} \leq c_n p [w]_{A_\infty} \|a_{u,w}\|_{D_p(w)} a_{u,w}(Q) \leq c_n p [w]_{A_\infty} a_{u,w}(Q).$$

Then we can pass to the strong norm by using the truncation method from Theorem 3.7 to get

$$\inf_{c \in \mathbb{R}} \left( \frac{1}{w(Q)} \int_Q |u - c|^p w dx \right)^{\frac{1}{p}} \leq c_n p [w]_{A_\infty} a_{u,w}(Q).$$

This concludes the proof of Theorem 2.4.  $\square$

We cannot get an  $L^{p^*}$  version as in the other theorems because of the lack of the fractional part  $\ell(Q)^\varepsilon$  in the definition of the functional (4.16) and hence we cannot prove a “smallness” type condition from Definition 4.1 part 1).

As a consequence of Theorem 2.4 we obtain Corollary 2.6, which is a global variant.

*Proof of Corollary 2.6.* By assumptions  $u \in L^1_{\text{loc}}(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n; w dx)$  and there exists an increasing sequence of cubes  $\{Q_j\}_{j \in \mathbb{N}}$  such that  $\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} Q_j$  and

$$\lim_{j \rightarrow \infty} \frac{1}{w(Q_j)} \int_{Q_j} u(x) w(x) dx = 0.$$

Write  $c_j = \frac{1}{w(Q_j)} \int_{Q_j} u(x) w(x) dx$  for all  $j \in \mathbb{N}$ . By (3.6),

$$\int_{Q_j} |u(x) - c_j|^p w(x) dx \leq 2^p \inf_{c \in \mathbb{R}} \int_{Q_j} |u(x) - c|^p w(x) dx.$$

Now, by Fatou’s lemma and Theorem 2.4 we get

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^p w(x) dx &= \int_{\mathbb{R}^n} \lim_{j \rightarrow \infty} |u(x) - c_j|^p \chi_{Q_j}(x) w(x) dx \\ &\leq 2^p \liminf_{j \rightarrow \infty} \inf_{c \in \mathbb{R}} \int_{Q_j} |u(x) - c|^p w(x) dx \\ &\leq c_n^p p^p (1 - \delta) [w]_{A_\infty}^p \liminf_{j \rightarrow \infty} \int_{Q_j} \int_{Q_j} \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy M_{\delta p, Q_j}(w)(x) dx \\ &\leq c_n^p p^p (1 - \delta) [w]_{A_\infty}^p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy M_{\delta p}(w)(x) dx. \end{aligned}$$

$\square$

*Proof of Corollaries 2.7 and 2.8.* We recall first the second part of Lemma 3.6, for  $0 \leq \alpha < n$  and let  $0 < \varepsilon \leq \frac{n}{n - \alpha}$ , then  $(M_\alpha \mu)^{\frac{n}{n - \alpha} - \varepsilon} \in A_1$  if  $M_\alpha \mu$  is finite almost everywhere. The  $A_1$  constant does not depend on  $\mu$ , more precisely,

$$[(M_\alpha \mu)^{\frac{n}{n - \alpha} - \varepsilon}]_{A_1} \leq 2^{\frac{n}{n - \alpha} - \varepsilon} \frac{2n}{n - \alpha} \frac{15^{n - \varepsilon(n - \alpha)}}{\varepsilon}.$$

Denote the weight on the left of (2.5) by  $w(x) := (M_{n - \frac{p\delta}{\beta}} \mu(x))^\beta$ , which is finite almost everywhere.

Let  $\alpha = n - \frac{p\delta}{\beta}$  and  $\varepsilon := \beta(\frac{n}{p\delta} - 1)$ . Since we assume that  $p < \frac{n}{\delta} \min\{1, \beta\}$ ,  $0 < \alpha < n$  and  $\varepsilon > 0$ . Then observe that with this definition,

$$w(x) = (M_\alpha \mu(x))^{\frac{n}{n - \alpha} - \varepsilon}$$

and hence  $w \in A_1$  with constant

$$[w]_{A_1} \leq 2^{\frac{n}{n - \alpha} - \varepsilon} \frac{2n\beta}{p\delta} \frac{15^{n - \varepsilon(n - \alpha)}}{\varepsilon} \leq 2^\beta \frac{2n\beta}{p\delta} \frac{15^n}{\varepsilon} = \frac{2^\beta}{n - p\delta} 2n15^n.$$

We apply now Theorem 2.4 to our weight  $w$ , since  $A_1 \subset A_\infty$  with  $[w]_{A_\infty} \leq [w]_{A_1}$ , for a cube  $Q$

$$\begin{aligned} \inf_{c \in \mathbb{R}} \left( \int_Q |u(x) - c|^p w dx \right)^{\frac{1}{p}} &\leq c_n p (1 - \delta)^{\frac{1}{p}} [w]_{A_1} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy M_{\delta p, Q}(w) dx \right)^{\frac{1}{p}} \\ &\leq c_n p (1 - \delta)^{\frac{1}{p}} \frac{2^\beta}{n - p\delta} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}} dy M_{\delta p}(w) dx \right)^{\frac{1}{p}}. \end{aligned} \quad (4.17)$$

To bound  $M_{\delta p}(w)$  we apply first part 3) and then part 1) of Lemma 3.6. Indeed, using the same notation as before with  $\alpha := n - \frac{p\delta}{\beta}$  and recalling that  $p\delta < n$ , we get

$$\begin{aligned} M_{\delta p} \left[ (M_\alpha \mu)^\beta \right] (x) &\leq \left( \frac{n}{\delta p} \right)' \| (M_\alpha \mu)^\beta \|_{L^{\frac{n}{\delta p}, \infty}(\mathbb{R}^n)} \\ &= \frac{n}{n - \delta p} \| M_\alpha \mu \|_{L^{\frac{n\beta}{\delta p}, \infty}(\mathbb{R}^n)}^\beta = \frac{n}{n - \delta p} \| M_\alpha \mu \|_{L^{\frac{n}{n - \alpha}, \infty}(\mathbb{R}^n)}^\beta \\ &\leq \frac{n}{n - \delta p} 5^{(n - \alpha)\beta} \mu(\mathbb{R}^n)^\beta = \frac{n 5^{\delta p}}{n - \delta p} \mu(\mathbb{R}^n)^\beta \\ &\leq \frac{n 5^n}{n - \delta p} \mu(\mathbb{R}^n)^\beta. \end{aligned}$$

Inserting this in (4.17) will give Corollary 2.7.

Corollary 2.8 follows from Corollary 2.7 after an application of Fatou's lemma.  $\square$

**4.4. Proof of Theorem 2.9.** We begin as before using (1.8) as the starting point,

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \leq c_n (1 - \delta)^{\frac{1}{p}} \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy dx \right)^{\frac{1}{p}},$$

which holds for every cube  $Q$  and any  $0 < \delta < 1$ .

a) We bound the integral term in the right-hand side as follows. Recall that  $0 < \varepsilon \leq \delta$

$$\begin{aligned} \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy dx \right)^{\frac{1}{p}} \\ = \ell(Q)^\delta \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy \frac{M_{(\delta - \varepsilon)p, Q}(w\chi_Q)(x)}{M_{(\delta - \varepsilon)p, Q}(w\chi_Q)(x)} dx \right)^{\frac{1}{p}} \\ \leq \ell(Q)^\varepsilon \left( \frac{1}{w(Q)} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}} dy M_{(\delta - \varepsilon)p, Q}(w\chi_Q)(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

and define now the new functional

$$a_{w, u}(Q) := \lambda_\delta \ell(Q)^\varepsilon \left( \frac{1}{w(Q)} \int_Q \int_Q g(x, y) dy A(w)(x) dx \right)^{\frac{1}{p}}, \quad (4.18)$$

where  $g(x, y) := \frac{|u(x) - u(y)|^p}{|x - y|^{n + p\delta}}$ ,  $A(w) := M_{(\delta - \varepsilon)p, Q}(w\chi_Q)$  and  $\lambda_\delta = c_n (1 - \delta)^{\frac{1}{p}}$ .

The key point is that this functional satisfies the property in Definition 4.1 part 1), namely

$$\left( \sum_i a_{w, u}(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \leq \left( \frac{|\cup_i Q_i|}{|Q|} \right)^{\frac{\varepsilon}{n}} a_{w, u}(Q) \quad (4.19)$$

and hence  $\|a_{w,u}\|_{SD_p^{\frac{n}{\varepsilon}}(w)} \leq 1$ . This property is very easy to verify. Indeed, let  $\{Q_i\}$  be a disjoint family of dyadic cubes in  $Q$ . Then

$$\begin{aligned} \sum_i a_{w,u}(Q_i)^p \frac{w(Q_i)}{w(Q)} &= \sum_i \lambda_\delta^p \ell(Q_i)^{p\varepsilon} \frac{1}{w(Q_i)} \int_{Q_i} \int_{Q_i} g(x,y) dy A(w)(x) dx \frac{w(Q_i)}{w(Q)} \\ &= \frac{\lambda_\delta^p}{w(Q)} \sum_i |Q_i|^{\frac{p\varepsilon}{n}} \int_{Q_i} \int_{Q_i} g(x,y) dy A(w)(x) dx \\ &\leq \frac{\lambda_\delta^p}{w(Q)} \left( \left| \bigcup_i Q_i \right| \right)^{\frac{p\varepsilon}{n}} \int_Q \int_Q g(x,y) dy A(w)(x) dx \\ &= \left( \frac{|\bigcup_i Q_i|}{|Q|} \right)^{\frac{p\varepsilon}{n}} a_{w,u}(Q)^p. \end{aligned}$$

Then, by Theorem 4.3 with  $\|a_{w,u}\|_{SD_p^{\frac{n}{\varepsilon}}(w)} \leq 1$  and  $s = \frac{n}{\varepsilon}$ ,

$$\|u - u_Q\|_{L^p(Q, \frac{w dx}{w(Q)})} \leq \frac{c_n}{\varepsilon} a_{w,u}(Q). \quad (4.20)$$

This yields (2.9).

b) To finish the proof of the theorem we prove now (2.10). Indeed, by part a) with  $\frac{1}{2} < \varepsilon = \delta < 1$ , there is a positive dimensional constant  $c_n$  such that for any weight  $w$ ,

$$\begin{aligned} &\left( \int_Q |u(x) - u_Q|^p w dx \right)^{1/p} \\ &\leq c_n \frac{(1-\delta)^{\frac{1}{p}}}{\delta} \ell(Q)^\delta \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy M(w\chi_Q)(x) dx \right)^{1/p}. \end{aligned}$$

This gives (2.10) of the theorem, since  $\delta > \frac{1}{2}$ . For (2.11) we recall that from Definition 3.2,  $M(w)(x) \leq [w, v]_{A_1} v(x)$  and hence we continue with,

$$\begin{aligned} \left( \int_Q |u(x) - u_Q|^p w dx \right)^{1/p} &\leq c_n (1-\delta)^{\frac{1}{p}} \ell(Q)^\delta \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy M(w)(x) dx \right)^{1/p} \\ &\leq c_n (1-\delta)^{\frac{1}{p}} \ell(Q)^\delta [w, v]_{A_1}^{\frac{1}{p}} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} dy v(x) dx \right)^{1/p}. \end{aligned}$$

This concludes the proof of part b) and hence the proof of Theorem 2.9.  $\square$

## 5. WEIGHTED FRACTIONAL ISOPERIMETRIC ESTIMATES AND REPRESENTATION FORMULA

In this section we will discuss the proofs of the weighted Hardy and Poincaré–Sobolev type inequalities stated in Section 2.3 using this time representation formulas instead of the general self-improving type arguments in Section 4. At the end we will derive some global inequalities as a consequence of these results.

First we will present some local representation formulas which follow from general Poincaré type inequalities. Then we will introduce some consequences of these representation formulas.

The following lemma is a key in our arguments. It essentially follows from [FLW96], but it can also be obtained by following the proof of [HSV13, Lemma 4.10], since cubes are examples of John domains. We are interested in the tracking of the constants involved in our estimates and so we will provide the proof here for the sake of clarity.

**Lemma 5.1.** *Let  $Q_0$  be a cube in  $\mathbb{R}^n$ . Assume that  $0 < \alpha < n$  and consider  $0 < \eta < n - \alpha$  and  $1 \leq r < \infty$ . Let  $u \in L^1(Q_0)$  and let  $g$  be a non-negative measurable function on  $Q_0$  such that for a finite constant  $\kappa$ ,*

$$\int_Q |u(x) - u_Q| dx \leq \kappa \ell(Q)^\alpha \left( \int_Q g(x)^r dx \right)^{\frac{1}{r}} \quad (5.1)$$

for every cube  $Q \subset Q_0$ . Then there exists a dimensional constant  $c_n$  such that

$$|u(x) - u_{Q_0}| \leq c_n \frac{\kappa}{\alpha^{1/r'} \eta^{1/r}} \ell(Q_0)^{\alpha/r'} (I_\alpha(g^r \chi_{Q_0})(x))^{\frac{1}{r}}$$

for almost every  $x \in Q_0$ . In the particular case that  $\alpha = \eta < \frac{n}{2}$ ,

$$|u(x) - u_{Q_0}| \leq c_n \frac{\kappa}{\alpha} \ell(Q_0)^{\alpha/r'} (I_\alpha(g^r \chi_{Q_0})(x))^{\frac{1}{r}}$$

*Proof.* This result is well known but we need to be precise with the main parameters involved. We adapt the main ideas from [FLW96] in the case  $r = 1$  and [FH00] when  $r > 1$  and we refer also to [LP02]. Let  $E \subset Q_0$  be the complement of the set of Lebesgue points of  $u$  in  $Q_0$ . Then  $|E| = 0$ . For a fixed  $x \in Q_0 \setminus E$ , there exists a chain  $\{Q_k\}_{k \in \mathbb{N}}$  of nested dyadic subcubes of  $Q_0$  such that  $Q_1 = Q_0$ ,  $Q_{k+1} \subset Q_k$  with  $|Q_k| = 2^n |Q_{k+1}|$  for all  $k \in \mathbb{N}$  and

$$\{x\} = \bigcap_{k \in \mathbb{N}} Q_k.$$

Then,

$$|u(x) - u_{Q_0}| = \left| \lim_{k \rightarrow \infty} u_{Q_k} - u_{Q_1} \right| \leq \sum_{k \in \mathbb{N}} |u_{Q_{k+1}} - u_{Q_k}|.$$

Now, using the dyadic structure of the chain and the assumption (5.1), we obtain that

$$\begin{aligned} \sum_{k \in \mathbb{N}} |u_{Q_{k+1}} - u_{Q_k}| &\leq \sum_{k \in \mathbb{N}} \frac{1}{|Q_{k+1}|} \int_{Q_{k+1}} |u(y) - u_{Q_k}| dy \\ &\leq 2^n \sum_{k \in \mathbb{N}} \frac{1}{|Q_k|} \int_{Q_k} |u(y) - u_{Q_k}| dy \\ &\leq 2^n \kappa \sum_{k \in \mathbb{N}} \ell(Q_k)^\alpha \left( \frac{1}{|Q_k|} \int_{Q_k} g(y)^r dy \right)^{1/r} \\ &\leq 2^n \kappa \left( \sum_{k \in \mathbb{N}} \ell(Q_k)^\alpha \right)^{1/r'} \left( \sum_{k \in \mathbb{N}} \frac{\ell(Q_k)^\alpha}{|Q_k|} \int_{Q_k} g(y)^r dy \right)^{1/r} \\ &\leq \frac{2^n}{(1 - 2^{-\alpha})^{1/r'}} \kappa \ell(Q_0)^{\alpha/r'} \left( \int_{Q_0} g(y)^r \sum_{k \in \mathbb{N}} \ell(Q_k)^{\alpha-n} \chi_{Q_k}(y) dy \right)^{1/r}. \end{aligned}$$

Note that the immediate estimate  $|x - y| \leq \sqrt{n} \ell(Q_k)$  produces an extra unwanted log factor when summing the series. We instead proceed as follows. Fix  $y \in Q_0 \setminus \{x\}$  and pick  $0 < \eta < n - \alpha$ . Write  $k_0(y) = \max\{j \in \mathbb{N} : 2^{j-1} \leq \sqrt{n} \frac{\ell(Q_0)}{|x-y|}\}$ . Then

$$\begin{aligned} \sum_{k \in \mathbb{N}} \ell(Q_k)^{\alpha-n} \chi_{Q_k}(y) &\leq \frac{c_n}{|x-y|^{n-\alpha-\eta}} \sum_{k=1}^{k_0(y)} \ell(Q_k)^{-\eta} \chi_{Q_k}(y) \\ &\leq \frac{c_n}{|x-y|^{n-\alpha-\eta} \ell(Q_0)^\eta} \sum_{k=1}^{k_0(y)} 2^{(k-1)\eta} \\ &\leq \frac{c_n 2^{\eta k_0(y)}}{|x-y|^{n-\alpha-\eta} \ell(Q_0)^\eta (1 - 2^{-\eta})} \\ &\leq \frac{c_n}{|x-y|^{n-\alpha} (1 - 2^{-\eta})}. \end{aligned}$$

We conclude that the desired inequality

$$\begin{aligned} |u(x) - u_{Q_0}| &\leq \frac{c_n \kappa}{(1 - 2^{-\alpha})^{1/r'} (1 - 2^{-\eta})^{1/r}} \ell(Q_0)^{\alpha/r'} \left( \int_{Q_0} \frac{g(y)^r}{|x-y|^{n-\alpha}} dy \right)^{\frac{1}{r}} \\ &\leq \frac{c_n \kappa}{\alpha^{1/r'} \eta^{1/r}} \ell(Q_0)^{\alpha/r'} (I_\alpha(g^r \chi_{Q_0})(x))^{1/r} \end{aligned}$$

holds for almost every  $x \in Q_0$  since  $\frac{1}{1-2^{-t}} < \frac{c_n}{t}$ ,  $0 < t < n$ .  $\square$

As a main application of the representation formula we can derive the following lemma.

**Lemma 5.2.** *Let  $Q_0$  be a cube in  $\mathbb{R}^n$ . Assume that  $0 < \alpha < n$ . Let  $u \in L^1(Q_0)$  and let  $g$  be a non-negative measurable function on  $Q_0$  such that the following representation formula holds*

$$|u(x) - u_{Q_0}| \leq \rho I_\alpha(g\chi_{Q_0})(x), \quad \text{a.e. } x \in Q_0.$$

There exists a constant  $c_n > 0$  such that, for any weight  $w$

$$\|(u - u_{Q_0})\chi_{Q_0}\|_{L^{\frac{n}{n-\alpha},\infty}(w)} \leq \frac{c_n \rho}{\alpha} \int_{Q_0} g(x) (Mw(x))^{\frac{n-\alpha}{n}} dx.$$

*Proof.* Recall that  $\|\cdot\|_{L^{p,\infty}(w)}$  is not a norm when  $p > 1$  but it is well known that there is a norm  $\|\|\cdot\|\|_{L^{p,\infty}(w)}$  such that

$$\|\|\cdot\|\|_{L^{p,\infty}(w)} \leq \|\cdot\|_{L^{p,\infty}(w)} \leq p' \|\|\cdot\|\|_{L^{p,\infty}(w)}.$$

Hence, combining this together with Minkowski's integral inequality (this is a known result but we refer to [BKS09] for sharp bounds), we have

$$\begin{aligned} \|(u - u_{Q_0})\chi_{Q_0}\|_{L^{\frac{n}{n-\alpha},\infty}(w)} &\leq \rho \|I_\alpha(g\chi_{Q_0})\chi_{Q_0}\|_{L^{\frac{n}{n-\alpha},\infty}(w)} \\ &\leq \frac{n\rho}{\alpha} \int_{Q_0} \left\| \frac{1}{|\cdot - y|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha},\infty}(w)} g(y) dy. \end{aligned}$$

But

$$\begin{aligned} \left\| \frac{1}{|\cdot - y|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha},\infty}(w)} &\leq c_n \sup_{r>0} (|Q(y,r)|^{-1} w(Q(y,r)))^{\frac{n-\alpha}{n}} \\ &= c_n (M^c w(y))^{\frac{n-\alpha}{n}}, \end{aligned}$$

where  $Q(y,r)$  is the cube with midpoint  $y$  and side-length  $2r$ . By collecting all estimates,

$$\|(u - u_{Q_0})\chi_{Q_0}\|_{L^{\frac{n}{n-\alpha},\infty}(w)} \leq \frac{c_n \rho}{\alpha} \int_{Q_0} g(y) (Mw(y))^{\frac{n-\alpha}{n}} dy.$$

$\square$

A combination of the two lemmata gives the proof of Theorem 2.10.

*Proof of Theorem 2.10.* Making use of (1.8) in the case  $p = 1$ , for any cube  $Q$

$$\int_Q |u(x) - u_Q| dx \leq c_n (1 - \delta) \ell(Q)^\delta \int_Q \int_Q \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy dx.$$

Then apply Lemma 5.1 with  $\alpha = \delta$ ,  $r = 1$ ,  $\kappa = c_n(1 - \delta)$  and

$$g(x) = \left( \int_{Q_0} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy \right) \chi_{Q_0}(x).$$

Then there exists a constant  $c = c(n)$  such that

$$|u(x) - u_{Q_0}| \leq c_n (1 - \delta) I_\delta(g\chi_{Q_0})(x)$$

for almost every  $x \in Q_0$ . By Lemma 5.2 we have

$$\|(u - u_{Q_0})\chi_{Q_0}\|_{L^{\frac{n}{n-\delta},\infty}(w)} \leq c_n (1 - \delta) \int_{Q_0} \int_{Q_0} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx$$

since  $\delta \geq \frac{1}{2}$ . To finish we use the fractional truncation method, see Theorem 3.7.  $\square$

Similarly we prove next a global type version of Corollary 2.11.

*Proof of Corollary 2.11.* We proceed as in the proof of Corollary 2.6. By assumption, there exists an increasing sequence of cubes  $\{Q_j\}_{j \in \mathbb{N}}$  such that  $\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} Q_j$  such that

$$\lim_{j \rightarrow \infty} \frac{1}{w(Q_j)} \int_{Q_j} u(x)w(x)dx = 0.$$

Write  $c_j = \frac{1}{w(Q_j)} \int_{Q_j} u(x)w(x)dx$  for all  $j \in \mathbb{N}$ . Using (3.6), Fatou's lemma combined with Theorem 2.10, yields

$$\begin{aligned} \left( \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} &= \left( \int_{\mathbb{R}^n} \liminf_{j \rightarrow \infty} \chi_{Q_j}(x) |u(x) - c_j|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \\ &\leq \liminf_{j \rightarrow \infty} \left( \int_{\mathbb{R}^n} \chi_{Q_j}(x) |u(x) - c_j|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \\ &= \liminf_{j \rightarrow \infty} \left( \int_{Q_j} |u(x) - c_j|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \\ &\leq c_n (1 - \delta) \liminf_{j \rightarrow \infty} \int_{Q_j} \int_{Q_j} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx \\ &\leq c_n (1 - \delta) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} dy (Mw(x))^{\frac{n-\delta}{n}} dx, \end{aligned}$$

which is the claimed inequality.  $\square$

As a corollary, we obtain the following lower bound for a weighted fractional perimeter.

*Proof of Corollary 2.12.* This follows by applying Corollary 2.11 to the characteristic function of  $E$  since  $L_c^\infty(\mathbb{R}^n) \subset \mathcal{F}_w$  for any weight  $w$  such that  $w \notin L^1(\mathbb{R}^n)$ .  $\square$

*Proof of Corollary 2.13.* Fix a cube  $Q$  and let  $E$  be a measurable subset of  $Q$ . Then we can apply Theorem 2.10 using (3.6) with  $u = \chi_E$  and  $\frac{1}{2} \leq \delta < 1$ , namely

$$\begin{aligned} &\left( \frac{1}{w(Q)} \int_Q |u(x) - u_{Q,w}|^{\frac{n}{n-\delta}} w(x) dx \right)^{\frac{n-\delta}{n}} \\ &\leq \frac{c_n(1-\delta)}{w(Q)^{\frac{n-\delta}{n}}} \left( \int_E \int_E \frac{|u(x) - u(y)| Mw(x)^{\frac{n-\delta}{n}}}{|x - y|^{n+\delta}} dy dx \right. \\ &\quad \left. + \int_E \int_{Q \setminus E} \cdots dy dx + \int_{Q \setminus E} \int_E \cdots dy dx + \int_{Q \setminus E} \int_{Q \setminus E} \cdots dy dx \right) \\ &= \frac{c_n(1-\delta)}{w(Q)^{\frac{n-\delta}{n}}} \left( \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x - y|^{n+\delta}} dy dx + \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x - y|^{n+\delta}} dy dx \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{1}{w(Q)} \int_Q |u(x) - u_{Q,w}|^{\frac{n}{n-\delta}} w(x) dx \\ &\geq \frac{1}{w(Q)} \int_{Q \setminus E} \left| \chi_E(x) - \frac{w(E)}{w(Q)} \right|^{\frac{n}{n-\delta}} w(x) dx = \left( \frac{w(E)}{w(Q)} \right)^{\frac{n}{n-\delta}} \frac{w(Q \setminus E)}{w(Q)}, \end{aligned}$$

and hence

$$\begin{aligned} &\left( \frac{w(E)}{w(Q)} \right)^{\frac{n}{n-\delta}} \frac{w(Q \setminus E)}{w(Q)} \\ &\leq \frac{1}{w(Q)} \left( c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x - y|^{n+\delta}} dy dx + c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x - y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}}. \end{aligned}$$



Repeating the same argument, but replacing  $E$  by  $Q \setminus E$ , we also have

$$\begin{aligned} & \left( \frac{w(Q \setminus E)}{w(Q)} \right)^{\frac{n}{n-\delta}} \frac{w(E)}{w(Q)} \\ & \leq \frac{1}{w(Q)} \left( c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx + c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}}. \end{aligned}$$

Then, taking the maximum

$$\begin{aligned} M &= \max \left\{ \left( \frac{w(Q \setminus E)}{w(Q)} \right)^{\frac{n}{n-\delta}} \frac{w(E)}{w(Q)}, \left( \frac{w(E)}{w(Q)} \right)^{\frac{n}{n-\delta}} \frac{w(Q \setminus E)}{w(Q)} \right\} \\ &\leq \frac{1}{w(Q)} \left( c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx + c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}}. \end{aligned}$$

The maximum simplifies as follows

$$\begin{aligned} M &= \max \left\{ \frac{w(Q \setminus E)}{w(Q)} \left( \frac{w(Q \setminus E)}{w(Q)} \right)^{\frac{\delta}{n-\delta}} \frac{w(E)}{w(Q)}, \frac{w(E)}{w(Q)} \left( \frac{w(E)}{w(Q)} \right)^{\frac{\delta}{n-\delta}} \frac{w(Q \setminus E)}{w(Q)} \right\} \\ &= \frac{w(Q \setminus E)}{w(Q)} \frac{w(E)}{w(Q)} \max \left\{ \left( \frac{w(Q \setminus E)}{w(Q)} \right)^{\frac{\delta}{n-\delta}}, \left( \frac{w(E)}{w(Q)} \right)^{\frac{\delta}{n-\delta}} \right\} \\ &= \frac{w(Q \setminus E)}{w(Q)} \frac{w(E)}{w(Q)} \max \left\{ \frac{w(Q \setminus E)}{w(Q)}, \frac{w(E)}{w(Q)} \right\}^{\frac{\delta}{n-\delta}} \end{aligned}$$

and hence

$$\begin{aligned} & \frac{w(Q \setminus E)}{w(Q)} \frac{w(E)}{w(Q)} \max \left\{ \frac{w(Q \setminus E)}{w(Q)}, \frac{w(E)}{w(Q)} \right\}^{\frac{\delta}{n-\delta}} \\ & \leq \frac{1}{w(Q)} \left( c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx + c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}}. \end{aligned}$$

Now, since  $\max\{1-\alpha, \alpha\} \geq \frac{1}{2}$  with  $\alpha = \frac{w(E)}{w(Q)} \in [0, 1]$ , we get

$$\begin{aligned} & \frac{w(Q \setminus E)}{w(Q)} \frac{w(E)}{w(Q)} \frac{1}{2^{\frac{\delta}{n-\delta}}} \\ & \leq \frac{1}{w(Q)} \left( c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx + c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}} \end{aligned}$$

and then

$$\begin{aligned} & w(Q \setminus E)w(E) \\ & \leq w(Q) \left( 2^{\frac{\delta}{n}} c_n(1-\delta) \int_E \int_{Q \setminus E} \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx + 2^{\frac{\delta}{n}} c_n(1-\delta) \int_{Q \setminus E} \int_E \frac{Mw(x)^{\frac{n-\delta}{n}}}{|x-y|^{n+\delta}} dy dx \right)^{\frac{n}{n-\delta}}. \end{aligned}$$

This finishes the proof of Corollary 2.13 by translating  $\delta$  to  $\varepsilon$ .  $\square$

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