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Author(s): Liu, Jiayin; Zhang, Shijin; Zhou, Yuan

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# A quantitative second order estimate for (weighted) $p$-harmonic functions in manifolds under curvature-dimension condition ${ }^{\text {th }}$ 

Jiayin Liu ${ }^{\text {a,* }}$, Shijin Zhang ${ }^{\text {b }}$, Yuan Zhou ${ }^{\text {c }}$<br>${ }^{\text {a }}$ Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35<br>(MaD), FI-40014, Jyväskylä, Finland<br>${ }^{\text {b }}$ School of Mathematical Science, Beihang University, Changping District Shahe<br>Higher Education Park South Third Street No. 9, Beijing 102206, PR China<br>${ }^{c}$ School of Mathematical Science, Beijing Normal University, Haidian District Xinjiekou Waidajie No. 19, Beijing 10875, PR China

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A B S T R A C T

We build up a quantitative second-order Sobolev estimate of $\ln w$ for positive $p$-harmonic functions $w$ in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted $p$-harmonic functions $w$ in weighted manifolds under the Bakry-Émery curvature-dimension condition.
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## 1. Introduction

Let $\left(M^{n}, g\right)$ be a complete non-compact Riemannian manifold with dimension $n \geq 2$. Suppose that the Ricci curvature is bounded from below, that is, Ric $_{g} \geq-\kappa$ for some $\kappa \geq 0$. For any positive harmonic function $w$ in a domain $\Omega \subset M^{n}$, Cheng-Yau [2] established the following famous gradient estimate:

$$
\begin{equation*}
|\nabla \ln w|=\frac{|\nabla w|}{w} \leq C(n) \frac{1+\sqrt{\kappa} r}{r} \quad \text { in } B(z, r) \subset B(z, 2 r) \subset \Omega \tag{1.1}
\end{equation*}
$$

Recall that a harmonic function $w$ in $\Omega$ is a weak solution to the Laplace equation

$$
\Delta w:=\operatorname{div}(\nabla w)=0 \text { in } \Omega
$$

We also refer to [17, Theorem 1.3] for a quantitative $W_{\text {loc }}^{2,2}$-regularity of harmonic functions.

Motivated by the application in the inverse mean curvature flow (see [11,15]), ChengYau type gradient estimate was extended by $[16,11,21,15]$ to $p$-harmonic functions in $\Omega$ for $1<p<\infty$, that is, weak solutions to the $p$-Laplace equation

$$
\Delta_{p} w=\operatorname{div}\left(|\nabla w|^{p-2} \nabla w\right)=0 \text { in } \Omega
$$

Precisely, if $\left(M^{n}, g\right)$ is flat (that is, the Euclidean space $\left.\mathbb{R}^{n}\right)$ or its sectional curvature is bounded from below by $-\kappa$, via Cheng-Yau's approach Moser [16] and Kotschwar-Ni [11] showed that any positive $p$-harmonic function $w$ in $\Omega$ satisfies

$$
\begin{equation*}
|\nabla \ln w| \leq C(n) \frac{1+\sqrt{\kappa} r}{r} \quad \text { in } B(z, r) \subset B(z, 2 r) \subset \Omega \tag{1.2}
\end{equation*}
$$

where the constant $C(n)>0$ is independent of $p \in(1, \infty)$. Under the Ricci curvature lower bound Ric $_{g} \geq-\kappa$, it was asked in [11] whether (1.2) holds or not. Some progress was made as below. Based on Cheng-Yau's argument, Wang-Zhang [21] proved that

$$
\begin{equation*}
|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W_{\text {loc }}^{1,2} \text { with } \gamma<0 \tag{1.3}
\end{equation*}
$$

and the following weaker revision of (1.2):

$$
\begin{equation*}
|\nabla \ln w| \leq C(n, p) \frac{1+\sqrt{\kappa} r}{r} \quad \text { in } B(z, r) \subset B(z, 2 r) \subset \Omega, \tag{1.4}
\end{equation*}
$$

where the constant $C(n, p)>0$ blows up as $p \rightarrow 1$. Recently, with the aid of the fake distance coming from capacity, $C(n, p)$ was proved by Mari-Rigoli-Setti [15] to be bounded by $\frac{n-1}{p-1}$ as $p \rightarrow 1$. Moreover, (1.3) and (1.4) were generalized to weighted manifolds $\left(M^{n}, g, e^{-h} d \operatorname{vol}_{g}\right)$. A weighted $p$-harmonic function $w$ in a domain $\Omega \subset M^{n}$ is a weak solution to the weighted $p$-harmonic equation

$$
\Delta_{p, h} w:=e^{h} \operatorname{div}\left(e^{-h}|\nabla w|^{p-2} \nabla w\right)=0 \text { in } \Omega
$$

Under the Bakry-Émery curvature-dimension condition $\operatorname{Ric}_{h}^{N} \geq-\kappa$ for some $N \in[n, \infty)$ and $\kappa \geq 0$ (see Section 2 for details), Dung-Dat [5] showed that if $w>0$, then $|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W_{\text {loc }}^{1,2}$ with $\gamma<0$ and also

$$
\begin{equation*}
|\nabla \ln w| \leq C(n, N, p) \frac{1+\sqrt{\kappa} r}{r} \quad \text { in } B(z, r) \subset B(z, 2 r) \subset \Omega \tag{1.5}
\end{equation*}
$$

The main aim of this paper is to build up a quantitative second-order Sobolev estimate of $\ln w$ for positive $p$-harmonic functions $w$ in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted $p$-harmonic functions $w$ in weighted manifolds under the Bakry-Émery curvature-dimension condition. See Theorem 1.1 and Theorem 1.2 separately. These improve the corresponding second-order Sobolev regularity in [21,5] mentioned above.

To be precise, under the Ricci curvature lower bound, we have the following result. For convenience, below we write $f_{E} f d m$ as the average of $f$ in the set $E$ with respect to the measure $m$, that is, $f_{E} f d m=\frac{1}{m(E)} \int_{E} f d m$. We use $C\left(a_{1}, \cdots, a_{m}\right)$ to denote a positive constant depending on absolute constants $a_{1}, \cdots, a_{m}$.

Theorem 1.1. Suppose that $\left(M^{n}, g\right)$ satisfies Ric $_{g} \geq-\kappa$ for some $\kappa \geq 0$. Let $1<p<\infty$ and $\gamma<3+\frac{p-1}{n-1}$. For any positive $p$-harmonic function $w$ in a domain $\Omega \subset M$, we have $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W_{\mathrm{loc}}^{1,2}(\Omega)$ and

$$
\begin{equation*}
\int_{B(z, r)}\left|\nabla\left[|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w\right]\right|^{2} d \operatorname{vol}_{g} \leq C(n, p, \gamma)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{p-\gamma+4} e^{\sqrt{\kappa} r} \tag{1.6}
\end{equation*}
$$

whenever $B(z, 4 r) \Subset \Omega$.
In particular, if $1<p<3+\frac{2}{n-2}$, then $\nabla^{2} \ln w \in L_{\text {loc }}^{2}(\Omega)$ and

$$
\begin{equation*}
f_{B(z, r)}\left|\nabla^{2} \ln w\right|^{2} d \operatorname{vol}_{g} \leq C(n, p)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{4} e^{\sqrt{\kappa} r} \tag{1.7}
\end{equation*}
$$

whenever $B(z, 4 r) \Subset \Omega$.
Here and throughout the paper for domains $A$ and $B$, the notation $A \Subset B$ stands for that $A$ is a bounded subdomain of $B$ and its closure $A \subset B$.

Recall that if $\left(M^{n}, g\right)$ is flat, that is, the Euclidean space $\mathbb{R}^{n}$, $p$-harmonic functions $w$ in a domain $\Omega \subset \mathbb{R}^{n}$ are proved to satisfy $|\nabla w|^{\frac{p-\gamma}{2}} \nabla w \in W_{\text {loc }}^{1,2}(\Omega)$ with some quantitative bound whenever $\gamma<3+\frac{p-1}{n-1}$ see $[13,9,4,14]$ and also the references therein for some earlier partial results. In particular, if $1<p<3+\frac{2}{n-2}$, noting $p<3+\frac{p-1}{n-1}$ and taking $\gamma=p$, one has $w \in W_{\mathrm{loc}}^{2,2}(\Omega)$. When $n \geq 3$ and $p \geq 3+\frac{2}{n-2}$, it is not clear whether
$w \in W_{\text {loc }}^{2,2}(\Omega)$ or not. When $n=2$, the range $\gamma<3+\frac{p-1}{n-1}=p+2$ is optimal as witnessed by some construction in [9].

Moreover, we extend Theorem 1.1 to weighted manifolds satisfying Bakry-Émery curvature-dimension condition,

Theorem 1.2. Let $\left(M^{n}, g, e^{-h} \operatorname{vol}_{g}\right)$ be a weighted manifold with Ric ${ }_{h}^{N} \geq-\kappa$ for some $n \leq N<\infty$ and $\kappa \geq 0$. Let $1<p<\infty$ and $\gamma<3+\frac{p-1}{N-1}$. For any positive weighted p-harmonic function $w$ in a domain $\Omega \subset M$, we have $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W_{\operatorname{loc}}^{1,2}(\Omega)$ and

$$
\begin{equation*}
f_{B(z, r)}\left|\nabla\left[|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w\right]\right|^{2} d \operatorname{vol}_{h} \leq C(n, N, p, \gamma)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{p-\gamma+4} e^{\sqrt{\kappa} r} \tag{1.8}
\end{equation*}
$$

whenever $B(z, 4 r) \Subset \Omega$.
In particular, if $p \in\left(1,3+\frac{2}{N-2}\right)$, then $\nabla^{2} \ln w \in L_{\mathrm{loc}}^{2}(\Omega)$ and

$$
\begin{equation*}
\int_{B(z, r)}\left|\nabla^{2} \ln w\right|^{2} d \operatorname{vol}_{h} \leq C(n, N, p)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{4} e^{\sqrt{\kappa} r} \tag{1.9}
\end{equation*}
$$

whenever $B(z, 4 r) \Subset \Omega$.
As a consequence of Theorem 1.1 and Theorem 1.2, one gets that $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{\text {loc }}^{1,2}$ for $\gamma<3+\frac{p-1}{n-1}$ or $\gamma<3+\frac{p-1}{N-1}$, while in [21,5], one has $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{\text {loc }}^{1,2}$ for all $\gamma<2$ (see (1.3) and the line above (1.5)). Thus our range for $\gamma$ obviously improves the one obtained in [21,5] respectively.

Now we sketch the ideas to prove Theorem 1.1 and Theorem 1.2. Note that when $N=n$ and $h \equiv 1$, we have $\operatorname{Ric}_{h}^{N}=R i c_{g}$, and hence Theorem 1.1 corresponds to the special case $N=n$ and $h \equiv 1$ in Theorem 1.2. We only need to prove Theorem 1.2. As usual, we approximate $u=-(p-1) \ln w$ by smooth solution $u^{\epsilon}$ to the standard approximation/regularized equation (3.3), that is,

$$
e^{h} \operatorname{div}\left(e^{-h}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-2}{2}} \nabla u^{\epsilon}\right)=\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-2}{2}}\left|\nabla u^{\epsilon}\right|^{2} .
$$

(i) Using Bochner formula and the approximation equation (3.3), for $0<\eta<1 / 2$ we bound the integral of

$$
\begin{equation*}
(1-\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \tag{1.10}
\end{equation*}
$$

from above by the integral of

$$
\operatorname{Ric}_{g}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)+\left\langle\nabla^{2} h \nabla u^{\epsilon}, \nabla u^{\epsilon}\right\rangle
$$

and other first order terms, where all integrals are taken against $\left[\left|\nabla u^{\epsilon}\right|^{2}+\right.$ $\epsilon]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g}$ where $\phi \in C_{c}^{\infty}(U)$ is a test function and $U \Subset \Omega$; see Lemma 3.2. Here in (1.10) and in what follows, for any $C^{2}$ function $f, \Delta_{\infty} f:=\left\langle\nabla^{2} f \nabla f, \nabla f\right\rangle$.
(ii) If $\gamma<3+\frac{p-1}{N-1}$, via a fundamental inequality given in Lemma 2.1 and the approximation equation (3.3), for sufficiently small $\eta>0$ we bound (1.10) as below
$(1.10) \geq \eta\left|\nabla^{2} u^{\epsilon}\right|^{2}-\frac{\left\langle\nabla h, \nabla u^{\epsilon}\right\rangle^{2}}{N-n}-C \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{4} \quad$ everywhere;
see Lemma 3.4. This is crucial to get Theorem 1.2. Note that the approach in $[21,5]$ could not give Lemma 3.4; see Remark 3.8 for details.
(iii) Combining (i)\&(ii) together, the integral of $\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}$ is bounded from above by the integral of $-R i c_{h}^{N}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)$ and other first order terms, where all integrals are taken against $\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g}$; see Corollary 3.6.
Under the assumption $\operatorname{Ric} c_{h}^{N} \geq-\kappa$, in Lemma 3.7 we obtain an upper $L_{\text {loc }}^{2}$ bound for $\nabla\left[\left|\nabla u^{\epsilon}\right|^{\frac{p-\gamma}{2}} \nabla u^{\epsilon}\right] \phi$ by the integral of some first order terms, where all integrals are against $e^{-h} d \mathrm{vol}_{g}$. A standard argument then leads to the proof of Theorem 1.2.

Finally, we also notice that the Cheng-Yau gradient estimate (1.1) was generalized to positive harmonic functions $w$ in Alexandrov spaces with curvature bounded from below by Zhang-Zhu in [22], where the authors showed $|\nabla \ln w|^{2} \in W_{\text {loc }}^{1,2}(\Omega)$ as a key step. Furthermore, one could study the regularity of $p$-harmonic functions in more general metric measure spaces. In these spaces, a natural generalization of the (weighted) Ricci curvature bound is the curvature-dimension condition $R C D(\kappa, N)$ in the sense of BakryÉmery or Ambrosio-Gigli-Savaré. The two senses turned out to be equivalent by the work of Erbar-Kuwada-Sturm [6] (in the finite dimensional case) and Ambrosio-Gigli-Savaré [1] and the spaces satisfying one of the two equivalent conditions are known as $R C D(\kappa, N)$ spaces. Some progress was made in $R C D(\kappa, N)$ spaces. The Cheng-Yau gradient estimate was established by Jiang in [10] for positive harmonic functions $w$ in $R C D(\kappa, N)$ spaces; recently, Gigli-Violo in [7] established $|\nabla \ln w|^{\beta / 2} \in W_{\text {loc }}^{1,2}(\Omega)$ under $R C D(0, N)$ spaces if $\beta>\frac{N-2}{N-1}$. However, when $p \neq 2$, it remains open to prove the Cheng-Yau type gradient estimates for positive $p$-harmonic functions in Alexandrov spaces and also $R C D(\kappa, N)$ spaces.

## 2. Preliminaries

Let $n \geq 2$ and $M^{n}$ be a Riemannian manifold, and $g$ be the Riemannian metric. By abuse of notation we also write $|\xi|^{2}=g(\xi, \xi)$ and $\langle\xi, \eta\rangle=g(\xi, \eta)$ for all $\xi, \eta \in T_{x} M^{n}$. The corresponding Riemannian volume measure is written as $d \mathrm{vol}_{g}$, and the volume of a set $E$ is written as $\operatorname{vol}_{g}(E)$. Denote by $R i c_{g}$ the Ricci curvature 2-tensor and write $\operatorname{Ric}_{g} \geq-\kappa$ if $\operatorname{Ric}_{g}(\xi, \xi) \geq-\kappa|\xi|^{2}$ for all $\xi \in T_{x} M^{n}$.

For $1<p<\infty$, the $p$-Laplace operator $\Delta_{p}$ in $M^{n}$ is given by

$$
\Delta_{p} f=\operatorname{div}\left(|\nabla f|^{p-2} \nabla f\right) \quad \forall f \in C^{2}\left(M^{n}\right)
$$

Obviously, $\Delta_{2}$ is exactly the Laplace-Beltrami operator $\Delta$ in $\left(M^{n}, g\right)$. A function $w$ defined in a domain $\Omega \subset M^{n}$ is called $p$-harmonic if $w \in W_{\text {loc }}^{1, p}(\Omega)$ is a weak solution to the $p$-Laplace equation $\Delta_{p} w=0$ in $\Omega$, that is,

$$
\int_{\Omega}|\nabla w|^{p-2}\langle\nabla w, \nabla \phi\rangle d \operatorname{vol}_{g}=0 \quad \forall \phi \in C_{c}^{\infty}(\Omega)
$$

Note that 2-harmonic functions are the well-known harmonic functions.
Next we recall some basic facts of weighted Riemannian manifolds ( $M^{n}, g, e^{-h} d \mathrm{vol}_{g}$ ), where the weight $h$ is a positive smooth function in $M^{n}$. The weighted measure $d \mathrm{vol}_{h}=$ $e^{-h} d \mathrm{vol}_{g}$ can be viewed as the volume form of a suitable conformal change of the metric $g$. Denote by $\operatorname{vol}_{h}(E)$ the weighted volume of a set $E$. For $n \leq N<\infty$, the corresponding $N$-Bakry-Émery curvature tensor is

$$
R i c_{h}^{N}=R i c_{g}+\nabla^{2} h-\frac{\nabla h \otimes \nabla h}{N-n}
$$

where when $N=n$, by convention, $h$ is a constant function and hence $R i c_{h}^{N}=\operatorname{Ric}_{g}$. We say that ( $M^{n}, g, e^{-h} d \mathrm{vol}_{g}$ ) satisfies the Bakry-Émery curvature-dimension condition $R i c_{h}^{N} \geq-\kappa$ if

$$
\operatorname{Ric}_{h}^{N}(\xi, \xi)=\operatorname{Ric}_{g}(\xi, \xi)+\left\langle\nabla^{2} h \xi, \xi\right\rangle-\frac{\langle\nabla h, \xi\rangle^{2}}{N-n} \geq-\kappa\langle\xi, \xi\rangle \forall \xi \in T_{x} M^{n}
$$

By [18], under $R i c_{h}^{N} \geq-\kappa$, one has the following volume comparison result

$$
\begin{equation*}
\operatorname{vol}_{h}\left(B_{2 r}(x)\right) \leq C(N) e^{\sqrt{\kappa} r} \operatorname{vol}_{h}\left(B_{r}(x)\right) \quad \forall x \in M, r>0 \tag{2.1}
\end{equation*}
$$

For $1<p<\infty$, the weighted $p$-Laplacian $\Delta_{h, p}$ is defined as

$$
\Delta_{p, h} f=e^{h} \operatorname{div}\left(e^{-h}|\nabla f|^{p-2} \nabla f\right)=\Delta_{p} f-|\nabla f|^{p-2}\langle\nabla f, \nabla h\rangle \quad \forall f \in C^{2}\left(M^{n}\right)
$$

In the case $p=2$, one writes $\Delta_{2, h}$ as $\Delta_{h}$, and hence

$$
\Delta_{h} f=\Delta f-\langle\nabla h, \nabla f\rangle
$$

A function $w$ in a domain $\Omega \subset M^{n}$ is called as a weighted $p$-harmonic function if $w \in W_{\text {loc }}^{1, p}(\Omega)$ is a weak solution to the weighted $p$-harmonic equation $\Delta_{p, h} w=0$ in $\Omega$, that is,

$$
\begin{equation*}
\int_{\Omega}|\nabla w|^{p-2}\langle\nabla w, \nabla \phi\rangle e^{-h} d \operatorname{vol}_{g}=0 \quad \forall \phi \in C_{c}^{\infty}(\Omega) \tag{2.2}
\end{equation*}
$$

By a density argument, we can relax $\phi \in C_{c}^{\infty}(\Omega)$ to $\phi \in W_{0}^{1, p}(\Omega)$ in (2.2).
We also recall the following Bochner formula in $\left(M^{n}, g, e^{-h} d \mathrm{vol}_{g}\right)$ :

$$
\begin{equation*}
\frac{1}{2} \Delta_{h}|\nabla f|^{2}=\left|\nabla^{2} f\right|^{2}+\left\langle\nabla f, \nabla \Delta_{h} f\right\rangle+\operatorname{Ric}_{g}(\nabla f, \nabla f)+\left\langle\nabla^{2} h \nabla f, \nabla f\right\rangle \quad \forall f \in C^{3}(M) \tag{2.3}
\end{equation*}
$$

which will be used in Section 3.
Finally, we recall the following fundamental inequality; see for example [21,5,14]. For the reader's convenience we include it here. Recall that $\Delta_{\infty} f=\left\langle\nabla^{2} f \nabla f, \nabla f\right\rangle$.

Lemma 2.1. Let $n \geq 2$ and $\Omega$ be a domain of $M^{n}$. For any $f \in C^{2}(\Omega)$, we have

$$
\begin{equation*}
|\nabla f|^{4}\left|\nabla^{2} f\right|^{2} \geq 2|\nabla f|^{2}\left|\nabla^{2} f \nabla f\right|^{2}+\frac{\left[|\nabla f|^{2} \Delta f-\Delta_{\infty} f\right]^{2}}{n-1}-\left(\Delta_{\infty} f\right)^{2} \text { in } \Omega, \tag{2.4}
\end{equation*}
$$

where when $n=2, " \geq$ " becomes $"=$ ".

Proof. It suffices to prove that for any symmetric $n \times n$ matrix $A$ one has

$$
\begin{equation*}
|A|^{2}|\xi|^{4} \geq \frac{1}{n-1}\left(\operatorname{tr} A|\xi|^{2}-\langle A \xi, \xi\rangle\right)^{2}+2|A \xi|^{2}|\xi|^{2}-\langle A \xi, \xi\rangle^{2} \quad \forall \xi \in \mathbb{R}^{n} \tag{2.5}
\end{equation*}
$$

Note that if $\xi=0,(2.5)$ holds obviously. Below assume that $\xi \neq 0$. Up to a scaling we may assume $|\xi|=1$. By a change of coordinates, we may further assume $\xi=e_{n}=(0, \cdots, 0,1)$; in this case, (2.5) reads as

$$
|A|^{2} \geq \frac{1}{n-1}\left(\operatorname{tr} A-\left\langle A e_{n}, e_{n}\right\rangle\right)^{2}+2\left|A e_{n}\right|^{2}-\left\langle A e_{n}, e_{n}\right\rangle^{2}
$$

Denoting by $A_{n-1}$ the $(n-1)$ order principal submatrix of $A$, one has

$$
|A|^{2}=\left|A_{n-1}\right|^{2}+2\left|A e_{n}\right|^{2}-\left\langle A e_{n}, e_{n}\right\rangle^{2}
$$

Noting that

$$
\left|A_{n-1}\right|^{2} \geq \frac{1}{n-1}\left(\operatorname{tr} A_{n-1}\right)^{2}=\frac{1}{n-1}\left(\operatorname{tr} A-\left\langle A e_{n}, e_{n}\right\rangle\right)^{2}
$$

where when $n=2$, one has $\left|A_{n-1}\right|^{2}=\left(\operatorname{tr} A_{n-1}\right)^{2}$, one concludes (2.4).

## 3. Proof of Theorem 1.2

Let $w$ be a positive weighted $p$-harmonic function in a domain $\Omega$. Set $u=-(p-1) \ln w$. Then $u$ is a weak solution to the equation

$$
\begin{equation*}
\Delta_{p} u-|\nabla u|^{p-2}\langle\nabla u, \nabla h\rangle=|\nabla u|^{p} \quad \text { in } \Omega, \tag{3.1}
\end{equation*}
$$

that is,

$$
-\int_{\Omega}|\nabla u|^{p-2}\langle\nabla u, \nabla \phi\rangle e^{-h} d \operatorname{vol}_{g}=\int_{\Omega}|\nabla u|^{p} \phi e^{-h} d \operatorname{vol}_{g} \quad \forall \phi \in C_{c}^{\infty}(\Omega)
$$

Given any smooth domain $U \Subset \Omega$ and $\epsilon \in(0,1]$, consider the approximation/regularized equation defined by

$$
\begin{equation*}
e^{h} \operatorname{div}\left(e^{-h}\left[|\nabla v|^{2}+\epsilon\right]^{\frac{p-2}{2}} \nabla v\right)=\left[|\nabla v|^{2}+\epsilon\right]^{\frac{p-2}{2}}|\nabla v|^{2} \quad \text { in } U ; v=u \text { on } \partial U . \tag{3.2}
\end{equation*}
$$

It is well known that if $u$ is the solution to (3.1), then $u \in C^{1, \alpha}(\Omega)$ for some $\alpha \in(0,1)$; see $[3,12,19,20]$. Moreover, in the following lemma, we summarize some properties of the solution $u$ to (3.1) and $u^{\epsilon}$ to (3.3), which result from [3] as a special case. See also [19].

Lemma 3.1. For any $\epsilon \in(0,1]$, there exists a unique solution $u^{\epsilon} \in C^{\infty}(U) \cap C^{0}(\bar{U})$ to (3.3), and moreover, $u^{\epsilon} \rightarrow u$ in $C^{0}(\bar{U})$ and $u^{\epsilon} \rightarrow u$ in $C^{1, \alpha}(V)$ uniformly in $\epsilon>0$ as $\epsilon \rightarrow 0$ for all $V \Subset U$ where $u$ is the solution to (3.1).

To show Lemma 3.1, we just need to check that equations (3.1) and (3.3) are special cases of those considered in [3]. We put this verification in the appendix.

By Lemma 3.1, the solution $u^{\epsilon}$ to (3.2) is $C^{\infty}$, which implies that $u^{\epsilon}$ satisfies (3.2) pointwise. Hence by a direct computation, (3.2) is equivalent to

$$
\begin{equation*}
\Delta_{h} u^{\epsilon}+(p-2) \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}=\left|\nabla u^{\epsilon}\right|^{2} \quad \text { in } U ; u^{\epsilon}=u \text { on } \partial U . \tag{3.3}
\end{equation*}
$$

To prove Theorem 1.2 we first build up the following upper bound.
Lemma 3.2. Let $u^{\epsilon}$ be the solution to (3.3). For any $\gamma \in \mathbb{R}, \eta>0$ and $\phi \in C_{c}^{\infty}(U)$, we have

$$
\begin{aligned}
& \int_{U}\left\{(1-\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}\right\} \\
& \times\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \leq-\int_{U}\left[R i c_{g}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)+\left\langle\nabla^{2} h \nabla u^{\epsilon}, \nabla u^{\epsilon}\right\rangle\right]\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g}
\end{aligned}
$$

$$
\begin{equation*}
+C(p, \gamma) \frac{1}{\eta} \int_{U}\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2}+\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2}\right) e^{-h} d \operatorname{vol}_{g} \tag{3.4}
\end{equation*}
$$

To prove this, we need the following identity.
Lemma 3.3. For any $v \in C^{3}(U)$ and $\psi \in C_{c}^{\infty}(U)$, one has

$$
\begin{align*}
\int_{U}\left|\nabla^{2} v\right|^{2} \psi e^{-h} d \mathrm{vol}_{g}= & -\int_{U}\left\langle\nabla^{2} v \nabla v-\Delta_{h} v \nabla v, \nabla \psi\right\rangle e^{-h} d \mathrm{vol}_{g}+\int_{U}\left(\Delta_{h} v\right)^{2} \psi e^{-h} d \mathrm{vol}_{g} \\
& -\int_{U}\left[\operatorname{Ric}_{g}(\nabla v, \nabla v)+\left\langle\nabla^{2} h \nabla v, \nabla v\right\rangle\right] \psi e^{-h} d \mathrm{vol}_{g} \tag{3.5}
\end{align*}
$$

Proof. Applying the Bochner formula to $v$, one has

$$
\left|\nabla^{2} v\right|^{2}+\operatorname{Ric}_{g}(\nabla v, \nabla v)=\frac{1}{2} \Delta_{h}|\nabla v|^{2}-\left\langle\nabla v, \nabla \Delta_{h} v\right\rangle-\left\langle\nabla^{2} h \nabla v, \nabla v\right\rangle
$$

and hence

$$
\begin{aligned}
\left|\nabla^{2} v\right|^{2}= & {\left[\frac{1}{2} \Delta_{h}|\nabla v|^{2}-\left(\Delta_{h} v\right)^{2}-\left\langle\nabla v, \nabla \Delta_{h} v\right\rangle\right]+\left(\Delta_{h} v\right)^{2} } \\
& -\left[\operatorname{Ric}_{g}(\nabla v, \nabla v)+\left\langle\nabla^{2} h \nabla v, \nabla v\right\rangle\right]
\end{aligned}
$$

By this, to get (3.5), it suffices to show the following identity

$$
\begin{align*}
\int_{U} & {\left[\frac{1}{2} \Delta_{h}|\nabla v|^{2}-\left(\Delta_{h} v\right)^{2}-\left\langle\nabla v, \nabla \Delta_{h} v\right\rangle\right] \psi e^{-h} d \operatorname{vol}_{g} } \\
& =-\int_{U}\left\langle\nabla^{2} v \nabla v-\Delta_{h} v \nabla v, \nabla \psi\right\rangle e^{-h} d \operatorname{vol}_{g} \tag{3.6}
\end{align*}
$$

Note that

$$
\begin{aligned}
-\left[\left(\Delta_{h} v\right)^{2}+\left\langle\nabla v, \nabla\left(\Delta_{h} v\right)\right\rangle\right] & =-e^{h} \operatorname{div}\left(e^{-h} \nabla v\right)\left(\Delta_{h} v\right)-e^{h}\left\langle e^{-h} \nabla v, \nabla\left(\Delta_{h} v\right)\right\rangle \\
& =-e^{h} \operatorname{div}\left(e^{-h} \nabla v \Delta_{h} v\right) .
\end{aligned}
$$

Via integration by parts, one has

$$
\begin{aligned}
-\int_{U}\left[\left(\Delta_{h} v\right)^{2}+\left\langle\nabla v, \nabla\left(\Delta_{h} v\right)\right\rangle\right] \psi e^{-h} d \operatorname{vol}_{g} & =-\int_{U} \operatorname{div}\left(e^{-h} \nabla v \Delta_{h} v\right) \psi d \operatorname{vol}_{g} \\
& =\int_{U}\left\langle\Delta_{h} v \nabla v, \nabla \psi\right\rangle e^{-h} d \operatorname{vol}_{g}
\end{aligned}
$$

Similarly, via integration by parts one also has

$$
\begin{aligned}
\frac{1}{2} \int_{U} \Delta_{h}|\nabla v|^{2} \psi e^{-h} d \operatorname{vol}_{g} & =\int_{U} \frac{1}{2} \operatorname{div}\left(e^{-h} \nabla|\nabla v|^{2}\right) \psi d \operatorname{vol}_{g} \\
& \left.=-\left.\int_{U} \frac{1}{2}\left\langle e^{-h} \nabla\right| \nabla v\right|^{2}, \nabla \psi\right\rangle d \operatorname{vol}_{g} \\
& =-\int_{U}\left\langle\nabla^{2} v \nabla v, \nabla \psi\right\rangle e^{-h} d \operatorname{vol}_{g}
\end{aligned}
$$

Combining together we obtain (3.6) and hence, (3.5) as desired.
We are ready prove Lemma 3.2 as below.
Proof of Lemma 3.2. Taking $v=u^{\epsilon}$ and $\psi=\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right] \frac{p-\gamma}{2} \phi^{2}$ in (3.5) we get

$$
\begin{align*}
& \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
&=-\int_{U}\left\langle\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}-\Delta_{h} u^{\epsilon} \nabla u^{\epsilon}, \nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2}\right]\right\rangle e^{-h} d \operatorname{vol}_{g} \\
&+\int_{U}\left(\Delta_{h} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
&-\int_{U}\left[\operatorname{Ric}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)+\left\langle\nabla^{2} h \nabla u^{\epsilon}, \nabla u^{\epsilon}\right\rangle\right]\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \tag{3.7}
\end{align*}
$$

To bound the second term in the right-hand side in (3.7), recalling (3.3), that is,

$$
\begin{equation*}
\Delta_{h} u^{\epsilon}=\left|\nabla u^{\epsilon}\right|^{2}-(p-2) \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon} \tag{3.8}
\end{equation*}
$$

by Cauchy-Schwarz's inequality one has

$$
\left(\Delta_{h} u^{\epsilon}\right)^{2} \leq(p-2)^{2} \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+\frac{\eta}{4}\left|\nabla^{2} u^{\epsilon}\right|^{2}+C(p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{4}
$$

where $0<\eta<1$ is any constant. Thus

$$
\begin{aligned}
\int_{U}\left(\Delta_{h} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \leq & (p-2)^{2} \int_{U}\left(\Delta_{\infty} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}-2} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
& +\frac{\eta}{4} \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g}
\end{aligned}
$$

$$
\begin{equation*}
+\frac{C(p)}{\eta} \int_{U}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2} e^{-h} d \operatorname{vol}_{g} \tag{3.9}
\end{equation*}
$$

The first term in the right-hand side in (3.7) is further written as

$$
\begin{align*}
&-\int_{U}\left\langle\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}-\Delta_{h} u^{\epsilon} \nabla u^{\epsilon}, \nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2}\right]\right\rangle e^{-h} d \operatorname{vol}_{g} \\
&=-(p-\gamma) \int_{U} \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
&+(p-\gamma) \int_{U} \Delta_{h} u^{\epsilon} \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
&-\int_{U}\left\langle\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}, \nabla \phi^{2}\right\rangle\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \operatorname{vol}_{g} \\
&+\int_{U}\left\langle\Delta_{h} u^{\epsilon} \nabla u^{\epsilon}, \nabla \phi^{2}\right\rangle\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \operatorname{vol}_{g} . \tag{3.10}
\end{align*}
$$

Using (3.8) and Cauchy-Schwarz's inequality, we obtain the following upper bound for the second term in (3.10):

$$
\begin{align*}
&(p-\gamma) \int_{U} \Delta_{h} u^{\epsilon} \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&=-(p-\gamma)(p-2) \int_{U}\left(\Delta_{\infty} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}-2} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&+(p-\gamma) \int_{U} \Delta_{\infty} u^{\epsilon}\left|\nabla u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}-1} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
& \leq-(p-\gamma)(p-2) \int_{U}\left(\Delta_{\infty} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}-2} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&+\frac{\eta}{4} \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&+\frac{C(p)}{\eta}|p-\gamma|^{2} \int_{U}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2} e^{-h} d \mathrm{vol}_{g} . \tag{3.11}
\end{align*}
$$

For the third term in the right-hand side of (3.10), by Cauchy-Schwarz's inequality, one has

$$
\begin{align*}
& \left|\int_{U}\left\langle\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}, \nabla \phi^{2}\right\rangle\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \operatorname{vol}_{g}\right| \\
& \quad \leq \frac{\eta}{4} \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g}+C \frac{1}{\eta} \int_{U}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2} e^{-h} d \operatorname{vol}_{g} . \tag{3.12}
\end{align*}
$$

For the fourth term in the right-hand side of (3.10), in a similar way, using (3.8), one has

$$
\begin{align*}
& \left|\int_{U}\left\langle\Delta_{h} u^{\epsilon} \nabla u^{\epsilon}, \nabla \phi^{2}\right\rangle\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \mathrm{vol}_{g}\right| \\
& \left.=\left|\int_{U}\langle | \nabla u^{\epsilon}\right|^{2} \nabla u^{\epsilon}-(p-2) \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon} \nabla u^{\epsilon}, \nabla \phi^{2}\right\rangle \left.\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \mathrm{vol}_{g} \right\rvert\, \\
& \leq \frac{\eta}{4} \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
& \quad+C(p) \frac{1}{\eta} \int_{U}\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2}+\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2}\right) e^{-h} d \mathrm{vol}_{g} . \tag{3.13}
\end{align*}
$$

From (3.13), (3.12), (3.11) and (3.10) we attain

$$
\begin{align*}
&- \int_{U}\left\langle\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}-\Delta_{h} u^{\epsilon} \nabla u^{\epsilon}, \nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2}\right]\right\rangle e^{-h} d \mathrm{vol}_{g} \\
&= \frac{3}{4} \eta \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&-(p-\gamma) \int_{U} \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \mathrm{vol}_{g} \\
&-(p-\gamma)(p-2) \int_{U}\left(\Delta_{\infty} u^{\epsilon}\right)^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}-2} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \quad+\frac{C(p)}{\eta} \int_{U}\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2}+\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2}\right) e^{-h} d \mathrm{vol}_{g} \tag{3.14}
\end{align*}
$$

Obviously from (3.14), (3.9) and (3.7) we conclude (3.4).
If $\gamma<3+\frac{p-1}{N-1}$, we get the following pointwise lower bound. Recall that when $N=n$, we always assume that $h$ is a constant function and $\frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n}=0$.

Lemma 3.4. Let $u^{\epsilon}$ be the solution to (3.3). If $\gamma<3+\frac{p-1}{N-1}$ for some $N \geq n$, then for sufficiently small $\eta>0$ we have

$$
\begin{align*}
& (1-\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \\
& \quad \geq \eta\left|\nabla^{2} u^{\epsilon}\right|^{2}-\frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n}-C(n, N, p, \gamma) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{4} \tag{3.15}
\end{align*}
$$

To prove this, we need the following pointwise lower bound for $\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}$.
Lemma 3.5. Let $u^{\epsilon}$ be the solution to (3.3). If $N \geq n$, then for $0<\eta<1$ we have

$$
\begin{align*}
(1+\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4} \geq & 2\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{2} \\
& +\left(\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}-1\right)\left(\Delta_{\infty} u^{\epsilon}\right)^{2} \\
& -(1+\eta) \frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n}\left|\nabla u^{\epsilon}\right|^{4}-C(n, N, p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{8} . \tag{3.16}
\end{align*}
$$

Proof. Applying (2.4) to $u^{\epsilon}$ one has

$$
\begin{equation*}
\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4} \geq 2\left|\nabla u^{\epsilon}\right|^{2}\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}+\frac{\left[\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}\right]^{2}}{n-1}-\left(\Delta_{\infty} u^{\epsilon}\right)^{2} \tag{3.17}
\end{equation*}
$$

By (3.8) and $\Delta u^{\epsilon}=\Delta_{h} u^{\epsilon}+\left\langle\nabla h, \nabla u^{\epsilon}\right\rangle$, we have

$$
\Delta u^{\epsilon}=\left|\nabla u^{\epsilon}\right|^{2}+\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle-(p-2) \frac{\Delta_{\infty} u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon} .
$$

Thus

$$
\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}=\left|\nabla u^{\epsilon}\right|^{2}\left(\left|\nabla u^{\epsilon}\right|^{2}+\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle\right)-\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right] \Delta_{\infty} u^{\epsilon}
$$

and hence,

$$
\begin{align*}
{\left[\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}\right]^{2}=} & {\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}\left(\Delta_{\infty} u^{\epsilon}\right)^{2} } \\
& +\left|\nabla u^{\epsilon}\right|^{4}\left(\left|\nabla u^{\epsilon}\right|^{2}+\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle\right)^{2} \\
& -2\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]\left|\nabla u^{\epsilon}\right|^{4} \Delta_{\infty} u^{\epsilon} \\
& -2\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]\left|\nabla u^{\epsilon}\right|^{2} \Delta_{\infty} u^{\epsilon}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle \\
= & I_{1}+I_{2}+I_{3}+I_{4} \tag{3.18}
\end{align*}
$$

Note that

$$
\begin{equation*}
\left|(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right|^{2} \leq 4 p^{2} \tag{3.19}
\end{equation*}
$$

which can be obtained by considering $p>2$ and $1<p<2$ separately. Using this, Cauchy-Schwarz inequality, for $0<\eta<1$, we have

$$
\begin{equation*}
I_{3} \geq-\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}-C(p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{8} \tag{3.20}
\end{equation*}
$$

If $h$ is a constant function and hence $\nabla h=0, I_{2} \geq 0$ and $I_{4}=0$, dividing by $n-1$ in both sides of (3.18), by (3.20) one has

$$
\begin{aligned}
& \frac{\left[\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}\right]^{2}}{n-1} \geq-\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4} \\
& \quad+\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2} \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{n-1}-\frac{C(p)}{\eta}\left|\nabla u^{\epsilon}\right|^{8}
\end{aligned}
$$

Plugging this in (3.17), noting $N=n$, and adding $\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}$ in both sides, one concludes (3.16).

If $h$ is not a constant function, set $\eta_{1}=\frac{N-n}{N-1}$. Then

$$
\begin{equation*}
1-\eta_{1}=\frac{n-1}{N-1}>0 \quad \text { and } \quad 1-\frac{1}{\eta_{1}}=-\frac{n-1}{N-n}<0 \tag{3.21}
\end{equation*}
$$

For any $0<\eta<1$ one has

$$
\begin{align*}
I_{2} & \geq\left|\nabla u^{\epsilon}\right|^{4}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}+2\left|\nabla u^{\epsilon}\right|^{6}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle \\
& \geq\left[1+\eta\left(1-\frac{1}{\eta_{1}}\right)\right]\left|\nabla u^{\epsilon}\right|^{4}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}-\frac{1}{\eta\left|1-\frac{1}{\eta_{1}}\right|}\left|\nabla u^{\epsilon}\right|^{8} . \tag{3.22}
\end{align*}
$$

Using Cauchy-Schwarz inequality, we have

$$
\begin{equation*}
I_{4} \geq-\eta_{1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}\left(\Delta_{\infty} u^{\epsilon}\right)^{2}-\frac{1}{\eta_{1}}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}\left|\nabla u^{\epsilon}\right|^{4} \tag{3.23}
\end{equation*}
$$

Dividing by $n-1$ in both sides of (3.18), by (3.20), (3.22) and (3.23) one has

$$
\begin{aligned}
\frac{\left[\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}\right]^{2}}{n-1} \geq & -\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}+\frac{1-\eta_{1}}{n-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}\left(\Delta_{\infty} u^{\epsilon}\right)^{2} \\
& +(1+\eta) \frac{1-\frac{1}{\eta_{1}}}{n-1}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}\left|\nabla u^{\epsilon}\right|^{4}-C(n, N, p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{8}
\end{aligned}
$$

By (3.21),

$$
\begin{aligned}
\frac{\left[\left|\nabla u^{\epsilon}\right|^{2} \Delta u^{\epsilon}-\Delta_{\infty} u^{\epsilon}\right]^{2}}{n-1} \geq & -\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}+\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}\left(\Delta_{\infty} u^{\epsilon}\right)^{2} \\
& -(1+\eta) \frac{1}{N-n}\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}\left|\nabla u^{\epsilon}\right|^{4}-C(n, N, p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{8}
\end{aligned}
$$

Plugging this in (3.17), and adding $\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}\left|\nabla u^{\epsilon}\right|^{4}$ in both sides, we conclude (3.16) as desired.

We now prove Lemma 3.4 by using Lemma 3.5.

Proof of Lemma 3.4. Given any point $x \in U$, if $\nabla u^{\epsilon}(x)=0$, then (3.15) holds trivially. Below we assume that $\nabla u^{\epsilon}(x) \neq 0$. At such point $x$, we already have (3.16) in Lemma 3.5. Dividing by $\left|\nabla u^{\epsilon}\right|^{4}$ in both sides of (3.16), for $0<\eta<1 / 2$ we obtain

$$
\begin{aligned}
(1+\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2} \geq & 2 \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}}+\left(\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}-1\right) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left|\nabla u^{\epsilon}\right|^{4}} \\
& -(1+\eta) \frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n}-\frac{C(n, N, p)}{\eta}\left|\nabla u^{\epsilon}\right|^{4}
\end{aligned}
$$

In both sides, multiplying by $\frac{1-2 \eta}{1+\eta}>0$ and adding

$$
\eta\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}},
$$

we get

$$
\begin{align*}
&(1-\eta)\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \\
& \geq \eta\left|\nabla^{2} u^{\epsilon}\right|^{2}+\left\{\frac{1-2 \eta}{1+\eta} 2+(p-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\right\} \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}} \\
&+\left\{\frac{1-2 \eta}{1+\eta}\left(\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}-1\right)\right. \\
&\left.+(p-2)(2-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}\right\} \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left|\nabla u^{\epsilon}\right|^{4}} \\
&-(1-2 \eta) \frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n}-C(n, N, p) \frac{1}{\eta}\left|\nabla u^{\epsilon}\right|^{4} \\
&= I_{1}+I_{2}+I_{3}+I_{4}+I_{5} . \tag{3.24}
\end{align*}
$$

Recall that if $N=n$ that is, $h$ is a constant function, $I_{4}=0$ by our convention. If $N>n$ that is, $h$ is not a constant, then by $1-2 \eta<1$, we have

$$
\begin{equation*}
I_{4} \geq-\frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n} \tag{3.25}
\end{equation*}
$$

To bound $I_{2}+I_{3}$ from below, since $\gamma<3+\frac{p-1}{N-1}$ and $N \geq 2$ implies

$$
p+2-\gamma>p+2-3-\frac{p-1}{N-1}=(p-1)\left(1-\frac{1}{N-1}\right) \geq 0
$$

we can find $0<\hat{\eta}(p, \gamma)<1 / 2$ such that for $0<\eta<\hat{\eta}$, one has $p+2 \frac{1-2 \eta}{1+\eta}-\gamma>0$. Thus the coefficient of $I_{2}$ satisfies

$$
(p-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+2 \frac{1-2 \eta}{1+\eta} \geq\left(p+2 \frac{1-2 \eta}{1+\eta}-\gamma\right) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+\frac{1-2 \eta}{1+\eta} \frac{\epsilon}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}>0
$$

Using this and observing

$$
\frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}} \geq \frac{\left|\Delta_{\infty} u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{4}}
$$

one has

$$
\begin{aligned}
& I_{2}+I_{3} \\
& \begin{aligned}
\geq & \left\{(p-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+2 \frac{1-2 \eta}{1+\eta}\right. \\
& +\frac{1-2 \eta}{1+\eta}\left(\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}-1\right) \\
& \left.+(p-2)(2-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}\right\} \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left|\nabla u^{\epsilon}\right|^{4}} \\
= & : H(\eta) \frac{\left(\Delta_{\infty} u^{\epsilon}\right)^{2}}{\left|\nabla u^{\epsilon}\right|^{4}}
\end{aligned}
\end{aligned}
$$

We claim that there exists $0<\bar{\eta}(n, N, p, \gamma)<\hat{\eta}$ such that $H(\eta)>0$ for all $0<\eta<\bar{\eta}$. Assuming this claim holds for the moment, for any $0<\eta<\bar{\eta}$, one has $I_{2}+I_{3}>0$. From this, (3.24) and (3.25) we conclude (3.15) as desired.

Finally we prove the above claim. It suffices to show that

$$
\begin{aligned}
& H(0):=(p-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]}+2+\left(\frac{1}{N-1}\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}-1\right) \\
&+(p-2)(2-\gamma) \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}
\end{aligned}
$$

$$
\begin{equation*}
>\delta(N, p, \gamma) \tag{3.26}
\end{equation*}
$$

where $\delta(N, p, \gamma)>0$ is a constant. Indeed, by (3.19), one has

$$
H(\eta) \geq H(0)-2\left[1-\frac{1-2 \eta}{1+\eta}\right]-\left[1-\frac{1-2 \eta}{1+\eta}\right]\left[\frac{4 p^{2}}{N-1}-1\right] \geq \delta(N, p, \gamma)-15 p^{2} \eta
$$

If $0<\eta<\bar{\eta}=: \min \left\{\hat{\eta}, \delta(N, p, \gamma) / 15 p^{2}\right\}$, one has $H(\eta)>0$ and hence the claim holds as desired.

We prove (3.26) as below. Since

$$
\left[(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+1\right]^{2}=(p-2)^{2} \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+2(p-2) \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]}+1
$$

we rewrite

$$
H(0)=(p-2)\left[2-\gamma+\frac{p-2}{N-1}\right] \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+\left[p-\gamma+\frac{2(p-2)}{N-1}\right] \frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]}+\frac{N}{N-1} .
$$

Observing

$$
\frac{\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]}=\frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+\frac{\epsilon\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}
$$

and

$$
1=\frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+2 \frac{\epsilon\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+\frac{\epsilon^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}},
$$

we further write

$$
\begin{aligned}
H(0)= & \left\{(p-2)\left[2-\gamma+\frac{p-2}{N-1}\right]+\left[p-\gamma+\frac{2(p-2)}{N-1}\right]+\frac{N}{N-1}\right\} \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \\
& +\left\{\left[p-\gamma+\frac{2(p-2)}{N-1}\right]+2 \frac{N}{N-1}\right\} \frac{\epsilon\left|\nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \\
& +\frac{N}{N-1} \frac{\epsilon^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} .
\end{aligned}
$$

By a direct calculation, $\gamma<3+\frac{p-1}{N-1}$ implies that

$$
\begin{aligned}
{\left[p-\gamma+\frac{2(p-2)}{N-1}\right]+2 \frac{N}{N-1} } & >p+\frac{2(p-2)}{N-1}+2 \frac{N}{N-1}-3-\frac{p-1}{N-1} \\
& =p-1+\frac{2(p-2)+2-(p-1)}{N-1} \\
& =(p-1) \frac{N}{N-1}
\end{aligned}
$$

$$
>0
$$

Moreover, $\gamma<3+\frac{p-1}{N-1}$ also implies that

$$
\begin{aligned}
& (p-2)\left[2-\gamma+\frac{p-2}{N-1}\right]+\left[p-\gamma+\frac{2(p-2)}{N-1}\right]+\frac{N}{N-1} \\
& =3(p-1)+\frac{(p-2)^{2}+2(p-2)+1}{N-1}-(p-1) \gamma \\
& =3(p-1)+\frac{(p-1)^{2}}{N-1}-(p-1) \gamma \\
& =(p-1)\left[3+\frac{p-1}{N-1}-\gamma\right] \\
& >0 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
H(0) & >(p-1)\left[3+\frac{p-1}{N-1}-\gamma\right] \frac{\left|\nabla u^{\epsilon}\right|^{4}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}+\frac{N}{N-1} \frac{\epsilon^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}} \\
& \geq \frac{1}{2} \min \left\{(p-1)\left[3+\frac{p-1}{N-1}-\gamma\right], \frac{N}{N-1}\right\} \\
& =: \delta(N, p, \gamma) \\
& >0
\end{aligned}
$$

that is, (3.26) holds.
Combining (3.15) and (3.4) we have the following. Recall that

$$
\operatorname{Ric}_{h}^{N}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)=\operatorname{Ric}_{g}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)+\left\langle\nabla^{2} h \nabla u^{\epsilon}, \nabla u^{\epsilon}\right\rangle-\frac{\left\langle\nabla u^{\epsilon}, \nabla h\right\rangle^{2}}{N-n} .
$$

Corollary 3.6. Let $u^{\epsilon}$ be the solution to (3.3). If $\gamma<3+\frac{p-1}{N-1}$ for some $N \geq n$, then for sufficiently small $\eta>0$ one has

$$
\begin{align*}
& \eta \int_{U}\left|\nabla^{2} u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \leq-\int_{U} R i c_{h}^{N}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right)\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
&+C(n, N, p, \gamma, \eta) \int_{U}\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2}+\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2}\right) e^{-h} d \operatorname{vol}_{g} \tag{3.27}
\end{align*}
$$

Under the Bakry-Émery curvature-dimension assumption, we have the following uniform upper bound.

Lemma 3.7. Let $u^{\epsilon}$ be the solution to (3.3). If $\gamma<3+\frac{p-1}{N-1}$ and $\operatorname{Ric}_{h}^{N} \geq-\kappa$, then one has

$$
\begin{aligned}
& \int_{U}\left|\nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right]\right|^{2} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \leq C(n, N, p, \gamma) \int_{U} \kappa\left|\nabla u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \quad+C(n, N, p, \gamma) \int_{U}\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+1}|\nabla \phi|^{2}+\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}+2} \phi^{2}\right) e^{-h} d \operatorname{vol}_{g}(3.28)
\end{aligned}
$$

Proof. By $\operatorname{Ric}_{h}^{N} \geq-\kappa$ we know that

$$
-\operatorname{Ric}_{h}^{N}\left(\nabla u^{\epsilon}, \nabla u^{\epsilon}\right) \leq \kappa\left|\nabla u^{\epsilon}\right|^{2}
$$

Thus the first term in the right-hand side of (3.27) is bounded from above by

$$
\kappa \int_{U}\left|\nabla u^{\epsilon}\right|^{2}\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} \phi^{2} e^{-h} d \operatorname{vol}_{g}
$$

On the other hand, a direct calculation leads to

$$
\begin{aligned}
& \left|\nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right]\right|^{2} \\
& \quad=\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}}\left|\nabla^{2} u^{\epsilon}+\frac{p-\gamma}{2} \frac{\nabla u^{\epsilon} \otimes \nabla^{2} u^{\epsilon} \nabla u^{\epsilon}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}\right|^{2} \\
& \quad=\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}}\left[\left|\nabla^{2} u^{\epsilon}\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left|\nabla u^{\epsilon}\right|^{2}+\epsilon}+\frac{(p-\gamma)^{2}}{4} \frac{\left|\nabla u^{\epsilon}\right|^{2}\left|\nabla^{2} u^{\epsilon} \nabla u^{\epsilon}\right|^{2}}{\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{2}}\right] \\
& \quad \leq C(n, p, \gamma)\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}}\left|\nabla^{2} u^{\epsilon}\right|^{2} .
\end{aligned}
$$

Thus, up to a constant multiplier, the left-hand side of (3.27) is bounded by

$$
\int_{U}\left|\nabla\left[\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right]\right|^{2} e^{-h} d \operatorname{vol}_{g}
$$

We therefore conclude (3.28) from (3.27).

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. Let $w \in W_{\mathrm{loc}}^{1, p}(\Omega)$ be any positive weighted $p$-harmonic function in the domain $\Omega$ and $u=-(p-1) \ln w$. Given any smooth domain $U \Subset \Omega$, for each $\epsilon \in(0,1]$, let $u^{\epsilon} \in C^{\infty}(U)$ be the solution to (3.3). By Lemma 3.1, we know that $u^{\epsilon} \rightarrow$ $u \in C^{1, \alpha}(U)$, for some $\alpha \in(0,1)$ uniformly in $\epsilon>0$ as $\epsilon \rightarrow 0$. Using this and choosing suitable test functions $\phi \in C_{c}^{\infty}(U)$ in (3.28), one concludes $\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon} \in W_{\text {loc }}^{1,2}(U)$ uniformly in $\epsilon \in(0,1]$.

Next, we claim that

$$
\begin{equation*}
|\nabla u|^{\frac{p-\gamma}{2}} \nabla u \in W_{\mathrm{loc}}^{1,2}(U) \tag{3.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right) \rightarrow \nabla\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right) \text { weakly in } L_{\mathrm{loc}}^{2}\left(U, \mathbb{R}^{n \times n}\right) \text { as } \epsilon \rightarrow 0 . \tag{3.30}
\end{equation*}
$$

To see this, for any subdomain $V \Subset U$, by Lemma 3.7, we already have

$$
\sup _{\epsilon \in(0,1]}\left\|\nabla\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right)\right\|_{L^{2}\left(V, \mathbb{R}^{n \times n}\right)}<C(\kappa, n, N, p, \gamma, V) .
$$

For any subsequence $\left\{\epsilon_{j}\right\}_{j \in \mathbb{N}}$ which converges to 0 , by the weak compactness of $W^{2,2}(V)$, up to some subsequence one has $\nabla\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_{j}}\right) \rightarrow z$ weakly in $L^{2}\left(V, \mathbb{R}^{n \times n}\right)$ for some function $z \in L^{2}\left(V, \mathbb{R}^{n \times n}\right)$. Let $\left\{e_{1}, \cdots, e_{n}\right\} \subset T_{x} U$ be a local orthonormal frame at each $x \in U$. Notice that the $n \times n$ matrix

$$
\nabla\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_{j}}\right)=\left(\nabla_{e_{l}}\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla_{e_{k}} u^{\epsilon_{j}}\right)\right)_{1 \leq k, l \leq n}
$$

Recalling from Lemma 3.1 that $\nabla u^{\epsilon} \rightarrow \nabla u$ in $C^{\alpha}(U)$ and $V \Subset U$, for any $\phi \in C_{c}^{\infty}(U)$ with $\left.\phi\right|_{V}=1$ and $1 \leq k, l \leq n$, we have

$$
\begin{aligned}
& \lim _{j \rightarrow 0} \int_{U} \nabla_{e_{l}}\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla_{e_{k}} u^{\epsilon_{j}}\right) \phi e^{-h} d \operatorname{vol}_{g} \\
&=-\lim _{j \rightarrow 0} \int_{U}\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla_{e_{k}} u^{\epsilon_{j}}\right) \nabla_{e_{l}}\left(\phi e^{-h}\right) d \operatorname{vol}_{g} \\
&=-\int_{U}\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla_{e_{k}} u\right) \nabla_{e_{l}}\left(\phi e^{-h}\right) d \operatorname{vol}_{g} \\
&=\int_{U} \nabla_{e_{l}}\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla_{e_{k}} u\right) \phi e^{-h} d \operatorname{vol}_{g}
\end{aligned}
$$

This shows that in the distributional sense

$$
\nabla\left(\left[\left|\nabla u^{\epsilon_{j}}\right|^{2}+\epsilon_{j}\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_{j}}\right) \rightarrow \nabla\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right) .
$$

Thus $z=\left.\nabla\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right)\right|_{V} \in L^{2}\left(V, \mathbb{R}^{n \times n}\right)$ in distributional sense. We therefore have $\left.|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right|_{V} \in W^{1,2}(V)$, which gives (3.29).

Moreover, by the arbitrariness of subsequence $\left\{\epsilon_{j}\right\}$, we have

$$
\nabla\left(\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon}\right) \rightarrow \nabla\left(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right)
$$

weakly in $L^{2}\left(V, \mathbb{R}^{n \times n}\right)$ as $\epsilon \rightarrow 0$. Hence by the arbitrariness of $V \Subset U,(3.30)$ holds.
Letting $\epsilon \rightarrow 0$ in (3.28) and using the convergence in the above verified claim, we obtain

$$
\begin{align*}
& \int_{U}\left|\nabla\left[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right]\right|^{2} \phi^{2} e^{-h} d \operatorname{vol}_{g} \\
& \leq \\
& \quad C(n, N, p, \gamma) \kappa \int_{U}|\nabla u|^{p-\gamma+2} \phi^{2} e^{-h} d \operatorname{vol}_{g}  \tag{3.31}\\
& \quad+C(n, N, p, \gamma) \int_{U}\left(|\nabla u|^{p-\gamma+2}|\nabla \phi|^{2}+|\nabla u|^{p-\gamma+4} \phi^{2}\right) e^{-h} d \operatorname{vol}_{g}
\end{align*}
$$

Let $\phi \in C_{c}^{\infty}\left(B_{2 r}\right)$, where $B_{4 r} \subset U$, such that $\phi=1$ in $B_{r}$ and $|\nabla \phi| \leq \frac{C}{r}$. Then (3.31) becomes

$$
\begin{aligned}
& \int_{B_{r}}\left|\nabla\left[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right]\right|^{2} e^{-h} d \mathrm{vol}_{g} \\
& \quad \leq C(n, N, p, \gamma) \int_{B_{2 r}}\left[\left(\frac{1}{r^{2}}+\kappa\right)|\nabla u|^{p-\gamma+2}+|\nabla u|^{p-\gamma+4}\right] e^{-h} d \operatorname{vol}_{g} .
\end{aligned}
$$

Recalling from (1.5) the Cheng-Yau type gradient estimate that $|\nabla u| \leq C(n, N, p) \frac{1+\sqrt{\kappa} r}{r}$ and noting that $\gamma<3+\frac{p-1}{N-1}$ guarantees $p-\gamma+2>0$, we deduce

$$
|\nabla u|^{p-\gamma+2} \leq C(n, N, p, \gamma)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{p-\gamma+2}
$$

Together with $\frac{1}{r^{2}}+\kappa \leq\left(\frac{1+\sqrt{\kappa} r}{r}\right)^{2}$, we conclude

$$
\int_{B_{r}}\left|\nabla\left[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u\right]\right|^{2} e^{-h} d \operatorname{vol}_{g} \leq C(n, N, p, \gamma) \operatorname{vol}_{h}\left(B_{2 r}\right)\left[\frac{1+\sqrt{\kappa} r}{r}\right]^{p-\gamma+4}
$$

Dividing both sides by $\operatorname{vol}_{h}\left(B_{r}\right)$, noting $\operatorname{vol}_{h}\left(B_{2 r}\right) \leq e^{\sqrt{\kappa} r} \operatorname{vol}_{h}\left(B_{r}\right)$ from the volume comparison (2.1), and recalling $u=-(p-1) \ln w$, we conclude (1.8).

Note that (1.9) is just the special case $\gamma=p$ of (1.8), where $p<3+\frac{2}{N-2}$ guarantees $p<3+\frac{p-1}{N-1}$ and hence one can take $\gamma=p$ in (1.8).

Finally, we compare our proof with [21,5], in particular, the crucial pointwise lower bound given in Lemma 3.4 and Lemma 3.5.

Remark 3.8. (i) It was well known that a positive (weighted) $p$-harmonic function $w$, and hence $\ln w$, is always smooth outside of the null set $E_{w}$ of $\nabla \ln w$. In $\Omega \backslash E_{w}$, the proof of Lemma 3.5 works for $\ln w$ so to get (3.16) with $u^{\epsilon}$ replaced by $\ln w$ and $\epsilon=0$, dividing both sides of which by $|\nabla \ln w|^{4}$, for $0<\eta<1 / 2$ one gets

$$
\begin{align*}
(1+\eta)\left|\nabla^{2} \ln w\right|^{2} \geq 2 & \frac{\left|\nabla^{2} \ln w \nabla \ln w\right|^{2}}{|\nabla \ln w|^{2}}+\left(\frac{(p-1)^{2}}{N-1}-1\right) \frac{\left(\Delta_{\infty} \ln w\right)^{2}}{|\nabla \ln w|^{4}} \\
& -(1+\eta) \frac{\langle\nabla \ln w, \nabla h\rangle^{2}}{N-n}-C(n, N, p) \frac{1}{\eta}|\nabla \ln w|^{4} . \tag{3.32}
\end{align*}
$$

If $\gamma<3+\frac{p-1}{N-1}$, using (3.32) and noting that the proof of Lemma 3.4 works for $\ln w$, we get (3.15) with $u^{\epsilon}$ replaced by $\ln w$ and $\epsilon=0$, that is, for $\eta>0$ sufficiently small,

$$
\begin{align*}
& (1-\eta)\left|\nabla^{2} \ln w\right|^{2}+(p-\gamma) \frac{\left|\nabla^{2} \ln w \nabla \ln w\right|^{2}}{|\nabla \ln w|^{2}}+(p-2)(2-\gamma) \frac{\left(\Delta_{\infty} \ln w\right)^{2}}{|\nabla \ln w|^{4}} \\
& \quad \geq \eta\left|\nabla^{2} \ln w\right|^{2}-\frac{\langle\nabla \ln w, \nabla h\rangle^{2}}{N-n}-C(n, N, p, \gamma) \frac{1}{\eta}|\nabla \ln w|^{4} \tag{3.33}
\end{align*}
$$

From the proof, we see that both of the coefficient 2 of $\frac{\left|\nabla^{2} \ln w \nabla \ln w\right|^{2}}{|\nabla \ln w|^{2}}$ and the coefficient $\frac{(p-1)^{2}}{N-1}-1$ of $\left(\Delta_{\infty} \ln w\right)^{2}$ in (3.32) are critical to guarantee the existence of sufficiently small $\eta>0$ in (3.33) when $\gamma<3+\frac{p-1}{N-1}$.

On the other hand, instead of (3.32), recall the following lower bound obtained in [5] by using Lemma 2.1 and the equation (3.1):

$$
\begin{equation*}
\left|\nabla^{2} \ln w\right|^{2} \geq \frac{\left|\nabla^{2} \ln w \nabla \ln w\right|^{2}}{|\nabla \ln w|^{2}}-2 \frac{p-1}{n-1} \Delta_{\infty} \ln w+\frac{1}{N-1}|\nabla \ln w|^{2}-\frac{\langle\nabla \ln w, \nabla h\rangle^{2}}{N-n} \tag{3.34}
\end{equation*}
$$

and also, when $N=n$ and $h \equiv 1$, recall the following lower bound derived in [21] via Lemma 2.1 and (3.1):

$$
\begin{equation*}
\left|\nabla^{2} \ln w\right|^{2} \geq\left[1+\min \left\{\frac{(p-1)^{2}}{n-1}, 1\right\}\right] \frac{\left|\nabla^{2} \ln w \nabla \ln w\right|^{2}}{|\nabla \ln w|^{2}}-2 \frac{p-1}{n-1} \Delta_{\infty} \ln w+\frac{1}{n-1}|\nabla \ln w|^{2} \tag{3.35}
\end{equation*}
$$

From (3.34) and (3.35), via a direct check one can conclude $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{\text {loc }}^{1,2}$ for $\gamma<2$, but NOT for all $\gamma<3+\frac{p-1}{N-1}$.
(ii) Moreover, unlike $[21,5]$ where the authors differentiate the equation (3.1) for $\ln w$, we directly derive an upper bound from Bochner formula for the left-hand side of (3.33) with respect to $\left[\left|\nabla u^{\epsilon}\right|^{2}+\epsilon\right]^{\frac{p-\gamma}{2}} e^{-h} d \operatorname{vol}_{g}$.

## Data availability

No data was used for the research described in the article.

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## Appendix A. Proof of Lemma 3.1

In the appendix, we show Lemma 3.1 by checking equations (3.1) and (3.3) are special cases considered in [3]. To this end, we recall the result in [3].

Let $\Omega$ be a domain of $M^{n}$. Consider the equation

$$
\begin{equation*}
-\operatorname{div} \vec{a}(x, \nabla u)+b(x, \nabla u)=0 \quad \text { in } \Omega \tag{A.1}
\end{equation*}
$$

where $\vec{a}$ is a map from $\Omega \times \mathbb{R}^{n}$ to $\mathbb{R}^{n}$ and $b$ maps $\Omega \times \mathbb{R}^{n}$ to $\mathbb{R}$. Let $\left\{e_{1}, \cdots, e_{n}\right\} \subset T_{x} \Omega$ be a local orthonormal frame at each $x \in \Omega$. By a weak solution of (A.1) we mean a function $u \in W_{\text {loc }}^{1, p}(\Omega)$ such that

$$
\begin{equation*}
\int_{\Omega}[\langle\vec{a}(x, \nabla u), \nabla \phi\rangle+b(x, \nabla u) \phi] d \operatorname{vol}_{g}=0 \quad \forall \phi \in C_{c}^{\infty}(\Omega) . \tag{A.2}
\end{equation*}
$$

Assume the following holds for $\vec{a}=\left(a_{1}, \cdots, a_{n}\right)$ and $b$.

$$
\begin{gather*}
\sum_{i, j=1}^{n} \frac{\partial a_{j}}{\partial \eta_{i}}(x, \eta) \xi_{i} \xi_{j} \geq \gamma_{0}|\eta|^{p-2}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{n}, p>1  \tag{1}\\
\left|\frac{\partial a_{j}}{\partial \eta_{i}}\right| \leq \gamma_{1}|\eta|^{p-2}, \quad 1 \leq i, j \leq n  \tag{2}\\
\left|\nabla_{e_{i}} a_{j}(x, \eta)\right| \leq \gamma_{1}|\eta|^{p-1}, \quad 1 \leq i, j \leq n  \tag{3}\\
|b(x, \eta)| \leq \gamma_{1}|\eta|^{p} \tag{4}
\end{gather*}
$$

and

$$
\begin{equation*}
\left|\nabla_{e_{i}} b(x, \eta)\right| \leq \gamma_{1}|\eta|^{p},\left|\frac{\partial b}{\partial \eta_{i}}(x, \eta)\right| \leq \gamma_{1}|\eta|^{p-1}, \quad 1 \leq i \leq n \tag{B}
\end{equation*}
$$

for all $\eta \in \mathbb{R}^{n}$, where $\gamma_{i}$ are positive constants, $i=0,1$.

For any smooth domain $U \Subset \Omega$ and $\epsilon \in(0,1]$, consider the regularized equation

$$
\begin{equation*}
-\operatorname{div} \overrightarrow{a^{\epsilon}}\left(x, \nabla u^{\epsilon}\right)+b^{\epsilon}\left(x, \nabla u^{\epsilon}\right)=0 \quad \text { in } U \text { and } u^{\epsilon}=u \text { on } \partial U \tag{A.3}
\end{equation*}
$$

where $\overrightarrow{a^{\epsilon}}$ is a map from $U \times \mathbb{R}^{n}$ to $\mathbb{R}^{n}$ and $b^{\epsilon}$ maps $U \times \mathbb{R}^{n}$ to $\mathbb{R}$ such that

$$
\lim _{\epsilon \rightarrow 0} \overrightarrow{a^{\epsilon}}(x, \eta)=\vec{a}(x, \eta) \text { and } \lim _{\epsilon \rightarrow 0} b^{\epsilon}(x, \eta)=b(x, \eta) \quad \forall(x, \eta) \in \Omega \times \mathbb{R}^{n}
$$

The weak solution of (A.3) is defined similarly as (A.2). Assume the following holds for $\overrightarrow{a^{\epsilon}}=\left(a_{1}^{\epsilon}, \cdots, a_{n}^{\epsilon}\right)$ and $b^{\epsilon}$.

$$
\begin{gather*}
\sum_{i, j=1}^{n} \frac{\partial a_{j}^{\epsilon}}{\partial \eta_{i}}(x, \eta) \xi_{i} \xi_{j} \geq \gamma_{0}\left(\epsilon+|\eta|^{2}\right)^{\frac{p-2}{2}}|\xi|^{2}, \quad \xi \in \mathbb{R}^{n}, p>1, \\
\left|\frac{\partial a_{j}^{\epsilon}}{\partial \eta_{i}}\right| \leq \gamma_{1}\left(\epsilon+|\eta|^{2}\right)^{\frac{p-2}{2}}, \quad 1 \leq i, j \leq n, \\
\left|\nabla_{e_{i}} a_{j}^{\epsilon}(x, \eta)\right| \leq \gamma_{1}\left(\epsilon+|\eta|^{2}\right)^{\frac{p-1}{2}}, \quad 1 \leq i, j \leq n, \\
\left|\mathrm{~b}^{\epsilon}(x, \eta)\right| \leq \gamma_{1}\left(\epsilon+|\eta|^{2}\right)^{\frac{p}{2}}
\end{gather*}
$$

for all $\eta \in \mathbb{R}^{n} \backslash\{0\}$.
We recall the results in [3] as follows.
Theorem A.1. Let $\epsilon \in(0,1]$ and $U \Subset \Omega$. Assume $\left(\mathrm{A}_{1}\right)-\left(\mathrm{A}_{4}\right)$, (B) and $\left(\mathrm{A}_{1, \epsilon}\right)-\left(\mathrm{A}_{4, \epsilon}\right)$ hold. Then there exists a unique solution $u^{\epsilon} \in C^{\infty}(U) \cap C^{0}(\bar{U})$ to (A.3), and moreover, $u^{\epsilon} \rightarrow u$ in $C^{0}(\bar{U})$ and $u^{\epsilon} \rightarrow u$ in $C^{1, \alpha}(V)$ uniformly in $\epsilon>0$ as $\epsilon \rightarrow 0$ for all $V \Subset U$ where $u$ is the solution to (A.1). As a consequence, $u \in C^{1, \alpha}(\Omega)$.

Theorem A. 1 is a combination of Theorem 1 and Theorem 2 in [3] and several intermediate results in the proof of these two theorems in [3]. Indeed, the existence, uniqueness and $C^{\infty}$-regularity of $u^{\epsilon}$ is by elliptic theory in PDE; see for example [8]. Based on these facts, in [3], the author first showed that under $\left(\mathrm{A}_{1}\right)-\left(\mathrm{A}_{4}\right),(\mathrm{B})$ and $\left(\mathrm{A}_{1, \epsilon}\right)-\left(\mathrm{A}_{4, \epsilon}\right), u^{\epsilon} \rightarrow u$ in $W^{1, p}(U)$ uniformly in $\epsilon>0$ in section 2 . Moreover, $\left\|u^{\epsilon}\right\|_{L^{\infty}(U)} \leq \max _{x \in \partial U}\{|u(x)|\}$. Thus recalling that $\left.u^{\epsilon}\right|_{\partial U}=\left.u\right|_{\partial U}$, we know $u^{\epsilon} \rightarrow u$ in $C^{0}(\bar{U})$. See the discussion around (2.7) in [3]. Then the author showed that $\left\|u^{\epsilon}\right\|_{C^{1, \alpha}(V)}$ is uniformly bounded independently of $\epsilon \in(0,1]$ and finally showed that $u^{\epsilon} \rightarrow u$ in $C^{1, \alpha}(V)$ and $u \in C^{1, \alpha}(U)$ for all $V \Subset U$. By the arbitrariness of $U \Subset \Omega$, one has $u \in C^{1, \alpha}(\Omega)$.

Proof of Lemma 3.1. It suffices to check equations (3.1) and (3.2) are special ones of (A.1) and (A.3) respectively. To this end, let $\vec{a}(x, \eta)=e^{-h(x)}|\eta|^{p-2} \eta, b(x, \eta)=$ $-e^{-h(x)}|\eta|^{p}, \overrightarrow{a^{\epsilon}}(x, \eta)=e^{-h(x)}\left(|\eta|^{2}+\epsilon\right)^{\frac{p-2}{2}} \eta$, and $b^{\epsilon}(x, \eta)=-e^{-h(x)}\left(|\eta|^{2}+\epsilon\right)^{\frac{p-2}{2}}|\eta|^{2}$ for all $x \in U$ and $\eta \in \mathbb{R}^{n}$. Then in the weak sense, the equations

$$
\int_{\Omega}[\langle\vec{a}(x, \nabla u), \nabla \phi\rangle+b(x, \nabla u) \phi] d \operatorname{vol}_{g}=0, \quad \forall \phi \in C_{c}^{\infty}(\Omega)
$$

and

$$
\int_{\Omega}\left[\left\langle\vec{a}^{\epsilon}(x, \nabla u), \nabla \phi\right\rangle+b^{\epsilon}(x, \nabla u) \phi\right] d \operatorname{vol}_{g}=0, \quad \forall \phi \in C_{c}^{\infty}(\Omega)
$$

are exactly (3.1) and (3.2) respectively.
We show $\vec{a}$ satisfies $\left(\mathrm{A}_{1}\right)$. Noting that $a_{j}(x, \eta)=e^{-h(x)}|\eta|^{p-2} \eta_{j}$, we compute

$$
\frac{\partial a_{j}}{\partial \eta_{i}}(x, \eta)=e^{-h(x)}\left[(p-2)|\eta|^{p-4} \eta_{i} \eta_{j}+\delta_{i j}|\eta|^{p-2}\right], \quad \forall 1 \leq i, j \leq n
$$

where $\delta_{i j}=1$ if $i=j$ and $\delta_{i j}=0$ if $i \neq j$. Thus

$$
\begin{aligned}
\sum_{i, j=1}^{n} \frac{\partial a_{j}}{\partial \eta_{i}}(x, \eta) \xi_{i} \xi_{j} & =e^{-h(x)} \sum_{i, j=1}^{n}\left[\left((p-2)|\eta|^{p-4} \eta_{i} \eta_{j}+\delta_{i j}|\eta|^{p-2}\right) \xi_{i} \xi_{j}\right] \\
& =e^{-h(x)}|\eta|^{p-4}\left[(p-2)\left(\sum_{i=1}^{n} \eta_{i} \xi_{i}\right)^{2}+|\eta|^{2}|\xi|^{2}\right], \quad \forall \xi \in \mathbb{R}^{n}
\end{aligned}
$$

If $1<p<2$, we have

$$
\sum_{i, j=1}^{n} \frac{\partial a_{j}}{\partial \eta_{i}}(x, \eta) \xi_{i} \xi_{j} \geq e^{-h(x)}(p-1)|\eta|^{p-2}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{n}
$$

And if $p \geq 2$, we have

$$
\sum_{i, j=1}^{n} \frac{\partial a_{j}}{\partial \eta_{i}}(x, \eta) \xi_{i} \xi_{j} \geq e^{-h(x)}|\eta|^{p-2}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{n}
$$

By taking $\gamma_{0}:=\min _{x \in \bar{U}}\left\{e^{-h(x)}\right\}$, we conclude that $a$ satisfies $\left(\mathrm{A}_{1}\right)$. By direct computations, one can also check $\vec{a}, \overrightarrow{a^{\epsilon}} \in C^{\infty}\left(U \times \mathbb{R}^{n}, \mathbb{R}^{n}\right), b, b^{\epsilon} \in C^{\infty}\left(U \times \mathbb{R}^{n}\right)$ satisfy ( $\left.\mathrm{A}_{2}\right)-\left(\mathrm{A}_{4}\right)$, (B) and $\left(\mathrm{A}_{1, \epsilon}\right)-\left(\mathrm{A}_{4, \epsilon}\right)$ respectively. We omit the details. Thus by Theorem A.1, we get the desired result.

## References

[1] L. Ambrosio, N. Gigli, G. Savaré, Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds, Ann. Probab. 43 (1) (2015) 339-404.
[2] S.Y. Cheng, S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Commun. Pure Appl. Math. 28 (3) (1975) 333-354.
[3] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983) 827-850.
[4] H. Dong, F. Peng, Y. Zhang, Y. Zhou, Hessian estimates for elliptic and parabolic equations involving $p$-Laplacian via a fundamental inequality, Adv. Math. 370 (2020) 107212.
[5] N.T. Dung, N.D. Dat, Weighted $p$ - harmonic functions and rigidity of smooth metric measure spaces, J. Math. Anal. Appl. 443 (2) (2016) 959-980.
[6] M. Erbar, K. Kuwada, K.T. Sturm, On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces, Invent. Math. 201 (2015) 993-1071.
[7] N. Gigli, I. Violo, Monotonicity formulas for harmonic functions in $R C D(0, N)$ spaces, J. Geom. Anal. 33 (2023) 100.
[8] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1977.
[9] T. Iwaniec, J.J. Manfredi, Regularity of $p$-harmonic functions on the plane, Rev. Mat. Iberoam. 5 (1-2) (1989) 1-19.
[10] R. Jiang, Cheeger-harmonic functions in metric measure spaces revisited, J. Funct. Anal. 266 (2014) 1373-1394.
[11] B. Kotschwar, L. Ni, Local gradient estimates of p-harmonic functions, $1 / H$ flow, and an entropy formula, Ann. Sci. Éc. Norm. Supér. (4) 42 (1) (2009) 1-36.
[12] J. Lewis, Regularity of the derivatives of solutions to certain elliptic equations, Indiana Univ. Math. J. 32 (1983) 849-858.
[13] J.J. Manfredi, A. Weitsman, On the Fatou theorem for $p$-harmonic functions, Commun. Partial Differ. Equ. 13 (1988) 651-668.
[14] S. Sarsa, Note on an elementary inequality and its application to the regularity of $p$-harmonic functions, Ann. Fenn. Math. 47 (1) (2021) 139-153.
[15] L. Mari, M. Rigoli, A. Setti, On the $1 / H$-flow by $p$-Laplace approximation: new estimates via fake distances under Ricci lower bounds, Am. J. Math. 144 (3) (2022).
[16] R. Moser, The inverse mean curvature flow and p-harmonic functions, J. Eur. Math. Soc. 9 (2007) 77-83.
[17] L. Ni, Y. Shi, L. Tam, Poisson equation, Poincaré-lelong equation and curvature decay on complete Kähler manifolds, J. Differ. Geom. 57 (2001) 339-388.
[18] Z. Qian, Estimates for weighted volumes and applications, Q. J. Math. Oxford Ser. (2) 48 (1997) 235-242.
[19] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differ. Equ. 51 (1984) 126-150.
[20] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, Acta Math. 138 (1977) 219-240.
[21] X. Wang, L. Zhang, Local gradient estimate for $p$-harmonic functions on Riemannian manifolds, Commun. Anal. Geom. 19 (4) (2011) 759-771.
[22] H. Zhang, X. Zhu, Yau's gradient estimates on Alexandrov spaces, J. Differ. Geom. 91 (2012) 445-522.


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    * Corresponding author.

    E-mail addresses: jiayin.mat.liu@jyu.fi (J. Liu), shijinzhang@buaa.edu.cn (S. Zhang), yuan.zhou@bnu.edu.cn (Y. Zhou).

