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DIMENSION ESTIMATE FOR THE TWO-SIDED POINTS OF PLANAR SOBOLEV EXTENSION DOMAINS

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Abstract. In this paper we give an estimate for the Hausdorff dimension of the set of two-sided points of the boundary of bounded simply connected Sobolev $W^{1,p}$-extension domain for $1 < p < 2$. Sharpness of the estimate is shown by examples. We also prove the equivalence of different definitions of two-sided points.

1. Introduction

This paper is part of the study of the geometry of the boundary of Sobolev extension domains in Euclidean spaces. We investigate the size of the set of two-sided points of simply connected planar Sobolev extension domains. Recall that a domain Ω is a $W^{1,p}$-extension domain if there exists a bounded operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ with the property that $Eu|_\Omega = u$ for each $u \in W^{1,p}(\Omega)$. Here, for $p \in [1,\infty]$, we denote by $W^{1,p}(\Omega)$ the set of all functions in $L^p(\Omega)$ whose first distributional derivatives are in $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is normed by

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

When $p > 1$ operator E can be assumed linear [7]. For $p = 1$ the linearity is known for the planar bounded simply connected case [11].

Several classes of domains are known to be $W^{1,p}$-extension domains. For instance, Lipschitz domains [1], [18]. Jones [8] introduced a wider class of (ϵ,δ)-domains and proved that every (ϵ,δ)-domain is a $W^{1,p}$-extension domain. Notice that the Hausdorff dimension of the boundary of a Lipschitz domain is $n-1$ and the boundary is rectifiable. For an (ϵ,δ)-domain the Hausdorff dimension of the boundary may be strictly greater than $n-1$ and it may not be locally rectifiable (for example the Koch snowflake). However, an easy argument shows that the boundary of an (ϵ,δ)-domain can not self-intersect.

The case we study in this paper is with $\Omega \subset \mathbb{R}^2$ bounded and simply connected. In this case, the $W^{1,p}$-extendability has been characterized. As we will see, from the characterizations it follows that the only relevant case for us is with $p < 2$. Firstly, for $p = 2$, from the results in [4], [5], [6], [8], we know that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,2}$-extension domain if and only if Ω is a quasidisk, or equivalently a uniform domain.

For $2 < p < \infty$ and a finitely connected bounded planar domain Ω, Shvartsman [17] proved that Ω is a Sobolev $W^{1,p}$-extension domain if and only if for some $C > 1$ the following condition is satisfied: for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\int_\gamma \operatorname{dist}(z,\partial\Omega)^{\frac{1}{1-p}}\,\mathrm{d}s(z) \le C\|x-y\|^{\frac{p-2}{p-1}}.$$

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In particular, when $2 \leq p < \infty$, a finitely connected bounded $W^{1,p}$-extension domain Ω is quasiconvex, meaning that there exists a constant $C \geq 1$ such that any pair of points in $z_1, z_2 \in \Omega$ can be connected with a rectifiable curve $\gamma \subset \Omega$ whose length satisfies $\ell(\gamma) \leq C|z_1 - z_2|$. Let us point out that (uniformly locally) quasiconvex domains are exactly the $W^{1,\infty}$-extension domains [19], [7].

In paper [10] the case $1 < p < 2$ was characterized: a bounded simply connected $\Omega \subset \mathbb{R}^2$ is a Sobolev $W^{1,p}$-extension domain if and only if there exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting z_1 and z_2 and satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq C\|z_1 - z_2\|^{2-p}. \tag{1.1}$$

The above geometric characterizations give bounds for the size of the boundary of Sobolev extension domains. The following estimate for the Hausdorff dimension of the boundary for simply connected $W^{1,p}$-extension domain Ω in the case $p \in (1,2)$ was given in [12] :

$$\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - \frac{M}{C},$$

where C is the constant in (1.1) and $M > 0$ is an universal constant. Recall, that for $s > 0$, the *s-dimensional Hausdorff measure* of a subset $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A) = \inf\left\{\sum_i \operatorname{diam}(E_i)^s : A \subset \bigcup_i E_i, \operatorname{diam}(E_i) \leq \delta\right\}$, and $\mathcal{H}^0$ is the counting measure. The *Hausdorff dimension* of a set $\emptyset \neq A \subset \mathbb{R}^n$ is then given by

$$\dim_{\mathcal{H}}(A) = \inf\{t \geq 0 : \mathcal{H}^t(A) < \infty\}.$$

For notational convenience, we set $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

In this paper, we are interested in the case $1 < p < 2$, when the boundary of Ω may self-intersect, (for examples see [9, Example 2.5], [2], and Section 4). More accurately, we study the size of the set of two-sided points. Our motivation is to obtain more concrete measurements differentiating general simply-connected Sobolev extension domains from (ϵ, δ)-domains.

In the literature domains with self-intersecting boundary have been studied in relation to mixed boundary value problems (see [x], [y]). Note that as an immediate consequence of the curve condition (1.1) we see that at most one of the boundary parts intersecting any given two-sided point can have a well-defined normal vector, allowing the Neumann boundary condition.

Before giving the definition of two-sided points let us briefly mention the cases where p is not in the interval $(1,2)$. In the case of $2 \leq p \leq \infty$, there are no two-sided points which can be seen from the quasiconvexity. The case $p = 1$ has been characterized in [11] as a variant of quasiconvexity of the complement. In this case the dimension of the set of two-sided points does not depend on the constant in quasiconvexity.

Let us now define what we mean by a two-sided point. Here we give a definition which generalizes to $\mathbb{R}^n$, but the proof of our main theorem will use an equivalent formulation based on conformal maps, see Section 2.

Definition 1.1 (Two-sided points of the boundary of a domain)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. A point $x \in \partial\Omega$ is called* two-sided*, if there exists $R > 0$ such that for all $r \in (0, R)$ there exist connected components Ω^1_r and Ω^2_r of $\Omega \cap B(x, r)$ that are nested: $\Omega^i_r \subset \Omega^i_s$ for $0 < r < s < R$ and $i \in \{1, 2\}$.*

We denote by $\mathcal{T}$ the two-sided points of Ω. Note that the nestedness condition in Definition 1.1 for the connected components Ω_r^i implies that $x \in \partial\Omega_r^i$. We establish the following dimension estimate for $\mathcal{T}$ for simply connected planar $W^{1,p}$-extension domains.

Theorem 1.2. *Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ a simply connected, bounded Sobolev $W^{1,p}$-extension domain. Let $\mathcal{T}$ be the set of two-sided points of Ω. Then*

$$\dim_{\mathcal{H}}(\mathcal{T}) \leq 2 - p + \log_2\left(1 - \frac{2^{p-1} - 1}{2^{5-2p}C}\right) \leq 2 - p - \frac{M_1(p)}{C}, \tag{1.2}$$

where $M_1(p) = \frac{2^{p-1}-1}{2^{5-2p}\log 2}$ and $C \geq 1$ is the constant in (1.1)*.*

Remark 1.3. If p and C are such that the right-hand side in (1.2) is strictly less than 0, then $\mathcal{T} = \emptyset$.

We divide the proof Theorem 1.2 in two parts. In Proposition 2.4 we show that $\mathcal{T}$ is covered by countably many curves satisfying (1.1) and in Lemma 3.1 we show that on each such curve we have the dimension bound (1.2).

In Section 4 we show the sharpness of Theorem 1.2 by proving the following existence of examples.

Theorem 1.4. *Let $1 < p < 2$. There exist constants $M_2 > 0$ and $C(p) \geq 1$ such that for each $C > C(p)$ there exists Sobolev $W^{1,p}$-extension domain $\Omega_{p,C}$ satisfying* (1.1) *with C, and*

$$\dim_{\mathcal{H}}(\mathcal{T}_{\Omega_{p,C}}) \geq 2 - p - \frac{M_2}{C}. \tag{1.3}$$

2. Equivalent definitions for two-sided points

In this section we give equivalent conditions for the set of two-sided points in the case that the domain is John. Although the equivalence stated in Theorem 2.1 is of independent interest, the main motivation for us is Proposition 2.4, where using one of the equivalent definitions for two-sided points we show the existence of a countable collection of curves covering $\mathcal{T}$ such that each of the curves fulfills a slightly refined version of (1.1).

We note that a bounded simply connected planar domain satisfying the condition (1.1) is John (this follows from [5, Chapter 6 Theorem 3.5] with [14, Theorem 4.5]). Recall, that Ω is a J-John domain, if there exists a constant $J > 0$ and a point $x_0 \in \Omega$ so that for every $x \in \Omega$ there exists a unit speed curve $\gamma\colon [0, \ell(\gamma)] \to \Omega$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq Jt \qquad \text{for all } t \in [0, \ell(\gamma)]. \tag{2.1}$$

We denote the open unit disk of the plane by $\mathbb{D}$. For a bounded simply connected John domain $\Omega \subset \mathbb{R}^2$, a conformal map $f\colon \mathbb{D} \to \Omega$ can always be extended continuously to a map $f\colon \overline{\mathbb{D}} \to \overline{\Omega}$. This is because a John domain is finitely connected along its boundary [14] and a conformal map from the unit disk to Ω can be extended continuously onto the closure $\overline{\Omega}$ if and only if the domain is finitely connected along its boundary [15].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected John domain (especially, if Ω is a bounded simply connected $W^{1,p}$-extension domain for $1 < p < 2$). Let $f\colon \mathbb{D} \to \Omega$ be a conformal map extended continuously to a function $\overline{\mathbb{D}} \to \overline{\Omega}$ still denoted by f. Define*

$$E = \{x \in \partial\Omega : f^{-1}(\{x\}) \text{ disconnects } \partial\mathbb{D}\}$$

and

$$\tilde{E} = \{x \in \partial\Omega : \operatorname{card}(f^{-1}(\{x\})) > 1\}.$$

Then

$$\mathcal{T} = E = \tilde{E},$$

where $\mathcal{T}$ *is the set of two-sided points according to Definition 1.1.*

In the proof of Theorem 2.1 we need the following lemma.

Lemma 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected John domain, let* $x \in \partial\Omega$, *and* $r \in (0, \operatorname{diam}(\Omega))$. *Suppose that there exist two disjoint open sets* $U_1, U_2 \subset \Omega \cap B(x,r)$ *such that* $x \in \partial U_1 \cap \partial U_2$ *and both of the sets* U_1 *and* U_2 *are unions of connected components of* $\Omega \cap B(x,r)$. *Then there exist connected components* U_1' *and* U_2' *of* U_1 *and* U_2 *respectively, such that* $x \in \partial U_1' \cap \partial U_2'$.

Proof. Let us first show that there exists $N \in \mathbb{N}$ independent of x and r such that

$$\operatorname{card}\{\tilde{\Omega} : \tilde{\Omega} \text{ connected component of } \Omega \cap B(x,r) \text{ such that } \tilde{\Omega} \cap B(x, r/2) \neq \emptyset\} \leq N. \quad (2.2)$$

Take $M \in \mathbb{N}$ components $\tilde{\Omega}_i$ as in (2.2), and choose from each one a point $x_i \in \tilde{\Omega}_i \cap B(x, r/2)$. Let γ_i be a John curve connecting x_i to a fixed John center x_0 of Ω. For each i for which $x_0 \notin \tilde{\Omega}_i$, the curve γ_i must exit $B(x, 2r/3)$. For these i we consider points $y_i \in \gamma_i \cap S(x, 2r/3)$, which then exist for all but maybe one of the indexes i. By the John condition there exists balls $B_i = B(y_i, Jr/6) \subset \tilde{\Omega}_i$. As the balls B_i are disjointed and B_i covers an arc of $S(x, 2r/3)$ of length at least $Jr/3$, we have $(M-1)Jr/3 \leq \frac{4}{3}\pi r$, hence $M - 1 \leq (\frac{J}{4\pi})^{-1}$.

Next we show that (2.2) implies the claim of the lemma. Let us enumerate

$$\{A_j\}_{j=1}^k := \{\tilde{\Omega} \subset U_1 : \tilde{\Omega} \text{ connected component of } \Omega \cap B(x,r) \text{ such that } \tilde{\Omega} \cap B(x, r/2) \neq \emptyset\}.$$

By (2.2) we have $k \leq N$. Since U_1 consists of connected components of $\Omega \cap B(x,r)$, we have

$$U_1 \cap B(x, r/2) \subset \bigcup_{j=1}^k A_j.$$

Now, because $x \in \overline{\bigcup_{j=1}^k A_j} = \bigcup_{j=1}^k \overline{A_j}$ there exists j such that $x \in \overline{A_j}$. We call this A_j the set U_1'. Similarly we find U_2' for U_2. $\square$

Notice that Lemma 2.2 does not hold for general simply connected domain Ω, for example, consider the topologist's comb.

Proof of Theorem 2.1. We divide the proof into several claims. Showing that

$$\tilde{E} \subset E \subset \mathcal{T} \subset \tilde{E}.$$

Claim 1: $\tilde{E} \subset E$.

Let $z \in \partial\Omega$ such that $\operatorname{card}(f^{-1}(\{z\})) > 1$, and $A = \partial\mathbb{D} \setminus f^{-1}(\{z\})$. Let $x_1, x_2 \in f^{-1}(\{z\})$. By [16, Theorem 10.9], the set $f^{-1}(\{z\})$ has Hausdorff dimension zero. Therefore, we find points of A from both components of $\partial\mathbb{D} \setminus \{x_1, x_2\}$. Hence A is disconnected in $\partial\mathbb{D}$, and thus $z \in E$.

Claim 2: $\mathcal{T} \subset \tilde{E}$.

Let $z \in \mathcal{T}$. By assumption there exists $R > 0$ such that for each $0 < r < R$ there exists disjoint connected components $\Omega_r^1, \Omega_r^2 \subset \Omega \cap B(z,r)$ with the property that $\Omega_r^i \subset \Omega_s^i$ when $0 < r < s$. Towards a contradiction, assume that $f^{-1}(\{z\})$ is a singleton ($w = f^{-1}(z)$). By continuity of f (up to the boundary) there exists $\varepsilon > 0$ such that $f(B(w,\varepsilon)\cap\overline{\mathbb{D}}) \subset B(z,r)$. Being a continuous image of a connected set $f(B(w,\varepsilon)\cap\mathbb{D})$ is connected. We show that $f^{-1}(\Omega_r^j)\cap B(w,\varepsilon)\neq\emptyset$ for $j=1,2$ which gives a contradiction with Ω_r^j being the disjoint connected components of $B(z,r)\cap\Omega$. Let $(z_i^j)_{i=1}^\infty \subset \Omega_r^j$ be a sequence such that $z_i^j \to z$. By going to a subsequence, we may assume that $(f^{-1}(z_i^j))_{i=1}^\infty$ converges to a point $w^j \in \overline{f^{-1}(\Omega_r^j)}$. Since f is continuous, $f(w^j)=z$. But then $w^j = w$ by the uniqueness of the preimage of z. Hence, $f^{-1}(z_i^j)\to w$, meaning that for some i we have $f^{-1}(z_i^j)\in B(w,\varepsilon)$ showing $f^{-1}(\Omega_r^j)\cap B(w,\varepsilon)\neq\emptyset$. Therefore, $\Omega_r^j \cap f(B(w,\varepsilon)\cap\mathbb{D})\neq\emptyset$, connecting sets Ω_r^j. This completes the proof.

Claim 3: $E \subset \mathcal{T}$.

Let $z \in E$. We will show that $z \in \mathcal{T}$. We do this by first showing by induction that there exists $i_0 \in \mathbb{N}$ so that for all $i \geq i_0$ there exist connected components $\Omega^j_{2^{-i}}$ of $\Omega \cap B(z,2^{-i})$, $j \in \{1,2\}$, that are nested for fixed $j\in\{1,2\}$. At each step of the induction we will have to make sure that $z \in \partial\Omega^1_{2^{-i}} \cap \partial\Omega^2_{2^{-i}}$.

Initial step: Let us show that there exists $r>0$ such that $B(z,r)\cap\Omega$ may be written as union of two disjointed open sets such that z is contained in the boundary of both sets. First, since $f^{-1}(\{z\}) = \cap_{r>0} f^{-1}(B(z,r)\cap\partial\Omega)$, there exists $R>0$ such that $H = f^{-1}(B(z,R)\cap\partial\Omega)$ disconnects $\partial\mathbb{D}$. By the continuity of f, $K = f^{-1}(\bar{B}(z,R/2))$ is a closed set in the closed disk $\overline{\mathbb{D}}$. Let $y_1, y_2 \in \partial\mathbb{D}\setminus H$ such that y_1 and y_2 are in different connected components of $\partial\mathbb{D}\setminus H$. Define $e = \min(\operatorname{dist}(y_1,K), \operatorname{dist}(y_2,K))/2$. Now $K\setminus B(0,1-e)$ is disconnected in $\overline{\mathbb{D}}$. Next we notice that $\operatorname{dist}(f(\bar{B}(0,1-e)),\partial\Omega) = R' > 0$. Thus the original claim holds with the radius $r = \min(R,R')/2$. Let us now define $i_0\in\mathbb{N}$ to be the smallest integer for which $2^{-i_0}\leq r$. Call U_1 and U_2 the two disjoint open sets for which $z \in \partial U_1 \cap \partial U_2$ and $\Omega\cap B(x,2^{-i_0}) = U_1 \cup U_2$. By Lemma 2.2 we have connected components $\Omega^1_{2^{-i_0}} \subset U_1$ and $\Omega^2_{2^{-i_0}} \subset U_2$ of $\Omega\cap B(z,2^{-i_0})$ such that $z\in\partial\Omega^1_{2^{-i_0}}\cap\partial\Omega^2_{2^{-i_0}}$.

Induction step: Assume that for some $i\in\mathbb{N}$ there exist disjoint connected components $\Omega^1_{2^{-i}}$ and $\Omega^2_{2^{-i}}$ of $\Omega\cap B(z,2^{-i})$ such that $z\in\partial\Omega^1_{2^{-i}}\cap\partial\Omega^2_{2^{-i}}$. Let $U_1 = \Omega^1_{2^{-i}}\cap B(z,2^{-i-1})$.

Let us show that U_1 is some union of connected components of $\Omega\cap B(z,2^{-i-1})$. Let V be a connected component of U_1. It suffices to show that V is a connected component of $\Omega\cap B(z,2^{-i-1})$. Take a connected component $V'\supset V$ of $\Omega\cap B(z,2^{-i-1})$. There exists connected component W' of $\Omega\cap B(z,2^{-i})$ such that $W'\supset V'$. Since $\emptyset\neq V\subset W'\cap\Omega^1_{2^{-i}}$ we have $W' = \Omega^1_{2^{-i}}$. Furthermore $V'\subset\Omega^1_{2^{-i}}\cap B(z,2^{-i-1}) = U_1$. As V' is connected we have $V'=V$.

Similarly for U_2. Now, by Lemma 2.2 we may choose connected components $U_1'\subset U_1$ and $U_2'\subset U_2$ (of $\Omega\cap B(z,2^{-i-1})$) such that $z\in\partial U_1'\cap\partial U_2'$.

General $r\in(0,2^{-i_0})$: Let $2^{-i-1}\leq r < 2^{-i}$. Let Ω^1_r be the connected component of $\Omega\cap B(z,r)$ containing $\Omega^1_{2^{-i-1}}$. Since $\Omega^1_{2^{-i}}$ is connected component of $\Omega\cap B(z,2^{-i})$ containing $\Omega^1_{2^{-i-1}}$, we have $\Omega^1_r\subset\Omega^1_{2^{-i}}$. Let us show that $\Omega^1_r\subset\Omega^1_s$ for all $0<r<s$. Let $0<r<s$. We consider two cases: (1) If $2^{-i-1}\leq r<s<2^{-i}$ the sets Ω^1_r and Ω^1_s are connected components of $\Omega\cap B(z,r)$ and $\Omega\cap B(z,s)$, respectively, both containing $\Omega^1_{2^{-i-1}}$. Since $\Omega^1_r\subset\Omega\cap B(z,s)$ and Ω^1_r is connected we have $\Omega^1_r\subset\Omega^1_s$.

(2) If $2^{-i-1}\leq r\leq 2^{-i}\leq 2^{-j-1}\leq s\leq 2^{-j}$ sets Ω^1_r and Ω^1_s are connected components of $\Omega\cap B(z,r)$ and $\Omega\cap B(z,s)$ which contain $\Omega^1_{2^{-i-1}}$ and $\Omega^1_{2^{-j-1}}$, respectively. Similarly as in (1) we have $\Omega^1_r\subset\Omega^1_{2^{-i}}\subset\cdots\subset\Omega^1_{2^{-j-1}}\subset\Omega^1_s$. $\square$

The proof of the following lemma follows closely the proof of [10, Lemma 2.3]. We present it here for the convenience of the reader and to point out that the condition (1.1) improves to subcurves without increasing the constant C in (1.1).

Lemma 2.3. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain for which the following holds: There exists $C > 0$ such that for each $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting z_1, z_2 for which (1.1) holds. Then the following stronger statement holds: For each pair of points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists an injective curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting z_1 and z_2 such that for each subcurve $\gamma|_{[t_1,t_2]}$*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq C\|\gamma(t_1) - \gamma(t_2)\|^{2-p}, \tag{2.3}$$

where C is the constant in the assumption.

Proof. Let $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, and let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be a curve between z_1 and z_2 for which (1.1) holds. We then have the trivial estimate

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) &\geq \int_{d(z_1,\partial\Omega)}^{d(z_1,\partial\Omega)+\ell(\gamma)} x^{1-p} \, \mathrm{d}x \\ &= \frac{1}{2-p}((\ell(\gamma) + d(z_1, \partial\Omega))^{2-p} - d(z_1, \partial\Omega)^{2-p}).\end{aligned}$$

This, in combination with (1.1), gives an upper bound for the length of γ:

$$\ell(\gamma) \leq \left((2-p)C\|z_1 - z_2\|^{2-p} + d(z_1, \partial\Omega)^{2-p}\right)^{\frac{1}{2-p}} - d(z_1, \partial\Omega). \tag{2.4}$$

Let $\gamma_j \subset \mathbb{R}^2 \setminus \Omega$ be a sequence of curves joining z_1 and z_2 such that

$$\int_{\gamma_j} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq c_j\|z_1 - z_2\|^{2-p},$$

where $c_j \leq C$ converge to the infimum $c \leq C$ of such constants c_j for the pair z_1 and z_2. By the continuity of the distance function, and since $\sup_i \ell(\gamma_i) < \infty$ by (2.4), there exists (see for example [10, Lemma 2.1]) a sequence $j_i \to \infty$ and a limit curve γ so that $\gamma_{j_i}(t) \to \gamma(t)$ for all t as $i \to \infty$ and

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq \liminf_{i\to\infty} \int_{\gamma_{j_i}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq c\|z_1 - z_2\|^{2-p}.$$

Thus there exists a curve minimizing the integral in (1.1). Now, fix $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ and let $\gamma\colon [0, T] \to \mathbb{R}^2 \setminus \Omega$ be a minimizer for the integral in (1.1) for z_1 and z_2. We claim that for any $0 \leq t_1 < t_2 \leq T$ the subcurve $\gamma|_{[t_1,t_2]}$ of γ is also a minimizer between $\gamma(t_1)$ and $\gamma(t_2)$. Otherwise, let γ' be a minimizer between $\gamma(t_1)$ and $\gamma(t_2)$. Then by the linearity of the integral we have that

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) &= \left(\int_{\gamma|_{[0,t_1]}} + \int_{\gamma|_{[t_1,t_2]}} + \int_{\gamma|_{[t_2,T]}}\right) \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \\ &> \left(\int_{\gamma|_{[0,t_1]}} + \int_{\gamma'} + \int_{\gamma|_{[t_2,T]}}\right) \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \\ &= \int_{\gamma''} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z),\end{aligned}$$

where

$$\gamma'' = \gamma|_{[0,t_1]} * \gamma' * \gamma|_{[t_2,T]}$$

joins z_1 and z_2. This contradicts the minimality assumption on γ. Thus our claim follows. Lastly, the injectivity of the curve is given by [3, Lemma 3.1]. □

Following the ideas of [11, Lemma 4.6] we use the equivalent definition E of two-sided points from Theorem 2.1 to show that the set of two-sided points can be covered by a countable union of injective curves fulfilling condition (1.1) for each subcurve.

Proposition 2.4. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be bounded simply connected Sobolev $W^{1,p}$-extension domain. Then there exists a countable collection Γ of injective curves $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C\|\gamma(t_1) - \gamma(t_2)\|^{2-p}, \tag{2.5}$$

for each subcurve $\gamma|_{[t_1,t_2]}$, where C is the same constant as in (1.1)*, so that for the set $\mathcal{T}$ of two-sided points we have*

$$\mathcal{T} \subset \bigcup_{\gamma\in\Gamma} \gamma \cap \partial\Omega.$$

Proof. To prove the inclusion we use the equivalent definition E of two-sided points given by Theorem 2.1. Let $f : \overline{\mathbb{D}} \to \overline{\Omega}$ be continuous, and conformal in $\mathbb{D}$. Let $\{x_j\}_{j\in\mathbb{N}} \subset \partial\mathbb{D}$ be dense. For each pair (x_i, x_j), $i \ne j$ we define $\gamma_{i,j}$ as an injective curve connecting $f(x_i)$ and $f(x_j)$, with property (2.5) for each subcurve. The existence of such curves is given by Lemma 2.3.

Define $\Gamma = \{\gamma_{i,j} : i \ne j\}$, and let $z \in E$. By the definition of E there exist $x_a, x_b \in f^{-1}(\{z\})$, $x_a \ne x_b$, which divide $\partial\mathbb{D}$ into two components I_a and I_b, so that $f(I_a) \ne \{z\} \ne f(I_b)$. By the continuity of f there exist i, j, $i \ne j$, such that $x_i \in I_a$ and $x_j \in I_b$ and $f(x_i) \ne z \ne f(x_j) \ne f(x_i)$. Let $\gamma_{i,j} \in \Gamma$ be the curve connecting $f(x_i) =: z_i$ and $f(x_j) =: z_j$. Let $\tilde\gamma := f([x_i, 0] \cup [0, x_j])$. The curve $[x_i, 0] \cup [0, x_j]$ divides $\mathbb{D}$ into two components A and B. By interchanging A and B if necessary, we have $x_a \in \bar{A}$ and $x_b \in \bar{B}$, and by continuity $z \in \overline{f(A)} \cap \overline{f(B)}$.

Since the curve $\gamma_{i,j}$ is injective, and $z_i \ne z_j$, the curve $\tilde\gamma \cup \gamma_{i,j}$ is Jordan. Let $\tilde{A}$ and $\tilde{B}$ be the corresponding Jordan components. Since $f(A) \subset \tilde{A}$, $f(B) \subset \tilde{B}$ we have $z \in \overline{\tilde{A}} \cap \overline{\tilde{B}} = \gamma_{i,j} \cup \tilde\gamma$. Furthermore, since $\tilde\gamma \subset f(\mathbb{D}) \cup \{z_i, z_j\} = \Omega \cup \{z_i, z_j\}$, we have $z \in \gamma_{i,j}$. □

3. Proof of Theorem 1.2

By Proposition 2.4 the proof of Theorem 1.2 is now reduced to proving the following lemma.

Lemma 3.1. *Let $1 < p < 2$ and $\gamma \subset \mathbb{R}^2 \setminus \Omega$ an injective curve satisfying*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C\|\gamma(t_1) - \gamma(t_2)\|^{2-p},$$

for each subcurve $\gamma|_{[t_1,t_2]}$. Then

$$\dim_{\mathcal{H}}(\gamma \cap \partial\Omega) \le 2 - p + \log_2\left(1 - \frac{2^{p-1} - 1}{2^{5-2p}C}\right).$$

In particular, if p and C are such that the right-hand side in (1.2) *is strictly less than 0, then* $\gamma \cap \partial\Omega = \emptyset$.[1]

To prove Lemma 3.1 we need the following sufficient condition for an upper bound of the Hausdorff dimension.

Lemma 3.2. *Let $F \subset \mathbb{R}^d$, $s \in \mathbb{R}$, $0 < \lambda < 1$ and $i_0 \in \mathbb{N}$. Define for each $i \geq i_0$ a maximal λ^i-separated net*

$$\{x_k^i\}_{k \in I_i} \subset F.$$

Assume that the following holds: For each $i \geq i_0$ and $k \in I_i$ there exists $j > i$, such that

$$N_j < \lambda^{-(j-i)s},$$

where $N_j = \operatorname{card}(\{l \in I_j : B(x_l^j, \lambda^j) \cap B(x_k^i, \lambda^i) \neq \emptyset\})$. Then $\dim_{\mathcal{H}}(F) \leq s$.

In particular, if $s < 0$, then $F = \emptyset$.

Proof. Define $\mathcal{B}_{i_0} = \{B(x_k^{i_0}, \lambda^{i_0}) : k \in I_{i_0}\}$ and inductively for $n > i_0$ by

$$\mathcal{B}_n = \bigcup_{B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}} \{B(x_m^j, \lambda^j) : B(x_m^j, \lambda^j) \cap B(x_k^i, \lambda^i) \neq \emptyset\},$$

where $j = j(i,k) > i$ is given by the assumption. Clearly, $\mathcal{B}_n$ is a cover of F for each $n \geq i_0$, and for all $B \in \mathcal{B}_n$

$$\operatorname{diam}(B) \leq 2\lambda^n.$$

By assumption, for each $B = B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}$ and with $j = j(i,k)$ again given by the assumption

$$\sum_{B(x_m^j, \lambda^j) \cap B \neq \emptyset} \operatorname{diam}(B(x_m^j, \lambda^j))^s = N_j (2\lambda^j)^s < (2\lambda^i)^s = \operatorname{diam}(B)^s,$$

and therefore

$$\sum_{B \in \mathcal{B}_n} \operatorname{diam}(B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \operatorname{diam}(B)^s.$$

Let $\delta > 0$ and choose $n \in \mathbb{N}$ such that $2\lambda^n < \delta$. Now

$$\begin{aligned} \mathcal{H}_\delta^s(F) &\leq \sum_{B \in \mathcal{B}_n} \operatorname{diam}(B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \operatorname{diam}(B)^s \leq \dots \\ &\leq \sum_{B \in \mathcal{B}_{i_0}} \operatorname{diam}(B)^s \leq \operatorname{card}(I_{i_0})(2\lambda^{i_0})^s < \infty. \end{aligned}$$

By letting $\delta \to 0$, we get $\mathcal{H}^s(F) \leq \operatorname{card}(I_{i_0})(2\lambda^{i_0})^s < \infty$, and consequently $\dim_{\mathcal{H}}(F) \leq s$. $\square$

Proof of Lemma 3.1. Define the set

$$\{x_k^i\}_{k \in I_i} \subset \gamma \cap \partial\Omega$$

to be a maximal 2^{-i} separated net for all $i \in \mathbb{N}$. Let $s < \min(\dim_{\mathcal{H}}(\gamma \cap \partial\Omega), 2 - p)$. We make the extra assumption $s < 2 - p$ here to have convergence in (3.2). This has no consequence on the dimension argument as we will show that $s < 2 - p + \delta$ for some $\delta = \delta(C, p) < 0$.

[1]In order to make the estimate on the dimension formally correct we adopted the notational convention $\dim_{\mathcal{H}}(\emptyset) := -\infty$.

By Lemma 3.2, there exists $i \in \mathbb{N}$ and $k \in I_i$ such that $N_j \geq 2^{(j-i)s}$ for all $j > i$, where

$$N_j = \operatorname{card}\left(\{l \in I_j : B(x_l^j, 2^{-j}) \cap B(x_k^i, 2^{-i}) \neq \emptyset\}\right).$$

Note that, trivially also $N_i \geq 1$. Denote $B = B(x_k^i, 2^{-i+1})$. For all $j \geq i+1$ the ball B contains at least N_{j-1} pairwise disjoint balls $B(x_l^{j-1}, 2^{-j})$ centered at $\gamma \cap \partial\Omega$, and so we have

$$\mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\}) \geq N_{j-1} 2^{-j}. \tag{3.1}$$

Using (2.5), Cavalieri's principle, (3.1), and Lemma 3.2 we estimate

$$\begin{aligned}
C 2^{-(i-2)(2-p)} &\geq \int_{\gamma \cap B} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \\
&= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega)^{1-p} > t\}) \, \mathrm{d}t \\
&= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{\frac{1}{1-p}}\}) \, \mathrm{d}t \\
&= \sum_{j \in \mathbb{Z}} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{\frac{1}{1-p}}\}) \, \mathrm{d}t \\
&\geq \sum_{j=i+1}^{\infty} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\}) \, \mathrm{d}t \\
&\geq \sum_{j=i+1}^{\infty} 2^{-j(1-p)}(1 - 2^{1-p}) N_{j-1} 2^{-j} \\
&\geq \sum_{j=i+1}^{\infty} (2^{p-1} - 1) 2^{-(j-1)(1-p)} 2^{(j-1-i)s} 2^{-j},
\end{aligned}$$

which implies

$$\begin{aligned}
C &\geq (2^{p-1} - 1) 2^{2p-5} \sum_{j=i+1}^{\infty} 2^{(j-i-1)(s+p-2)} \\
&= (2^{p-1} - 1) 2^{2p-5} \frac{1}{1 - 2^{-(2-(p+s))}}.
\end{aligned} \tag{3.2}$$

A reordering of (3.2) gives

$$s \leq 2 - p + \log_2 \left(1 - \frac{2^{2p-5}(2^{p-1} - 1)}{C}\right).$$

Since $s < \min(\dim_{\mathcal{H}}(\gamma \cap \partial\Omega), 2 - p)$ was arbitrary, we have

$$\dim_{\mathcal{H}}(\gamma \cap \partial\Omega) \leq 2 - p + \log_2 \left(1 - \frac{2^{2p-5}(2^{p-1} - 1)}{C}\right).$$

$\square$

4. Sharpness of the dimension estimate

In this section we show the sharpness of the estimate given in Theorem 1.2. We do this by constructing a domain whose set of two-sided points contains a Cantor type set.

Let $0 < \lambda < 1/2$. Let $\mathcal{C}_\lambda$ be the standard Cantor set obtained as the attractor of the iterated function system $\{f_1 = \lambda x, f_2 = \lambda x + 1 - \lambda\}$. For later use we fix some notation. Let $I_0^1 = [0,1]$, and $\tilde{I}_1^1 := (\lambda, 1-\lambda)$ be the first removed interval. We denote by I_j^i the 2^j closed intervals left after j iterations of the construction of the Cantor set, and similarly the 2^{j-1} removed open intervals by $\tilde{I}_j^i$. The lengths of the intervals are

$$|I_j^i| = \lambda^j, \quad i = 1, \dots, 2^j, j = 0, 1, 2, \dots$$

and

$$|\tilde{I}_j^i| = (1-2\lambda)\lambda^{j-1}, \quad i = 1, \dots, 2^{j-1}, j = 1, 2, 3, \dots.$$

Recall that, $\mathcal{C}_\lambda$ is of zero $\mathcal{H}^1$-measure, and $\dim_{\mathcal{H}}(\mathcal{C}_\lambda) = \frac{\log 2}{-\log \lambda}$ (see e.g. [13, p.60–62]).

Define

$$\Omega_\lambda = (-1,1)^2 \setminus \{(x,y) : x \geq 0, |y| \leq d(x, \mathcal{C}_\lambda)\}.$$

Set Ω_λ is clearly a domain and the set of two-sided points is $\mathcal{C}_\lambda \setminus \{(0,0)\}$.

Lemma 4.1. *The domain Ω_λ above satisfies the curve condition* (1.1) *for $1 < p < 2 + \frac{\log 2}{\log \lambda}$. That is, for each $x, y \in \Omega_\lambda^c$ there exists rectifiable curve $\gamma\colon [0, l(\gamma)] \to \Omega_\lambda^c$ connecting x, y such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq C(p,\lambda) \|x-y\|^{2-p}. \tag{4.1}$$

Moreover, we have the estimate

$$C(p,\lambda) \leq \frac{c}{(2-p)(1-2\lambda^{2-p})},$$

where c is an absolute constant.

Proof. To prove the claim we construct a curve connecting x and y in $\mathbb{R}^2 \setminus \Omega_\lambda$ consisting of line segments either parallel to the coordinate axes or at an angle $\pm\frac{\pi}{4}$. To simplify the discussion, within the proof, we will call a *component of* Ω_λ^c the closure of an open connected component of $\operatorname{int}(\Omega_\lambda^c)$. Let us record the following observation: If $I \subset \mathbb{R}^2 \setminus \Omega_\lambda$ is a line segment which can be arclength parametrized by t in such a manner that

$$\operatorname{dist}(z, \partial\Omega_\lambda) \geq \frac{t}{\sqrt{2}} \text{ for all } z = z(t) \in I, \tag{4.2}$$

then

$$\int_I \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq 2^{\frac{p-1}{2}} \int_0^{|I|} t^{1-p} \, \mathrm{d}t = \frac{2^{\frac{p-1}{2}}}{2-p} |I|^{2-p}. \tag{4.3}$$

Note that any line segment $I \subset \mathbb{R}^2 \setminus (-1,1)$ with angle $\pm\frac{\pi}{4}$ can be decomposed into at most two subsegments on which (4.2) holds, and similarly for any I parallel to coordinate axes contained in a bounded component of Ω_λ^c. For such segments we have

$$\int_I \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq \frac{2^{\frac{p+1}{2}}}{2-p} |I|^{2-p}. \tag{4.4}$$

Let us assume first that x and y are in the same component of Ω^c. If x and y are in the unbounded component $\mathbb{R}^2 \setminus (-1,1)^2$ of Ω^c, x and y may be connected with at most 4 diagonal segments, two of which may have to be decomposed into two to fulfill (4.2). In case of x, y being in the same bounded component of Ω^c x and y may be connected with two segments parallel to coordinate axes (both of which we may again have to decompose into two) for which (4.4) holds. Let us now consider the case where x and y are in different bounded components of Ω_λ^c. By the above we may assume that x and y are on the real line. Let then $j \in \mathbb{N}$ be such that

$$\lambda^j < \|x-y\| \leq \lambda^{j-1}.$$

Now, $[x,y]$ intersects at most two of the intervals I_j^i and one $\tilde{I}_j^i$, where I_j^i and $\tilde{I}_j^i$ are the closed and open intervals, respectively, related to the jth step of the construction of the Cantor set. Interval $\tilde{I}_j^i$ is of the type considered above, so we have the estimate (4.4). Let us estimate the integral over I_j^i. By self-similarity we may consider the interval $[0,1]$ instead. Since the Cantor set in our construction has measure zero, the integral over $[0,1]$ is exactly the integral over all the removed intervals $\tilde{I}_j^i$. There are exactly 2^{j-1} of these with $|\tilde{I}_j^i| = (1-2\lambda)\lambda^{j-1}$ for $j \geq 1$, so

$$\begin{aligned} \int_{[0,1]} \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) &\leq \frac{2^{\frac{p+1}{2}}}{2-p}(1-2\lambda)^{2-p} \sum_{j=1}^{\infty} 2^{j-1}\lambda^{(j-1)(2-p)} \\ &= \frac{2^{\frac{p+1}{2}}}{2-p}(1-2\lambda)^{2-p} \frac{1}{1-2\lambda^{2-p}}. \end{aligned} \tag{4.5}$$

To get to the integral over I_j^i we multiply (4.5) by $|I_j^i|^{2-p}$.

Combining the above we get the following: Any two $x, y \in \Omega_\lambda^c$ can be joined using at most 6 line segments for which (4.4) holds and at most 2 segments to which (4.5), rescaled to the interval, applies. Calling the resulting path γ and the segments I_k, we have

$$\begin{aligned} \int_\gamma \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{6} |I_k|^{2-p} + \frac{(1-2\lambda)^{2-p}}{1-2\lambda^{2-p}} \sum_{k=7}^{8} |I_k|^{2-p}\right) \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8} |I_k|\right)^{2-p} \left(\sum_{k=1}^{6} 1 + \sum_{k=7}^{8} \Big(\frac{(1-2\lambda)^{2-p}}{1-2\lambda^{2-p}}\Big)^{1/(p-1)}\right)^{p-1} \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8} |I_k|\right)^{2-p} \left(\sum_{k=1}^{6} \Big(\frac{1}{1-2\lambda^{2-p}}\Big)^{1/(p-1)} + \sum_{k=7}^{8} \Big(\frac{1}{1-2\lambda^{2-p}}\Big)^{1/(p-1)}\right)^{p-1} \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8} |I_k|\right)^{2-p} \frac{8^{p-1}}{1-2\lambda^{2-p}} \end{aligned}$$

by Hölder's inequality. By the definition of Ω_λ, we may choose I_k's so that $\sum_{k=1}^{8} |I_k| = |\gamma| \leq c\|x-y\|$ for an absolute constant c. □

Proof of Theorem 1.4. We show the existence of constants $M_2 > 0$ and $C(p) > 0$ so that (1.3) holds for $C \geq C(p)$. Fix $p \in (1,2)$, and let $M_2 = \frac{2c}{\log 2}$ where c is the absolute constant from Lemma 4.1. In order to make estimates, we use the construction for $\lambda \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$. In

Lemma 4.1 we established that domain Ω_λ satisfies the curve condition with the constant

$$\frac{c}{(2-p)(1-2\lambda^{2-p})}. \tag{4.6}$$

Setting $\lambda = \frac{1}{2}2^{\frac{1}{p-2}}$ in (4.6) we define

$$C(p) = \frac{c}{(2-p)(1-2^{p-2})}.$$

Now, for $C \geq C(p)$, by the continuity of the constant in (4.6) as a function of λ and the fact that it tends to infinity as $\lambda \nearrow 2^{\frac{1}{p-2}}$, there exists $\lambda_C \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$ such that

$$C = \frac{c}{(2-p)(1-2\lambda_C^{2-p})}.$$

We show that

$$\dim_{\mathcal{H}} \mathcal{C}_{\lambda_C} = -\frac{\log 2}{\log \lambda_C} \geq 2 - p - \frac{M_2}{C}. \tag{4.7}$$

In order to see that (4.7) holds, we show that

$$f_p(\lambda) = 2 - p - \frac{M_2}{c}(2-p)(1-2\lambda^{2-p}) + \frac{\log 2}{\log \lambda}$$

is non-positive on the interval $[\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$. This follows from

$$\begin{aligned} \min_{\lambda \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}}]} f_p'(\lambda) &\geq 2\frac{M_2}{c}(2-p)^2(2^{\frac{1}{p-2}})^{1-p} - \frac{\log 2}{2^{-1}2^{\frac{1}{p-2}}\log^2(2^{\frac{1}{p-2}})} \\ &= \frac{(2-p)^2}{2^{\frac{1}{p-2}}}\left(\frac{M_2}{c} - \frac{2}{\log 2}\right) \geq 0, \end{aligned}$$

and

$$f_p(\lambda) \leq f_p(2^{\frac{1}{p-2}}) = 0.$$

Hence, (4.7) holds.

□

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