Chaotic Decompositions of the Lévy-Itô space

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Master's thesis in Mathematics

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February 2024

Tiivistelmä

Eetu Luuri, *Chaotic Decompositions of the Lévy-Itô space*, matematiikan pro gradu-tutkielma, 42 s., Jyväskylän yliopisto, Matematiikan ja tilastotieteen laitos, helmikuu 2024.

Tämän tutkielman aiheena ovat erilaiset kaoottiset hajotelmat Lévy prosessien funktionaaleille. Näillä hajotelmilla pyritään esittämään kyseiset funktionaalit iteroitujen integraalien summana tietyn, keskenään ortogonaalisten martingaalien joukon suhteen.

Ensimmäisenä käymme läpi hieman teoriaa, jonka pohjalle myöhemmin tutkielmassa esiintyvät hajotelmat pohjautuvat. Esittelemme joukon määritelmiä, jotka ovat tarpeen tässä tutkielmassa esiintyvän teorian ymmärtämiseksi. Näihin määritelmiin lukeutuu muun muassa Lévy processit, martingaalit ja stokastiset integraalit. Lisäksi esittelemme myöhemmissä todistuksissa tarvittavia epäyhtälöitä, lemmoja ja lauseita.

Kun olemme käsitelleet tarvittavat esitiedot, siirrymme kohti tutkielman keskeisintä lausetta. Tätä lausetta varten esittelemme niin kutsutut Teugelin martingaalit. Nämä martingaalit ovat käytännössä Lévy prosessin kompensoituja hyppyprosesseja. Näistä Teugelin martingaaleista muodostamme keskenään ortogonaalisen joukon, jota käytämme kaoottisen hajotelman määrittelemiseen. Tämä teoria ja kaoottinen hajotelma pohjautuvat David Nualartin ja Win Schoutensin artikkeliin *Chaotic and predictable representations for Lévy processes.* Käytämme tätä tutkielmamme keskeisimpänä lähteenä, jossa esiintyviä lauseita ja todistuksia tutkimme yksityiskohtaisemmin. Lisäksi esittelemme ja käsittelemme muita kirjallisuudessa esiintyviä kaoottisia hajotelmia.

Yksi näistä hajotelmista on Kyoshi Itôn ortogonaalinen hajotelma, jonka hän esitteli artikkelissaan *Spectral Type of the Shift Transformation of Differential Processes With stationary increments.* Tämä lause hyödyntää Wiener integraaleja Lévy prosessin avulla määritellyn kahdesti integroituvien satunnaismuuttujien avaruuden ortogonaalisen hajotelman määrittelyssä. Tämän hajotelman todistuksen käymme läpi ykityiskohtaisesti, jonka jälkeen hyödynnämme sitä toisen hajotelman todistamiseen.

Lopuksi esittelemme vielä hieman yleisempään tapaukseen soveltuvan hajotelman. Paolo Di Tellan ja Haus-Juergen Engelbertin, artikkelissa *The Chaotic Representation of Compensated-Covariation Stable Families of Martingales*, esittelemä hajotelma sopeutuu funktionaalejen esittämiseen iteroitujen Wiener integraalien avulla suhteessa ortogonaaliseen ja kompensoidun kovarianssin suhteen vakaiden martingaalien joukkoon.

Abstract

In the present thesis, we will study the chaotic representation properties for functionals on Lévy processes. These chaotic representation properties are a way to represent square integrable random variables as a sum of iterated integrals with respect to a certain set of orthogonal martingales.

We will first go over the basic settings and some preliminary theory we need in order to understand Lévy processes, martingale theory, stochastic integrals and the chaotic representation properties following later in the thesis. These preliminaries include some inequalities, lemmas and theorems used in the proofs of this thesis as well as the basic definitions.

The main result of this thesis characterizes a chaotic representation property using a pairwise strongly orthogonal family of so-called Teugels martingales. These Teugels martingales are, in fact, the compensated power jump processes of a Lévy process. This theorem covering the chaotic representation property for Teugels martingales was explored by David Nualart and Wim Schoutens in their article *Chaotic and predictable representations for Lévy processes.* We use this article as our main source for this thesis and expand upon it by providing more details and exploring alternative versions of chaotic representation properties found in the literature.

One of the chaotic representation properties we examine and prove in detail after our main theorem is Itô's orthogonal decomposition introduced in *Spectral Type of the Shift Transformation of Differential Processes With stationary increments* by Kyoshi Itô. This theorem uses multiple Wiener integrals to define an orthogonal decomposition of the space of square integrable random variables. After the proof, we use this theorem to formulate another, different orthogonal decomposition.

Finally we conclude our thesis by going over a more general decomposition. This chaotic representation property uses iterated integrals with respect to a family of compensated-covariance stable martingales. This property has been covered by Paolo Di Tella and Hans-Juergen Engelbert in *The Chaotic Representation of Compensated-Covariation Stable Families of Martingales.*

Contents

1. Introduction

In this thesis we are going to study different types of chaotic representation properties for functionals on Lévy processes. These representation properties are used in stochastic analysis and stochastic process theory. As an example, the Itô's orthogonal decomposition was used to investigate quantitative properties of stochastic processes in continuous time and to prove covariance relations and inequalities for general Poisson process ([**2**], page 8607).

We will start by going over general Lévy processes satisfying some moment conditions, their associated power jump processes and orthogonalized Teugels martingales. This orthogonalization will then be used to generate a system of iterated integrals in order to formulate the chaotic representation property introduced by Nualart and Schoutens ([**13**]).

After this we will be going over other chaotic representation properties. We start this by taking a look at the decomposition generated with the multiple Itô integrals in [**12**]. Then we will use this decomposition to prove another decomposition, that will be similar to the chaotic representation property of Nualart and Schoutens.

At the end we shall briefly consider one more decomposition. This will be the chaotic representation property of compensated-covariation stable families of martingales treated by Di Tella and Engelbert ([**5**]), which is constructed using iterated integrals with respect to a certain family of square integrable martingales. This is a more general result and the family of orthogonal Teugels martingales is an example of it.

2. Preliminaries

In this thesis we will assume that we are given a stochastic basis that satisfies the usual assumptions([**8**], Definition 2.4.11, page 36).

Definition 2.1. A stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\in I})$, where $I = [0, \infty)$, satisfies the usual conditions given that

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,
(2) $A \in \mathcal{F}_t$ for all $t \in I$, where $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$,
(3) the filtration $(\mathcal{F}_t)_{t\in I}$ is right-continuous, which means that $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, for all $t \in I$.

Definition 2.2 (Adapted process). A stochastic process $X = \{X_t, t \geq 0\}$ is called adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ (or $(\mathcal{F}_t)_{t\geq 0}$-adapted) if X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.

Given a process $X = (X_t)_{t>0}$, $X_t : \Omega \to \mathbb{R}$, in this thesis we use the augmentation of the natural filtration $\mathcal{F}_t := \mathcal{F}_t^X \cup \mathcal{N}$, where $\mathcal{F}_t^X := \sigma(X_s : s \in [0, t])$ is the natural filtration, also denoted simply by $\mathcal{F}^X$, and $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$. With the augmentation of the natural filtration, as well as with the natural filtration, the stochastic process X is always $(\mathcal{F}_t)_{t\geq 0}$-adapted.

We note that in this thesis we assume that the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfies the usual assumptions, which we will define next ([**14**], page 3).

Definition 2.3. A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfies the usual assumptions given that

(i) $\mathcal{N} \subset \mathcal{F}_0$, where $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$,
(ii) $\mathcal{F}_t$ is right continuous, which means that $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ for all $0 \leq t < \infty$.

Next we define what it means for a stochastic process to be càdlàg ([**14**], page 4). We need this for the definition of a Lévy process.

Definition 2.4. A stochastic process $X = \{X_t : t \geq 0\}$ is called càdlàg if all the trajectories $[0, \infty) \ni t \to X_t(\omega) \in \mathbb{R}$ are right continuous, with left limits. Similarly, a stochastic process X is called càglàd if the trajectories are left continuous with right limits.

These acronyms càdlàg and càglàd come from the French language and stand for continu à droite, limites à gauche and continu à gauche, limites à droite, respectively.

Now we can define the notion of a Lévy process.

Definition 2.5 (Lévy process). A real-valued stochastic process $X = \{X_t : t \geq 0\}$ is called a Lévy process if it satisfies the following properties:

(1) $X_0 \equiv 0$.
(2) Independent increments: The random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent for any $0 = t_0 < t_1 < ... < t_n < \infty$.
(3) Stationary increments: $X_t - X_s$ and X_{t-s} have the same distribution for any $0 \leq s \leq t \leq \infty$.
(4) All trajectories $[0, \infty) \ni t \to X_t(\omega) \in \mathbb{R}$ are càdlàg.

If we replace the condition (4) with stochastic continuity we obtain a Lévy process in law ([**15**], page 3).

Next we want to define semi-martingales. Before we can do that we need to give definitions for stopping times ([**1**], page 91), martingales ([**1**], page 84), stochastic processes of bounded variation ([**8**], Theorem 3.2.3, page 68) and local martingales ([**1**], page 92).

Definition 2.6 (Stopping times). A random variable $\tau : \Omega \to [0, \infty]$ is called a stopping time provided that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition 2.7. Let τ be a stopping time and define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leqq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

$\mathcal{F}_\tau$ is a σ-algebra and it is called the σ-algebra generated by τ.

Definition 2.8 (Martingale). An adapted process $M = \{M_t, t \geq 0\}$ is called a martingale if it is integrable, i.e. $E|M_t| < \infty$ for all $t \geq 0$, and for all $0 \leq s < t < \infty$

we have that

$E(M_t|\mathcal{F}_s) = M_s$ a.s.

DEFINITION 2.9. A stochastic process $X = \{X_t, t \geq 0\}$ is said to be of bounded variation if, $X_0 \equiv 0$ and for all $\omega \in \Omega$ and all $t \geq 0$,

$$V_t^1(X(\omega)) := \sup_{\substack{n\in\mathbb{N} \\ t_0,\ldots,t_n \text{ with} \\ 0=t_0\leq\ldots\leq t_n=t}} \sum_{k=1}^{n} |X_{t_k}(\omega) - X_{t_{k-1}}(\omega)| < \infty.$$

DEFINITION 2.10 (Local martingale). A càdlàg process $M = \{M_t, t \geq 0\}$, with $M_0 \equiv 0$, is called a local martingale if there exists a sequence of stopping times such that $\tau_1 \leq ... \leq \tau_n \to \infty$ on Ω and each of the processes $\{M_{t\wedge\tau_n}, t \geq 0\}$ is a martingale.

Now we can define semi-martingales ([**1**], page 137).

DEFINITION 2.11 (Semi-martingale). An adapted càdlàg process $N = \{N_t : t \geq 0\}$ is called a semi-martingale provided that there is a local martingale $M = \{M_t, t \geq 0\}$ and an adapted càdlàg process of bounded variation $C = \{C_t, t \geq 0\}$, such that one has

$N_t = N_0 + M_t + C_t, t \geq 0$, almost surely.

We note that every Lévy process is a semi-martingale ([**1**], Proposition 2.7.1, page 137).

The jump size at time t will be part of many of our upcoming formulas so we shall define that next:

DEFINITION 2.12 (Jump size at time t). The jump size of a Lévy process at time $t > 0$ is defined by $\Delta X_t := X_t - X_{t-}$, where $X_{t-} = \lim_{s\to t-} X_s$ is the left limit if $t > 0$ and $X_{0-} := 0$.

In this thesis we also use stochastic integrals ([**14**], page 58). To define stochastic integrals we need first to define simple predictable processes ([**14**], page 51).

DEFINITION 2.13. A process $L = (L_t)_{t\geq 0}$ is called a simple process if there exists an $n \in \mathbb{N}$ and a finite sequence of stopping times $0 = \tau_0 \leq ... \leq \tau_n < \infty$, and $\mathcal{F}_{\tau_i}$-measurable random variables $v_i : \Omega \to \mathbb{R}$, $i = 0, 1, ..., n$, with $\sup_{i,\omega} |v_i(\omega)| < \infty$, such that

$L_t(\omega) = \sum_{i=1}^{n} \chi_{(\tau_{i-1}(\omega),\tau_i(\omega)]}(t)v_{i-1}(\omega)$.

The class of these processes is denoted by $\mathcal{L}_0$.

DEFINITION 2.14. For $L \in \mathcal{L}_0$, with the representation

$L_t(\omega) = \sum_{i=1}^{n} \chi_{(\tau_{i-1}(\omega),\tau_i(\omega)]}(t)v_{i-1}(\omega)$, and a càdlàg process X the mapping

$J_t^X(L) \coloneqq \sum_{i=1}^n v_{i-1}(X_{t\wedge\tau_i} - X_{t\wedge\tau_{i-1}})$

is called the stochastic integral of L with respect to X and is also denoted by $\int_{(0,t]} L_s dX_s$.

Now we want to extend this mapping to all left continuous adapted processes. In order to do so we have to introduce the *ucp* topology. We define this in the space of càglàd adapted processes which is denoted by $\mathbb{L}$.

Definition 2.15. It is said that a sequence of processes $(H^n)_{n\geq 1} \subseteq \mathbb{L}$ converges to a process $H \in \mathbb{L}$ uniformly on compacts in probability (abbreviated *ucp*) if, for each $t \geq 0$, $\sup_{0\leq s\leq t} |H_s^n - H_s|$ converges to 0 in probability. This is also denoted by $H^n \to H$ in *ucp*.

We check that $\sup_{0\leq s\leq t} |H_s^n - H_s|$ is indeed measurable, so that the above definition is well-posed. We know that there exists a countable and dense subset S of $[0,t]$ with $0 \in S$, for example $[0,t] \cap \mathbb{Q}$. Since $H, H^n \in \mathbb{L}$ for all $n \geq 1$, we know that these processes have left continuous paths. Combining the facts that H and H^n have left continuous paths and S is dense in $[0,t]$, we get that $\sup_{0\leq s\leq t} |H_s^n - H_s| = \sup_{s\in S} |H_s^n - H_s|$. Since S is countable we have that $\sup_{s\in S} |H_s^n - H_s|$ is measurable, which in turn implies that $\sup_{0\leq s\leq t} |H_s^n - H_s|$ is measurable.

In Protter ([**14**], page 57) it is stated that $H^n \to H$ in *ucp* if $(H^n - H)_t^* \to 0$ in probability for each $t > 0$, where $H_t^* = \sup_{0\leq s\leq t} |H_s|$. We also know that the space $\mathbb{L}$ is metrizable with the *ucp* topology. For example one suitable metric for $X, Y \in \mathbb{L}$ is given by

$$d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} E[\min(1, (X-Y)_n^*)].$$

Now with the *ucp* topology we get that $\mathcal{L}_0$ is dense in $\mathbb{L}$ ([**14**], Theorem 10, page 57) and the integration operator is sequentially continuous in $\mathcal{L}_0$ ([**14**], Theorem 11, page 58), which allows us to extend the mapping to all left continuous adapted processes. So in this thesis we can consider a stochastic integral as a map from $\mathbb{L}$ to the space of càdlàg and adapted processes that are vanishing in zero. Here by the deterministic cáglád processes generate $\mathcal{B}([0,T])$, which means that a deterministic Borel-measurable process is predictable ([**14**], page 156).

Next we define the bracket processes ([**14**], page 66).

Definition 2.16. Let M and N be semi-martingales. Then the quadratic variation, $[M,M] = ([M,M]_t)_{t\geq 0}$, is defined by

$[M,M]_t \coloneqq M_t^2 - 2\int_{(0,t]} M_{s-} dM_s.$

The bracket process (also known as quadratic covariation) of M and N, $[M,N] = ([M,N]_t)_{t\geq 0}$, is defined by

$[M,N] := \frac{1}{2}([M+N,M+N] - [M,M] - [N,N])$.

We also note that for the bracket process it holds that:

$[M,N]_t := M_tN_t - \int_{(0,t]} N_{s-}dM_s - \int_{(0,t]} M_{s-}dN_s, t \geq 0$, a.s.

In this thesis we take for $[M,M]$ the version that is càdlàg, $[M,M]_0 = M_0^2$ and all paths are non-decreasing ([**14**], Theorem 22, page 66).

We can also separate $[M,M]$ into its continuous part $[M,M]^c$ and its jump part $\sum_{0\leq s\leq t}(\Delta M_s)^2$ ([**14**], page 70). So we have that

$[M,M]_t = [M,M]_t^c + \sum_{0\leq s\leq t}(\Delta M_s)^2$, $t \geq 0$ pathwise.

From this we get for the bracket process of M and N that, pathwise,

$$\begin{aligned}
[M,N]_t =& \frac{1}{2}([M+N,M+N]_t - [M,M]_t - [N,N]_t) \\
=& \frac{1}{2}([M+N,M+N]_t^c + \sum_{0\leq s\leq t}(\Delta(M+N)_s)^2 \\
& - [M,M]_t^c - \sum_{0\leq s\leq t}(\Delta M_s)^2 - [N,N]_t^c - \sum_{0\leq s\leq t}(\Delta N_s)^2) \\
=& \frac{1}{2}([M+N,M+N]_t^c - [M,M]_t^c - [N,N]_t^c \\
& + \sum_{0\leq s\leq t}(\Delta(M+N)_s)^2 - \sum_{0\leq s\leq t}(\Delta M_s)^2 - \sum_{0\leq s\leq t}(\Delta N_s)^2) \\
=& [M,N]_t^c + \frac{1}{2}(\sum_{0\leq s\leq t}(\Delta(M+N)_s)^2 - \sum_{0\leq s\leq t}(\Delta M_s)^2 - \sum_{0\leq s\leq t}(\Delta N_s)^2),
\end{aligned}$$

where $[M,N]_t^c = \frac{1}{2}([M+N,M+N]_t^c - [M,M]_t^c - [N,N]_t^c)$ denotes the continuous part of $[M,N]_t$. We denote $\Omega_0 = \{\omega \in \Omega : \sum_{0\leq s\leq t}[(\Delta M_s(\omega))^2 + (\Delta N_s(\omega))^2) + (\Delta(M+N)_s(\omega))^2] < \infty\}$. For each $\omega \in \Omega$ there are only countably many $s \in [0,t]$ with $\Delta M_s(\omega) \neq 0$ or $\Delta N_s(\omega) \neq 0$ or $\Delta(M+N)_s(\omega) \neq 0$, which means that $\mathbb{P}(\Omega_0) = 1$. From this we get that

$$\begin{aligned}
(1) \qquad [M,N]_t =& [M,N]_t^c + \frac{1}{2}(\sum_{0\leq s\leq t}(\Delta(M+N)_s)^2 - \sum_{0\leq s\leq t}(\Delta M_s)^2 - \sum_{0\leq s\leq t}(\Delta N_s)^2) \\
=& [M,N]_t^c + \frac{1}{2}\sum_{0\leq s\leq t}\Delta M_s\Delta N_s \text{ pathwise.}
\end{aligned}$$

Next we will define Itô's formula on Lévy processes that we shall use in this thesis. The proof for this formula can be found in [**14**] (page 78, Theorem 32).

THEOREM 2.17 (Itô's formula). *Let N be a semi-martingale and f: $\mathbb{R} \to \mathbb{R}$ be a twice differentiable function, where the second derivative is continuous, also known as a $C^2(\mathbb{R})$ function. Then $f(N) = (f(N_t))_{t\geq 0}$ is also a semi-martingale, and for all $t_0 \in [0,\infty)$ one has:*

$$\begin{aligned} f(N_t) - f(N_{t_0}) = &\int_{(t_0,t]} f'(N_{s-})dN_s + \frac{1}{2}\int_{(t_0,t]} f''(N_{s-})d[N,N]^c_s \\ &+ \sum_{t_0<s\leq t} \{f(N_s) - f(N_{s-}) - f'(N_{s-})\Delta N_s\} \end{aligned}$$

for all $t \geq t_0$ a.s.

Now we define the Kunita-Watanabe Inequality ([**14**], Theorem 25, page 69).

THEOREM 2.18 (Kunita-Watanabe Inequality). *If X and Y are semi-martingales and $H, K \in \mathbb{L}$, then*

$\int_0^\infty |H_s||K_s||d[X,Y]_s| \leq (\int_0^\infty H_s^2 d[X]_s)^{1/2}(\int_0^\infty K_s^2 d[Y]_s)^{1/2}$ *a.s.*

The following theorem ([**14**], Corollary 2, page 68) will also be used later.

THEOREM 2.19 (Integration by parts). *Let X and Y be semi-martingales. Then XY is also a semi-martingale and*

$X_tY_t = X_0Y_0 + \int_{(0,t]} X_{s-}dY_s + \int_{(0,t]} Y_{s-}dX_s + [X,Y]_t,\ t \geq 0,$ *a.s.*

It is also known that the special case of $Y_t \equiv t$ leads to $X_tY_t = \int_{(0,t]} X_{s-}dY_s + \int_{(0,t]} Y_{s-}dX_s + X_0Y_0 = \int_{(0,t]} X_{s-}dY_s + \int_{(0,t]} Y_{s-}dX_s$, $t \geq 0$, a.s. This follows from the Kunita-Watanabe inequality, which gives us

$$\begin{aligned} |[X,Y]_T| =&| \int_{(0,t]} d[X,Y]_s| \leq (\int_{(0,t]} d[X,X]_s)^{\frac{1}{2}}(\int_{(0,t]} d[Y,Y]_s)^{\frac{1}{2}} \\ =&(\int_{(0,t]} d[X,X]_s)^{\frac{1}{2}} \times 0 = 0. \end{aligned}$$

From the quadratic covariation we can define a random measure

$d[X,X]_t(\omega) \cong \mu(dt,\omega),$

where $t \to [X,X]_t(\omega)$ is the distribution function of $\mu(\cdot,\omega)$. This random measure and the fact that for $H \in \mathbb{L}$ the map $\omega \to \int_{(0,t]} H_s(\omega)\mu(ds,\omega)$ is $\mathcal{F}_t$-measurable are used to formulate the following theorem ([**14**], Theorem 29, page 75).

THEOREM 2.20. *If X and Y are semi-martingales and $H, K \in \mathbb{L}$, then*

$$\left[\int_{(0,\cdot]} H_s dX_s, \int_{(0,\cdot]} K_s dY_s\right]_t = \int_{(0,t]} H_s K_s d[X,Y]_s, t \geq 0, \ a.s.$$

Now we define square integrable martingales ([**14**], page 178) and introduce the definitions for the angle bracket process ([**11**], Definition 6.24, page 185) and bracket process ([**11**], Definition 6.27, page 186) for such martingales.

DEFINITION 2.21. $\mathcal{M}_0^2$ denotes the space of square integrable martingales M on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, which means that $E(M_t^2) < \infty$, for all $t \geq 0$, with $M_0 \equiv 0$.

DEFINITION 2.22. Let $M, N \in \mathcal{M}^2$. There exists a unique predictable increasing process, denoted by $\langle M, M\rangle$ or $\langle M\rangle$, such that $M^2 - \langle M\rangle \in \mathcal{M}_0$, where $\mathcal{M}_0$ stands for the martingales with $M_0 \equiv 0$. The process $\langle M\rangle$ is called the predictable quadratic variation or the angle bracket process of M. We also set $\langle M, N\rangle = \frac{1}{2}[\langle M+N\rangle - \langle M\rangle - \langle N\rangle]$ and call $\langle M, N\rangle$ the predictable quadratic covariation or the angle bracket process of M and N.

Next we use Theorems 2.18, 2.19 and 2.20 to prove a form of Itô's isometry.

THEOREM 2.23 (Itô isometry). *Let X and Y be square integrable martingales with $X_0 \equiv Y_0 \equiv 0$, i.e. $X, Y \in \mathcal{M}_0^2$. Assume $H, K \in \mathbb{L}$ with*

$$E\int_{(0,T]} H_s^2 d[X]_s + E\int_{(0,T]} K_s^2 d[Y]_s < \infty.$$

Then

(1) $E[\int_{(0,T]} H_s dX_s \int_{(0,T]} K_s dY_s] = E\int_{(0,T]} H_s K_s d[X,Y]_s$, *for* $T \geq 0$.
(2) *If additionally,*

$$E\int_{(0,T]} H_s^2 d\langle X\rangle_s + E\int_{(0,T]} K_s^2 d\langle Y\rangle_s < \infty,$$

then

$$E\Big[\int_{(0,T]} H_s dX_s \int_{(0,T]} K_s dY_s\Big] = E\int_{(0,T]} H_s K_s d\langle X, Y\rangle_s, \ for\ T \geq 0.$$

Proof. We only prove part (1). By Theorem 2.20 we get that $E|\int_0^\infty H_s dX_s|^2 < \infty$ and $E|\int_0^\infty K_s dY_s|^2 < \infty$ which gives us that $\int_0^\infty H_s dX_s \int_0^\infty K_s dY_s \in \mathcal{L}_1$. When we combine this with the Theorem 2.18 we get $E\int_0^\infty |H_s K_s| d|[X,Y]|_s < \infty$ and because of this the terms in (1) are well-defined.

From [**11**] (Theorem 6.28, page 167) we get that $X, Y \in \mathcal{M}_0^2$ implies that $(X_t Y_t - [X,Y]_t)_{t\geq 0} \in \mathcal{M}_0$. This means that $EX_tY_t = E[X,Y]_t$. By combining all of this with Theorem 2.20 we get that

$$\begin{aligned} E[\int_{(0,T]} H_s dX_s \int_{(0,T]} K_s dY_s] =& E\Big[\int_{(0,\cdot]} H_s dX_s, \int_{(0,\cdot]} K_s dY_s\Big]_T \\ =& E\int_{(0,T]} H_s K_s d[X,Y]_s. \end{aligned}$$

□

The stochastic integral also has a property called associativity that we will need later ([**14**], Theorem 19, page 62).

Theorem 2.24. *Let $(Y_t)_{t\geq 0}$ be a process, where $Y_t = (H\cdot X)_t := \int_{(0,t]} H_s dX_s$, is a semi-martingale and assume that $G \in \mathbb{L}$. Then we have that $G\cdot Y = G\cdot(H\cdot X) = (GH)\cdot X$.*

Now we define uniform integrability ([**15**], Definition 36.1, page 245).

Definition 2.25. A family of random variables $\{X_i : i \in I\}$ is called uniformly integrable if $\sup_{i\in I} E|X_i|1_{\{|X_i|>a\}} = \sup_{i\in I}\int_{\{|X_i|>a\}} |X_i| d\mathbb{P} \to 0$ when $a \to \infty$.

The following theorem ([**14**], Theorem 13, page 9) can be used to check if a martingale is uniformly integrable.

Theorem 2.26. *A càdlàg martingale $(X_t)_{t\geq 0}$ is uniformly integrable if and only if $Y = \lim_{t\to\infty} X_t$ exists a.s, $E(|Y|) < \infty$ and $(X_t)_{0\leq t\leq\infty}$ is a martingale, where $Y = X_\infty$.*

Next we define a Π-system ([**1**], page 4) and a Π-step-function.

Definition 2.27 (Π-system). A collection Π of subsets of Ω is called a Π−system if $A\cap B \in \Pi$ for all $A, B \in \Pi$.

Definition 2.28 (Π-step-function). Let $\Pi \subseteq 2^\Omega$ be a non-empty system of subsets. A function $h : \Omega \to \mathbb{R}$ is called a Π-step-function, provided that $h = \sum_{k=1}^n \alpha_k \chi_{A_k}$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ and $A_1, ..., A_n \in \Pi$.

Now we consider a Lévy process $X = (X_t)_{t\geq 0}$ and a Π-system where

$$\begin{aligned} \Pi := \{A = \{\omega \in \Omega : X_{s_2} - X_{s_1} \in \mathcal{B}_1, ..., X_{s_m} - X_{s_{m-1}} \in \mathcal{B}_m\} : \\ 0 \leq s_1 \leq s_2 \leq ... \leq s_m, m \in \mathbb{N}, \mathcal{B}_1, ..., \mathcal{B}_m \in \mathcal{B}(\mathbb{R})\}. \end{aligned}$$

With this we get that $\sigma(\Pi) = \sigma(X_s : s \geq 0)$ and by definition $\emptyset, \Omega \in \Pi$. This is used to apply the following theorem ([**10**], Theorem A.8, page 101) later on Π-step-functions.

THEOREM 2.29. *Let Π be a system of subsets of a non-empty Ω for which*
i) $A \cap B \in \Pi$ *when* $A, B \in \Pi$,
ii) $\Omega \in \Pi$.
When $p \in [1, \infty)$ and $\mathcal{F} := \sigma(\Pi)$, then for all $f \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ there exists Π-step-functions $f_n : \Omega \to \mathbb{R}$ such that $\lim_n ||f - f_n||_p = 0$.

The following proposition ([**7**], Proposition 4.1.4, page 48) will also be used in this thesis, but first we need to define step functions ([**7**], Proposition 4.1.1, page 48).

DEFINITION 2.30 (step function). A function $f : \Omega \to \mathbb{R}$ is called a step function given that

$f(\omega) := \sum_{i=1}^{n} \alpha_i \chi_{A_i}(\omega)$,

where $n \in \mathbb{N}$, $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and $A_1, ..., A_n \in \mathcal{F}$.

PROPOSITION 2.31. *Let $f : \Omega \to \mathbb{R}$ be a function in a measurable space $(\Omega, \mathcal{F})$. Then the following properties are equivalent:*
(1) *There exists a sequence $(f_n)_{n=1}^{\infty}$ of step functions $f_n : \Omega \to \mathbb{R}$, such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for all $\omega \in \Omega$.*
(2) *f is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))-$measurable.*

Now we introduce the Hölder's inequality ([**7**], Theorem 6.12.5, page 106) and Lebesgue's dominated convergence theorem ([**1**], Theorem 1.1.4, page 8).

THEOREM 2.32 (Hölder's inequality). *Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be measurable maps. For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ one has that*

$$\int_\Omega |fg| d\mu \leq (\int_\Omega |f|^p d\mu)^{\frac{1}{p}} (\int_\Omega |g|^p d\mu)^{\frac{1}{q}}.$$

THEOREM 2.33 (Lebesgue's dominated convergence theorem). *Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $g, f, f_1, f_2, ... : \Omega \to \mathbb{R}$ be measurable with $|f_n| \leq g$ a.e. If $\int_\Omega |g| d\mu < \infty$ and $f = \lim_{n\to\infty} f_n$ a.e., then f is integrable and*

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

We also use the following uniqueness theorem for Fourier transforms ([**4**], Proposition 5.1.11, page 193) later in this thesis.

DEFINITION 2.34 (Fourier transform). Let $f \in L_1(\mathbb{R}^d, \mathbb{C})$, then

$$\hat{f}(u) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i(u,x)} f(x) dx$$

denotes the Fourier transform of f for all $u \in \mathbb{R}^d$.

THEOREM 2.35. *If* $f \in L_1(\mathbb{R}^d, \mathbb{C})$ *and* $\hat{f}(u) = 0$ *on* $\mathbb{R}^d$, *then* $f(x) = 0$ *a.e.*

Next we introduce Urysohn's lemma for metric spaces ([**16**]).

LEMMA 2.36 (Urysohn's lemma). *Let* (M, d) *be a metric space and let* F_0 *and* F_1 *be non-empty closed sets such that* $F_0 \cap F_1 = \emptyset$. *Then there exists a continuous function* $f : M \to [0, 1]$ *such that* $f(x) = i$ *for* $x \in F_i$.

3. Orthogonal decompositions of the Lévy-Itô space by Teugels martingales

Here we explore the orthogonal decomposition of the Lévy-Itô space using a strongly orthogonal set of Teaugels martingales. This chaotic representation property, or CRP, has been explored by Nualart and Schoutens ([**13**]). We will expand on their results in this section.

3.1. Notation. Before we can take a look at the CRP we must go over some definitions and notation. Now let $X = (X_t)_{t\geq 0}$ be a Lévy process and $\mathcal{F}$ be the σ-algebra generated by it ([**15**], page 6), which we will define next.

DEFINITION 3.1. Let T be an arbitrary index-set and $X = \{X_t : t \in T\}$ be a family of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the sub-$\sigma$-algebra $\mathcal{G} = \sigma(X_t : t \in T)$ is called the σ-algebra generated by X if

(1) X_t is $\mathcal{G}$-measurable for every $t \in T$,
(2) $\mathcal{G}$ is the smallest that satisfises (1).

Let $N : \mathcal{B}([0, \infty) \times \mathbb{R}) \times \Omega \to \{\infty, 0, 1, 2, ...\}$ be the Poisson random measure associated with X, where

$N(E) := \#\{t \in [0, \infty) : (t, \Delta X_t) \in E\}$

for $E \in \mathcal{B}([0, \infty) \times \mathbb{R})$. For $B \in \mathcal{B}(\mathbb{R})$, with $B \cap (-\epsilon, \epsilon) = \emptyset$ for some $\epsilon > 0$, we set

$\nu(B) := \mathbb{E}N([0, 1] \times B)$.

By letting $\epsilon \to 0$ we obtain the Lévy measure ν on $\mathcal{B}(\mathbb{R})$ that satisfies $\nu(\{0\}) = 0$ and $\int_{-\infty}^{+\infty} (1 \wedge x^2)\nu(dx) < \infty$. In this section we assume that the Lévy measure ν satisfies the following moment conditions for some $\epsilon > 0$ and $\lambda > 0$:

$\int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\nu(dx) < \infty$.

This also means that

$\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty, \quad i \geq 2,$

which in turn implies that X_t has moments of all orders ([**15**], Theorem 25.3, page 159). Now we can formulate the Lévy-Khintchine formula ([**14**], Theorem 43, page 32), which describes uniquely the distribution of a Lévy process.

Theorem 3.2 (Lévy-Khintchine formula). *Let $X = (X_t)_{t\geq 0}$ be a Lévy process and ν be the associated Lévy measure. Then*

$$E(e^{iuX_t}) = e^{-t\psi(u)},$$

where $\psi(u) = \frac{\sigma^2}{2}u^2 - i\alpha u + \int_{\{|x|\geq 1\}}(1 - e^{iux})d\nu(x) + \int_{\{|x|<1\}}(1 - e^{iux} + iux)d\nu(x)$ is the characteristic function of X, $\alpha \in \mathbb{R}$ and $\sigma \geq 0$. Moreover every Lévy process is uniquely described by the triplet (α, σ^2, ν).

Next we will define some important transformations of a Lévy process X that will be needed for our analysis.

$X_t^{(i)} := \sum_{0<s\leq t}(\Delta X_s)^i, \;\; i \geq 2$ and $X_t^{(1)} = X_t$.

These processes $X^{(i)} = \{X_t^{(i)}, t \geq 0\}, \;\; i = 1, 2, ...$ are called the power jump processes and they jump at the same points as the original process. We know, that the sums used to define these power jump processes are well defied, since the càdlàg process $t \to X_t(\omega)$ has only countably many jumps. Otherwise there would exist an $\epsilon > 0$, such that X_t would have uncountably many jumps of size larger than ϵ, which is a contradiction ([**3**], Lemma 1, page 122). Now we recall that $E|X_t^{(i)}| < \infty$. From here we get the compensated power jump process of order i, also known as the Teugels martingales ([**13**]):

$Y_t^{(i)} := X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - m_i t, i = 1, 2, 3, ...,$

where $m_i = \int_{\mathbb{R}} x^i \nu(dx)$. We prove that $E[X_t^{(i)}] = m_i t = t\int_{\mathbb{R}} x^i \nu(dx)$, by taking a $B \in \mathcal{B}(\mathbb{R})$ with $0 \notin \bar{B}$. From here we get that

$$\begin{aligned}\nu(B) =& E\#\{\Delta X_s \in B : s \in (0, 1]\}\\ =& \frac{1}{t}E\#\{\Delta X_s \in B : s \in (0, t]\}\\ =& \frac{1}{t}E\sum_{s\in(0,t]} 1_B(\Delta X_s).\end{aligned}$$

This implicates that

$$t\int_{\mathbb{R}} 1_B(x)d\nu(x) = E\sum_{s\in(0,t]} 1_B(\Delta X_s),$$

and therefore

$$t\int_{\mathbb{R}} \gamma(x)d\nu(x) = E\sum_{s\in(0,t]} \gamma(\Delta X_s),$$

for any Borel function $\gamma:\mathbb{R}\to\mathbb{R}$, such that $\int_{\mathbb{R}}|\gamma(x)|d\nu(x)<\infty$. Now one can see that

$$tm_i = EX_t^{(i)} = E\sum_{s\in(0,t]}(\Delta X_s)^i = t\int_{\mathbb{R}} x^i d\nu(x). \tag{2}$$

Next we give the definition of strong orthogonality ([**14**], page 179).

Definition 3.3 (Strongly orthogonal martingales). Two martingales $N, M\in\mathcal{M}_0^2$ are strongly orthogonal if and only if their product MN is a martingale. We denote this by $M \perp\!\!\!\perp N$.

It is also known that strong orthogonality implies that $EM_tN_t = 0$. This is because MN is a martingale, when $M \perp\!\!\!\perp N$, which gives us $EM_tN_t = EM_0N_0 = 0$. We also have the following proposition ([**11**], Corollary 6.30, page 187).

Proposition 3.4. *For $M, N\in\mathcal{M}_0^2$ the following assertions are equivalent:*
(1) $M \perp\!\!\!\perp N$ *and* $\Delta M\Delta N = 0$,
(2) $[M, N] = 0$.

Later in this thesis we shall need a set of pairwise strongly orthogonal martingales $\{H^{(i)}, i\ge 1\}$ for which each $H^{(i)}$ is a linear combination of $Y^{(j)}, j = 1, 2, ..., i$ with the leading coefficient equal to 1. This means that

$$H^{(i)} = Y^{(i)} + a_{i,i-1}Y^{(i-1)} + ... + a_{i,1}Y^{(1)}, i\ge 1,$$

and the $\{H^{(i)}, i\ge 1\}$ are pairwise strongly orthogonal.

Because of the Equation (1) and since $X_t^{(i)}$ has no continuous part for $i > 1$ we get that

$[X^{(k)}, X^{(j)}]_t = \sum_{0<s\leq t}(\Delta X_s)^k(\Delta X_s)^j = \sum_{0<s\leq t}(\Delta X_s)^{k+j} = X_t^{(k+j)}$, for $j > 1$ and $k \geq 1$.

Now we have that

$$\begin{aligned}[H^{(i)}, Y^{(j)}]_t =& \sum_{k=1}^{i} a_{i,k}[Y^{(k)}, Y^{(j)}]_t = \sum_{k=1}^{i} a_{i,k}[X^{(k)}, X^{(j)}]_t \\ =& X_t^{(i+j)} + a_{i,i-1}X_t^{(i+j-1)} + \cdots + a_{i,1}X_t^{(1+j)} + a_{i,1}\sigma^2 t 1_{\{j=1\}}.\end{aligned}$$

The term $a_{i,1}\sigma^2 t 1_{\{j=1\}}$ comes from the Brownian motion part $a_{i,1}[X^{(1)}, X^{(1)}]_t = a_{i,1}[X, X]_t = a_{i,1}([X, X]_t^c + \sum_{0<s\leq t}(\Delta X_s)^2) = a_{i,1}X_t^{(2)} + a_{i,1}\sigma^2 t$, which only appears in the above sum expression for the bracket process if $j = 1$. From the expression above we also get that

$$E[H^{(i)}, Y^{(j)}]_t = t(m_{i+j} + a_{i,i-1}m_{i+j-1} + \cdots + a_{i,1}m_{1+j} + a_{i,1}\sigma^2 1_{\{j=1\}}).$$

So $E[H^{(i)}, Y^{(j)}]_t = tE[H^{(i)}, Y^{(j)}]_1$. Which means that $[H^{(i)}, Y^{(j)}]_t = 0$, $\forall\ t \geq 0$, if and only if $[H^{(i)}, Y^{(j)}]_1 = 0$. This implies that when $[H^{(i)}, Y^{(j)}]_1 = 0$ we have that $H^{(i)} \perp\!\!\!\perp Y^{(j)}$.

Now let S_1 be the space of all real polynomials on the positive real line. On this space we apply a scalar product $\langle ., . \rangle_1$ given by

$$\langle P(x), Q(x)\rangle_1 := \int_{-\infty}^{+\infty} P(x)Q(x)x^2 v(dx) + \sigma^2 P(0)Q(0).$$

From this and equation 2 we get that

$$\begin{aligned}\langle x^{i-1}, x^{j-1}\rangle_1 =& \int_{-\infty}^{+\infty} x^{i-1}x^{j-1}x^2\nu(dx) + \sigma^2 0^{i-1}0^{j-1} \\ =& \int_{-\infty}^{+\infty} x^{i+j}\nu(dx) + \sigma^2 1_{\{i=j=1\}} \\ =& m_{i+j} + \sigma^2 1_{\{i=j=1\}}, \quad i, j \geq 1.\end{aligned}$$

Then we denote the space of all linear transformations of the compensated power jump processes by $S_2 := \{\sum_{i=1}^n a_i Y^{(i)} : n \in \{1, 2, ...\}, a_i \in \mathbb{R}, \forall\, i = 1, ..., n\}$ and apply on it the scalar product $\langle ., . \rangle_2$ given by

$$\langle Y^{(i)}, Y^{(j)}\rangle_2 := E([Y^{(i)}, Y^{(j)}]_1) = m_{i+j} + \sigma^2 1_{\{i=j=1\}}, \quad i, j \geq 1. \tag{3}$$

Now we see that $x^{i-1} \longleftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. This means that an orthogonalization of $\{1, x^1, x^2, ...\}$ also gives us an orthogonalization of $\{Y^{(1)}, Y^{(2)}, ...\}$.

From this we can establish for the rest of the thesis that $\{H^{(i)}, i = 1, 2, ...\}$ is the set of pairwise strongly orthogonal martingales that we were looking for, given by the orthogonalization of $\{Y^{(1)}, Y^{(2)}, ...\}$.

3.2. Main result. We want to prove the following chaotic representation property:

THEOREM 3.5 (Chaotic representation property (CRP)). *Let $F \in L^2(\Omega, \mathcal{F})$. Then F has a representation of the form*

$$\begin{aligned} F =& E[F] \\ &+ \sum_{j=1}^{\infty} \sum_{i_1,...,i_j \geq 1} \int_{(0,\infty)} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1, ..., t_j) dH_{t_j}^{(i_j)} ... dH_{t_2}^{(i_2)} dH_{t_1}^{(i_1)} \ a.s., \end{aligned}$$

where the functions $f_{(i_1,...,i_j)}$ belong to $L^2(\mathbb{R}_+^j)$.

3.3. Preparation for the proof of Theorem 3.5. Before we can prove Theorem 3.5 we need several lemmas and propositions, which we shall present in this subsection. First we prove a representation of the power of an increment of a Lévy process, $(X_{t+t_0} - X_{t_0})^k, k = 1, 2, 3, ...,$ as a sum of stochastic integrals with respect to the compensated power jump processes $Y^{(j)}, j = 1, ..., k$.

LEMMA 3.6. *The power of an increment of a Lévy process, $(X_{t+t_0} - X_{t_0})^k$, has a representation of the form*

$$\begin{aligned} (X_{t+t_0} - X_{t_0})^k =& f^{(k)}(t) + \sum_{j=1}^{k} \sum_{(i_1,...,i_j) \in \{1,...,k\}^j} \int_{(t_0,t+t_0]} \int_{t_0}^{t_1-} \cdots \\ & \int_{t_0}^{t_{j-1}-} f_{(i_1,...,i_j)}^{(k)}(t_0, t_1, ..., t_j) dY_{t_j}^{(i_j)} ... dY_{t_2}^{(i_2)} dY_{t_1}^{(i_1)}, \end{aligned}$$

where the $f_{(i_1,...,i_j)}^{(k)}$ are deterministic functions in $L^2(\mathbb{R}_+^j)$.

Proof. Using Itô's formula and integration by parts we get for the function $f(x) = x^k, k \geq 2$ and the Lévy process $Y_t = X_{t+t_0} - X_{t_0}$, a.s.,

$$
\begin{aligned}
(X_{t+t_0} - X_{t_0})^k =& (X_{t+t_0} - X_{t_0})^k - (X_{0+t_0} - X_{t_0})^k = f(Y_t) - f(Y_0) \\
=& \int_{(0,t]} k(Y_{s-})^{k-1} dY_s \\
&+ \frac{1}{2}\int_{(0,t]} k(k-1)(Y_{s-})^{k-2} d[Y,Y]^c_s \\
&+ \sum_{0<s\le t} [(Y_s)^k - (Y_{s-})^k - k(Y_{s-})^{k-1}\Delta Y_s] \\
=& \int_{(0,t]} k(X_{(s+t_0)-} - X_{t_0})^{k-1} d(X_{s+t_0} - X_{t_0}) \\
&+ \frac{\sigma^2}{2}\int_{(0,t]} k(k-1)(X_{(s+t_0)-} - X_{t_0})^{k-2} ds \\
&+ \sum_{0<s\le t} [(X_{s+t_0} - X_{t_0})^k - (X_{(s+t_0)-} - X_{t_0})^k \\
&- k(X_{(s+t_0)-} - X_{t_0})^{k-1}\Delta X_{s+t_0}] \\
=& \int_{(t_0,t+t_0]} k(X_{u-} - X_{t_0})^{k-1} dX^{(1)}_u \\
&+ \frac{\sigma^2}{2} k(k-1)\Big((X_{t+t_0} - X_{t_0})^{k-2} t - \int_{(0,t]} s d(X_{s+t_0} - X_{t_0})^{k-2}\Big) \\
&+ \sum_{0<s\le t} [(X_{(s+t_0)-} + \Delta X_{s+t_0} - X_{t_0})^k - (X_{(s+t_0)-} - X_{t_0})^k \\
&- k(X_{(s+t_0)-} - X_{t_0})^{k-1}\Delta X_{s+t_0}] \\
=& \int_{(t_0,t_0+t]} k(X_{u-} - X_{t_0})^{k-1} dX^{(1)}_u \\
&+ \frac{\sigma^2}{2} k(k-1)\Big((X_{t+t_0} - X_{t_0})^{k-2} t - \int_{(0,t]} s d(X_{s+t_0} - X_{t_0})^{k-2}\Big) \\
&+ \sum_{0<s\le t} \sum_{j=2}^{k} \binom{k}{j} (X_{(s+t_0)-} - X_{t_0})^{k-j} (\Delta X_{s+t_0})^j \\
=& \int_{(t_0,t_0+t]} k(X_{u-} - X_{t_0})^{k-1} dX^{(1)}_u \\
&+ \frac{\sigma^2}{2} k(k-1)\Big((X_{t+t_0} - X_{t_0})^{k-2} t - \int_{(0,t]} s d(X_{s+t_0} - X_{t_0})^{k-2}\Big) \\
&+ \sum_{t_0<u\le t+t_0} \sum_{j=2}^{k} \binom{k}{j} (X_{u-} - X_{t_0})^{k-j} (\Delta X_u)^j \\
=& \sum_{j=1}^{k} \binom{k}{j} \int_{(t_0,t+t_0]} (X_{u-} - X_{t_0})^{k-j} dX^{(j)}_u \qquad (4)
\end{aligned}
$$

$$+\frac{\sigma^2}{2}k(k-1)\Big((X_{t+t_0}-X_{t_0})^{k-2}t-\int_{(0,t]}sd(X_{s+t_0}-X_{t_0})^{k-2}\Big)$$

Next we rewrite the sum from (4) by using integration by parts again:

$$\begin{aligned}
&\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dX_s^{(j)}\\
&=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}d(Y_s^{(j)}+m_js)\\
&=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k}\binom{k}{j}m_j\int_{(t_0,t+t_0]}(X_s-X_{t_0})^{k-j}ds\\
&=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k}\binom{k}{j}m_j\int_{(0,t]}(X_{u+t_0}-X_{t_0})^{k-j}du\\
&=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k}\binom{k}{j}m_j\Big(t(X_{t+t_0}-X_{t_0})^{k-j}\\
&\quad-\int_{(0,t]}ud(X_{u+t_0}-X_{t_0})^{k-j}\Big)\\
&=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k-1}\binom{k}{j}m_j\Big(t(X_{t+t_0}-X_{t_0})^{k-j}\\
&\quad-\int_{(t_0,t+t_0]}(s-t_0)d(X_s-X_{t_0})^{k-j}\Big)\\
&\quad+\binom{k}{k}m_k\Big(t(X_{t+t_0}-X_{t_0})^{k-k}-\int_{(0,t]}ud(X_{u+t_0}-X_{t_0})^{k-k}\Big)
\end{aligned}$$

$$\begin{aligned}
(5)\quad &=\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k-1}\binom{k}{j}m_jt(X_{t+t_0}-X_{t_0})^{k-j}\\
&\quad-\sum_{j=1}^{k-1}\binom{k}{j}m_j\int_{(t_0,t+t_0]}(s-t_0)d(X_s-X_{t_0})^{k-j}+m_kt.
\end{aligned}$$

By combining (4) and (5) we get, a.s., that

$$\begin{aligned}
(6)\quad &(X_{t+t_0}-X_{t_0})^k\\
&=\frac{\sigma^2}{2}k(k-1)\Big((X_{t+t_0}-X_{t_0})^{k-2}t-\int_{(0,t]}sd(X_{s+t_0}-X_{t_0})^{k-2}\Big)\\
&\quad+\sum_{j=1}^{k}\binom{k}{j}\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}+\sum_{j=1}^{k-1}\binom{k}{j}m_jt(X_{t+t_0}-X_{t_0})^{k-j}
\end{aligned}$$

$$-\sum_{j=1}^{k-1}\binom{k}{j}m_j\int_{(t_0,t+t_0]} s-t_0 d(X_s-X_{t_0})^{k-j}+m_k t.$$

Now we finish the proof by induction.

If $k=1$, then we have that, a.s., $X_{t+t_0}-X_{t_0}=\int_{(t_0,t+t_0]}dX_s=\int_{(t_0,t+t_0]}d(Y_s^{(1)}+m_1 s)=\int_{(t_0,t+t_0]}dY_s^{(1)}+m_1 t$, where $m_1 t$ is linear in t. This gives us that the lemma holds for $k=1$.

As the induction hypothesis we assume that for $l\in\{1,...,k-1\}$, where $k\in\{2,3,...\}$, it holds a.s. that

$$\begin{aligned}(X_{t+t_0}-X_{t_0})^l =& f^{(l)}(t)+\sum_{j=1}^{l}\sum_{(i_1,...,i_j)\in\{1,...,l\}^j}\int_{(t_0,t+t_0]}\int_{t_0}^{t_1-}\cdots\\ &\int_{t_0}^{t_{j-1}-} f^{(l)}_{(i_1,...,i_j)}(t_0,t_1,...,t_j)dY_{t_j}^{(i_j)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)},\end{aligned}$$

where the $f^{(l)}_{(i_1,...,i_j)}$ are deterministic functions in $L^2(\mathbb{R}_+^j)$. We also know that $f^{(l)}(t)$ is a polynom in t, because of the term $m_1 t$, in case $k=1$, that is linear in t.

Then we want to represent the Equation (6) as a sum of deterministic functions and iterated integrals of the form shown in the lemma. We do this by applying the induction hypothesis to the equation. For this there are 3 types of terms, with $j=1,...,k$, that we need to consider:

i) $(X_{t+t_0}-X_{t_0})^{k-j}$
ii) $\int_{(t_0,t+t_0]}(X_{s-}-X_{t_0})^{k-j}dY_s^{(j)}$
iii) $\int_{(t_0,t+t_0]} s d(X_s-X_{t_0})^{k-j}$.

After that the linearity of the integral mapping finishes the proof.

i) In the first case we have that, a.s.,

$$\begin{aligned}&(X_{t+t_0}-X_{t_0})^{k-j}\\ &\quad = f^{(k-j)}(t)+\sum_{n=1}^{k-j}\sum_{(i_1,...,i_n)\in\{1,...,k-j\}^n}\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\\ &\quad\int_{t_0}^{t_{n-1}-} f^{(k-j)}_{(i_1,...,i_n)}(t_0,t_1,...,t_n)dY_{t_n}^{(i_n)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)}.\end{aligned}$$

ii) In the second case we get a.s. that

$$\int_{(t_0,t+t_0]}(X_s - X_{t_0})^{k-j}dY_s^{(j)}$$
$$= \int_{(0,t]}(X_{s+t_0} - X_{t_0})^{k-j}dY_{s+t_0}^{(j)}$$
$$= \int_{(0,t]}\Big(f^{(k-j)}(s) + \sum_{n=1}^{k-j}\sum_{(i_1,...,i_n)\in\{1,...,k-j\}^n}\int_{t_0}^{(s+t_0)-}\int_{t_0}^{t_1-}\cdots$$
$$\int_{t_0}^{t_{n-1}-} f_{(i_1,...,i_n)}^{(k-j)}(t_0,t_1,...,t_n)dY_{t_n}^{(i_n)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)}\Big)dY_{s+t_0}^{(j)}$$
$$= \int_{(0,t]} f^{(k-j)}(s)dY_{s+t_0}^{(j)}$$
$$+ \sum_{n=1}^{k-j}\sum_{(i_1,...,i_n)\in\{1,...,k-j\}^n}\int_{(0,t]}\int_{t_0}^{(s+t_0)-}\int_{t_0}^{t_1-}\cdots$$
$$\int_{t_0}^{t_{n-1}-} f_{(i_1,...,i_n)}^{(k-j)}(t_0,t_1,...,t_n)dY_{t_n}^{(i_n)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)}dY_{s+t_0}^{(j)}$$
$$= \int_{(t_0,t+t_0]} f^{(k-j)}(s-t_0,t_0)dY_s^{(j)}$$
$$+ \sum_{n=1}^{k-j}\sum_{(i_1,...,i_n)\in\{1,...,k-j\}^n}\int_{(t_0,t+t_0]}\int_{t_0}^{(s+t_0)-}\int_{t_0}^{t_1-}\cdots$$
$$\int_{t_0}^{t_{n-1}-} f_{(i_1,...,i_n)}^{(k-j)}(t_0,t_1,...,t_n)dY_{t_n}^{(i_n)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)}dY_s^{(j)}.$$

iii) For the final case we apply Theorem 2.24, to get a.s. that

$$\int_{(t_0,t+t_0]} sd(X_s - X_{t_0})^{k-j}$$
$$= \int_{(0,t]} sd(X_{s+t_0} - X_{t_0})^{k-j}$$
$$= \int_{(0,t]} sd\Big(f^{(k-j)}(s) + \sum_{n=1}^{k-j}\sum_{(i_1,...,i_n)\in\{1,...,k-j\}^n}\int_{(t_0,s+t_0]}\int_{t_0}^{t_1-}\cdots$$
$$\int_{t_0}^{t_{n-1}-} f_{(i_1,...,i_n)}^{(k-j)}(t_0,t_1,...,t_n)dY_{t_n}^{(i_n)}...dY_{t_2}^{(i_2)}dY_{t_1}^{(i_1)}\Big).$$

So we have shown that each term in the Equation (6) can be formulated as the type of sum the representation of the Lemma 3.6 consists of. Now one can finally apply the linearity of the integral mapping to this sum and thus the lemma is proven. □

LEMMA 3.7. *The power of an increment of a Lévy process, $(X_{t+t_0} - X_{t_0})^k$, has a representation of the form*

$$(X_{t+t_0} - X_{t_0})^k = f^{(k)}(t) + \sum_{j=1}^{k} \sum_{(i_1,...,i_j)\in\{1,...,k\}^j} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{j-1}-} h^{(k)}_{(i_1,...,i_j)}(t, t_0, t_1, ..., t_j) dH^{(i_j)}_{t_j} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1},$$

where the $h^{(k)}_{(i_1,...,i_j)}$ are deterministic functions in $L^2(\mathbb{R}^j_+)$.

Proof. Since we have that

$$H^{(i)} = Y^{(i)} + a_{i,i-1}Y^{(i-1)} + ... + a_{i,1}Y^{(1)}, i \geq 1,$$

we get that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1} & 1 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \\ Y^{(3)} \\ \vdots \\ Y^{(i)} \end{pmatrix} = \begin{pmatrix} H^{(1)} \\ H^{(2)} \\ H^{(3)} \\ \vdots \\ H^{(i)} \end{pmatrix}.$$

Now since

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1} & 1 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & 1 \end{pmatrix}$$

is a lower triangular matrix where all the diagonal elements are 1, we get that $\det(A) = 1 \neq 0$. This means that the matrix A is invertible. Because of this the lemma follows from the representation in Lemma 3.6 by switching from the $Y^{(i)}$ to the $H^{(i)}$ by a linear transformation. □

We denote

$$\mathcal{H}^{(i_1,...,i_j)} := \Big\{ F \in L^2(\Omega) : \\ F = \int_0^\infty \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f(t_1,...,t_j) dH^{(i_j)}_{t_j} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1}, f \in L^2(\mathbb{R}^j_+) \Big\},$$

where $\{H^{(i)}, i = 1, 2, ...\}$ is the set of pairwise strongly orthogonal martingales that we established in Section 3.

Definition 3.8. Two multi-indexes $(i_1, ..., i_k)$ and $(j_1, ..., j_l)$ are different if $k \neq l$ or $k = l$ and for some $1 \leq n \leq k = l$ we have that $j_n \neq i_n$. This is denoted by $(i_1, ..., i_k) \neq (j_1, ..., j_l)$.

Proposition 3.9. *If* $(i_1, ..., i_k) \neq (j_1, ..., j_l)$*, then* $\mathcal{H}^{(i_1,...,i_k)} \perp \mathcal{H}^{(j_1,...,j_l)}$ *i.e.* $K \perp L$ *when* $K \in \mathcal{H}^{(i_1,...,i_k)}$ *and* $L \in \mathcal{H}^{(j_1,...,j_l)}$.

Proof. First we prove the case $l = k$ and after that we consider the case $l \neq k$. For this we use induction, starting with the case $l = k = 1$. This means that $j_1 \neq i_1$. We also assume the following representations:

$K = \int_0^\infty f(t_1) dH^{(i_1)}_{t_1}$ and $L = \int_0^\infty g(t_1) dH^{(j_1)}_{t_1}$.

Since we know that $H^{(i_1)} \perp\!\!\!\perp H^{(j_1)}$ and stochastic integrals with respect to strongly orthogonal martingales are orthogonal ([**14**], Theorem 36 and Lemma 2, page 180), we get that $K \perp L$ and therefore the case $l = k = 1$ holds.

For the induction hypothesis we assume that the proposition holds for $1 \leq k = l \leq n-1$ and prove then the case $k = l = n$. We assume the following representations:

$$K = \int_0^\infty \int_0^{t_1-} \cdots \int_0^{t_{n-1}-} f(t_1,...,t_n) dH^{(i_n)}_{t_n} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1} = \int_0^\infty \alpha_{t_1} dH^{(i_1)}_{t_1},$$
$$L = \int_0^\infty \int_0^{t_1-} \cdots \int_0^{t_{n-1}-} g(t_1,...,t_n) dH^{(j_n)}_{t_n} ... dH^{(j_2)}_{t_2} dH^{(j_1)}_{t_1} = \int_0^\infty \beta_{t_1} dH^{(j_1)}_{t_1}.$$

for this we have to consider two cases:

i) $i_1 \neq j_1$
ii) $i = i_1 = j_1$.

In case i) we again apply the fact that $H^{(i_1)} \perp\!\!\!\perp H^{(j_1)}$ and stochastic integrals with respect to strongly orthogonal martingales are orthogonal ([**14**], Theorem 36 and Lemma 2, page 180). This means that $K \perp L$.

For ii) we must have $(i_2, ..., i_n) \neq (j_2, ..., j_n)$ and therefore by the induction hypothesis $\alpha_{t_1} \perp \beta_{t_1}$. From this we get by applying Theorem 2.23 that

$$\begin{aligned}
E[KL] &= \lim_{T\to\infty} E\Big[\int_{(0,T]} \alpha_{t_1} dH^{(i)}_{t_1} \int_{(0,T]} \beta_{t_1} dH^{(i)}_{t_1}\Big] \\
&= \lim_{T\to\infty} E\Big[\int_{(0,T]} \alpha_s\beta_s d\langle H^{(i)}, H^{(i)}\rangle_s\Big] \\
&= \lim_{T\to\infty} \int_{(0,T]} E[\alpha_s\beta_s] d\langle H^{(i)}, H^{(i)}\rangle_s \\
&= \int_0^\infty 0 d\langle H^{(i)}, H^{(i)}\rangle_s = 0,
\end{aligned}$$

because $(\langle H^{(i)}, H^{(i)}\rangle_s)_{s\geq 0}$ is deterministic based on equation (3), which means that $K \perp L$.

Now we are left with the case $k \neq l$, and because of symmetry we can assume that $k < l$. Now we again consider two cases:

i) $(i_1, ..., i_k) \neq (j_1, ..., j_k)$
ii) $(i_1, ..., i_k) = (j_1, ..., j_k)$.

In case i) we can just apply the first part of the proof and that stochastic integrals with respect to strongly orthogonal martingales are orthogonal ([**14**], Theorem 36 and Lemma 2, page 180) for representations

$$\begin{aligned}
K &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} f(t_1, ..., t_k) dH^{(i_k)}_{t_k} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1}, \\
L &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} ... \int_0^{t_{l-1}-} g(t_1, ..., t_k, ..., t_l) dH^{(j_l)}_{t_l} ... dH^{(j_k)}_{t_k} ... dH^{(j_2)}_{t_2} dH^{(j_1)}_{t_1} \\
&= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} \beta(t_1, ..., t_k) dH^{(j_k)}_{t_k} ... dH^{(j_2)}_{t_2} dH^{(j_1)}_{t_1} \text{ a.s.}
\end{aligned}$$

which gives us that $K \perp L$.

For the case ii) we have the following representations:

$$\begin{aligned}
K &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} f(t_1, ..., t_k) dH^{(i_k)}_{t_k} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1}, \\
L &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} ... \int_0^{t_{l-1}-} g(t_1, ..., t_k, ..., t_l) dH^{(j_l)}_{t_l} ... dH^{(j_k)}_{t_k} ... dH^{(j_2)}_{t_2} dH^{(j_1)}_{t_1} \\
&= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{k-1}-} \beta(t_1, ..., t_k) dH^{(i_k)}_{t_k} ... dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1} \text{ a.s.}
\end{aligned}$$

Next we use induction to finish the proof. First lets assume that k=1. This gives us that

$$
\begin{aligned}
K &= \int_0^\infty f(t_1)dH_{t_1}^{(i_1)}, \\
L &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{l-1}-} g(t_1, ..., t_l)dH_{t_l}^{(j_l)}...dH_{t_2}^{(j_2)}dH_{t_1}^{(j_1)} \\
&= \int_0^\infty \beta(t_1)dH_{t_1}^{(i_1)} \text{ a.s.}
\end{aligned}
$$

Now since $\int_0^{t_1-} ... \int_0^{t_{l-1}-} g(t_1, ..., t_l)dH_{t_l}^{(j_l)}...dH_{t_2}^{(j_2)}$ has mean 0 and $f(t_1)$ is a deterministic function we get that

$$
\begin{aligned}
E[KL] &= \lim_{T\to\infty} E\Big[\int_{(0,T]} f(t_1)dH_{t_1}^{(i_1)} \int_{(0,T]} \beta(t_1)dH_{t_1}^{(i_1)}\Big] \\
&= \lim_{T\to\infty} E\Big[\int_{(0,T]} f(s)\beta(s)d\langle H^{(i_1)}, H^{(i_1)}\rangle_s\Big] \\
&= \lim_{T\to\infty} \int_{(0,T]} E[f(s)\beta(s)]d\langle H^{(i_1)}, H^{(i_1)}\rangle_s \\
&= \int_0^\infty 0d\langle H^{(i_1)}, H^{(i_1)}\rangle_s = 0,
\end{aligned}
$$

which proves the case $k = 1$. For the induction hypothesis we assume that when $k \in \{1, 2, ..., n-1\}$ we have that $K \perp L$ and use this to prove the case $k = n$. Now the induction hypothesis gives us that, when

$$
\begin{aligned}
K &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{n-1}-} f(t_1, ..., t_n)dH_{t_n}^{(i_n)}...dH_{t_2}^{(i_2)}dH_{t_1}^{(i_1)} = \int_0^\infty \alpha(t_1)dH_{t_1}^{(i_1)} \text{ and} \\
L &= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{n-1}-} ... \int_0^{t_{l-1}-} g(t_1, ..., t_n, ..., t_l)dH_{t_l}^{(j_l)}...dH_{t_n}^{(j_n)}...dH_{t_2}^{(j_2)}dH_{t_1}^{(j_1)} \\
&= \int_0^\infty \int_0^{t_1-} ... \int_0^{t_{n-1}-} \beta(t_1, ..., t_n)dH_{t_n}^{(i_n)}...dH_{t_2}^{(i_2)}dH_{t_1}^{(i_1)} = \int_0^\infty \gamma(t_1)dH_{t_1}^{(i_1)},
\end{aligned}
$$

we have that $\alpha(t_1) \perp \gamma(t_1)$. From this we can once again get that

$$
\begin{aligned}
E[KL] &= \lim_{T\to\infty} E\Big[\int_{(0,T]} \alpha(t_1)dH_{t_1}^{(i_1)} \int_{(0,T]} \gamma(t_1)dH_{t_1}^{(i_1)}\Big] \\
&= \lim_{T\to\infty} E\Big[\int_{(0,T]} \alpha(s)\gamma(s)d\langle H^{(i_1)}, H^{(i_1)}\rangle_s\Big] \\
&= \lim_{T\to\infty} \int_{(0,T]} E[\alpha(s)\gamma(s)]d\langle H^{(i_1)}, H^{(i_1)}\rangle_s \\
&= \int_0^\infty 0d\langle H^{(i_1)}, H^{(i_1)}\rangle_s = 0
\end{aligned}
$$

and this completes the proof. □

For the next proposition we define a total family.

Definition 3.10. Let $A \subset L^2(\Omega, \mathcal{F})$. A is called a total family if $f = 0$ a.s. for any $f \in L^2(\Omega, \mathcal{F})$, such that $\langle f, g\rangle = \int fgd\mathbb{P} = 0$, for every $g \in A$.

Now we use this definition to get a proposition, so that we are ready to prove the main theorem.

Proposition 3.11. *Let*

$$\mathcal{P} := \{X_{t_1}^{k_1}(X_{t_2} - X_{t_1})^{k_2} ... (X_{t_n} - X_{t_{n-1}})^{k_n} : \\ n \geq 0, 0 \leq t_1 < t_2 < ... < t_n, k_1, ..., k_n \geq 1\}.$$

Then $\mathcal{P}$ is a total family in $L^2(\Omega, \mathcal{F})$.

Proof. We need to show that $\mathcal{P}$ is a total family in $L^2(\Omega, \mathcal{F})$. We do this by taking a $Z \in L^2(\Omega, \mathcal{F})$, such that $Z \perp \mathcal{P}$, and showing that $Z = 0$ a.s.

Let $\epsilon > 0$. We recall that $\mathcal{F}$ is the σ-algebra generated by the Lévy process X and because of this we can apply Theorem 2.29. So there exists a set $\{0 < s_1 < ... < s_m\}$ and a square integrable random variable $Z_\epsilon \in L^2(\Omega, \sigma(X_{s_1}, X_{s_2}, ..., X_{s_m}))$ such that

$E(Z - Z_\epsilon)^2 < \epsilon$.

This means that there exists a Borel function f such that

$Z_\epsilon = f_\epsilon(X_{s_1}, (X_{s_2} - X_{s_1}), ..., (X_{s_m} - X_{s_{m-1}}))$.

Since $\int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\nu(dx) < \infty$ applies for the Lévy measure ν, the polynomials are dense in $L^2(\mathbb{R}, \mathbb{P} \circ X_t^{-1})$ for each $t > 0$ ([**6**], Theorem 3.2.18, page 69). This means that we can approximate $Z_\epsilon \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with polynomials in $(X_t - X_s)$, so that $E[ZZ_\epsilon] = 0$, since $Z \perp \mathcal{P}$. From this and, by Hölder's inequality, we get that

$|E[Z(Z - Z_\epsilon)]| \leq \sqrt{E[Z^2]E[(Z - Z_\epsilon)^2]}$, which finally gives us

$E[Z^2] = E[Z^2] - E[ZZ_\epsilon] = E[Z(Z - Z_\epsilon)] \leq \sqrt{E[Z^2]E[(Z - Z_\epsilon)^2]} < \sqrt{\epsilon E[Z^2]}$.

Now it's clear that $\epsilon E[Z^2] \to 0$, when $\epsilon \to 0$, and thus $Z = 0$ a.s. □

3.4. Proof of Theorem 3.5. Now we are prepared to give a proof for the main theorem of this thesis.

From Proposition 3.11 we get that $\mathcal{P}$ is a total family in $L^2(\Omega, \mathbb{R})$. This means that it is sufficient for us to show that the theorem applies to every element of $\mathcal{P}$.

Lemma 3.7 implies that any product $(X_s - X_t)^k (X_v - X_u)^l$, where $k, l \geq 1$ and $0 \leq t < s \leq u < v$, can be represented as a sum of products of the form AB where

$$A = \int_t^s \int_t^{t_1-} \dots \int_t^{t_{n-1}-} h^{(k)}_{(i_1,\dots,i_n)}(t, t_1, \dots, t_n) dH^{(i_n)}_{t_n} \dots dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1}$$

and

$$B = \int_u^v \int_u^{u_1-} \dots \int_u^{u_{m-1}-} h^{(l)}_{(j_1,\dots,j_m)}(u, u_1, \dots, u_m) dH^{(j_m)}_{u_m} \dots dH^{(j_2)}_{u_2} dH^{(j_1)}_{u_1}.$$

From this we get that

$$\begin{aligned} AB = & \int_u^v \int_u^{u_1-} \dots \int_u^{u_{m-1}-} \int_t^s \int_t^{t_1-} \dots \int_t^{t_{n-1}-} h^{(l)}_{(j_1,\dots,j_m)}(u, u_1, \dots, u_m) \\ & \times h^{(k)}_{(i_1,\dots,i_n)}(t, t_1, \dots, t_n) dH^{(i_n)}_{t_n} \dots dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1} dH^{(j_m)}_{u_m} \dots dH^{(j_2)}_{u_2} dH^{(j_1)}_{u_1} \\ = & \int_0^\infty \int_0^{u_1-} \dots \int_0^{u_{m-1}-} \int_0^{u_m-} \int_0^{t_1-} \dots \int_0^{t_{n-1}-} 1_{(u,v]}(u_1) \\ & \times 1_{(u,u_1)}(u_2) \dots 1_{(u,u_{m-1})}(u_m) 1_{(t,s]}(t_1) 1_{(t,t_1)}(t_2) \dots 1_{(t,t_{n-1})}(t_n) \\ & \times h^{(l)}_{(j_1,\dots,j_m)}(u, u_1, \dots, u_m) h^{(k)}_{(i_1,\dots,i_n)}(t, t_1, \dots, t_n) \\ & \times dH^{(i_n)}_{t_n} \dots dH^{(i_2)}_{t_2} dH^{(i_1)}_{t_1} dH^{(j_m)}_{u_m} \dots dH^{(j_2)}_{u_2} dH^{(j_1)}_{u_1}. \end{aligned}$$

Now we note that this is an integral of the form that is presented in the Theorem 3.4. This means that the product $(X_s - X_t)^k (X_v - X_u)^l$ can be represented as a sum of integrals which corresponds to the form shown in the Theorem 3.4. We also recall that every element of $\mathcal{P}$ is of the form $X^{k_1}_{t_1}(X_{t_2} - X_{t_1})^{k_2} \dots (X_{t_n} - X_{t_{n-1}})^{k_n}$, where $n \geq 0$, $0 \leq t_1 < t_2 < \dots < t_n$, $k_1, \dots, k_n \geq 1$, which means that they can also be represented in the desired way and this completes the proof.

4. Itô's chaos decomposition

Now we move on to other types of decompositions of a Lévy-Itô space and compare them to the result of Nualart and Schoutens. We start with Itô's decomposition ([**12**], Theorem 2, page 257) using multiple Wiener integrals. In this section we have a change of setting from $t \in [0, \infty)$ to finite time $t \in [0, T]$.

4.1. Notation and preliminaries. We start by going over the necessary notation for this chapter and defining the multiple Wiener integrals, so that we can introduce the decomposition from [**12**]. First we define the compensated Poisson random measure $\tilde{N} := N - \lambda \otimes \nu$ on the ring $E \in \mathcal{B}([0,T] \times \mathbb{R})$ with $\mathbb{m}(E) < \infty$. Next we formulate the Lévy-Itô-decomposition ([**1**], Theorem 2.4.16, page 126).

THEOREM 4.1 (The Lévy-Itô decomposition). *For a Lévy process X there exists a constant $\beta \in \mathbb{R}$, a Brownian motion B and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ such that, for each $t \geq 0$ and some $\sigma \geq 0$,*

$$X(t) = \beta t + \sigma B(t) + \int_{|x|<1} x\tilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx).$$

The triplet (β, σ^2, ν), where σ^2 is the variance function of B and ν is the Lévy measure associated with X, is called the characteristics of X.

Next we use the Lévy measure ν introduced in Section 3 and $\sigma \geq 0$, which is the parameter of the Brownian motion part of X, to define two σ-finite measures

$$d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x) \text{ and } d\mathbb{m}(t,x) := d(\lambda \otimes \mu)(t,x)$$

,

on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}([0,T] \times \mathbb{R})$ respectively. In this thesis we assume that $\sigma \equiv 0$ which gives us $d\mu(x) = x^2 d\nu(x)$.

We show that the measure μ is σ-finite. First we recall that a measure μ on $\mathcal{B}(\mathbb{R})$ is called σ-finite given that there exist sets $B_n \in \mathcal{B}(\mathbb{R})$, such that $\cup_{n\in\mathbb{N}} B_n = \mathbb{R}$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$. We set $B_n = [-n,n]$, which means that

$$\cup_{n\in\mathbb{N}} B_n = \cup_{n\in\mathbb{N}}[-n,n] = \mathbb{R}$$

and for $n \in \mathbb{N}$ we have that

$$\begin{aligned}\int_{[-n,n]\setminus\{0\}} d\mu(x) &= \int_{[-n,n]} x^2 d\nu(x) = \int_{[-n,n]} (x^2 \wedge n^2) d\nu(x) \\ &\leq \int_{\mathbb{R}} (n^2x^2 \wedge n^2) d\nu(x) = n^2 \int_{\mathbb{R}} (x^2 \wedge 1) d\nu(x) < \infty.\end{aligned}$$

This gives us $\mu([-n,n]) < \infty$ for every $n \in \mathbb{N}$.

For an $E \in \mathcal{B}([0,T] \times \mathbb{R})$ with $\mathbb{m}(E) < \infty$ we introduce

$$M(E) \coloneqq \lim_{N\to\infty} \int_{E\cap([0,T]\times\{\frac{1}{N}<|x|<N\})} x d\tilde{N}(t,x),$$

where the limit is taken in L_2. We set

$L_2^n \coloneqq L_2\big(([0,T]\times\mathbb{R})^n, \mathcal{B}(([0,T]\times\mathbb{R})^n), \mathbb{m}^{\otimes n}\big)$.

Now let $E_1, ..., E_n \in \mathcal{B}([0,T]\times\mathbb{R})$ be pairwise disjoint with $\mathbb{m}(E_i) < \infty$ and

$f_n((t_1,x_1),...,(t_n,x_n)) \coloneqq 1_{E_1}(t_1,x_1)\cdots 1_{E_n}(t_n,x_n)$,

then we define the multiple integral by

$I_n(f_n) \coloneqq M(E_1)\cdots M(E_n)$.

We also note that if

$f_n((t_1,x_1),...,(t_n,x_n)) = f_n((t_{\pi(1)},x_{\pi(1)}),...,(t_{\pi(n)},x_{\pi(n)}))$

for all $(t_1,x_1),...,(t_n,x_n)$ and $\pi \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the set of all permutations of $\{1,...,n\}$, then the kernel f_n is called symmetric. Also the symmetrisation of an $f_n \in L_2^n$ is given by

$\tilde{f}_n((t_1,x_1),...,(t_n,x_n)) \coloneqq \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} f_n((t_{\pi(1)},x_{\pi(1)}),...,(t_{\pi(n)},x_{\pi(n)}))$.

Next we take a function

$$g_n = \sum_{l=1}^{L} \alpha_l 1_{E_1^l\times\cdots\times E_n^l},$$

where each $E_1^l,...,E_n^l \in \mathcal{B}([0,T]\times\mathbb{R})$, with $l = 1,...,L$, is pairwise disjoint and $\alpha_l \in \mathbb{R}$ for every $l = 1,...,L$. For such a function g_n we get that $I_n(g_n)$ is well defined by linearity. For such g_n, we also have the following properties: $I_n(g_n) = I_n(\tilde{g}_n)$, where $\tilde{g}_n$ is the symmetrisation of g_n, and $E|I_n(\tilde{g}_n)|^2 = n!||\tilde{g}_n||^2_{L_2^n}$.

Now we shall define $Diag(n,s)$.

Definition 4.2. $Diag(n,s)$, with $n, s \in \mathbb{N}$, is the set of all n-cuboids with edges $(\frac{k-1}{s}T, \frac{k}{s}T]$, where $k \in \{1,2,...,s\}$ and at least two edges are the same.

We denote with S_s^n the set of $\mathcal{B}\big(([0,T]\times\mathbb{R})^n\big)$-measurable functions $f_n : \big([0,T]\times\mathbb{R}\big)^n \to \mathbb{R}$, such that f_n takes only finitely many values, $f_n((t_1,x_1),...,(t_n,x_n)) = 0$ if $(t_1,...,t_n) \in Diag(n,s)$ and $f_n((t_1,x_1),...,(t_n,x_n)) = f_n((s_1,x_1),...,(s_n,x_n))$ if s_i and

t_i belong to the same $(\frac{k-1}{s}T, \frac{k}{s}T]$ for every $i = 1, ..., n$. We can expand the definition of I_n for such an S_s^n by the above properties for the symmetrisation and the multiple integral. Our next goal is to prove the following lemma.

LEMMA 4.3. *For Diag(n, s), with $n, s \in \mathbb{N}$, we have that*

$$\mathbb{m}^{\otimes n}\Big(Diag(n, s) \times \mathbb{R}^n\Big) \to 0,$$

when $s \to \infty$.

Proof. We prove this lemma by taking a large enough s, for $\epsilon > 0$, so that we have the approximation $|\frac{T}{s}| < \epsilon$. We also note that there are $\binom{n}{2}$ possibilities to arrange 2 equal edges of the n-cuboid and the remaining $n-2$ edges can be any $(\frac{k-1}{s}T, \frac{k}{s}T]$, with $k \in \{1, 2, ..., s\}$, which gives us the approximation

$$\mathbb{m}^{\otimes n}(Diag(n, s) \times \mathbb{R}^n) \leq \mu(\mathbb{R})^n \binom{n}{2} s \Big(\frac{T}{s}\Big)^2 \Big(T\Big)^{n-2} = \frac{1}{s}\binom{n}{2} T^n \mu(\mathbb{R})^n.$$

With this approximation one can see that $\mathbb{m}^{\otimes n}\Big(Diag(n, s) \times \mathbb{R}^n\Big) \to 0$, when $\epsilon \to 0$.

□

From this lemma one can see that the family $\cup_{s>0} S_s^n$ is dense in L_2^n, which in turn lets us extend the multiple integral into $I_n : L_2^n \to L_2(\mathcal{F}^X)$, where $\mathcal{F}^X$ is the natural filtration.

Next we introduce a proposition cowering some properties for the multiple integral ([**2**], page 8614).

PROPOSITION 4.4. *Let $f_n \in L_2^n$ and $f_m \in L_2^m$. Then one has that*

(1) *$I_n(f_n)$ and $I_m(f_m)$ are orthogonal for any kernels f_n and f_m, provided that $n \neq m$,*
(2) *$I_n(f_n) = I_n(\tilde{f}_n)$ a.s.,*
(3) *$\|I_n(\tilde{f}_n)\|_{L_2(\mathcal{F}^X)} = \sqrt{n!}\|\tilde{f}_n\|_{L_2^n}$.*

Before we move on we prove one lemma.

LEMMA 4.5. *Let*

$$S := \{Y = f(N(E_1), ..., N(E_n)) : n = 1, 2, ..., \quad E_i \text{ are pairwise disjoint and } f \text{ is bounded and continuous}\}.$$

Then S is dense in $L_2(\mathcal{F}^X)$.

Proof. Let

$$\mathcal{F}_T := \mathcal{F}_T^X \cup \mathcal{N} = \sigma(N(E) : \mathbb{m}(E) < \infty) \cup \mathcal{N},$$

where $\mathcal{N}$ is the family of empty sets.

We also introduce the Π-system

$$\begin{aligned}\mathbf{P} := \{&\{N(E_1) \in A_1, ..., N(E_n) \in A_n\} : n \in \mathbb{N},\\ &\mathbb{m}(E_i) < \infty \text{ and } A_i \in \mathcal{B}(\mathbb{R}) \text{ for every } i = 1, ..., n\}.\end{aligned}$$

We know that the linear span of step functions

$$\text{span}\Big\{ \sum_{i=1}^{I} \alpha_i 1_{p_i} : \alpha_i \in \mathbb{R}, I \in \mathbb{N}, p_i \in \mathbf{P} \Big\}$$

is dense in $\mathcal{L}_2(\mathcal{F}_T)$ ([**10**], Theorem A.9, page 101). We also note that every step function is a linear combination of functions 1_p, where $p \in \mathbf{P}$. This means that it is enough to show that, for any $\epsilon > 0$ and $p = \{N(E_1) \in A_1, ..., N(E_n) \in A_n\} = \{(N(E_1), ..., N(E_n)) \in A_1 \times \cdots \times A_n\} \in \mathbf{P}$, there exists a $f \in C_b$, where C_b denotes continuous and bounded functions, such that

$$E|1_p - f(N(E_1), ..., N(E_n))|^2 < \epsilon^2.$$

We denote

$$\mu_n := \text{law}(N(E_1), ..., N(E_n)) \in \mathcal{M}_1^+(\mathbb{R}^n),$$

which means that μ_n is outer regular ([**3**], Theorem 1.1, page 7) with respect to open sets, that is

$$\mu_n(B) = \inf\{\mu_n(C) : C \supseteq B,\ C \text{ is open}\}.$$

Because of the outer regularity, there exists an open set $G \supseteq A_1 \times \cdots \times A_n$, with $|\mu_n(G) - \mu_n(A_1 \times \cdots \times A_n)| < \epsilon$, and it is sufficient to have

$$E|1_{\{(N(E_1),...,N(E_n))\in G\}} - f(N(E_1),...,N(E_n))|^2 < \epsilon^2$$

for some $f \in C_b$ to complete the proof. For $G = \mathbb{R}^n$ this is clear so let us assume $G \neq \mathbb{R}^n$.

Let

$$F_m := \Big\{x \in \mathbb{R}^n : x \in G, \mathrm{d}(x, \partial G) \geq \frac{1}{m}\Big\},$$

where $\mathrm{d}(x, \partial G) = \inf\{\mathrm{d}(x, y) = |x - y| : y \in \partial G\}$ and ∂G denotes the boundary of G, which means that $y \in \partial G$ if and only if $\{x \in \mathbb{R}^n : \mathrm{d}(x, y) < \epsilon\} \cap G \neq \emptyset$ and $\{x \in \mathbb{R}^n : \mathrm{d}(x, y) < \epsilon\} \cap G^c \neq \emptyset$ for every $\epsilon > 0$.

Next we want to apply Urysohn's lemma on sets F_m and G^c. This means that we need to show that F_m and G^c are closed and $F_m \cap G^c = \emptyset$. We start by showing that F_m is closed.

If $x \in G$ such that $\mathrm{d}(x, \partial G) < \frac{1}{m}$, then there exists an $\epsilon > 0$ such that $\mathrm{d}(x, \partial G) + \epsilon < \frac{1}{m}$. This means that $\{y \in \mathbb{R}^n : \mathrm{d}(x, y) < \epsilon\} \cap F_m = \emptyset$, which implies that $x \notin \partial F_m$.

On the other hand, if $x \in G$ such that $\mathrm{d}(x, \partial G) > \frac{1}{m}$, then there exists an $\epsilon > 0$ such that $\mathrm{d}(x, \partial G) - \epsilon > \frac{1}{m}$. From this we get that $\{y \in \mathbb{R}^n : \mathrm{d}(x, y) < \epsilon\} \cap F_m^c = \emptyset$, which implies again that $x \notin \partial F_m$.

Now one can see that if $x \in \partial F_m$ then $\mathrm{d}(x, \partial G) = \frac{1}{m}$. This implies that $\partial F_m \subset F_m$, which in turn indicates that F_m is closed.

We also note that G^c is closed since G is by definition open. For any $x \in F_m$ we have that $x \in G$, and that is why $F_m \cap G^c = \emptyset$. Now we can apply Urysohn's lemma to get a function $f_m \in C_b$ such that $f_m(x) = 1$ for every $x \in F_m$ and $f_m(y) = 0$ for every $y \in G^c$.

Now to finish the proof we have to show that $\cup_{m=1}^{\infty} F_m = G$. By definition of F_m we have that $\cup_{m=1}^{\infty} F_m \subset G$. Since G is open we have that $\mathrm{d}(x, \partial G) > 0$ for every $x \in G$. This also means that there is an $M \in \mathbb{N}$ such that $\mathrm{d}(x, \partial G) > \frac{1}{M}$, which means that $x \in F_M$. From this we get that $G \subset \cup_{m=1}^{\infty} F_m$ and finally $\cup_{m=1}^{\infty} F_m = G$.

From here one can see that by dominated convergence

$$\lim_{m\to\infty} f_m(x) = 1_G(x).$$

This means that for every $\epsilon > 0$ there is an $m_\epsilon \in \mathbb{N}$ such that

$$E|1_{\{(N(E_1),...,N(E_n))\in G\}} - f_{m_\epsilon}(N(E_1),...,N(E_n))|^2 < \epsilon^2,$$

which completes the proof.

□

4.2. Itô's decomposition. Now we have the necessary definition for multiple integrals, so we can move forward to Itô´s decomposition. For a multiple Wiener integral of nth degree denoted by I_n we get the following theorem ([**12**], Theorem 2, page 257):

THEOREM 4.6. *For the space $L_2(\mathcal{F}^X)$ and $L_2^n := L_2(\mathbb{R}^n, \mathrm{m}^{\otimes n})$ it holds that*

$$L_2(\mathcal{F}^X) = \oplus_{n=0}^{\infty} I_n(L_2^n).$$

Before we can prove this theorem we need some lemmas, which we present next.

LEMMA 4.7. *Let*

$$\begin{aligned} R := \{&N(E_1)^{p_1} \cdots N(E_n)^{p_n} : n \in \{1,2,...\}, p_i \in \{0,1,2,...\}, \\ &E_i \in [0,T] \times \mathbb{R} \text{ are pairwise disjoint with } \lambda \otimes \nu(E_i) < \infty\}. \end{aligned}$$

Then R is a total family in $L_2(\mathcal{F}^X)$.

Proof. First we notice that every element of R has a finite norm and thus belongs to $L_2(\mathcal{F}^X)$. Lemma 4.5 gives us that

$$\begin{aligned} S = \{Y =&f(N(E_1),...,N(E_n)) : n = 1,2,..., \\ &E_i \text{ are pairwise disjoint and } f \text{ is bounded and continuous}\} \end{aligned}$$

is dense in $L_2(\mathcal{F}^X)$. This means that it is enough to show, that if $F \in S$, with $E(FY) = 0$ for every $Y \in R$, then $F = 0$ a.s. We will show this only for $n = 1$, which means that $F = f(N(E))$. Let σ_1 denote the distribution of $N(E)$ and assume, for all $p = 0,1,2,...$, that

$$E(FN(E)^p) = \int_{\mathbb{R}} f(x)x^p d\sigma_1(x) = 0. \tag{7}$$

We first show that the integrals do exist. Because σ_1 is a Poisson distribution with parameter λ, we have that

$$\int_{\mathbb{R}} e^{|2tx|} d\sigma_1(x) = \sum_{k=0}^{\infty} e^{2k} \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} \frac{(e^2\lambda)^k}{k!} e^{-\lambda} < \infty.$$

Combining this with Hölder's inequality we get that, for $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{|tx|} |f(x)| d\sigma_1(x) \leq \Big(\int_{\mathbb{R}} |f(x)|^2 d\sigma_1(x) \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} e^{2|tx|} d\sigma_1(x) \Big)^{\frac{1}{2}} < \infty,$$

which shows that the integrals exist.

By using Lebesgue's dominated convergence theorem and equation (7), we get that

$$\int_{\mathbb{R}} f(x) e^{itx} d\sigma_1(x) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} f(x) \frac{1}{k!} (it)^k x^k d\sigma_1 = 0,$$

for all $t \in \mathbb{R}$. This gives us that $F = f(N(E)) = 0$ a.s. by the uniqueness theorem for Fourier transforms.

□

Now we have another lemma to prove.

Lemma 4.8. *Let*

$$\begin{aligned} P := \{ & N(E_1) \cdots N(E_n) : n = 1, 2, ..., \\ & E_i \in \mathcal{B}([0,T] \times \mathbb{R}) \text{ are pairwise disjoint with } (\lambda \otimes \nu)(E_i) < \infty \}. \end{aligned}$$

Then P is a total family in $L_2(\mathcal{F}^X)$.

Proof. By Lemma 4.7 it is enough to show that, for any $Y \in R$, we have that Y belongs to the closed linear span of P. For Y we assume the form $Y = N(E_1)^{p_1} \cdots N(E_n)^{p_n}$, with $n \in \{1, 2, ...\}$, $p_i \in \{1, 2, ...\}$ and $E_i \in \mathcal{B}([0,T] \times \mathbb{R})$ are pairwise disjoint with $(\lambda \otimes \nu)(E_i) < \infty$. We take a subdivision $\{F_i\}$, $i = 1, ..., s$ of $\{E_i\}$, $i = 1, ..., n$, such that $(\lambda \otimes \nu)(F_i) < \epsilon$, where $\epsilon > 0$, and $E := \cup_{i=1}^n E_i = \cup_{i=1}^s F_i$. From here we get the expression

$$Y = N(E_1)^{p_1} \cdots N(E_n)^{p_n} = \sum N(F_{i(1)})^{j_1} \cdots N(F_{i(r)})^{j_r} \text{ a.s.,}$$

with $i(1) < i(2) < ... < i(r)$.

We know that $N(F_i) \in \{0, 1, 2, ...\}$, which gives us

$$Y \geq \sum N(F_{i(1)}) \cdots N(F_{i(r)}) \equiv Y_\epsilon.$$

Now, since there are $\binom{n}{2}$ possibilities to arrange 2 equal elements over n positions and the remaining $n-2$ positions can hold any value out of $\{1, 2, ..., s\}$, we get that

$$P(Y \neq Y_\epsilon) \leq \binom{n}{2} \sum_{i=1}^{s} (\lambda \otimes \nu)(F_i)^2 \Big(\sum_{i=1}^{s} (\lambda \otimes \nu)(F_i) \Big)^{n-2} \leq \binom{n}{2} \epsilon (\lambda \otimes \nu)(E)^{n-1}.$$

This means that $Y_\epsilon \to Y$ in probability as $\epsilon \to 0$. Consequently we can take a sequence $(\epsilon_n)_{n \in \mathbb{N}}$, such that $\epsilon_n \to 0$ as $n \to \infty$, so that $Y_{\epsilon_n} \to Y$ for almost every ω. Since we also have that $0 \leq Y_{\epsilon_n} \leq Y$ and $Y \in L^2(\mathcal{F}^X)$, we can see that $||Y_{\epsilon_n} - Y||_{L_2} \to 0$ and therefore the lemma is proved.

□

We prove one more lemma before we move on to the proof of the Itô's decomposition.

LEMMA 4.9. *Let $\mathcal{P} \subseteq H_0 \subseteq H$, $\mathcal{P}$ total in H and H_0 be a closed subspace of H. Then $H_0 = H$.*

Proof. Let us assume that $X \in H \setminus H_0$ with $||X|| = 1$ and $X \perp\!\!\!\perp H_0$. This means that $X \perp\!\!\!\perp \mathcal{P}$, because $\mathcal{P} \subseteq H_0$. This would mean that $\mathcal{P}$ is not total in H, so such an X cannot exist and therefore $H = H_0$. □

4.3. Proof of Theorem 4.6. Now we are prepared to prove Itô's decomposition by showing that $L_2(\mathcal{F}^X) = \oplus_{n=0}^{\infty} I_n(L_2^n)$. Since $\oplus_{n=0}^{\infty} I_n(L_2^n) \subseteq L_2(\mathcal{F}^X)$ are Hilbert spaces, it is enough to show that $\oplus_{n=0}^{\infty} I_n(L_2^n)$ is closed and for any $f \in L_2(\mathcal{F}^X)$, with $f \perp\!\!\!\perp g$ for every $g \in \oplus_{n=0}^{\infty} I_n(L_2^n)$, we have that $f = 0$ a.s. Because of Lemma 4.8 and Lemma 4.9, it suffices to show that $\oplus_{n=0}^{\infty} I_n(L_2^n)$ is closed and $P \subseteq \oplus_{n=0}^{\infty} I_n(L_2^n)$.

First we show that $I_n(L_2^n)$ is closed. We note that L_2^n is closed and the subspace of symmetric functions is also closed. Now since we have the isometry

$$\|I_n(\tilde{f}_n)\|_{L_2(\mathcal{F}^X)} = \sqrt{n!} \|\tilde{f}_n\|_{L_2^n},$$

we get that the subspace $I_n(L_2^n)$ of $L_2(\Omega)$ is also closed.

Next we show that this leads to $\oplus_{n=0}^{\infty} I_n(L_2^n)$ being closed. Let $H \coloneqq (H_n)_{n=0}^{\infty}$ be a sequence of separable Hilbert spaces. We equip this space with the norm

$$||(X_n)_{n=0}^{\infty}||_H \coloneqq (\sum_{n=0}^{\infty} ||X_n||_{H_n}^2)^{\frac{1}{2}}$$

and inner product

$$\langle (X_n)_{n=0}^{\infty}, (Y_n)_{n=0}^{\infty} \rangle_H \coloneqq \sum_{n=0}^{\infty} \langle X_n, Y_n \rangle_{H_n}.$$

We also note that $(X_n)_{n=0}^{\infty} \in H$ if and only if $||(X_n)_{n=0}^{\infty}||_H < \infty$. Let $((X_n^k)_{n=0}^{\infty})_{k=1}^{\infty}$ be a Cauchy sequence in H. This means that for every $\epsilon > 0$ there exists a $k_\epsilon \in \mathbb{N}$ such that $||(X_n^k)_{n=0}^{\infty} - (X_n^l)_{n=0}^{\infty}||_H < \epsilon$ for every $k, l \geq k_\epsilon$. Now to show that H is closed, which infers that $\oplus_{n=0}^{\infty} I_n(L_2^n)$ is closed, we need to show that $((X_n^k)_{n=0}^{\infty})_{k=1}^{\infty}$ has a limit in H.

Now let $\epsilon > 0$ and $k_\epsilon \in \mathbb{N}$ such that $||(X_n^k)_{n=0}^{\infty} - (X_n^l)_{n=0}^{\infty}||_H < \epsilon$ for every $k, l \geq k_\epsilon$. This means that $||X_{n_0}^k - X_{n_0}^l||_{H_{n_0}} \leq ||(X_n^k)_{n=0}^{\infty} - (X_n^l)_{n=0}^{\infty}||_H < \epsilon$, for every $k, l \geq k_\epsilon$, and since H_{n_0} is a separable Hilbert space, we also know that $X_{n_0} \coloneqq \lim_{k\to\infty} X_{n_0}^k$ in H_{n_0}. From this and $\epsilon^2 \geq \sum_{n=0}^{\infty} ||X_n^k - X_n^l||_{H_n} \geq \sum_{n=0}^{N} ||X_n^k - X_n^l||_{H_n}$ we get that $\sum_{n=0}^{N} ||X_n^k - X_n^l||_{H_n} \to \sum_{n=0}^{N} ||X_n^k - X_n||_{H_n}$ when $l \to \infty$. Now we see that for every $N \geq 0$ and $k \geq k_\epsilon$ we have that $\sum_{n=0}^{N} ||X_n^k - X_n||_{H_n} \leq \epsilon^2$, and with $N \to \infty$ we get that $\sum_{n=0}^{\infty} ||X_n^k - X_n||_{H_n} \leq \epsilon^2$ for every $k \geq k_\epsilon$. We also have that $(X_n^k)_{n=0}^{\infty} \in H$ and $(X_n^k - X_n)_{n=0}^{\infty} \in H$, which means that $(X_n)_{n=0}^{\infty} \in H$. This gives us that $(X_n^k)_{n=0}^{\infty} \to (X_n)_{n=0}^{\infty}$, when $k \to \infty$, in H.

Now for the final part we define $\tilde{L}_2^n$ as the space of functions, such that if $f_n \in L_2^n$, then $x_1 \cdots x_n f_n \cong \tilde{f}_n \in \tilde{L}_2^n$. We also denote

$$\begin{aligned}
&\|f_n\|_{L_2^n} \\
&\quad = \int_{(0,T]\times\mathbb{R}} \cdots \int_{(0,T]\times\mathbb{R}} |f_n(t_1, ..., t_n, x_1, ..., x_n)|^2 x_1^2 \cdots x_n^2 \\
&\qquad \times d(\lambda \otimes \nu)(t_1, x_1) \cdots d(\lambda \otimes \nu)(t_n, x_n) \\
&\quad = \int_{(0,T]\times\mathbb{R}} \cdots \int_{(0,T]\times\mathbb{R}} |\tilde{f}_n(t_1, ..., t_n, x_1, ..., x_n)|^2 d(\lambda \otimes \nu)(t_1, x_1) \cdots d(\lambda \otimes \nu)(t_n, x_n) \\
&\quad = \|\tilde{f}_n\|_{\tilde{L}_2^n},
\end{aligned}$$

which means that $\oplus_{n=0}^{\infty} I_n(L_2^n) = \oplus_{n=0}^{\infty} \tilde{I}_n(\tilde{L}_2^n)$. We also have that $\tilde{I}_n(1_{E_1 \times \cdots \times E_n}) = \tilde{N}(E_1) \cdots \tilde{N}(E_n) = (N(E_1) - c_1) \cdots (N(E_n) - c_n)$ with $c_i \coloneqq EN(E_i)$. This shows that $\tilde{I}_n(1_{E_1\times\cdots\times E_n})$ is a linear combination of terms $Y \in P$, where P is as in Lemma 4.8. The proof is completed by showing that $P \subseteq \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$ with an induction over n.

For $n = 1$ we see that $\tilde{I}_1(1_{E_1}) = \tilde{N}(E_1) = N(E_1) - c_1$ which gives us that $N(E_1) \in \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$. Next we assume that $N(E_1)\cdots N(E_{n-1}) \in \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$ for $n-1$, with $n \geq 2$, as an induction hypothesis. Also $\tilde{I}_n(1_{E_1\times\cdots\times E_n}) = \tilde{N}(E_1)\cdots\tilde{N}(E_n) = (N(E_1)-c_1)\cdots(N(E_n)-c_n) = N(E_1)\cdots N(E_n)+C$, where C is a linear combination of terms $Y = N(E_{i_1})\cdots N(E_{i_j}) \in P$ with $j = 1, ..., n-1$ and $i_k \in \{1, 2, ..., n\}$, with $i_1, ..., i_j$ being pair-wise different. When we apply the induction hypothesis to this we get that $C \in \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$, and since $\tilde{I}_n(1_{E_1\times\cdots\times E_n}) \in \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$, we also get that $N(E_1)\cdots N(E_n) \in \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$. This means that $P \subseteq \oplus_{n=0}^{\infty}\tilde{I}_n(\tilde{L}_2^n)$ which completes the proof.

5. General orthogonal decomposition of the Lévy-Itô space in terms of an orthonormal basis

In this section we examine another orthogonal decomposition for the Lévy-Itô space. We will derive this decomposition, which will be more similar to the orthogonal decomposition from Section 3, using Itô's decomposition from Theorem 4.6.

First we note that we have a change of setting in this section, where we are operating on a finite time interval $t \in [0, 1]$ and otherwise use the notation from Section 4.1. In this section we also only consider functions of the form $F = h(X_1) \in L_2(\mathcal{F}^X)$ while constructing the orthogonal decomposition. Now we prove the following lemma.

Lemma 5.1. *The space $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is separable.*

Proof. We recall that a space is called separable if it contains a dense, countable subset. Every open set $G \subset \mathbb{R}$ can be represented as a countable union of open intervals of the form (a, b), for $a, b \in \mathbb{Q}$. This means that the countable set

$$\mathcal{G} := \{G \subset \mathbb{R} : G = \cup_{i=1}^{n}(a_i, b_i),\ \text{where } a_i, b_i \in \mathbb{Q} \text{ for all } i \in \{1, 2, ..., n\}\}$$

is a dense subset of the family of all open sets of $\mathbb{R}$. This means that, for any open $B \subset \mathbb{R}$ and $\epsilon > 0$, one has a set $G \in \mathcal{G}$ for which $\mu(B - G) \leq \epsilon$. Now we get that

$$S := \{\sum_{i=1}^{n}\alpha_i 1_{G_i} : G_i \in \mathcal{G}, \alpha_i \in \mathbb{Q}, n \in \mathbb{N}\} \subseteq L_2(\mathbb{R}, \mu)$$

is countable. Next we note that the measure μ is outer regular.

Since any Borel function $f : \mathbb{R} \to \mathbb{R}$, with $\int_{\mathbb{R}} |f|^2 d\mu < \infty$, can be split into negative- and non-negative parts we can assume for the following part of our proof that $f \geq 0$. From the Proposition 2.31 we get that there exists a series of step functions $(h_n)_{n\in\mathbb{N}}$ such that

$0 \leq h_n \to f$ on $\mathbb{R}$.

Since f and h_n, $n \in \mathbb{N}$, are measurable, we can modify each

$h_n = \sum_{i=1}^{K_n} \alpha_i^{(n)} \chi_{A_i^{(n)}}$,

where $K_n \in \mathbb{N}$, $\alpha_1^{(n)}, ..., \alpha_k^{(n)} \in \mathbb{R}$ and $A_1^{(n)}, ..., A_n^{(n)} \in \mathcal{B}(\mathbb{R})$, into

$f_n = \sum_{i=1}^{K_n} \alpha_i^{(n)} \chi_{A_i^{(n)} \setminus D_i^{(n)}} + \sum_{i=1}^{K_n} \delta_i^{(n)} \chi_{D_i^{(n)}}$,

where $D_i^{(n)} := \{x \in A_i^{(n)} : h_n(x) > f(x)\}$ and $\delta_i^{(n)} := \inf_{x\in D_i^{(n)}} f(x)$. This means that $0 < f_n(x) \leq f(x)$, for all $x \in \mathbb{R}$. Now by Dominated Convergence ([**7**], Theorem 6.5.2, page 89) we have that

$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f|^2 d\mu = 0$,

which is well defined since

$|f_n - f|^2 \leq 2(|f_n|^2 + |f|^2) \leq 4|f|^2 \in L_1(\mathbb{R}, \mu)$.

Now we recall that f_n are step functions for all $n \in \mathbb{N}$, so they can be represented in the form $f_n = \sum_{l=1}^{N_n} \beta_l 1_{B_l}$. Also since μ is outer regular there exists for every $\epsilon > 0$ an open $C_l^\epsilon \supseteq B_l$ such that

$\mu(C_l^\epsilon \setminus B_l) < \frac{\epsilon}{2N_n}$.

We combine this with the fact that every open set can be approximated as a countable union of open intervals to get a $G_l^\epsilon \in \mathcal{G}$ for which $G_l^\epsilon \subseteq C_l^\epsilon$ and

$\mu(C_l^\epsilon \setminus G_l^\epsilon) < \frac{\epsilon}{2N_n}$.

Now all of this gives us a $g_n = \sum_{l=1}^{N_n} \gamma_l 1_{G_l^\epsilon} \in S$, where γ_l is chosen such that $|\beta_l - \gamma_l| \leq \frac{\epsilon}{N_n}$, so that

$$\begin{aligned} \int_{\mathbb{R}} |f_n - g_n|^2 d\mu = \int_{\mathbb{R}} |\sum_{l=1}^{N_n} \beta_l 1_{B_l} - \gamma_l 1_{G_l^\epsilon}|^2 d\mu \leq \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} |\beta_l 1_{B_l} - \gamma_l 1_{G_l^\epsilon}| \Big)^2 d\mu \\ \leq \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} |\beta_l||1_{B_l} - 1_{G_l^\epsilon}| + |1_{G_l^\epsilon}||\beta_l - \gamma_l| \Big)^2 d\mu. \end{aligned}$$

Now with $M_n = \max_{l=1,...,N_n} |\beta_l|$ we get that

$$\begin{aligned}
\int_{\mathbb{R}} |f_n - g_n|^2 d\mu \leq & \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} |\beta_l||1_{B_l} - 1_{G_l^\epsilon}| + |1_{G_l^\epsilon}||\beta_l - \gamma_l| \Big)^2 d\mu \\
\leq & \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} M_n|1_{B_l} - 1_{G_l^\epsilon}| + |1_{G_l^\epsilon}||\beta_l - \gamma_l| \Big)^2 d\mu \\
\leq & \int_{\mathbb{R}} \Big(\epsilon + \sum_{l=1}^{N_n} M_n|1_{B_l} - 1_{G_l^\epsilon}| \Big)^2 d\mu \\
= & \epsilon^2 + 2\epsilon \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} M_n|1_{B_l} - 1_{G_l^\epsilon}| \Big) d\mu + \int_{\mathbb{R}} \Big(\sum_{l=1}^{N_n} M_n|1_{B_l} - 1_{G_l^\epsilon}| \Big)^2 d\mu \\
\leq & \epsilon^2 + 2\epsilon M_n \epsilon + M_n^2 \Big(N_n \frac{\epsilon}{N_n} \Big)^2 = \epsilon^2 + 2\epsilon M_n \epsilon + M_n^2 \epsilon^2.
\end{aligned}$$

This in turn implies that S is dense in $L_2(\mathbb{R}, \mu)$.

□

Next we recall that a Hilbert space is separable if and only if there exists a countable orthonormal basis. When we equip $\mathscr{H} := L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ with the inner product $\langle f, g \rangle_{\mathscr{H}} := \int_{\mathbb{R}} fg d\mu$, for $f, g \in \mathscr{H}$, we get a Hilbert space. This with the fact that $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is separable gives us that there exists a countable orthonormal basis $(D_j)_{j \in J}$.

Let $f_n \in L_2(\mathbb{R}^n, \mu^n)$ and $F = g(X_1) \in L_2(\Omega)$, where $(X_t)_{t \in [0,1]}$ is the Lévy process, which gives us $g \in L_2(\mathbb{R}, \mathbb{P}_{X_1})$ and $E|g(X_1)|^2 < \infty$. Because of [**2**] (page 8615) we can let $f_n((t_1, x_1), ..., (t_n, x_n)) = \bar{f}_n(x_1, ..., x_n)$. We also assume that $\bar{f}_n$ is symmetric. This means that $g(X_1) = \sum_{n=0}^{\infty} I_n(\bar{f}_n(x_1, ..., x_n) 1_{(0,1]}^{\otimes n})$. We also note that $(D_{j_1} \otimes \cdots \otimes D_{j_n})(x_1, ..., x_n) = D_{j_1}(x_1) \cdots D_{j_n}(x_n)$, where $j_1, ..., j_n \in J$ and $(D_{j_1} \otimes \cdots \otimes D_{j_n})_{j_1, ..., j_n \in J}$ is a countable orthonormal basis of $L_2(\mathbb{R}^n, \mu^n)$. We also set $\alpha_{j_1, ..., j_n} := \int_{\mathbb{R}^n} (\bar{f}_n D_{j_1} \cdots D_{j_n}) d\mu^n(x_1, ..., x_n)$. Now we can formulate the decomposition we want to prove in this section.

THEOREM 5.2. *For $F = h(X_1) \in L_2(\mathcal{F}^X)$ we have an orthogonal decomposition of the form*

$$\begin{aligned}
F = & EF + \sum_{n=1}^{\infty} \sum_{j_1, ..., j_n \in J^n} \alpha_{j_1, ..., j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \\
= & EF + \sum_{n=1}^{\infty} \sum_{j_1 \leq ... \leq j_n \in J^n} \kappa_{j_1, ..., j_n} \alpha_{j_1, ..., j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}),
\end{aligned}$$

where $\kappa_{j_1,...,j_n}$ is the number of different orderings of $j_1, j_2, ..., j_n$.

Proof. Using the orthonormal basis we get that

$$\bar{f}_n \underset{L_2(\mathbb{R}^n,\mu^n)}{=} \sum_{(j_1,...,j_n)\in J^n} (D_{j_1} \otimes \cdots \otimes D_{j_n}) \int_{\mathbb{R}^n} (\bar{f}_n D_{j_1} \cdots D_{j_n}) d\mu^n(x_1, ..., x_n).$$

Now we want to show that

$$I_n(\bar{f}_n 1_{(0,1]}^{\otimes n}) \underset{L_2}{=} \sum_{(j_1,...,j_n)\in J^n} \alpha_{j_1,...,j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}),$$

where $\alpha_{j_1,...,j_n} = \int_{\mathbb{R}^n} (\bar{f}_n D_{j_1} \cdots D_{j_n}) d\mu^n(x_1, ..., x_n)$.

Since $\bar{f}_n$ are symmetric, we get for a permutation $\pi \in \mathcal{S}_n$ that

$$\begin{aligned}
\alpha_{j_1,...,j_n} &= \int_{\mathbb{R}^n} (\bar{f}_n(x_1, ..., x_n) D_{j_1}(x_1) \cdots D_{j_n}(x_n)) d\mu^n(x_1, ..., x_n) \\
&= \int_{\mathbb{R}^n} (\bar{f}_n(x_{\pi^{-1}(1)}, ..., x_{\pi^{-1}(n)}) D_{j_1}(x_{\pi^{-1}(1)}) \cdots D_{j_n}(x_{\pi^{-1}(n)})) d\mu^n(x_1, ..., x_n) \\
&= \int_{\mathbb{R}^n} (\bar{f}_n(x_1, ..., x_n) D_{j_1}(x_{\pi^{-1}(1)}) \cdots D_{j_n}(x_{\pi^{-1}(n)})) d\mu^n(x_1, ..., x_n) \\
&= \int_{\mathbb{R}^n} (\bar{f}_n(x_1, ..., x_n) D_{j_{\pi(1)}}(x_1) \cdots D_{j_{\pi(n)}}(x_n)) d\mu^n(x_1, ..., x_n) \\
&= \alpha_{j_{\pi(1)},...,j_{\pi(n)}},
\end{aligned}$$

which means that $\alpha_{j_1,...,j_n}$ is symmetric.

We consider the case $n = 1$ and $J = \mathbb{N}$. By using the finite additivity of I_1 we get that

$$\begin{aligned}
\sum_{j\in J} \alpha_j I_1(D_j \otimes 1_{(0,1]}) &= L_2 - \lim_{L\to\infty} \sum_{j=1}^{L} \alpha_j I_1(D_j \otimes 1_{(0,1]}) \\
&= L_2 - \lim_{L\to\infty} I_1\Big(\sum_{j=1}^{L} \alpha_j D_j \otimes 1_{(0,1]} \Big).
\end{aligned}$$

From the fact that $\sum_{j=1}^{L}\alpha_j D_j \underset{L\to\infty}{\longrightarrow} \sum_{j\in J}\alpha_j D_j = f_1$ in $L_2(\mathbb{R}\times(0,1],\mu\otimes\lambda)$, we get that

$$\sum_{j\in J}\alpha_j I_1(D_j\otimes 1_{(0,1]}) = I_1(f_1 1_{(0,1]}).$$

Now we investigate the case $n\in\mathbb{N}$. Here we assume that $j_1\le ...\le j_n$ and $k_1\le ...\le k_n$ are different, i.e. there exists $l\in\{1,...,n\}$ such that $j_l\ne k_l$. Now by applying the polarization formula, which states that $ab=\frac{1}{4}[(a+b)^2-(a-b)^2]$ for $a,b\in\mathbb{R}$, and the isometry from Section 4, which gives us $\|I_n(\tilde{f}_n)\|^2_{L_2(\mathcal{F}^X)} = \sqrt{n!}\|\tilde{f}_n\|^2_{L_2^n}$, we get for symmetric functions $\tilde{f},\tilde{g}\in L_2(\mathbb{R}^n,\mu^n)$ that

$$\begin{aligned}
EI_n(\tilde{f}_n)I_n(\tilde{g}_n) =& \frac{1}{4}E\Big[[I_n(\tilde{f}_n)+I_n(\tilde{g}_n)]^2-[I_n(\tilde{f}_n)-I_n(\tilde{g}_n)]^2\Big]\\
=&\frac{1}{4}\Big[E[I_n(\tilde{f}_n+\tilde{g}_n)]^2-E[I_n(\tilde{f}_n-\tilde{g}_n)]^2\Big]\\
=&\frac{1}{4}n!\Big[\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}(\tilde{f}_n+\tilde{g}_n)^2 d\mu^n(x_1,...,x_n)\\
&-\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}(\tilde{f}_n-\tilde{g}_n)^2 d\mu^n(x_1,...,x_n)\Big]\\
=&\frac{n!}{4}\Big[\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}(\tilde{f}_n+\tilde{g}_n)^2-(\tilde{f}_n-\tilde{g}_n)^2 d\mu^n(x_1,...,x_n)\Big]\\
=&\frac{n!}{4}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}4\tilde{f}_n\tilde{g}_n d\mu^n(x_1,...,x_n)\\
=&n!\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\tilde{f}_n\tilde{g}_n d\mu^n(x_1,...,x_n).
\end{aligned}$$

By using this property, we get that

$$\begin{aligned}
&EI_n\Big(D_{j_1}\otimes\cdots\otimes D_{j_n}\otimes 1_{(0,1]}^{\otimes n}\Big)I_n\Big(D_{k_1}\otimes\cdots\otimes D_{k_n}\otimes 1_{(0,1]}^{\otimes n}\Big)\\
=&EI_n\Big((D_{j_1}\otimes\cdots\otimes D_{j_n})^s\otimes 1_{(0,1]}^{\otimes n}\Big)I_n\Big((D_{k_1}\otimes\cdots\otimes D_{k_n})^s\otimes 1_{(0,1]}^{\otimes n}\Big)\\
=&n!\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}(D_{j_1}\otimes\cdots\otimes D_{j_n})^s(D_{k_1}\otimes\cdots\otimes D_{k_n})^s d\mu^n(x_1,...,x_n)\\
=&n!\frac{1}{n!}\frac{1}{n!}\sum_{\pi\in\mathcal{S}^n}\sum_{\sigma\in\mathcal{S}^n}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}D_{j_{\pi(1)}}\cdots D_{j_{\pi(n)}}D_{k_{\sigma(1)}}\cdots D_{k_{\sigma(n)}}d\mu^n(x_1,...,x_n),
\end{aligned}$$

in which

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} D_{j_{\pi(1)}} \cdots D_{j_{\pi(n)}} D_{k_{\sigma(1)}} \cdots D_{k_{\sigma(n)}} d\mu^n(x_1, ..., x_n) = \prod_{l=1}^{n} \int_{\mathbb{R}} D_{j_{\pi(l)}} D_{k_{\sigma(l)}} d\mu.$$

Here we note that when $j_{\pi(l)} \neq k_{\sigma(l)}$, we have that $\int_{\mathbb{R}} D_{j_{\pi(l)}} D_{k_{\sigma(l)}} d\mu = 0$. Since $j_1 \leq ... \leq j_n$ and $k_1 \leq ... \leq k_n$ are different we get that at least one factor in $\prod_{l=1}^{n} \int_{\mathbb{R}} D_{j_{\pi(l)}} D_{k_{\sigma(l)}} d\mu$ is equal to 0. This gives us that

$$EI_n\Big(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}\Big) I_n\Big(D_{k_1} \otimes \cdots \otimes D_{k_n} \otimes 1_{(0,1]}^{\otimes n}\Big) = 0.$$

From here we get that

$$\begin{aligned} I_n(\bar{f}_n 1_{(0,1]}^{\otimes n}) \underset{L_2}{=} & I_n\Big(\sum_{(j_1,...,j_n) \in J^n} (D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \\ & \times \int_{\mathbb{R}^n} (\bar{f}_n D_{j_1} \cdots D_{j_n}) d\mu^n(x_1, ..., x_n)\Big) \\ = & I_n\Big(\sum_{(j_1,...,j_n) \in J^n} (D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \alpha_{j_1,...,j_n}\Big) \\ = & \sum_{(j_1,...,j_n) \in J^n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \alpha_{j_1,...,j_n} \\ = & \sum_{j_1 \leq ... \leq j_n} \kappa_{j_1,...,j_n} \alpha_{j_1,...,j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \end{aligned}$$

where $\kappa_{j_1,...,j_n}$ is the number of different orderings.

This decomposition together with Theorem 4.6 gives us that for $F \in L_2(\mathcal{F}^X)$ we have an orthogonal decomposition of the form

$$\begin{aligned} F = & EF + \sum_{n=1}^{\infty} \sum_{j_1,...,j_n \in J^n} \alpha_{j_1,...,j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \\ = & EF + \sum_{n=1}^{\infty} \sum_{j_1 \leq ... \leq j_n \in J^n} \kappa_{j_1,...,j_n} \alpha_{j_1,...,j_n} I_n(D_{j_1} \otimes \cdots \otimes D_{j_n} \otimes 1_{(0,1]}^{\otimes n}) \end{aligned}$$

□

6. Connections to compensated-covariation stable families of martingales

Di Tella and Engelbert show in their article [**5**], that a version of the chaotic representation property, formulated using iterated integrals, applies on certain families of square integrable martingales. They introduce the notion of compensated-covariation stability of these families and use it as a requirement for the families they define the CRP on. Their main theorem ([**5**], Theorem 5.8, page 20) is a more general result than the chaotic representation property for the Teugels martingales, explored by Nualart and Schoutens. After the proof of the main theorem, they use it to prove another theorem focused on Lévy processes ([**5**], Theorem 6.8, page 25). This application has a close resemblance to other decompositions explored in this thesis.

6.1. Compensated-covariation stable families. In this section, just like in the article of Di Tella and Engelbert, we are going to assume a finite time interval $[0, T]$ for the square integrable martingales. We start by defining the compensated-covariation process of square integrable martingales starting at zero $X^{(\alpha)}$, $X^{(\beta)} \in \{X^{(\alpha)}, \alpha \in A\}$:

$$X^{(\alpha,\beta)} := [X^{(\alpha)}, X^{(\beta)}] - \langle X^{(\alpha)}, X^{(\beta)} \rangle.$$

Now we can define the compensated-covariation stability of a family of square integrable martingales ([**5**], Definition 4.1, page 12).

Definition 6.1. A family of square integrable martingales $\mathcal{X} := \{X^{(\alpha)}, \alpha \in A\}$ is called compensated-covariation stable given that for all $\alpha, \beta \in A$ the compensated-covariation process $X^{(\alpha,\beta)}$ belongs to $\mathcal{X}$.

We also note that for $\alpha_1, ..., \alpha_m \in A$, with $m \geq 0$, the process $X^{(\alpha_1,...,\alpha_m)}$ is defined recursively by

$$X^{(\alpha_1,...,\alpha_m)} := [X^{(\alpha_1,...,\alpha_{m-1})}, X^{(\alpha_m)}] - \langle X^{(\alpha_1,...,\alpha_{m-1})}, X^{(\alpha_m)} \rangle.$$

Also, if $\mathcal{X}$ is a compensated-covariation stable family, then $X^{(\alpha_1,...,\alpha_m)} \in \mathcal{X}$ for every $\alpha_1, ..., \alpha_m \in A$.

We define the family $\mathcal{K}$ for $\mathcal{X} = \{X^{(\alpha)}, \alpha \in A\}$ by

$$\mathcal{K} := \Big\{ \prod_{i=1}^{m} X_{t_i}^{(\alpha_i)}, \alpha_i \in A, t_i \in [0, T], i = 1, ..., m; m \geq 2 \Big\}.$$

Now we define what it means for the CRP to hold for a compensated-covariation stable family ([**5**], Definition 3.6, page 8).

Definition 6.2. The chaotic representation property (CRP) holds for a compensated-covariation stable family $\mathcal{X} = \{X^{(\alpha)}, \alpha \in A\}$ on the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ if the linear space of terminal variables of iterated integrals from the space of iterated integrals generated by $\mathcal{X}$ is equal to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Now we are prepared to formulate the main result of Di Tella and Engelbert ([**5**], Theorem 5.8, page 20):

Theorem 6.3. *Let $\mathcal{X} = \{X^{(\alpha)}, \alpha \in A\}$ be a compensated-covariation stable family of square integrable martingales. If $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic for every $\alpha, \beta \in A$ and the family $\mathcal{K}$ is a total family in $L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, then the CRP holds for $\mathcal{X}$.*

6.2. Application on Lévy processes. We apply Theorem 6.3 on Lévy processes to see the connection between this theorem and the other decompositions introduced in this thesis.

From the characteristics of X, given by the Lévy-Itô decomposition, we derive a measure

$$\mu := \sigma^2 \delta_0 + \nu,$$

where δ_0 denotes the Dirac measure in the origin. Now we can introduce the decomposition for Lévy processes derived from Theorem 6.3 ([**5**], Theorem 6.8, page 25) using martingales $(X^{(f_1)}, ..., X^{(f_n)})$, $n \geq 1$, introduced in [**5**] ((39), page 22).

Theorem 6.4. *Let X be a Lévy process with the characteristics (β, σ^2, ν) and $\mathcal{I} = \{f_n, n \geq 1\}$ be a complete orthogonal system in $L^2(\mu)$. Then the associated family $\mathcal{X} = \mathcal{X}_{\mathcal{I}}$ has the CRP on $L^2(\Omega, \mathcal{F}^X, \mathbb{P})$ and the following decomposition holds:*

$$L^2(\Omega, \mathcal{F}^X, \mathbb{P}) = \mathbb{R} \bigoplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1, ..., j_n) \in \mathbb{N}^n} \mathscr{I}_n(f_{j_1}, ..., f_{j_n}),$$

where $\mathscr{I}_n(f_{j_1}, ..., f_{j_n})$ denotes the linear space of n-fold iterated integral for $f_1, ..., f_n \in \mathcal{I}$, with respect to $(X^{(f_1)}, ..., X^{(f_n)})$, $n \geq 1$.

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